DESY 09-022
SFB/CPP-09-16
LTH 821

DESY 09-022
SFB/CPP-09-16
LTH 821

February 2009

arXiv:0902.2342 [hep-ph]

arXiv:0902.2342v1 [hep-ph] 13 Feb 2009

Threshold Resummation of the Structure Function F_L

S. Moch[a] and A. Vogt[b]

[a]*Deutsches Elektronensynchrotron DESY*
Platanenallee 6, D–15738 Zeuthen, Germany

[c]*Department of Mathematical Sciences, University of Liverpool*
Liverpool L69 3BX, United Kingdom

Abstract

The behaviour of the quark coefficient function for the longitudinal structure function F_L in deep-inelastic scattering is investigated for large values of the Bjorken variable x. We combine a highly plausible conjecture on the large-x limit of the physical evolution kernel for this quantity with our explicit three-loop results to derive the coefficients of the three leading large-x logarithms, $\alpha_s^n \ln^{2n-1-k}(1-x)$, $k = 1, 2, 3$, to all orders in the strong coupling constant α_s. Corresponding results are derived for the non-C_F part of the gluon coefficient function suppressed by a factor $1-x$, and for the analogous subleading $(1-x)\ln^k(1-x)$ contributions in the quark case. Our results appear to indicate an obstacle for an exponentiation with a higher logarithmic accuracy.

Structure functions in deep-inelastic scattering (DIS) provide an important (and accessible, via forward Compton amplitudes) laboratory for studying higher-order effects in perturbative QCD. Indeed, they are presently the only observables depending on a dimensionless variable (Bjorken-x in the case at hand) for which Feynman diagram calculations have been extended to the third order in the strong coupling α_s [1–5]. Such calculations are not only relevant phenomenologically, but also open up ways to new results for different quantities. For instance, a direct line runs from an observation on subleading large-x logarithms at three loop [1,2] via its interpretation in Ref. [6] to first results on the third-order splitting functions for the final-state parton fragmentation [7,8].

In the present letter we study the same class of large-x contributions, $\alpha_s^n \ln^k(1-x)$, to the higher-order quark coefficient functions [3,4,9–11] for the longitudinal structure function F_L (an analogous investigation of F_2 and F_3 will be presented elsewhere [12]) where these logarithms form the leading terms at $x \to 1$. Such contributions have been addressed before in Refs. [13–16], but no explicit all-order predictions have been presented so far for any coefficient function beyond the leading logarithms. This situation for the leading large-x behaviour of F_L is in striking contrast to that for F_2 and F_3 where the soft-gluon exponentiation [17,18] is known to the next-to-next-to-next-to-leading logarithmic accuracy and predicts the leading seven term to all orders in α_s [19].

The (flavour non-singlet) quark coefficient functions $C_{a,\rm ns}$ provide the connection between the structure functions $F_{a,\rm ns}$ and the corresponding quark distributions $q_{\rm ns}$,

$$\begin{aligned}
\mathcal{F}_{a=2,L}(x,Q^2) &\equiv x^{-1}F_{a,\rm ns}(x,Q^2) = C_{a,\rm ns}(x,\alpha_s)\otimes q_{\rm ns}(x,Q^2) \\
&= \left[(1-\delta_{aL})\delta(1-x) + \sum_{n=1} a_s^n\, c_{a,q}^{(n)}(x)\right]\otimes q_{\rm ns}(x,Q^2)\;, \qquad (1)
\end{aligned}$$

where $\otimes$ stands for the standard Mellin convolution. The renormalization and factorization scales μ_r and μ_f have been set to the physical hard scale Q^2 in Eq. (1), and the expansion parameter is normalized as $a_s = \alpha_s/(4\pi)$. The large-x expansion of the $\overline{\rm MS}$ coefficient function for F_L reads

$$\begin{aligned}
C_{L,\rm ns}(\alpha_s,x) &= \sum_{n=1} a_s^n\, c_{L,q}^{(n)}(x) \\
&= \sum_{n=1} a_s^n \left\{ \sum_{k=0}^{2n-2} \ln^k(1-x)\left[\bar{c}_{L,k}^{(n)} + (1-x)\,\bar{d}_{L,k}^{(n)} + O\left((1-x)^2\right)\right]\right\} \\
&\stackrel{\rm M-trf}{=} \frac{1}{N}\sum_{n=1} a_s^n \left\{ \sum_{k=0}^{2n-2} \ln^k N\left[c_{L,k}^{(n)} + \frac{1}{N} d_{L,k}^{(n)} + O\left(\frac{1}{N^2}\right)\right]\right\}. \qquad (2)
\end{aligned}$$

