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Integrability of scattering amplitudes in $N = 4$ SUSY *

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Abstract

We argue, that the multi-particle scattering amplitudes in $N = 4$ SUSY at large N_c and in the multi-Regge kinematics for some physical regions have the high energy behavior appearing from the contribution of the Mandelstam cuts in the corresponding t-channel partial waves. The Mandelstam cuts correspond to gluon composite states in the adjoint representation of the gauge group $SU(N_c)$. The hamiltonian for these states in the leading logarithmic approximation coincides with the local hamiltonian of an integrable open spin chain. We construct the corresponding wave functions using the integrals of motion and the Baxter-Sklyanin approach.

1 Introduction

At high energies $s \gg -t$ in QCD the elastic scattering amplitude for the process $AB \to A'B'$ in the leading logarithmic approximation (LLA)

$$\alpha_s \ln s \sim 1\,,\ \alpha_s \ll 1 \tag{1}$$

has the Regge form [1]

$$A_{2\to 2} = 2\,g\delta_{\lambda_A\lambda_{A'}} T^c_{AA'} \frac{s^{1+\omega(t)}}{t}\, g\, T^c_{BB'}\, \delta_{\lambda_B\lambda_{B'}}\,,\ t = -\vec{q}^{\,2}\,. \tag{2}$$

Here T^c are the generators of the gauge group $SU(N_c)$, λ_r are the particle helicities and $j(t) = 1+\omega(t)$ is the gluon Regge trajectory for the space-time dimension $D = 4 - 2\epsilon$

$$\omega(-\vec{q}^{\,2}) = -\frac{\alpha_s N_c}{(2\pi)^2}\,(2\pi\mu)^{2\epsilon} \int d^{2-2\epsilon}k\, \frac{\vec{q}^{\,2}}{\vec{k}^2(\vec{q}-\vec{k})^2} \approx -\,a\,\left(\ln\frac{\vec{q}^{\,2}}{\mu^2} - \frac{1}{\epsilon}\right)\,. \tag{3}$$

In the framework of the dimensional regularization the parameter μ is the renormalization point for the 't Hooft coupling constant and

$$a = \frac{\alpha_s\, N_c}{2\pi}\,\left(4\pi e^{-\gamma}\right)^{\epsilon}\,,\ \gamma = -\psi(1)\,, \tag{4}$$

where $\gamma = -\psi(1)$ is the Euler constant and $\psi(x) = (\ln\Gamma(x))'$. The gluon trajectory $j(t)$ was calculated also in the next-to-leading approximation in QCD [2] and in the SUSY gauge models [3].

In LLA gluons with momenta k_r (r=1,..,n) are produced in the multi-Regge kinematics

$$s = (p_A + p_B)^2 \gg s_r = (k_r + k_{r-1})^2 \gg -t_r = \mathbf{q}_r^2\,,\ k_r = q_{r+1} - q_r\,, \tag{5}$$

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where the amplitude has the factorized form

$$A_{2\to 2+n} = 2\,s\,\delta_{\lambda_A\lambda_{A'}} g\,T^{c_1}_{AA'} \frac{s_1^{\omega(-\vec{q}_1^{\,2})}}{\vec{q}_1^{\,2}} g C_\mu(q_2,q_1) e^*_\mu(k_1) T^{d_1}_{c_2c_1} \frac{s_2^{\omega(-\vec{q}_2^{\,2})}}{\vec{q}_2^{\,2}} \ldots \frac{s_{n+1}^{\omega(-\vec{q}_{n+1}^{\,2})}}{\vec{q}_{n+1}^{\,2}}\, g\,T^{c_{n+1}}_{BB'}\,\delta_{\lambda\lambda_{B'}}\,. \tag{6}$$

Here $C_\mu(q_2,q_1)$ is the effective Reggeon-Reggeon-gluon vertex. In the case when the polarization vector $e_\mu(k_1)$ describes the gluon with a positive helicity in its c.m. system with the particle A' one can obtain [4]

$$C \equiv C_\mu(q_2,q_1)\,e^*_\mu(k_1) = \sqrt{2}\,\frac{q_2^* q_1}{k_1}\,, \tag{7}$$

where the complex notation $q = q_x + iq_y$ for the two-dimensional transverse vectors $\vec{q}$ was used.

The elastic scattering amplitude with vacuum quantum numbers in the t-channel is calculated in terms of the production amplitude $A_{2\to 2+n}$ with the use of the s-channel unitarity [1]. In this approach the Pomeron appears as a composite state of two Reggeized gluons. It is convenient to present the gluon transverse coordinates in the complex form together with their canonically conjugated momenta [4, 5]

$$\rho_k = x_k + iy_k\,,\ \rho_k^* = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial\rho_k}\,,\ p_k^* = i\frac{\partial}{\partial\rho_k^*}\,. \tag{8}$$

In this case the homogeneous Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for the Pomeron wave function can be written as follows [1]

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\ \min\,E\,, \tag{9}$$

where Δ is the Pomeron intercept entering in the asymptotic expression for the total cross-section $\sigma_t \sim s^\Delta$. The BFKL Hamiltonian has a rather simple operator representation [5]

$$H_{12} = \ln\,|p_1p_2|^2 + \frac{1}{p_1p_2^*}(\ln\,|\rho_{12}|^2)\,p_1p_2^* + \frac{1}{p_1^*p_2}(\ln\,|\rho_{12}|^2)\,p_1^*p_2 - 4\psi(1) \tag{10}$$

with $\rho_{12} = \rho_1 - \rho_2$. The kinetic energy is proportional to the sum of two gluon Regge trajectories $\omega(-|p_i|^2)$ $(i = 1,2)$. The potential energy $\sim \ln\,|\rho_{12}|^2$ is obtained by the Fourier transformation from the product of two gluon production vertices C_μ. This Hamiltonian is invariant under the Möbius transformation [6]

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,, \tag{11}$$

where a, b, c and d are complex parameters. The eigenvalues of the corresponding Casimir operators are expressed in terms of the conformal weights

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\ \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}\,, \tag{12}$$

where ν and n are respectively real and integer numbers for the principal series of unitary representations of the Möbius group $SL(2,C)$. The eigenvalues of H_{12} depend on these parameters [6]

$$E_{m,\widetilde{m}} = \psi(m) + \psi(1-m) + \psi(\widetilde{m}) + \psi(1-\widetilde{m}) - 4\psi(1)\,. \tag{13}$$

The Pomeron intercept in LLA is positive

$$\Delta = 4\,\frac{\alpha_s}{\pi} N_c\,\ln 2 > 0 \tag{14}$$

and therefore the Froissart bound $\sigma_t < c\ln^2 s$ for the total cross-section is violated [1]. To restore the broken s-channel unitarity one should take into account the contributions of diagrams corresponding to the t-channel exchange of an arbitrary number of reggeized gluons in the t-channel. The wave

function of the colorless state constructed from n reggeized gluons can be obtained in LLA as a solution of the Bartels-Kwiecinski-Praszalowicz (BKP) equation [7]

$$E\,\Psi = H^{(0)}\,\Psi\,,\ \Delta = -\frac{\alpha_s N_c}{4\pi}\,\min\,E \tag{15}$$

In the $N_c \to \infty$ limit the color structure is simplified and the corresponding Hamiltonian has the property of the holomorphic separability [8]

$$H^{(0)} = \sum_{k=1}^{n} H_{k,k+1} = h^{(0)} + h^{(0)*}\,,\ [h^{(0)}, h^{(0)*}] = 0\,. \tag{16}$$

It is a consequence of the similar property for the pair BFKL hamiltonian H_{12} (10) and the energy $E_{m,\widetilde{m}}$ (13).

The holomorphic Hamiltonian in the multi-color QCD can be written as follows (cf. (10))

$$h^{(0)} = \sum_k h^{(0)}_{k,k+1}\,,\ h^{(0)}_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\,(\ln\rho_{12})\,p_1 + \frac{1}{p_2}\,(\ln\rho_{12})\,p_2 - 2\psi(1)\,, \tag{17}$$

where $\psi(x) = (\ln\Gamma(x))'$. As a result, the wave function Ψ has the holomorphic factorization [8]

$$\Psi = \sum_{r,\widetilde{r}} a_{r,\widetilde{r}}\,\Psi^r(\rho_1, ..., \rho_n)\,\Psi^{\widetilde{r}}(\rho_1^*, ..., \rho_n^*)\,, \tag{18}$$

which in the case of two-dimensional conformal field theories is a consequence of the infinite dimensional Virasoro group. Moreover, the holomorphic hamiltonian $h^{(0)}$ is invariant under the duality transformation [9]

$$p_i \to \rho_{i,i+1} \to p_{i+1}\,, \tag{19}$$

combined with its transposition.

Further, there are integrals of motion q_r commuting among themselves and with $h^{(0)}$ [5, 10]:

$$q_r^{(0)} = \sum_{k_1<k_2<...<k_r} \rho_{k_1 k_2}\rho_{k_2 k_3}\cdots\rho_{k_2 k_3}\cdots\rho_{k_r k_1}\,p_{k_1}p_{k_2}\cdots p_{k_r}\,,\ [q_r, h] = 0\,. \tag{20}$$

The integrability of the BFKL dynamics in LLA was established in Ref. [10]. This remarkable property is related to the fact that h coincides with the local Hamiltonian of an integrable Heisenberg spin model [11]. Eigenvalues and eigenfunctions of this hamiltonian were constructed in Refs. [12, 13] in the framework of the Baxter-Sklyanin approach [14].

In the next-to-leading approximation the integral kernel for the BFKL equation was constructed in Refs. [3, 15]. In QCD the eigenvalue of the kernel contains the Kroniker symbols $\delta_{n,0}$ and $\delta_{n,2}$ but in $N = 4$ SUSY it is an analytic function of the conformal spin and having the property of the maximal transcendentality [3, 16]. This extended supersymmetric theory appears in the framework of the AdS/CFT correspondence [17, 18, 19]. It is important, that the one-loop anomalous dimension for twist-2 operators in $N = 4$ SUSY is proportional to the expression $\psi(1) - \psi(j-1)$, which is related to the integrability of evolution equations for the quasi-partonic operators in this model [20]. The integrability persists also for some operators in QCD [21]. The maximal transcendentality principle suggested in Ref. [16] gave a possibility to extract the universal anomalous dimension up to three loops in $N = 4$ SUSY [22, 23] from the corresponding QCD results [24]. The integrability of the $N = 4$ model was verified also for other operators, large coupling constants and in higher loops [25, 26, 27]. The asymptotic Bethe ansatz and integrability allowed to calculate the anomalous dimensions in four loops [28]. The result is in an agreement with the next-to-leading BFKL predictions after taking into account the wrapping effects [29]. The maximal transcendentality was helpful for finding a closed integral equation for the cusp anomalous dimension in this model [30, 31] with the use of the 4-loop result [32].

