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Third-order QCD corrections to the charged-current structure function F_3

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Abstract

We compute the coefficient function for the charge-averaged $W^{\pm}$-exchange structure function F_3 in deep-inelastic scattering (DIS) to the third order in massless perturbative QCD. Our new three-loop contribution to this quantity forms, at not too small values of the Bjorken variable x, the dominant part of the next-to-next-to-next-to-leading order corrections. It thus facilitates improved determinations of the strong coupling α_s and of $1/Q^2$ power corrections from scaling violations measured in neutrino-nucleon DIS. The expansion of F_3 in powers of α_s is stable at all values of x relevant to measurements at high scales Q^2. At small x the third-order coefficient function is dominated by diagrams with the colour structure $d^{abc}d_{abc}$ not present at lower orders. At large x the coefficient function for F_3 is identical to that of F_1 up to terms vanishing for $x \to 1$.

1 Introduction

Structure functions in inclusive deep-inelastic lepton-nucleon scattering (DIS) are among the most important observables probing QCD, the theory of the strong interaction. Present data on these quantities can provide accurate information about the quark and gluon momentum distributions in the proton down momentum fractions $x \approx 10^{-4}$, see Ref. [1]. These distributions, in turn, are indispensable ingredients for the analysis of all hard (high scale/mass) scattering processes at proton–(anti-)proton colliders, cf. Refs. [2, 3], such as TEVATRON and the LHC which will form the high-energy frontier of particle physics for at least the next fifteen years. Structure functions are also very well suited for precision determinations of the strong coupling constant α_s, one of the fundamental parameters of our description of nature, in the framework of perturbative QCD.

Inclusive quantities such as structure functions are the observables best accessible to field-theoretic calculations to 'high orders' (today: two loops and beyond) in the expansion in powers of the coupling constant. Indeed, the second-order partonic cross sections (coefficient functions) for inclusive DIS were completed as early as 1991/2 [4–6], while corresponding quantities for jet shapes in e^+e^- collisions have been presented only very recently [7–9]. At large x (with hindsight: $x > 10^{-2}$) the former quantities are the dominant part of the next-to-next-to-leading order (NNLO) contributions in renormalization-group improved perturbation theory. They are not sufficient, though, for NNLO analyses over the full x-range opened up by HERA and required for the LHC.

The three-loop corrections for inclusive DIS – required to complete the NNLO framework and to provide the dominant next-to-next-to-next-to-leading order (N^3LO) corrections at large-x enabling a perturbative accuracy of 1% for α_s determinations – have been the subject of a long-term research program, which started from sum rules [10,11], and proceeded via low integer Mellin moments [12–14] to the computation of the exact expressions for all NNLO splitting functions [15, 16] and the third-order coefficient functions for the structure functions F_L and F_2 [17, 18]. In the present article, we extend the latter results to the most important structure function not covered by Ref. [18], the vector–axial-vector interference structure function F_3 in charged-current, specifically $(W^+ + W^-)$-exchange DIS measured with high precision in neutrino-nucleon DIS [1].

The remainder of this article is organized as follows: In Section 2 we briefly recall the general formalism for the calculation of the coefficient function for F_3 and discuss some aspects specific to the present computation. We then write down the coefficient functions for $F_3^{W^+ + W^-}$ in Section 3. The second- and third-order quantities are presented via compact and accurate parametrizations. We also address the behaviour of the third-order coefficient functions at large and small x, stressing the importance of the $d^{abc}d_{abc}$ contribution not present at lower orders. This and other numerical effects are then illustrated in Section 4 where we assess the size of the higher-order corrections. Finally we summarize our results in Section 5 and close with a brief outlook on possible future improvements on the present accuracy. Appendix A contains the very lengthy exact expression of our new third-order coefficient function, and Appendix B complements the discussion of F_3 in Section 3 by providing the subleading large-x logarithms for the case of F_2. It turns out that the quark coefficient functions for F_3 and $2xF_1 \equiv F_2 - F_L$ in charged-current DIS are the same in the large-x limit, specifically that $C_3(x) = C_1(x) + O(1-x)$ holds to (at least) order α_s^3.

2 Formalism and calculation

We are interested in unpolarized charged-current deep-inelastic scattering (DIS), i.e., the reaction

$$l(k) + \mathrm{nucl}(p) \;\to\; l'(k') + X \;, \tag{2.1}$$

where l, l' and 'nucl' denote a charged lepton, its (anti-)neutrino (in this or the opposite order) and a nucleon with respective momenta k, k' and p. X stands for all hadronic states allowed by quantum number conservation. The inclusive cross section for the process (2.1) can be written as $d\sigma \sim L^{\mu\nu}W_{\mu\nu}$ in terms of leptonic and hadronic tensors $L_{\mu\nu}$ and $W_{\mu\nu}$. The former tensor is well documented in the literature for both pure electromagnetic and electroweak gauge-boson exchange, see, e.g., Ref. [1]. Thus there is no need to consider it here.

Our focus is on the (spin-averaged) hadronic tensor $W_{\mu\nu}$ which can be expanded to define the structure functions $F_{2,\,3\,,L}$, viz

$$\begin{aligned} W_{\mu\nu}(p,q) &= \frac{1}{4\pi}\int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z}\,\langle \mathrm{nucl,p}|J^\dagger_\mu(z)J_\nu(0)|\,\mathrm{nucl,p}\rangle \\ &= e_{\mu\nu}\,\frac{1}{2x}\,F_L(x,Q^2) + d_{\mu\nu}\,\frac{1}{2x}\,F_2(x,Q^2) + \mathrm{i}\,\varepsilon_{\mu\nu\alpha\beta}\,\frac{p^\alpha q^\beta}{p\cdot q}\,F_3(x,Q^2)\;. \end{aligned} \tag{2.2}$$

Here q denotes the momentum transferred by the W-boson, with $Q^2 \equiv -q^2 > 0$, and J_μ represents the weak current. The Bjorken variable is defined as $x = Q^2/(2\,p\cdot q)$ with $0 < x \leq 1$. We do not consider the functions $F_{2,\,L}$ corresponding to the symmetric $e_{\mu\nu}$ and $d_{\mu\nu}$ in this article, see Ref. [18]. Instead we address the structure function F_3 associated to the totally antisymmetric tensor $\varepsilon_{\mu\nu\alpha\beta}$ which arises from the vector/axial-vector interference of the two $V-A$ currents. Specifically we are interested here in the charged-averaged quantity $F_3^{W^+ + W^-}$ which has been accurately measured in neutrino-nucleon DIS off (almost) isoscalar targets [1] (recall that $F_3^{W^+ - W^-}$ is small in this case, being proportional to a flavour asymmetry in the quark sea).

As in the previous calculations in Refs. [15–20], we derive analytic expressions for the Mellin moments of the perturbative contribution to structure functions, which for the present case are defined as

$$F_3^{W^+ + W^-}(N,Q^2) \;=\; \int_0^1 dx\, x^{N-1}\, F_3^{W^+ + W^-}(x,Q^2)\;. \tag{2.3}$$

To this end we make use of the optical theorem relating the hadronic tensor in Eq. (2.2) to the imaginary part of the forward Compton amplitude $T_{\mu\nu}$ of virtual gauge-boson–nucleon scattering. This amplitude is expressed in terms of a time-ordered product of two local currents to which standard perturbation theory applies,

$$T_{\mu\nu}(p,q) \;=\; \mathrm{i}\int d^4z\, \mathrm{e}^{\mathrm{i}q\cdot z}\langle \mathrm{nucl},\,p|\,T\left(J^\dagger_\mu(z)J_\nu(0)\right)|\mathrm{nucl},\,p\rangle\;. \tag{2.4}$$

In $D = 4-2\varepsilon$ dimensions the structure function F_3 is obtained from $T_{\mu\nu}$ using the projection

$$P_3^{\mu\nu} \;=\; -\mathrm{i}\,\frac{1}{(1-2\varepsilon)(1-\varepsilon)}\,\varepsilon^{\mu\nu\alpha\beta}\,\frac{p_\alpha q_\beta}{2\,p\cdot q}\;. \tag{2.5}$$

As discussed in detail in the literature, see, e.g., Refs. [19, 20], the operator-product expansion (OPE) can be applied to the product of currents in Eq. (2.4) together with a dispersion relation. For the structure function considered here this procedure finally yields

$$\frac{1-(-1)^N}{2}\,F_3^{W^+ + W^-}(N,Q^2) \;=\; C_3^-\left(N,\frac{Q^2}{\mu^2},\alpha_s\right) A_{v,\,\mathrm{nucl}}(N,\mu^2)\;. \tag{2.6}$$

One thus obtains the odd-N moments (2.3) which uniquely fix all moments, and hence the complete x-dependence, by analytic continuation. The notation C_3^- for the coefficient function of $F_3^{W^+ + W^-}$ reflects this basis of odd moments. $A_{v,\,\mathrm{nucl}}(N,\mu^2)$ in Eq. (2.6) is the nucleon matrix elements of the combination O_v of local spin-N twist-two quark operators corresponding to the total valence quark distribution, renormalized at the scale μ. $1/Q^2$ corrections have been disregarded in Eq. (2.6).

As usual, we perform the renormalization in the modified [19] minimal subtraction scheme [21] in dimensional regularization. Hence also the renormalized coupling α_s in Eq.(2.6) refers to $D = 4-2\varepsilon$ dimensions, i.e., its scale dependence is given by

$$\frac{d}{d\ln\mu^2}\frac{\alpha_s}{4\pi} \;\equiv\; \frac{d\,a_s}{d\ln\mu^2} \;=\; -\varepsilon\,a_s - \beta_0\,a_s^2 - \beta_1\,a_s^3 - \beta_2\,a_s^4 - \beta_3\,a_s^5 - \ldots \tag{2.7}$$

where β_n denote the coefficients of the usual four-dimensional $\overline{\mathrm{MS}}$ beta function of QCD. The renormalization to the third order requires the coefficients up to β_2 [22, 23], and the consistent application of the resulting three-loop (N^{3}LO) coefficient functions in data analyses even β_3, the highest contribution to Eq. (2.7) computed so far [24, 25].

There is one delicate issue entering the operator renormalization which deserves special attention. This is the presence of γ_5 due to the axial-vector coupling of the W-boson. As before in Refs. [11, 14, 15, 20] we employ the so-called Larin scheme [11, 26] which, at the present level, is equivalent to, but computationally more convenient than the original prescription [27, 28] for consistently dealing with γ_5 in dimensional regularization. See also the discussion in Ref. [6]. Therefore we need to perform a special renormalization, based on relating vector and axial-vector currents, in order to restore the axial Ward-identities,

$$O_v \;\to\; Z_5\,Z_A\,O_v\;. \tag{2.8}$$

The constant Z_A for the axial renormalization and the finite renormalization Z_5 due to this treatment of γ_5 in the $\overline{\mathrm{MS}}$-scheme are known to three loops [11, 26]. The explicit expressions read

$$\begin{aligned}
Z_A \;=\;& 1 + a_s^2\frac{1}{\varepsilon}\Big[\frac{22}{3}C_AC_F - \frac{4}{3}C_Fn_f\Big] + a_s^3\Big[\frac{1}{\varepsilon^2}\Big(-\frac{484}{27}C_FC_A^2 + \frac{176}{27}C_FC_An_f - \frac{16}{27}C_Fn_f^2\Big)\\
&+\frac{1}{\varepsilon}\Big(-\frac{308}{9}C_F^2C_A + \frac{3578}{81}C_FC_A^2 + \frac{32}{9}C_F^2n_f - \frac{832}{81}C_FC_An_f + \frac{8}{81}C_Fn_f^2\Big)\Big]\;,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
Z_5 \;=\;& 1 + a_s\Big[-4C_F - 10\varepsilon C_F + \varepsilon^2C_F(-22+2\zeta_2)\Big] + a_s^2\Big[22C_F^2 - \frac{107}{9}C_FC_A + \frac{2}{9}C_Fn_f\\
&+\varepsilon\Big(C_F^2(132-48\zeta_3) + C_FC_A\Big(-\frac{7229}{54}+48\zeta_3\Big) + \frac{331}{27}C_Fn_f\Big)\Big]
\end{aligned}$$

$$+a_s^3\Big[C_F^3\Big(-\frac{370}{3}+96\zeta_3\Big)+C_F^2C_A\Big(\frac{5834}{27}-160\zeta_3\Big)+C_FC_A^2\Big(-\frac{2147}{27}+56\zeta_3\Big)$$
$$+C_F^2n_f\Big(-\frac{62}{27}-\frac{32}{3}\zeta_3\Big)+C_AC_Fn_f\Big(\frac{356}{81}+\frac{32}{3}\zeta_3\Big)+\frac{52}{81}C_Fn_f^2\Big]\,. \tag{2.10}$$

Both these expansions are given in terms of the renormalized coupling (2.7), and Z_5 is expressed to exactly the depth in ε implemented in the present calculation (see below Eq. (2.21) for a comment on the positive powers of ε absent in Ref. [11]).

We are now ready to briefly address the standard part of the renormalization of the spin-N operators $O_{\rm v}$ entering Eq. (2.6),

$$O_{\rm v}^{\rm ren} \;=\; Z_{\rm v}\,O_{\rm v}^{\rm bare}\,, \tag{2.11}$$

where (as in some equations below) we have suppressed the dependence on N, and exploited the fact that $O_{\rm v}$ does not mix with other operators under renomalization. The corresponding (non-singlet) anomalous dimensions $\gamma_{\rm v}$, governing the scale dependence of the operators $O_{\rm v}$ via

$$\frac{d}{d\ln\mu^2}\,O_{\rm v}^{\rm ren} \;=\; -\gamma_{\rm v}\,O_{\rm v}^{\rm ren}\;, \tag{2.12}$$

are connected to these renormalization constants in the standard way,

$$\gamma_{\rm v} \;=\; -\Big(\frac{d}{d\ln\mu^2}\,Z_{\rm v}\Big)\,Z_{\rm v}^{-1} \;=\; \sum_{l=0}^{\infty} a_s^{l+1}\gamma_{\rm v}^{(l)}\;. \tag{2.13}$$

The reader is referred to Ref. [15] for the relation of the expansion coefficients $\gamma_{\rm v}^{(l)}$ to those of other non-singlet combinations. In terms of these quantities, the $\overline{\rm MS}$ renormalization factor $Z_{\rm v}$ in Eq. (2.11) is given by the Laurent series

$$\begin{aligned}
Z_{\rm v} \;=\;& 1+a_{\rm s}\frac{1}{\varepsilon}\gamma_{\rm v}^{(0)}+a_{\rm s}^2\left[\frac{1}{2\varepsilon^2}\left\{\left(\gamma_{\rm v}^{(0)}-\beta_0\right)\gamma_{\rm v}^{(0)}\right\}+\frac{1}{2\varepsilon}\gamma_{\rm v}^{(1)}\right]\\
&+a_{\rm s}^3\left[\frac{1}{6\varepsilon^3}\left\{\left(\gamma_{\rm v}^{(0)}-2\beta_0\right)\left(\gamma_{\rm v}^{(0)}-\beta_0\right)\gamma_{\rm v}^{(0)}\right\}\right.\\
&\left.+\frac{1}{6\varepsilon^2}\left\{3\gamma_{\rm v}^{(0)}\gamma_{\rm v}^{(1)}-2\beta_0\gamma_{\rm v}^{(1)}-2\beta_1\gamma_{\rm v}^{(0)}\right\}+\frac{1}{3\varepsilon}\gamma_{\rm v}^{(2)}\right]\,.
\end{aligned} \tag{2.14}$$

Of course, the perturbative calculation of the anomalous dimension (2.12) and the coefficient function C_3^- cannot proceed via Eq. (2.6) including the non-perturbative nucleon states $|{\rm nucl,p}\rangle$. However, as the OPE represents an operator relation, the calculation can be performed using quark states instead. Hence we apply the Lorentz projector (2.5) to the forward virtual-W–quark Compton amplitude $\mathcal{T}_{\mu\nu}$ analogous to Eq. (2.4) and obtain the perturbative quantities $\mathcal{T}_3$ which can be expanded in powers of the renormalized coupling $a_{\rm s}\equiv\alpha_{\rm s}/(4\pi)$ at the scale $\mu^2=Q^2$,

$$\mathcal{T}_3(N) \;=\; \Big(\mathcal{T}_3^{(0)}(N)+a_s\,\mathcal{T}_3^{(1)}(N)+a_s^2\,\mathcal{T}_3^{(2)}(N)+a_s^3\,\mathcal{T}_3^{(3)}(N)+\ldots\Big)\,A_{\rm v}\;, \tag{2.15}$$

where the N-independent $A_{\rm v}$ denotes the quark operator matrix element. After performing the operator renormalization according to Eqs. (2.9) – (2.11) and (2.14), one thus arrives at explicit

expressions for the $\mathcal{T}_3^{(l)}$. The anomalous dimensions $\gamma_{\rm v}^{(l)}$ can then be read off from the poles in ε, while the coefficient function C_3^- is related to the finite terms in ε. During this procedure it is important know the expansion of the D-dimensional coefficient function

$$\mathcal{C}_3^-(\alpha_{\rm s},N) \;=\; \sum_{l=0}^{\infty} a_{\rm s}^l \Big(c_3^{(l)}(N) + \varepsilon a_3^{(l)}(N) + \varepsilon^2 b_3^{(l)}(N) + \ldots \Big) \tag{2.16}$$

to a sufficient depth (positive powers) in ε at lower orders. We normalize the $\alpha_{\rm s}^0$ contribution to Eq. (2.15) including $A_{\rm v}$, i.e.,

$$\mathcal{T}_{3,\pm}^{(0)}(N) \;=\; 1 \tag{2.17}$$

implies

$$c_{3,\pm}^{(0)}(N) \;=\; 1\,, \quad a_{3,\pm}^{(0)}(N) \;=\; b_{3,\pm}^{(0)}(N) \;=\; \ldots \;=\; 0\,. \tag{2.18}$$

In this normalization also the first-order expression is the same for $F_3^{W^+ + W^-}$ and $F_3^{W^+ - W^-}$. Again suppressing the N-dependence it reads

$$\mathcal{T}_{3,\pm}^{(1)} \;=\; \frac{1}{\varepsilon}\gamma_{\rm qq}^{(0)} + c_{3,\pm}^{(1)} + \varepsilon a_{3,\pm}^{(1)} + \varepsilon^2 b_{3,\pm}^{(1)} + O(\varepsilon^3)\,. \tag{2.19}$$

The corresponding second- and third-order results are given by

$$\begin{aligned}\mathcal{T}_{3,-}^{(2)} \;=\;& \frac{1}{2\varepsilon^2}\Big\{\Big(\gamma_{\rm qq}^{(0)} - \beta_0\Big)\gamma_{\rm qq}^{(0)}\Big\} + \frac{1}{2\varepsilon}\Big\{\gamma_{\rm v}^{(1)} + 2\,c_{3,\pm}^{(1)}\gamma_{\rm qq}^{(0)}\Big\} \\ & + c_{3,-}^{(2)} + a_{3,\pm}^{(1)}\gamma_{\rm qq}^{(0)} + \varepsilon\Big\{a_{3,-}^{(2)} + b_{3,\pm}^{(1)}\gamma_{\rm qq}^{(0)}\Big\}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\mathcal{T}_{3,-}^{(3)} \;=\;& \frac{1}{6\varepsilon^3}\Big\{\Big(\gamma_{\rm qq}^{(0)} - 2\beta_0\Big)\Big(\gamma_{\rm qq}^{(0)} - \beta_0\Big)\gamma_{\rm qq}^{(0)}\Big\} \\ & + \frac{1}{6\varepsilon^2}\Big\{3\gamma_{\rm v}^{(1)}\gamma_{\rm qq}^{(0)} - 2\beta_0\gamma_{\rm v}^{(1)} - 2\beta_1\gamma_{\rm qq}^{(0)} + 3c_{3,\pm}^{(1)}\Big(\gamma_{\rm qq}^{(0)} - \beta_0\Big)\gamma_{\rm qq}^{(0)}\Big\} \\ & + \frac{1}{6\varepsilon}\Big\{2\gamma_{\rm v}^{(2)} + 3c_{3,\pm}^{(1)}\gamma_{\rm v}^{(1)} + 6c_{3,-}^{(2)}\gamma_{\rm qq}^{(0)} + 3a_{3,\pm}^{(1)}\Big(\gamma_{\rm qq}^{(0)} - \beta_0\Big)\gamma_{\rm qq}^{(0)}\Big\} \\ & + c_{3,-}^{(3)} + \frac{1}{2}a_{3,\pm}^{(1)}\gamma_{\rm v}^{(1)} + a_{3,-}^{(2)}\gamma_{\rm qq}^{(0)} + \frac{1}{2}b_{3,\pm}^{(1)}\Big(\gamma_{\rm qq}^{(0)} - \beta_0\Big)\gamma_{\rm qq}^{(0)}\,.\end{aligned} \tag{2.21}$$

Consequently the two- and three-loop coefficient functions can be read off from the ε-independent parts of $\mathcal{T}_{3,-}^{(2)}$ and $\mathcal{T}_{3,-}^{(3)}$ after subtracting the respective contributions due to the lower-order ε and ε^2 quantities in Eq. (2.16).

We are now in a position to give the comment announced above on the ε-terms in Eq. (2.10). These terms do not affect the extracted coefficient functions $c_{3,-}^{(2)}$ and $c_{3,-}^{(3)}$. Without them, however, the functions $a_{3,\pm}^{(1)}$, $b_{3,\pm}^{(1)}$ and $a_{3,-}^{(2)}$ would exhibit an unphysical behaviour of their N-independent terms (which should be the same as those for F_2), a feature irrelevant here but unwanted for more general applications such as the determination of time-like (fragmentation) coefficient functions via a suitable analytic continuation of the DIS results, cf. Refs. [29, 30]. In other words, Eq. (2.10) effectively restores the anticommutativity of γ_5 also for those ε and ε^2 contributions.

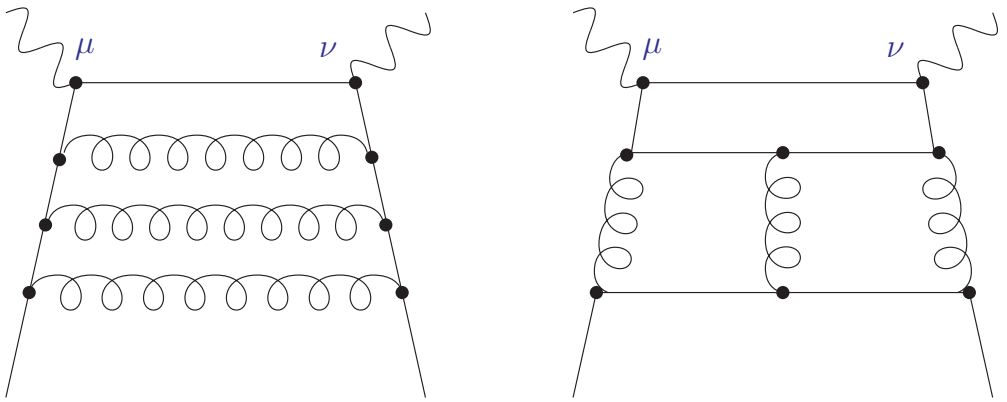
The actual computation of the Feynman diagrams for the contributions to Eq. (2.15) follows those of Refs. [15–18,31] in every respect, so we can be very brief here. The graphs have been generated automatically with the diagram generator QGRAF [32]. All further symbolic manipulations have been performed in FORM [33,34], using the SUMMER package [35] for the analytic evaluation of all nested sums. The calculation relied on a massive tabulation of intermediate integrals, and check of all intermediate and final results were performed for fixed values of N against the MINCER program [36,37] and the findings to Ref. [14]. The three-loop $1/\varepsilon$ terms of the present calculation have already been used to complete the set of non-singlet NNLO splitting functions [15].

There are two aspects which deserve special attention in the context of the present calculation. The first is the appearance of two functions g_1, g_2 in the final odd-N results for the coefficient functions (see Refs. [18] for very similar even-N functions in the results of F_2 and F_L) which fall outside the class of simple harmonic sums [35] sufficient at previous orders,

$$g_n(N) \;=\; N^n \left(5\zeta_5 - 2S_{-5} + 4S_{-2}\zeta_3 - 4S_{-2,-3} + 8S_{-2,-2,1} + 4S_{3,-2} - 4S_{4,1} + 2S_5\right) . \quad (2.22)$$

Note that the bracketed combination of harmonic sums vanishes as $1/N^2$ for $N \to \infty$, hence $g_n(N)$ do not contribute to the leading $\ln^k N$, $k = 1, \ldots, 6$, behaviour of the coefficient function $c^{(3)}_{3,-}(N)$. The x-space expressions corresponding to Eqs. (2.22) can be found at the end of Appendix A.

