arXiv:1001.3848v2 [hep-th] 8 Mar 2010

March 9, 2010
DESY 09–218

Analytic result for the one-loop scalar pentagon integral with massless propagators

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Abstract

The method of dimensional recurrences proposed by one of the authors [1, 2] is applied to the evaluation of the pentagon-type scalar integral with on-shell external legs and massless internal lines. For the first time, an analytic result valid for arbitrary space-time dimension d and five arbitrary kinematic variables is presented. An explicit expression in terms of the Appell hypergeometric function F_3 and the Gauss hypergeometric function ${}_2F_1$, both admitting one-fold integral representations, is given. In the case when one kinematic variable vanishes, the integral reduces to a combination of Gauss hypergeometric functions ${}_2F_1$. For the case when one scalar invariant is large compared to the others, the asymptotic values of the integral in terms of Gauss hypergeometric functions ${}_2F_1$ are presented in $d = 2 - 2\varepsilon$, $4 - 2\varepsilon$, and $6 - 2\varepsilon$ dimensions. For multi-Regge kinematics, the asymptotic value of the integral in $d = 4 - 2\varepsilon$ dimensions is given in terms of the Appell function F_3 and the Gauss hypergeometric function ${}_2F_1$.

PACS numbers: 02.30.Gp, 02.30.Ks, 12.38.Bx, 12.40.Nn
Keywords: Feynman integrals, Appell hypergeometric functions, multi-Regge kinematics

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1 Introduction

Theoretical predictions for ongoing and future experiments at the CERN Large Hadron Collider (LHC) and an International e^+e^- Linear Collider (ILC) must include high-precision radiative corrections. The complexity of the evaluation of such radiative corrections is related, in particular, to the difficulties in calculating integrals corresponding to Feynman diagrams with many external legs depending on many kinematic variables. Purely numerical evaluation of such integrals cannot provide sufficiently high precision within reasonable computer time. The evaluation of one-loop integrals corresponding to diagrams with two, three, and four external legs was studied in numerous publications.

As for integrals associated with diagrams with five and more external legs, the situation is quite different. Not so many results for such integrals are available in the literature. Various authors [3] discussed the reduction of pentagon integrals to box integrals in space-time dimension $d = 4$. They were able to express these integrals as linear combinations of five different loop integrals with four external legs. Infrared divergences if any were supposed to be regulated by introducing a small fictitious mass. The representation of the dimensionally regularized pentagon integral in terms of box-type integrals was considered in Refs. [4, 5]. However, in the calculation of multi-loop radiative corrections, higher orders in $\varepsilon = (4 - d)/2$ are needed, and, therefore, one should extend such an expansion beyond the "box approximation." The first step in this direction was recently taken in Ref. [6], where an analytic result for the one-loop massless pentagon integral with on-shell external legs as well as several terms of its ε expansion were presented in the limit of multi-Regge kinematics. However, no analytic results for arbitrary kinematics are available until now. Practically nothing is known about the analytic structure of on-shell pentagon integrals with unconstrained kinematics in arbitrary space-time dimension d. For very simplified kinematics, a pentagon-type integral for arbitrary value of d was given in Ref. [7] in terms of Euler gamma functions. A representation of the pentagon integral in terms of a four-fold Mellin-Barnes integral may be found in Ref. [8].

Important applications that require the evaluation of Feynman integrals with massless propagators include the study of jet production in QCD [9], which allows for a high-precision extraction of the strong-coupling constant α_s, the investigation of the iterative structure of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) amplitudes [10], and tests of the scattering-amplitude/Wilson-loop duality [11].

In this paper, we perform a first analytic study of the on-shell pentagon integral with massless internal lines and arbitrary kinematic invariants. We use the method of dimensional recurrences proposed in Refs. [1, 2], which was already applied to the calculation of one- and two-loop integrals in Refs. [2, 12, 13] and, quite recently, also to the calculation of three- and four-loop integrals in Ref. [14].

This paper is organized as follows. In Section 2, we introduce our notations, explain the recurrence relation for the pentagon-type integral with respect to the space-time dimension d, and we present a detailed derivation of its solution in Section 3. In Section 4, we consider a particular case of the on-shell pentagon integral with one vanishing kinematic variable and present an analytic expression in terms of the Gauss hypergeometric function ${}_2F_1$. In Section 5, we specify the asymptotic values of the pentagon integral in $d = 2 - 2\varepsilon$, $d = 4 - 2\varepsilon$, and $d = 6 - 2\varepsilon$ space-time dimensions when one of the scalar invariants is much larger than the others. In Section 6, we present an analytic expression for the pentagon integral in the limit of multi-Regge kinematics in terms of the Appell function F_3 and the Gauss hypergeomteric function ${}_2F_1$. In the Conclusions, we summarize the accomplishments of the present paper and offer some perspectives for the application of the method of dimensional recurrences to six-point integrals. In Appendix A, we collect useful formulae for hypergeometric functions used in this paper. Appendix B contains intermediate results from Section 3.

2 Definitions and dimensional recurrences

We consider the following integral with five massless propagators:

$$I_5^{(d)}(s_{12}, s_{23}, s_{34}, s_{45}, s_{15};\ s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) = \int \frac{d^d q}{i\pi^{d/2}} \prod_{j=1}^{5} \frac{1}{D_j}, \tag{2.1}$$

where

$$D_j = (q - p_j)^2 + i\epsilon. \tag{2.2}$$

The pentagon diagram associated with the integral $I_5^{(d)}(\{s_{kr}\})$ is depicted in Fig. 1. The labeling of the momenta in Fig. 1. corresponds to Eq. (2.1). The Lorentz invariants are defined as:

$$s_{ij} = p_{ij}^2, \qquad p_{ij} = p_i - p_j. \tag{2.3}$$

In the present paper, we take the squares of the external momenta to be vanishing,

$$s_{12} = s_{23} = s_{34} = s_{45} = s_{15} = 0, \tag{2.4}$$

and work in the Euclidean region, $s_{ij} < 0$. In what follows, we only keep non-vanishing variables as arguments of $I_5^{(d)}$ and use the notation

$$I_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) \equiv I_5^{(d)}(0, 0, 0, 0, 0;\ s_{13}, s_{14}, s_{24}, s_{25}, s_{35}), \tag{2.5}$$

for the on-shell case of Eq. (2.4). Due to the symmetry with respect to permutations of all propagators,

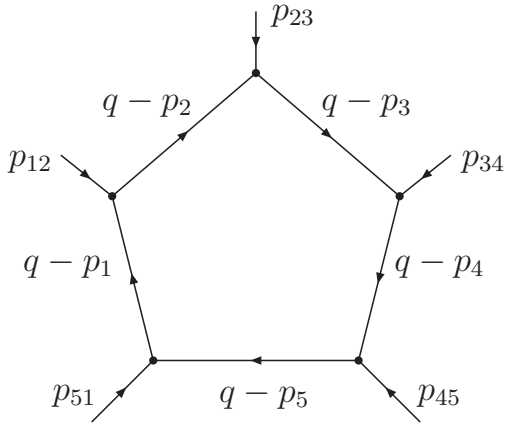

Figure 1: Feynman diagram corresponding the integral $I_5^{(d)}$.

the integral considered as a function of the kinematic variables s_{ij} must fulfill the following relations:

$$\begin{aligned}
&I_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) = I_5^{(d)}(s_{13}, s_{35}, s_{25}, s_{24}, s_{14}) \\
&= I_5^{(d)}(s_{14}, s_{13}, s_{35}, s_{25}, s_{24}) = I_5^{(d)}(s_{14}, s_{24}, s_{25}, s_{35}, s_{13}) \\
&= I_5^{(d)}(s_{24}, s_{25}, s_{35}, s_{13}, s_{14}) = I_5^{(d)}(s_{24}, s_{14}, s_{13}, s_{35}, s_{25}) \\
&= I_5^{(d)}(s_{25}, s_{24}, s_{14}, s_{13}, s_{35}) = I_5^{(d)}(s_{25}, s_{35}, s_{13}, s_{14}, s_{24}) \\
&= I_5^{(d)}(s_{35}, s_{13}, s_{14}, s_{24}, s_{25}) = I_5^{(d)}(s_{35}, s_{25}, s_{24}, s_{14}, s_{13}).
\end{aligned} \tag{2.6}$$

As was shown in Refs. [1, 5], the integrals $I_5^{(d+2)}$ and $I_5^{(d)}$ fulfill the following relation:

$$(d-4)s_{13}s_{24}s_{35}s_{14}s_{25}h_5 I_5^{(d+2)}(s_{13},s_{14},s_{24},s_{25},s_{35})$$

$$= 2s_{13}s_{24}s_{35}s_{14}s_{25} I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}) + P^{(d)}, \tag{2.7}$$

where

$$P^{(d)} = \sum_{k=1}^{5} \varkappa_k J_k, \tag{2.8}$$

$$\begin{aligned}
\varkappa_1 &= -s_{24}s_{35}(s_{24}s_{35} - s_{35}s_{14} - s_{13}s_{24} + s_{13}s_{25} + s_{14}s_{25}),\\
\varkappa_2 &= -s_{35}s_{14}(s_{35}s_{14} + s_{13}s_{24} - s_{24}s_{35} - s_{14}s_{25} + s_{13}s_{25}),\\
\varkappa_3 &= -s_{14}s_{25}(s_{14}s_{25} - s_{35}s_{14} - s_{13}s_{25} + s_{24}s_{35} + s_{13}s_{24}),\\
\varkappa_4 &= s_{13}s_{25}(s_{13}s_{24} - s_{13}s_{25} - s_{24}s_{35} - s_{35}s_{14} + s_{14}s_{25}),\\
\varkappa_5 &= s_{13}s_{24}(s_{13}s_{25} - s_{13}s_{24} - s_{35}s_{14} + s_{24}s_{35} - s_{14}s_{25}),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
J_1 &= I_4^{(d)}(0,0,0,s_{25};\ s_{24},s_{35}), \qquad J_2 = I_4^{(d)}(0,0,0,s_{13};\ s_{14},s_{35}),\\
J_3 &= I_4^{(d)}(0,0,0,s_{24};\ s_{25},s_{14}), \qquad J_4 = I_4^{(d)}(0,0,0,s_{35};\ s_{13},s_{25}),\\
J_5 &= I_4^{(d)}(0,0,0,s_{14};\ s_{24},s_{13}),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
h_5 \;=\; & \frac{1}{s_{13}s_{24}s_{35}s_{14}s_{25}}\left[s_{24}^2s_{35}^2 - 2s_{24}s_{13}^2s_{25} + s_{13}^2s_{24}^2 + s_{13}^2s_{25}^2 + s_{14}^2s_{35}^2 + s_{14}^2s_{25}^2\right.\\
& +2s_{13}s_{24}s_{35}s_{14} - 2s_{24}s_{14}s_{35}^2 - 2s_{13}s_{35}s_{24}^2 + 2s_{24}s_{13}s_{35}s_{25} + 2s_{24}s_{35}s_{14}s_{25}\\
& \left.-2s_{35}s_{14}^2s_{25} + 2s_{13}s_{35}s_{14}s_{25} - 2s_{13}s_{14}s_{25}^2 + 2s_{24}s_{13}s_{14}s_{25}\right].
\end{aligned} \tag{2.11}$$

Here, $I_4^{(d)}$ are integrals corresponding to Feynman diagrams with four external legs defined as

$$I_4^{(d)}(s_{nj},s_{jk},s_{kl},s_{nl};\ s_{jl},s_{nk}) = \int \frac{d^dq}{i\pi^{d/2}}\,\frac{1}{D_nD_jD_kD_l}, \tag{2.12}$$

where D_j are defined in Eq. (2.2). An analytic expression for this integral with arbitrary kinematics in arbitrary space-time dimension d was recently obtained in Ref. [15].