There is another region of investigation, in which remarkable properties of the N=4 SUSY are also found. Namely, Bern, Dixon and Smirnov (BDS) suggested a simple ansatz for the gluon scattering amplitudes in this model [33]. This ansatz was verified for the elastic amplitude in the strong coupling regime using the AdS/CFT correspondence [34]. But the BDS hypothesis does not agree in this regime with the calculation of the multi-particle amplitude [35]. The property of the conformal invariance of the BDS amplitudes in the momentum space was discussed in Ref. [36] and the relation with the Wilson loop approach was suggested in Ref. [37] generalizing the results of the strong coupling calculations of Ref. [34]. The BDS amplitudes A_n for $n \geq 6$ in the multi-Regge kinematics do not have correct analytic properties compatible with the Steinman relations [38]. It is a consequence of the fact, that these amplitudes do not include the Mandelstam cuts [38]. This cut contribution was obtained from the BFKL-like equation for the amplitude with the t-channel exchange in the adjoint representation of the gauge group [38]. This equation was solved in LLA and the two-loop expression for the 6-point scattering amplitude in the multi-Regge kinematics was derived [39]. Recently the two-loop correction was calculated numerically for some values of external momenta in an agreement with expectations based on the Wilson loop approach [40].

In this paper we demonstrate, that in the multi-color limit for the production amplitudes the contributions of the Mandelstam cuts generated by the multi-Reggeon t-channel exchange can be expressed in terms of the solution of the BKP-like equation for the composite states of several reggeized gluons in the adjoint representation. It turns out, that in LLA the corresponding Hamiltonian coincides with the local Hamiltonian of an integrable open Heisenberg spin chain. These results partly were presented at the conferences [41, 42].

2 Mandelstam cuts

A planar amplitude for the production of two gluons in the multi-Regge kinematics $s \gg |s_1| \sim |s_2| \sim |s_3| \gg |t_1| \sim |t_2| \sim |t_3|$ has the multi-Regge form almost in all physical kinematical regions. But in the physical region where $s_1,\, s_3 < 0;\, s > 0,\, s_2 > 0$ the amplitude contains also the Mandelstam cut [43] in the angular momentum plane j_2 of the crossing channel $t_2 = -q^2$ [38] in the adjoint representation of the color group. The cut appears as a result of the exchange of two reggeized gluons with the momenta $p_1 = k$ and $p_2 = q - k$, respectively [39] (see Appendix A for more details). In the region $s_1, s_3 < 0;\, s_2 > 0$ the integrals over the Sudakov variables $\alpha = kp_A/p_Ap_B$ and $\beta = kp_B/p_Ap_B$ do not vanish as in other regions because the integrand contains singularities situated above and below the corresponding integration contours in an accordance with the Mandelstam requirements [43]. These singularities lead to simultaneous discontinuities of the amplitude in the invariants s_2 and s.

For the planar amplitude with six external particles only diagrams with two reggeons in the t_2-channel give a non-vanishing contribution because for a larger number of reggeons the Mandelstam conditions for singularities in other Sudakov variables are not fulfilled. However in the case of a larger number of external particles the exchange of several reggeons with momenta p_l gives also a non-vanishing contribution to the amplitude constructed from planar diagrams. For the Mandelstam cut resulting from an exchange of n reggeons one needs at least k=2+2n external particles to have simultaneous singularities in upper and lower complex semi-planes for the Sudakov parameters α'_l, β'_l of the reggeon momenta p_l, as it is demonstrated in Appendix A.

Let us discuss such composite state of n reggeized gluons in the adjoint representation (cf. a similar approach for the simple case $n = 4$ in Ref. [39]). One can write the homogeneous BKP equation for its wave function described by an amplitude with amputated propagators in the form (see Appendix A)

$$H\Psi = E\Psi\,,\;\; \Delta_n = -\frac{g^2 N_c}{16\pi^2}\, E\,. \tag{21}$$

Here H is a redefined hamiltonian obtained after subtraction of the gluon Regge trajectory $\omega(t)$

containing infrared divergencies. Namely, the Regge trajectory of the composite state is [41, 42]

$$\omega_n(t) = a\left(\frac{1}{\epsilon} - \ln\frac{-t}{\mu^2}\right) + \Delta_n\,,\ a = \frac{g^2 N_c}{8\pi^2}\left(4\pi e^{-\gamma}\right)^{\epsilon}\,, \tag{22}$$

where Δ_n is the infrared stable quantity expressed in terms of the energy E.

The hamiltonian H in the multi-color limit can be written in the holomorphically separable form (see Appendix A) (cf. [39])

$$H = h + h^*\,,\ h = \ln\frac{p_1\,p_n}{q^2} + \sum_{r=1}^{n-1} h^t_{r,r+1}\,,\ q = \sum_1^n p_r\,, \tag{23}$$

where the pair hamiltonian $h^t_{r,r+1}$ is transposed to the corresponding unamputated operator (17)

$$h^t_{r,r+1} = \ln(p_r p_{r+1}) + p_r\ln(\rho_{r,r+1})\,\frac{1}{p_r} + p_{r+1}\ln(\rho_{r,r+1})\,\frac{1}{p_{r+1}} + 2\gamma\,. \tag{24}$$

It is seen from eq. (23) that the holomorphic hamiltonian for the composite state in the adjoint representation differs from the corresponding expression for the singlet case $h^{(0)}$ (17) after its transposition only by the substitution

$$h_{n,1} \to \ln\frac{p_1\,p_n}{q^2}\,, \tag{25}$$

which is related to the fact, that the planar Feynman diagrams have the topology of a strip and the infrared divergencies in the Regge trajectories of the particles 1 and n are not compensated by the contribution from the pair potential energy $V_{n,1}$.

It turns out, that the eigenvalues E do not depend on $|q|^2$ due to the scale invariance of H, as it will be demonstrated below. As a result, the t-dependence of $\omega_n(t)$ is the same as in the gluon Regge trajectory.

The transposed holomorphic hamiltonian is related to the initial hamiltonian by the similarity transformation

$$h^t = \left(\prod_{r=1}^n p_r\right)^{-1} h \left(\prod_{r=1}^n p_r\right)\,, \tag{26}$$

which leads to the following hermicity property of the total hamiltonian H

$$H^+ = \left(\prod_{r=1}^n |p_r|^2\right)^{-1} H \left(\prod_{r=1}^n |p_r|^2\right)\,. \tag{27}$$

The last relation is compatible with the normalization condition for the wave function

$$||\Psi||^2 = \int \prod_{r=1}^{n-1} d^2p_r\ \Psi^* \prod_{s=1}^n |p_s|^{-2}\Psi\,,\ \sum_{s=1}^n p_s = q\,. \tag{28}$$

Using the duality transformation (cf. [9])

$$p_1 = z_{0,1}\,,\ p_r = z_{r-1,r}\,,\ q = z_{0,n}\,,\ \rho_{r,r+1} = i\frac{\partial}{\partial z_r} = i\partial_r\,, \tag{29}$$

the holomorphic hamiltonian can be rewritten as follows

$$h = \ln\frac{z_{0,1}\,z_{n-1,n}}{z^2_{0,n}} + \sum_{r=1}^{n-1} h^t_{r,r+1}\,, \tag{30}$$

where

$$h^t_{r,r+1} = 2\ln(\partial_r) + \frac{1}{\partial_r}\,\frac{1}{z_{r-1,r}} + \frac{1}{\partial_r}\,\frac{1}{z_{r+1,r}} + \ln(z_{r,r+1}\,z_{r-1,r}) + 2\gamma\,. \tag{31}$$

Here and later we neglect the pure imaginary contribution $2\ln(i)$ because it is cancelled in the total hamiltonian H. Note, that for the colorless composite state and $q = z_{0,n} = 0$ the transformation (29) is indeed reduced to the usual duality substitution of Ref. [9].

To simplify h one can use the relations [5, 9]

$$\ln\partial = -\ln x + \frac{1}{2}(\psi(x\partial + 1) + \psi(-x\partial))\,,$$

$$\ln(x^2\partial) = \ln x + \frac{1}{2}(\psi(x\partial) + \psi(-x\partial + 1))\,,$$

$$\ln\partial = \ln(x^2\partial) - 2\ln x + \frac{1}{\partial}\,\frac{1}{x}\,. \tag{32}$$

Then $h^t_{r,r+1}$ can be presented as follows

$$h^t_{r,r+1} = \ln(z^2_{r,r+1}\partial_r) + \ln(z^2_{r-1,r}\partial_r) - \ln z_{r,r+1} - \ln z_{r-1,r} + 2\gamma\,. \tag{33}$$

Further, by regrouping its terms we can write the holomorphic hamiltonian in another form

$$h = -2\ln z_{0,n} + \ln(z^2_{0,1}\partial_1) + \ln(z^2_{n-1,n}\partial_{n-1}) + 2\gamma + \sum_{r=1}^{n-2} h'_{r,r+1}\,, \tag{34}$$

where

$$h'_{r,r+1} = \ln(z^2_{r,r+1}\partial_r) + \ln(z^2_{r,r+1}\partial_{r+1}) - 2\ln z_{r,r+1} + 2\gamma$$

$$= \ln(\partial_r) + \ln(\partial_{r+1}) + \frac{1}{\partial_r}\,\ln z_{r,r+1}\,\partial_r + \frac{1}{\partial_{r+1}}\,\ln z_{r,r+1}\,\partial_{r+1} + 2\gamma\,. \tag{35}$$

The pair hamiltonian $h'_{r,r+1}$ coincides in fact after the substitution $z_r \to \rho_r$ with the corresponding hamiltonian in the coordinate representation (17) acting on the wave function with non-amputated propagators.

In particular, for $n = 2$ one obtains (cf. [39])

$$h = -2\ln z_{0,2} + \ln(z^2_{0,1}\,\partial_1) + \ln(z^2_{1,2}\,\partial_1) + 2\gamma\,. \tag{36}$$

It is important, that h (34) is invariant under the Möbius transformations

$$z_k \to \frac{az_k + b}{cz_k + d} \tag{37}$$

and does not contain the derivatives ∂_0 and ∂_n. Therefore we can put

$$z_0 = 0\,,\ \ z_n = \infty\,, \tag{38}$$

which leads to the simplified expression for h

$$h \to h' = \ln(z_1^2\partial_1) + \ln(\partial_{n-1}) + 2\gamma + \sum_{r=1}^{n-2} h'_{r,r+1}\,. \tag{39}$$

To return to initial variables in the final expression for the wave function one should perform the following substitution of z_k

$$z_k \to \frac{z_k - z_0}{z_k - z_n} = \frac{\sum_{r=1}^{k} p_r}{q - \sum_{r=1}^{k} p_r}\,. \tag{40}$$

According to the above representation (30) for h, its transposed part h^t can be obtained from h by the similarity transformation which can be written in terms of h' as follows

$$h'^{\,t} = z_1^{-1} \left(\prod_{r=1}^{n-2} z_{r,r+1} \right)^{-1} h' \; z_1 \left(\prod_{r=1}^{n-2} z_{r,r+1} \right) , \tag{41}$$

which is compatible with the following normalization condition for the wave function in the full two-dimensional space

$$||\Psi||_1^2 = \int \frac{d^2 z_{n-1}}{|z_1|^2} \prod_{r=1}^{n-2} \frac{d^2 z_r}{|z_{r,r+1}|^2} \, |\Psi|^2 \, . \tag{42}$$

On the other hand, from the expression (39) for h' we obtain another relation for $h'^{\,t}$

$$h'^{\,t} = \left(\prod_{r=1}^{n-1} \partial_r \right) \, h' \, \left(\prod_{r=1}^{n-1} \partial_r \right)^{-1} , \tag{43}$$

corresponding to the second normalization condition for Ψ compatible with the hermicity properties of the total hamiltonian

$$||\Psi||_2^2 = \int \prod_{r=1}^{n-1} d^2 z_r \; \Psi^* \prod_{r=1}^{n-1} |\partial_r|^2 \; \Psi \, . \tag{44}$$

