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Reduction Method for One-loop Tensor 5- and 6-point Integrals Revisited

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A complete analytical reduction of general one-loop Feynman integrals with five legs for tensors up to rank $R = 3$ and six legs for tensors up to rank 4 is reviewed [1]. An elegant formalism with extensive use of signed minors was developed for the cancellation of leading inverse Gram determinants. The resulting compact formulae allow both for a study of analytical properties and for efficient numerical programming. Here some special numerical examples are presented.

1 Introduction

The Feynman integrals for reactions with up to four external particles have been systematically studied and evaluated in numerous studies. It is needed to be mentioned here the seminal papers [2] and [3] and the Fortran packages `FF` [4] and `LoopTools` [5], which evidently show the situation so far. The treatment of Feynman integrals with a higher multiplicity than four becomes quite involved if questions of efficiency and stability become vital, as it happens with the calculational problems related to high-dimensional phase space integrals over sums of thousands of Feynman diagrams with internal loops.

What is reviewed here is an approach which reduces the tensor integrals algebraically to sums over a small set of scalar two-, three- and four-point functions, which are assumed to be known. To accomplish this methods of ref. [6, 7] are used. The present goal is to provide compact analytic formulas for the complete reduction of tensor pentagons and hexagons to scalar master integrals, which are free of leading inverse Gram determinants. For a study of gauge invariance and of the ultraviolet (UV) and infrared (IR) singularity structure of a set of Feynman diagrams, it is evident that a complete reduction is advantageous, and it may also be quite useful for a tuned, analytical study of certain regions of potential numerical instabilities.

The numerics are obtained with two independent implementations, one made in Mathematica, and another one in Fortran. The Mathematica program `hexagon.m` with the reduction formulae is made publicly available [8], see also [9] for a short description. For numerical applications, one has to link the package with a program for the evaluation of scalar one- to four-point functions, e.g. with `LoopTools` [5, 10, 4], `CutTools` [11, 12], `QCDLoop` [13].

2 Useful Notations

It is useful to introduce the notation for the loop integrals and also for certain determinants that occur in the recurrence relations and their solutions. The one-loop, N-point tensor integrals of rank R in d-dimensional space-time are defined as,

$$I^{(N)}_{\mu_1\dots\mu_R}(d;\nu_1,\dots,\nu_N) = \int \frac{d^dk}{i\pi^{d/2}} \frac{k_{\mu_1}\dots k_{\mu_R}}{D_1^{\nu_1}\dots D_N^{\nu_N}} \tag{1}$$

with propagator denominators

$$D_j = (k - q_j)^2 - m_j^2 + i\epsilon\,. \tag{2}$$

The determinant of an $(N+1)\times(N+1)$ matrix, known as the modified Cayley determinant is defined as: [14],

$$()_N \;\equiv\; \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \dots & Y_{NN} \end{vmatrix}, \tag{3}$$

with coefficients

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2\,, \quad (i,j = 1\dots N)\,. \tag{4}$$

All other determinants appearing are signed minors of $()_N$, constructed by deleting m rows and m columns from $()_N$, and multiplying with a sign factor. They will be denoted by

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \;\equiv\; (-1)^{\sum_l (j_l + k_l)} \\ \mathrm{sgn}_{\{j\}}\,\mathrm{sgn}_{\{k\}} \left| \begin{array}{c} \text{rows } j_1 \cdots j_m \text{ deleted} \\ \text{columns } k_1 \cdots k_m \text{ deleted} \end{array} \right|, \tag{5}$$

where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \cdots j_m$ and columns $k_1 \cdots k_m$ into ascending order.