3 Solution of the dimensional recurrence relation

Redefining the integral $I_5^{(d)}$ as

$$I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}) = \frac{{h_5}^{-\frac{d}{2}}}{\Gamma\left(\frac{d-4}{2}\right)}\overline{I}_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}), \tag{3.13}$$

we obtain the relation

$$\overline{I}_5^{(d+2)}(s_{13},s_{14},s_{24},s_{25},s_{35}) = \overline{I}_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}) + \frac{\Gamma\left(\frac{d-4}{2}\right)}{2s_{13}s_{24}s_{35}s_{14}s_{25}}h_5^{\frac{d}{2}}P^{(d)}, \tag{3.14}$$

which has the following solution:

$$\overline{I}_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}) = \Pi_m(d) + \sum_{r=0}^{\infty}\frac{\Gamma\left(\frac{d-2r-6}{2}\right)}{2s_{13}s_{24}s_{35}s_{14}s_{25}}h_5^{\frac{d}{2}-r-1}P^{(d-2r-2)}, \tag{3.15}$$

where $\Pi_m(d)$ is an arbitrary periodic function depending on the scalar invariants s_{ij} and satisfying the condition

$$\Pi_m(d+2) = \Pi_m(d). \tag{3.16}$$

Another solution of Eq. (3.14) reads:

$$\overline{I}_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) = \Pi_p(d) - \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{d+2r-4}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}+r} P^{(d+2r)}, \tag{3.17}$$

where

$$\Pi_p(d+2) = \Pi_p(d). \tag{3.18}$$

The correctness of both solutions, Eqs. (3.15) and (3.17), may be easily verified by direct substitution into Eq. (3.14). Thus, for example, using Eqs. (3.17) and (3.18), we have

$$\begin{aligned}
\overline{I}_5^{(d+2)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) &= \Pi_p(d+2) - \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{d+2+2r-4}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d+2}{2}+r} P^{(d+2+2r)} \\
&= \Pi_p(d) - \sum_{r=1}^{\infty} \frac{\Gamma\left(\frac{d+2r-4}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}+r} P^{(d+2r)} \\
&= \overline{I}_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) + \frac{\Gamma\left(\frac{d-4}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}} P^{(d)},
\end{aligned} \tag{3.19}$$

in agreement with Eq. (3.14). Furthermore, the solution in the form of Eq. (3.17) may be easily obtained from Eq. (3.15) by adding to and subtracting from the expression on the right-hand side of Eq. (3.15) the sum

$$\sum_{r=-1}^{-\infty} \frac{\Gamma\left(\frac{d-2r-6}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}-r-1} P^{(d-2r-2)}. \tag{3.20}$$

In fact, the combination

$$\begin{aligned}
&\sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{d-2r-6}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}-r-1} P^{(d-2r-2)} + \sum_{r=-1}^{-\infty} \frac{\Gamma\left(\frac{d-2r-6}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}-r-1} P^{(d-2r-2)} \\
&\qquad = \sum_{r=-\infty}^{\infty} \frac{\Gamma\left(\frac{d-2r-6}{2}\right)}{2 s_{13} s_{24} s_{35} s_{14} s_{25}} h_5^{\frac{d}{2}-r-1} P^{(d-2r-2)}
\end{aligned} \tag{3.21}$$

is invariant with respect to the change $d \to d \pm 2l$, where l is integer, so that this sum may be absorbed into the periodic constant that we denoted by $\Pi_p(d)$. Changing the summation index in the remaining sum as $r \to -r-1$, we obtain Eq. (3.17). To obtain the solution of the difference equation (3.14) in terms of convergent series, one may choose either Eq. (3.15) or Eq. (3.17) depending on the kinematics.

The dependence of the arbitrary periodic functions $\Pi_m(d)$ and $\Pi_p(d)$ on the scalar invariants s_{ij} may be constructed from a system of differential equations which follows from the one for the integral $I_5^{(d)}$. For the integral $I_5^{(d)}$, we derive a system consisting of 5 differential equations of the form

$$\begin{aligned}
s_{13} s_{14} s_{24} s_{25} s_{35}\ h_5\ s_{ij} \frac{\partial I_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35})}{\partial s_{ij}} &= (d-5) \sum_{k=1}^{5} J_k R_{ij}^{(k)} \\
&+ \left(\phi_{ij} - \frac{d}{2} s_{13} s_{14} s_{24} s_{25} s_{35} s_{ij} \frac{\partial h_5}{\partial s_{ij}} \right) I_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}),
\end{aligned} \tag{3.22}$$

where $(i,j) = (1,3), (1,4), (2,4), (2,5), (3,5)$ is not summed over, J_k are the box integrals defined in Eq. (2.10), and $R_{ij}^{(k)}$ and ϕ_{ij} are polynomials in s_{kr}. To derive this system of equations, we use

the method proposed in Ref. [16]. Some details of the derivation are presented in Appendix B. The derivation of such equations is done with the help of the computer program package Maple. Explicit expressions for $R_{ij}^{(k)}$ and ϕ_{ij} are given in Eqs. (10.76) and (10.77) in Appendix B, respectively. Using Eq. (3.13), one may obtain from Eq. (3.22) a system of equations for the integral $\overline{I}_5^{(d)}$. Substituting Eq. (3.15) into this system, we obtain the following system of equations for the periodic function $\Pi_m(d)$ after a rather tedious calculation:

$$
\begin{aligned}
s_{13}\frac{\partial \Pi_m(d)}{\partial s_{13}} &= \frac{\phi_{13}}{h_5 s_{13} s_{14} s_{24} s_{25} s_{35}} \Pi_m(d), \qquad & s_{14}\frac{\partial \Pi_m(d)}{\partial s_{14}} &= \frac{\phi_{14}}{h_5 s_{13} s_{14} s_{24} s_{25} s_{35}} \Pi_m(d), \\
s_{24}\frac{\partial \Pi_m(d)}{\partial s_{24}} &= \frac{\phi_{24}}{h_5 s_{13} s_{14} s_{24} s_{25} s_{35}} \Pi_m(d), \qquad & s_{25}\frac{\partial \Pi_m(d)}{\partial s_{25}} &= \frac{\phi_{25}}{h_5 s_{13} s_{14} s_{24} s_{25}^2 s_{35}} \Pi_m(d), \\
s_{35}\frac{\partial \Pi_m(d)}{\partial s_{35}} &= \frac{\phi_{35}}{h_5 s_{13} s_{14} s_{24} s_{25} s_{35}} \Pi_m(d). &&
\end{aligned}
\tag{3.23}
$$

These differential equations do not depend explicitly on d and are, thus, much simpler than those for the integral $I_5^{(d)}$ itself. The solution of this system of differential equation with respect to s_{ij} for $\Pi_m(d)$ obtained with the help of the computer program package Maple reads:

$$
\Pi_m(d) = \frac{\kappa(d)\; h_5^{\frac{5}{2}}}{(s_{13} s_{14} s_{35} s_{24} s_{25})^3}, \tag{3.24}
$$

where $\kappa(d)$ is an arbitrary periodic constant,

$$
\kappa(d+2) = \kappa(d), \tag{3.25}
$$

which is independent of the scalar invariants s_{ij}. The system of differential equations for $\Pi_p(d)$ looks similar to Eq. (3.23). An arbitrary periodic constant $\kappa(d)$ may be determined from $I_5^{(d)}$ calculated for some particular kinematics. Usually, setting some of the scalar invariants s_{ij} to zero greatly simplifies the computation of the integral. But in our case, as may be seen from Eq. (3.24), $\kappa(d)$ cannot be determined from $I_5^{(d)}$ calculated for such kinematics because the term proportional to $\kappa(d)$ drops out. For the same reason, one cannot use $I_5^{(d)}$ calculated for kinematics with $h_5 = 0$. Instead, we determine the periodic function $\Pi_m(d)$ by comparing the limiting value of our analytic result for $|d| \to \infty$ with the analogous value obtained from the integral representation of Eq. (9.63) by exploiting the steepest-descent method described in details in Ref. [12].

Without loss of generality, we henceforth assume the following hierarchy between the scalar invariants:

$$
-s_{13} > -s_{14}, -s_{35} > -s_{25} > -s_{24}. \tag{3.26}
$$

For the case when the scalar invariants s_{ij} satisfy the conditions of Eq. (3.26) and additionally

$$
-s_{14} - s_{35} + s_{13} > 0, \tag{3.27}
$$

the integrals $I_4^{(d)}$ in Eq. (2.10) may be written as

$$
\begin{aligned}
I_4^{(d)}(0,0,0,s_{25};s_{24},s_{35}) &= \chi^{(d)}(s_{25},s_{25},s_{35},s_{24}) - \eta^{(d)}(s_{24},s_{25},s_{35},s_{24}) - \chi^{(d)}(s_{35},s_{25},s_{35},s_{24}), \\
I_4^{(d)}(0,0,0,s_{13};s_{14},s_{35}) &= \chi^{(d)}(s_{13},s_{13},s_{14},s_{35}) - \chi^{(d)}(s_{35},s_{13},s_{14},s_{35}) - \chi^{(d)}(s_{14},s_{13},s_{14},s_{35}) \\
&\quad - \frac{2\pi(d-3)}{s_{35}s_{14}\sin\frac{\pi d}{2}}\; I_2^{(d)}\left(\frac{-s_{35}s_{14}}{s_{35}+s_{14}-s_{13}}\right), \\
I_4^{(d)}(0,0,0,s_{24};s_{25},s_{14}) &= \eta^{(d)}(s_{24},s_{24},s_{25},s_{14}) - \chi^{(d)}(s_{14},s_{24},s_{25},s_{14}) - \chi^{(d)}(s_{25},s_{24},s_{25},s_{14}) \\
&\quad - \frac{4\pi(d-3)}{s_{14}s_{25}\tan\frac{\pi d}{2}}\; I_2^{(d)}\left(\frac{s_{25}s_{14}}{s_{25}+s_{14}-s_{24}}\right),
\end{aligned}
$$

$$I_4^{(d)}(0,0,0,s_{35};s_{13},s_{25}) = \chi^{(d)}(s_{35},s_{35},s_{13},s_{25}) - \eta^{(d)}(s_{25},s_{35},s_{13},s_{25}) - \chi^{(d)}(s_{13},s_{35},s_{13},s_{25}),$$
$$I_4^{(d)}(0,0,0,s_{14};s_{24},s_{13}) = \chi^{(d)}(s_{14},s_{14},s_{24},s_{13}) - \chi^{(d)}(s_{13},s_{14},s_{24},s_{13}) - \eta^{(d)}(s_{24},s_{14},s_{24},s_{13}), \tag{3.28}$$

where

$$\eta^{(d)}(s,s_{14},s_{24},s_{13}) = \frac{4(d-3)I_2^{(d)}(s)}{(d-4)s_{13}s_{24}}\,{}_2F_1\left[\begin{matrix}1,\frac{d}{2}-2\,;\\ \frac{d}{2}-1\,;\end{matrix}\ \frac{s(s_{13}+s_{24}-s_{14})}{s_{13}s_{24}}\right],$$

$$\chi^{(d)}(s,s_{14},s_{24},s_{13}) = \frac{-4(d-3)I_2^{(d)}(s)}{(d-6)s(s_{13}+s_{24}-s_{14})}\,{}_2F_1\left[\begin{matrix}1,3-\frac{d}{2}\,;\\ 4-\frac{d}{2}\,;\end{matrix}\ \frac{s_{13}s_{24}}{s(s_{13}+s_{24}-s_{14})}\right], \tag{3.29}$$

and $I_2^{(d)}(p^2)$ is the one-loop massless propagator-type integral

$$I_2^{(d)}(p^2) = \frac{1}{i\pi^{d/2}}\int\frac{d^dq}{q^2(q-p)^2} = \frac{-\pi^{\frac{3}{2}}\,(-p^2)^{\frac{d}{2}-2}}{2^{d-3}\Gamma\left(\frac{d-1}{2}\right)\sin\frac{\pi d}{2}}. \tag{3.30}$$

An explicit derivation of these results using the method of dimensional recurrences may be found in Ref. [15]. Results for these integrals were also obtained in Ref. [4] using a different method.