By comparing two above relations between h' and $h'^{\,t}$ one can conclude (cf. [10]), that the operator

$$A' = z_1 \prod_{s=1}^{n-2} z_{s,s+1} \prod_{r=1}^{n-1} \partial_r \tag{45}$$

commutes with the holomorphic hamiltonian

$$[A', h'] = 0 \, . \tag{46}$$

3 Integrable open spin chain

Let us verify, that the holomorphic hamiltonian h' (39) also commutes with the differential operator $D(u)$ being the matrix element T_{22} of the monodromy matrix (cf. [10])

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L_1(u) L_2(u) ... L_{n-1}(u) \, , \tag{47}$$

where the L-operator is defined by the relation

$$L_r(u) = \begin{pmatrix} u + i z_r \partial_r & i \partial_r \\ -i z_r^2 \partial_r & u - i z_r \partial_r \end{pmatrix} . \tag{48}$$

To prove the commutativity of h' and $D(u)$ one can use the following relation

$$[L_k(u) \, L_{k+1}(u), h'_{k,k+1}] = -i \left(L_k(u) - L_{k+1}(u) \right) , \tag{49}$$

valid due to the Möbius symmetry of the pair hamiltonian

$$[\vec{M}_{k,k+1}, h'_{k,k+1}] = 0 \, , \;\; \vec{M}_{k,k+1} = \vec{M}_k + \vec{M}_{k+1} \tag{50}$$

and the commutation relation (see [9])

$$[h'_{k,k+1}, [\vec{M}^2_{k,k+1}, \vec{N}_{k,k+1}]] = 4 \vec{N}_{k,k+1} \, , \;\; \vec{N}_{k,k+1} = \vec{M}_k - \vec{M}_{k+1} \, . \tag{51}$$

The last relation is a consequence of the fact, that the operator $\vec{N}_{k,k+1}$ has non-vanishing matrix elements only between the states $|m_{k,k+1}>$ and $|m_{k,k+1} \pm 1>$ in the representation, where the Casimir operator of the Möbius group is diagonal

$$\vec{M}^2_{k,k+1}|m_{k,k+1}>= m_{k,k+1}(m_{k,k+1}-1)|m_{k,k+1}> \ . \tag{52}$$

In this representation the commutation relation (51) is reduced to the recurrent relation for the eigenvalues $\epsilon(m_{k,k+1})$ of the hamiltonian $h'_{k,k+1}$ (35)

$$\epsilon(m+1)-\epsilon(m)=2/m\,, \tag{53}$$

fulfilled due to the well known representation of $\epsilon(m)$

$$\epsilon(m)=\psi(m)+\psi(1-m)+2\gamma\,. \tag{54}$$

Relation (49) leads to the equality

$$[T(u),\sum_{r=1}^{n-2} h'_{r,r+1}]=iL_2(u)L_3(u)...L_{n-1}(u)-iL_1(u)L_2(u)...L_{n-2}(u)\,. \tag{55}$$

On the other hand, one can easily verify, that

$$[T_{22}(u),\ln(z_1^2\partial_1)+\ln\partial_{n-1}]=(0\,,\ 1)\,[T(u),\ln(z_1^2\partial_1)+\ln\partial_{n-1}]\begin{pmatrix}0\\1\end{pmatrix}$$

$$=-i\,(0\,,\ 1)\,(L_2(u)L_3(u)...L_{n-1}(u)-L_1(u)L_2(u)...L_{n-2}(u))\begin{pmatrix}0\\1\end{pmatrix}\,, \tag{56}$$

which proves that the differential operator $D(u)=T_{22}(u)$ is an integral of motion

$$[D(u),h']=0\,. \tag{57}$$

Thus, our hamiltonian is the local hamiltonian for an open integrable Heisenberg spin model with the spins which are generators of the Möbius group[1].

With the use of the following decomposition of the L-operators

$$L_r(u)=\begin{pmatrix}u&0\\0&u\end{pmatrix}+\begin{pmatrix}1\\-z_r\end{pmatrix}(z_r\,,\ 1)\ i\partial_r \tag{58}$$

one can construct the matrix element $T_{22}=D(u)$ in an explicit way

$$D(u)=\sum_{k=0}^{n-1} u^{n-1-k}\,q'_k\,, \tag{59}$$

where

$$q'_0=1\,,\ q'_1=-i\sum_{r=1}^{n-1} z_r\,\partial_r\,. \tag{60}$$

In a general case the integrals of motion q'_k are given below

$$q'_k=-\sum_{0<r_1<r_2<...<r_k<n} z_{r_1}\prod_{s=1}^{k-1} z_{r_s,r_{s+1}}\prod_{t=1}^{k} i\partial_{r_t}\,. \tag{61}$$

[1] I thank L. D. Faddeev for the fruitful discussion in which he suggested, that the operator $D(u)$ could be an integral of motion for this open spin chain.

In particular, we obtain, that q'_{n-1} is proportional to the integral of motion A' (45)

$$q'_{n-1} = -i^{n-1} z_1 \prod_{s=1}^{n-2} z_{s,s+1} \prod_{t=1}^{n-1} \partial_t = -i^{n-1} A' \,. \tag{62}$$

Note, that one can parameterize the monodromy matrix in another form

$$T(u) = \begin{pmatrix} j_0(u) + j_3(u) & j_+(u) \\ j_-(u) & j_0(u) - j_3(u) \end{pmatrix} \,,\; j_\pm(u) = j_1(u) \pm i j_2(u) \,. \tag{63}$$

In this case the Yang-Baxter equations for the currents j_μ have the Lorentz-invariant representation [9]

$$[j_\mu(u), j_\nu(v)] = \frac{\epsilon_{\mu\nu\rho\sigma}}{2(u-v)} \left(j^\rho(u) j^\sigma(v) - j^\rho(v) j^\sigma(u) \right) \,. \tag{64}$$

Here $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetric tensor in the four-dimensional Minkowski space and $\epsilon_{1230} = 1$, $g_{\mu\nu} = (1, -1, -1, -1)$.

In particular, we obtain from the Yang-Baxter equations the relation

$$[j_0(u) - j_3(u), j_0(v) - j_3(v)] = [j_0(v), j_3(u)] - [j_0(u), j_3(v)] = 0 \tag{65}$$

and therefore the integrals of motion q'_k are independent operators and commute each with others

$$[q'_k, q'_l] = 0 \,. \tag{66}$$

4 Composite states of two and three gluons

In the case $n = 2$ we have only one non-trivial integral of motion

$$q'_1 = -i z_1 \, \partial_1 \,. \tag{67}$$

Taking into account the normalization condition for the eigenfunction in the two-dimensional space

$$||\Psi||^2 = \int \frac{d^2 z_1}{|z_1|^2} \, |\Psi|^2 \,, \tag{68}$$

we find the orthonormalized and complete set of eigenfunctions

$$\Psi^{(2)}_{m,\widetilde{m}} = z_1^{-\frac{1}{2}+m} \, (z_1^*)^{-\frac{1}{2}+\widetilde{m}} \,,\; m = \frac{1+n}{2} + i\nu \,,\; \widetilde{m} = \frac{1+n}{2} - i\nu \,, \tag{69}$$

satisfying the single-valuedness requirement. Note, that using the substitution (40) one can reproduce the wave functions of two gluon composite states in the momentum space (see [39]).

For the case $n = 3$ the operator $D(u)$ is given below

$$D_3(u) = u^2 - iu(z_1 \, \partial_1 + z_2 \, \partial_2) + z_1 z_{1,2} \, \partial_1 \partial_2 \,. \tag{70}$$

With taking into account the normalization condition

$$||\Psi||^2 = \int \frac{d^2 z_1 \, d^2 z_2}{|z_1|^2 |z_{1,2}|^2} \, |\Psi|^2 \,, \tag{71}$$

one can search the holomorphic eigenfunction of this operator in the form

$$\Psi^{(3)}_m = z_2^{-\frac{1}{2}+m} \, f\left(\frac{z_2}{z_1}\right) \,. \tag{72}$$

The function $f(x)$ satisfies the equation

$$\left(x(1-x)\partial^2 + (\frac{1}{2}+m)(1-x)\partial + \lambda\right) f = 0\,,\ x = \frac{z_2}{z_1}\,, \tag{73}$$

where λ is the eigenvalue of the operator $z_1 z_{1,2}\partial_1\partial_2$. Two independent solutions of this equation can be expressed in terms of the hypergeometric function F

$$f_1(x) = F(a_1, a_2; 1+a_1+a_2; x)\,,\ f_2(x) = x^{a_1+a_2}\, F(-a_2, -a_1; 1-a_1-a_2; x)\,, \tag{74}$$

where the parameters a_1 and a_2 are obtained from the set of equations

$$a_1 + a_2 = -\frac{1}{2} + m\,,\ a_1 a_2 = -\lambda\,. \tag{75}$$

The solutions near the point $x = 1$ can be also expressed in terms of hypergeometric functions and are expanded as follow

$$\frac{\Gamma(a_1)\,\Gamma(a_2)}{\Gamma(1+a_1+a_2)} f_1(x)_{|_{x\to 1}} = \frac{1}{a_1 a_2} - (x-1)\left(\ln(1-x) - \psi(1) - \psi(2) + \psi(1+a_1) + \psi(1+a_2)\right)$$

and

$$\frac{\Gamma(-a_1)\,\Gamma(-a_2)}{\Gamma(1-a_1-a_2)} f_2(x)_{|_{x\to 1}} = \frac{1}{a_1 a_2} - (x-1)\left(\ln(1-x) - \psi(1) - \psi(2) + \psi(1-a_1) + \psi(1-a_2)\right)$$

Analogously one can find the large-x behavior of the functions f_1 and f_2

$$f_1(x)_{|_{x\to\infty}} = \frac{\Gamma(a_1+a_2+1)\Gamma(a_2-a_1)}{\Gamma(a_2)\,\Gamma(1+a_2)}(-x)^{-a_1} + \frac{\Gamma(a_1+a_2+1)\Gamma(a_1-a_2)}{\Gamma(a_1)\,\Gamma(1+a_1)}(-x)^{-a_2}\,,$$

$$f_2(x)_{|_{x\to\infty}} = \frac{\Gamma(1-a_1-a_2)\Gamma(a_1-a_2)}{\Gamma(-a_2)\,\Gamma(1-a_2)}(-x)^{-a_2} + \frac{\Gamma(1-a_1-a_2)\Gamma(a_2-a_1)}{\Gamma(-a_1)\,\Gamma(1-a_1)}(-x)^{-a_1}\,.$$

To construct the wave function Ψ with the property of the single-valuedness in the two-dimensional subspaces $\vec{z}_1$ and $\vec{x}$ we should write a bilinear combination of the functions $f_i(x)$ and the corresponding functions in the anti-holomorphic subspace $\widetilde{f}_i(x^*)$ taking into account that in the second pair of functions one should perform the substitution

$$a_1 \to \widetilde{a}_1\,,\ a_2 \to \widetilde{a}_2\,,\ m \to \widetilde{m} = \frac{1-n}{2} + i\nu\,. \tag{76}$$