The second interesting point is the presence of a new colour structure, a contribution proportional to the higher SU(n_c) group invariant $d^{abc}d_{abc}$ in the three-loop coefficient function $c^{(3)}_{3,-}$. This new colour factor is related to Feynman diagrams of a particular topology concerning the fermion flow through the diagram. We distinguish the two cases (so-called flavour classes) fl_2 and fl_{02} depending on whether both W-bosons are attached to the open fermion line of the initial and final state quark (fl_2) or whether both gauge bosons are coupled to a closed fermion loop (fl_{02}). Sample diagrams for the two cases are displayed in Fig. 1. The impact of the new fl_{02} contributions with $d^{abc}d_{abc}$ on the coefficient function will be discussed in the next sections.



+ 348 other diagrams + 11 other diagrams

Figure 1: Typical Feynman diagrams for the contributions without (left) and with (right) $d^{abc}d_{abc}$ to the charged-current DIS coefficient function C_3^- up to the third order in the strong coupling α_s.

3 Results and discussion

We are now ready to present the charged-current coefficient function C_3^- to the third order in the strong coupling $a_s = \alpha_s/(4\pi)$, i.e., the coefficients $c_{3,-}^{(l\leq 3)}$ in

$$F_3^{W^+ + W^-} \;=\; C_3^- \otimes q_{\rm val} \;=\; \left(\delta(1-x) + a_s\, c_{3,-}^{(1)} + a_s^2\, c_{3,-}^{(2)} + a_s^3\, c_{3,-}^{(3)} + \ldots\right) \otimes q_{\rm val}\,. \tag{3.1}$$

Here $q_{\rm val}$ represents the total (flavour summed) valence quark distribution of the hadron, $q_{\rm val} = \sum_{i=1}^{n_f}(q_i - \bar{q}_i)$ where n_f is the number of effectively massless flavours. $\otimes$ denotes the Mellin convolution which turns into a simple multiplication in N-space. All results below will be given in the $\overline{\rm MS}$ scheme for the standard choice $\mu_r^2 = \mu_f^2 = Q^2$ of the renormalization and factorization scales. The complete expressions for the dependence on μ_r and μ_f to the third order can be found, for example, in Eqs. (2.16) – (2.18) of Ref. [38].

As discussed above, our calculation via the optical theorem and a dispersion relation directly determines the coefficient function (3.1) for all odd-integer moments N in terms of harmonic sums [35, 39, 40]. From these functions the x-space expressions can be reconstructed algebraically [20, 43] in terms of harmonic polylogarithms [41–43]. As in the case of $F_{2,L}$ in electromagnetic DIS presented before [17, 18], the exact third-order expressions are unpleasantly long in both N-space and x-space. We therefore refrain from writing down the former in this article, and defer the latter to Appendix A.

For the convenience of the reader we first recall the known results up to the second order. The first-order contribution to Eq. (3.1) is identical to that for the $W^+ - W^-$ case C_3^+ and given by [19]

$$\begin{aligned} c_{3,\pm}^{(1)}(x) \;&=\; C_F\{4\,\mathcal{D}_1 - 3\,\mathcal{D}_0 - (9 + 4\,\zeta_2)\,\delta(x_1) - 2\,(1+x)(L_1 - L_0) \\ &\quad - 4\,x_1^{-1} L_0 + 4 + 2\,x\} \end{aligned} \tag{3.2}$$

with $C_F = (n_c^2 - 1)/(2n_c) = 4/3$ and $n_c = C_A = 3$ in QCD. Here and below we use the abbreviations

$$x_1 \;=\; 1 - x\,, \quad L_0 \;=\; \ln x\,, \quad L_1 \;=\; \ln x_1\,, \quad \mathcal{D}_k \;=\; [\,x_1^{-1} L_1^k\,]_+\,. \tag{3.3}$$

As usual, the +-distributions are defined via

$$\int_0^1 dx\, a(x)_+ f(x) \;=\; \int_0^1 dx\, a(x)\,\{f(x) - f(1)\} \tag{3.4}$$

for regular functions $f(x)$. Convolutions with the distributions $\mathcal{D}_k$ in Eq. (3.3) can be written as

$$x[\mathcal{D}_k \otimes f](x) \;=\; \int_x^1 dy\, \frac{\ln^k(1-y)}{1-y} \left\{\frac{x}{y} f\left(\frac{x}{y}\right) - x f(x)\right\} + x f(x)\, \frac{\ln^{k+1}(1-x)}{k+1}\,. \tag{3.5}$$

Already at two loops the coefficient functions involve polylogarithms and Nielsen functions, hence it is convenient to have at one's disposal accurate parametrizations in terms of the elementary functions in Eq. (3.3). After inserting the QCD values of the colour factors C_F and C_A, the exact two-loop coefficient function (A.3) first obtained in Refs. [6, 20] can be represented by

$$
\begin{aligned}
c_{3,-}^{(2)}(x) \;\cong\;& 128/9\,\mathcal{D}_3 - 184/3\,\mathcal{D}_2 - 31.1052\,\mathcal{D}_1 + 188.641\,\mathcal{D}_0 - 338.572\,\delta(x_1) \\
& - 16.40\,L_1^3 + 78.46\,L_1^2 - 470.6\,L_1 - 149.75 - 693.2\,x + 0.218\,xL_0^4 \\
& + L_0L_1(33.62\,L_0 - 117.8\,L_1) - 49.30\,L_0 - 94/3\,L_0^2 - 104/27\,L_0^3 \\
+\;& n_f\,\{\,16/9\,\mathcal{D}_2 - 232/27\,\mathcal{D}_1 + 6.34888\,\mathcal{D}_0 + 46.8464\,\delta(x_1) + 0.066\,L_1^3 \\
& - 0.663\,L_1^2 + 24.86\,L_1 - 5.738 - 5.845\,x - 10.235\,x^2 - 0.190\,xL_0^3 \\
& + 4.265\,L_0L_1 + 20/9\,L_0(4 + L_0)\,\} \qquad (3.6)
\end{aligned}
$$

with an error well below 0.1% at all values of x. This expression is slightly less compact, but considerably more accurate than the (practically sufficient) previous parametrization in Ref. [44]. At this order the coefficient functions for the $W^+ + W^-$ and $W^+ - W^-$ cases are different. The two-loop coefficient function $c_{3,+}^{(2)}$ for the latter can be evaluated via Eq. (3.6) above and Eq. (2.9) in Ref. [45] for $\delta c_3^{(2)} \equiv c_{3,+}^{(2)} - c_{3,-}^{(2)}$.

The coefficients of the +-distributions $\mathcal{D}_k$ in Eq. (3.6) and Eq. (3.7) below are exact up to a truncation of the values of the Riemann ζ-function. Also exact are those coefficients of $L_0^k \equiv \ln^k x$ and $L_1^k \equiv \ln^k(1-x)$ given as fractions. Most of the remaining coefficients have been obtained by fits to the exact coefficient functions at $10^{-6} \leq x \leq 1 - 10^{-6}$. Finally the coefficients of $\delta(1-x)$ have been adjusted very slightly from their exact values using the lowest integer moments, thus fine-tuning the convolution with the quark distributions to maximal accuracy (cf. the discussion at the end of Section 4 of Ref. [16].

We now turn to our new three-loop results. Inserting, as in Eq. (3.6) above, the numerical QCD values of the n_f-independent SU(n_c) colour factors, the third-order contribution (A.4) to Eq. (3.1) can be approximated by

$$
\begin{aligned}
c_{3,-}^{(3)}(x) \;\cong\;& 512/27\,\mathcal{D}_5 - 5440/27\,\mathcal{D}_4 + 501.099\,\mathcal{D}_3 + 1171.54\,\mathcal{D}_2 - 7328.45\,\mathcal{D}_1 \\
& + 4442.76\,\mathcal{D}_0 - 9172.68\,\delta(x_1) - 512/27\,L_1^5 + 8896/27\,L_1^4 - 1396\,L_1^3 \\
& + 3990\,L_1^2 + 14363\,L_1 - 1853 - 5709\,x + xx_1(5600 - 1432\,x) \\
& - L_0L_1(4007 + 1312\,L_0) - 0.463\,xL_0^6 - 293.3\,L_0 - 1488\,L_0^2 - 496.95\,L_0^3 \\
& - 4036/81\,L_0^4 - 536/405\,L_0^5 \\
+\;& n_f\,\{\,640/81\,\mathcal{D}_4 - 6592/81\,\mathcal{D}_3 + 220.573\,\mathcal{D}_2 + 294.906\,\mathcal{D}_1 - 729.359\,\mathcal{D}_0 \\
& + 2575.46\,\delta(x_1) - 640/81\,L_1^4 + 32576/243\,L_1^3 - 660.7\,L_1^2 + 959.1\,L_1 \\
& + 516.1 - 465.2\,x + xx_1(635.3 + 310.4\,x) + 31.95\,x_1L_1^4 \\
& + L_0L_1\,(1496 + 270.1\,L_0 - 1191\,L_1) - 1.200\,xL_0^4 + 366.9\,L_0 + 305.32\,L_0^2 \\
& + 48512/729\,L_0^3 + 304/81\,L_0^4\,\} \\
+\;& n_f^2\,\{\,64/81\,\mathcal{D}_3 - 464/81\,\mathcal{D}_2 + 7.67505\,\mathcal{D}_1 + 1.00830\,\mathcal{D}_0 - 103.2602\,\delta(x_1) \\
& - 64/81\,L_1^3 + 992/81\,L_1^2 - 49.65\,L_1 + 11.32 - xx_1(44.52 + 11.05\,x) \\
& + 51.94\,x + 0.0647\,xL_0^4 - L_0L_1\,(39.99 + 5.103\,L_0 - 16.30\,L_1) - 16.00\,L_0 \\
& - 2848/243\,L_0^2 - 368/243\,L_0^3\,\}
\end{aligned}
$$

$$+\; fl_{02}\, n_f \{2.147\, L_1^2 - 24.57\, L_1 + 48.79 - x_1(242.4 - 150.7\, x) - L_0 L_1\, (81.70 + 9.412\, L_1) + xL_0\, (218.1 + 82.27\, L_0^2) - 477.0\, L_0 - 113.4\, L_0^2 + 17.26\, L_0^3 - 16/27\, L_0^5 \}\, x_1 \;. \tag{3.7}$$

Here the factor fl_{02} ($= 1$ for the numerical evaluation) indicates the $d^{abc}d_{abc}$ contribution entering at this order for the first time, cf. Ref. [11]. Also Eq. (3.7), first presented in Ref. [46], deviates by much less than one part in a thousand from the corresponding exact expression which we evaluated using a weight-five extension of the FORTRAN package [47] for the harmonic polylogarithms. Eqs. (3.6) and (3.7) can be readily transformed to Mellin space at complex values of N (using, e.g., the appendix of Ref. [40] for the moments of $\ln x \ln^2(1-x)$ etc) for use with N-space programs, such as QCD-PEGASUS [48], for the evolution of parton densities and structure functions.

The $x \to 1$ and $x \to 0$ end-point behaviour of the higher-order contributions to Eq. (3.1) is of special interest, both theoretically and phenomenologically. We first address the large-x limit. Here the leading terms of $c_{3,\pm}^{(n)}(x)$ are the soft-gluon +-distributions $\mathcal{D}_k$ with $k = 0, \ldots, 2n-1$. To order α_s^3 the corresponding coefficients are identical to those for the electromagnetic and charged-current structure functions F_2 which can be found in Eqs. (4.14) – (4.19) of Ref. [18].[1]

The terms with $\delta(1-x)$ (arising from virtual corrections and soft-gluon contributions) are the same for all four charged-current coefficient functions $c_{2,3,\pm}^{(3)}$ and differ from the corresponding photon-exchange quantity of Ref. [18] only by the obvious absence of contributions from the fl_{11} flavour classes (where the photons couple to different quark loops) in W-exchange processes. For the convenience of the reader we here collect the third-order $\delta(1-x)$ terms scattered over the 13 pages of Eq. (A.4):

$$\begin{aligned} c_{3,\pm}^{(3)}\Big|_{\delta(x_1)} \;=\; & C_A^2 C_F \left[-\frac{1909753}{1944} - \frac{143255}{81}\,\zeta_2 + \frac{105712}{81}\,\zeta_3 + \frac{25184}{135}\,\zeta_2^2 - \frac{416}{3}\,\zeta_5 \right. \\ & \left. + 540\,\zeta_2\zeta_3 - \frac{248}{3}\,\zeta_2^3 - \frac{3512}{63}\,\zeta_3^2 \right] \;+\; C_A C_F^2 \left[\frac{9161}{12} + \frac{104117}{54}\,\zeta_2 \right. \\ & \left. - \frac{6419}{3}\,\zeta_3 + \frac{87632}{135}\,\zeta_2^2 - \frac{4952}{9}\,\zeta_5 - \frac{6644}{9}\,\zeta_2\zeta_3 + \frac{1016}{3}\,\zeta_2^3 - \frac{33556}{315}\,\zeta_3^2 \right] \\ & + C_F^3 \left[-\frac{7255}{24} - \frac{3379}{6}\,\zeta_2 - 318\,\zeta_3 - \frac{2148}{5}\,\zeta_2^2 + 1240\,\zeta_5 + 808\,\zeta_2\zeta_3 \right. \\ & \left. - \frac{304}{3}\,\zeta_2^3 + \frac{4184}{315}\,\zeta_3^2 \right] \;+\; C_F n_f^2 \left[-\frac{9517}{486} - \frac{860}{27}\,\zeta_2 - \frac{152}{81}\,\zeta_3 - \frac{32}{27}\,\zeta_2^2 \right] \\ & + C_A C_F n_f \left[\frac{142883}{486} + \frac{40862}{81}\,\zeta_2 - \frac{18314}{81}\,\zeta_3 - \frac{2488}{135}\,\zeta_2^2 + \frac{8}{3}\,\zeta_5 - \frac{56}{3}\,\zeta_2\zeta_3 \right] \\ & + C_F^2 n_f \left[-\frac{341}{36} - \frac{5491}{27}\,\zeta_2 + \frac{1348}{3}\,\zeta_3 - \frac{16472}{135}\,\zeta_2^2 - \frac{592}{9}\,\zeta_5 - \frac{352}{9}\,\zeta_2\zeta_3 \right] \;. \end{aligned} \tag{3.8}$$

[1] The sign of the term $-232\,\zeta_5$ in the first line of Eq. (4.19) has been misprinted in the hep-ph version of Ref. [18]. The journal version is correct.

Finally we consider the subleading (integrable) large-x logarithms which have attracted renewed theoretical interest recently [49–51]. Terms up to $\ln^{2n-1}(1-x)$ occur in the n-th order coefficient functions. Their coefficients are the same for the $W^+ + W^-$ and $W^+ - W^-$ cases. Unlike the +-distributions, however, these contributions differ (except for the highest power in each colour factor, where the coefficient is minus that of the highest +-distributions) between F_2 and F_3. For the present structure function the two-loop contributions read

$$c_{3,\pm}^{(2)}\Big|_{L_1^3} = -8C_F^2 \tag{3.9}$$

$$c_{3,\pm}^{(2)}\Big|_{L_1^2} = \frac{22}{3}C_A C_F + 52C_F^2 - \frac{4}{3}C_F n_f \tag{3.10}$$

$$c_{3,\pm}^{(2)}\Big|_{L_1} = -C_A C_F\left[\frac{640}{9} - 8\zeta_2\right] - C_F^2[16 - 32\zeta_2] + \frac{124}{9}C_F n_f\,. \tag{3.11}$$

The corresponding coefficients for the third-order coefficient functions are given by

$$c_{3,\pm}^{(3)}\Big|_{L_1^5} = -8C_F^3 \tag{3.12}$$

$$c_{3,\pm}^{(3)}\Big|_{L_1^4} = \frac{220}{9}C_A C_F^2 + 84C_F^3 - \frac{40}{9}C_F^2 n_f \tag{3.13}$$

$$\begin{aligned} c_{3,\pm}^{(3)}\Big|_{L_1^3} = {} & -\frac{484}{27}C_A^2 C_F - C_A C_F^2\left[\frac{9056}{27} - 32\zeta_2\right] - C_F^3[110 - 96\zeta_2] \\ & + \frac{176}{27}C_A C_F n_f + \frac{1640}{27}C_F^2 n_f - \frac{16}{27}C_F n_f^2 \end{aligned} \tag{3.14}$$

$$\begin{aligned} c_{3,\pm}^{(3)}\Big|_{L_1^2} = {} & C_A^2 C_F\left[\frac{7580}{27} - \frac{98}{3}\zeta_2\right] + C_A C_F^2\left[\frac{12031}{9} - 372\zeta_2 - 240\zeta_3\right] \\ & - C_F^3\left[\frac{1097}{3} + 656\zeta_2 + 16\zeta_3\right] - C_A C_F n_f\left[\frac{2734}{27} - \frac{16}{3}\zeta_2\right] \\ & - C_F^2 n_f\left[\frac{2098}{9} - \frac{112}{3}\zeta_2\right] + \frac{248}{27}C_F n_f^2 \end{aligned} \tag{3.15}$$

$$\begin{aligned} c_{3,\pm}^{(3)}\Big|_{L_1} = {} & -C_A^2 C_F\left[\frac{138598}{81} - \frac{4408}{9}\zeta_2 - 272\zeta_3 + \frac{176}{5}\zeta_2^2\right] - C_A C_F^2\left[\frac{69833}{162}\right. \\ & \left. - \frac{12568}{9}\zeta_2 - \frac{1904}{3}\zeta_3 + \frac{764}{5}\zeta_2^2\right] + C_F^3\left[\frac{1741}{6} + \frac{1220}{3}\zeta_2\right. \\ & \left. + 480\zeta_3 - \frac{376}{5}\zeta_2^2\right] + C_A C_F n_f\left[\frac{45260}{81} - 108\zeta_2 - 16\zeta_3\right] \\ & + C_F^2 n_f\left[\frac{9763}{81} - \frac{2224}{9}\zeta_2 - \frac{112}{3}\zeta_3\right] - C_F n_f^2\left[\frac{3520}{81} - \frac{32}{9}\zeta_2\right]\,. \end{aligned} \tag{3.16}$$

The extraction of some of these coefficients from Eqs. (A.3) and (A.4) is far from trivial. We therefore provide the corresponding results for F_2, which we did not include in Ref. [18], in Appendix B where we also discuss an unexpected (to us) relation between the large-x coefficient functions.

We now turn to the small-x limit of the second- and third-order coefficient function in Eq. (3.1). The leading terms in this case are the 'double-logarithms' $L_0^k \equiv \ln^k x$ with $k = 1, \ldots, 2n-1$ at the n-th order in α_s. The two-loop coefficients are

$$c_{3,-}^{(2)}\Big|_{L_0^3} = -2C_AC_F + \frac{7}{3}C_F^2 \tag{3.17}$$

$$c_{3,-}^{(2)}\Big|_{L_0^2} = -\frac{103}{6}C_AC_F + 21C_F^2 + \frac{5}{3}C_Fn_f \tag{3.18}$$

$$c_{3,-}^{(2)}\Big|_{L_0} = -C_AC_F\left[\frac{122}{3} - 8\zeta_2\right] + C_F^2[21 + 8\zeta_2] + \frac{20}{3}C_Fn_f\ . \tag{3.19}$$

The results for the corresponding $W^+ - W^-$ coefficient function $c_{3,+}^{(2)}$ can be obtained by combining Eqs. (3.17) – (3.19) with Eq. (2.7) in Ref. [45]. Also that equation for $\delta c_3^{(2)} \equiv c_{3,+}^{(2)} - c_{3,-}^{(2)}$ contains terms up to $\ln^3 x$, hence $c_{3,-}^{(2)}$ and $c_{3,+}^{(2)}$ already differ in the leading logarithm. On the other hand, the leading logarithms (but only these) are the same for the even-N based two-loop coefficient functions $c_{2,+}^{(2)}$ and $c_{3,+}^{(2)}$ for F_2 and F_3.

At the third order in α_s, the coefficients of $\ln^k x$ in Eq. (3.1) are given by

$$c_{3,-}^{(3)}\Big|_{L_0^5} = +\frac{2}{5}C_A^2C_F - \frac{29}{15}C_AC_F^2 + \frac{53}{30}C_F^3 - \frac{32}{15}\frac{d^{abc}d_{abc}}{n_c} \tag{3.20}$$

$$c_{3,-}^{(3)}\Big|_{L_0^4} = -\frac{166}{27}C_A^2C_F + \frac{43}{36}C_AC_F^2 + \frac{89}{12}C_F^3 + \frac{46}{27}C_AC_Fn_f - \frac{31}{18}C_F^2n_f \tag{3.21}$$

$$\begin{aligned} c_{3,-}^{(3)}\Big|_{L_0^3} = {} & -C_A^2C_F\left[\frac{9799}{81} + \frac{64}{9}\zeta_2\right] + C_AC_F^2\left[\frac{24245}{162} + \frac{220}{3}\zeta_2\right] \\ & - C_F^3\left[\frac{49}{3} + \frac{710}{9}\zeta_2\right] + \frac{772}{27}C_AC_Fn_f - \frac{2179}{81}C_F^2n_f - \frac{92}{81}C_Fn_f^2 \\ & - \frac{d^{abc}d_{abc}}{n_c}\left[\frac{704}{9} - \frac{256}{3}\zeta_2\right] \end{aligned} \tag{3.22}$$

$$\begin{aligned} c_{3,-}^{(3)}\Big|_{L_0^2} = {} & -C_A^2C_F\left[\frac{38642}{81} - \frac{226}{9}\zeta_2 + \frac{212}{3}\zeta_3\right] + C_AC_Fn_f\left[\frac{4216}{27} - \frac{112}{9}\zeta_2\right] \\ & + C_AC_F^2\left[\frac{13297}{54} + 401\zeta_2 + 328\zeta_3\right] - C_F^2n_f\left[\frac{2687}{27} + \frac{62}{3}\zeta_2\right] \\ & + C_F^3\left[\frac{1085}{6} - \frac{1373}{3}\zeta_2 - 286\zeta_3\right] - \frac{712}{81}C_Fn_f^2 \\ & - \frac{d^{abc}d_{abc}}{n_c}\left[1056 - 96\zeta_2 - \frac{1216}{3}\zeta_3\right] \end{aligned} \tag{3.23}$$

$$c_{3,-}^{(3)}\Big|_{L_0} = -C_A^2C_F\left[\frac{53650}{81} - \frac{11260}{27}\zeta_2 + \frac{1696}{9}\zeta_3 + \frac{416}{5}\zeta_2^2\right] - C_AC_F^2\left[\frac{404119}{324}\right.$$

$$
\begin{aligned}
&-\frac{16273}{27}\,\zeta_2-\frac{9206}{9}\,\zeta_3-\frac{3416}{15}\,\zeta_2^2\Bigg] + C_F^3\left[\frac{5553}{4}-\frac{2308}{3}\,\zeta_2\right.\\
&\left.-\frac{1978}{3}\,\zeta_3-\frac{4748}{15}\,\zeta_2^2\right] + C_A C_F\,n_f\left[\frac{26084}{81}-\frac{3112}{27}\,\zeta_2+\frac{304}{9}\,\zeta_3\right]\\
&- C_F^2\,n_f\left[\frac{271}{162}+\frac{718}{27}\,\zeta_2+\frac{1028}{9}\,\zeta_3\right] - C_F\,n_f^2\left[\frac{1684}{81}-\frac{16}{3}\,\zeta_2\right]\\
&- \frac{d^{abc}d_{abc}}{n_c}\left[1600-\frac{1024}{3}\,\zeta_2-\frac{1088}{3}\,\zeta_3+\frac{2176}{5}\,\zeta_2^2\right]\,. \qquad (3.24)
\end{aligned}
$$

Inserting $C_A = 3$, $C_F = 4/3$, $d^{abc}d_{abc}/n_c = 5n_f/18$ and the numerical values of the ζ-function, Eqs. (3.20) – (3.24) yield

$$
\begin{aligned}
c_{3,-}^{(3)}|_{L_0^5} &\cong -1.32346 - 0.59259\,n_f\,fl_{02}\\
c_{3,-}^{(3)}|_{L_0^4} &\cong -49.8272 + 3.75309\,n_f\\
c_{3,-}^{(3)}|_{L_0^3} &\cong -496.842 + 66.5460\,n_f - 1.51440\,n_f^2 + 17.2626\,n_f\,fl_{02}\\
c_{3,-}^{(3)}|_{L_0^2} &\cong -1485.12 + 305.353\,n_f - 11.7202\,n_f^2 - 114.125\,n_f\,fl_{02}\\
c_{3,-}^{(3)}|_{L_0^1} &\cong -274.294 + 367.303\,n_f - 160.229\,n_f^2 - 494.486\,n_f\,fl_{02} \qquad (3.25)
\end{aligned}
$$

where, as in Eq. (3.6) above, fl_{02} marks the $d^{abc}d_{abc}$ contribution.