Here and below $\stackrel{\rm M-trf}{=}$ indicates that the right-hand-side is the Mellin transform of the previous expression. The leading x- and N-space coefficients $\bar{c}_{L,k}^{(n)}$ and $c_{L,k}^{(n)}$ in Eq. (2) are related via

$$\begin{aligned}
(-1)^k \int_0^1 dx\, x^{N-1} \ln^k(1-x) &= k!\,\frac{1}{N} S_{\underbrace{1,\ldots,1}_{k}}(N) \\
&= \frac{1}{N}\ln^k\widetilde{N} + \sum_{l=2}^{k} \frac{k!}{l(k-l)!}\,\bar{\zeta}_l\,\frac{1}{N}\,\ln^{k-l}\widetilde{N} + O\Big(\frac{1}{N^2}\ln^{k-1}N\Big) \qquad (3)
\end{aligned}$$

with $\widetilde{N} = N e^{\gamma_E}$ and the Riemann-zeta combinations $\bar{\zeta}_{2,3} = \zeta_{2,3}$, $\bar{\zeta}_4 = \zeta_4 + \frac{1}{2}\zeta_2^2$, $\bar{\zeta}_5 = \zeta_5 + \frac{5}{6}\zeta_2\zeta_3$ etc. See Refs. [20,21] for the notation and properties of the harmonic sums $S_{m_1,\ldots,m_k}(N)$.

It is convenient, both phenomenologically (especially for F_2 and F_3) and theoretically, to express the scaling violations of non-singlet structure functions in terms of these structure functions themselves. This explicitly eliminates any dependence on the factorization scheme and the scale μ_f. The corresponding 'physical evolution kernels' K_a can be derived for $\mu_r^2 = Q^2$ by differentiating Eq. (1) with respect to Q^2 by means of the evolution equations for $a_s = \alpha_s/(4\pi)$ and q_{ns},

$$\frac{d\,a_s}{d\ln Q^2} \;=\; \beta(a_s) \;=\; -\beta_0\,a_s^2 - \beta_1\,a_s^3 - \ldots\,, \qquad \beta_0 \;=\; \frac{11}{3}C_A - \frac{2}{3}n_f\,, \tag{4}$$

$$\frac{d\,q_{ns}}{d\ln Q^2} \;=\; P_{ns}\otimes q_{ns} \;=\; \sum_{n=1} a_s^n\,A_n[1-x]_+^{-1}\otimes q_{ns} + \ldots \;\stackrel{\rm M-trf}{=}\; -\sum_{n=1} a_s^n\,A_n \ln N + \ldots\,. \tag{5}$$

The 'cusp anomalous dimension' $A(a_s) = A_1\,a_s + A_2\,a_s^2 + \ldots$ with $A_1 = 4C_F$ has been calculated to order α_s^3 [1]. Finally using the inverse of Eq. (1) to eliminate q_{ns} leads to the evolution equations

$$\frac{d}{d\ln Q^2}\,\mathcal{F}_a \;=\; \left\{P_{ns}(a_s) + \beta(a_s)\,\frac{d}{da_s}\ln C_a(a_s)\right\}\otimes\,\mathcal{F}_a \;=\; K_a\otimes\,\mathcal{F}_a \;\equiv\; \sum_{n=1} a_s^n\,K_a^{(n)}\otimes\,\mathcal{F}_a\,. \tag{6}$$

Inserting the coefficients known from Refs. [3,4], the same leading-logarithmic behaviour for both F_2 and F_L, viz

$$\begin{aligned} K_a^{(n)}(x) \;&=\; A_1(-\beta_0)^{n-1}\left[\frac{\ln^{n-1}(1-x)}{1-x}\right]_+ + O\left(\left[\frac{\ln^{n-2}(1-x)}{1-x}\right]_+\right) \\ &\stackrel{\rm M-trf}{=}\; -\frac{A_1\beta_0^{n-1}}{n}\,\ln^n N \,+\, O\left(\ln^{n-1}N\right)\,, \end{aligned} \tag{7}$$

is established to $n = 4$ for F_2 and $n = 3$ for F_L. For F_2 the soft-gluon resummation [17–19],

$$C_{2,ns}(N,a_s) \;=\; g_2^{(0)}(a_s)\,\exp\left[L g_2^{(1)}(a_s L) + g_2^{(2)}(a_s L) + \ldots\right]\,, \quad g_2^{(i)}(\lambda) \;=\; \sum_j g_{2j}^{(i)}\,\lambda^j\,, \tag{8}$$

($L \equiv \ln N$) guarantees Eq. (7) to all orders in α_s [22]. It is crucial that the physical kernel, unlike the coefficient functions, receives only this single-logarithmic higher-order enhancement for $x \to 1$.