Due to the single-valuedness of the wave function near $x = 0$ we obtain for it the following expression

$$\Psi = |z_2|^{2i\nu}\left(\frac{z_2}{z_2^*}\right)^{\frac{n}{2}} \Psi_{m,\widetilde{m}}(\vec{x})\,,\ \Psi_{m,\widetilde{m}}(\vec{x}) = f_1(x)\,\widetilde{f}_1(x^*) + C\, f_2(x)\,\widetilde{f}_2(x^*)\,, \tag{77}$$

where the constant C should be fixed from the requirement, that the analytic continuation of Ψ in the neighborhood of the points $x = 1$ and $x = \infty$ leads also to a single-valued expression. The condition, that near $x = 1$ the terms proportional to $|x-1|^2\ln(1-x)\ln(1-x^*)$ are absent, gives the relation

$$\frac{\Gamma(a_1+a_2+1)}{\Gamma(a_1)\,\Gamma(a_2)}\,\frac{\Gamma(\widetilde{a_1}+\widetilde{a_2}+1)}{\Gamma(\widetilde{a_1})\,\Gamma(\widetilde{a_2})} + C\,\frac{\Gamma(1-a_1-a_2)}{\Gamma(-a_1)\,\Gamma(-a_2)}\,\frac{\Gamma(1-\widetilde{a_1}-\widetilde{a_2})}{\Gamma(-\widetilde{a_1})\,\Gamma(-\widetilde{a_2})} = 0\,. \tag{78}$$

Providing, that the constant C is fixed by this equality, the behavior of the total wave function at $x \to 1$ is simplified

$$\lim_{x\to 1}\Psi_{m,\widetilde{m}}(\vec{x}) \sim \left(\psi(1+a_1) + \psi(1+a_2) - \psi(-a_1) - \psi(-a_2)\right)|1-x|^2\ln(1-x^*)$$

$$+\left(\psi(1+\widetilde{a_1})+\psi(1+\widetilde{a_2})-\psi(-\widetilde{a_1})-\psi(-\widetilde{a_2})\right)|1-x|^2\ln(1-x)\,. \tag{79}$$

Thus, the single-valuedness condition at $x\to 1$ leads to the additional equation

$$\cot(\pi a_1)+\cot(\pi a_2)=\cot(\pi\widetilde{a_1})+\cot(\pi\widetilde{a_2})\,. \tag{80}$$

A stronger constraint can be obtained from the single-valuedness condition for Ψ at $x\to\infty$. Indeed, its consequence for the bilinear combinations

$$(-x)^{-a}(-x^*)^{-\widetilde{a_1}}\,,\ (-x)^{-a_2}(-x^*)^{-\widetilde{a_2}}$$

leads to the relations

$$a_1-\widetilde{a_1}=N_{a_1}\,,\ a_2-\widetilde{a_2}=N_{a_2}\,, \tag{81}$$

where N_{a_1}, N_{a_2} are integers. Further, the absence of the interference terms

$$(-x)^{-a_1}(-x^*)^{-\widetilde{a_2}}\,,\ (-x)^{-a_2}(-x^*)^{-\widetilde{a_1}}$$

is fulfilled due to the above relation (77) for C.

One can write the integral representation for the wave function satisfying the above constraints

$$\Psi\sim z_2^{a_1+a_2}\,(z_2^*)^{\widetilde{a_1}+\widetilde{a_2}}\int\frac{d^2y}{|y|^2}\,y^{-a_2}(y^*)^{-\widetilde{a_2}}\,\left(\frac{y-1}{y-x}\right)^{a_1}\,\left(\frac{y^*-1}{y^*-x^*}\right)^{\widetilde{a_1}}\,,\ x=\frac{z_2}{z_1}\,, \tag{82}$$

where the integration is performed over the two-dimensional plane $\vec{y}$. Note, that the integrand has no ambiguity in the points $y=0,1,x$ due to the derived relations between $a_1,\widetilde{a_1}$ and $a_2,\widetilde{a_2}$. Moreover, the function Ψ near the points $x=0,1,\infty$ can be presented in terms of the sum of products of above hypergeometric functions.

There is another basis for the holomorphic solutions

$$\Psi_1(z_1,z_2)=z_1^{a_1}\,z_2^{a_2}\,F(a_1,-a_2,1+a_1-a_2;\frac{z_1}{z_2})\,,$$

$$\Psi_2(z_1,z_2)=z_1^{a_2}\,z_2^{a_1}\,F(a_2,-a_1,1+a_2-a_1;\frac{z_1}{z_2}) \tag{83}$$

allowing to construct an equivalent representation for the total wave function Ψ. Note, that these functions can be written in terms of the Mellin-Barnes integrals

$$\Psi_1(z_1,z_2)\sim\int_{-i\infty}^{i\infty}\frac{\Gamma(a_1+s)\,\Gamma(-a_2+s)\,\Gamma(-s)}{\Gamma(a_1-a_2+1+s)}\,(-z_1)^{a_1+s}(-z_2)^{a_2-s}\,d\,s\,,$$

$$\Psi_2(z_1,z_2)\sim\int_{-i\infty}^{i\infty}\frac{\Gamma(a_2+s)\,\Gamma(-a_1+s)\,\Gamma(-s)}{\Gamma(a_2-a_1+1+s)}\,(-z_1)^{a_2+s}(-z_2)^{a_1-s}\,d\,s\,. \tag{84}$$

Here it is assumed, that the poles of $\Gamma(-s)$ are situated to the right from the integration contour whereas all other poles lie to the left of it.

5 Hamiltonian and integrals of motion

The holomorphic hamiltonian for composite states of two reggeized gluons can be written as follows

$$\widetilde{h}=\ln(z_1^2\partial_1)+\ln(\partial_1)+2\gamma=\psi(z_1\partial_1)+\psi(-z_1\partial_1)+2\gamma\,. \tag{85}$$

Acting by $\widetilde{h}$ on the function z_1^δ we obtain

$$\widetilde{h}z_1^\delta=\epsilon(\delta)\,z_1^\delta\,,\ \epsilon(\delta)=\psi(\delta)+\psi(-\delta)+2\gamma\,. \tag{86}$$

In the case of wave function (69) satisfying the single-valuedness and orthonormality conditions in the two-dimensional space one derives the following expression for the total energy [39]

$$E_{m,\widetilde{m}} = \epsilon_m + \epsilon_{\widetilde{m}}\,,\ \epsilon_m = \psi(-\frac{1}{2}+m) + \psi(\frac{1}{2}-m) + 2\gamma\,. \tag{87}$$

Note, that it does not coincide with the corresponding result (13) for the Pomeron state.

The holomorphic hamiltonian for composite states of three gluons has the form

$$h' = \ln(z_1^2\partial_1) + \ln(\partial_2) + \ln(z_{1,2}^2\partial_1) + \ln(z_{1,2}^2\partial_2) - 2\ln z_{1,2} + 4\gamma\,. \tag{88}$$

In the region

$$z_1 \ll z_2 \tag{89}$$

it is a sum of two independent pair hamiltonians

$$h' = \psi(z_1\partial_1) + \psi(-z_1\partial_1) + \psi(z_2\partial_1) + \psi(-z_2\partial_1) + 4\gamma\,. \tag{90}$$

Because the limit $x = z_2/z_1 \to \infty$ in solution (72) corresponds to this kinematics, we obtain

$$\epsilon = \epsilon(a_1) + \epsilon(a_2)\,, \tag{91}$$

where a_1 and a_2 are parameters of the three-gluon composite state (see (75)). The eigenvalues of the integrals of motion are also expressed in terms of these parameters. Due to the normalizability condition these quantities together with the parameters $\widetilde{a_1}, \widetilde{a_2}$ of the wave function in the anti-holomorphic space should be chosen as follows

$$\begin{aligned} a_1 &= i\nu_{a_1} + \frac{n_{a_1}}{2}\,,\ a_2 = i\nu_{a_2} + \frac{n_{a_2}}{2}\,, \\ \widetilde{a_1} &= i\nu_{a_1} - \frac{n_{a_1}}{2}\,,\ \widetilde{a_2} = i\nu_{a_2} - \frac{n_{a_2}}{2}\,, \end{aligned} \tag{92}$$

where ν_r are real and n_r are integer numbers.

Note, that

$$\nu = \nu_{a_1} + \nu_{a_2}\,,\ n = n_{a_1} + n_{a_2}\,,\ a_1a_2 = -\lambda\,,\ \widetilde{a_1}\widetilde{a_2} = -\widetilde{\lambda} \tag{93}$$

and the eigenvalues of two integrals of motion q_k' can be obtained as coefficients of the polynomials

$$P_2(u) = (u - ia_1)(u - ia_2)\,,\ \widetilde{P}_2(u) = (u - i\widetilde{a_1})(u - i\widetilde{a_2})\,. \tag{94}$$

Generally for the composite state of n reggeized gluons the situation is similar. Namely, the holomorphic wave function in the region

$$z_1 \ll z_2 \ll z_3 \ll ... \ll z_{n-1}\,. \tag{95}$$

is factorized

$$\Psi_{a_1,a_2,...,a_{n-1}} = \prod_{r=1}^{n-1} z_r^{a_r}\,. \tag{96}$$

The energy for this solution is the sum of the particle energies

$$\epsilon = \sum_{r=1}^{n-1} \epsilon(a_r)\,. \tag{97}$$

The eigenvalues of integrals of motion q_k' can be expressed in terms of the coefficients of the polynomial

$$P_n(u) = \prod_{r=1}^{n-1} (u - ia_r)\,. \tag{98}$$

Due to the condition of the normalizability the parameters should have the form

$$a_r = i\nu_r + \frac{n_r}{2}\,, \tag{99}$$

where ν_r is real and n_r is an integer number. The energies and eigenvalues of the integrals of motion in the anti-holomorphic space are given by the same expressions with the corresponding substitution of parameters

$$a_r \to \widetilde{a}_r = i\nu_r - \frac{n_r}{2}\,. \tag{100}$$

The holomorphic wave function satisfies a set of differential equations following from the eigenvalue equation for the operator $D(u)$

$$D(u)\,\Psi_{a_1,a_2,...,a_{n-1}} = \prod_{r=1}^{n-1}(u - ia_r)\,\Psi_{a_1,a_2,...,a_{n-1}}\,. \tag{101}$$

This equation can be solved with the use of the Taylor expansion

$$\Psi_{a_1,..a_{n-1}} = \prod_{r=1}^{n-1} z_r^{a_r} \sum_{s_2=0}^{\infty}\left(\frac{z_1}{z_2}\right)^{s_2} ... \sum_{s_{n-1}}^{\infty}\left(\frac{z_{n-2}}{z_{n-1}}\right)^{s_{n-1}} c(s_2,...,s_{n-1})\,, \tag{102}$$

where the coefficients $c(s_2,...,s_{n-1})$ are calculated in a recurrent way. The recurrent relations obtained from the eigenvalue equations for different operators q_r' are compatible due to their commutativity. The obtained solution has the singularities at $z_{kl} = 0$. But, if we consider $(n-1)!$ functions $\Psi_{a_{i_1},..a_{i_{n-1}}}$ obtained by all possible permutations of parameters a_r and multiply them on the corresponding functions in the anti-holomorphic subspace, it is possible to construct the wave function having the single-valuedness property in two-dimensional spaces $\vec{z}_r$

$$\Psi(\vec{z}_1,...,\vec{z}_{n-1}) = \sum_{\{i_1,i_2,...,i_{n-1}\}} C_{\{i_1,...,i_{n-1}\}}\,\Psi_{a_{i_1},a_{i_2},...,a_{i_{n-1}}}\,\Psi_{\widetilde{a}_{i_1},\widetilde{a}_{i_2},...,\widetilde{a}_{i_{n-1}}}\,. \tag{103}$$