3 Pentagons

In this chapter final results are provided of the reduction concerning ranks up to 3. More about these can be found in [15]. For the **scalar** 5-point function the recursion relation for the limit of $d = 4$ is,

$$E \equiv I_5 \;=\; \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s, \tag{6}$$

Similarly, for the tensor integral of **rank 1** (vector) in the limit $d \to 4$ we obtain:

$$I_5^\mu \;=\; \sum_{i=1}^{4} q_i^\mu I_{5,i}, \tag{7}$$

$$I_{5,i} \equiv E_i \;=\; -\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \begin{pmatrix} 0 & i \\ 0 & s \end{pmatrix}_5 I_4^s, \tag{8}$$

The tensor integral of **rank 2** can be written:

$$I_5^{\mu\,\nu} = \sum_{i,j=1}^{4} q_i^\mu\, q_j^\nu E_{ij} + g^{\mu\nu} E_{00}, \tag{9}$$

$$E_{ij} = \sum_{s=1}^{5} S_{ij}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ij}^{3,st} I_3^{st}, \tag{10}$$

$$E_{00} = -\frac{1}{2}\frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \frac{\binom{s}{0}_5}{\binom{s}{s}_5} \left[\binom{0 \quad s}{0 \quad s}_5 I_4^s - \sum_{t=1}^{5} \binom{t \quad s}{0 \quad s}_5 I_3^{st} \right]. \tag{11}$$

Finally the tensor integral of **rank 3**

$$I_5^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{4} q_i^\mu\, q_j^\nu\, q_k^\lambda E_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{12}$$

$$E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^{stu}, \tag{13}$$

$$E_{00k} = \sum_{s=1}^{5} S_{00k}^{4,s} I_4^s + \sum_{s,t=1}^{5} S_{00k}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{00k}^{2,stu} I_2^{stu}. \tag{14}$$

All coefficients,also those not explicitly defined here $S_{ijk}^{4,s}, S_{ijk}^{3,st}, S_{ijk}^{2,stu}$, $S_{00k}^{4,s}$, $S_{00k}^{3,st}$, $S_{00k}^{2,stu}$, $S_{ij}^{4,s}$, $S_{ij}^{3,st}$ (see [15]), are free of the leading Gram determinants.

4 Hexagons

If the external momenta of a hexagon are 4-dimensional, their Gram determinant vanishes: $()_6 = 0$, and a linear relation between the propagators D_j exists:

$$1 = \sum_{j=1}^{6} \frac{\binom{0}{j}_6}{\binom{0}{0}_6} D_j\,. \tag{15}$$

With this relation, any hexagon integral can trivially be reduced to pentagons. For example, for the scalar hexagon, one obtains the well-known result [14]:

$$I_6 = \sum_{r=1}^{6} \frac{\binom{0}{r}_6}{\binom{0}{0}_6}\; I_5^r\,, \tag{16}$$

where the scalar pentagon I_5^r on the right hand side is obtained by removing line r from the hexagon I_6. In the same way, tensor hexagons of rank R can be reduced to tensor pentagons

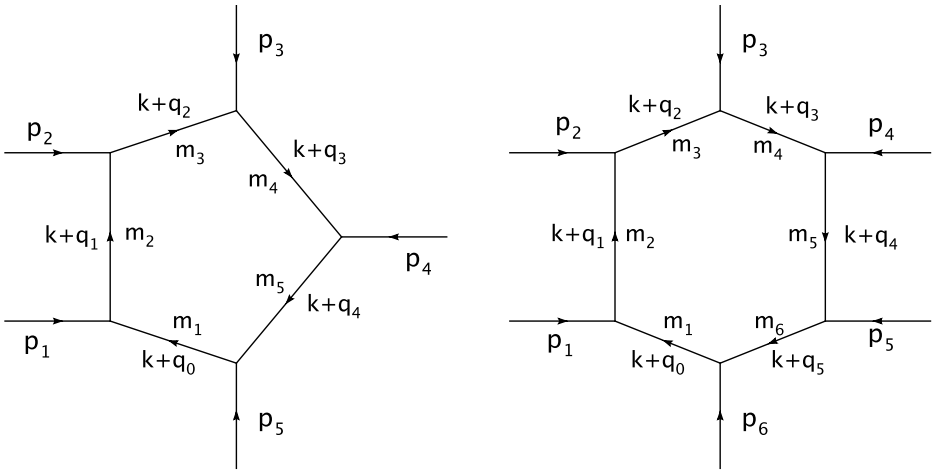

Figure 1: Momenta flow used in the numerical examples for six- and five-point integrals.

