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Differential reduction of generalized hypergeometric functions from Feynman diagrams: One-variable case

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Abstract

The differential-reduction algorithm, which allows one to express generalized hypergeometric functions with parameters of arbitrary values in terms of such functions with parameters whose values differ from the original ones by integers, is discussed in the context of evaluating Feynman diagrams. Where this is possible, we compare our results with those obtained using standard techniques. It is shown that the criterion of reducibility of multiloop Feynman integrals can be reformulated in terms of the criterion of reducibility of hypergeometric functions. The relation between the numbers of master integrals obtained by differential reduction and integration by parts is discussed.

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1 Introduction

It is commonly accepted that any multiloop and/or multileg Feynman diagram in covariant gauge within dimensional regularization [1] may be treated as a generalized hypergeometric function[1] [3]. Starting from its α representation, any Feynman diagram may be written in the form of a Mellin-Barnes integral [4,5],

$$\Phi(n) \sim \int_{-i\infty}^{+i\infty} \prod_{a,b,c} \frac{\Gamma(\sum_{i=1}^{m} A_{ai} z_i + B_a)}{\Gamma(\sum_{j=1}^{r} C_{bj} z_j + D_b)} dz_c Y^{z_c - 1} \; , \tag{1}$$

where Y_a are algebraic functions of external kinematic invariants and A, B, C, D are some matrices depending in a linear way on the dimension of space-time n, which is an arbitrary complex number, and the powers of the propagators.

By the application of Cauchy's theorem,[2] this integral can be rewritten as a linear combination of multiple series,[3]

$$\Phi(n, \vec{x}) \sim \sum_{k_1, \cdots, k_{r+m}=0}^{\infty} \prod_{a,b} \frac{\Gamma(\sum_{i=1}^{m} \tilde{A}_{ai} k_i + \tilde{B}_a)}{\Gamma(\sum_{j=1}^{r} \tilde{C}_{bj} k_j + \tilde{D}_b)} x_1^{k_1} \cdots x_{r+m}^{k_{r+m}} \; . \tag{2}$$

For real diagrams, some of the variables x_k may be complex number. We call this type of variable a "hidden variable" and the corresponding index of summation a "hidden index of summation." In all existing examples, the representation of Eq. (2) belongs to a Horn-type hypergeometric series [8] if the hidden index of summation is considered as an independent variable.

For the reader's convenience, we recall that the multiple series $\sum_{\vec{m}=0}^{\infty} C(\vec{m}) \vec{x}^{\vec{m}}$ is called *Horn-type hypergeometric* if, for each $i = 1, \ldots, r$, the ratio $C(\vec{m} + \vec{e}_i)/C(\vec{m})$ is a rational function in the index of summation $(m_1, \cdots, m_r)$ [8,9]. The coefficients of such a series have the general form

$$C(\vec{m}) = \prod_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \frac{\prod_{j=1}^{N} \Gamma(\mu_j(\vec{m}) + \gamma_j)}{\prod_{k=1}^{M} \Gamma(\nu_k(\vec{m}) + \delta_k)} \; , \tag{3}$$

where $N, M \geq 0$, $\lambda_j, \delta_j, \gamma_j$ are arbitrary complex numbers, $\mu_j, \nu_k : Z^r \to Z$ are arbitrary integer-valued linear maps, and R is an arbitrary rational function [8,10].

However, to our knowledge, the proof that any Feynman diagram can be described by a Horn-type series does not exist. There is another way to proof this statement. The classical α representation of a Feynman diagram is a particular case of the generalized Euler integral representation of the Gel'fand-Kapranov-Zelevinski system [8] that is related to the Horn-type series representation. We call the Feynman diagram representation of Eq. (1) or the equivalent representation of Eq. (2) a hypergeometric representation [8].

For Horn-type hypergeometric functions, there are so-called step-up (step-down) operators H_λ^+ (H_λ^-) [9,11]. These are differential operators which, upon application

[1] This statement is also valid for phase-space integrals (see, e.g., Ref. [2]).

[2] This is true if all arguments of the Γ functions are different. Otherwise, an additional regularization for each propagator (or the introduction of extra masses) is necessary. See, e.g., Ref. [6].