To evaluate $I_5^{(d)}$, we use the solution in the form of Eq. (3.15). This solution may be used if

$$\left|\frac{4}{h_5 s_{ij}}\right| \leq 1. \tag{3.31}$$

As we shall see later, the quantities $4/(h_5 s_{ij})$ emerge as expansion parameters in the resulting hypergeometric series. If condition (3.31) is fulfilled, then we may obtain a convergent series using Eq. (3.15). If this condition is not fulfilled, then we may use Eq. (3.17). In this case, the inverse quantities, $h_5 s_{ij}/4$, will be the expansion parameters in the resulting hypergeometric series.

In the following, we obtain an analytic result assuming that Eq. (3.31) is fulfilled for all scalar invariants s_{ij}. If this condition is not fulfilled for a particular term in $P^{(d)}$, then an analytic result may be obtained by performing analytic continuations of the hypergeometric functions in the final result. Another possibility to obtain the analytic result in this case is to repeat the calculation using the solution of the form of Eq. (3.17). It should be noted that, if Eq. (3.27) is not satisfied, then the arguments of all hypergeometric functions generated by the integral $I_4^{(d)}(0,0,0,s_{13};s_{14},s_{35})$ exceed unity. Analytic continuation of the result for this integral given in Eq. (3.28) yields

$$I_4^{(d)}(0,0,0,s_{13};s_{14},s_{35}) = \eta^{(d)}(s_{13},s_{13},s_{14},s_{35}) - \eta^{(d)}(s_{35},s_{13},s_{14},s_{35}) - \eta^{(d)}(s_{14},s_{13},s_{14},s_{35}). \tag{3.32}$$

Adopting Eq. (3.31), exploiting Eq. (3.28) and, if $-s_{14}-s_{35}+s_{13}<0$, also Eq. (3.32) for J_2, we obtain a result in which each term is real.

The infinite sums resulting from Eq. (3.15) may be written in terms of known hypergeometric functions. As is evident from explicit expressions for the $I_4^{(d)}$ integrals, we must compute three different types of sums. The first one is related to the function $\chi^{(d)}$,

$$\begin{aligned}
&\sum_{r=0}^{\infty}\Gamma\left(\frac{d-2r-6}{2}\right)h_5^{\frac{d}{2}-r-1}\chi^{(d-2r-2)}(s,a,b,c)\\
&= \frac{-4h_5^{\frac{d}{2}-1}}{s\ Q}\sum_{r=0}^{\infty}I_2^{(d-2r-2)}(s)\frac{\Gamma\left(\frac{d}{2}-r-3\right)}{h_5^r}\frac{(d-2r-5)}{(d-2r-8)}\,{}_2F_1\left[\begin{matrix}1,r+4-\frac{d}{2}\,;\\ r+5-\frac{d}{2}\,;\end{matrix}\ x\right]\\
&= \frac{4(d-3)(d-5)\Gamma\left(\frac{d}{2}-4\right)}{Q\ s^2(x-1)}\ h_5^{\frac{d}{2}-1}\ I_2^{(d)}(s)\ F_3\left(1,1,\frac{7-d}{2},1,\frac{10-d}{2};\frac{4}{sh_5},\frac{x}{x-1}\right),
\end{aligned} \tag{3.33}$$

where

$$Q = c + b - a, \qquad x = \frac{cb}{sQ}. \tag{3.34}$$

The result of the summation of a term including the $\eta^{(d)}$ function reads:

$$\begin{aligned}
&\sum_{r=0}^{\infty} \Gamma\left(\frac{d-2r-6}{2}\right) h_5^{\frac{d}{2}-r-1} \eta^{(d-2r-2)}(s,a,b,c) \\
&\quad = \frac{8(d-3)(d-5)}{bcs} h_5^{\frac{d}{2}-1} \Gamma\left(\frac{d-6}{2}\right) I_2^{(d)}(s) \\
&\quad \times \sum_{r=0}^{\infty} \frac{\left(\frac{7-d}{2}\right)_r}{\left(\frac{8-d}{2}\right)_r} \left(\frac{4}{s\ h_5}\right)^r \frac{1}{(d-2r-6)}\, {}_2F_1\left[\begin{matrix} 1, \frac{d-6}{2} - r\,; \\ \frac{d-4}{2} - r\,; \end{matrix}\ \frac{s(c+b-a)}{bc}\right].
\end{aligned} \tag{3.35}$$

The simplest type of infinite series originates from the two terms in Eq. (3.28) without ${}_2F_1$ functions. The contribution arising from the term proportional to $\varkappa_2$ reads:

$$\begin{aligned}
&\sum_{r=0}^{\infty} \Gamma\left(\frac{d-2r-6}{2}\right)\ h_5^{\frac{d}{2}-r-1}\ \frac{(-2\pi)(d-2r-5)}{s_{35}s_{14}\ \sin\frac{\pi(d-2r-2)}{2}} I_2^{(d-2r-2)}\left(\frac{-s_{35}s_{14}}{s_{35}+s_{14}-s_{13}}\right) \\
&= \frac{8\pi(d-3)(d-5)}{(d-6)\sin\frac{\pi d}{2}} \frac{(s_{14}+s_{35}-s_{13})}{s_{35}^2 s_{14}^2} I_2^{(d)}\left(\frac{-s_{35}s_{14}}{s_{35}+s_{13}-s_{13}}\right) \\
&\times h_5^{\frac{d}{2}-1} \Gamma\left(\frac{d-4}{2}\right)\ {}_2F_1\left[\begin{matrix} 1, \frac{7-d}{2}\,; \\ \frac{8-d}{2}\,; \end{matrix}\ \frac{4(s_{35}+s_{14}-s_{13})}{s_{35}s_{14}h_5}\right],
\end{aligned} \tag{3.36}$$

and the one related to $\varkappa_3$ reads:

$$\begin{aligned}
&\sum_{r=0}^{\infty} \Gamma\left(\frac{d-2r-6}{2}\right)\ h_5^{\frac{d}{2}-r-1}\ \frac{(-4\pi)(d-2r-5)}{s_{14}s_{25}\tan\frac{\pi(d-2r-2)}{2}}\ I_2^{(d-2r-2)}\left(\frac{s_{14}s_{25}}{s_{25}+s_{14}-s_{24}}\right) \\
&= \frac{16\pi(d-3)(d-5)}{(d-6)\tan\frac{\pi d}{2}} \frac{(s_{14}+s_{25}-s_{24})}{s_{35}^2 s_{14}^2}\ h_5^{\frac{d}{2}-1} \Gamma\left(\frac{d-4}{2}\right)\ I_2^{(d)}\left(\frac{s_{14}s_{25}}{s_{25}+s_{14}-s_{24}}\right) \\
&\quad \times\ {}_2F_1\left[\begin{matrix} 1, \frac{7-d}{2}\,; \\ \frac{8-d}{2}\,; \end{matrix}\ \frac{4(s_{14}+s_{25}-s_{24})}{s_{25}s_{14}h_5}\right].
\end{aligned} \tag{3.37}$$

Inserting the sums of Eqs. (3.33), (3.35), (3.36), and (3.37) in Eq. (3.15), we obtain the following

expression for the integral $I_5^{(d)}$:

$$
\begin{aligned}
I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}) &= \frac{-\pi^2\ \Gamma\left(3-\frac{d}{2}\right)}{\sin^2\frac{\pi d}{2}}\ \frac{(-h_5)^{\frac{5-d}{2}}}{\sqrt{-s_{13}s_{14}s_{35}s_{24}s_{25}}} \\
&+ \frac{\varkappa_2(s_{35}+s_{14}-s_{13})(d-8)\pi K}{2h_5 s_{35}^2 s_{14}^2\ \sin\frac{\pi d}{2}} I_2^{(d)}\left(\frac{-s_{35}s_{14}}{s_{35}+s_{14}-s_{13}}\right)\ {}_2F_1\left[\begin{matrix}1,\frac{7-d}{2};\\ \frac{8-d}{2};\end{matrix}\ \frac{4(s_{35}+s_{14}-s_{13})}{s_{35}s_{14}h_5}\right] \\
&+ \frac{\varkappa_3(s_{14}+s_{25}-s_{24})(d-8)\pi K}{h_5 s_{14}^2 s_{25}^2\tan\frac{\pi d}{2}}\ I_2^{(d)}\left(\frac{s_{14}s_{25}}{s_{14}+s_{25}-s_{24}}\right)\ {}_2F_1\left[\begin{matrix}1,\frac{7-d}{2};\\ \frac{8-d}{2};\end{matrix}\ \frac{4(s_{14}+s_{25}-s_{24})}{s_{25}s_{14}h_5}\right] \\
&+ \frac{(d-8)K}{h_5 s_{24}} I_2^{(d)}(s_{24})\left[\frac{\varkappa_1}{s_{35}s_{24}} H^{(d)}\left(\frac{4}{h_5 s_{24}},\frac{(s_{35}+s_{24}-s_{25})}{s_{35}}\right)\right. \\
&\left. - \frac{\varkappa_3}{s_{14}s_{25}} H^{(d)}\left(\frac{4}{h_5 s_{24}},\frac{s_{24}(s_{14}+s_{25}-s_{24})}{s_{14}s_{25}}\right) + \frac{\varkappa_5}{s_{13}s_{24}} H^{(d)}\left(\frac{4}{h_5 s_{24}},\frac{s_{13}+s_{24}-s_{14}}{s_{13}}\right)\right] \\
&- \frac{K}{h_5 s_{13}} I_2^{(d)}(s_{13})\left[\frac{\varkappa_2}{(s_{13}-s_{35})(s_{13}-s_{14})}\Phi^{(d)}\left(\frac{4}{h_5 s_{13}},\frac{s_{35}s_{14}}{(s_{13}-s_{35})(s_{13}-s_{14})}\right)\right. \\
&\left. + \frac{\varkappa_4}{(s_{13}-s_{35})s_{13}}\Phi^{(d)}\left(\frac{4}{h_5 s_{13}},\frac{-s_{25}}{s_{13}-s_{35}}\right) + \frac{\varkappa_5}{(s_{13}-s_{14})s_{13}}\Phi^{(d)}\left(\frac{4}{h_5 s_{13}},\frac{-s_{24}}{s_{13}-s_{14}}\right)\right] \\
&+ \frac{K}{h_5 s_{14}} I_2^{(d)}(s_{14})\left[\frac{\varkappa_2}{(s_{13}-s_{14})s_{14}}\Phi^{(d)}\left(\frac{4}{h_5 s_{14}},\frac{s_{35}}{s_{13}-s_{14}}\right) - \frac{\varkappa_3}{(s_{14}-s_{24})s_{14}}\Phi^{(d)}\left(\frac{4}{h_5 s_{14}},\frac{-s_{25}}{s_{14}-s_{24}}\right)\right. \\
&\left. + \frac{\varkappa_5}{(s_{14}-s_{24})(s_{13}-s_{14})}\Phi^{(d)}\left(\frac{4}{h_5 s_{14}},\frac{-s_{13}s_{24}}{(s_{14}-s_{24})(s_{13}-s_{14})}\right)\right] \\
&+ \frac{K}{h_5 s_{25}} I_2^{(d)}(s_{25})\left[\frac{\varkappa_1}{(s_{25}-s_{35})(s_{24}-s_{25})}\Phi^{(d)}\left(\frac{4}{h_5 s_{25}},\frac{-s_{24}s_{35}}{(s_{25}-s_{35})(s_{24}-s_{25})}\right)\right. \\
&\left. + \frac{\varkappa_4(d-8)}{s_{25}s_{13}} H^{(d)}\left(\frac{4}{h_5 s_{25}},\frac{s_{25}+s_{13}-s_{35}}{s_{13}}\right) + \frac{\varkappa_3}{(s_{24}-s_{25})s_{25}}\Phi^{(d)}\left(\frac{4}{h_5 s_{25}},\frac{s_{14}}{s_{24}-s_{25}}\right)\right] \\
&+ \frac{K}{h_5 s_{35}} I_2^{(d)}(s_{35})\left[\frac{\varkappa_1}{(s_{25}-s_{35})s_{35}}\Phi^{(d)}\left(\frac{4}{h_5 s_{35}},\frac{s_{24}}{s_{25}-s_{35}}\right) + \frac{\varkappa_2}{(s_{13}-s_{35})s_{35}}\Phi^{(d)}\left(\frac{4}{h_5 s_{35}},\frac{s_{14}}{s_{13}-s_{35}}\right)\right. \\
&\left. - \frac{\varkappa_4}{(s_{25}-s_{35})(s_{13}-s_{35})}\Phi^{(d)}\left(\frac{4}{h_5 s_{35}},\frac{s_{25}s_{13}}{(s_{25}-s_{35})(s_{13}-s_{35})}\right)\right], \qquad (3.38)
\end{aligned}
$$

where

$$
K = \frac{-8(d-3)(d-5)}{(d-6)(d-8)s_{13}s_{14}s_{24}s_{25}s_{35}}, \qquad (3.39)
$$

$$
\begin{aligned}
\Phi^{(d)}(w,z) &= \sum_{r=0}^{\infty}\frac{\left(\frac{7-d}{2}\right)_r}{\left(\frac{10-d}{2}\right)_r} w^r\ {}_2F_1\left[\begin{matrix}1,1;\\ r+5-\frac{d}{2};\end{matrix}\ z\right] \\
&= F_3\left(1,1,\frac{7-d}{2},1,\frac{10-d}{2};w,z\right), \qquad (3.40) \\
H^{(d)}(w,z) &= \sum_{r=0}^{\infty}\frac{\left(\frac{7-d}{2}\right)_r}{\left(\frac{8-d}{2}\right)_r}\ \frac{w^r}{(d-2r-6)}\ {}_2F_1\left[\begin{matrix}1,\frac{d-6}{2}-r;\\ \frac{d-4}{2}-r;\end{matrix}\ z\right] \\
&= \frac{1}{(d-6)}\sum_{l=0}^{\infty} z^l \frac{\left(\frac{d-6}{2}\right)_l}{\left(\frac{d-4}{2}\right)_l}\ {}_3F_2\left[\begin{matrix}1,\frac{7-d}{2},\frac{6-d}{2}-l;\\ \frac{8-d}{2},\frac{8-d}{2}-l;\end{matrix}\ w\right], \qquad (3.41)
\end{aligned}
$$

and the quantities $\varkappa_i$ are defined in Eq. (2.9). The functions $H^{(d)}$ and $\Phi^{(d)}$ are related as

$$H^{(d)}(w,z) = \frac{1}{(1-z)(d-8)}\Phi^{(d)}\left(w, \frac{1}{1-z}\right) - \frac{\pi(-z)^{3-\frac{d}{2}}}{2\sin\frac{\pi d}{2}}\,{}_2F_1\left[\begin{array}{c} 1, \frac{7-d}{2}\,; \\ \frac{8-d}{2}\,; \end{array} wz\right]. \tag{3.42}$$

The Appell function F_3 admits the following one-fold integral representation:

$$F_3\left(1,1,\frac{7-d}{2},1,\frac{10-d}{2};w,z\right) = \frac{\Gamma\left(\frac{10-d}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7-d}{2}\right)}\frac{1}{\sqrt{z}}\int_0^1 \frac{(1-u)^{\frac{5-d}{2}}}{1-w+uw}\,\frac{\arcsin\sqrt{uz}}{\sqrt{1-uz}}du. \tag{3.43}$$

It is interesting to note that changing the space-time dimension d in Eq. (3.43) by one unit, $d \to d+1$, we obtain the Appell function F_3, which we already encountered in the calculation of the one-loop master integral entering the calculation of radiative corrections to Bhabha scattering [17]. As a matter of fact, we observed in Ref. [17] that, at the one-loop level, the set of hypergeometric functions appearing in the results for n-point integrals with massive propagators also appear (up to the change $d \to d+1$) in the calculation of $(n+1)$-point integrals with all propagators being massless. For example, the one-loop propagator integral with two different masses is expressible in terms of two Gauss hypergeometric functions ${}_2F_1(1,(d-1)/2;d/2;z)$ with different arguments, while the result for the one-loop vertex integral with massless propagators is expressible in terms of the ${}_2F_1(1,(d-2)/2;(d-1)/2;z)$ function with different arguments. The result for the one-loop vertex integral with arbitrary masses and external momenta [12] is expressible in terms of the Appell function $F_1((d-2)/2,1,1/2,d/2;x,y)$ and the Gauss hypergeometric function ${}_2F_1(1,(d-2)/2;(d-1)/2;z)$ with different arguments, while the result for the box integral with massless propagators and arbitrary external momenta is expressible in terms of the same functions up to the shift $d \to d+1$, as was observed in Ref. [15]. We stress that the number of terms with hypergeometric functions appearing in the expressions for one-loop integrals with massive propagators and the arguments of these hypergeometric functions are different from those for their counterparts with massless propagators, but the sets of hypergeometric functions are the same up to the shift in d mentioned above.

The analytic continuation of the result in Eq. (3.38) amounts to the analytic continuation of the hypergeometric functions F_3 and ${}_2F_1$ and the factors in front of these functions. From Eq. (3.38), we may obtain the value of the integral in any region by using the usual $i\epsilon$ prescription and observing that $I_5^{(d)}$ is manifestly real in the Euclidean region where all scalar invariants s_{ij} are negative. The analytic continuation of the Gauss hypergeometric function ${}_2F_1$ is well understood [18]. Several useful formulae are given in Appendix A. The analytic continuation of the Appell function F_3 may be obtained from the series representation of Eq. (3.40) by the analytic continuation of the ${}_2F_1$ function. We notice that the well-known formula for the analytic continuation of the F_3 function [18] in terms of the Appell function F_2 is not applicable if both arguments of the F_3 function in Eq. (3.40) are large. In this case, one may proceed by analytically continuing the functions ${}_2F_1$ and ${}_3F_2$ in Eqs. (3.40) and (3.41). For example, the analytic continuation of the ${}_3F_2$ function in Eq. (3.41) is

$$\begin{aligned}
{}_3F_2\left[\begin{array}{c} 1, \frac{7-d}{2}, \frac{6-d}{2}-l\,; \\ 4-\frac{d}{2}, 4-\frac{d}{2}-l\,; \end{array} w\right] &= \frac{(6-d)(d-6+2l)}{w(d-5)(d-4+2l)}\;{}_3F_2\left[\begin{array}{c} 1, \frac{d-4}{2}, \frac{d-4}{2}+l\,; \\ \frac{d-3}{2}, \frac{d-2}{2}+l\,; \end{array} \frac{1}{w}\right] \\
&+ \frac{\Gamma\left(\frac{8-d}{2}\right)(d+2l-6)\Gamma\left(\frac{d-5}{2}\right)}{\sqrt{\pi}\,(2l+1)}(-w)^{\frac{d-7}{2}}\;{}_2F_1\left[\begin{array}{c} \frac{1}{2}, l+\frac{1}{2}\,; \\ l+\frac{3}{2}\,; \end{array} \frac{1}{w}\right] \\
&- \frac{\pi(d+2l-6)\Gamma\left(\frac{8-d}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}{2\,\sin\frac{\pi d}{2}\,\Gamma\left(\frac{7-d}{2}\right)l!}(-1)^l(-w)^{\frac{d}{2}+l-3}.
\end{aligned} \tag{3.44}$$

Substituting this formula into Eq. (3.41), we obtain the following representation for the $H^{(d)}(w,z)$

function for $|w| > 1$:

$$H^{(d)}(w,z) = \frac{(6-d)}{(d-5)(d-4)w}\,\phi^{(d-1)}\left(\frac{1}{w},z\right) - \frac{\pi^{\frac{3}{2}}\,\Gamma\left(\frac{8-d}{2}\right)}{2\,\sin\frac{\pi d}{2}\,\Gamma\left(\frac{7-d}{2}\right)}\,\frac{(-w)^{\frac{d-6}{2}}}{\sqrt{1-wz}}$$
$$- \frac{(-w)^{\frac{d-6}{2}}}{\sqrt{\pi\,(wz-1)}}\Gamma\left(4-\frac{d}{2}\right)\Gamma\left(\frac{d-5}{2}\right)\arcsin\sqrt{\frac{(1-wz)}{w(1-z)}}, \qquad (3.45)$$

where

$$\phi^{(d)}(x,y) = F^{1;2;1}_{1;1;0}\left[\begin{matrix}\frac{d-3}{2}:\ \frac{d-3}{2},\ 1; & 1; \\ \frac{d-1}{2}: & \frac{d-2}{2};\ - ;\end{matrix}\ x,y\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_{r+s}}{\left(\frac{d-1}{2}\right)_{r+s}}\,\frac{\left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r}\,x^r y^s$$
$$= \sum_{r=0}^{\infty}\frac{\left(\frac{d-3}{2}\right)_r\left(\frac{d-3}{2}\right)_r}{\left(\frac{d-2}{2}\right)_r\left(\frac{d-1}{2}\right)_r}\,x^r\,{}_2F_1\left[\begin{matrix}1,\frac{d-3}{2}+r\,; \\ \frac{d-1}{2}+r\,;\end{matrix}\ y\right], \qquad (3.46)$$

and $F^{1;2;1}_{1;1;0}$ is the Kampé de Fériet function [19]. To analytically continue the $H^{(d)}(w,z)$ function into the region $w,z>1$, one may use Eq. (3.45). The analytic continuation of the $\phi^{(d)}(w,z)$ function into the region $|z|>1$ may be obtained by the analytic continuation of the ${}_2F_1$ function in Eq. (3.46) using Eq. (9.66) from Appendix A.