The qualitative pattern in Eq. (3.25) is the same as found for the three-loop splitting functions in Eqs. (4.16) and (4.18) of Ref. [15] and the non-singlet photon-exchange coefficient function for F_2 in Eq. (4.26) of Ref. [18]: The coefficients show a dramatic rise until the fourth term, thus precluding any meaningful approximation by (up to) the three highest contributions (3.20) – (3.22) at practically relevant values of x. It is also worthwhile to note that the new $d^{abc}d_{abc}$ colour structure actually forms the largest contribution to the first line of Eq. (3.6) for physical values of the number of flavours. Hence it is not possible to obtain, or reliably estimate, even the leading $\alpha_s^n \ln^{2n-1} x$ terms at higher orders n by any resummation of lower-order structures.

For the corresponding $W^+ - W^-$ coefficient function $c_{3,+}^{(3)}$ only the six lowest even-integer moments have been computed so far [52, 53]. As observed below Eq. (3.19), the leading-log coefficients are the same for the even-N coefficient functions for F_2 and F_3 to two loops. Assuming that this relations holds at three loops as well, we can employ Eq. (3.20) above and Eq. (4.14) in Ref. [18] to derive the conjecture

$$
\delta c_3^{(3)}\Big|_{L_0^5} = \frac{1}{3}\,(C_A - 2C_F)\,C_F^2 - \frac{2}{5}\,(C_A - 2C_F)^2\,C_F\,. \qquad (3.26)
$$

The relation is supported by the fact that, as the even-integer moments [45] of $\delta c_3^{(3)} \equiv c_{3,+}^{(3)} - c_{3,-}^{(3)}$ (with the C_3^--specific $d^{abc}d_{abc}$ contribution removed from the latter quantity), also the coefficient (3.26) shows the characteristic $1/n_c^2$ suppression in the limit of a large number of colours n_c predicted in Ref. [54]. We expect to be able to verify the above conjecture by a complete calculation of the coefficient function $c_{3,+}^{(3)}(x)$ in the not too distant future.

4 Numerical implications

We start this section by graphically illustrating our new third-order coefficient function $c^{(3)}_{3,-}(x)$ and its convolution with a schematic but sufficiently characteristic quark distribution,

$$xf(x) \;=\; x^{0.5}(1-x)^3\,. \tag{4.1}$$

All curves in the corresponding two figures are scaled down from the normalization (3.1) by a factor $2000 \simeq (4\pi)^3$, effectively switching back to the normal-sized expansion parameter α_s.

On the left side of Fig. 2 we compare the exact x-shape given by Eqs. (3.7) and (A.4) to two approximations indicating the previous uncertainty band [55]. This band was constrained by the lowest seven odd-integer moments $N = 1\ldots13$ computed in Ref. [14] and the four leading +-distributions [56] fixed by the next-to-leading logarithmic threshold resummation matched to the second-order coefficient function of Ref. [6]. We see that this estimate was indeed reliable, and sufficiently accurate at $x \gtrsim 0.2$. Thus a future determination of the fourth-order effects can be started reliably from fixed-N moments combined with the seven leading +-distributions now fixed by the next-to-next-to-next-to-leading logarithmic threshold resummation performed in Ref. [57]. We refer the reader to this article for a discussion of higher-order effects at very large values of x.

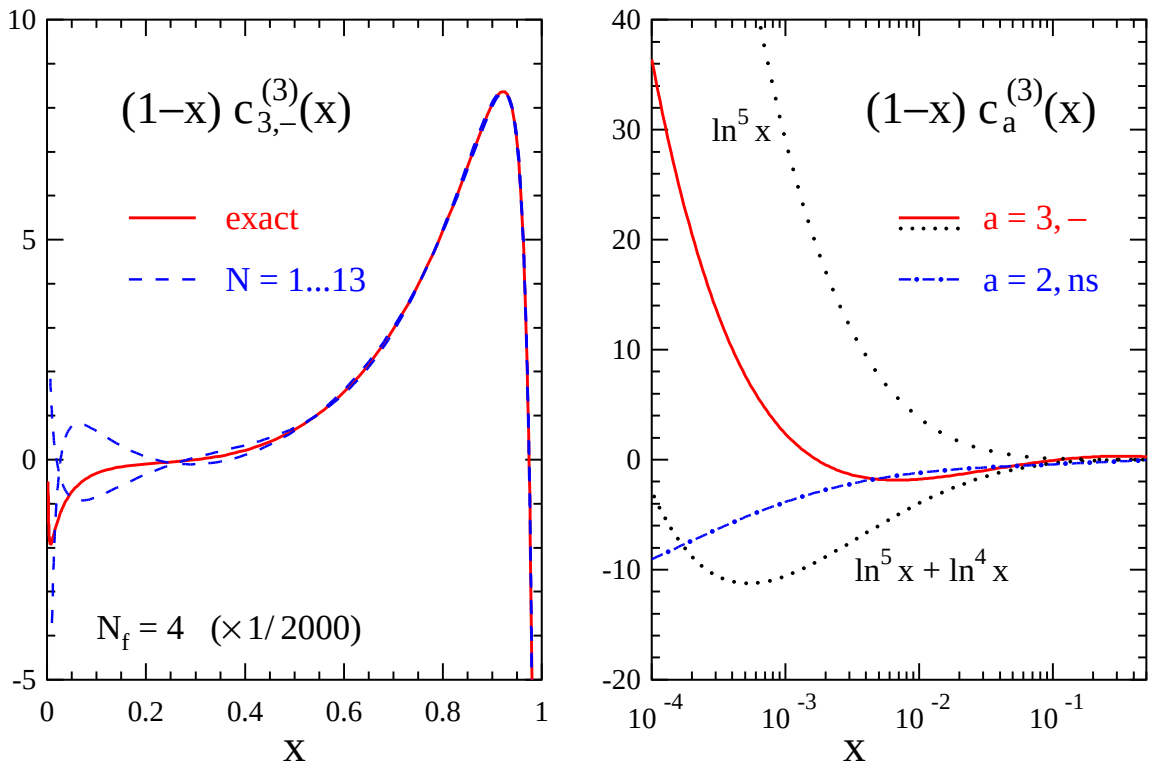


Figure 2: The third-order coefficient function $c^{(3)}_{3,-}(x)$ for four flavours, multiplied by $(1-x)/2000$ for display purposes. Also shown (left) are the previous uncertainty band [55] and (right) the corresponding contribution to the $W^+ + W^-$ structure function $F_{2,\mathrm{ns}}$ (from Ref. [18] for $fl_{11} = 0$) and the small-x approximations by the leading and next-to-leading logarithms (3.20) and (3.21).

The right side of Fig. 2 focuses on medium to small values of x. A huge low-x rise is found from $x \simeq 10^{-3}$. This is totally different from the behaviour of the corresponding electromagnetic and $W^+ + W^-$ coefficient function for F_2 also shown in the figure. In fact, this rise can be attributed entirely to the new $d^{abc}d_{abc}$ colour structure in Eqs. (3.7) and (3.25): If this contribution were removed from the coefficient function, then the solid line (red in the archive version) would not end at 36.4 for $x = 10^{-4}$, but at -4.3, in qualitative agreement with the behaviour of $c^{(3)}_{2,\mathrm{ns}}(x)$.

What physically matters, of course, is not the distribution (aka generalized function) $c^{(3)}_{3,-}(x)$ itself but the resulting contribution to the structure function (3.1), obtained by the convolution (3.5) and its obvious counterpart for the regular terms with the valence quark distribution. This convolution is shown in Fig. 3 for the typical quark distribution (4.1) (the normalization is irrelevant as we display all results normalized to f, thus suppressing large but trivial variations over the chosen wide range in x). For the sake of a direct comparison the same shape is used in Fig. 3 for the (different) quark distribution entering $F_{2,\mathrm{ns}}$.

A comparison between the result for $c^{(3)}_{3,-}(x)$ and $c^{(3)}_{2,\mathrm{ns}}(x)$ illustrates the 'x-shifting' power of the convolution integrals: The former coefficient function (for $fl_{02} = 0$) is larger than the latter at all $x < 0.8$, yet its convolution result is larger only for $x < 10^{-3}$. Note also that the small-x rise for F_3, already delayed in Fig. 2 by about one order of magnitude in x to $x \lesssim 10^{-3}$ by non-leading logarithms, is confined to $x \lesssim 10^{-4}$ after the convolution.

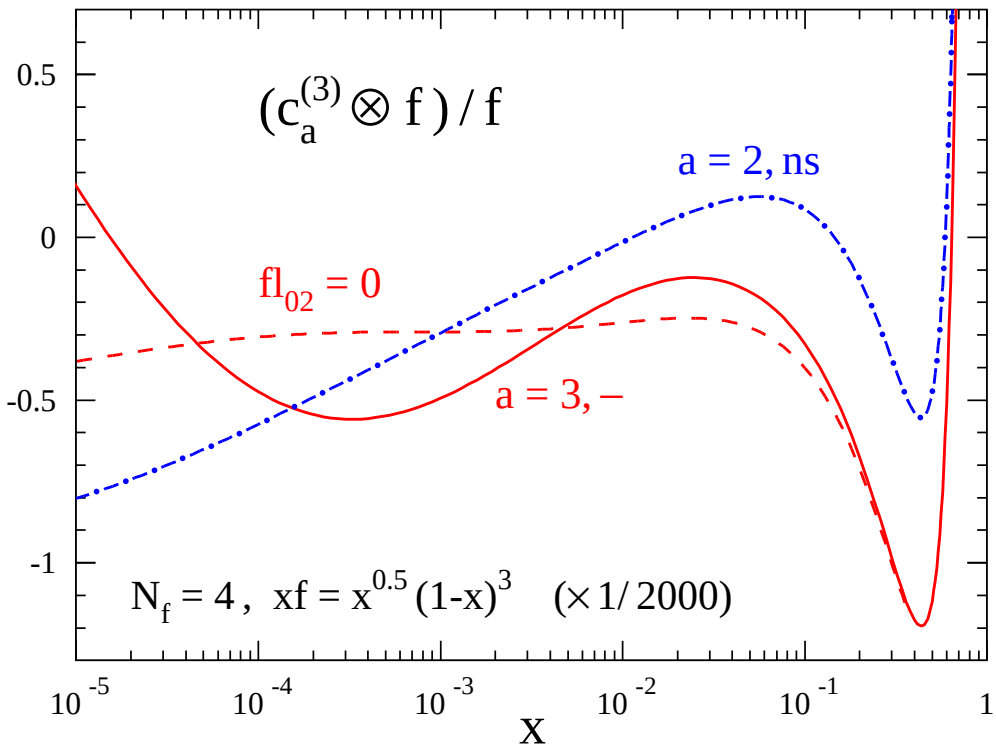


Figure 3: Convolution of the $W^+ + W^-$ charged-current coefficient functions $c^{(3)}_{3,-}$ and $c^{(3)}_{2,\mathrm{ns}}$ for $n_f = 4$ with a simple shape typical of non-singlet quark distributions of the nucleon. Also shown (dashed) is the result obtained for the former if the $d^{abc}d_{abc}$ part of Eqs. (3.7) and (A.4) is left out.

The normalized convolution $(c_{3,-}^{(3)} \otimes f)/f$ of Fig. 2, i.e., the coefficient of α_s^3 in $F_3^{W^+ + W^-}/q_{\rm val}$ for $xq_{\rm val} = x^{0.5}(1-x)^3$, exceeds three, thus 2.5% of the lowest order for $\alpha_s \simeq 0.2$, only at $x > 0.75$ (where we run out of measurements far from the resonance region anyway) and $x < 1.5 \cdot 10^{-8}$. Hence the perturbative expansion for this structure function appears stable at all practically relevant values of x, including the very low values $x \gtrsim 10^{-8}$ probed at $Q^2 \approx 10^4\,\mathrm{GeV}^2$ by the scattering of ultra-high energy cosmic neutrinos, see, e.g., Ref. [58].

We finally assemble Eqs. (3.2), (3.6) and (3.7) to illustrate the perturbative expansion (3.1) of the structure function $F_3^{W^+ + W^-}$. In order to easily compare the contributions of the various orders, we use an order-independent value of the strong coupling, $\alpha_s(\mu_r^2 = \mu_f^2 = Q^2) = 0.2$, corresponding to a scale $Q^2 \approx 30 \ldots 50\,\mathrm{GeV}^2$. For the same reason we use the same quark distribution at all orders of Eq. (3.1), starting with the model shape (4.1) already employed above.

In Figs. 4 and 5 we show the total results at $\mathrm{N}^l\mathrm{LO}$ as well as the relative effects of the l-loop contributions to the structure function. The higher-order (NNLO and $\mathrm{N}^3\mathrm{LO}$) corrections are small (1% or less at $\mathrm{N}^3\mathrm{LO}$) except for the soft-gluon rise towards $x = 1$. Note that with increasing order this rise becomes steeper, but starts at larger values of x. For instance, the relative l-loop corrections exceed 5% at $x > 0.46,\ 0.7,\ 0.83$ for $l = 1,\ 2,\ 3$, respectively. The latter value corresponds to an invariant mass $W \simeq 3$ GeV for $Q^2 = 40\,\mathrm{GeV}^2$.

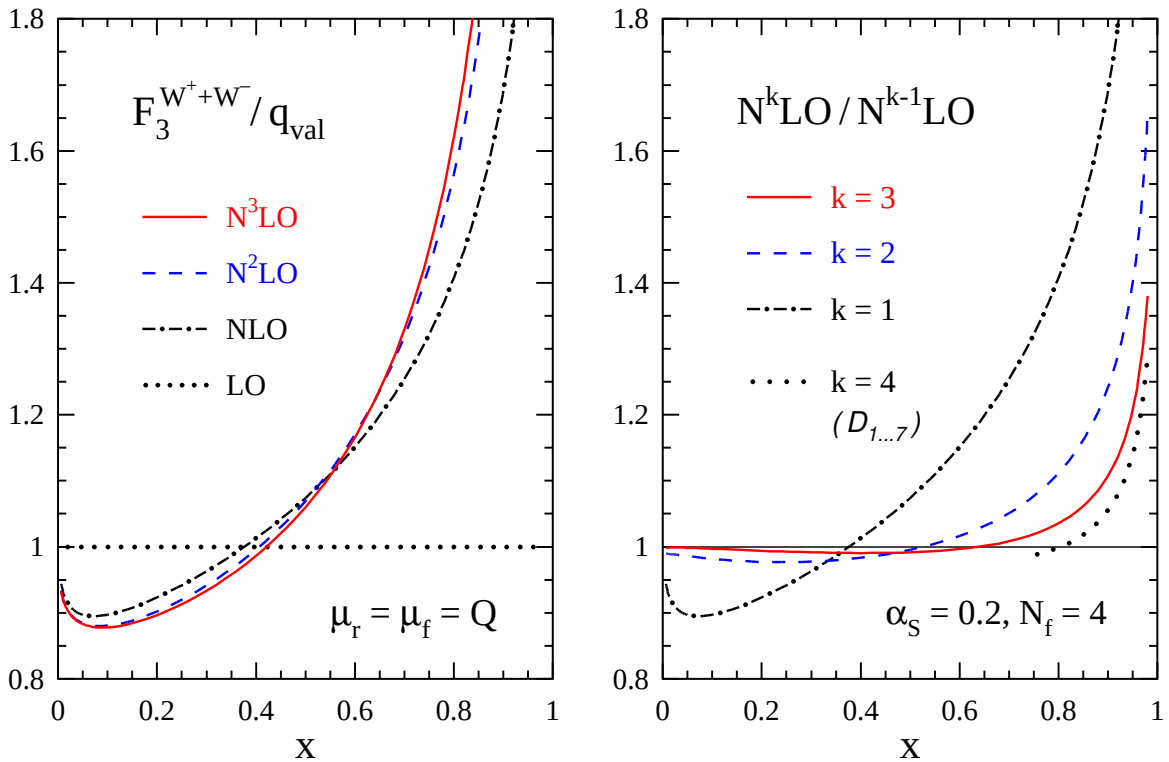


Figure 4: The structure function (3.1) to order α_s^3 ($\mathrm{N}^3\mathrm{LO}$) for a fixed value of α_s and the schematic valence quark distribution (4.1). Next to the cumulative effect of the known corrections on the left, we show on the right the relative corrections due to the k-loop coefficient functions, including an $\mathrm{N}^4\mathrm{LO}$ large-x estimate by the +-distributions $\mathcal{D}_{1,\ldots,7}$ known from the threshold resummation [57].

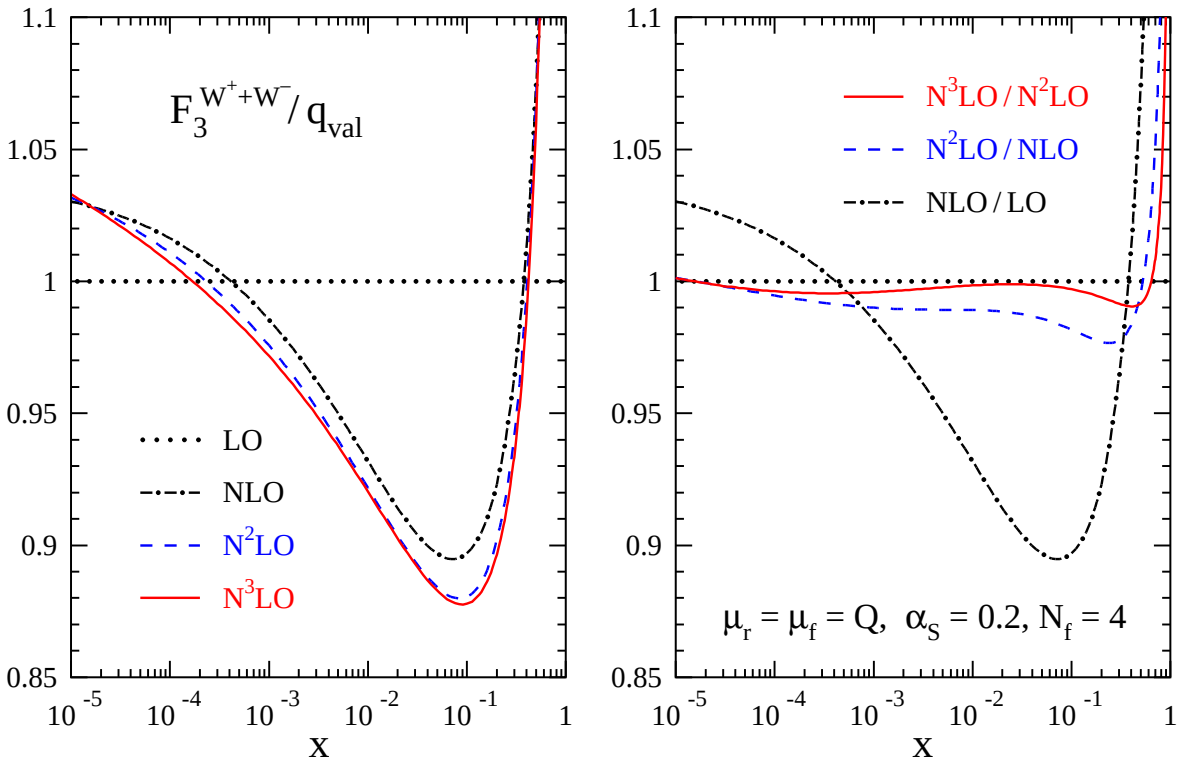


Figure 5: As Fig. 4, but for the higher-order corrections for $F_3^{W^+ + W^-}$ down to small values of x.

An estimate of the large-x fourth-order coefficient function by the seven +-distributions fixed the next-to-next-to-next-to-leading logarithmic soft-gluon (threshold) resummation [57] strongly suggests that this trend will continue at even higher orders. As shown in the right part of Fig. 4, this estimate leads to $x \gtrsim 0.9$ for a relative four-loop correction exceeding 5%. All in all, the N^{3}LO expansion can be considered safe to, at least, $x \simeq 0.8$ at $Q^2 \approx 40\,\text{GeV}^2$ (the safe range, of course, widens (shrinks) with increasing (decreasing) Q^2 due to the scale dependence of α_s).

Specific numbers as given in the last two paragraphs depend on the quark distribution. This is illustrated in Fig. 6, where the Reggeon $\times$ counting-rule ansatz (4.1), cf. Ref. [59], for xq_{val} is modified at large x (left) or small x (right). Suppressing q_{val} in either region leads to larger corrections mainly in the same region, hence Fig. 6 focuses on the large-x region in the former case, and small values of x in the latter. Again we show the results for $\alpha_s = 0.2$ and four flavours.

Suppressing xq_{val} by two powers of $(1-x)$ at large-x leads to a considerable widening of the region of large soft-gluon corrections, with the NLO, NNLO and N^{3}LO 5% x-values listed above Fig. 4 all reduced by about 0.1. A reduction of the small x-quark distribution by a factor $x^{0.2}$, i.e., a factor of 10 at $x = 10^{-5}$ with respect to Eq. (4.1) has a rather dramatic effect, especially at NLO, as shown in the right part of Fig. 6. However, while the relative NLO correction is as large as 24% at $x = 10^{-5}$ in this case, the higher-order corrections remain small, with the N^{3}LO contribution exceeding 1% only at $x \leq 2 \cdot 10^{-5}$. Hence, even under these conditions, the perturbative expansion to N^{3}LO proves sufficient for (more than) all practically relevant situations.

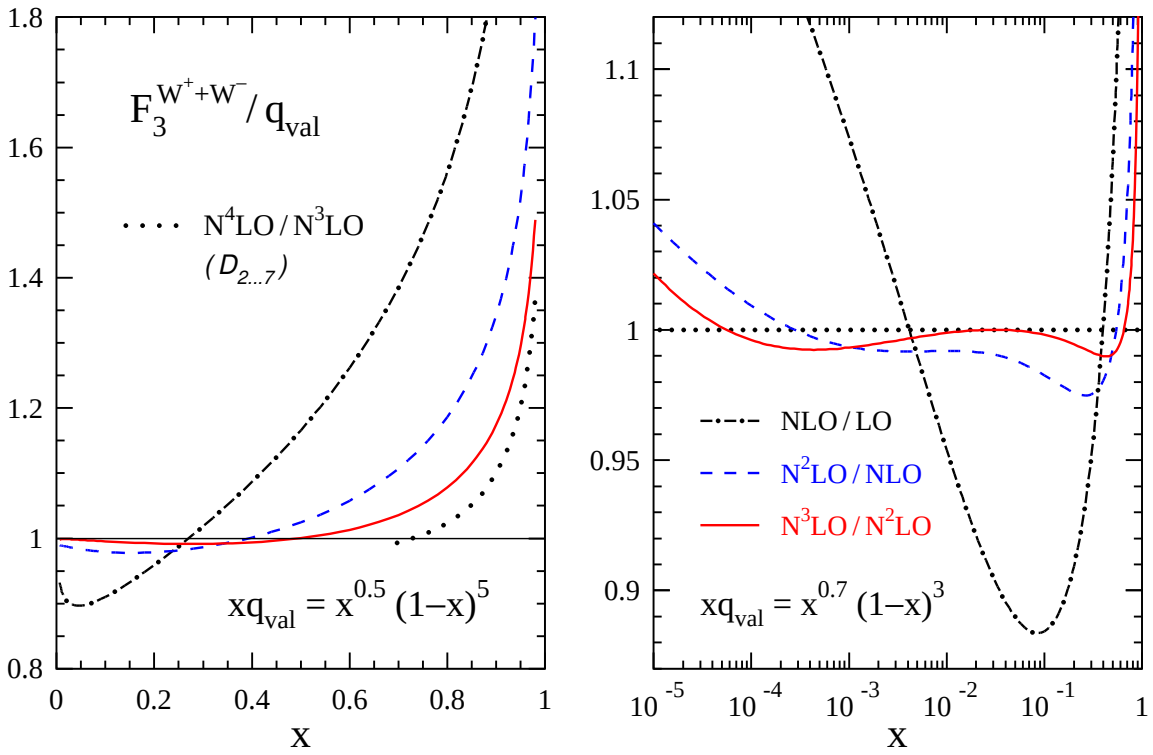


Figure 6: As the right parts of Figs. 4 and 5, but for different choices of the quark distribution $q_{\rm val}$.

Like the large-x rise in the right part of the figure, the small-x rise on the left appears to move closer to the end-point as the order in α_s is increased. Unlike at large x, though, there is no way of estimating the fourth-order contributions at values $x \lesssim 10^{-4}$, as there was no way predicting the size of the $d^{abc}d_{abc}$ three-loop correction from lower-order information. Also fixed moments cannot provide constraints at such values of x, therefore the present (fortunately satisfactory) status will remain unless/until someone (else) calculates the exact fourth-order coefficient function.