We are now, finally, in a position to state the conjecture announced in the abstract. It is **(a)** that this single-logarithmic enhancement remains true for F_L beyond order α_s^3 and **(b)** that Eq. (7) holds to (at least) $n = 4$ also for F_L. **(a)** implies that that there is an exponentiation as Eq. (8) (but, of course, with an overall prefactor N^{-1}) also for F_L with some functions $g_L^{(i)}$. **(b)** additionally requires that the leading logarithmic functions $g_a^{(1)}$ are actually the same for $a = 2$ and $a = L$ to (at least) order α_s^3. We consider the results of Refs. [13–15] as sufficient evidence for these natural assumptions generalizing our fixed-order results. In particular, it may be expected that the new approach of Ref. [15] will facilitate a full proof in the future.

Inserting Eqs. (2), (4) and (5) into Eq. (6) and imposing the vanishing of the resulting $\alpha_s^n \ln^{2n-2}$ and $\alpha_s^n \ln^{2n-3}$ contributions to $K_L^{(n)}$ at $n \geq 4$ fixes the coefficients of the two highest logarithms in Eq. (2) to all orders n in α_s (with $\theta_{nj} = 1$ for $n \geq j$ and $\theta_{nj} = 0$ else):

$$c_{L,\,2n-2}^{(n)} \;=\; 2(2C_F)^n\,\frac{1}{(n-1)!}\,, \tag{9}$$

$$c_{L,\,2n-3}^{(n)} \;=\; c_{L,\,1}^{(2)}\,(2C_F)^{n-2}\,\frac{\theta_{n2}}{(n-2)!} + \frac{2\beta_0}{3}\,(2C_F)^{n-1}\,\frac{\theta_{n3}}{(n-3)!}\,. \tag{10}$$

We have conjectured Eq. (9) before [3] on the basis of the explicit calculations for $n \leq 3$ and the results of Refs. [13, 14]. To the best of our knowledge, Eq. (10) has not been written down before. Furthermore the vanishing of the $\alpha_s^n \ln^{2n-4}$ contributions to $K_L^{(n)}$ at $n \geq 5$ yields

$$\begin{aligned} c_{L,2n-4}^{(n)} &= c_{L,2}^{(3)} (2C_F)^{n-3} \frac{\theta_{n3}}{(n-3)!} + \frac{\beta_0}{3} c_{L,1}^{(2)} (2C_F)^{n-3} \frac{\theta_{n4}}{(n-4)!} - c_{L,0}^{(2)} (2C_F)^{n-2} \frac{(n-3)\theta_{n4}}{(n-2)!} \\ &\quad + \frac{\beta_0^2}{9} (2C_F)^{n-2} \frac{\theta_{n5}}{(n-5)!} - \frac{2}{3\beta_0} K_L^{(4)}\Big|_{\ln^4 N} (2C_F)^{n-3} \frac{\theta_{n4}}{(n-4)!} \,. \end{aligned} \tag{11}$$

The last line includes the leading term of the physical kernel at order α_s^4, i.e., we have not included conjecture **(b)** in the derivation of Eq. (11). After inserting Eq. (7) for $a = L$ and $n = 4$, i.e., applying also **(b)**, we arrive at a definite prediction also for the third tower (11) of logarithms, thus reaching the predictive power of a next-to-leading logarithmic exponentiation, cf. Ref. [19]. The other coefficients in Eqs. (9) – (11) can be extracted from the loop calculations in Refs. [3, 4, 9–11],

$$c_{L,0}^{(1)} = 4C_F \tag{12}$$

$$c_{L,1}^{(2)} = C_F C_A \left[\frac{92}{3} - 16\zeta_2\right] - C_F^2 [36 - 32\zeta_2 - 16\gamma_e] - \frac{8}{3} C_F n_f \tag{13}$$

$$\begin{aligned} c_{L,0}^{(2)} &= -C_F^2 \left[34 + 40\zeta_2 - 48\zeta_3 + 36\gamma_e - 32\gamma_e\zeta_2 - 8\gamma_e^2\right] \\ &\quad + C_F C_A \left[\frac{430}{9} + 16\zeta_2 - 24\zeta_3 + \frac{92}{3}\gamma_e - 16\gamma_e\zeta_2\right] - C_F n_f \left[\frac{76}{9} + \frac{8}{3}\gamma_e\right] \end{aligned} \tag{14}$$

$$\begin{aligned} c_{L,2}^{(3)} &= -C_F^3 \left[34 - 16\zeta_2 + 32\zeta_3 + 216\gamma_e - 192\gamma_e\zeta_2 - 48\gamma_e^2\right] + \frac{16}{9} C_F n_f^2 \\ &\quad - C_F^2 C_A \left[\frac{530}{9} - 80\zeta_2 - 80\zeta_3 - \frac{640}{3}\gamma_e + 96\gamma_e\zeta_2\right] - C_F C_A n_f \left[\frac{320}{9} - 16\zeta_2\right] \\ &\quad + C_F C_A^2 \left[\frac{1276}{9} - 56\zeta_2 - 32\zeta_3\right] + C_F^2 n_f \left[\frac{92}{9} - 32\zeta_2 - \frac{64}{3}\gamma_e\right] \,. \end{aligned} \tag{15}$$