There are several relations for the F_3 function which may be useful for its analytic continuation and performing its ε expansion. We present here two formulae for the analytic continuation of the F_3 function with large first argument x and $y<0$. One such relation follows from the results given in Ref. [17] and reads:

$$F_3\left(1,1,\frac{7-d}{2},1,\frac{10-d}{2},x,y\right) = \frac{(d-6)(d-8)}{(d-3)(d-5)(xy-y-x)x}\left\{{}_2F_1\left[\begin{matrix}1,\frac{1}{2}\,; \\ \frac{d-1}{2}\,;\end{matrix}\ \frac{y}{x+y-xy}\right]\right.$$
$$\left.+\frac{(d-4)}{x(d-1)}F_3\left(\frac{1}{2},1,1,\frac{d-2}{2},\frac{d+1}{2},\frac{y}{x+y-xy},\frac{1}{x}\right)\right\}$$
$$+\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{d-5}{2}\right)\Gamma\left(\frac{10-d}{2}\right)\sqrt{1-x}\,(-x)^{\frac{d-8}{2}}\,{}_2F_1\left[\begin{matrix}1,1\,; \\ \frac{3}{2}\,;\end{matrix}\ \frac{y(x-1)}{x}\right]$$
$$-\frac{(d-8)}{(d-5)\,x(1-y)}\,{}_2F_1\left[\begin{matrix}1,\frac{d-4}{2}\,; \\ \frac{d-3}{2}\,;\end{matrix}\ \frac{y}{x(y-1)}\right]\,{}_2F_1\left[\begin{matrix}1,3-\frac{d}{2}\,; \\ 4-\frac{d}{2}\,;\end{matrix}\ \frac{y}{y-1}\right]. \qquad (3.47)$$

The F_3 function with appropriate parameters admits the one-fold integral representation

$$F_3\left(\frac{1}{2},1,1,\frac{d-2}{2},\frac{d+1}{2};\ w,z\right) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-2}{2}\right)}\frac{1}{w}\int_0^1\frac{(1-v)^{\frac{d-4}{2}}}{1-z+vz}\,\ln\frac{1+\sqrt{wv}}{1-\sqrt{wv}}\,dv, \qquad (3.48)$$

which may also be used for the analytic continuation. Another formula for the analytic continuation into the region of large x and $y<0$ connects the F_3 function and the Horn function H_2,

$$F_3\left(1,1,\frac{7-d}{2},1,\frac{10-d}{2};x,y\right) = \frac{(8-d)}{x(d-5)}H_2\left(\frac{d-6}{2},1,1,1,\frac{d-3}{2};\frac{1}{x},-y\right)$$
$$+\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{d-5}{2}\right)\Gamma\left(\frac{10-d}{2}\right)(-x)^{\frac{d-8}{2}}\sqrt{1-x}\,{}_2F_1\left[\begin{matrix}1,1\,; \\ \frac{3}{2}\,;\end{matrix}\ \frac{y(x-1)}{x}\right]. \qquad (3.49)$$

We checked Eq. (3.38) by comparing its numerical values calculated for some specific values of d and s_{ij} satisfying the condition of Eq. (3.26) with the result of the direct numerical evaluation of the four-fold integral representation given by Eq. (9.63) in Appendix A using the computer algebra program

package Maple. In all cases, we found perfect agreement to all valid digits of the numerical results (usually about 12). For example, we performed a numerical calculation for $d = 21/2$ and $d = 10 - 2\varepsilon$, where $\varepsilon = 1/10000$, setting $s_{13} = -1000 + \delta$, $s_{14} = -65 + \delta$, $s_{35} = -190 + \delta$, $s_{24} = -1/20 + \delta$, and $s_{25} = -1/10 + \delta$ with $\delta = i10^{-20}$.

4 Case of one vanishing variable

In practical applications, such as the analytic continuation of the pentagon integral with the help of functional equations [17, 20], we need the value of the integral $I_5^{(d)}$ with one or more vanishing kinematic variables. As was shown in Ref. [20], the one-loop vertex integral with arbitrary masses and external momenta may be expressed in terms of a vertex integral, in which two masses and the square of one external momentum vanish, with the help of functional equations. The formulae for the analytic continuation of the vertex integral with arbitrary masses contain integrals with all masses vanishing. The relation between the master integrals of Bhabha scattering and heavy-quark production derived in Ref. [17] includes an integral with massless propagators. In all these cases, relations connecting integrals with different kinematical variables include integrals with simpler kinematics, in which either some masses or some squared momenta are equal to zero. We observed a rather similar situation for the pentagon integral. Functional equations relevant for the analytic continuation of a considered integral include the value of this integral with one scalar invariant taken to be zero. For this reason, we wish to present here the value of the integral $I_5^{(d)}$ with one scalar invariant taken to be zero. Due to the symmetry (2.6) of this integral, it is sufficient to consider the case when $s_{24} = 0$. A detailed discussion of functional equations for pentagon-type integrals will be presented in a separate publication [15].

For $s_{24} = 0$, we obtain the following result:

$$
\begin{aligned}
&I_5^{(d)}(s_{13}, s_{14}, 0, s_{25}, s_{35}) \\
&= \frac{-8(d-3)(d-5)}{(d-6)(d-8)(s_{14}s_{35} - s_{14}s_{25} + s_{13}s_{25})} \left\{ \frac{\pi(d-8)(s_{14}+s_{25})}{s_{14}s_{25}\tan\frac{\pi d}{2}} I_2^{(d)}\left(\frac{s_{14}s_{25}}{s_{14}+s_{25}}\right) \right. \\
&+ \frac{s_{35}s_{14}}{(s_{13}-s_{35})(s_{13}-s_{14})s_{13}} I_2^{(d)}(s_{13})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} \frac{s_{35}s_{14}}{(s_{13}-s_{35})(s_{13}-s_{14})} \right] \\
&+ \frac{s_{25}}{s_{13}(s_{13}-s_{35})} I_2^{(d)}(s_{13})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} \frac{s_{25}}{s_{35}-s_{13}} \right] \\
&- \frac{s_{35}}{s_{14}(s_{13}-s_{14})} I_2^{(d)}(s_{14})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} \frac{s_{35}}{s_{13}-s_{14}} \right] - \frac{s_{25}}{s_{14}^2} I_2^{(d)}(s_{14})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} -\frac{s_{25}}{s_{14}} \right] \\
&- \frac{s_{14}}{s_{25}^2} I_2^{(d)}(s_{25})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} -\frac{s_{14}}{s_{25}} \right] - \frac{(d-8)}{(d-6)s_{25}} I_2^{(d)}(s_{25})\, {}_2F_1\left[\begin{matrix} 1, \frac{d}{2}-3; \\ \frac{d}{2}-2; \end{matrix} \frac{s_{25}+s_{13}-s_{35}}{s_{13}} \right] \\
&- \frac{s_{14}}{s_{35}(s_{13}-s_{35})} I_2^{(d)}(s_{35})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} \frac{s_{14}}{s_{13}-s_{35}} \right] \\
&+ \frac{s_{13}s_{25}}{s_{35}(s_{25}-s_{35})(s_{13}-s_{35})} I_2^{(d)}(s_{35})\, {}_2F_1\left[\begin{matrix} 1, 1; \\ 5-\frac{d}{2}; \end{matrix} \frac{s_{13}s_{25}}{(s_{25}-s_{35})(s_{13}-s_{35})} \right] \\
&\left. + \frac{\pi(s_{13}-s_{35}-s_{14})(d-8)}{2\sin\frac{\pi d}{2}\, s_{14}s_{35}} I_2^{(d)}\left(\frac{s_{35}s_{14}}{s_{13}-s_{35}-s_{14}}\right) \right\}. \qquad (4.50)
\end{aligned}
$$

We observe that this result is significantly simplified compared to Eq. (3.38). In fact, only ${}_2F_1$ functions remain.

5 Asymptotic values of the integral $I_5^{(d)}$ for $|s_{13}| \to \infty$

If the scalar invariants fulfill the hierarchy of Eq. (3.26) with $|s_{13}|$ being much larger than the other scalar invariants and $s_{24} \neq s_{25}$, then we have

$$\begin{aligned} h_5 \;\cong\; & \frac{(s_{24}-s_{25})^2}{s_{24}s_{35}s_{14}s_{25}} s_{13} \\ & + \frac{2(s_{14}s_{24}s_{35} - s_{14}s_{25}^2 - s_{24}^2 s_{35} + s_{24}s_{35}s_{25} + s_{14}s_{25}s_{35} + s_{14}s_{24}s_{25})}{s_{24}s_{35}s_{14}s_{25}} + O\left(\frac{1}{s_{13}}\right). \end{aligned} \tag{5.51}$$

As one can see from the explicit expression of Eq. (3.38), the first argument of the two functions $\Phi^{(d)}$ and $H^{(d)}$ is always proportional to $1/|s_{13}|$, so that it is sufficient to keep only the first terms of the series in Eqs. (3.40) and (3.41) in order to find the asymptotically leading term of the integral $I_5^{(d)}$. In $d = 4 - 2\varepsilon$ dimensions, this value reads:

$$\begin{aligned} & I_5^{(4-2\varepsilon)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35})\Big|_{|s_{13}|\to\infty} \\ & = \frac{\pi(1-2\varepsilon)}{\sin\pi\varepsilon} \frac{(s_{25}+s_{24})}{s_{14}s_{35}s_{24}s_{25}} I_2^{(4-2\varepsilon)}\left(\frac{s_{35}s_{14}}{s_{13}}\right) \left\{1 - \frac{(s_{25}-s_{24})^2}{s_{25}(s_{25}+s_{24})} {}_2F_1\left[\begin{matrix} 1, 1-\varepsilon\,; \\ 1+\varepsilon\,; \end{matrix} \frac{s_{24}}{s_{25}}\right]\right\} \\ & \quad - \frac{\pi^2\Gamma(1+\varepsilon)}{\sin^2\pi\varepsilon} \frac{(-s_{13})^{\varepsilon}[(s_{24}-s_{25})^2]^{\frac{1}{2}+\varepsilon}}{(s_{14}s_{35}s_{24}s_{25})^{1+\varepsilon}} + O\left(\frac{1}{|s_{13}|^{1-\varepsilon}}\right). \end{aligned} \tag{5.52}$$

In $d = 2 - 2\varepsilon$ dimensions, the result is

$$\begin{aligned} & I_5^{(2-2\varepsilon)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35})\Big|_{|s_{13}|\to\infty} = \frac{4\pi(1-4\varepsilon^2)(3+2\varepsilon)(s_{25}+s_{24})s_{13}}{s_{14}^2 s_{35}^2 (s_{24}-s_{25})^2 (2+\varepsilon)\sin\pi\varepsilon} \\ & \quad \times I_2^{(4-2\varepsilon)}\left(\frac{s_{35}s_{14}}{s_{13}}\right) {}_2F_1\left[\begin{matrix} 1, \frac{5}{2}+\varepsilon\,; \\ 3+\varepsilon\,; \end{matrix} \frac{-4s_{24}s_{25}}{(s_{24}-s_{25})^2}\right] \\ & \quad - \left(\frac{-s_{13}(s_{24}-s_{25})^2}{s_{14}s_{35}s_{24}s_{25}}\right)^{\frac{3}{2}+\varepsilon} \frac{\pi^2\Gamma(2+\varepsilon)}{\sqrt{-s_{13}s_{14}s_{24}s_{25}s_{35}}\,\sin^2\pi\varepsilon} + O\left(\frac{1}{|s_{13}|^{-\varepsilon}}\right). \end{aligned} \tag{5.53}$$

We observe that, when one of the scalar invariants is large compared to the others, the hypergeometric functions of two variables collapse to hypergeometric function of just one variable.