For this purpose one should adjust the coefficients $C_{\{i_1,...,i_{n-1}\}}$ in an appropriate way presumably without additional constraints on the parameters a_r and $\widetilde{a}_r$. The composite state of $n-1$ gluons has the following total energy

$$E = \epsilon + \widetilde{\epsilon}\,,\ \epsilon = \sum_{r=1}^{n-1}\epsilon(a_r)\,,\ \widetilde{\epsilon} = \sum_{r=1}^{n-1}\epsilon(\widetilde{a}_r)\,. \tag{104}$$

6 Baxter-Sklyanin approach

To find a solution of the Yang-Baxter equation for the open spin chain one can use the Bethe ansatz. For this purpose it is convenient to work in the transposed representation for the monodromy matrix

$$T^t(u) = \begin{pmatrix} j_0^t(u) + j_3^t(u) & j_+^t(u) \\ j_-^t(u) & j_0^t(u) - j_3^t(u) \end{pmatrix} = L_1^t(u)L_2^t(u)...L_{n-1}^t(u)\,, \tag{105}$$

where the L-operator can be chosen as follows

$$L_r^t(u) = \begin{pmatrix} u + i\partial_r z_r & i\partial_r \\ -i\partial_r z_r^2 & u - i\partial_r z_r \end{pmatrix}\,. \tag{106}$$

The pseudo-vacuum state is defined as a solution of the equation

$$j_-^t(u)\Psi_0 = 0\,. \tag{107}$$

It can be written in the form [11]

$$\Psi_0 = \prod_{r=1}^{n-1} z_r^{-2} \,. \tag{108}$$

Note, that the function $|\Psi_0|^2$ does not belong to the principal series of the unitary representations. As a result, the states constructed in the framework of the Bethe ansatz by applying the product of the operators $j_+^r(u_r)$ to Ψ_0

$$\Psi_k^t = \prod_{r=1}^{k} j_+^t(u_r)\, \Psi_0 \tag{109}$$

are non-physical. Nevertheless, these states are eigenfunctions of the integral of motion

$$D^t(u)\Psi_k^t = (j_0^t(u) - j_3^t(u))\Psi_k^t = \Lambda(u)\Psi_k^t \tag{110}$$

providing that

$$\Lambda(u) = (u+i)^{n-1} \prod_{t=1}^{k} \frac{u-u_t+i}{u-u_t} \equiv (u+i)^{n-1} \frac{Q(u+i)}{Q(u)} \tag{111}$$

is a polynomial, which leads to a quantization condition for the Bethe roots u_t. If we parameterize this polynomial as follows

$$\Lambda(u) = \prod_{l=1}^{n-1} (u - ia_l) \,, \tag{112}$$

the above defined Baxter function $Q(u)$ can be calculated

$$Q(u) = \phi(u) \prod_{l=1}^{n-1} \frac{\Gamma(-iu - a_l)}{\Gamma(-iu+1)} \,. \tag{113}$$

Here for generality we included the factor $\phi(u)$ which is an arbitrary periodic function

$$\phi(u) = \phi(u+i) \,. \tag{114}$$

In the case of a finite number of the multipliers $j_+^t(u_r)$ in the Bethe ansatz for the wave function Ψ_k the expression $Q(u)$ is also a polynomial

$$Q(u) = \prod_{r=1}^{k} (u - u_r) \,. \tag{115}$$

For such solutions the parameters $a_l = -k_l - 1$ are negative integer numbers satisfying the condition

$$\sum_{l=1}^{n-1} k_l = k \,. \tag{116}$$

The corresponding Baxter functions can be written as follows

$$Q(u) = \prod_{l=1}^{n-1} \prod_{t=1}^{k_l} (u + it) = \prod_{p=1}^{\max_t k_t} (u+ip)^{r_p} \,, \tag{117}$$

where r_p is the number of k_t satisfying the condition $k_t \geq p$.

As it was mentioned above, the polynomial solutions for $Q(u)$ are non-physical, because the corresponding wave functions Ψ do not belong to the principal series of unitary representations of the Möbius group. We should find a set of non-polynomial solutions $Q_s(u)$ satisfying this physical requirement.

According to E. Sklyanin [14] the correct variables in which the dynamics of the Heisenberg spin model is drastically simplified are the zeroes $\hat{b}_r$ of the operator $B(u) = j_+^t(u)$ entering in the monodromy matrix

$$B(u) = P_{n-1} \prod_{k=1}^{n-2} (u - \hat{b}_r)\,, \;\; P_{n-1} = i \sum_{r=1}^{n-1} \partial_r\,, \tag{118}$$

where the operators $\hat{b}_r$ and P_{n-1} commute each with others

$$[\hat{b}_r, \hat{b_s}] = [\hat{b}_r, P_{n-1}] = 0\,. \tag{119}$$

It is convenient to pass from the coordinate representation $\vec{z}$ to the Baxter-Sklyanin representation [12], in which the currents $j_+^t(u)$ and $(j_+^t(u))^*$ (together with the operators $\hat{b}_r, \hat{b}_r^*$ and P_{n-1}, P_{n-1}^*) are diagonal. We denote the eigenvalues of the Sklyanin operators by b_r, b_r^*. The kernel of the unitary transformation to the Baxter-Sklyanin representation is known explicitly for the cases $n = 2$, $n = 3$ and $n = 4$ [12]. For general n this integral operator can be presented as a multi-dimensional integral [13].

In the Baxter-Sklyanin representation the wave function in the holomorphic subspace can be expressed as a product of the pseudo-vacuum state in this representation $\Psi_0(P_{n-1}, b_1, b_2, ..., b_{n-2})$ and the Baxter functions $Q(u_t)$

$$\Psi^t(P_{n-1}; b_1, ..., b_{n-2}) = P_{n-1}^{-\frac{n-1}{2}-m} \prod_{k=1}^{n-2} Q(b_k)\, \Psi_0(P_{n-1}, b_1, ..., b_{n-2})\,, \tag{120}$$

where the power-like behavior in the variable P_{n-1} is in an agreement with the normalization condition.

The analogous representation is valid for the total wave function

$$\Psi^+(\vec{P}_{n-1}; \vec{b}_1, ..., \vec{b}_{n-2}) = P_{n-1}^{-\frac{n-1}{2}-m}\, (P_{n-1}^*)^{-\frac{n-1}{2}-\tilde{m}} \prod_{k=1}^{n-2} Q(\vec{b}_r)\, \Psi_0(\vec{P}_{n-1}; \vec{b}_1, ..., \vec{b}_{n-2}) \tag{121}$$

with the use of the generalized Baxter function $Q(\vec{u})$ being a bilinear combination of the usual Baxter functions in the holomorphic and anti-holomorphic subspaces

$$Q(\vec{u}) = \sum_{s,t} d_{s,t}\, Q_s(u)\, Q_t(u^*)\,. \tag{122}$$

Here $Q_s(u)$ are different solutions of the Baxter equation with the same eigenvalue $\Lambda(u)$. The coefficients $d_{s,t}$ are chosen from the requirement, that the function $Q(\vec{u})$ satisfies the normalization condition everywhere including the points where the functions $Q_s(u)$ and $Q_t(u^*)$ have the poles [12, 13]. For the periodic spin chain this condition leads to the quantization of the eigenvalue of the operator $A(u) + B(u)$ although a simpler method of quantization is based on the requirement, that all Baxter functions corresponding to the same eigenvalue should have the same holomorphic energy [12]. In the case of the open Heisenberg spin model the situation is simpler and will be discussed below.

7 Baxter-Sklyanin representation for two and three gluon states

Let us consider the composite states constructed from two and three reggeons in the framework of the Baxter-Sklyanin approach. In the case $n = 2$ we have the following integral of motion in the transposed space

$$D^t(u) = j_0 - j_3 = u - i\partial_1 z_1 \tag{123}$$

and its eigenstates in accordance with the Sklyanin approach are given by the expression

$$\Psi^t \sim p_1^{-\frac{1}{2}-m}\, z_1^{-2} \sim z_1^{-\frac{3}{2}+m}\,. \tag{124}$$

The corresponding transposed hamiltonian is presented below

$$h^t = \ln(\partial_1 z_1^2) + \ln \partial_2 + 2\gamma \,. \tag{125}$$

Its eigenvalue calculated on the above eigenfunction Ψ^t is

$$\epsilon_m = \psi(-\frac{1}{2} + m) + \psi(\frac{1}{2} - m) + 2\gamma \,. \tag{126}$$

For the states composed from three reggeized gluons the transposed integral of motion in the holomorphic subspace is

$$D_3^t(u) = u^2 - iu(\partial_1 z_1 + \partial_2 z_2) + \partial_1 \partial_2 z_1 z_{1,2} \tag{127}$$

and the operator j_+^t is given below

$$j_+^t = iu(\partial_1 + \partial_2) - \partial_1 \partial_2 z_{12} = i(\partial_1 + \partial_2)\,(u - \hat{b}_1)\,, \tag{128}$$

where

$$\hat{b}_1 = -i \frac{\partial_1 \partial_2}{\partial_1 + \partial_2}\, z_{12} \,. \tag{129}$$

The operator j_+^t is easily diagonalized after a transition to the momentum representation, where

$$i\partial_1 f_{p_1,p_2} = p_1 f_{p_1,p_2} \,, \; i\partial_2 f_{p_1,p_2} = p_2 f_{p_1,p_2} \,. \tag{130}$$

In this case the eigenvalue equation for j_-^t has the form

$$\left(u(p_1 + p_2) - i\, p_1 p_2 (\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2})\right) f = (p_1 + p_2)(u - b_1)\, f \,, \tag{131}$$

where b_1 is the eigenvalue of $\hat{b}_1$. Its solution is given below

$$f = \chi(p_1 + p_2, b_1) \, \left(\frac{p_1}{p_2}\right)^{-ib_1} \,, \tag{132}$$

where χ is an arbitrary function of $p_1 + p_2$ and b_1. The dependence of Ψ^t from $p_1 + p_2$ is fixed by the normalization condition

$$\Psi^t \sim (p_1 + p_2)^{-a_1 - a_2} \,. \tag{133}$$

On the other hand, the eigenvalue equation for the integral of motion in the momentum space can be written in the form

$$p_1 p_2 \frac{\partial}{\partial p_1} \, \left(\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}\right) \Psi(p_1, p_2) = a_1 a_2 \Psi(p_1, p_2) \,. \tag{134}$$

Using the anzatz

$$\Psi(p_1, p_2) = (p_1 + p_2)^{-a_1 - a_2}\, \eta(y) \,, \; y = \frac{p_2}{p_1} \,, \tag{135}$$

we obtain the following equation for the function $\eta(y)$

$$\left(y^2\, \partial^2 + (a_1 + a_2 + 1)\, y\, \partial - a_1 a_2\right) \eta(y) = \left(-y^3\, \partial^2 - 2\, y^2\, \partial\right) \eta(y) \,. \tag{136}$$