Inserting Eqs. (12) – (15) into Eqs. (9) – (11) and transforming back to x-space, one arrives at the four-loop prediction (using $L_x \equiv \ln(1-x)$ for brevity)

$$\begin{aligned} c_{L,q}^{(4)}(x) &= \frac{16}{3} C_F^4 L_x^6 + \left\{[72 - 64\zeta_2] C_F^4 - \left[\frac{728}{9} - 32\zeta_2\right] C_F^3 C_A + \frac{80}{9} C_F^3 n_f\right\} L_x^5 \\ &\quad + \left\{[32\zeta_2 - 160\zeta_3] C_F^4 - \left[\frac{904}{3} - \frac{1856}{9}\zeta_2 - 208\zeta_3\right] C_F^3 C_A + \left[\frac{160}{3} - \frac{704}{9}\zeta_2\right] C_F^3 n_f \right. \\ &\quad \left. + \left[\frac{3388}{9} - \frac{1360}{9}\zeta_2 - 64\zeta_3\right] C_F^2 C_A^2 - \left[\frac{880}{9} - \frac{352}{9}\zeta_2\right] C_F^2 C_A n_f + \frac{16}{3} C_F^2 n_f^2\right\} L_x^4 \\ &\quad + O(L_x^3) \,. \end{aligned} \tag{16}$$

This result will become useful also outside the large-x region in combination with a future generalization of Ref. [23] to low fixed-N moments at order α_s^4, since fewer moments will be needed for a useful x-space approximation analogous to Ref. [22].

For future applications and possible extensions to next-to-next-to-leading logarithmic accuracy, it is useful to reformulate Eqs. (12) – (15) in terms of the exponentiation coefficients $g_{ij} \equiv g_{Lj}^{(i)}$. For this purpose we adapt Eq. (14) of Ref. [24] to the present case with $g_L^{(0)} = N^{-1}[a_s c_{L,0}^{(1)} + O(a_s^2)]$ instead of $g_2^{(0)} = 1 + O(a_s)$, yielding

$$c_{L,2n-2}^{(n)}/(4C_F) = \frac{g_{11}^{n-1}}{(n-1)!} , \tag{17}$$

$$c_{L,2n-3}^{(n)}/(4C_F) = \frac{\theta_{n2}\, g_{11}^{n-2}}{(n-2)!}\, g_{21} + \frac{\theta_{n3}\, g_{11}^{n-3}}{(n-3)!}\, g_{12} , \tag{18}$$

$$\begin{aligned} c_{L,2n-4}^{(n)}/(4C_F) &= \frac{\theta_{n2}\, g_{11}^{n-2}}{(n-2)!}\, g_{01} + \frac{\theta_{n3}\, g_{11}^{n-3}}{(n-3)!}\left(g_{22} + \frac{1}{2} g_{21}^2\right) + \frac{\theta_{n4}\, g_{11}^{n-4}}{(n-4)!}\left(g_{13} + g_{12} g_{21}\right) \\ &\quad + \frac{\theta_{n5}\, g_{11}^{n-5}}{2(n-5)!}\, g_{12}^2 . \end{aligned} \tag{19}$$

Eqs. (9) and (10) are obviously compatible with Eqs. (17) and (18). Also Eq. (11) can be recast in the form (19) by suitably combining the first and the last term in the first line. The comparison of the two sets of expressions then leads to

$$g_{11} = 2C_F , \quad g_{12} = \frac{2}{3}\beta_0 C_F , \quad g_{13} = \frac{1}{3}\beta_0^2 C_F , \tag{20}$$

$$g_{21} = \beta_0 + 4\gamma_e C_F - C_F + (4 - 4\zeta_2)(C_A - 2C_F) \tag{21}$$

and

$$\begin{aligned} g_{22} &= -32 C_F^2 \left[1 - 3\zeta_2 + \zeta_3 + \zeta_2^2\right] + C_F C_A \left[\frac{547}{18} - \frac{256}{3}\zeta_2 + 32\zeta_3 + 32\zeta_2^2 + \frac{22}{3}\gamma_e\right] + \frac{2}{9} n_f^2 \\ &\quad + C_A^2 \left[\frac{109}{18} + \frac{50}{3}\zeta_2 - 8\zeta_3 - 8\zeta_2^2\right] + C_F n_f \left[\frac{7}{9} - \frac{8}{3}\zeta_2 - \frac{4}{3}\gamma_e\right] - C_A n_f \left[\frac{34}{9} - \frac{4}{3}\zeta_2\right] \\ &= \frac{1}{2}(\beta_0 g_{21} + A_2) - 8(C_A - 2C_F)^2 \left(1 - 3\zeta_2 + \zeta_3 + \zeta_2^2\right) \end{aligned} \tag{22}$$

with the two-loop cusp anomalous dimension $A_2 = 8 C_F K,\ K = (67/18 - \zeta_2) C_A - 5 n_f/9$ [25].