5 Summary and outlook

We have extended our previous computations [17, 18] of exact third-order coefficient functions in inclusive deep-inelastic scattering to the charged-current structure function $F_3^{W^+ + W^-}$. Hence the next-to-next-to-next-to-leading order coefficient functions are now known, in massless perturbative QCD, for all structure functions for which precision measurements have been performed in fixed-target DIS and/or at HERA, enabling improved analyses of such data at $x > 0.01$.

Also the present calculation has been performed in Mellin-N space, obtaining an analytic formula in terms of harmonic sums [35] for all odd-N moments (as in Refs. [19] and [20] at first and second order) using the optical theorem and a dispersion relation in the Bjorken variable x. From this result, which we will make available but did not write down in this article for brevity, we have reconstructed the equally lengthy exact x-dependence in terms of harmonic polylogarithms [43] presented in Eq. (A.3). A compact and accurate parametrization is provided by Eq. (3.7).

The singular large-x terms, $(1-x)_+^{-1}\ln^k(1-x)$ and $\delta(1-x)$, are the same for $F_3^{W^+\pm W^-}$ and $F_2^{W^+\pm W^-}$. The latter coefficient function, in turn, is identical to that for $F_2^{\rm e.m.}$ of Ref. [18] up to the obvious absence of the fl_{11} diagram class which contributes to the coefficient of $\delta(1-x)$ at three-loop in the photon- and Z-exchange cases. Even at the largest values of x practically relevant at large scales Q^2 and invariant masses W^2, the observable-specific non-leading contributions to the coefficient functions are non-negligible. We have presented, in particular for use in theoretical studies of subleading terms, explicit expressions of the $\ln^k(1-x)$ contributions to both F_2 and F_3 which turn out to be related via the corresponding terms for F_L already presented in Ref. [18].

At small-x the N^{l}LO non-singlet coefficient functions include potentially large logarithmic terms up to $\alpha_s^l\ln^{2l-1}x$. For $F_2^{\rm e.m.}$ we found that the prefactors of the third-order terms are such that a small-x rise only occurs at irrelevantly low values of x [18]. The results for the vector–axial-vector interference quantity $F_3^{W^++W^-}$ would be similar, were it not for the $d^{abc}d_{abc}\,fl_{02}$ diagram class (absent in F_2 due to Furry's theorem) which occurs at the third order for the first time. These diagrams dominate the small-x limit and lead to a rise of the coefficient function at $x\lesssim 10^{-3}$ before and $x\lesssim 10^{-4}$ after the convolution with the valence quark distribution. Nevertheless, the corrections remain very small at x-values accessible to colliders and unproblematic down even to the very low values of x of interest in DIS of ultra-high energy cosmic neutrinos at $Q^2\approx M_W^2$.

Progress beyond our present three-loop accuracy for $F_3^{W^++W^-}$ would be possible at large-x if the fixed-N results of Ref. [14] could be extended to the fourth order, by combining those results with the substantial constraints from the threshold resummation [57] into approximations analogous to those of Ref. [55]. Given that a first four-loop moment has been computed already, the splitting function for quark combinations such as $u+\bar{u}-(d+\bar{d})$ at $N=2$ for three flavours [60], this seems a not entirely unrealistic perspective.

On the other hand, any attempt of reliably inferring the small-x behaviour of $F_3^{W^++W^-}$ via a resummation of lower-order information appears doomed to failure by the hierarchy of the small-x coefficients, recall Eq. (3.25), and the possible occurrence of dominant new colour factors such as $d^{abc}d_{abc}$ in the present three-loop case, see also Ref. [15]. Hence the rather forbidding extension of the present all-N computation to the fourth order would be required for progress at small x. Fortunately the size of the three-loop corrections does not call urgently for such a calculation.

FORM files of our results in both N-space and x-space, and FORTRAN subroutines of the exact and approximate coefficient functions can be obtained from the preprint server `http://arXiv.org` by downloading the source of this article. Furthermore they are available from us upon request.

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Appendix A: The exact x-space results

Here we present the exact expressions for the coefficient functions up to the third order in terms of harmonic polylogarithms $\mathrm{H}_{m_1,\ldots,m_w}(x)$, $m_j = 0, \pm 1$ [43]. Functions up to weight (number of indices) $2n-1$ contribute at order α_s^n. For 'brevity' we suppress the argument x and define

$$p_{\rm qq}(x) \;=\; 2\,(1-x)^{-1} - 1 - x\,. \tag{A.1}$$

All divergences for $x \to 1$ in Eq. (A.1) and below are to be read as $+$-distributions, see Eqs. (3.4) and (3.5) above. In this notation the one-loop coefficient function (3.2) for F_3 can be written as

$$c_{3,\pm}^{(1)}(x) \;=\; C_F \Big(\frac{1}{2}(5+x) - \frac{1}{2} p_{\rm qq}(x)(3 + 4\mathrm{H}_0 + 4\mathrm{H}_1) - \delta(1-x)(9 + 4\zeta_2) \Big)\,. \tag{A.2}$$

The exact two-loop result corresponding to Eq. (3.6) reads

$$\begin{aligned}
c_{3,-}^{(2)}(x) \;=\; & C_F^2 \Big(-4 p_{\rm qq}(-x)(7\zeta_3 - 8\mathrm{H}_{-1}\zeta_2 - 2\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 8\mathrm{H}_{-1,-1,0} + 10\mathrm{H}_{-1,0,0} \\
& + 4\mathrm{H}_{-1,0,1} + 6\mathrm{H}_{0,-1,0} - 3\mathrm{H}_{0,0,0} - 2\mathrm{H}_{0,0,1}) + \frac{1}{4} p_{\rm qq}(x)(51 + 128\zeta_3 + 48\zeta_2 + 122\mathrm{H}_0 + 96\mathrm{H}_0\zeta_2 \\
& + 54\mathrm{H}_1 + 32\mathrm{H}_1\zeta_2 - 12\mathrm{H}_{0,0} - 48\mathrm{H}_{0,1} - 72\mathrm{H}_{1,0} - 72\mathrm{H}_{1,1} + 96\mathrm{H}_{0,-1,0} - 32\mathrm{H}_{0,0,0} - 96\mathrm{H}_{0,0,1} \\
& - 96\mathrm{H}_{0,1,0} - 112\mathrm{H}_{0,1,1} - 48\mathrm{H}_{1,0,0} - 96\mathrm{H}_{1,0,1} - 128\mathrm{H}_{1,1,0} - 96\mathrm{H}_{1,1,1}) + 4(3-x)\zeta_3 \\
& - 8(1 + 2x^{-1} + x + 2x^2)\mathrm{H}_{-1,0} + 8(1-x)(\mathrm{H}_1\zeta_2 - \mathrm{H}_{1,0,0}) - 2(1+x)(4\mathrm{H}_{-1}\zeta_2 + 4\mathrm{H}_0\zeta_2 \\
& - 3\mathrm{H}_{1,0} + 8\mathrm{H}_{-1,-1,0} - 4\mathrm{H}_{-1,0,0} - 5\mathrm{H}_{0,0,0} - 4\mathrm{H}_{0,0,1} - 2\mathrm{H}_{0,1,0} - 2\mathrm{H}_{0,1,1}) + 2(1+5x)\mathrm{H}_{1,1} \\
& - \frac{1}{2}(3 + 79x)\mathrm{H}_0 - 8(4 + 5x + 2x^2)\zeta_2 + \frac{1}{2}(39 - 17x)\mathrm{H}_1 + (45 + 49x + 16x^2)\mathrm{H}_{0,0} \\
& + 8(3+5x)\mathrm{H}_{0,1} - \frac{1}{4}(233 - 131x) + 16\mathrm{H}_{0,-1,0}x + \Big\{ \frac{331}{8} - 78\zeta_3 + 69\zeta_2 + 6{\zeta_2}^2 \Big\}\delta(1-x) \Big) \\
& + C_F C_A \Big(-8\zeta_3 + 2 p_{\rm qq}(-x)(7\zeta_3 - 8\mathrm{H}_{-1}\zeta_2 - 2\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 8\mathrm{H}_{-1,-1,0} + 10\mathrm{H}_{-1,0,0} \\
& + 4\mathrm{H}_{-1,0,1} + 6\mathrm{H}_{0,-1,0} - 3\mathrm{H}_{0,0,0} - 2\mathrm{H}_{0,0,1}) - \frac{1}{108} p_{\rm qq}(x)(3155 - 216\zeta_3 - 1584\zeta_2 + 4302\mathrm{H}_0 \\
& - 432\mathrm{H}_0\zeta_2 + 2202\mathrm{H}_1 - 1296\mathrm{H}_1\zeta_2 + 1980\mathrm{H}_{0,0} + 1584\mathrm{H}_{0,1} + 1296\mathrm{H}_{0,-1,0} + 648\mathrm{H}_{0,0,0} \\
& + 792\mathrm{H}_{1,0} + 792\mathrm{H}_{1,1} + 432\mathrm{H}_{0,0,1} + 864\mathrm{H}_{1,0,0} + 432\mathrm{H}_{1,0,1} - 432\mathrm{H}_{1,1,0}) - \frac{9}{2}(3-5x)\mathrm{H}_1 \\
& + 4(1 + 2x^{-1} + x + 2x^2)\mathrm{H}_{-1,0} - 4(1-x)(\mathrm{H}_1\zeta_2 - \mathrm{H}_{1,0,0}) - 4(4 + 3x + 2x^2)\mathrm{H}_{0,0} \\
& + 4(1+x)(\mathrm{H}_{-1}\zeta_2 - 2\mathrm{H}_{0,1} + 2\mathrm{H}_{-1,-1,0} - \mathrm{H}_{-1,0,0}) + 4(3 + 2x + 2x^2)\zeta_2 - \frac{1}{6}(29 - 271x)\mathrm{H}_0 \\
& + \frac{5}{36}(91 + 167x) - 8\mathrm{H}_{0,-1,0}x + \Big\{ -\frac{5465}{72} + \frac{140}{3}\zeta_3 - \frac{251}{3}\zeta_2 + \frac{71}{5}{\zeta_2}^2 \Big\}\delta(1-x) \Big) \\
& + C_F n_f \Big(\frac{1}{54} p_{\rm qq}(x)(247 - 144\zeta_2 + 342\mathrm{H}_0 + 174\mathrm{H}_1 + 180\mathrm{H}_{0,0} + 144\mathrm{H}_{0,1} + 72\mathrm{H}_{1,0} + 72\mathrm{H}_{1,1}) \\
& + \frac{1}{3}(1 - 11x)\mathrm{H}_0 + (1-3x)\mathrm{H}_1 + \frac{1}{18}(5 - 119x) + \Big\{ \frac{457}{36} + \frac{38}{3}\zeta_2 + \frac{4}{3}\zeta_3 \Big\}\delta(1-x) \Big)\,.
\end{aligned} \tag{A.3}$$

Finally full three-loop coefficient function for $F_3^{W^+ + W^-}$ approximated by Eq. (3.7) is given by

$$
\begin{aligned}
c_{3,-}^{(3)}(x) \;=\; & \frac{d^{abc}d_{abc}}{n_c}\,16\left(\frac{8}{3}(1-5x)\mathrm{H}_{0,0,0,-1,0}+\frac{56}{3}(1-2x)\mathrm{H}_{0,0,-1,0,0}-\frac{1}{27}(1-x)(4376\right.\\
&+3865\mathrm{H}_1+684\mathrm{H}_{0,-1}\zeta_3+1728\mathrm{H}_{1,0}+2730\mathrm{H}_{1,1}-720\mathrm{H}_{0,-1,-1}\zeta_2+432\mathrm{H}_{0,-1,0}\zeta_2\\
&+318\mathrm{H}_{1,0,0}-198\mathrm{H}_{1,0,1}+198\mathrm{H}_{1,1,0}+216\mathrm{H}_{1,0,-1,0}-54\mathrm{H}_{1,0,1,0}-720\mathrm{H}_{0,-1,-1,-1,0}\\
&+1044\mathrm{H}_{0,-1,-1,0,0}+360\mathrm{H}_{0,-1,-1,0,1}+504\mathrm{H}_{0,-1,0,-1,0}-504\mathrm{H}_{0,-1,0,0,0}-252\mathrm{H}_{0,-1,0,0,1}\\
&+72\mathrm{H}_{0,-1,0,1,0}+936\mathrm{H}_{0,0,-1,-1,0}-72\mathrm{H}_{0,0,1,0,1}+72\mathrm{H}_{0,0,1,1,0})+\frac{2}{9}(1+x)(168\mathrm{H}_{0,1}\zeta_3\\
&+12\mathrm{H}_{0,1,0}\zeta_2-205\mathrm{H}_{0,1,1}+60\mathrm{H}_{0,1,1}\zeta_2+18\mathrm{H}_{0,1,0,1}-18\mathrm{H}_{0,1,1,0}-54\mathrm{H}_{0,0,0,1,0}-90\mathrm{H}_{0,0,0,1,1}\\
&-6\mathrm{H}_{0,0,1,0,0}-24\mathrm{H}_{0,1,0,-1,0}+24\mathrm{H}_{0,1,0,0,0}+18\mathrm{H}_{0,1,0,0,1}+6\mathrm{H}_{0,1,0,1,0}-54\mathrm{H}_{0,1,1,0,0})\\
&+\frac{4}{3}(1+35x)\mathrm{H}_{0,0,-1}\zeta_2-\frac{2}{3}(3-56x^{-1}+3x-56x^2)\mathrm{H}_{-1,-1,0,0}-\frac{10}{9}(3-27x-16x^2)\mathrm{H}_{0,0,1,1}\\
&+\frac{40}{9}(3-4x^{-1}+3x-4x^2)\mathrm{H}_{-1,-1,-1,0}-\frac{80}{9}(3+2x^{-1}+3x+2x^2)\mathrm{H}_{-1,0,1,1}\\
&+\frac{4}{9}(3+4x^{-1}-3x-4x^2)(14\mathrm{H}_1\zeta_3+5\mathrm{H}_{1,1}\zeta_2)-\frac{16}{3}(3+7x^{-1}+3x+7x^2)\mathrm{H}_{-1,-1}\zeta_2\\
&-\frac{28}{9}(3+8x^{-1}+3x+8x^2)\mathrm{H}_{-1,0,0,0}-\frac{2}{3}(3-29x-16x^2)\mathrm{H}_{0,0,1,0}+8(4+3x)\mathrm{H}_{0,0,0}\zeta_2\\
&-\frac{8}{9}(6-7x^{-1}-6x+7x^2)\mathrm{H}_{1,0}\zeta_2-\frac{8}{9}(6-12x-35x^2)\mathrm{H}_{0,0,0,1}-\frac{4}{27}(6+261x-268x^2)\mathrm{H}_{0,0,0}\\
&-\frac{8}{3}(7+11x)\mathrm{H}_{0,0,-1,0,1}-\frac{4}{3}(9-8x^{-1}+9x-8x^2)\mathrm{H}_{-1,0,-1,0}-\frac{8}{3}(11+9x)\mathrm{H}_{0,0,0,0,1}\\
&-\frac{14}{3}(9+8x^{-1}+9x+8x^2)\mathrm{H}_{-1,0,0,1}+\frac{4}{3}(9-8x-26x^2)\mathrm{H}_{0,0}\zeta_2+\frac{4}{3}(11+15x)\mathrm{H}_{0,0,1}\zeta_2\\
&+\frac{8}{9}(15-4x^{-1}-15x+4x^2)\mathrm{H}_{1,0,0,0}-\frac{4}{3}(15+8x^{-1}+15x+8x^2)\mathrm{H}_{-1,0,1,0}\\
&+\frac{8}{3}(15+16x^{-1}+15x+16x^2)\mathrm{H}_{-1,0}\zeta_2+\frac{4}{9}(15-3x-8x^2)\mathrm{H}_{0,0,-1,0}-\frac{8}{3}(28+15x)\zeta_5\\
&-\frac{4}{9}(21+48x^{-1}-9x+68x^2)\mathrm{H}_{0,-1,0,0}+\frac{2}{3}(27+16x^{-1}+9x-8x^2)\mathrm{H}_{0,1}\zeta_2-\frac{4}{3}(29+5x)\zeta_2\zeta_3\\
&-\frac{8}{27}(27+134x^{-1}+27x+134x^2)\mathrm{H}_{-1,0,0}-\frac{4}{3}(33-16x^{-1}-15x-8x^2)\mathrm{H}_{0,-1,-1,0}\\
&+\frac{2}{9}(33-4x^{-1}-33x+4x^2)\mathrm{H}_{1,0,0,1}+\frac{2}{9}(33+39x+28x^2)\mathrm{H}_{0,1,0,0}-\frac{2}{3}(41+36x)\mathrm{H}_{0,1,0}\\
&-\frac{4}{3}(39+16x^{-1}-5x+16x^2)\mathrm{H}_{0,-1,0,1}-\frac{2}{9}(39+28x^{-1}-39x-28x^2)\mathrm{H}_{1,1,0,0}\\
&+\frac{2}{3}(45+48x^{-1}+5x+40x^2)\mathrm{H}_{0,-1}\zeta_2+\frac{4}{9}(51+64x^{-1}+51x+64x^2)\mathrm{H}_{-1,-1,0,1}\\
&+\frac{4}{9}(51-9x-110x^2)\mathrm{H}_0\zeta_3-\frac{8}{15}(51+20x)\mathrm{H}_0{\zeta_2}^2+\frac{4}{27}(99-150x-670x^2)\zeta_3\\
&-\frac{8}{27}(111-134x^{-1}+111x-134x^2)\mathrm{H}_{-1,-1,0}+\frac{2}{9}(111+188x^{-1}+111x+188x^2)\mathrm{H}_{-1}\zeta_3\\
&-\frac{4}{27}(111+69x-268x^2)\mathrm{H}_{0,0,1}+\frac{2}{27}(123+268x^{-1}-123x-268x^2)\mathrm{H}_1\zeta_2\\
&+\frac{4}{9}(177+134x^{-1}+177x+134x^2)\mathrm{H}_{-1}\zeta_2-\frac{4}{27}(186-69x+536x^2)\mathrm{H}_0\zeta_2
\end{aligned}
$$

$$
-\frac{2}{27}(221-18x^{-1}+221x-18x^2)\mathrm{H}_{-1,0}-\frac{2}{45}(291+201x-364x^2){\zeta_2}^2
$$
$$
-\frac{4}{27}(297-639x+268x^2)\mathrm{H}_{0,-1,0}-\frac{8}{27}(321+134x^{-1}+321x+134x^2)\mathrm{H}_{-1,0,1}
$$
$$
+\frac{2}{27}(1172-1019x+18x^2)\zeta_2-\frac{2}{27}(1342-77x+18x^2)\mathrm{H}_{0,0}-\frac{2}{27}(1393-1019x)\mathrm{H}_{0,1}
$$
$$
-\frac{1}{27}(4192-3681x)\mathrm{H}_0+\frac{8}{3}(19\mathrm{H}_{0,0}\zeta_3-6\mathrm{H}_{0,0,0,0,0})+\frac{64}{3}\mathrm{H}_{0,0,0,0}x^2\Bigg)
$$
$$
+C_F^3\Bigg(-\frac{2}{15}p_{\mathrm{qq}}(-x)(4960\zeta_5+150\zeta_3-140\zeta_2-7320\zeta_2\zeta_3+816{\zeta_2}^2-4110\mathrm{H}_{-1}\zeta_3
$$
$$
-280\mathrm{H}_{-1}\zeta_2+952\mathrm{H}_{-1}{\zeta_2}^2+1225\mathrm{H}_0-10\mathrm{H}_0\zeta_3+555\mathrm{H}_0\zeta_2+772\mathrm{H}_0{\zeta_2}^2+30600\mathrm{H}_{-1,-1}\zeta_3
$$
$$
+3960\mathrm{H}_{-1,-1}\zeta_2+240\mathrm{H}_{-1,0}-18600\mathrm{H}_{-1,0}\zeta_3-3180\mathrm{H}_{-1,0}\zeta_2-22000\mathrm{H}_{0,-1}\zeta_3
$$
$$
-2220\mathrm{H}_{0,-1}\zeta_2+550\mathrm{H}_{0,0}+5720\mathrm{H}_{0,0}\zeta_3-1070\mathrm{H}_{0,0}\zeta_2+260\mathrm{H}_{0,1}-3080\mathrm{H}_{0,1}\zeta_3
$$
$$
-1830\mathrm{H}_{0,1}\zeta_2-30240\mathrm{H}_{-1,-1,-1}\zeta_2+1920\mathrm{H}_{-1,-1,0}+35760\mathrm{H}_{-1,-1,0}\zeta_2+22560\mathrm{H}_{-1,0,-1}\zeta_2
$$
$$
-1580\mathrm{H}_{-1,0,0}-17880\mathrm{H}_{-1,0,0}\zeta_2+1240\mathrm{H}_{-1,0,1}+1640\mathrm{H}_{-1,0,1}\zeta_2+22680\mathrm{H}_{0,-1,-1}\zeta_2
$$
$$
-1700\mathrm{H}_{0,-1,0}-23680\mathrm{H}_{0,-1,0}\zeta_2-14000\mathrm{H}_{0,0,-1}\zeta_2+1195\mathrm{H}_{0,0,0}+4520\mathrm{H}_{0,0,0}\zeta_2
$$
$$
-1220\mathrm{H}_{0,0,1}-3280\mathrm{H}_{0,0,1}\zeta_2-240\mathrm{H}_{0,1,0}-2120\mathrm{H}_{0,1,0}\zeta_2-240\mathrm{H}_{0,1,1}-1720\mathrm{H}_{0,1,1}\zeta_2
$$
$$
+2160\mathrm{H}_{-1,-1,-1,0}-1800\mathrm{H}_{-1,-1,0,0}-2880\mathrm{H}_{-1,-1,0,1}-900\mathrm{H}_{-1,0,-1,0}-270\mathrm{H}_{-1,0,0,0}
$$
$$
+3000\mathrm{H}_{-1,0,0,1}+1440\mathrm{H}_{-1,0,1,0}+1920\mathrm{H}_{-1,0,1,1}-840\mathrm{H}_{0,-1,-1,0}+120\mathrm{H}_{0,-1,0,0}
$$
$$
+1800\mathrm{H}_{0,-1,0,1}-280\mathrm{H}_{0,0,-1,0}+1490\mathrm{H}_{0,0,0,0}+390\mathrm{H}_{0,0,0,1}-630\mathrm{H}_{0,0,1,0}-960\mathrm{H}_{0,0,1,1}
$$
$$
-60\mathrm{H}_{0,1,0,1}+60\mathrm{H}_{0,1,1,0}-5760\mathrm{H}_{-1,-1,-1,-1,0}+25680\mathrm{H}_{-1,-1,-1,0,0}+27360\mathrm{H}_{-1,-1,-1,0,1}
$$
$$
+5520\mathrm{H}_{-1,-1,0,-1,0}-21800\mathrm{H}_{-1,-1,0,0,0}-31120\mathrm{H}_{-1,-1,0,0,1}-9920\mathrm{H}_{-1,-1,0,1,0}
$$
$$
-12480\mathrm{H}_{-1,-1,0,1,1}+5760\mathrm{H}_{-1,0,-1,-1,0}-18800\mathrm{H}_{-1,0,-1,0,0}-19680\mathrm{H}_{-1,0,-1,0,1}
$$
$$
-6560\mathrm{H}_{-1,0,0,-1,0}+9480\mathrm{H}_{-1,0,0,0,0}+15720\mathrm{H}_{-1,0,0,0,1}+7000\mathrm{H}_{-1,0,0,1,0}+9040\mathrm{H}_{-1,0,0,1,1}
$$
$$
+3040\mathrm{H}_{-1,0,1,0,0}+2160\mathrm{H}_{-1,0,1,0,1}+1360\mathrm{H}_{-1,0,1,1,0}+1440\mathrm{H}_{-1,0,1,1,1}+6000\mathrm{H}_{0,-1,-1,-1,0}
$$
$$
-18840\mathrm{H}_{0,-1,-1,0,0}-19680\mathrm{H}_{0,-1,-1,0,1}-4800\mathrm{H}_{0,-1,0,-1,0}+13920\mathrm{H}_{0,-1,0,0,0}
$$
$$
+20120\mathrm{H}_{0,-1,0,0,1}+5760\mathrm{H}_{0,-1,0,1,0}+7280\mathrm{H}_{0,-1,0,1,1}-5920\mathrm{H}_{0,0,-1,-1,0}+11200\mathrm{H}_{0,0,-1,0,0}
$$
$$
+11040\mathrm{H}_{0,0,-1,0,1}+3840\mathrm{H}_{0,0,0,-1,0}-2040\mathrm{H}_{0,0,0,0,0}-4200\mathrm{H}_{0,0,0,0,1}-2440\mathrm{H}_{0,0,0,1,0}
$$
$$
-3240\mathrm{H}_{0,0,0,1,1}-1280\mathrm{H}_{0,0,1,0,0}-1040\mathrm{H}_{0,0,1,0,1}-720\mathrm{H}_{0,0,1,1,0}-720\mathrm{H}_{0,0,1,1,1}
$$
$$
-160\mathrm{H}_{0,1,0,-1,0}+480\mathrm{H}_{0,1,0,0,0}+720\mathrm{H}_{0,1,0,0,1}-160\mathrm{H}_{0,1,0,1,0}+480\mathrm{H}_{0,1,1,0,0}+80\mathrm{H}_{0,1,1,0,1}
$$
$$
-80\mathrm{H}_{0,1,1,1,0})-\frac{1}{240}p_{\mathrm{qq}}(x)(15015+214400\zeta_5-34320\zeta_3+35480\zeta_2+74240\zeta_2\zeta_3
$$
$$
+15200\mathrm{H}_1\zeta_2+48140\mathrm{H}_0-167040\mathrm{H}_0\zeta_3+64880\mathrm{H}_0\zeta_2+59520\mathrm{H}_0{\zeta_2}^2-311040\mathrm{H}_{0,-1,-1}\zeta_2
$$
$$
-3136{\zeta_2}^2+54528\mathrm{H}_1{\zeta_2}^2+263680\mathrm{H}_{0,-1}\zeta_3+130240\mathrm{H}_{0,-1}\zeta_2-13480\mathrm{H}_{0,0}-35840\mathrm{H}_{0,0}\zeta_3
$$
$$
-113280\mathrm{H}_{0,0}\zeta_2-31640\mathrm{H}_{0,1}+148480\mathrm{H}_{0,1}\zeta_3-45120\mathrm{H}_{0,1}\zeta_2-47640\mathrm{H}_{1,0}+175360\mathrm{H}_{1,0}\zeta_3
$$
$$
-84480\mathrm{H}_{1,0}\zeta_2-33480\mathrm{H}_{1,1}+253440\mathrm{H}_{1,1}\zeta_3-51840\mathrm{H}_{1,1}\zeta_2+11220\mathrm{H}_1-17280\mathrm{H}_1\zeta_3
$$
$$
+11840\mathrm{H}_{0,-1,0}+231680\mathrm{H}_{0,-1,0}\zeta_2+220160\mathrm{H}_{0,0,-1}\zeta_2-16240\mathrm{H}_{0,0,0}-35840\mathrm{H}_{0,0,0}\zeta_2
$$