There are two independent solutions of this equation

$$\eta_1(y) = \sum_{k=1}^{\infty} \frac{\Gamma(k - a_1)\, \Gamma(k - a_2)\, (-1)^{k-1}\, y^{-k}}{\Gamma(k+1)\Gamma(k)\Gamma(1 - a_1)\Gamma(1 - a_2)} =$$

$$\frac{1}{y}\,F(1-a_1,1-a_2,2;-\frac{1}{y}) = \frac{\Gamma(a_1-a_2)\,y^{-a_1}}{\Gamma(1-a_2)\,\Gamma(1+a_1)}\,F(-a_1,1-a_1,1+a_2-a_1;-y)$$
$$+\,\frac{\Gamma(a_2-a_1)\,y^{-a_2}}{\Gamma(1-a_1)\,\Gamma(1+a_2)}\,F(-a_2,1-a_2,1+a_1-a_2;-y) \tag{137}$$

and

$$\eta_2(y) = \frac{1}{a_1\,a_2} + \sum_{k=1}^{\infty}\frac{\Gamma(k-a_1)\,\Gamma(k-a_2)\,(-1)^{k-1}\,y^{-k}}{\Gamma(k+1)\Gamma(k)\Gamma(1-a_1)\Gamma(1-a_2)}\,(\ln y + c_k(a_1,a_2))$$
$$= -\frac{\Gamma(-a_1)\,\Gamma(+a_2)}{\Gamma(1+a_2-a_1)}\,y^{-a_1}\,F(-a_1,1-a_1,1+a_2-a_1;-y)\,, \tag{138}$$

where

$$c_k(a_1,a_2) = \psi(k)+\psi(k+1)-\psi(k-a_1)-\psi(1-k+a_2)\,. \tag{139}$$

One can construct the bilinear combination of these solutions having the single-valuedness property at $\vec{y}=\infty$

$$\eta(\vec{y}) \sim \eta_1(y)\,\widetilde{\eta}_2(y^*) + \eta_2(y)\,\widetilde{\eta}_1(y^*) + \widetilde{C}\,\eta_1(y)\,\widetilde{\eta}_1(y^*)\,. \tag{140}$$

On the other hand let us use the above expression for $\eta_1(y)$ and $\eta_2(y)$ expressed in terms of the hypergeometric function regular at $y=0$. To cancel the interference terms violating the single-valuedness condition at $y\to 0$ in the above bilinear combination for $\eta(\vec{y})$ we should fix $\widetilde{C}$ as follows

$$\widetilde{C} = -\frac{\sin(a_1\pi)\,\sin(a_2\pi)}{\pi\sin((a_1-a_2)\pi)} = -\frac{\sin(\widetilde{a}_1\pi)\,\sin(\widetilde{a}_2\pi)}{\pi\sin((\widetilde{a}_1-\widetilde{a}_2)\pi)}\,. \tag{141}$$

Finally with the use of the integral representation for the hypergeometric function the wave function Ψ^t in the momentum space can be written as follows

$$\Psi^t(\vec{p}_1,\vec{p}_2) = (p_1+p_2)^{-a_1-a_2}\,(p_1^*+p_2^*)^{-\widetilde{a}_1-\widetilde{a}_2}\,\phi(\vec{y})\,, \tag{142}$$

where $\phi(y)$ is given below

$$\phi(\vec{y}) = \int d^2t\,\left(\frac{1}{t\,y}+1\right)^{a_1}\,\left(\frac{1}{t^*\,y^*}+1\right)^{\widetilde{a}_1}\,(1-t)^{a_2-1}\,(1-t^*)^{\widetilde{a}_2-1} \tag{143}$$

and satisfies the single valuedness condition in the $\vec{y}$-space due to the quantization conditions (92).

The transition to the Baxter-Sklyanin representation $(u,\widetilde{u})$ corresponds to the Mellin-type transformation of $\phi(\vec{y})$

$$\phi(u,\widetilde{u}) = \int\frac{d^2y}{|y|^2}\,y^{-iu}\,(y^*)^{-i\widetilde{u}}\phi(\vec{y}) = \int d^2t\,(1-t)^{a_2-1}\,(1-t^*)^{\widetilde{a}_2-1}\,\chi(\vec{t})\,, \tag{144}$$

where

$$-iu = i\nu_u + \frac{N_u}{2}\,,\ -i\widetilde{u} = i\nu_u - \frac{N_u}{2}\,. \tag{145}$$

Here ν_u is a real number and $N_u = 0,\pm1,\pm2,....$ The function χ is given below

$$\chi(\vec{t}) = \int\frac{d^2y}{|y|^2}\,y^{-iu}\,(y^*)^{-i\widetilde{u}}\,\left(\frac{1}{t\,y}+1\right)^{a_1}\,\left(\frac{1}{t^*\,y^*}+1\right)^{\widetilde{a}_1} = t^{iu}(t^*)^{i\widetilde{u}}\,c_1\,. \tag{146}$$

The corresponding integrals can be calculated explicitly

$$c_1 = \pi\,\frac{\Gamma(1+\widetilde{a}_1)}{\Gamma(-a_1)}\,\frac{\Gamma(iu)\,\Gamma(-iu-a_1)}{\Gamma(1-i\widetilde{u})\,\Gamma(1+i\widetilde{u}+\widetilde{a}_1)}\,,$$

$$c_2 = \int \frac{d^2t}{|1-t|^2}\,(1-t)^{a_2}\,(1-t^*)^{\widetilde{a}_2}\,t^{iu}(t^*)^{i\widetilde{u}} = \pi\,\frac{\Gamma(a_2)}{\Gamma(1-\widetilde{a}_2)}\,\frac{\Gamma(1+i\widetilde{u})\,\Gamma(-iu-a_2)}{\Gamma(-iu)\,\Gamma(1+i\widetilde{u}+\widetilde{a}_2)}\,.$$

Therefore we obtain for $\phi(u,\widetilde{u})$ the following expression

$$\phi(u,\widetilde{u}) = \frac{\pi^2\Gamma(1+\widetilde{a}_1)\Gamma(a_2)}{\Gamma(-a_1)\,\Gamma(1-\widetilde{a}_2)}\,\frac{\Gamma(iu)\Gamma(1+i\widetilde{u})}{\Gamma(-iu)\,\Gamma(1-i\widetilde{u})}\,\frac{\Gamma(-iu-a_1)\,\Gamma(-iu-a_2)}{\Gamma(1+i\widetilde{u}+\widetilde{a}_1)\Gamma(1+i\widetilde{u}+\widetilde{a}_2)}\,. \tag{147}$$

The inverse transformation corresponds to the Baxter-Sklyanin representation for the wave function

$$\Psi^t(\vec{p}_1,\vec{p}_2) = (p_1+p_2)^{-a_1-a_2}(p_1^*+p_2^*)^{-\widetilde{a}_1-\widetilde{a}_2}\,\int d^2u\,\phi(u,\widetilde{u})\,\left(\frac{p_1}{p_2}\right)^{-iu}\,\left(\frac{p_1^*}{p_2^*}\right)^{-i\widetilde{u}}\,, \tag{148}$$

where

$$-iu = i\nu_u + \frac{N_u}{2}\,,\;\; -i\widetilde{u} = i\nu_u - \frac{N_u}{2}\,,\;\; \int d^2u \equiv \int_{-\infty}^{\infty} d\nu_u \sum_{N_u=-\infty}^{\infty}\,. \tag{149}$$

One can interpret the wave function $\phi(u,\widetilde{u})$ in the Baxter-Sklyanin representation as a product of the pseudo-vacuum state $u\,\widetilde{u}$ and the total Baxter function

$$\phi(u,\widetilde{u}) = u\,\widetilde{u}\,Q(u,\widetilde{u})\,, \tag{150}$$

where

$$Q(u,\widetilde{u}) \sim \frac{\Gamma(iu)\Gamma(i\widetilde{u})}{\Gamma(1-iu)\,\Gamma(1-i\widetilde{u})}\,\frac{\Gamma(-iu-a_1)\,\Gamma(-iu-a_2)}{\Gamma(1+i\widetilde{u}+\widetilde{a}_1)\Gamma(1+i\widetilde{u}+\widetilde{a}_2)}\,. \tag{151}$$

This expression for $Q(u,\widetilde{u})$ is symmetric to the substitution

$$(u,a_1,a_2) \leftrightarrow (\widetilde{u},\widetilde{a}_1,\widetilde{a}_2) \tag{152}$$

and can be written in the factorized form

$$Q(u,\widetilde{u}) \sim Q(u,a_1,a_2)\,Q(\widetilde{u},\widetilde{a}_1,\widetilde{a}_2)\,, \tag{153}$$

where

$$Q(u,a_1,a_2) = \frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma^2(1-iu)}\,\Phi(u)\,, \tag{154}$$

$$\Phi(u) = \sqrt{\frac{\sin(\pi(-iu-a_1))\,\sin(\pi(-iu-a_2))}{\sin^2(-i\pi u)}}\,. \tag{155}$$

The expression $Q(u,a_1,a_2)$ differs from the Baxter function in the holomorphic space

$$Q(u) = \frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma^2(1-iu)} \tag{156}$$

only by the periodic function $\Phi(u)$ and therefore it can be considered also as a Baxter function. Note, however, that the function $\Phi(u)$ contains a square root singularity and, as a result, the recurrence relation for the function $Q(u,\widetilde{u})$ differs from the similar relation for $Q(u)$ by a sign in its right hand side

$$Q(u+i,\widetilde{u}) = -\frac{(u-ia_1)(u-ia_2)}{(u+i)^2}\,Q(u,\widetilde{u})\,. \tag{157}$$

To overcome this problem we can write $Q(u,\widetilde{u})$ as follows

$$Q(u,\widetilde{u}) = Q(u)\,Q(\widetilde{u})\,\Phi(u,\widetilde{u})\,, \tag{158}$$

where the function Φ is given below

$$\Phi(u,\widetilde{u}) = \frac{\sin(\pi(i\widetilde{u}+\widetilde{a}_1))\,\sin(\pi(i\widetilde{u}+\widetilde{a}_2))}{\sin(i\pi u)\,\sin(i\pi\widetilde{u})}\,. \tag{159}$$

This additional factor $\Phi(u,\widetilde{u})$ can be included in the definition of a new pseudo-vacuum state

$$\Psi_0 = \Phi(u,\widetilde{u})\,u\,\widetilde{u}\,. \tag{160}$$

Really this pseudo-vacuum state can be considered as the additional factor for the wave function in the Baxter-Sklyanin representation providing correct hermicity properties of the hamiltonian and integrals of motion in this representation[2] (see also Ref. [13]). We shall return to this problem in our future publications.

8 Conclusion

In this paper we established, that the gluon production amplitudes in the planar approximation could have the Mandelstam cut contributions in the multi-regge kinematics at some physical regions. For the cut corresponding to the composite states of n reggeized gluons the number of external particles should be $k \geq 2+2n$. The wave functions of these states in the adjoint representation satisfy the BFKL-like equation integrable in LLA and have the property of the holomorphic factorization. The corresponding holomorphic hamiltonian coincides with the local hamiltonian for an integrable open Heisenberg spin model. The Baxter equation for this model is reduced to a simple recurrent relation and can be solved in terms of the product of the Γ-functions. We constructed the wave functions of composite states of 2 and 3 gluons explicitly.