Some comments are in order here: As expected from the discussion below Eq. (8) the relations (20), with the value of g_{13} due to conjecture **(b)**, are identical to Eq. (9) in Ref. [24]. The third and first term of Eq. (21) are identical, up to a trivial normalization factor, to $\gamma_{J'}$ in Eq. (16) of Ref. [13] — see also Eq. (48) of Ref. [14] and note that the presence of γ_e in Eq. (21) results from our use of $L \equiv \ln N$ instead of $\ln\widetilde{N}$ in Eq. (8) (keeping γ_e facilitates some easy checks).

As shown by the last line of Eq. (22), the coefficient $g_{22} \equiv g_{L2}^{(2)}$ in the expansion of $g_L^{(2)}(a_s L)$ does not follow the pattern of the resummation for F_2 which would demand $g_{22} = 1/2\,(\beta_0 g_{21} + A_2)$ (cf., e.g., Eq. (10) of Ref. [24]), i.e., the absence of the 'non-planar' $(C_A - 2C_F)^2$ part in Eq. (22). Hence also $g_{L3}^{(2)}$ cannot be predicted at this point (if at all – consider the ζ_3 contributions to Eqs. (14), (15) and (22)) from lower-order information. Consequently $c_{L,1}^{(3)}$, known from Ref. [4], can be used to derive $g_{L1}^{(3)}$, but not (yet) the fourth tower $c_{L,2n-4}^{(n)}$ of logarithms at orders $n \geq 4$.

Let us briefly turn to the gluon coefficient function $C_{L,g}$ for the structure function F_L which is suppressed by an(other) order in $(1-x)$. From the third-order results in Refs. [3,4] we extract that

$$C_{L,g}(\alpha_s,x) \;=\; \sum_{n=1} a_s^n \biggl\{ 8 n_f \,\frac{(2C_A)^{n-1}}{(n-1)!}\,\frac{1}{N^2}\,\ln^{2n-2} N + O\Bigl(\frac{1}{N^2}\,\ln^{2n-3} N\Bigr) \biggr\} \tag{23}$$

holds for the first three terms of the expansion in powers of α_s (for $n=1$ one obviously has $O(N^{-3})$ instead of the last term in Eq.(23)). The generalization to all n can be obtained via the physical kernel for the 'non-singlet' (no gluons emitted from quarks, i.e., only the $C_A^k n_f^{n-k}$ terms are kept) gluon contribution to F_L (cf. also Ref. [8]). In fact, in this unphysical limit our whole previous treatment of the (non-singlet – the pure-singlet part does not contribute at the present accuracy) quark coefficient function can be carried over to the gluon case in an obvious manner. Since they might be of theoretical interest at some point, we present here the relations corresponding to Eqs. (20) – (22):

$$g_{11,g} \;=\; 2C_A\;, \qquad g_{12,g} \;=\; \frac{2}{3}\beta_0 C_A\;, \qquad g_{13,g} \;=\; \frac{1}{3}\beta_0^2 C_A\;, \tag{24}$$

$$g_{21,g} \;=\; (8+4\gamma_e)C_A\;, \quad g_{22,g} \;=\; \frac{1}{2}\,[\beta_0\, g_{21,g} + A_{2,g}] + C_A^2\,(18-8\zeta_2) \tag{25}$$

with $A_{2,g} = C_A/C_F\,A_2$. The situation for $g_{22,g}$ is analogous, if simpler in terms of the ζ-function, to that in Eq. (22) discussed above.

Returning to the quark case, we note that also the subleading $\ln^k(1-x)$ or $N^{-1}\ln^k N$ contributions to the physical kernel (6) show only a single logarithmic higher-order enhancement, again in contrast to the corresponding $(1-x)\ln^k(1-x)$ or $N^{-2}\ln^k N$ terms in the coefficient function. I.e.,

$$K_a^{(n)}(x)\Big|_{\ln^k(1-x)} \;\stackrel{\text{M}-\text{trf}}{=}\; K_a^{(n)}(N)\Big|_{N^{-1}\ln^k N} \;=\; 0 \quad \text{for} \quad k \,\geq\, n \tag{26}$$

where, as before, n stands for the order in α_s. Also Eq. (26) is the result of the fixed-order calculations [3–5] at $n \leq 4$ for $a = 2,\,3$ – the missing four-loop splitting function does not contribute at this logarithmic level [6] – and at $n \leq 3$ for $a = L$. It appears almost obvious that also this result holds to all orders. Hence we can predict, completely analogous to Eqs. (9) – (11), the three sub-leading coefficients $d_{L,2n-1-k}^{(n)}$, $k = 1,\,2,\,3$ in Eq. (2) at all higher orders, and 'postdict' $d_{L,2}^{(2)}$ (and $d_{L,4}^{(3)}$ and $d_{L,3}^{(3)}$) from first- (and second-)order coefficients. Due to $d_{L,0}^{(1)} = -c_{L,0}^{(1)}$ the overall signs are opposite to those in Eqs. (9) – (11), and the most compact representation of the results is obtained via the sum of the corresponding $\ln^k N$ and $N^{-1}\ln^k N$ coefficients. It reads