$$
\begin{aligned}
&- 86160\mathrm{H}_{0,0,1} - 29440\mathrm{H}_{0,0,1}\zeta_2 - 88800\mathrm{H}_{0,1,0} - 14080\mathrm{H}_{0,1,0}\zeta_2 - 60160\mathrm{H}_{0,1,1} \\
&- 51200\mathrm{H}_{0,1,1}\zeta_2 - 55040\mathrm{H}_{1,0,-1}\zeta_2 - 44400\mathrm{H}_{1,0,0} + 57600\mathrm{H}_{1,0,0}\zeta_2 - 45920\mathrm{H}_{1,0,1} \\
&- 46080\mathrm{H}_{1,0,1}\zeta_2 - 59520\mathrm{H}_{1,1,0} + 2560\mathrm{H}_{1,1,0}\zeta_2 - 25920\mathrm{H}_{1,1,1} - 46080\mathrm{H}_{1,1,1}\zeta_2 \\
&+ 120960\mathrm{H}_{0,-1,-1,0} - 64640\mathrm{H}_{0,-1,0,0} - 69760\mathrm{H}_{0,-1,0,1} - 34560\mathrm{H}_{0,0,-1,0} + 30400\mathrm{H}_{0,0,0,0} \\
&+ 91520\mathrm{H}_{0,0,0,1} + 28800\mathrm{H}_{0,0,1,0} + 61440\mathrm{H}_{0,0,1,1} - 6720\mathrm{H}_{0,1,0,0} + 58560\mathrm{H}_{0,1,0,1} \\
&+ 72000\mathrm{H}_{0,1,1,0} + 77760\mathrm{H}_{0,1,1,1} - 17280\mathrm{H}_{1,0,-1,0} + 17280\mathrm{H}_{1,0,0,0} + 90240\mathrm{H}_{1,0,0,1} \\
&+ 76800\mathrm{H}_{1,0,1,0} + 77760\mathrm{H}_{1,0,1,1} + 2880\mathrm{H}_{1,1,0,0} + 86400\mathrm{H}_{1,1,0,1} + 103680\mathrm{H}_{1,1,1,0} \\
&+ 86400\mathrm{H}_{1,1,1,1} - 104960\mathrm{H}_{0,-1,-1,-1,0} + 268800\mathrm{H}_{0,-1,-1,0,0} + 258560\mathrm{H}_{0,-1,-1,0,1} \\
&+ 84480\mathrm{H}_{0,-1,0,-1,0} - 130560\mathrm{H}_{0,-1,0,0,0} - 186880\mathrm{H}_{0,-1,0,0,1} - 23040\mathrm{H}_{0,-1,0,1,0} \\
&- 23040\mathrm{H}_{0,-1,0,1,1} + 276480\mathrm{H}_{0,0,-1,-1,0} - 230400\mathrm{H}_{0,0,-1,0,0} - 81920\mathrm{H}_{0,0,-1,0,1} \\
&- 102400\mathrm{H}_{0,0,0,-1,0} - 7680\mathrm{H}_{0,0,0,0,0} + 46080\mathrm{H}_{0,0,0,0,1} + 84480\mathrm{H}_{0,0,0,1,0} + 116480\mathrm{H}_{0,0,0,1,1} \\
&+ 44800\mathrm{H}_{0,0,1,0,0} + 124160\mathrm{H}_{0,0,1,0,1} + 142080\mathrm{H}_{0,0,1,1,0} + 142080\mathrm{H}_{0,0,1,1,1} \\
&+ 74240\mathrm{H}_{0,1,0,-1,0} - 53760\mathrm{H}_{0,1,0,0,0} + 128000\mathrm{H}_{0,1,0,0,1} + 139520\mathrm{H}_{0,1,0,1,0} \\
&+ 140800\mathrm{H}_{0,1,0,1,1} + 75520\mathrm{H}_{0,1,1,0,0} + 147200\mathrm{H}_{0,1,1,0,1} + 172800\mathrm{H}_{0,1,1,1,0} \\
&+ 138240\mathrm{H}_{0,1,1,1,1} + 243200\mathrm{H}_{1,0,-1,-1,0} - 48640\mathrm{H}_{1,0,-1,0,0} + 176640\mathrm{H}_{1,0,-1,0,1} \\
&- 71680\mathrm{H}_{1,0,0,-1,0} - 65280\mathrm{H}_{1,0,0,0,0} + 11520\mathrm{H}_{1,0,0,0,1} + 131840\mathrm{H}_{1,0,0,1,0} \\
&+ 135680\mathrm{H}_{1,0,0,1,1} + 67840\mathrm{H}_{1,0,1,0,0} + 138240\mathrm{H}_{1,0,1,0,1} + 158720\mathrm{H}_{1,0,1,1,0} \\
&+ 126720\mathrm{H}_{1,0,1,1,1} + 120320\mathrm{H}_{1,1,0,-1,0} - 94720\mathrm{H}_{1,1,0,0,0} + 145920\mathrm{H}_{1,1,0,0,1} \\
&+ 161280\mathrm{H}_{1,1,0,1,0} + 126720\mathrm{H}_{1,1,0,1,1} + 88320\mathrm{H}_{1,1,1,0,0} + 138240\mathrm{H}_{1,1,1,0,1} \\
&+ 161280\mathrm{H}_{1,1,1,1,0} + 115200\mathrm{H}_{1,1,1,1,1}) + \frac{128}{3}(1-9x)\mathrm{H}_{0,0,0,-1,0} - 72(1-3x)\mathrm{H}_{1,1,1,1} \\
&+ \frac{416}{3}(1-2x)\mathrm{H}_{0,0,-1,0,0} - 8(1-x)(15\mathrm{H}_1{\zeta_2}^2 - 38\mathrm{H}_{1,0}\zeta_3 - 6\mathrm{H}_{1,1}\zeta_3 + 28\mathrm{H}_{1,0,-1}\zeta_2 \\
&- 60\mathrm{H}_{1,0,0}\zeta_2 - 12\mathrm{H}_{1,0,1}\zeta_2 - 56\mathrm{H}_{1,1,0}\zeta_2 - 12\mathrm{H}_{1,1,1}\zeta_2 - 8\mathrm{H}_{1,0,-1,-1,0} - 12\mathrm{H}_{1,0,-1,0,0} \\
&- 32\mathrm{H}_{1,0,-1,0,1} - 4\mathrm{H}_{1,0,0,-1,0} + 20\mathrm{H}_{1,0,0,0,0} + 48\mathrm{H}_{1,0,0,0,1} + 4\mathrm{H}_{1,0,0,1,0} + 4\mathrm{H}_{1,0,0,1,1} \\
&- 12\mathrm{H}_{1,0,1,0,0} - 16\mathrm{H}_{1,1,0,-1,0} + 24\mathrm{H}_{1,1,0,0,0} + 36\mathrm{H}_{1,1,0,0,1} - 28\mathrm{H}_{1,1,1,0,0}) \\
&+ \frac{8}{15}(1+x)(177\mathrm{H}_{-1}{\zeta_2}^2 + 1530\mathrm{H}_{-1,-1}\zeta_3 - 630\mathrm{H}_{-1,0}\zeta_3 - 1860\mathrm{H}_{-1,-1,-1}\zeta_2 \\
&+ 1440\mathrm{H}_{-1,-1,0}\zeta_2 + 1140\mathrm{H}_{-1,0,-1}\zeta_2 - 840\mathrm{H}_{-1,0,0}\zeta_2 + 60\mathrm{H}_{-1,0,1}\zeta_2 - 360\mathrm{H}_{-1,-1,-1,-1,0} \\
&+ 1500\mathrm{H}_{-1,-1,-1,0,0} + 1680\mathrm{H}_{-1,-1,-1,0,1} + 360\mathrm{H}_{-1,-1,0,-1,0} - 720\mathrm{H}_{-1,-1,0,0,0} \\
&- 1140\mathrm{H}_{-1,-1,0,0,1} - 120\mathrm{H}_{-1,-1,0,1,0} - 120\mathrm{H}_{-1,-1,0,1,1} + 360\mathrm{H}_{-1,0,-1,-1,0} \\
&- 900\mathrm{H}_{-1,0,-1,0,0} - 960\mathrm{H}_{-1,0,-1,0,1} - 300\mathrm{H}_{-1,0,0,-1,0} + 300\mathrm{H}_{-1,0,0,0,0} + 660\mathrm{H}_{-1,0,0,0,1} \\
&+ 60\mathrm{H}_{-1,0,0,1,0} + 60\mathrm{H}_{-1,0,0,1,1} - 120\mathrm{H}_{-1,0,1,0,0} + 360\mathrm{H}_{0,0,1,1,1} + 250\mathrm{H}_{0,1,0,1,0} \\
&+ 180\mathrm{H}_{0,1,0,1,1} + 210\mathrm{H}_{0,1,1,0,1} + 240\mathrm{H}_{0,1,1,1,0} + 180\mathrm{H}_{0,1,1,1,1}) - \frac{64}{3}(2-9x)\mathrm{H}_{1,0,1,0} \\
&+ 16(3-11x)\mathrm{H}_{0,1,1}\zeta_2 - 64(4+x)\mathrm{H}_{0,-1,0,-1,0} - 8(5-39x)\mathrm{H}_{0,1,0,0,1} - 8(5-27x)\mathrm{H}_{0,1,0,0,0}
\end{aligned}
$$

$$
-\frac{16}{3}(5-11x)\mathrm{H}_{0,0,-1}\zeta_2+\frac{32}{3}(5+x)\mathrm{H}_{0,-1,0,1,0}+64(5+2x)\mathrm{H}_{0,-1,-1,-1,0}
-4(5+57x)\mathrm{H}_{0,0,0}\zeta_2+24(7-17x)\mathrm{H}_{0,1,0}\zeta_2-\frac{88}{3}(7-13x)\zeta_2\zeta_3-\frac{32}{3}(7+5x)\mathrm{H}_{0,0,-1,0,1}
-\frac{32}{3}(9+16x)\mathrm{H}_{0,0,1}\zeta_2+\frac{64}{3}(10+11x)\mathrm{H}_{0,0,1,1,0}-\frac{64}{3}(10+30x+9x^2)\mathrm{H}_{0,0,-1,0}
+\frac{40}{3}(11-x)\mathrm{H}_{0,1,1,0,0}+\frac{16}{3}(14+7x^{-1}-44x-7x^2)\mathrm{H}_{1,0}\zeta_2-\frac{32}{3}(17+79x)\mathrm{H}_{0,-1,-1,0,1}
-\frac{8}{3}(19-22x^{-1}-75x+22x^2)\mathrm{H}_{1,0,0,1}-\frac{16}{3}(19-47x)\mathrm{H}_{1,1,0,1}-\frac{32}{3}(19-x)\mathrm{H}_{0,0,-1,-1,0}
+6(19+3x)\mathrm{H}_{1,1,1}+12(19+55x)\mathrm{H}_{0,1,1,1}-\frac{8}{3}(23-97x)\mathrm{H}_{1,1,1,0}-4(23-45x)\mathrm{H}_{0,0,0,0,0}
+\frac{8}{3}(25-24x^{-1}-x+24x^2)\mathrm{H}_{1,0,0,0}-32(25+2x^{-1}+25x+2x^2)\mathrm{H}_{-1,0,1,1}
+16(27+x)\mathrm{H}_{0,-1,0,0,0}+\frac{16}{3}(29+41x)\mathrm{H}_{0,0,0,1,0}+\frac{4}{3}(31-361x)\mathrm{H}_{0,0}\zeta_3
-\frac{4}{3}(31-149x)\mathrm{H}_{1,0,1,1}+\frac{16}{3}(31+33x)\mathrm{H}_{0,0,1,0,0}+\frac{32}{3}(32+85x)\mathrm{H}_{0,-1,-1}\zeta_2
+\frac{16}{3}(37+33x)\mathrm{H}_{0,0,1,0,1}+\frac{16}{3}(43+83x)\mathrm{H}_{0,-1,0,0,1}+\frac{16}{3}(44+116x+11x^2)\mathrm{H}_{0,1,0,0}
-\frac{8}{3}(50-24x^{-1}+259x+12x^2)\mathrm{H}_{0,1}\zeta_2+\frac{4}{3}(47+171x)\mathrm{H}_{0,0,0,0,1}+\frac{8}{3}(61+85x)\mathrm{H}_{0,0,0,1,1}
-8(43+87x)\mathrm{H}_{0,-1}\zeta_3-\frac{32}{3}(55+6x^{-1}+55x+6x^2)\mathrm{H}_{-1,0,1,0}-\frac{16}{3}(67+101x)\mathrm{H}_{0,-1,0}\zeta_2
+\frac{32}{3}(64+78x^{-1}+73x+78x^2)\mathrm{H}_{-1,-1,0,1}+\frac{16}{3}(81-12x^{-1}+43x-12x^2)\mathrm{H}_{-1,-1,-1,0}
-\frac{8}{3}(95-48x^{-1}-33x+48x^2)\mathrm{H}_{1,0,-1,0}+\frac{16}{3}(68+59x)\mathrm{H}_{0,1,0}-\frac{16}{3}(95+91x)\mathrm{H}_{0,-1,-1,0,0}
-\frac{8}{3}(113-24x^{-1}+59x-24x^2)\mathrm{H}_{-1,0,-1,0}+\frac{8}{3}(115+216x^{-1}+203x+216x^2)\mathrm{H}_{-1,-1,0,0}
+\frac{8}{3}(119+12x^{-1}-137x-12x^2)\mathrm{H}_{1,1}\zeta_2+\frac{8}{3}(122-65x)\mathrm{H}_{1,0,1}-\frac{16}{3}(103+120x)\zeta_5
-\frac{8}{3}(145-48x^{-1}-95x-24x^2)\mathrm{H}_{0,-1,-1,0}+\frac{4}{3}(153+116x^{-1}-281x-116x^2)\mathrm{H}_1\zeta_3
-\frac{8}{3}(165+48x^{-1}+253x+216x^2)\mathrm{H}_{0,-1,0,0}-\frac{8}{3}(175+324x^{-1}+249x+324x^2)\mathrm{H}_{-1,-1}\zeta_2
+\frac{4}{15}(129-151x)\mathrm{H}_0{\zeta_2}^2+\frac{4}{3}(199+495x)\mathrm{H}_{0,1,1,0}-\frac{4}{15}(203-720x-148x^2){\zeta_2}^2
-\frac{4}{3}(227+44x^{-1}-269x-44x^2)\mathrm{H}_{1,1,0,0}-\frac{8}{9}(261-488x^{-1}+231x-488x^2)\mathrm{H}_{-1,-1,0}
+\frac{8}{3}(204+463x+24x^2)\mathrm{H}_{0,0,1,0}+\frac{8}{3}(253+597x+24x^2)\mathrm{H}_{0,0,1,1}+\frac{2}{3}(101+29x)\mathrm{H}_{1,1,0}
-\frac{8}{9}(267+488x^{-1}+276x+488x^2)\mathrm{H}_{-1,0,0}-\frac{16}{3}(275+108x^{-1}+284x+108x^2)\mathrm{H}_{-1,0,0,1}
+\frac{8}{3}(294+264x^{-1}+359x+264x^2)\mathrm{H}_{-1}\zeta_3+\frac{20}{9}(394-427x)\mathrm{H}_1+\frac{4}{3}(245+613x)\mathrm{H}_{0,1,0,1}
-\frac{8}{3}(455+48x^{-1}+111x+312x^2)\mathrm{H}_{0,-1,0,1}-\frac{4}{9}(471-488x^{-1}-159x+488x^2)\mathrm{H}_1\zeta_2
+\frac{4}{3}(531+488x^{-1}+541x+488x^2)\mathrm{H}_{-1}\zeta_2+\frac{8}{3}(541+252x^{-1}+555x+252x^2)\mathrm{H}_{-1,0}\zeta_2
$$

$$
-\frac{4}{3}(599+192x^{-1}+641x+192x^2)\mathrm{H}_{-1,0,0,0}-\frac{4}{9}(669-1629x+976x^2)\mathrm{H}_{0,-1,0}
+\frac{2}{3}(755+1383x+480x^2)\mathrm{H}_{0,0,0,0}+\frac{4}{3}(765+144x^{-1}+317x+648x^2)\mathrm{H}_{0,-1}\zeta_2
+\frac{1}{3}(817-763x)\mathrm{H}_{1,0,0}+\frac{2}{3}(851+615x)\mathrm{H}_{0,1,1}-\frac{8}{9}(927+488x^{-1}+927x+488x^2)\mathrm{H}_{-1,0,1}
-\frac{4}{9}(1045+462x^{-1}+877x+186x^2)\mathrm{H}_{-1,0}+\frac{2}{3}(1975+2847x+776x^2)\mathrm{H}_{0,0,0,1}
-\frac{2}{3}(2035+2271x+1288x^2)\mathrm{H}_0\zeta_3-\frac{2}{3}(2295+2847x+1064x^2)\mathrm{H}_{0,0}\zeta_2
+\frac{1}{9}(2823+1305x+3904x^2)\mathrm{H}_{0,0,0}+\frac{1}{6}(4143-5305x)\mathrm{H}_{1,0}-\frac{1}{9}(6705+3537x+9760x^2)\zeta_3
+\frac{1}{9}(7533+6969x+3904x^2)\mathrm{H}_{0,0,1}-\frac{1}{9}(10209+6969x+7808x^2)\mathrm{H}_0\zeta_2-128\mathrm{H}_{0,1,0,-1,0}x
+\frac{1}{6}(7129-7411x)\mathrm{H}_{1,1}+\frac{1}{18}(11171-29353x+1488x^2)\mathrm{H}_{0,0}+\frac{1}{18}(12137-30523x)\mathrm{H}_{0,1}
-\frac{1}{18}(20497-30523x+1488x^2)\zeta_2+\frac{1}{18}(28579+7263x)\mathrm{H}_0+\frac{1}{144}(481783-463765x)
+16(17\mathrm{H}_{0,1}\zeta_3+4\mathrm{H}_{0,-1,0,1,1})+\frac{4}{3}g_1(x)+\frac{4}{3}g_2(x)+\left\{-\frac{7255}{24}+1240\zeta_5-\frac{950}{3}\zeta_3
-\frac{304}{3}{\zeta_3}^2-\frac{1129}{2}\zeta_2+808\zeta_2\zeta_3-\frac{2148}{5}{\zeta_2}^2+\frac{4184}{315}{\zeta_2}^3\right\}\delta(1-x)\Bigg)
+C_F^2 n_f\Bigg(\frac{8}{405}p_{\mathrm{qq}}(-x)(2700\zeta_3+830\zeta_2+1026{\zeta_2}^2-1440\mathrm{H}_{-1}\zeta_3-3000\mathrm{H}_{-1}\zeta_2-1305\mathrm{H}_0
+990\mathrm{H}_0\zeta_3+345\mathrm{H}_0\zeta_2+1800\mathrm{H}_{-1,-1}\zeta_2+1660\mathrm{H}_{-1,0}-1260\mathrm{H}_{-1,0}\zeta_2-2340\mathrm{H}_{0,-1}\zeta_2
-1910\mathrm{H}_{0,0}+450\mathrm{H}_{0,0}\zeta_2-3600\mathrm{H}_{-1,-1,0}+4800\mathrm{H}_{-1,0,0}+1200\mathrm{H}_{-1,0,1}+3000\mathrm{H}_{0,-1,0}
-1245\mathrm{H}_{0,0,0}-600\mathrm{H}_{0,0,1}+3600\mathrm{H}_{-1,-1,-1,0}-6120\mathrm{H}_{-1,-1,0,0}-3960\mathrm{H}_{-1,0,-1,0}
+6120\mathrm{H}_{-1,0,0,0}+360\mathrm{H}_{-1,0,0,1}-720\mathrm{H}_{-1,0,1,1}-3960\mathrm{H}_{0,-1,-1,0}+6120\mathrm{H}_{0,-1,0,0}
+360\mathrm{H}_{0,-1,0,1}+4500\mathrm{H}_{0,0,-1,0}-2070\mathrm{H}_{0,0,0,0}-360\mathrm{H}_{0,0,0,1}+360\mathrm{H}_{0,0,1,1})
-\frac{1}{3240}p_{\mathrm{qq}}(x)(30045+580080\zeta_3+92280\zeta_2-89568{\zeta_2}^2+287700\mathrm{H}_0+420480\mathrm{H}_0\zeta_3
+505680\mathrm{H}_0\zeta_2+14940\mathrm{H}_1+187200\mathrm{H}_1\zeta_3+253440\mathrm{H}_1\zeta_2-138240\mathrm{H}_{0,-1}\zeta_2+64920\mathrm{H}_{0,0}
+262080\mathrm{H}_{0,0}\zeta_2-145400\mathrm{H}_{0,1}+92160\mathrm{H}_{0,1}\zeta_2-306680\mathrm{H}_{1,0}+43200\mathrm{H}_{1,0}\zeta_2-245880\mathrm{H}_{1,1}
+57600\mathrm{H}_{1,1}\zeta_2+224640\mathrm{H}_{0,-1,0}-196080\mathrm{H}_{0,0,0}-489360\mathrm{H}_{0,0,1}-333120\mathrm{H}_{0,1,0}
-371520\mathrm{H}_{0,1,1}-301200\mathrm{H}_{1,0,0}-368640\mathrm{H}_{1,0,1}-387360\mathrm{H}_{1,1,0}-302400\mathrm{H}_{1,1,1}
+276480\mathrm{H}_{0,-1,0,0}+138240\mathrm{H}_{0,-1,0,1}+276480\mathrm{H}_{0,0,-1,0}-129600\mathrm{H}_{0,0,0,0}-267840\mathrm{H}_{0,0,0,1}
-279360\mathrm{H}_{0,0,1,0}-290880\mathrm{H}_{0,0,1,1}-204480\mathrm{H}_{0,1,0,0}-218880\mathrm{H}_{0,1,0,1}-293760\mathrm{H}_{0,1,1,0}
-236160\mathrm{H}_{0,1,1,1}+138240\mathrm{H}_{1,0,-1,0}-152640\mathrm{H}_{1,0,0,0}-100800\mathrm{H}_{1,0,0,1}-264960\mathrm{H}_{1,0,1,0}
-195840\mathrm{H}_{1,0,1,1}-262080\mathrm{H}_{1,1,0,0}-172800\mathrm{H}_{1,1,0,1}-236160\mathrm{H}_{1,1,1,0}-172800\mathrm{H}_{1,1,1,1})
+\frac{8}{3}(1-39x+14x^2)\mathrm{H}_{0,1,0}+\frac{8}{3}(1-7x)\mathrm{H}_{0,1,0,0}-\frac{16}{3}(1-x)(3\mathrm{H}_1\zeta_3+7\mathrm{H}_{1,0}\zeta_2+5\mathrm{H}_{1,1}\zeta_2
-4\mathrm{H}_{0,-1,0,1}-4\mathrm{H}_{1,0,-1,0}-6\mathrm{H}_{1,0,0,0}-5\mathrm{H}_{1,0,0,1}+\mathrm{H}_{1,0,1,0}-\mathrm{H}_{1,1,0,1}+\mathrm{H}_{1,1,1,0})
$$