In $d = 6 - 2\varepsilon$ dimensions, the leading large-$|s_{13}|$ term of the integral $I_5^{(d)}$ is somewhat more

complicated, being

$$
\begin{aligned}
I_5^{(6-2\varepsilon)}(s_{13},s_{14},s_{24},s_{25},s_{35})\Big|_{|s_{13}|\to\infty} &= -\frac{\pi^2\Gamma(\varepsilon)}{\sin^2\pi\varepsilon}\frac{\left[\frac{-s_{13}(s_{25}-s_{24})^2}{s_{14}s_{35}s_{24}s_{25}}\right]^{\varepsilon}}{s_{13}(s_{25}-s_{24})} \\
&- \frac{(1-2\varepsilon)}{\varepsilon\ s_{13}(s_{25}-s_{24})} \\
&\times\left\{\frac{-\pi(s_{24}+s_{25})}{(s_{25}-s_{24})\sin\pi\varepsilon}I_2^{(4-2\varepsilon)}\left(\frac{s_{35}s_{14}}{s_{13}}\right)\,{}_2F_1\left[\begin{array}{l}1,\frac{1}{2}+\varepsilon\,;\\ 1+\varepsilon\,;\end{array}\ \frac{-4s_{24}s_{25}}{(s_{24}-s_{25})^2}\right]\right. \\
&+\frac{2\pi}{\tan\pi\varepsilon}\ I_2^{(4-2\varepsilon)}\left(\frac{s_{14}s_{25}}{s_{14}+s_{25}-s_{24}}\right)+\frac{s_{24}}{(1+\varepsilon)\ s_{14}}I_2^{(4-2\varepsilon)}(s_{14})\,{}_2F_1\left[\begin{array}{l}1,1+\varepsilon\,;\\ 2+\varepsilon\,;\end{array}\ \frac{s_{24}}{s_{14}}\right] \\
&-\frac{s_{25}}{(1+\varepsilon)(s_{14}+s_{25}-s_{24})}I_2^{(4-2\varepsilon)}(s_{14})\,{}_2F_1\left[\begin{array}{l}1,1+\varepsilon\,;\\ 2+\varepsilon\,;\end{array}\ \frac{s_{25}}{s_{14}+s_{25}-s_{24}}\right] \\
&-\frac{1}{\varepsilon}\ I_2^{(4-2\varepsilon)}(s_{24})\,{}_2F_1\left[\begin{array}{l}1,-\varepsilon\,;\\ 1-\varepsilon\,;\end{array}\ \frac{s_{35}+s_{24}-s_{25}}{s_{35}}\right] \\
&-\frac{1}{\varepsilon}I_2^{(4-2\varepsilon)}(s_{24})\,{}_2F_1\left[\begin{array}{l}1,-\varepsilon\,;\\ 1-\varepsilon\,;\end{array}\ \frac{s_{24}(s_{14}+s_{25}-s_{24})}{s_{14}s_{25}}\right]+I_2^{(4-2\varepsilon)}(s_{24})\ln\frac{s_{14}-s_{24}}{s_{13}} \\
&-I_2^{(4-2\varepsilon)}(s_{25})\ln\frac{s_{35}-s_{25}}{s_{13}}+\left[I_2^{(4-2\varepsilon)}(s_{24})-I_2^{(4-2\varepsilon)}(s_{25})\right][\gamma+\psi(-\varepsilon)] \\
&-\frac{s_{14}}{(1+\varepsilon)(s_{14}+s_{25}-s_{24})}I_2^{(4-2\varepsilon)}(s_{25})\,{}_2F_1\left[\begin{array}{l}1,1+\varepsilon\,;\\ 2+\varepsilon\,;\end{array}\ \frac{s_{14}}{s_{14}+s_{25}-s_{24}}\right] \\
&-\frac{s_{35}s_{24}}{(1+\varepsilon)s_{25}(s_{35}+s_{24}-s_{25})}I_2^{(4-2\varepsilon)}(s_{25})\,{}_2F_1\left[\begin{array}{l}1,1+\varepsilon\,;\\ 2+\varepsilon\,;\end{array}\ \frac{s_{24}s_{35}}{s_{25}(s_{35}+s_{24}-s_{25})}\right] \\
&+\frac{s_{24}}{(1+\varepsilon)\ (s_{35}+s_{24}-s_{25})}I_2^{(4-2\varepsilon)}(s_{35})\,{}_2F_1\left[\begin{array}{l}1,1+\varepsilon\,;\\ 2+\varepsilon\,;\end{array}\ \frac{s_{24}}{s_{35}+s_{24}-s_{25}}\right] \\
&\left.-\frac{s_{25}}{(1+\varepsilon)\ s_{35}}I_2^{(4-2\varepsilon)}(s_{35})\,{}_2F_1\left[\begin{array}{l}1,1+\varepsilon\,;\\ 2+\varepsilon\,;\end{array}\ \frac{s_{25}}{s_{35}}\right]\right\}+\mathcal{O}\left(\frac{1}{|s_{13}|^{2-\varepsilon}}\right),
\end{aligned}
\tag{5.54}
$$

where $\psi(x)$ is the logarithmic derivative of the Euler Γ function,

$$
\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}, \tag{5.55}
$$

and γ is the Euler-Mascheroni constant.

6 Limit of multi-Regge kinematics

In this section, we consider the pentagon integral $I_5^{(d)}$ with the somewhat peculiar kinematics,

$$
-s_{13}\gg -s_{14},-s_{35}\gg -s_{25},-s_{24}. \tag{6.56}
$$

In addition, we assume that the following relation among the scalar invariants s_{ij} holds:

$$
s_{14}s_{35}\sim s_{13}s_{24}\sim s_{13}s_{25}. \tag{6.57}
$$

Equivalently, one may define such an ordering by introducing the scaling parameter λ, putting

$$
s_{13}\to s_{13},\quad s_{14}\to\lambda s_{14},\quad s_{35}\to\lambda s_{35},\quad s_{25}\to\lambda^2 s_{25},\quad s_{24}\to\lambda^2 s_{24}, \tag{6.58}
$$

and taking the limit $\lambda\to 0$. In fact, this arrangement corresponds to the multi-Regge kinematics. Recently, the pentagon integral in $d=6-2\varepsilon$ dimensions in the multi-Regge kinematics defined by

Eq. (6.58) was considered in Ref. [6]. The integral $I_5^{(d)}$ in $d = 6 - 2\varepsilon$ dimensions is needed to determine the ε expansion of the integral $I_5^{(d)}$ in $d = 4 - 2\varepsilon$ dimensions.

Here, we consider the integral $I_5^{(d)}$ in the multi-Regge-kinematics limit of Eq. (6.56) directly in $d = 4 - 2\varepsilon$ dimensions. To obtain the leading contribution in this limit, we perform the scaling of Eq. (6.58) and retain the leading terms in the limit $\lambda \to 0$. The most divergent terms in Eq. (3.38) are of the order $\ln\lambda/\lambda^{4+2\varepsilon}$. As will be seen later, such terms cancel out in the sum, so that the leading asymptotic terms are of order $1/\lambda^{4+2\varepsilon}$. In order to find the leading terms in the limit $\lambda \to 0$, we must know the asymptotic behavior of the $H^{(d)}(w,z)$ function for $z \to 1$ and that of the $\Phi^{(d)}(w,z)$ function for $z \to \infty$. The leading and subleading terms of these functions in the respective limits may be obtained from the series representations of Eqs. (3.41) and (3.40) by retaining the first leading terms of the expansions of the ${}_2F_1$ functions. With the help of Eq. (9.66) in Appendix A, we obtain

$$H^{(d)}(w,z)\Big|_{z\to 1} = \frac{1}{2}\left\{\frac{\pi}{\tan\frac{\pi d}{2}} - \gamma - \psi\left(4-\frac{d}{2}\right) - \ln(1-z)\right\} {}_2F_1\left[\begin{array}{c} 1, \frac{7-d}{2}; \\ \frac{8-d}{2}; \end{array} w\right]$$
$$- \frac{1}{2}\frac{\partial}{\partial\alpha}\, {}_3F_2\left[\begin{array}{c} 1, \frac{7-d}{2}, 4-\frac{d}{2}+\alpha; \\ 4-\frac{d}{2}, 4-\frac{d}{2}; \end{array} w\right]\Bigg|_{\alpha=0} + O((1-z)\ln(1-z)). \qquad (6.59)$$

In a similar way, the asymptotic behavior of the $\Phi^{(d)}$ function for $z \to \infty$ may be derived. Keeping the logarithmic and constant terms of Eq. (9.68) in Appendix A, we arrive at the following result:

$$\Phi^{(d)}(w,z)\Big|_{z\to\infty} = \frac{8-d}{2z}\left\{\left[\gamma - \ln(-z) + \psi\left(4-\frac{d}{2}\right)\right] {}_2F_1\left[\begin{array}{c} 1, \frac{7-d}{2}; \\ \frac{8-d}{2}; \end{array} w\right]\right.$$
$$\left. + \frac{\partial}{\partial\alpha}\, {}_3F_2\left[\begin{array}{c} 1, \frac{7-d}{2}, 4-\frac{d}{2}+\alpha; \\ 4-\frac{d}{2}, 4-\frac{d}{2}; \end{array} w\right]\Bigg|_{\alpha=0}\right\} + O\left(\frac{\ln z}{z}\right). \qquad (6.60)$$