$$d_{L,2n-2}^{(n)} \;=\; -\,c_{L,2n-2}^{(n)}\;, \tag{27}$$

$$d_{L,2n-3}^{(n)} \;=\; -\,c_{L,2n-3}^{(n)} + \Bigl\{d_{L,1}^{(2)} + c_{L,1}^{(2)}\Bigr\}\,(2C_F)^{n-2}\,\frac{\theta_{n2}}{(n-2)!}\;, \tag{28}$$

$$\begin{aligned} d_{L,2n-4}^{(n)} \;=\; & -\,c_{L,2n-4}^{(n)} + \Bigl\{d_{L,2}^{(3)} + c_{L,2}^{(3)}\Bigr\}\,(2C_F)^{n-3}\,\frac{\theta_{n3}}{(n-3)!} \\ & + \Bigl\{d_{L,1}^{(2)} + c_{L,1}^{(2)}\Bigr\}\,\frac{\beta_0}{3}\,(2C_F)^{n-3}\,\frac{\theta_{n4}}{(n-4)!} \;-\; \Bigl\{d_{L,0}^{(2)} - c_{L,0}^{(2)}\Bigr\}\,(2C_F)^{n-2}\,\frac{(n-3)\theta_{n4}}{(n-2)!}\;. \end{aligned} \tag{29}$$

The lower-order coefficients entering these relations read

$$d_{L,1}^{(2)} = -c_{L,1}^{(2)} - 8C_F^2 \,, \tag{30}$$

$$d_{L,0}^{(2)} = -c_{L,0}^{(2)} + C_F^2\,[\,14+16\zeta_2-8\gamma_{\rm e}\,] - C_F C_A\,[\,14+8\zeta_2\,] + 4C_F n_f \,, \tag{31}$$

$$d_{L,2}^{(3)} = -c_{L,2}^{(3)} + C_F^3\,[\,148-32\zeta_2-48\gamma_{\rm e}\,] - C_F^2 C_A\,[\,104-16\zeta_2] + 16C_F^2 n_f \,. \tag{32}$$

Obviously it is possible to recast also Eqs. (27) – (29) into an exponential form analogous to Eqs. (17) – (22). The corresponding leading-logarithmic function $\tilde{g}_1$ is the same as in Eqs. (20), while $\tilde{g}_{21}$ and $\tilde{g}_{22}$ differ from their counterparts in Eqs. (21) and (22) by $2C_F$ and $C_F\beta_0 - 14C_F^2$, respectively. The ζ-function contributions, in particular, are the same. For comparison and use with future four-loop computations, we also carry out the inverse Mellin transform of these results for $n = 4$. This leads to

$$\begin{aligned} c_{L,q}^{(4)}(x) \;=\; & \text{Eqn. (16)} \\ & - (1-x)\Bigg(\frac{16}{3} C_F^4 L_x^6 + \left\{ [\,8-64\zeta_2]\, C_F^4 - \left[\frac{728}{9} - 32\zeta_2 \right] C_F^3 C_A + \frac{80}{9} C_F^3 n_f \right\} L_x^5 \\ & - \Bigg\{ [\,568-608\zeta_2+160\zeta_3]\, C_F^4 + \left[\frac{4544}{9} - \frac{736}{9}\zeta_2 + 208\zeta_3 \right] C_F^3 C_A \\ & + \left[\frac{3388}{9} - \frac{1360}{9}\zeta_2 - 64\zeta_3 \right] C_F^2 C_A^2 - \left[\frac{368}{9} + \frac{704}{9}\zeta_2 \right] C_F^3 n_f \\ & - \left[\frac{880}{9} - \frac{352}{9}\zeta_2 \right] C_F^2 C_A n_f + \frac{16}{3} C_F^2 n_f^2 \Bigg\} L_x^4 \;+\; O(L_x^3) \Bigg) \;+\; O\left((1-x)^2\right) \;. \end{aligned} \tag{33}$$

Finally we briefly illustrate the approximations of the N-space coefficient functions $c_{L,\rm ns}^{(n)}(N)$ in terms of the leading $N^{-1}\ln^k \widetilde{N}$ contributions (obtained from Eqs. (9) – (11) by nullifying the Euler-Mascheroni constant $\gamma_{\rm e}$ in Eqs. (13) – (15), recall $\ln \widetilde{N} = \ln N + \gamma_{\rm e}$). In the left part of Fig. 1 we compare the successive approximations obtained by including one (only the $\ln^4 \widetilde{N}$ term, the curve labeled '1' in the figure), two (that and the $\ln^3 \widetilde{N}$ term, curve 2) etc large-N logarithms to the complete result of Refs. [3, 4]. We see that including all four logarithms leads to a good approximation down to surprisingly low values of N, and that the highest three logarithms provide a reasonable first estimate at large N.