of rank R. However, it was noticed in ref. [6] that a reduction directly to tensor pentagons of rank $R-1$ is also possible:

$$I_6^{\mu_1\cdots\mu_R} = \sum_{r=1}^{6} v_r^{\mu_1} I_5^{\mu_2\cdots\mu_R\,,r}\,, \tag{17}$$

where

$$v_r^{\mu} \equiv -\frac{1}{\binom{0}{0}_6} \sum_{i=1}^{5} \binom{0 \quad i}{0 \quad r}_6 q_i^{\mu}\,. \tag{18}$$

A more general proof of this property was given in ref. [16]. By substituting the reduction formulas for tensor pentagons into eq. (17), we can immediately express tensor hexagons in terms of scalar master integrals. In this way using the formulas of the previous section we can provide results for integrals up to 4th rank for the hexagons (see [15]).

5 Numerical results

In order to illustrate the numerical results which can be obtained with the described approach, a representative collection of tensor components will be evaluated, for some special cases which are not included in [9]. The kinematics are visualized in Figure 1.

For the evaluation of the scalar two-, three- and four-point functions, which appear after the complete reduction, we have implemented two numerical libraries:

- For massive internal particles: `Looptools 2.2` [5, 10];
- If there are also massless internal particles: `QCDLoop-1.4` [13].

the first one in the published package `hexagon.m` [8] and both of them in the Fortran implimentation.

p_1	5.0	0.0	0.0	5.0
p_2	5.0	0.0	0.0	− 5.0
p_3	− 1.6554963633	1.2970338732	− 0.9062452085	− 0.4869198730
p_4	− 3.8970139847	0.0528728505	− 2.5360890226	2.9584074987
p_5	− 4.4474896520	− 1.3499067237	3.4423342311	− 2.4714876256
$m_1 = \cdots = m_5 = 0.0$				

Table 1: The external four-momenta for the five-point functions; all internal and external particles are massless.

	ϵ^0	$1/\epsilon$
E_0	0.49975096E-03 + i 0.12807271E-02	0.33696138E-03 − i 0.64416161E-03
E^1	− 0.50336057E-03 − i 0.10928553E-02	− 0.34786666E-03 + i 0.54767334E-03
E^{12}	− 0.11603164E-02 − i 0.17552616E-02	− 0.60899168E-03 + i 0.12327007E-02
E^{122}	− 0.43997517E-02 − i 0.34454891E-02	− 0.10597882E-02 + i 0.36519758E-02

	$1/\epsilon^2$
E_0	− 0.15779987E-03+ i 0.00000000E+00
E^1	0.17432984E-03+ i 0.00000000E+00
E^{12}	0.39238082E-03+ i 0.00000000E+00
E^{122}	0.11624600E-02+ i 0.00000000E+00

Table 2: Selected tensor components of five-point tensor functions with massless particles; kinematics defined in Table 1 (Cross checked with `golem95` [17]).

p_1	5.0	0.0	0.0	5.0
p_2	5.0	0.0	0.0	− 5.0
p_3	− 0.7623942818	0.5390582570	− 0.5220507689	0.1346262645
p_4	− 3.3298826057	− 1.0349623069	− 1.1048040197	2.9658690580
p_5	− 2.8267285956	− 1.4136906402	2.3189438782	− 0.7838192500
p_6	− 3.0809945169	1.9095946901	− 0.6920890895	− 2.3166760725
$m_1 = 1.0,\ m_2 = 1.2,\quad m_3 = 1.4,\quad m_4 = 1.6,\quad m_5 = 1.8,\quad m_6 = 2.0$				

Table 3: The external four-momenta for the six-point functions; all external legs massless and the internal massive.

F_0	0.54701021E-04 − i 0.67031213E-04
F^3	− 0.32082506E-04 + i 0.24545301E-03
F^{11}	0.13862332E-04 − i 0.12247788E-03
F^{112}	− 0.22452724E-04 − i 0.39826579E-04
F^{0121}	0.15817785E-03 + i 0.26882173E-03

Table 4: Selected tensor components of six-point tensor functions produced by the phase space point of Table 3.

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