Using Eqs. (6.59) and (6.60), and retaining only terms contributing at orders $\ln\lambda/\lambda^{4+2\varepsilon}$ and $1/\lambda^{4+2\varepsilon}$, we find from Eq. (3.38) in the limit of multi-Regge kinematics the leading term,

$$I_5^{(4-2\varepsilon)}(s_{13},s_{14},s_{24},s_{25},s_{35}) = \frac{-\pi^2\Gamma(1+\varepsilon)(-\overline{h}_5)^{\frac{1}{2}+\varepsilon}}{\sin^2\pi\varepsilon\,\sqrt{-s_{13}s_{14}s_{24}s_{25}s_{35}}} - \frac{2(1-4\varepsilon^2)}{(1+\varepsilon)s_{13}s_{14}s_{35}s_{25}s_{24}\overline{h}_5}$$
$$\times\left\{\frac{2(s_{13}s_{24}-s_{14}s_{35}-s_{13}s_{25})}{s_{24}} I_2^{(4-2\varepsilon)}(s_{24}) H^{(4-2\varepsilon)}\left(\frac{4}{s_{24}\overline{h}_5}, \frac{s_{24}}{s_{25}}\right)\right.$$
$$- \frac{(s_{14}s_{35}+s_{13}s_{24}-s_{13}s_{25})}{s_{24}} I_2^{(4-2\varepsilon)}(s_{24}) \ln Y\, {}_2F_1\left[\begin{array}{c} 1, \frac{3}{2}+\varepsilon; \\ 2+\varepsilon; \end{array} \frac{4}{s_{24}\overline{h}_5}\right]$$
$$+ \frac{s_{24}(s_{14}s_{35}+s_{13}s_{24}-s_{13}s_{25})}{(2+\varepsilon)(s_{24}-s_{25})s_{25}} I_2^{(4-2\varepsilon)}(s_{25}) \Phi^{(4-2\varepsilon)}\left(\frac{4}{s_{25}\overline{h}_5}, \frac{s_{24}}{s_{24}-s_{25}}\right)$$
$$+ \frac{(s_{13}s_{24}-s_{14}s_{35}-s_{13}s_{25})}{s_{25}} I_2^{(4-2\varepsilon)}(s_{25}) \left(\ln Y + \frac{\pi}{\tan\pi\varepsilon}\right) {}_2F_1\left[\begin{array}{c} 1, \frac{3}{2}+\varepsilon; \\ 2+\varepsilon; \end{array} \frac{4}{s_{25}\overline{h}_5}\right]$$
$$\left. - \frac{\pi s_{13}(s_{14}s_{35}+s_{13}s_{24}+s_{13}s_{25})}{\sin\pi\varepsilon\; s_{14}s_{35}} I_2^{(4-2\varepsilon)}\left(\frac{s_{35}s_{14}}{s_{13}}\right) {}_2F_1\left[\begin{array}{c} 1, \frac{3}{2}+\varepsilon; \\ 2+\varepsilon; \end{array} \frac{-4s_{13}}{s_{14}s_{35}\overline{h}_5}\right]\right\}, \qquad (6.61)$$

where

$$Y = \frac{s_{13}(s_{25}-s_{24})}{s_{14}s_{35}},$$
$$\overline{h}_5 = \frac{s_{14}^2s_{35}^2 + s_{13}^2s_{24}^2 + s_{13}^2s_{25}^2 - 2s_{24}s_{13}^2s_{25} + 2s_{13}s_{35}s_{14}s_{25} + 2s_{13}s_{24}s_{35}s_{14}}{s_{13}s_{24}s_{35}s_{14}s_{25}}. \qquad (6.62)$$

As mentioned above, the terms with $\ln\lambda$ cancel, as well as those involving the function $\psi(x)$, γ and the derivative of the ${}_3F_2$ function with respect to a parameter.

The value of the integral $I_5^{(d)}$ for $d = 6 - 2\varepsilon$ in the limit of multi-Regge kinematics may be derived from Eq. (2.7) using Eq. (6.61) and the asymptotic values of integrals $I_4^{(d)}$ given in Eq. (3.28). We just note that it is again expressed in terms of F_3 and ${}_2F_1$ functions because the integrals $I_4^{(d)}$ add only ${}_2F_1$ functions. An analytic expression for the integral $I_5^{(d)}$ in $d = 6 - 2\varepsilon$ dimensions was recently obtained in Ref. [6]. The result of Ref. [6] is given in terms of derivatives of the Kampé de Fériet function [19] with respect to a parameter. For a direct comparison of our result with that of Ref. [6], one needs an expression of the F_3 function in terms of derivatives of the Kampé de Fériet function, which, to our knowledge, is not currently available.

7 Conclusions

In this paper, we evaluated the one-loop scalar pentagon integral in arbitrary space-time dimension d with on-shell external legs, massless internal lines, and otherwise arbitrary scalar invariants. Exploiting the method of dimensional recurrences, we obtained a result in terms of the hypergeometric functions F_3 and ${}_2F_1$. In our case, both functions admit one-fold integral representations suitable for ε expansions. Using the methods of Ref. [6], the on-shell pentagon integral may be represented in terms of four-fold hypergeometric series, while the method of dimensional recurrences advocated here just yields two-fold series.

The method of dimensional recurrences may also be applied to the evaluation of the hexagon integral, which is needed for the calculation of the $\mathcal{O}(\varepsilon)$ contributions to one-loop maximally-helicity-violating amplitudes at one loop in $\mathcal{N} = 4$ SYM theory. To simplify derivations in this case, one may start from the dimensional recurrences written in the limit of multi-Regge kinematics. In our opinion, the method of dimensional recurrences is quite efficient to go beyond the "box approximation" for the n-point one-loop integrals.

The ε expansion of our results in Eqs. (3.38) and (6.61) will be presented in a future publication [15].

We expect that the results presented in this paper and our forthcoming one [15] may be conveniently incorporated in program packages for the automated analytic computation of one-loop integrals in massless theories, similar to the package for numerical calculations presented in Ref. [21].

8 Acknowledgments

This work was supported in part by the German Federal Ministry for Education and Research BMBF through Grant No. 05H09GUE, by the German Research Foundation DFG through Grant No. KN 365/3–2, and by the Helmholtz Association HGF through Grant No. HA 101.

9 Appendix A

9.1 Integral representation of $I_5^{(d)}$

For numerical checks of the result for the integral $I_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35})$ in Eq. (3.38), we use the Feynman parameterization

$$I_5^{(d)}(s_{13}, s_{14}, s_{24}, s_{25}, s_{35}) = -\Gamma\left(5 - \frac{d}{2}\right) \int_0^1 ... \int_0^1 dx_1 dx_2 dx_3 dx_4 \; x_1^3 x_2^2 x_3 \; H_5^{\frac{d}{2}-5}, \tag{9.63}$$

where

$$H_5 = x_1x_2\left[(1-x_3)(1-x_1)s_{13} + x_1x_3x_4(1-x_2)s_{25} + x_3x_4(1-x_1)s_{35} + x_1x_3(1-x_4)(1-x_2)s_{24} + x_1x_2x_3(1-x_4)(1-x_3)s_{14}\right]. \tag{9.64}$$

9.2 Useful formulae for the Gauss hypergeometric function ${}_2F_1$

The Gauss hypergeometric function ${}_2F_1$ has the following integral representation:

$${}_2F_1\left[\begin{matrix}\alpha,\beta\,;\\ \gamma\,;\end{matrix}\ x\right] = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\quad \int_0^1 du\, u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-ux)^{-\alpha},$$

$$\mathrm{Re}(\beta)>0, \quad \mathrm{Re}(\gamma-\beta)>0. \tag{9.65}$$

The following formulae are useful for the analytic continuation of the ${}_2F_1$ function:

$$\begin{aligned}
&{}_2F_1\left[\begin{matrix}a,b\,;\\ a+b+m\,;\end{matrix}\ z\right]\frac{1}{\Gamma(a+b+m)} = \\
&\frac{\Gamma(m)}{\Gamma(a+m)\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{(1-m)_n n!}(1-z)^n \\
&+\frac{(1-z)^m(-1)^m}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{(n+m)!n!}[h_n^{''}-\ln(1-z)](1-z)^n,
\end{aligned} \tag{9.66}$$

$$-\pi<\arg(1-z)<\pi, \qquad a,b\neq 0,-1,2,\ldots,$$

where

$$h_n^{''} = \psi(n+1)+\psi(n+m+1)-\psi(a+n+m)-\psi(b+n+m), \tag{9.67}$$

and

$$\begin{aligned}
&\Gamma(a+m)[\Gamma(c)]^{-1}\ {}_2F_1\ (a,a+m;c;z) \\
&\quad = \frac{(-z)^{-a-m}}{\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(1-c+a)_{n+m}}{n!(n+m)!}z^{-n}[\ln(-z)+h_n] \\
&\qquad +(-z)^{-a}\sum_{n=0}^{m-1}\frac{(a)_n\Gamma(m-n)}{\Gamma(c-a-n)n!}z^{-n},
\end{aligned} \tag{9.68}$$

$$|\arg(-z)|<\pi, \qquad a\neq 0,-1,-2,\ldots, \qquad m=0,1,\ldots,$$

where

$$h_n = \psi(1+m+n)+\psi(1+n)-\psi(a+m+n)-\psi(c-a-m-n). \tag{9.69}$$

Here it is understood that the sum $\sum_0^{m-1}$ is empty for $m=0$.

9.3 Useful formulae for the Appell function F_3

The Appell function F_3 has the following series representations:

$$\begin{aligned}
F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{(\alpha')_n(\beta')_n}{(\gamma)_n\ n!}y^n\ {}_2F_1\left[\begin{matrix}\alpha,\beta\,;\\ \gamma+n\,;\end{matrix}\ x\right] \\
&= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n\ n!}x^n\ {}_2F_1\left[\begin{matrix}\alpha',\beta'\,;\\ \gamma+n\,;\end{matrix}\ y\right].
\end{aligned} \tag{9.70}$$

The integral representation of the F_3 function used for the derivation of the one-fold integral representations of the Φ function reads:

$$F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)} \int_0^1 \frac{u^{\gamma-\beta-1}(1-u)^{\beta-1}}{(1-x+ux)^{\alpha}} \, {}_2F_1\left[\begin{array}{c} \alpha', \beta'; \\ \gamma-\beta; \end{array} uy \right] \, du. \tag{9.71}$$

This integral representation follows from Eq. (20) in Ref. [22].