$$+\frac{4}{9}(1+x)(48\mathrm{H}_{-1}\zeta_3-48\mathrm{H}_{-1,-1}\zeta_2+24\mathrm{H}_{-1,0}\zeta_2+69\mathrm{H}_{0,0}\zeta_2-96\mathrm{H}_{-1,-1,-1,0}$$
$$+144\mathrm{H}_{-1,-1,0,0}+96\mathrm{H}_{-1,0,-1,0}-72\mathrm{H}_{-1,0,0,0}+96\mathrm{H}_{0,-1,-1,0}-91\mathrm{H}_{0,0,0,0}-69\mathrm{H}_{0,0,0,1}$$
$$-42\mathrm{H}_{0,0,1,0}-42\mathrm{H}_{0,0,1,1}-18\mathrm{H}_{0,1,0,1}-18\mathrm{H}_{0,1,1,0}-18\mathrm{H}_{0,1,1,1})-\frac{32}{3}(1+7x)\mathrm{H}_{0,-1,0,0}$$
$$-\frac{8}{3}(1+21x+14x^2)\mathrm{H}_{1,0,1}+\frac{64}{3}(2-x^{-1}+2x-x^2)\mathrm{H}_{-1,-1,0}+\frac{1}{9}(591-1133x)\mathrm{H}_{1,1}$$
$$-\frac{4}{3}(3-13x)\mathrm{H}_0\zeta_3+\frac{8}{9}(5-117x)\mathrm{H}_{0,1,1}-\frac{8}{3}(7+4x^{-1}-29x-18x^2)\mathrm{H}_1\zeta_2$$
$$+\frac{64}{9}(8+3x^{-1}+8x+3x^2)\mathrm{H}_{-1,0,1}+\frac{64}{9}(8+9x^{-1}+8x+9x^2)\mathrm{H}_{-1,0,0}$$
$$-\frac{32}{9}(10+9x^{-1}+10x+9x^2)\mathrm{H}_{-1}\zeta_2+\frac{16}{9}(15+24x^{-1}-11x+12x^2)\mathrm{H}_{0,-1,0}$$
$$-\frac{8}{3}(5-11x)\mathrm{H}_{0,1}\zeta_2-\frac{4}{15}(23+63x){\zeta_2}^2-\frac{16}{3}(37+43x+12x^2)\mathrm{H}_{0,0,0}+\frac{2}{3}(59-113x)\mathrm{H}_{1,0,0}$$
$$+\frac{8}{9}(75-106x)\mathrm{H}_1+\frac{4}{9}(103-133x+84x^2)\mathrm{H}_{1,1,0}-\frac{8}{9}(108+221x+66x^2)\mathrm{H}_{0,0,1}$$
$$+\frac{8}{9}(138+221x+90x^2)\mathrm{H}_0\zeta_2+\frac{16}{27}(158+141x^{-1}+158x+141x^2)\mathrm{H}_{-1,0}$$
$$-\frac{4}{9}(129-325x+132x^2)\zeta_3-\frac{1}{27}(341+5617x)\mathrm{H}_{0,1}+\frac{64}{3}(2\mathrm{H}_{0,-1}\zeta_2-3\mathrm{H}_{0,0,-1,0})x$$
$$+\frac{1}{27}(625-3283x)\mathrm{H}_{1,0}+\frac{1}{27}(2869+5617x+2256x^2)\zeta_2+\frac{1}{81}(4105-2911x)\mathrm{H}_0$$
$$-\frac{1}{81}(16627+13429x+6768x^2)\mathrm{H}_{0,0}+\frac{1}{216}(32905-20971x)+\frac{32}{3}(2-5x)\mathrm{H}_{1,1,1}$$
$$+\left\{-\frac{341}{36}-\frac{592}{9}\zeta_5+\frac{1348}{3}\zeta_3-\frac{5491}{27}\zeta_2-\frac{352}{9}\zeta_2\zeta_3-\frac{16472}{135}{\zeta_2}^2\right\}\delta(1-x)\Bigg)$$
$$+C_F n_f^2\Bigg(-\frac{1}{729}p_{\mathrm{qq}}(x)(4357-864\zeta_3-7236\zeta_2+11610\mathrm{H}_0-3888\mathrm{H}_0\zeta_2+4230\mathrm{H}_1$$
$$-1296\mathrm{H}_1\zeta_2+10980\mathrm{H}_{0,0}+7236\mathrm{H}_{0,1}+3132\mathrm{H}_{1,0}+3132\mathrm{H}_{1,1}+4968\mathrm{H}_{0,0,0}+3888\mathrm{H}_{0,0,1}$$
$$+2592\mathrm{H}_{0,1,0}+2592\mathrm{H}_{0,1,1}+1296\mathrm{H}_{1,0,0}+1296\mathrm{H}_{1,0,1}+1296\mathrm{H}_{1,1,0}+1296\mathrm{H}_{1,1,1})$$
$$+\frac{2}{9}(1-3x)(10\zeta_2-19\mathrm{H}_1-10\mathrm{H}_{0,1}-6\mathrm{H}_{1,0}-6\mathrm{H}_{1,1})-\frac{4}{27}(17-59x)\mathrm{H}_{0,0}-\frac{1}{27}(171-577x)$$
$$-\frac{2}{81}(197-803x)\mathrm{H}_0+\left\{-\frac{9517}{486}-\frac{152}{81}\zeta_3-\frac{860}{27}\zeta_2-\frac{32}{27}{\zeta_2}^2\right\}\delta(1-x)\Bigg)$$
$$+C_F^2 C_A\Bigg(\frac{1}{405}p_{\mathrm{qq}}(-x)(197640\zeta_5-98550\zeta_3-92260\zeta_2-338040\zeta_2\zeta_3+33102{\zeta_2}^2$$
$$-47610\mathrm{H}_{-1}\zeta_3+97080\mathrm{H}_{-1}\zeta_2+35856\mathrm{H}_{-1}{\zeta_2}^2+139905\mathrm{H}_0-99990\mathrm{H}_0\zeta_3-17655\mathrm{H}_0\zeta_2$$
$$+38124\mathrm{H}_0{\zeta_2}^2+1608120\mathrm{H}_{-1,-1}\zeta_3+27720\mathrm{H}_{-1,-1}\zeta_2-78680\mathrm{H}_{-1,0}-860760\mathrm{H}_{-1,0}\zeta_3$$
$$-33660\mathrm{H}_{-1,0}\zeta_2-1088640\mathrm{H}_{0,-1}\zeta_3+43020\mathrm{H}_{0,-1}\zeta_2+137620\mathrm{H}_{0,0}+232200\mathrm{H}_{0,0}\zeta_3$$
$$-133470\mathrm{H}_{0,0}\zeta_2+52920\mathrm{H}_{0,1}-182520\mathrm{H}_{0,1}\zeta_3-107730\mathrm{H}_{0,1}\zeta_2-1594080\mathrm{H}_{-1,-1,-1}\zeta_2$$
$$+244800\mathrm{H}_{-1,-1,0}+1864080\mathrm{H}_{-1,-1,0}\zeta_2+1134000\mathrm{H}_{-1,0,-1}\zeta_2-276180\mathrm{H}_{-1,0,0}$$
$$-854280\mathrm{H}_{-1,0,0}\zeta_2+25320\mathrm{H}_{-1,0,1}+113400\mathrm{H}_{-1,0,1}\zeta_2+1143720\mathrm{H}_{0,-1,-1}\zeta_2$$
$$-208860\mathrm{H}_{0,-1,0}-1185840\mathrm{H}_{0,-1,0}\zeta_2-654480\mathrm{H}_{0,0,-1}\zeta_2+143385\mathrm{H}_{0,0,0}+191160\mathrm{H}_{0,0,0}\zeta_2$$

$$
\begin{aligned}
&- 28860\mathrm{H}_{0,0,1} - 183600\mathrm{H}_{0,0,1}\zeta_2 - 6480\mathrm{H}_{0,1,0} - 126360\mathrm{H}_{0,1,0}\zeta_2 - 6480\mathrm{H}_{0,1,1}\\
&- 98280\mathrm{H}_{0,1,1}\zeta_2 - 100080\mathrm{H}_{-1,-1,-1,0} + 220680\mathrm{H}_{-1,-1,0,0} - 77760\mathrm{H}_{-1,-1,0,1}\\
&+ 149940\mathrm{H}_{-1,0,-1,0} - 273330\mathrm{H}_{-1,0,0,0} + 65160\mathrm{H}_{-1,0,0,1} + 38880\mathrm{H}_{-1,0,1,0} + 83520\mathrm{H}_{-1,0,1,1}\\
&+ 151560\mathrm{H}_{0,-1,-1,0} - 272520\mathrm{H}_{0,-1,0,0} + 32760\mathrm{H}_{0,-1,0,1} - 183960\mathrm{H}_{0,0,-1,0}\\
&+ 173970\mathrm{H}_{0,0,0,0} + 84690\mathrm{H}_{0,0,0,1} - 17010\mathrm{H}_{0,0,1,0} - 41760\mathrm{H}_{0,0,1,1} + 6480\mathrm{H}_{0,1,0,0}\\
&- 1620\mathrm{H}_{0,1,0,1} + 1620\mathrm{H}_{0,1,1,0} - 155520\mathrm{H}_{-1,-1,-1,-1,0} + 1134000\mathrm{H}_{-1,-1,-1,0,0}\\
&+ 1516320\mathrm{H}_{-1,-1,-1,0,1} + 123120\mathrm{H}_{-1,-1,0,-1,0} - 916920\mathrm{H}_{-1,-1,0,0,0} - 1643760\mathrm{H}_{-1,-1,0,0,1}\\
&- 509760\mathrm{H}_{-1,-1,0,1,0} - 682560\mathrm{H}_{-1,-1,0,1,1} + 142560\mathrm{H}_{-1,0,-1,-1,0} - 792720\mathrm{H}_{-1,0,-1,0,0}\\
&- 1062720\mathrm{H}_{-1,0,-1,0,1} - 194400\mathrm{H}_{-1,0,0,-1,0} + 381240\mathrm{H}_{-1,0,0,0,0} + 735480\mathrm{H}_{-1,0,0,0,1}\\
&+ 294840\mathrm{H}_{-1,0,0,1,0} + 416880\mathrm{H}_{-1,0,0,1,1} + 142560\mathrm{H}_{-1,0,1,0,0} + 79920\mathrm{H}_{-1,0,1,0,1}\\
&+ 15120\mathrm{H}_{-1,0,1,1,0} + 38880\mathrm{H}_{-1,0,1,1,1} + 162000\mathrm{H}_{0,-1,-1,-1,0} - 795960\mathrm{H}_{0,-1,-1,0,0}\\
&- 1062720\mathrm{H}_{0,-1,-1,0,1} - 120960\mathrm{H}_{0,-1,0,-1,0} + 568080\mathrm{H}_{0,-1,0,0,0} + 1011960\mathrm{H}_{0,-1,0,0,1}\\
&+ 280800\mathrm{H}_{0,-1,0,1,0} + 369360\mathrm{H}_{0,-1,0,1,1} - 142560\mathrm{H}_{0,0,-1,-1,0} + 453600\mathrm{H}_{0,0,-1,0,0}\\
&+ 583200\mathrm{H}_{0,0,-1,0,1} + 125280\mathrm{H}_{0,0,0,-1,0} - 81000\mathrm{H}_{0,0,0,0,0} - 178200\mathrm{H}_{0,0,0,0,1}\\
&- 91800\mathrm{H}_{0,0,0,1,0} - 130680\mathrm{H}_{0,0,0,1,1} - 56160\mathrm{H}_{0,0,1,0,0} - 36720\mathrm{H}_{0,0,1,0,1} - 10800\mathrm{H}_{0,0,1,1,0}\\
&- 19440\mathrm{H}_{0,0,1,1,1} - 12960\mathrm{H}_{0,1,0,-1,0} + 30240\mathrm{H}_{0,1,0,0,0} + 49680\mathrm{H}_{0,1,0,0,1} - 12960\mathrm{H}_{0,1,0,1,0}\\
&+ 30240\mathrm{H}_{0,1,1,0,0} + 6480\mathrm{H}_{0,1,1,0,1} - 6480\mathrm{H}_{0,1,1,1,0}) + \frac{1}{6480}p_{\mathrm{qq}}(x)(2292435 + 4812480\zeta_5\\
&+ 2680080\zeta_3 + 1050360\zeta_2 - 1347840\zeta_2\zeta_3 - 1746432{\zeta_2}^2 + 5132100\mathrm{H}_0 - 1625760\mathrm{H}_0\zeta_3\\
&+ 4330560\mathrm{H}_0\zeta_2 + 271296\mathrm{H}_0{\zeta_2}^2 + 1001340\mathrm{H}_1 - 4821120\mathrm{H}_{1,0,-1}\zeta_2 + 25920\mathrm{H}_{0,-1,0,0}\\
&+ 1565568\mathrm{H}_1{\zeta_2}^2 + 6048000\mathrm{H}_{0,-1}\zeta_3 + 2967840\mathrm{H}_{0,-1}\zeta_2 + 1045320\mathrm{H}_{0,0} + 2056320\mathrm{H}_{0,0}\zeta_3\\
&- 188640\mathrm{H}_{0,0}\zeta_2 - 1679800\mathrm{H}_{0,1} + 7568640\mathrm{H}_{0,1}\zeta_3 + 1260000\mathrm{H}_{0,1}\zeta_2 - 3855640\mathrm{H}_{1,0}\\
&+ 9849600\mathrm{H}_{1,0}\zeta_3 + 501120\mathrm{H}_{1,0}\zeta_2 - 3033000\mathrm{H}_{1,1} + 10108800\mathrm{H}_{1,1}\zeta_3 + 1229760\mathrm{H}_{1,1}\zeta_2\\
&- 7724160\mathrm{H}_{0,-1,-1}\zeta_2 + 1853280\mathrm{H}_{0,-1,0} + 4371840\mathrm{H}_{0,-1,0}\zeta_2 + 2695680\mathrm{H}_{0,0,-1}\zeta_2\\
&- 851520\mathrm{H}_{0,0,0} + 1192320\mathrm{H}_{0,0,0}\zeta_2 - 5074800\mathrm{H}_{0,0,1} + 1762560\mathrm{H}_{0,0,1}\zeta_2 - 4427520\mathrm{H}_{0,1,0}\\
&+ 1866240\mathrm{H}_{0,1,0}\zeta_2 - 4648320\mathrm{H}_{0,1,1} + 1503360\mathrm{H}_{0,1,1}\zeta_2 + 1838880\mathrm{H}_1\zeta_3 + 2321280\mathrm{H}_1\zeta_2\\
&+ 4008960\mathrm{H}_{1,0,0}\zeta_2 - 4279680\mathrm{H}_{1,0,1} + 1520640\mathrm{H}_{1,0,1}\zeta_2 - 5107680\mathrm{H}_{1,1,0} + 1831680\mathrm{H}_{1,1,0}\zeta_2\\
&- 3741120\mathrm{H}_{1,1,1} + 1313280\mathrm{H}_{1,1,1}\zeta_2 + 3499200\mathrm{H}_{0,-1,-1,0} - 3673200\mathrm{H}_{1,0,0}\\
&- 1218240\mathrm{H}_{0,-1,0,1} + 1382400\mathrm{H}_{0,0,-1,0} - 99360\mathrm{H}_{0,0,0,0} - 591840\mathrm{H}_{0,0,0,1} - 2904480\mathrm{H}_{0,0,1,0}\\
&- 2888640\mathrm{H}_{0,0,1,1} - 2897280\mathrm{H}_{0,1,0,0} - 2472480\mathrm{H}_{0,1,0,1} - 3114720\mathrm{H}_{0,1,1,0} - 2597760\mathrm{H}_{0,1,1,1}\\
&+ 1391040\mathrm{H}_{1,0,-1,0} - 2158560\mathrm{H}_{1,0,0,0} - 1005120\mathrm{H}_{1,0,0,1} - 2914560\mathrm{H}_{1,0,1,0}\\
&- 3841920\mathrm{H}_{1,1,0,0} - 2030400\mathrm{H}_{1,1,0,1} - 2364480\mathrm{H}_{1,1,1,0} - 1900800\mathrm{H}_{1,1,1,1} - 2257920\mathrm{H}_{1,0,1,1}\\
&- 2937600\mathrm{H}_{0,-1,-1,-1,0} + 6462720\mathrm{H}_{0,-1,-1,0,0} + 6255360\mathrm{H}_{0,-1,-1,0,1} + 2246400\mathrm{H}_{0,-1,0,-1,0}\\
&- 2453760\mathrm{H}_{0,-1,0,0,0} - 3144960\mathrm{H}_{0,-1,0,0,1} + 241920\mathrm{H}_{0,-1,0,1,0} + 380160\mathrm{H}_{0,-1,0,1,1}
\end{aligned}
$$

$$+7050240\mathrm{H}_{0,0,-1,-1,0}-3525120\mathrm{H}_{0,0,-1,0,0}+829440\mathrm{H}_{0,0,-1,0,1}-1589760\mathrm{H}_{0,0,0,-1,0}$$
$$-1296000\mathrm{H}_{0,0,0,0,0}-984960\mathrm{H}_{0,0,0,0,1}-535680\mathrm{H}_{0,0,0,1,0}-535680\mathrm{H}_{0,0,0,1,1}$$
$$-1866240\mathrm{H}_{0,0,1,0,0}-622080\mathrm{H}_{0,0,1,0,1}-138240\mathrm{H}_{0,0,1,1,0}-311040\mathrm{H}_{0,0,1,1,1}$$
$$+3628800\mathrm{H}_{0,1,0,-1,0}-4320000\mathrm{H}_{0,1,0,0,0}+915840\mathrm{H}_{0,1,0,0,1}-120960\mathrm{H}_{0,1,0,1,0}$$
$$-328320\mathrm{H}_{0,1,0,1,1}-2488320\mathrm{H}_{0,1,1,0,0}-207360\mathrm{H}_{0,1,1,0,1}+535680\mathrm{H}_{0,1,1,1,0}$$
$$+6186240\mathrm{H}_{1,0,-1,-1,0}+1900800\mathrm{H}_{1,0,-1,0,0}+7914240\mathrm{H}_{1,0,-1,0,1}+691200\mathrm{H}_{1,0,0,-1,0}$$
$$-4337280\mathrm{H}_{1,0,0,0,0}-2108160\mathrm{H}_{1,0,0,0,1}+241920\mathrm{H}_{1,0,0,1,0}+414720\mathrm{H}_{1,0,0,1,1}$$
$$-2661120\mathrm{H}_{1,0,1,0,0}-380160\mathrm{H}_{1,0,1,0,1}+103680\mathrm{H}_{1,0,1,1,0}-311040\mathrm{H}_{1,0,1,1,1}$$
$$+5080320\mathrm{H}_{1,1,0,-1,0}-6272640\mathrm{H}_{1,1,0,0,0}+1693440\mathrm{H}_{1,1,0,0,1}+103680\mathrm{H}_{1,1,0,1,0}$$
$$-241920\mathrm{H}_{1,1,0,1,1}-2937600\mathrm{H}_{1,1,1,0,0}-69120\mathrm{H}_{1,1,1,0,1}+622080\mathrm{H}_{1,1,1,1,0})$$
$$-4(1+8x^{-1}-26x-4x^2)\mathrm{H}_{0,1}\zeta_2-\frac{8}{3}(1+40x^{-1}-2x-40x^2)\mathrm{H}_{1,0,0,1}-4(1-3x)\mathrm{H}_{1,0,1,0}$$
$$-\frac{320}{3}(1-2x)\mathrm{H}_{0,0,-1,0,0}+4(1-x)(7\mathrm{H}_1{\zeta_2}^2-94\mathrm{H}_{1,0}\zeta_3-22\mathrm{H}_{1,1}\zeta_3+92\mathrm{H}_{1,0,-1}\zeta_2$$
$$-108\mathrm{H}_{1,0,0}\zeta_2-12\mathrm{H}_{1,0,1}\zeta_2-72\mathrm{H}_{1,1,0}\zeta_2-12\mathrm{H}_{1,1,1}\zeta_2-8\mathrm{H}_{1,0,-1,-1,0}-44\mathrm{H}_{1,0,-1,0,0}$$
$$-96\mathrm{H}_{1,0,-1,0,1}-20\mathrm{H}_{1,0,0,-1,0}+36\mathrm{H}_{1,0,0,0,0}+80\mathrm{H}_{1,0,0,0,1}+4\mathrm{H}_{1,0,0,1,0}+4\mathrm{H}_{1,0,0,1,1}$$
$$-12\mathrm{H}_{1,0,1,0,0}-48\mathrm{H}_{1,1,0,-1,0}+40\mathrm{H}_{1,1,0,0,0}+36\mathrm{H}_{1,1,0,0,1}-28\mathrm{H}_{1,1,1,0,0})$$
$$-\frac{4}{15}(1+x)(153\mathrm{H}_{-1}{\zeta_2}^2+2970\mathrm{H}_{-1,-1}\zeta_3-1350\mathrm{H}_{-1,0}\zeta_3-3780\mathrm{H}_{-1,-1,-1}\zeta_2$$
$$+2640\mathrm{H}_{-1,-1,0}\zeta_2+2100\mathrm{H}_{-1,0,-1}\zeta_2-1680\mathrm{H}_{-1,0,0}\zeta_2+60\mathrm{H}_{-1,0,1}\zeta_2+195\mathrm{H}_{0,1,1,1}$$
$$-360\mathrm{H}_{-1,-1,-1,-1,0}+2460\mathrm{H}_{-1,-1,-1,0,0}+3600\mathrm{H}_{-1,-1,-1,0,1}+360\mathrm{H}_{-1,-1,0,-1,0}$$
$$-960\mathrm{H}_{-1,-1,0,0,0}-2100\mathrm{H}_{-1,-1,0,0,1}-120\mathrm{H}_{-1,-1,0,1,0}-120\mathrm{H}_{-1,-1,0,1,1}+360\mathrm{H}_{-1,0,-1,-1,0}$$
$$-1380\mathrm{H}_{-1,0,-1,0,0}-1920\mathrm{H}_{-1,0,-1,0,1}-540\mathrm{H}_{-1,0,0,-1,0}+540\mathrm{H}_{-1,0,0,0,0}+1260\mathrm{H}_{-1,0,0,0,1}$$
$$+60\mathrm{H}_{-1,0,0,1,0}+60\mathrm{H}_{-1,0,0,1,1}-240\mathrm{H}_{-1,0,1,0,0}+40\mathrm{H}_{0,1,0,1,0}-60\mathrm{H}_{0,1,0,1,1}+60\mathrm{H}_{0,1,1,1,0})$$
$$+\frac{8}{3}(1+2x)\mathrm{H}_{1,1,0,1}+\frac{32}{3}(1+19x)\mathrm{H}_{0,1,0,-1,0}+\frac{16}{3}(2-23x)\mathrm{H}_{0,0,-1}\zeta_2-\frac{32}{3}(2+x)\mathrm{H}_{0,-1,0,1,0}$$
$$+\frac{8}{3}(3+11x)(\mathrm{H}_{0,0,1,0,1}-\mathrm{H}_{0,0,1,1,0})-\frac{4}{3}(5+4x^{-1}-9x-4x^2)\mathrm{H}_{1,0}\zeta_2+\frac{4}{3}(19-35x)\mathrm{H}_{1,1,1,0}$$
$$-\frac{16}{3}(5-41x)\mathrm{H}_{0,0,0,-1,0}+\frac{32}{3}(5-13x)\mathrm{H}_{0,1,0,0,0}+\frac{8}{3}(5+3x)(6\mathrm{H}_{0,0,0,1,0}+\mathrm{H}_{0,0,1,0,0})$$
$$+\frac{16}{3}(7-26x)\mathrm{H}_{0,1,0,0,1}-\frac{8}{3}(7+5x)(5\mathrm{H}_{0,0,1}\zeta_2-6\mathrm{H}_{0,0,0,1,1})+\frac{32}{3}(7+8x)\mathrm{H}_{0,0,-1,0,1}$$
$$-\frac{4}{3}(21-31x)\mathrm{H}_{1,0,1,1}-24(9+4x)\mathrm{H}_{0,0,0}\zeta_2+\frac{32}{3}(11+85x)\mathrm{H}_{0,-1,-1,0,1}-\frac{64}{3}(11-6x)\mathrm{H}_{0,0}\zeta_3$$
$$-16(14+57x)\mathrm{H}_{0,-1,-1}\zeta_2+\frac{32}{3}(16-7x)\mathrm{H}_{0,0,-1,-1,0}-\frac{4}{3}(121+362x+24x^2)\mathrm{H}_{0,0,1,0}$$
$$+\frac{16}{3}(8+23x)\mathrm{H}_{0,1,1,0,0}-\frac{16}{3}(25+86x)\mathrm{H}_{0,-1,0,0,1}+\frac{16}{3}(31-x)\mathrm{H}_{0,-1,0,-1,0}$$
$$-4(37+4x^{-1}-23x-4x^2)\mathrm{H}_{1,1}\zeta_2+\frac{32}{3}(37-10x)\zeta_2\zeta_3-\frac{8}{3}(43-89x)\mathrm{H}_{0,1,0}\zeta_2$$
$$+\frac{32}{15}(43+45x)\mathrm{H}_0{\zeta_2}^2+\frac{4}{3}(59+80x^{-1}-13x-80x^2)\mathrm{H}_{1,1,0,0}+\frac{16}{3}(62+79x)\mathrm{H}_{0,-1,-1,0,0}$$