Our new predictions (17) – (19) for the three highest logarithms at order $\alpha_{\rm s}^4$ are shown in the same manner in the right part of Fig. 1. Comparing the shape and relative size of these terms with those of the three-loop contributions, one has to conclude that three leading logarithms alone are insufficient for a quantitative prediction of the unknown coefficient function for F_L. One may expect that the complete coefficient function exceeds the three-logarithm result in Fig. 1 by a factor of about 1.5 to 3 at $N \simeq 15 \ldots 30$. This is consistent with the fourth-order Padé predictions, e.g.,

$$C_{L,\rm ns}(N=20) \;=\; 0.0202\,\alpha_{\rm s} + 0.108\,\alpha_{\rm s}^2 + 0.465\,\alpha_{\rm s}^3 + 2.0_{\,[1/1]\,\text{Padé}}\,\alpha_{\rm s}^4 + \ldots \,. \tag{34}$$

Hence the present results are compatible with (but of course not conclusive of) a fourth-order continuation of the very slow large-N convergence of F_L already discussed at order $\alpha_{\rm s}^3$ in Ref. [4].

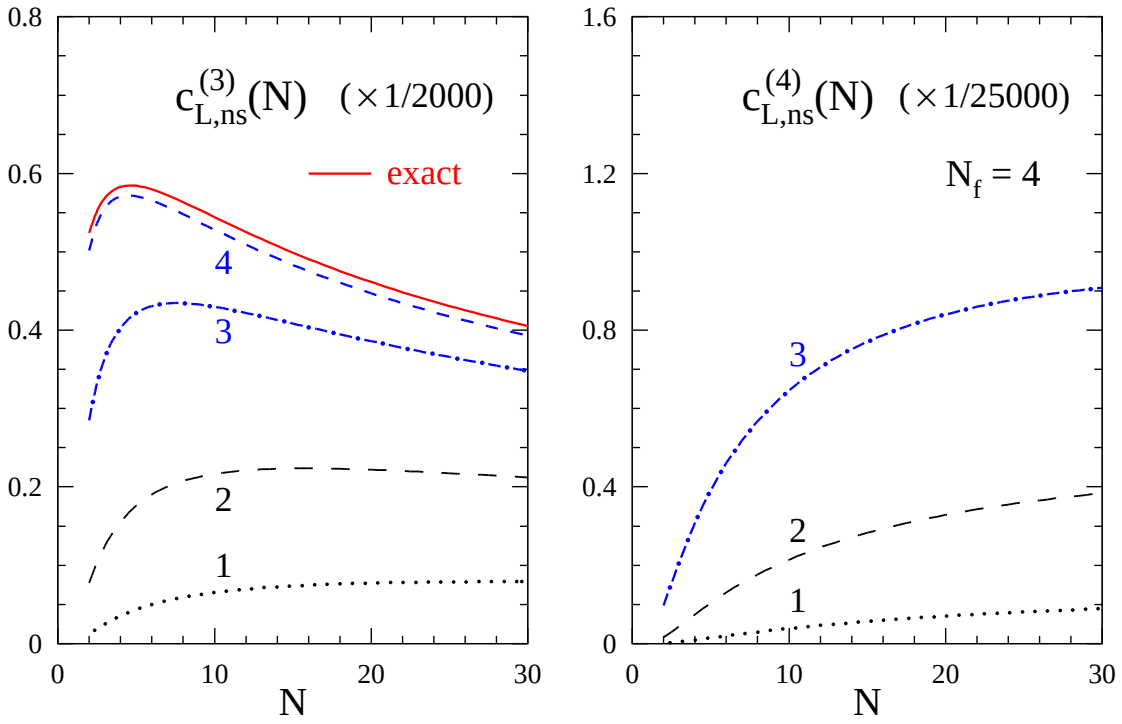

Figure 1: Successive large-N approximations by the leading 1, 2, 3 and (left) 4 large-N logarithms $\ln^k \widetilde{N} \equiv (\ln N + \gamma_e)^k$ for the third- and fourth-order quark coefficient function of F_L for four flavours. Also shown is the complete third-order all-N result computed in Refs [3,4]. The curves have been scaled to correspond to the expansion parameter α_s instead of $a_s = \alpha_s/(4\pi)$ used in our formulae.