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A Mandelstam cuts in planar diagrams

Here we discuss an appearance of the Mandelstam cuts [43] in the crossing channels having adjoint representations of the color group $SU(N_c)$ for the planar Feynman diagrams in the t'Hooft limit $\alpha \ll 1$, $\alpha_c N_c \sim 1$ and calculate the impact factors corresponding to the multi-reggeon exchange. To begin with, let us consider the elastic amplitude $A(s,t)$ for the gluon-gluon scattering in the Regge kinematics. It is well known, that in the leading logarithmic approximation the corresponding t-channel partial wave contains only one reggeized gluon pole. The contribution from the Pomeron exchange with color singlet quantum numbers is suppressed at large N_c. The BFKL Pomeron appears as a composite state of two reggeized gluons and corresponds to the Mandelstam cut in the j-plane of the crossing channel. For the elastic amplitude the cuts in the adjoint representation appear in non-planar diagrams and are also suppressed at large N_c. Indeed, according to S. Mandelstam these contributions should have the following form

$$A(s,t) \sim \int \frac{d^2k_\perp}{(2\pi)^2 i\,s}\,\frac{(-s)^{j(-\vec{k}^2)}}{\vec{k}^2}\,\frac{(-s)^{j(-(\vec{q}-\vec{k})^2)}}{(\vec{q}-\vec{k})^2}\,\Phi_1(k_\perp,q)\,\Phi_2(k_\perp,q)\,, \tag{A.1}$$

where $j(t)$ are the Regge trajectories. The impact factors Φ_r are the integrals from the particle-reggeon scattering amplitudes f_r (including the reggeon residues) over the invariants s_r in the direct channel

$$\Phi_r(k_\perp,q_\perp) = \int_L \frac{ds_r}{2\pi i} f_r(p_r,k,q)\,,\ s_1 = (p_A-k)^2\,,\ s_2 = (p_B+k)^2\,. \tag{A.2}$$

[2] I thank Prof. F. Smirnov for discussions related to this important interpretation of the pseudo-vacuum state.

Here the integration contour L goes along the real axis above the right singularities of f_r and below left ones according to the Feynman prescription. Only when the amplitude f_r is constructed from the diagrams having both these singularities simultaneously the result of the integration is non-zero because in an opposite case we can shift the contour L from the real axis to infinity with a vanishing result. The Mandelstam cuts are absent also for the planar amplitude with five external particles.

However, in the case of the six point amplitude there are planar diagrams in which the Mandelstam cuts are present. Let us denote the momenta of initial gluons by p_A, p_B and the momenta of final particles by $p_{A'}, k_1, k_2, p_{B'}$ in an accordance with the order of multiplication of the corresponding color matrices T_r. Then this cut appears in the physical region, where

$$s = (p_A + p_B)^2 > 0\,,\ s_1 = (p_{A'} + k_1)^2 < 0\,,\ s_2 = (k_1 + k_2)^2 > 0\,,\ s_3 = (k_2 + p_{B'})^2 < 0\,. \tag{A.3}$$

This region corresponds to the transition of four particles with their momenta $p_A, -k_1, -k_2$ and p_B to the two particles with the momenta $p_{A'}$ and $p_{B'}$. In the multi-Regge kinematics, where the corresponding Sudakov parameters are strongly ordered $1 \gg -\beta_1 \gg -\beta_2\,,\ -\alpha_1 \ll -\alpha_2 \ll 1$, the integrands in the impact factors Φ_r

$$\Phi_1(\vec{k}, \vec{k}_1, \vec{q}_2,) = \int_L \frac{s\, d\alpha}{2\pi i}\, f_1(p_A, k, k_1, q_2)\,,\ \Phi_2(\vec{k}, \vec{k}_2, \vec{q}_2,) = \int_L \frac{s\, d\beta}{2\pi i}\, f_2(p_B, k, k_2, q_2) \tag{A.4}$$

in the simplest case have only the poles in the integration variables $\alpha \approx 2kp_A/s$ and $\beta \approx 2kp_B/s$.

$$f_1 = \frac{1}{(p_A - k)^2 + i\epsilon}\,\frac{1}{(k_1 + q_2 - k)^2 + i\epsilon} = \frac{1}{-s\alpha - \vec{k}^2 + i\epsilon}\,\frac{1}{-s\alpha\beta_1 - (\vec{k}_1 + \vec{q}_2 - \vec{k})^2 + i\epsilon}\,,$$

$$f_2 = \frac{1}{(p_B + k)^2 + i\epsilon}\,\frac{1}{(k_2 - q_2 + k)^2 + i\epsilon} = \frac{1}{s\beta - \vec{k}^2 + i\epsilon}\,\frac{1}{s\alpha_2\beta - (\vec{k}_2 - \vec{q}_2 + \vec{k})^2 + i\epsilon}\,. \tag{A.5}$$

These poles are situated above and below the integration contours L due to the inequalities $\beta_1 < 0$, $\alpha_2 < 0$ valid in the considered kinematical region where $s_1 < 0$, $s_3 < 0$. Therefore the integrals are non-zero and can be calculated by residues

$$\Phi_1(\vec{k}, \vec{k}_1, \vec{q}_2) = \frac{1}{(\vec{k}_1 + \vec{q}_2 - \vec{k})^2}\,,\ \Phi_2(\vec{k}, \vec{k}_2, \vec{q}_2) = \frac{1}{(\vec{k}_2 - \vec{q}_2 + \vec{k})^2}\,. \tag{A.6}$$

In the case of production of two gluons with the same helicity at the multi-Regge kinematics in the physical region where $s_1 < 0$, $s_2 > 0$, $s_3 < 0$ the amplitude is proportional to the Born expression

$$A_{2\to4} = f_{2\to4}\, 2s\, g\, T^{c_1}_{A'A}\, \frac{1}{|q_1|^2}\, g\, C(q_2, q_1)\, T^{d_1}_{c_2c_1}\, \frac{1}{|q_2|^2}\, g\, C(q_3, q_2)\, T^{d_2}_{c_3c_2}\, \frac{1}{|q_3|^2}\, g\, T^{c_3}_{B'B}\,, \tag{A.7}$$

where the Reggeon-Reggeon gluon vertex C is given above (see (7)). In the lowest order approximation the corresponding proportionality factor f_{LO} for the Mandelstam cut contribution in the t_2-channel contains some additional multipliers from the effective vertices C in comparison with the above result (see [4, 39])

$$f_{LO} = i\,\frac{g^2\, N_c}{4\pi}\,\int \frac{\mu^{2\epsilon} d^{2-2\epsilon}k}{(2\pi)^{1-2\epsilon}}\,\frac{(-s_2)^{j(-\vec{k}^2)-1}}{|k|^2}\,\frac{(-s_2)^{j(-(\vec{q}_2-\vec{k})^2)-1}}{|q_2 - k|^2}\,\widetilde{\Phi}_1\,\widetilde{\Phi}_2\,, \tag{A.8}$$

where

$$\widetilde{\Phi}_1 = q_1^*(k_1 + q_2 - k)\,(q_2^* - k^*)\,\Phi_1(\vec{k}, \vec{k}_1, \vec{q}_2) = \frac{q_1^*(q_2^* - k^*)}{k_1^* + q_2^* - k^*}\,, \tag{A.9}$$

$$\widetilde{\Phi}_2 = q_3(q_2^* - k_2^* - k^*)\,(q_2 - k)\,\Phi_2(\vec{k}, \vec{k}_2, \vec{q}_2) = \frac{q_3(q_2 - k)}{q_2 - k_2 - k}\,. \tag{A.10}$$

In the weak coupling limit $j = 1$ and at $\epsilon \to 0$ the amplitude f_{LO} is

$$\lim_{j\to 1} f_{LO} = \pi i\, a \left(\ln \frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon} \right) ,\ a = \frac{g^2 N_c}{8\pi^2} \left(4\pi\, e^{-\gamma}\right)^{\epsilon} , \tag{A.11}$$

which coincides in this limit with the logarithm of the factor C introduced in Ref. [38]. This factor violates the Regge factorization of the BDS amplitude in the considered kinematical region due to the presence of the Mandelstam cut [38].

One can take into account the gluon reggeization in the channels t_1 and t_3 using the following substitution in the above expressions

$$\frac{1}{-s\alpha - \vec{k}^2 + i\epsilon} \to -(s\alpha - i\epsilon)^{j(t_1)-2} ,\ \frac{1}{s\beta - \vec{k}^2 + i\epsilon} \to -(-s\beta - i\epsilon)^{j(t_3)-2} . \tag{A.12}$$

It would lead to the multiplication of the integrand with the real factor

$$R = \left(\frac{-(\vec{k}_1 + \vec{q}_2 - \vec{k})^2}{\beta_1} \right)^{j(t_1)-1} \left(\frac{-(\vec{k}_2 - \vec{q}_2 + \vec{k})^2}{\alpha_2} \right)^{j(t_3)-1} \approx (-s_1)^{j(t_1)-1}\ (-s_3)^{j(t_3)-1} . \tag{A.13}$$

We can include also the diagrams with the reggeized gluon scattering in the crossing channel. It leads to the following expression for the Mandelstam contribution in LLA

$$f_{LLA}^{Mand} = i\, R\, \frac{g^2\, N_c}{4\pi} \int \frac{\mu^{2\epsilon} d^{2-2\epsilon}k}{(2\pi)^{1-2\epsilon}} \frac{\mu^{2\epsilon} d^{2-2\epsilon}k'}{(2\pi)^{1-2\epsilon}} \frac{1}{|k|^2} \frac{1}{|q_2-k|^2}\, G(\vec{k},\vec{k}',\vec{q}_2;\ln(-s_2))\, \widetilde{\Phi}_1\, \widetilde{\Phi}_2 , \tag{A.14}$$

where G is the Green function satisfying the BFKL-like equation for the octet quantum numbers in t_2-channel

$$\frac{\partial}{\partial \ln s_2}\, G(\vec{k},\vec{k}',\vec{q}_2;\ln(-s_2)) = K\, G(\vec{k},\vec{k}',\vec{q}_2;\ln(-s_2)) ,$$

$$G(\vec{k},\vec{k}',\vec{q}_2;0) = \frac{(2\pi)^{1-2\epsilon}}{\mu^{2\epsilon}}\, \delta^{2-2\epsilon}(k-k') . \tag{A.15}$$

Here the operator K in LLA can be expressed in terms of the Hamiltonian H which does not contain infrared divergencies

$$K = \omega(t_2) - \frac{g^2 N_c}{16\pi^2}\, H ,\ \omega(t) = a \left(\frac{1}{\epsilon} - \ln \frac{-t}{\mu_2} \right) ,$$

$$H = 2 \ln \frac{|p_1|^2 |p_2|^2}{|q_2|^2} + p_1 p_2^*\, \ln|\rho_{12}|^2\, \frac{1}{p_1 p_2^*} + p_1^* p_2\, \ln|\rho_{12}|^2\, \frac{1}{p_1^* p_2} , \tag{A.16}$$

where $p_1 = k$, $p_2 = q - k$.