10 Appendix B

In this appendix, we present the system of differential equations for the integral $I_5^{(d)}$ to be used for the derivation of the differential equations for the periodic constants $P_p(d)$ and $P_m(d)$. To obtain this system of differential equations for $I_5^{(d)}$, we exploit the method described in Ref. [16]. According to this method, derivatives with respect to s_{ij} may be expressed in terms of integrals with shifted space-time dimensions. Explicit expressions for such derivatives may be derived from the integral representation with α parameters,

$$I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35}) = \frac{1}{i^{(d+2)/2}} \int\limits_0^\infty d\alpha_1 \ldots \int\limits_0^\infty d\alpha_5 \frac{\exp\{i\frac{Q}{D}\}}{D^{\frac{d}{2}}}, \tag{10.72}$$

where

$$\begin{aligned} Q &= \alpha_1\alpha_3 s_{13} + \alpha_1\alpha_4 s_{14} + \alpha_2\alpha_4 s_{24} + \alpha_2\alpha_5 s_{25} + \alpha_3\alpha_5 s_{35}, \\ D &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5. \end{aligned} \tag{10.73}$$

From Eq. (10.72), it follows that

$$\begin{aligned} \frac{\partial I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35})}{\partial s_{13}} &= \int \frac{d^{d+2}q}{i\pi^{\frac{d+2}{2}}} \frac{P}{D_1 D_3}, \\ \frac{\partial I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35})}{\partial s_{14}} &= \int \frac{d^{d+2}q}{i\pi^{\frac{d+2}{2}}} \frac{P}{D_1 D_4}, \\ \frac{\partial I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35})}{\partial s_{24}} &= \int \frac{d^{d+2}q}{i\pi^{\frac{d+2}{2}}} \frac{P}{D_2 D_4}, \\ \frac{\partial I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35})}{\partial s_{25}} &= \int \frac{d^{d+2}q}{i\pi^{\frac{d+2}{2}}} \frac{P}{D_2 D_5}, \\ \frac{\partial I_5^{(d)}(s_{13},s_{14},s_{24},s_{25},s_{35})}{\partial s_{35}} &= \int \frac{d^{d+2}q}{i\pi^{\frac{d+2}{2}}} \frac{P}{D_3 D_5}, \end{aligned} \tag{10.74}$$

where

$$P = \frac{1}{D_1 D_2 D_3 D_4 D_5}, \tag{10.75}$$

and D_j are defined in Eq. (2.2). In order to reduce the $(d+2)$-dimensional integrals on the right-hand sides of the relations in Eq. (10.74) to a set of basic integrals, we use recurrence relations [23]. All calculations are performed with the help of computer program package Maple. The resulting 5 differential equations for the integral $I_5^{(d)}$ all have the form of Eq. (3.22). The polynomials $R_{ij}^{(k)}$ and

ϕ_{ij} occuring therein are found to be

$$
\begin{aligned}
R_{13}^{(1)} &= (-s_{25}+s_{24}+s_{35})(s_{14}s_{25}-s_{35}s_{14}-s_{13}s_{25}+s_{24}s_{35}+s_{13}s_{24}),\\
R_{13}^{(2)} &= -(s_{14}^2s_{25}-s_{14}^2s_{35}+s_{35}s_{14}s_{25}-s_{13}s_{14}s_{25}+2s_{13}s_{35}s_{14}-s_{14}s_{35}^2+s_{13}s_{24}s_{14}\\
&\quad +s_{24}s_{35}s_{14}+s_{25}s_{13}s_{35}+s_{24}s_{35}^2-s_{24}s_{13}s_{35}),\\
R_{13}^{(3)} &= -(-s_{24}s_{35}+s_{35}s_{14}+s_{13}s_{24}-s_{13}s_{25}-s_{14}s_{25})(s_{14}+s_{25}-s_{24}),\\
R_{13}^{(4)} &= s_{13}(-s_{14}s_{25}+s_{35}s_{14}-2s_{25}^2+2s_{24}s_{25}-s_{13}s_{25}+2s_{25}s_{35}+s_{13}s_{24}-s_{24}s_{35}),\\
R_{13}^{(5)} &= -s_{13}(s_{14}s_{25}-s_{35}s_{14}-2s_{14}s_{24}-s_{13}s_{25}-2s_{24}s_{25}+s_{13}s_{24}+s_{24}s_{35}+2s_{24}^2),\\
R_{14}^{(1)} &= (s_{13}s_{25}-s_{13}s_{24}+s_{24}s_{35}+s_{35}s_{14}-s_{14}s_{25})(s_{24}+s_{35}-s_{25}),\\
R_{14}^{(2)} &= s_{14}(s_{14}s_{25}-s_{24}s_{35}+2s_{13}s_{35}-s_{35}s_{14}-s_{13}s_{25}+s_{13}s_{24}+2s_{25}s_{35}-2s_{35}^2),\\
R_{14}^{(3)} &= s_{14}(s_{35}s_{14}-s_{14}s_{25}-2s_{25}^2+2s_{24}s_{25}-s_{13}s_{25}+2s_{25}s_{35}+s_{13}s_{24}-s_{24}s_{35}),\\
R_{14}^{(4)} &= (s_{35}-s_{13}-s_{25})(s_{35}s_{14}-s_{24}s_{35}+s_{13}s_{24}-s_{13}s_{25}-s_{14}s_{25}),\\
R_{14}^{(5)} &= (s_{13}s_{24}^2-s_{24}s_{13}s_{35}+s_{24}s_{35}s_{14}-2s_{13}s_{24}s_{14}-s_{24}s_{13}s_{25}-s_{35}s_{24}^2\\
&\quad +s_{13}^2s_{24}-s_{24}s_{14}s_{25}+s_{13}s_{14}s_{25}-s_{13}s_{35}s_{14}-s_{13}^2s_{25}),\\
R_{24}^{(1)} &= s_{24}(s_{14}s_{25}-s_{24}s_{35}+2s_{13}s_{35}-s_{35}s_{14}-s_{13}s_{25}+s_{13}s_{24}+2s_{25}s_{35}-2s_{35}^2),\\
R_{24}^{(2)} &= (-s_{13}s_{25}+s_{13}s_{24}-s_{24}s_{35}-s_{35}s_{14}+s_{14}s_{25})(-s_{35}-s_{14}+s_{13}),\\
R_{24}^{(3)} &= -(-s_{14}^2s_{25}+s_{14}^2s_{35}-s_{14}s_{25}^2+s_{13}s_{14}s_{25}+s_{35}s_{14}s_{25}\\
&\quad +2s_{24}s_{14}s_{25}+s_{13}s_{24}s_{14}-s_{24}s_{35}s_{14}+s_{13}s_{25}^2+s_{24}s_{35}s_{25}-s_{24}s_{13}s_{25}),\\
R_{24}^{(4)} &= (s_{13}+s_{25}-s_{35})(s_{35}s_{14}+s_{13}s_{24}-s_{24}s_{35}-s_{14}s_{25}+s_{13}s_{25}),\\
R_{24}^{(5)} &= -s_{24}(-s_{24}s_{35}+s_{13}s_{24}-2s_{13}s_{14}-2s_{13}s_{35}+s_{13}s_{25}+2s_{13}^2-s_{14}s_{25}+s_{35}s_{14}),\\
R_{25}^{(1)} &= -(-s_{24}s_{35}^2+2s_{24}s_{35}s_{25}-s_{24}s_{13}s_{25}+s_{25}s_{13}s_{35}\\
&\quad +s_{24}s_{13}s_{35}+s_{13}s_{24}^2+s_{24}s_{14}s_{25}-s_{35}s_{24}^2+s_{14}s_{35}^2-s_{35}s_{14}s_{25}+s_{24}s_{35}s_{14}),\\
R_{25}^{(2)} &= (-s_{13}s_{25}+s_{13}s_{24}+s_{35}s_{14}-s_{24}s_{35}+s_{14}s_{25})(s_{35}+s_{14}-s_{13}),\\
R_{25}^{(3)} &= s_{25}(-2s_{14}^2-s_{14}s_{25}+2s_{13}s_{14}-s_{35}s_{14}+2s_{14}s_{24}+s_{13}s_{25}-s_{13}s_{24}+s_{24}s_{35}),\\
R_{25}^{(4)} &= -s_{25}(-s_{24}s_{35}+s_{13}s_{24}-2s_{13}s_{14}-2s_{13}s_{35}+s_{13}s_{25}+2s_{13}^2-s_{14}s_{25}+s_{35}s_{14}),\\
R_{25}^{(5)} &= (s_{13}+s_{24}-s_{14})(s_{35}s_{14}+s_{13}s_{24}-s_{24}s_{35}-s_{14}s_{25}+s_{13}s_{25}),\\
R_{35}^{(1)} &= -s_{35}(s_{14}s_{25}-s_{35}s_{14}-2s_{14}s_{24}-s_{13}s_{25}-2s_{24}s_{25}+s_{13}s_{24}+s_{24}s_{35}+2s_{24}^2),\\
R_{35}^{(2)} &= s_{35}(-2s_{14}^2-s_{14}s_{25}+2s_{13}s_{14}-s_{35}s_{14}+2s_{14}s_{24}+s_{13}s_{25}-s_{13}s_{24}+s_{24}s_{35}),\\
R_{35}^{(3)} &= (s_{14}+s_{25}-s_{24})(-s_{13}s_{25}+s_{13}s_{24}+s_{35}s_{14}-s_{24}s_{35}+s_{14}s_{25}),\\
R_{35}^{(4)} &= -(s_{14}s_{25}^2-s_{35}s_{14}s_{25}+s_{13}s_{14}s_{25}+s_{13}s_{35}s_{14}-s_{13}s_{25}^2-s_{13}^2s_{25}\\
&\quad +s_{24}s_{35}s_{25}+s_{24}s_{13}s_{25}+2s_{25}s_{13}s_{35}-s_{24}s_{13}s_{35}+s_{13}^2s_{24}),\\
R_{35}^{(5)} &= (s_{14}s_{25}-s_{35}s_{14}-s_{13}s_{25}+s_{24}s_{35}+s_{13}s_{24})(s_{13}+s_{24}-s_{14}),
\end{aligned}
\tag{10.76}
$$

and

$$
\begin{aligned}
\phi_{13} &= -3s_{14}^2 s_{25}^2 + 6s_{25}s_{14}(s_{14}-s_{24})s_{35} - s_{14}s_{25}(s_{24}-s_{25})s_{13} \\
&\quad - 3(s_{14}-s_{24})^2 s_{35}^2 + 2(s_{24}-s_{25})^2 s_{13}^2 + (s_{24}^2 - s_{14}s_{25} - s_{14}s_{24} - s_{24}s_{25})s_{13}s_{35}, \\
\phi_{14} &= -3s_{13}^2 s_{25}^2 + (s_{35}^2 - s_{25}s_{35} - s_{13}s_{35} - s_{13}s_{25})s_{14}s_{24} \\
&\quad + s_{13}s_{25}(s_{25}-s_{35})s_{14} + 6s_{13}s_{25}(s_{13}-s_{35})s_{24} - 3(s_{13}-s_{35})^2 s_{24}^2 + 2(s_{25}-s_{35})^2 s_{14}^2, \\
\phi_{24} &= -3s_{14}^2 s_{35}^2 - 6s_{35}s_{14}(s_{13}-s_{14})s_{25} - s_{35}s_{14}(s_{13}-s_{35})s_{24} \\
&\quad + (s_{13}^2 - s_{35}s_{14} - s_{13}s_{14} - s_{13}s_{35})s_{24}s_{25} + 2(s_{13}-s_{35})^2 s_{24}^2 - 3(s_{13}-s_{14})^2 s_{25}^2, \\
\phi_{25} &= -3s_{14}^2 s_{35}^2 - s_{35}s_{14}(s_{13}-s_{14})s_{25} - 6s_{35}s_{14}(s_{13}-s_{35})s_{24} \\
&\quad + (s_{13}^2 - s_{35}s_{14} - s_{13}s_{14} - s_{13}s_{35})s_{24}s_{25} - 3(s_{13}-s_{35})^2 s_{24}^2 + 2(s_{13}-s_{14})^2 s_{25}^2, \\
\phi_{35} &= -3s_{14}^2 s_{25}^2 + s_{25}s_{14}(s_{14}-s_{24})s_{35} - 6s_{14}s_{25}(s_{24}-s_{25})s_{13} + 2(s_{14}-s_{24})^2 s_{35}^2 \\
&\quad - 3(s_{24}-s_{25})^2 s_{13}^2 + (s_{24}^2 - s_{14}s_{25} - s_{14}s_{24} - s_{24}s_{25})s_{13}s_{35},
\end{aligned}
\tag{10.77}
$$

respectively.

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