$$+\frac{8}{3}(71+36x)\mathrm{H}_{0,0,0,0,1}+\frac{8}{3}(79+209x)\mathrm{H}_{0,-1,0}\zeta_2+\frac{4}{3}(121-24x^{-1}+79x-24x^2)\mathrm{H}_{-1,0,-1,0}$$
$$-\frac{8}{3}(80-137x)\mathrm{H}_{1,1,1}-\frac{8}{3}(95+13x)\mathrm{H}_{0,-1,0,0,0}+\frac{4}{3}(107-144x^{-1}-53x+144x^2)\mathrm{H}_{1,0,-1,0}$$
$$+\frac{8}{3}(107+147x+84x^2)\mathrm{H}_{0,0,-1,0}-\frac{8}{3}(109-12x^{-1}+73x-12x^2)\mathrm{H}_{-1,-1,-1,0}$$
$$+\frac{16}{3}(127+6x^{-1}+127x+6x^2)\mathrm{H}_{-1,0,1,0}-\frac{4}{3}(140+164x^{-1}-95x-164x^2)\mathrm{H}_1\zeta_3$$
$$-4(165+136x^{-1}+187x+136x^2)\mathrm{H}_{-1,-1,0,0}-\frac{4}{3}(193+617x+24x^2)\mathrm{H}_{0,0,1,1}$$
$$+\frac{16}{3}(175+6x^{-1}+175x+6x^2)\mathrm{H}_{-1,0,1,1}-\frac{16}{3}(200+174x^{-1}+209x+174x^2)\mathrm{H}_{-1,-1,0,1}$$
$$-\frac{4}{3}(197+15x+80x^2)\mathrm{H}_{0,1,0,0}-\frac{4}{3}(237-408x-154x^2)\mathrm{H}_{1,0,1}-\frac{32}{3}(20+x)\mathrm{H}_{0,-1,-1,-1,0}$$
$$-\frac{2}{3}(211+311x)\mathrm{H}_{0,1,0,1}+\frac{4}{9}(379+94x^{-1}+406x+94x^2)\mathrm{H}_{-1,0,0}-\frac{4}{3}(227+77x)\mathrm{H}_{0,1}\zeta_3$$
$$-\frac{4}{3}(273-216x+154x^2)\mathrm{H}_{0,1,0}+\frac{4}{3}(217-48x^{-1}-155x-24x^2)\mathrm{H}_{0,-1,-1,0}$$
$$-4(429+164x^{-1}+433x+164x^2)\mathrm{H}_{-1,0}\zeta_2-(455-661x)\mathrm{H}_{1,0,0}+2(269+201x)\zeta_5$$
$$+\frac{2}{9}(465-622x^{-1}-1581x-302x^2)\mathrm{H}_1\zeta_2+\frac{2}{9}(549-1056x^{-1}-3917x+1244x^2)\mathrm{H}_{0,-1,0}$$
$$-\frac{2}{3}(563-144x^{-1}-581x+144x^2)\mathrm{H}_{1,0,0,0}+\frac{2}{9}(1743+5148x-188x^2)\mathrm{H}_{0,0,0}$$
$$+\frac{8}{3}(647+204x^{-1}+656x+204x^2)\mathrm{H}_{-1,0,0,1}+\frac{4}{3}(689+48x^{-1}+289x+408x^2)\mathrm{H}_{0,-1,0,0}$$
$$+\frac{4}{3}(691+708x^{-1}+763x+708x^2)\mathrm{H}_{-1,-1}\zeta_2-\frac{4}{3}(898+552x^{-1}+961x+552x^2)\mathrm{H}_{-1}\zeta_3$$
$$-\frac{2}{9}(929-1415x+924x^2)\mathrm{H}_{1,1,0}+\frac{4}{9}(957-622x^{-1}+867x-622x^2)\mathrm{H}_{-1,-1,0}$$
$$+\frac{2}{3}(1187+288x^{-1}+1205x+288x^2)\mathrm{H}_{-1,0,0,0}-\frac{2}{9}(1735+223x+1296x^2)\mathrm{H}_{0,0,0,0}$$
$$-\frac{2}{3}(1313+1469x+656x^2)\mathrm{H}_{0,0,0,1}+\frac{4}{3}(1351+48x^{-1}+75x+696x^2)\mathrm{H}_{0,-1,0,1}$$
$$+\frac{2}{15}(1617+192x+796x^2){\zeta_2}^2-\frac{4}{9}(1270-633x)\mathrm{H}_{0,1,1}-\frac{2}{3}(21-71x)\mathrm{H}_{0,1,1,0}$$
$$+\frac{2}{3}(1741+1469x+992x^2)\mathrm{H}_{0,0}\zeta_2-\frac{8}{9}(582+145x+80x^2)\mathrm{H}_{0,0,1}-\frac{16}{3}(20-x)\mathrm{H}_{0,1,1}\zeta_2$$
$$+\frac{2}{3}(2281+559x+1432x^2)\mathrm{H}_0\zeta_3-\frac{2}{3}(2485+144x^{-1}+305x+1416x^2)\mathrm{H}_{0,-1}\zeta_2$$
$$+\frac{4}{9}(2503+622x^{-1}+2503x+622x^2)\mathrm{H}_{-1,0,1}+\frac{2}{9}(2877+580x+1564x^2)\mathrm{H}_0\zeta_2$$
$$+\frac{2}{9}(3219+1714x+5882x^2)\zeta_3-\frac{2}{9}(4049+1866x^{-1}+4139x+1866x^2)\mathrm{H}_{-1}\zeta_2$$
$$-\frac{2}{27}(9253+3882x^{-1}+9973x+4926x^2)\mathrm{H}_{-1,0}+\frac{1}{162}(28121+235007x+59112x^2)\mathrm{H}_{0,0}$$
$$-\frac{1}{18}(29169-34843x)\mathrm{H}_{1,1}+\frac{1}{54}(40879-108101x-19704x^2)\zeta_2+\frac{4}{3}(167+511x)\mathrm{H}_{0,-1}\zeta_3$$
$$-\frac{1}{54}(49721-81983x)\mathrm{H}_{1,0}-\frac{1}{18}(50101-52955x)\mathrm{H}_1-\frac{1}{54}(77891-108101x)\mathrm{H}_{0,1}$$

$$
-\frac{2}{81}(80219-78011x)\mathrm{H}_0-\frac{1}{432}(1449755-1225601x)-8(4\mathrm{H}_{0,-1,0,1,1}-21\mathrm{H}_{0,0,0,0,0})
-\frac{4}{3}g_1(x)-\frac{4}{3}g_2(x)+\left\{\frac{9161}{12}-\frac{4952}{9}\zeta_5-2141\zeta_3+\frac{1016}{3}{\zeta_3}^2+\frac{104189}{54}\zeta_2
-\frac{6644}{9}\zeta_2\zeta_3+\frac{87632}{135}{\zeta_2}^2-\frac{33556}{315}{\zeta_2}^3)\delta(1-x)\right)
+C_FC_A^2\left(-\frac{1}{405}p_{\mathrm{qq}}(-x)(31860\zeta_5-51300\zeta_3-44240\zeta_2-70200\zeta_2\zeta_3+5535{\zeta_2}^2
+31680\mathrm{H}_{-1}\zeta_3+52320\mathrm{H}_{-1}\zeta_2+5076\mathrm{H}_{-1}{\zeta_2}^2+53415\mathrm{H}_0-49860\mathrm{H}_0\zeta_3-16320\mathrm{H}_0\zeta_2
+8640\mathrm{H}_0{\zeta_2}^2+390960\mathrm{H}_{-1,-1}\zeta_3-39600\mathrm{H}_{-1,-1}\zeta_2-42580\mathrm{H}_{-1,0}-179280\mathrm{H}_{-1,0}\zeta_3
+26100\mathrm{H}_{-1,0}\zeta_2-247320\mathrm{H}_{0,-1}\zeta_3+51480\mathrm{H}_{0,-1}\zeta_2+61385\mathrm{H}_{0,0}+38880\mathrm{H}_{0,0}\zeta_3
-52290\mathrm{H}_{0,0}\zeta_2+22950\mathrm{H}_{0,1}-49680\mathrm{H}_{0,1}\zeta_3-29160\mathrm{H}_{0,1}\zeta_2-388800\mathrm{H}_{-1,-1,-1}\zeta_2
+96480\mathrm{H}_{-1,-1,0}+449280\mathrm{H}_{-1,-1,0}\zeta_2+262440\mathrm{H}_{-1,0,-1}\zeta_2-116760\mathrm{H}_{-1,0,0}
-185760\mathrm{H}_{-1,0,0}\zeta_2-4080\mathrm{H}_{-1,0,1}+34560\mathrm{H}_{-1,0,1}\zeta_2+265680\mathrm{H}_{0,-1,-1}\zeta_2-81480\mathrm{H}_{0,-1,0}
-273240\mathrm{H}_{0,-1,0}\zeta_2-138240\mathrm{H}_{0,0,-1}\zeta_2+55560\mathrm{H}_{0,0,0}+34560\mathrm{H}_{0,0,0}\zeta_2+2040\mathrm{H}_{0,0,1}
-47520\mathrm{H}_{0,0,1}\zeta_2-34560\mathrm{H}_{0,1,0}\zeta_2-25920\mathrm{H}_{0,1,1}\zeta_2-79200\mathrm{H}_{-1,-1,-1,0}+134640\mathrm{H}_{-1,-1,0,0}
+87120\mathrm{H}_{-1,0,-1,0}-133020\mathrm{H}_{-1,0,0,0}-7920\mathrm{H}_{-1,0,0,1}+15840\mathrm{H}_{-1,0,1,1}+87120\mathrm{H}_{0,-1,-1,0}
-137880\mathrm{H}_{0,-1,0,0}-7920\mathrm{H}_{0,-1,0,1}-88200\mathrm{H}_{0,0,-1,0}+66870\mathrm{H}_{0,0,0,0}+37080\mathrm{H}_{0,0,0,1}
-7920\mathrm{H}_{0,0,1,1}+3240\mathrm{H}_{0,1,0,0}+220320\mathrm{H}_{-1,-1,-1,0,0}+388800\mathrm{H}_{-1,-1,-1,0,1}
-12960\mathrm{H}_{-1,-1,0,-1,0}-164160\mathrm{H}_{-1,-1,0,0,0}-401760\mathrm{H}_{-1,-1,0,0,1}-120960\mathrm{H}_{-1,-1,0,1,0}
-172800\mathrm{H}_{-1,-1,0,1,1}-6480\mathrm{H}_{-1,0,-1,-1,0}-142560\mathrm{H}_{-1,0,-1,0,0}-265680\mathrm{H}_{-1,0,-1,0,1}
-8640\mathrm{H}_{-1,0,0,-1,0}+62640\mathrm{H}_{-1,0,0,0,0}+155520\mathrm{H}_{-1,0,0,0,1}+52920\mathrm{H}_{-1,0,0,1,0}
+86400\mathrm{H}_{-1,0,0,1,1}+30240\mathrm{H}_{-1,0,1,0,0}+10800\mathrm{H}_{-1,0,1,0,1}-10800\mathrm{H}_{-1,0,1,1,0}
-143640\mathrm{H}_{0,-1,-1,0,0}-265680\mathrm{H}_{0,-1,-1,0,1}+4320\mathrm{H}_{0,-1,0,-1,0}+96120\mathrm{H}_{0,-1,0,0,0}
+234360\mathrm{H}_{0,-1,0,0,1}+62640\mathrm{H}_{0,-1,0,1,0}+86400\mathrm{H}_{0,-1,0,1,1}+8640\mathrm{H}_{0,0,-1,-1,0}
+75600\mathrm{H}_{0,0,-1,0,0}+142560\mathrm{H}_{0,0,-1,0,1}+10800\mathrm{H}_{0,0,0,-1,0}-12960\mathrm{H}_{0,0,0,0,0}
-32400\mathrm{H}_{0,0,0,0,1}-12960\mathrm{H}_{0,0,0,1,0}-21600\mathrm{H}_{0,0,0,1,1}-10800\mathrm{H}_{0,0,1,0,0}-4320\mathrm{H}_{0,0,1,0,1}
+4320\mathrm{H}_{0,0,1,1,0}-4320\mathrm{H}_{0,1,0,-1,0}+8640\mathrm{H}_{0,1,0,0,0}+15120\mathrm{H}_{0,1,0,0,1}-4320\mathrm{H}_{0,1,0,1,0}
+8640\mathrm{H}_{0,1,1,0,0}+2160\mathrm{H}_{0,1,1,0,1}-2160\mathrm{H}_{0,1,1,1,0})-\frac{1}{7290}p_{\mathrm{qq}}(x)(2996875+651240\zeta_5
-797580\zeta_3-3438810\zeta_2-252720\zeta_2\zeta_3-20412{\zeta_2}^2+5966955\mathrm{H}_0-1296000\mathrm{H}_0\zeta_3
-1808460\mathrm{H}_0\zeta_2+252720\mathrm{H}_0{\zeta_2}^2+2281005\mathrm{H}_1-324000\mathrm{H}_1\zeta_3-1694520\mathrm{H}_1\zeta_2
+1183896\mathrm{H}_1{\zeta_2}^2+1399680\mathrm{H}_{0,-1}\zeta_3+680400\mathrm{H}_{0,-1}\zeta_2+4599450\mathrm{H}_{0,0}+699840\mathrm{H}_{0,0}\zeta_3
-1607040\mathrm{H}_{0,0}\zeta_2+3055590\mathrm{H}_{0,1}+2099520\mathrm{H}_{0,1}\zeta_3-1125900\mathrm{H}_{0,1}\zeta_2+1062450\mathrm{H}_{1,0}
+3285360\mathrm{H}_{1,0}\zeta_3-1474200\mathrm{H}_{1,0}\zeta_2+1255230\mathrm{H}_{1,1}+2799360\mathrm{H}_{1,1}\zeta_3-891000\mathrm{H}_{1,1}\zeta_2
-1982880\mathrm{H}_{0,-1,-1}\zeta_2+952560\mathrm{H}_{0,-1,0}+699840\mathrm{H}_{0,-1,0}\zeta_2-155520\mathrm{H}_{0,0,-1}\zeta_2
$$

$$+2502900\mathrm{H}_{0,0,0}-77760\mathrm{H}_{0,0,0}\zeta_2+1551420\mathrm{H}_{0,0,1}-77760\mathrm{H}_{0,0,1}\zeta_2+1070820\mathrm{H}_{0,1,0}$$
$$-213840\mathrm{H}_{0,1,0}\zeta_2+784080\mathrm{H}_{0,1,1}+38880\mathrm{H}_{0,1,1}\zeta_2-2293920\mathrm{H}_{1,0,-1}\zeta_2+887760\mathrm{H}_{1,0,0}$$
$$+252720\mathrm{H}_{1,0,0}\zeta_2+826200\mathrm{H}_{1,0,1}-97200\mathrm{H}_{1,0,1}\zeta_2-42120\mathrm{H}_{1,1,0}-349920\mathrm{H}_{1,1,0}\zeta_2$$
$$+392040\mathrm{H}_{1,1,1}+77760\mathrm{H}_{1,1,1}\zeta_2+1049760\mathrm{H}_{0,-1,-1,0}+505440\mathrm{H}_{0,-1,0,0}-155520\mathrm{H}_{0,-1,0,1}$$
$$+1040040\mathrm{H}_{0,0,-1,0}+1203660\mathrm{H}_{0,0,0,0}+1333260\mathrm{H}_{0,0,0,1}+306180\mathrm{H}_{0,0,1,0}+362880\mathrm{H}_{0,0,1,1}$$
$$+220320\mathrm{H}_{0,1,0,0}+341820\mathrm{H}_{0,1,0,1}-312660\mathrm{H}_{0,1,1,0}+913680\mathrm{H}_{1,0,-1,0}+541080\mathrm{H}_{1,0,0,0}$$
$$+1146960\mathrm{H}_{1,0,0,1}-291600\mathrm{H}_{1,0,1,0}+320760\mathrm{H}_{1,0,1,1}-463320\mathrm{H}_{1,1,0,0}+178200\mathrm{H}_{1,1,0,1}$$
$$-498960\mathrm{H}_{1,1,1,0}-855360\mathrm{H}_{0,-1,-1,-1,0}+1594080\mathrm{H}_{0,-1,-1,0,0}+1555200\mathrm{H}_{0,-1,-1,0,1}$$
$$+622080\mathrm{H}_{0,-1,0,-1,0}-388800\mathrm{H}_{0,-1,0,0,0}-349920\mathrm{H}_{0,-1,0,0,1}+311040\mathrm{H}_{0,-1,0,1,0}$$
$$+388800\mathrm{H}_{0,-1,0,1,1}+1866240\mathrm{H}_{0,0,-1,-1,0}-233280\mathrm{H}_{0,0,-1,0,0}+1088640\mathrm{H}_{0,0,-1,0,1}$$
$$-116640\mathrm{H}_{0,0,0,-1,0}-233280\mathrm{H}_{0,0,0,0,0}+116640\mathrm{H}_{0,0,0,0,1}+116640\mathrm{H}_{0,0,0,1,0}$$
$$+194400\mathrm{H}_{0,0,0,1,1}-427680\mathrm{H}_{0,0,1,0,0}+1477440\mathrm{H}_{0,1,0,-1,0}-1263600\mathrm{H}_{0,1,0,0,0}$$
$$+913680\mathrm{H}_{0,1,0,0,1}-58320\mathrm{H}_{0,1,0,1,0}+19440\mathrm{H}_{0,1,0,1,1}-1088640\mathrm{H}_{0,1,1,0,0}-38880\mathrm{H}_{0,1,1,0,1}$$
$$+19440\mathrm{H}_{0,1,1,1,0}+1632960\mathrm{H}_{1,0,-1,-1,0}+1438560\mathrm{H}_{1,0,-1,0,0}+3110400\mathrm{H}_{1,0,-1,0,1}$$
$$+933120\mathrm{H}_{1,0,0,-1,0}-1127520\mathrm{H}_{1,0,0,0,0}+291600\mathrm{H}_{1,0,0,0,1}+505440\mathrm{H}_{1,0,0,1,0}$$
$$+777600\mathrm{H}_{1,0,0,1,1}-1049760\mathrm{H}_{1,0,1,0,0}+38880\mathrm{H}_{1,0,1,0,1}-38880\mathrm{H}_{1,0,1,1,0}$$
$$+1944000\mathrm{H}_{1,1,0,-1,0}-2021760\mathrm{H}_{1,1,0,0,0}+1205280\mathrm{H}_{1,1,0,0,1}-116640\mathrm{H}_{1,1,0,1,0}$$
$$+77760\mathrm{H}_{1,1,0,1,1}-1555200\mathrm{H}_{1,1,1,0,0}-77760\mathrm{H}_{1,1,1,0,1})-16(1+9x)\mathrm{H}_{0,-1,0}\zeta_2$$
$$+\frac{56}{3}(1-2x)\mathrm{H}_{0,0,-1,0,0}+\frac{4}{3}(1-x)(12\mathrm{H}_1{\zeta_2}^2+84\mathrm{H}_{1,0}\zeta_3+24\mathrm{H}_{1,1}\zeta_3-96\mathrm{H}_{1,0,-1}\zeta_2$$
$$+72\mathrm{H}_{1,0,0}\zeta_2+24\mathrm{H}_{1,1,0}\zeta_2-7\mathrm{H}_{1,1,1,0}+20\mathrm{H}_{0,-1,-1,-1,0}-14\mathrm{H}_{0,-1,0,-1,0}-2\mathrm{H}_{0,-1,0,1,0}$$
$$-26\mathrm{H}_{0,0,-1,-1,0}+2\mathrm{H}_{0,0,1,0,1}-2\mathrm{H}_{0,0,1,1,0}+48\mathrm{H}_{1,0,-1,0,0}+96\mathrm{H}_{1,0,-1,0,1}+24\mathrm{H}_{1,0,0,-1,0}$$
$$-24\mathrm{H}_{1,0,0,0,0}-48\mathrm{H}_{1,0,0,0,1}+48\mathrm{H}_{1,1,0,-1,0}-24\mathrm{H}_{1,1,0,0,0})-\frac{4}{15}(1+x)(12\mathrm{H}_{-1}{\zeta_2}^2$$
$$-720\mathrm{H}_{-1,-1}\zeta_3+360\mathrm{H}_{-1,0}\zeta_3+960\mathrm{H}_{-1,-1,-1}\zeta_2-600\mathrm{H}_{-1,-1,0}\zeta_2-480\mathrm{H}_{-1,0,-1}\zeta_2$$
$$+420\mathrm{H}_{-1,0,0}\zeta_2-50\mathrm{H}_{0,1,1}\zeta_2+720\mathrm{H}_{-1,0,1,0}+1000\mathrm{H}_{-1,0,1,1}+10\mathrm{H}_{1,0,1,0}-480\mathrm{H}_{-1,-1,-1,0,0}$$
$$-960\mathrm{H}_{-1,-1,-1,0,1}+120\mathrm{H}_{-1,-1,0,0,0}+480\mathrm{H}_{-1,-1,0,0,1}+240\mathrm{H}_{-1,0,-1,0,0}+480\mathrm{H}_{-1,0,-1,0,1}$$
$$+120\mathrm{H}_{-1,0,0,-1,0}-120\mathrm{H}_{-1,0,0,0,0}-300\mathrm{H}_{-1,0,0,0,1}+60\mathrm{H}_{-1,0,1,0,0}+45\mathrm{H}_{0,0,0,1,0}$$
$$+75\mathrm{H}_{0,0,0,1,1}+5\mathrm{H}_{0,0,1,0,0}-75\mathrm{H}_{0,1,0,0,1}-5\mathrm{H}_{0,1,0,1,0}+105\mathrm{H}_{0,1,1,0,0})+\frac{16}{3}(1+7x)\mathrm{H}_{0,1,0,0,0}$$
$$-\frac{8}{3}(2+5x)\mathrm{H}_{-1,0,-1,0}-\frac{16}{3}(1+13x)\mathrm{H}_{0,1,0,-1,0}-\frac{2}{3}(1+33x)\mathrm{H}_{0,1}\zeta_2+\frac{4}{3}(1+35x)\mathrm{H}_{0,0,-1}\zeta_2$$
$$-8(6-5x)\mathrm{H}_{0,-1,-1,0}+4(41-8x^{-1}-39x+8x^2)\mathrm{H}_{1,0,0,0}$$
$$-\frac{4}{3}(2+3x)\mathrm{H}_{1,0,1,1}+\frac{8}{3}(1-5x)\mathrm{H}_{0,0,0,-1,0}-\frac{8}{3}(3-24x^{-1}-5x+24x^2)\mathrm{H}_{1,0,-1,0}$$
$$+\frac{4}{3}(4-27x^{-1}-14x+27x^2)\mathrm{H}_{1,0}\zeta_2-\frac{4}{3}(5-51x^{-1}-14x+51x^2)\mathrm{H}_{1,0,0,1}$$
$$+\frac{16}{3}(5+43x)\mathrm{H}_{0,-1,-1}\zeta_2-\frac{8}{3}(5+91x)\mathrm{H}_{0,-1,-1,0,1}+8(4+3x)\mathrm{H}_{0,0,0}\zeta_2+\frac{4}{3}(5-11x)\mathrm{H}_{1,1}\zeta_2$$