To summarize, we have derived an explicit all-order resummation of the leading and subleading large Mellin-N contributions, $\alpha_s^n N^{-l} \ln^k N$ for $l = 1$ and $l = 2$, to the quark coefficient function for the longitudinal structure function F_L in deep-inelastic scattering. The resummation is performed 'bottom-up' by exploiting the absence (established to $n = 3$ by the complete results of Refs. [3, 4], and conjectured for all higher orders) of double-logarithmic contributions to the physical evolution kernel for the flavour non-singlet part of F_L. Specifically we obtain the three highest logarithms at each order $n \geq 4$, i.e., the terms $\alpha_s^n (1-x)^{l-1} \ln^{2n-1-k}(1-x)$ for $l = 1, 2$ and $k = 1, 2, 3$ after transformation to Bjorken-x space. These contributions alone are not relevant for phenomenology, but will become useful in conjunction with future higher-order calculations of, e.g., some integer-N moments of this coefficient function.

With three terms per order, our present resummation has the predictive power of a next-to-leading logarithmic exponentiation, cf. Ref. [19]. However, writing the results in a manner analogous to the well-known exponentiation of the $\alpha_s^n \ln^k N$ contributions to, e.g., the structure function F_2, we notice a peculiar behaviour of the next-to-leading function $g_2(\alpha_s \ln N)$ in the exponent: the second Taylor-coefficient is not, as for F_2, a simple function of the first and the α_s^2 cusp anomalous dimension, and hence the third coefficient cannot be predicted at this point (if at all). If that coefficient could be derived in a 'top-down' approach complementary to that of this letter, then a forth tower of logarithms would be calculable via matching to the known (but presently unused) $\alpha_s^3 \ln N$ coefficient. It might even become possible to achieve a full next-to-next-to-leading logarithmic accuracy which would provide a realistic estimate of the fourth-order large-x coefficient function.

Acknowledgments: Some of our symbolic manipulations have been carried out in FORM [26]. S.M. acknowledges support by the Helmholtz Gemeinschaft under contract VH-NG-105 and in part by the Deutsche Forschungsgemeinschaft (DFG) in Sonderforschungsbereich/Transregio 9. The research of A.V. has been supported by the UK Science & Technology Facilities Council (STFC) under grant numbers PP/E007414/1 and ST/G00062X/1.

References

[1] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B688 (2004) 101, hep-ph/0403192

[2] A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B691 (2004) 129, hep-ph/0404111

[3] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B606 (2005) 123, hep-ph/0411112

[4] J.A.M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B724 (2005) 3, hep-ph/0504242

[5] S. Moch, J.A.M. Vermaseren and A. Vogt, arXiv:0812.451768 [hep-ph] (Nucl. Phys. B, in press)

[6] Y.L. Dokshitzer, G. Marchesini and G.P. Salam, Phys. Lett. B634 (2006) 504, hep-ph/0511302

[7] A. Mitov, S. Moch and A. Vogt, Phys. Lett. B638 (2006) 61, hep-ph/0604053

[8] S. Moch and A. Vogt, Phys. Lett. B659 (2008) 290, arXiv:0709.3899 [hep-ph]

[9] J.S. Guillen et al., Nucl. Phys. B353 (1991) 337

[10] W.L. van Neerven and E.B. Zijlstra, Phys. Lett. B272 (1991) 127

[11] S. Moch and J.A.M. Vermaseren, Nucl. Phys. B573 (2000) 853, hep-ph/9912355

[12] S. Moch and A. Vogt, to appear

[13] R. Akhoury, M.G. Sotiropoulos and G. Sterman, Phys. Rev. Lett. 81 (1998) 3819, hep-ph/9807330

[14] R. Akhoury and M.G. Sotiropoulos, hep-ph/0304131

[15] E. Laenen, G. Stavenga and C.D. White, arXiv:0811.2067 [hep-ph]

[16] G. Grunberg, arXiv:0710.5693 [hep-ph]

[17] G. Sterman, Nucl. Phys. B281 (1987) 310

[18] S. Catani and L. Trentadue, Nucl. Phys. B327 (1989) 323; ibid. B353 (1991) 183

[19] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B726 (2005) 317, hep-ph/0506288

[20] J.A.M. Vermaseren, Int. J. Mod. Phys. A14 (1999) 2037, hep-ph/9806280

[21] J. Blümlein and S. Kurth, Phys. Rev. D60 (1999) 014018, hep-ph/9810241

[22] W.L. van Neerven and A. Vogt, Nucl. Phys. B603 (2001) 42, hep-ph/0103123

[23] S. Larin, P. Nogueira, T. van Ritbergen, J. Vermaseren, Nucl. Phys. B492 (1997) 338, hep-ph/9605317

[24] A. Vogt, Phys. Lett. B471 (1999) 97, hep-ph/9910545

[25] J. Kodaira and L. Trentadue, Phys. Lett. B112 (1982) 66

[26] J.A.M. Vermaseren, math-ph/0010025