Let us consider now the Mandelstam cuts constructed from several reggeons. The non-vanishing contribution from the exchange of $r+1$ reggeons appears in the planar diagrams only if the number of the external lines is $n \geq 2r+4$. For the inelastic transition $2 \to 2+2r$ with the initial and final momenta p_A, p_B and $p_{A'}, k_1, k_2, ..., k_{2r}, p_{B'}$, respectively, the cut exists in the crossing channel with the momentum

$$q = p_A - p_{A'} - \sum_{l=1}^{r} k_l = p_{B'} - p_B + \sum_{l=r+1}^{2r} k_l = \sum_{l=1}^{r+1} q_l' , \tag{A.17}$$

where q_l' are momenta of reggeons forming the composite state. The corresponding amplitude has the form

$$A(p_A, p_{A'}, k_1, ..., k_{2r}, p_{B'}, p_B) \sim \int \prod_{t=1}^{r} \frac{d^2 q_t'}{2\pi\, s} \prod_{l=1}^{r+1} \frac{(-s)^{j(-\vec{q}_l'^{\,2})}}{|q_l'|^2} \Phi_1(\vec{q}_1', ..., \vec{q}_{r+1}') \Phi_2(\vec{q}_1', ..., \vec{q}_{r+1}') . \tag{A.18}$$

The impact factors $\Phi_{1,2}$ are given in terms of the integrals over the Sudakov parameters $\alpha'_l = 2q'_l p_A/s$, $\beta'_l = 2q'_l p_B/s$ from the reggeon-particle scattering amplitudes $f_{1,2}$

$$\Phi_1 = \prod_{l=1}^{r-1} \int_L \frac{s\, d\alpha'_l}{2\pi i}\, f_1\,,\ \Phi_2 = \prod_{l=1}^{r-1} \int_L \frac{s\, d\beta'_l}{2\pi i}\, f_2\,. \tag{A.19}$$

In QCD the tree expressions for $f_{1,2}$ appearing in the planar diagrams are given below

$$f_1 = I_1\, \frac{1}{(p_A - q'_1)^2}\, \frac{1}{(p_A - k_0 - q'_1)^2} ... \frac{1}{(p_A - \sum_{l=1}^{r} q'_l - \sum_{l=0}^{r-2} k_l)^2}\, \frac{1}{(p_A - \sum_{l=1}^{r} q'_l - \sum_{l=0}^{r-1} k_l)^2}\,,$$

$$f_2 = I_2\, \frac{1}{(p_B + q'_1)^2}\, \frac{1}{(p_B - k_{2r+1} + q'_1)^2} ... \frac{1}{(p_B + \sum_{l=1}^{r} q'_l - \sum_{l=r+3}^{2r+1} k_l)^2}\, \frac{1}{(p_B + \sum_{l=1}^{r} q'_l - \sum_{l=r+2}^{2r+1} k_l)^2}\,,$$

where $k_0 = p_{A'}$, $k_{2r+1} = p_{B'}$. The additional factors $I_{1,2}$ contain effective reggeon vertices for the production and scattering of the gluons with the same helicity. They can be written in the multi-Regge kinematics (5) as follows (cf. [4])

$$I_1 = \prod_{l=1}^{r} \frac{q'^*_{l+1}(Q - \sum_{t=1}^{l} q'_t - \sum_{t=1}^{l-1} k_t)}{(Q^* - \sum_{t=1}^{l+1} q'^*_t - \sum_{t=1}^{l-1} k^*_t)}\, \prod_{l=1}^{r} \beta_r\,,$$

$$I_2 = \prod_{l=1}^{r} \frac{q'_{l+1}(\widetilde{Q}^* + \sum_{t=1}^{l} q'^*_t - \sum_{t=1}^{l-1} k^*_{2r-t+1})}{(\widetilde{Q} + \sum_{t=1}^{l+1} q'_t - \sum_{t=1}^{l-1} k_{2r-t+1})}\, \prod_{l=1}^{r} \alpha_r\,, \tag{A.20}$$

where $Q = p_A - p_{A'}$, $\widetilde{Q} = p_B - p_{B'}$ and the Sudakov variables of the produced particles $\alpha_l = 2k_l p_A/s$, $\beta_l = 2k_l p_B/s$ are strongly ordered

$$1 \gg |\beta_1| \gg |\beta_2| ... \gg |\beta_{2k}|\,,\ |\alpha_1| \ll |\alpha_2| \ll ... |\alpha_{2k}| \ll 1\,. \tag{A.21}$$

In these variables the functions $f_{1,2}$ are given below

$$f_1 = I_1\, \frac{1}{-s\alpha'_1 + i\epsilon}\, \frac{1}{-s\beta_1\alpha'_1 - |Q - q'_1|^2 + i\epsilon}\, \frac{1}{-s\beta_1\alpha'_2 + i\epsilon}\, \frac{1}{-s\beta_2\alpha'_2 - |Q - q'_1 - q'_2 - k_1|^2 + i\epsilon} ...\,,$$

$$f_2 = I_2\, \frac{1}{s\beta'_1 + i\epsilon}\, \frac{1}{s\alpha_{2r}\beta'_1 - |\widetilde{Q} + q'_1|^2 + i\epsilon}\, \frac{1}{s\alpha_{2r}\beta'_2 + i\epsilon}\, \frac{1}{s\alpha_{2r-1}\beta'_2 - |\widetilde{Q} + q'_1 + q'_2 - k_{2r}|^2 + i\epsilon} ...\,,$$

where we took into account, that in the essential region of integration

$$\alpha'_l \sim \frac{|Q|^2}{s\beta_l}\,,\ \beta'_l \sim \frac{|\widetilde{Q}|^2}{s\alpha_{2r-l+1}}\,. \tag{A.22}$$

In the physical region, where the signs of the Sudakov parameters of momenta k_l alternate with the index l

$$\beta_1,\, \alpha_{2r} < 0\,;\ \beta_2,\, \alpha_{2r-1} > 0\,;\ \beta_3,\, \alpha_{2r-2} < 0\,; ...\,, \tag{A.23}$$

which is equivalent to the following constraints on the invariants

$$s_1 < 0, s_2 < 0, ..., s_r < 0, s_{r+1} > 0, s_{r+2} < 0, s_{r+3} < 0, ..., s_{2r+1} < 0, s > 0\,, \tag{A.24}$$

the integrands in expressions for $\Phi_{1,2}$ contain poles above and below the integration contours L over all variables α'_l, β'_l. Therefore $\Phi_{1,2}$ are non-zero and can be calculated by taking residues from the poles in $f_{1,2}$

$$\Phi_1(\vec{q}'_1, ..., \vec{q}'_{r+1}) = \prod_{l=1}^{r} \frac{q'^*_{l+1}}{(Q^* - \sum_{s=1}^{l} q'^*_s - \sum_{s=1}^{l-1} k^*_s)\,(Q^* - \sum_{t=1}^{l+1} q'^*_t - \sum_{t=1}^{l-1} k^*_t)} \tag{A.25}$$

$$\Phi_2(\vec{q}_1',...,\vec{q}_{r+1}') = \prod_{l=1}^{r} \frac{q_{l+1}'}{(\widetilde{Q} + \sum_{s=1}^{l} q_s' - \sum_{s=1}^{l-1} k_{2r-s+1})\,(\widetilde{Q} + \sum_{t=1}^{l+1} q_t' - \sum_{t=1}^{l-1} k_{2r-t+1})}\,. \qquad (A.26)$$

In the case of the production of $2r$ gluons with the same helicity the amplitude in $N = 4$ SUSY is proportional to the Born expression containing the effective Reggeon-Reggeon-gluon vertices C (7). The proportionality factor $f_{2\to 2+2r}$ for the Mandelstam cut constructed from $r+1$ reggeized gluons can be written as follow

$$f_{LO}^{2\to 2+2r} = \left(i\,\frac{g^2\,N_c}{4\pi}\right)^r\, Q^*\widetilde{Q} \int \prod_{l=1}^{r} \frac{\mu^{2\epsilon} d^{2-2\epsilon} q_l'}{(2\pi)^{1-2\epsilon}} \prod_{l=1}^{r+1} \frac{(-s_{r+1})^{j(-|q_l'|^2)-1}}{|q_l'|^2}\,\Phi_1\,\Phi_2 \prod_{t=1}^{r} k_t^* k_{2r-t}\,. \qquad (A.27)$$

In the leading logarithmic approximation the proportionality factor has the form

$$f_{LLA}^{2\to 2+2r}$$

$$= \left(i\,\frac{g^2\,N_c}{4\pi}\right)^r\, Q^*\widetilde{Q} \int \prod_{l=1}^{r} \frac{\mu^{2\epsilon} d^{2-2\epsilon} p_l}{(2\pi)^{1-2\epsilon}}\,\frac{\mu^{2\epsilon} d^{2-2\epsilon} p_l'}{(2\pi)^{1-2\epsilon}} \prod_{l=1}^{r+1} \frac{1}{|p_l|^2}\,G(p,p';s_{r+1})\Phi_1\,\Phi_2 \prod_{t=1}^{r} k_t^* k_{2r-t}\,, \qquad (A.28)$$

where we introduce the new notation p_l for the reggeon momenta q_l'. The Green function satisfies the equation

$$\frac{\partial}{\partial \ln s_{r+1}}\,G(\vec{p},\vec{p}';s_{r+1}) = K\,G(\vec{p},\vec{p}';s_{r+1})\,,\ G(\vec{p},\vec{p}';0) = \prod_{l=1}^{r} \frac{(2\pi)^{1-2\epsilon}}{\mu^{2\epsilon}}\,\delta^{2-2\epsilon}(p_l - p_l')\,. \qquad (A.29)$$

Here the operator K in LLA can be expressed in terms of the Hamiltonian H which does not contain infrared divergencies

$$K = \omega(t) - \frac{g^2 N_c}{16\pi^2}\,H\,,\ \omega(t) = a\left(\frac{1}{\epsilon} - \ln\frac{-t}{\mu_2}\right)\,,\ t = -|q|^2\,,$$

$$H = \ln\frac{|p_1|^2 |p_{r+1}|^2}{|q|^4} + \sum_{l=1}^{r} H_{l,l+1}\,, \qquad (A.30)$$

where

$$H_{l,l+1} = \ln|p_l|^2 + \ln|p_{l+1}|^2 + p_l\,p_{l+1}^*\,\ln|\rho_{l,l+1}|^2\,\frac{1}{p_l\,p_{l+1}^*} + p_l^*\,p_{l+1}\,\ln|\rho_{l,l+1}|^2\,\frac{1}{p_l^*\,p_{l+1}}\,. \qquad (A.31)$$

Note, that the above hamiltonian has the property of the holomorphic separability

$$H = h + h^*\,,\ h = \ln\frac{p_1\,p_{r+1}}{q^2} + \sum_{l=1}^{r} h_{l,l+1}\,, \qquad (A.32)$$

where

$$h_{l,l+1} = \ln p_l + \ln p_{l+1} + p_l\,\ln\rho_{l,l+1}\,\frac{1}{p_l} + p_{l+1}\,\ln\rho_{l,l+1}\,\frac{1}{p_{l+1}}\,. \qquad (A.33)$$

One can take into account also the enhanced contributions in the impact factors leading to the Regge-type dependence of the amplitude from other invariants s_i $(i \neq r+1)$.

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