$$+\frac{8}{3}(7+5x)\mathrm{H}_{0,-1,0,0,0}-\frac{8}{3}(7+11x)\mathrm{H}_{0,0,-1,0,1}-\frac{8}{3}(5+17x)\mathrm{H}_{0,1,0}\zeta_2+\frac{32}{3}(5+2x)\mathrm{H}_{0,1}\zeta_3$$
$$+\frac{4}{3}(7+89x)\mathrm{H}_{0,-1,0,0,1}+\frac{4}{3}(9-4x)\mathrm{H}_{1,1,0,1}-\frac{8}{3}(11+9x)\mathrm{H}_{0,0,0,0,1}+\frac{4}{3}(11+15x)\mathrm{H}_{0,0,1}\zeta_2$$
$$+\frac{8}{3}(14+15x)\mathrm{H}_{-1,-1,-1,0}+\frac{64}{3}(17+12x^{-1}+17x+12x^2)\mathrm{H}_{-1,-1,0,1}+\frac{4}{3}(17+180x)\mathrm{H}_{0,0,1,1}$$
$$-\frac{4}{3}(19+125x)\mathrm{H}_{0,-1}\zeta_3+\frac{2}{9}(21-1315x+128x^2)\mathrm{H}_0\zeta_2-\frac{2}{27}(4363-5244x+2742x^2)\mathrm{H}_{0,0}$$
$$-\frac{4}{3}(29+67x)\mathrm{H}_{0,-1,-1,0,0}+\frac{4}{9}(30+264x^{-1}+572x-67x^2)\mathrm{H}_{0,-1,0}+2(7+73x)\mathrm{H}_{0,0,1,0}$$
$$-16(31+8x^{-1}+31x+8x^2)\mathrm{H}_{-1,0,0,1}+\frac{4}{3}(25-51x^{-1}-29x+51x^2)\mathrm{H}_{1,1,0,0}$$
$$+\frac{4}{3}(34+75x^{-1}-20x-75x^2)\mathrm{H}_1\zeta_3-\frac{4}{9}(56-197x^{-1}+65x-197x^2)\mathrm{H}_{-1,0,0}$$
$$+\frac{2}{9}(39+1315x-262x^2)\mathrm{H}_{0,0,1}+\frac{8}{3}(34-37x)\mathrm{H}_{1,0,0}+\frac{2}{3}(37+63x)\mathrm{H}_{0,1,0,1}-\frac{8}{3}(49+15x)\zeta_5$$
$$-4(49+8x^{-1}+47x+8x^2)\mathrm{H}_{-1,0,0,0}-\frac{8}{15}(51+20x)\mathrm{H}_0{\zeta_2}^2-\frac{4}{3}(67+27x+48x^2)\mathrm{H}_{0,0,-1,0}$$
$$-\frac{4}{81}(76-12481x)-\frac{4}{9}(77+38x+198x^2)\mathrm{H}_{1,0,1}+\frac{4}{9}(77+41x+198x^2)\mathrm{H}_{1,1,0}$$
$$-\frac{4}{3}(83+5x)\zeta_2\zeta_3+\frac{4}{3}(84-16x+51x^2)\mathrm{H}_{0,1,0,0}-\frac{8}{3}(131+9x+48x^2)\mathrm{H}_{0,-1,0,0}$$
$$+\frac{4}{3}(137+48x^2)\mathrm{H}_{0,0,0,0}+\frac{4}{9}(148+145x)\mathrm{H}_{0,1,1}+\frac{4}{3}(190+96x^{-1}+179x+96x^2)\mathrm{H}_{-1,-1,0,0}$$
$$-\frac{8}{3}(224-9x+96x^2)\mathrm{H}_{0,-1,0,1}+\frac{2}{9}(237+271x+396x^2)\mathrm{H}_{0,1,0}+\frac{1}{9}(2533-1915x)\mathrm{H}_{1,1}$$
$$-\frac{4}{3}(258+192x^{-1}+257x+192x^2)\mathrm{H}_{-1,-1}\zeta_2+\frac{4}{3}(302+144x^{-1}+301x+144x^2)\mathrm{H}_{-1}\zeta_3$$
$$+\frac{4}{27}(3097+1317x^{-1}+3151x+1371x^2)\mathrm{H}_{-1,0}-\frac{4}{9}(348-67x^{-1}+318x-67x^2)\mathrm{H}_{-1,-1,0}$$
$$-\frac{4}{9}(788+67x^{-1}+788x+67x^2)\mathrm{H}_{-1,0,1}+\frac{4}{3}(373+120x^{-1}+372x+120x^2)\mathrm{H}_{-1,0}\zeta_2$$
$$-\frac{4}{9}(408+482x+197x^2)\mathrm{H}_{0,0,0}+\frac{4}{3}(430-3x+192x^2)\mathrm{H}_{0,-1}\zeta_2+\frac{1}{9}(977-1223x)\mathrm{H}_{1,0}$$
$$-\frac{2}{3}(449+401x+186x^2)\mathrm{H}_{0,0}\zeta_2+\frac{2}{3}(315+401x+90x^2)\mathrm{H}_{0,0,0,1}-\frac{2}{3}(31+63x)\mathrm{H}_{0,1,1,0}$$
$$-\frac{2}{9}(721+2035x+1523x^2)\zeta_3-\frac{1}{15}(1095-619x+1296x^2){\zeta_2}^2+\frac{1}{162}(20601-5183x)\mathrm{H}_0$$
$$+\frac{2}{9}(1228+201x^{-1}+1258x+201x^2)\mathrm{H}_{-1}\zeta_2+\frac{2}{9}(502+67x^{-1}-242x+329x^2)\mathrm{H}_1\zeta_2$$
$$-\frac{4}{3}(367+5x+219x^2)\mathrm{H}_0\zeta_3+\frac{1}{27}(3459-2365x+5484x^2)\zeta_2+\frac{1}{27}(8929+2365x)\mathrm{H}_{0,1}$$
$$+\frac{1}{54}(40039-23821x)\mathrm{H}_1+\frac{8}{3}(19\mathrm{H}_{0,0}\zeta_3-6\mathrm{H}_{0,0,0,0,0})+\frac{1}{3}g_1(x)+\frac{1}{3}g_2(x)+\left\{-\frac{1909753}{1944}\right.$$
$$\left.-\frac{416}{3}\zeta_5+\frac{105739}{81}\zeta_3-\frac{248}{3}{\zeta_3}^2-\frac{143282}{81}\zeta_2+540\zeta_2\zeta_3+\frac{25184}{135}{\zeta_2}^2-\frac{3512}{63}{\zeta_2}^3\right\}\delta(1-x)\Bigg)$$
$$+C_FC_An_f\left(-\frac{4}{405}p_{\mathrm{qq}}(-x)(2700\zeta_3+830\zeta_2+1026{\zeta_2}^2-1440\mathrm{H}_{-1}\zeta_3-3000\mathrm{H}_{-1}\zeta_2\right.$$

$$
\begin{aligned}
&- 1305\mathrm{H}_0 + 990\mathrm{H}_0\zeta_3 + 345\mathrm{H}_0\zeta_2 + 1800\mathrm{H}_{-1,-1}\zeta_2 + 1660\mathrm{H}_{-1,0} - 1260\mathrm{H}_{-1,0}\zeta_2 \\
&- 2340\mathrm{H}_{0,-1}\zeta_2 - 1910\mathrm{H}_{0,0} + 450\mathrm{H}_{0,0}\zeta_2 - 3600\mathrm{H}_{-1,-1,0} + 4800\mathrm{H}_{-1,0,0} + 1200\mathrm{H}_{-1,0,1} \\
&+ 3000\mathrm{H}_{0,-1,0} - 1245\mathrm{H}_{0,0,0} - 600\mathrm{H}_{0,0,1} + 3600\mathrm{H}_{-1,-1,-1,0} - 6120\mathrm{H}_{-1,-1,0,0} \\
&- 3960\mathrm{H}_{-1,0,-1,0} + 6120\mathrm{H}_{-1,0,0,0} + 360\mathrm{H}_{-1,0,0,1} - 720\mathrm{H}_{-1,0,1,1} - 3960\mathrm{H}_{0,-1,-1,0} \\
&+ 6120\mathrm{H}_{0,-1,0,0} + 360\mathrm{H}_{0,-1,0,1} + 4500\mathrm{H}_{0,0,-1,0} - 2070\mathrm{H}_{0,0,0,0} - 360\mathrm{H}_{0,0,0,1} + 360\mathrm{H}_{0,0,1,1}) \\
&+ \frac{1}{3645}p_{\mathrm{qq}}(x)(402265 + 66690\zeta_3 - 521010\zeta_2 + 27216{\zeta_2}^2 + 895905\mathrm{H}_0 + 119880\mathrm{H}_0\zeta_3 \\
&- 210870\mathrm{H}_0\zeta_2 + 338895\mathrm{H}_1 + 12960\mathrm{H}_1\zeta_3 - 168480\mathrm{H}_1\zeta_2 - 77760\mathrm{H}_{0,-1}\zeta_2 + 731250\mathrm{H}_{0,0} \\
&- 74520\mathrm{H}_{0,0}\zeta_2 + 491130\mathrm{H}_{0,1} - 103680\mathrm{H}_{0,1}\zeta_2 + 193320\mathrm{H}_{1,0} - 139320\mathrm{H}_{1,0}\zeta_2 + 209520\mathrm{H}_{1,1} \\
&- 81000\mathrm{H}_{1,1}\zeta_2 + 126360\mathrm{H}_{0,-1,0} + 318060\mathrm{H}_{0,0,0} + 220050\mathrm{H}_{0,0,1} + 174150\mathrm{H}_{0,1,0} \\
&+ 142560\mathrm{H}_{0,1,1} + 119880\mathrm{H}_{1,0,0} + 103680\mathrm{H}_{1,0,1} + 38880\mathrm{H}_{1,1,0} + 71280\mathrm{H}_{1,1,1} \\
&+ 155520\mathrm{H}_{0,-1,0,0} + 77760\mathrm{H}_{0,-1,0,1} + 155520\mathrm{H}_{0,0,-1,0} + 74520\mathrm{H}_{0,0,0,0} + 71280\mathrm{H}_{0,0,0,1} \\
&+ 29160\mathrm{H}_{0,0,1,0} + 35640\mathrm{H}_{0,0,1,1} + 22680\mathrm{H}_{0,1,0,0} + 32400\mathrm{H}_{0,1,0,1} - 32400\mathrm{H}_{0,1,1,0} \\
&+ 77760\mathrm{H}_{1,0,-1,0} + 51840\mathrm{H}_{1,0,0,0} + 106920\mathrm{H}_{1,0,0,1} - 29160\mathrm{H}_{1,0,1,0} + 29160\mathrm{H}_{1,0,1,1} \\
&- 42120\mathrm{H}_{1,1,0,0} + 16200\mathrm{H}_{1,1,0,1} - 45360\mathrm{H}_{1,1,1,0}) + \frac{8}{3}(1-x)(5\mathrm{H}_{1,1}\zeta_2 - 5\mathrm{H}_{1,0,0} - 4\mathrm{H}_{0,-1,0,1} \\
&- 4\mathrm{H}_{1,0,-1,0} - 6\mathrm{H}_{1,0,0,0} + \mathrm{H}_{1,0,1,0} - \mathrm{H}_{1,1,0,1} + \mathrm{H}_{1,1,1,0}) - \frac{8}{3}(5 + 2x^{-1} - 5x - 2x^2)\mathrm{H}_{1,0,0,1} \\
&- \frac{8}{9}(1+x)(12\mathrm{H}_{-1}\zeta_3 - 12\mathrm{H}_{-1,-1}\zeta_2 + 6\mathrm{H}_{-1,0}\zeta_2 - 7\mathrm{H}_{0,1,1} - 24\mathrm{H}_{-1,-1,-1,0} + 36\mathrm{H}_{-1,-1,0,0} \\
&+ 24\mathrm{H}_{-1,0,-1,0} - 18\mathrm{H}_{-1,0,0,0} + 24\mathrm{H}_{0,-1,-1,0}) + \frac{8}{15}(1 + 4x + 8x^2){\zeta_2}^2 + \frac{16}{3}(1+7x)\mathrm{H}_{0,-1,0,0} \\
&- \frac{32}{3}(2 - x^{-1} + 2x - x^2)\mathrm{H}_{-1,-1,0} + \frac{8}{3}(3 - 2x^{-1} - 3x + 2x^2)\mathrm{H}_1\zeta_3 - \frac{2}{9}(3 - 35x + 72x^2)\mathrm{H}_{0,1,0} \\
&+ \frac{8}{3}(4 - 5x)\mathrm{H}_{0,1}\zeta_2 + \frac{8}{3}(4 + 3x + 2x^2)\mathrm{H}_0\zeta_3 + \frac{2}{81}(5447 - 13721x) - \frac{16}{3}(1 - x + x^2)\mathrm{H}_{0,1,0,0} \\
&+ \frac{8}{3}(7 + 2x^{-1} - 7x - 2x^2)\mathrm{H}_{1,0}\zeta_2 - \frac{32}{9}(8 + 3x^{-1} + 8x + 3x^2)\mathrm{H}_{-1,0,1} + \frac{4}{9}(20 - 77x)\mathrm{H}_{1,1} \\
&- \frac{32}{9}(8 + 9x^{-1} + 8x + 9x^2)\mathrm{H}_{-1,0,0} + \frac{16}{9}(10 + 9x^{-1} + 10x + 9x^2)\mathrm{H}_{-1}\zeta_2 + \frac{4}{9}(37 - 61x)\mathrm{H}_{1,0} \\
&- \frac{4}{9}(11 - 12x^{-1} + 20x + 48x^2)\mathrm{H}_1\zeta_2 + \frac{2}{9}(169 - 113x + 96x^2)\zeta_3 + \frac{2}{27}(667 - 671x)\mathrm{H}_{0,1} \\
&+ \frac{4}{9}(35 - 4x + 36x^2)(\mathrm{H}_{1,0,1} - \mathrm{H}_{1,1,0}) + \frac{8}{9}(81 + 52x + 36x^2)\mathrm{H}_{0,0,0} + \frac{16}{3}(x^{-1} - x^2)\mathrm{H}_{1,1,0,0} \\
&- \frac{8}{27}(158 + 141x^{-1} + 158x + 141x^2)\mathrm{H}_{-1,0} - \frac{8}{9}(15 + 24x^{-1} - 11x + 12x^2)\mathrm{H}_{0,-1,0} \\
&+ \frac{2}{9}(207 + 149x + 120x^2)\mathrm{H}_{0,0,1} - \frac{2}{9}(267 + 149x + 168x^2)\mathrm{H}_0\zeta_2 + \frac{1}{81}(7651 - 23577x)\mathrm{H}_0 \\
&- \frac{2}{27}(1299 - 671x + 564x^2)\zeta_2 + \frac{2}{27}(1685 - 1219x + 564x^2)\mathrm{H}_{0,0} + \frac{1}{27}(1855 - 4693x)\mathrm{H}_1 \\
&- \frac{8}{3}(x + 2x^2)(\mathrm{H}_{0,0}\zeta_2 - \mathrm{H}_{0,0,0,1}) - \frac{8}{3}(8\mathrm{H}_{0,-1}\zeta_2 - 12\mathrm{H}_{0,0,-1,0} + \mathrm{H}_{0,0,1,0} - \mathrm{H}_{0,1,0,1} + \mathrm{H}_{0,1,1,0})x \\
&+ \left\{\frac{142883}{486} + \frac{8}{3}\zeta_5 - \frac{18314}{81}\zeta_3 + \frac{40862}{81}\zeta_2 - \frac{56}{3}\zeta_2\zeta_3 - \frac{2488}{135}{\zeta_2}^2\right\}\delta(1-x)\Bigg) \,. \qquad \text{(A.4)}
\end{aligned}
$$

Eq. (A.4) includes, with the colour factor $C_F(C_A-2C_F)^2$, the sum of the inverse Mellin transforms of the functions $g_1(N)$ and $g_2(N)$ in Eqs. (2.22). These functions are given by

$$\begin{aligned}
g_1(x) \;=\; & -2(1-(1-x)^{-1})(4\mathrm{H}_{-1}\zeta_2 - 2\mathrm{H}_{-1,0,0} - 4\mathrm{H}_{-1,0,1} + \mathrm{H}_{0,0,0} + 2\mathrm{H}_{0,0,1} - 3\mathrm{H}_0\zeta_2 - 3\zeta_3) \\
& + 2((1-x)^{-2} - (1-x)^{-1})\Big(4\mathrm{H}_{0,-1}\zeta_2 - 2\mathrm{H}_{0,-1,0,0} - 4\mathrm{H}_{0,-1,0,1} + 2\mathrm{H}_{0,0,0,1} - 3\mathrm{H}_{0,0}\zeta_2 \\
& - 3\mathrm{H}_0\zeta_3 + \mathrm{H}_{0,0,0,0} - \frac{2}{5}\zeta_2^2\Big) + 2(1-(1+x)^{-1})(2\mathrm{H}_{0,-1,0} - \mathrm{H}_{0,0,0} + \mathrm{H}_0\zeta_2 + 2\zeta_3) \\
& - ((1+x)^{-2} - (1+x)^{-1})\Big(4\mathrm{H}_{0,0,-1,0} + 2\mathrm{H}_{0,0}\zeta_2 + 4\mathrm{H}_0\zeta_3 - 2\mathrm{H}_{0,0,0,0} + \frac{21}{5}\zeta_2^2\Big)\;, \qquad \text{(A.5)}
\end{aligned}$$

$$\begin{aligned}
g_2(x) \;=\; & -2(2(1-x)^{-3} - 3(1-x)^{-2} + (1-x)^{-1})\Big(4\mathrm{H}_{0,-1}\zeta_2 + \mathrm{H}_{0,0,0,0} - 2\mathrm{H}_{0,-1,0,0} \\
& - 4\mathrm{H}_{0,-1,0,1} + 2\mathrm{H}_{0,0,0,1} - 3\mathrm{H}_{0,0}\zeta_2 - 3\mathrm{H}_0\zeta_3 - \frac{2}{5}\zeta_2^2\Big) - 2\zeta_2(1-(1-x)^{-1}) \\
& - 4((1-x)^{-2} - (1-x)^{-1})(4\mathrm{H}_{-1}\zeta_2 - 2\mathrm{H}_{-1,0,0} - 4\mathrm{H}_{-1,0,1} + 2\mathrm{H}_{0,0,1} + \mathrm{H}_{0,0,0} - 3\mathrm{H}_0\zeta_2 - 3\zeta_3) \\
& + (2(1+x)^{-3} - 3(1+x)^{-2} + (1+x)^{-1})\Big(4\mathrm{H}_{0,0,-1,0} - 2\mathrm{H}_{0,0,0,0} + 2\mathrm{H}_{0,0}\zeta_2 + 4\mathrm{H}_0\zeta_3 + \frac{21}{5}\zeta_2^2\Big) \\
& + 4((1+x)^{-2} - (1+x)^{-1})(2\mathrm{H}_{0,-1,0} - \mathrm{H}_{0,0,0} + \mathrm{H}_0\zeta_2 + 2\zeta_3) \\
& - 2(1-(1+x)^{-1})(2\mathrm{H}_{-1,0} + 2\mathrm{H}_{0,1} - \zeta_2) + (\zeta_2 - \zeta_3)\delta(1-x)\;. \qquad \text{(A.6)}
\end{aligned}$$

Note that the $1/(1-x)$ terms in Eqs. (A.5) and (A.6) do not imply the presence of +-distributions, but large-x cancellations between the harmonic polylogarithms in the round brackets. These cancellations become numerically problematic for $1-x \ll 1$, a region unavoidable in moment and Mellin convolution integrals, even with the high-accuracy code of Ref. [47]. In this region one can instead use

$$\begin{aligned}
(g_1+g_2)(x) \;=\; & (\zeta_2-\zeta_3)\,\delta(1-x) \,-\, (1-x)\left\{\ln(1-x) + \frac{\zeta_2}{2} - \zeta_3 - \frac{3}{4}\right\} \\
& - \frac{1}{2}\,(1-x)^2 \,+\, O[(1-x)^3]\;. \qquad \text{(A.7)}
\end{aligned}$$

Appendix B: Subleading large-x contributions to $F_{2,\mathrm{ns}}$

In this final appendix we consider the subleading (integrable) large-x logarithms $\ln^k(1-x)$ for the quark contributions to F_2 and their relation to the coefficients for F_3 given in Eqs. (3.9) – (3.16) above. The corresponding three-loop results for F_L (where these terms are the leading contributions) have been given in Eq. (4.31) - (4.34) of Ref. [18]. At two loops we have

$$c_{2,\mathrm{q}}^{(2)}\Big|_{L_1^3} \;=\; -\,8\,C_F^2 \qquad \text{(B.1)}$$

$$c_{2,\mathrm{q}}^{(2)}\Big|_{L_1^2} \;=\; \frac{22}{3}\,C_A C_F \,+\, 60\,C_F^2 \,-\, \frac{4}{3}\,C_F\,n_f \qquad \text{(B.2)}$$

$$c_{2,\mathrm{q}}^{(2)}\Big|_{L_1} \;=\; -C_A C_F\left[\frac{916}{9}-24\,\zeta_2\right]+20C_F^2+\frac{148}{9}C_F\,n_f\;. \tag{B.3}$$

The corresponding contributions to the third-order coefficient function are given by

$$c_{2,\mathrm{q}}^{(3)}\Big|_{L_1^5} \;=\; -8\,C_F^3 \tag{B.4}$$

$$c_{2,\mathrm{q}}^{(3)}\Big|_{L_1^4} \;=\; \frac{220}{9}C_A C_F^2+92\,C_F^3-\frac{40}{9}C_F^2\,n_f \tag{B.5}$$

$$\begin{aligned} c_{2,\mathrm{q}}^{(3)}\Big|_{L_1^3} \;=\;& -\frac{484}{27}C_A^2C_F-C_AC_F^2\left[\frac{10976}{27}-64\,\zeta_2\right]-C_F^3\,[38-32\,\zeta_2]\\ &+\frac{176}{27}C_AC_F\,n_f+\frac{1832}{27}C_F^2\,n_f-\frac{16}{27}C_F\,n_f^2 \end{aligned} \tag{B.6}$$

$$\begin{aligned} c_{2,\mathrm{q}}^{(3)}\Big|_{L_1^2} \;=\;& C_A^2C_F\left[\frac{11408}{27}-\frac{266}{3}\,\zeta_2-32\,\zeta_3\right]+C_AC_F^2\left[\frac{11501}{9}-292\,\zeta_2-160\,\zeta_3\right]\\ &-C_F^3\left[\frac{1199}{3}+688\,\zeta_2+48\,\zeta_3\right]-C_AC_F\,n_f\left[\frac{3694}{27}-\frac{64}{3}\,\zeta_2\right]\\ &-C_F^2\,n_f\left[\frac{2006}{9}-\frac{16}{3}\,\zeta_2\right]+\frac{296}{27}C_F\,n_f^2 \end{aligned} \tag{B.7}$$

$$\begin{aligned} c_{2,\mathrm{q}}^{(3)}\Big|_{L_1} \;=\;& -C_A^2C_F\left[\frac{215866}{81}-824\,\zeta_2-\frac{1696}{3}\,\zeta_3+\frac{304}{5}\,\zeta_2^2\right]+C_AC_F^2\left[\frac{126559}{162}\right.\\ &\left.+872\,\zeta_2+792\,\zeta_3-\frac{1916}{5}\,\zeta_2^2\right]+C_F^3\left[\frac{157}{6}+\frac{1268}{3}\,\zeta_2\right.\\ &\left.-272\,\zeta_3+488\,\zeta_2^2\right]+C_AC_F\,n_f\left[\frac{64580}{81}-\frac{1292}{9}\,\zeta_2-\frac{304}{3}\,\zeta_3\right]\\ &-C_F^2\,n_f\left[\frac{4445}{81}+208\,\zeta_2-\frac{208}{3}\,\zeta_3\right]-C_F\,n_f^2\left[\frac{4432}{81}-\frac{32}{9}\,\zeta_2\right]\;. \end{aligned} \tag{B.8}$$

A comparison of the large-x limit of the complete charged-current quark coefficient functions $C_{1,2,3,L}$, defined analogous to Eq. (3.1) above, now reveals that

$$C_1(x,\alpha_s)\;\equiv\;C_2(x,\alpha_s)-C_L(x,\alpha_s)\;=\;C_3(x,\alpha_s)+O(1-x) \tag{B.9}$$

to the third order in α_s. I.e., not only are the above log-enhanced terms of C_1 and C_3 related, but also the constants for $x\to 1$, leaving the difference vanish as $1/N^2$ in moment space. To the best of our knowledge, Eq. (B.9) has not been noted so far in the literature even on the level of the two-loop coefficient functions of Refs. [4–6]. In this context it appears worthwhile to recall that the non-singlet coefficient function for the polarized structure function g_1 [no relation to the quantity in Eq. (A.5)] is identical to C_3 up to the $d^{abc}d_{abc}$ term irrelevant at large x, and that an analogous $1/N^2$ suppression of the helicity flip for $x\to 1$ is present in polarized splitting functions, see Ref. [61] and references therein.

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