[3] One of the first examples of this type of representation for a Feynman diagram was given in Ref. [7].

to a hypergeometric function S_λ, shift the value of one its upper (lower) parameters by unity, as $H^+_\lambda S_\lambda = S_{\lambda+1}$ ($H^-_\lambda S_\lambda = S_{\lambda-1}$). Takayama [12] proposed an algorithm that allows one to construct inverse differential operators, the step-down (step-up) operators B^-_λ (B^+_λ), starting from direct operators and systems of differential equations for hypergeometric functions. These operators satisfy the relations $B^-_\lambda S_{\lambda+1} \to S_\lambda$ ($B^+_\lambda S_\lambda \to S_{\lambda+1}$). Takayama pointed out [12] that inverse operators are uniquely defined for any hypergeometric function with an irreducible monodromy group, which implies that the parameters and the differences between upper and lower parameters are not integer. By the action of such differential operators on a hypergeometric function, the value of any parameter can be shifted by an arbitrary integer. We call this procedure of applying differential operators to shift the parameters by integers *differential reduction*.[4] An important step of Takayama's algorithm is the construction of a differential Gröbner basis for the system of differential equations for hypergeometric functions.[5]

It is quite surprising that the technique for the reduction of Feynman diagrams advocated here, namely to split a given Feynman diagram into a linear combination of Horn-type hypergeometric functions with rational coefficients and to subsequently apply differential reduction, has never been elaborated and that its interrelation with the well-known integration-by-parts (IBP) technique [15] has never been discussed.

The aim of this paper is to demonstrate how the differential-reduction algorithm may be successfully applied to evaluate Feynman diagrams. For simplicity, we consider here only the particular case of Horn-type multiple hypergeometric functions, i.e. the functions ${}_{p+1}F_p$, and some Feynman diagrams with arbitrary powers of propagators, which are expressible in terms of these functions.

The structure of this paper is as follows. In Section 2, we discuss the differential-reduction algorithm for generalized hypergeometric functions ${}_{p+1}F_p$. This is a crucial step towards proving theorems on the construction of all-order ε expansions presented in Refs. [16–19]. Section 3 illustrates the application of differential reduction to several Feynman diagrams of phenomenological interest. In Section 4, we discuss how the counting of master integrals in the differential-reduction approach is related to that in the IBP technique. The results of our analysis are briefly summarized in Section 5. In Appendix A, we describe interrelations between a set of basis functions generated by the differential-reduction algorithm and a set of hypergeometric functions whose higher-order ε expansions were constructed in Ref. [20].

[4] There are various publications on contiguous relations for hypergeometric functions, i.e. algebraic relations between hypergeometric functions of several variables with shifted values of parameters, starting from the classical paper by Gauss [13]. To our knowledge, a closed algorithm for the algebraic reduction of Horn-type hypergeometric functions of several variables, i.e. an algorithm for solving these algebraic relations, does not exist.

[5] The idea of using the differential-Gröbner-basis technique directly for the reduction of off-shell Feynman diagrams, without spitting them into linear combinations of hypergeometric functions, was proposed by Tarasov [14].

2 Differential-reduction algorithm for the generalized hypergeometric function ${}_{p+1}F_p$

2.1 Notation

Let us consider the generalized hypergeometric function ${}_pF_q(a;b;z)$, defined by a series about $z=0$ as

$$ {}_pF_q(\vec{a};\vec{b};z) \equiv {}_pF_q\left(\left.\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} , \tag{4} $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. The sets $\vec{a} = (a_1, \cdots, a_p)$ and $\vec{b} = (b_1, \cdots, b_q)$ are called the upper and lower parameters of the hypergeometric function, respectively. In terms of the differential operator θ,

$$ \theta = z\frac{d}{dz} , \tag{5} $$

the differential equation for the hypergeometric function ${}_pF_q$ can be written as

$$ \left[z\prod_{i=1}^{p}(\theta + a_i) - \theta\prod_{i=1}^{q}(\theta + b_i - 1) \right] {}_pF_q(\vec{a};\vec{b};z) = 0. \tag{6} $$

Hypergeometric functions which differ by ± 1 in the value of one of their parameters are called contiguous, and linear relations between contiguous functions and their derivatives are called contiguous relations.[6] In particular, the following differential identities between contiguous functions are universal [11]:

$$ \begin{aligned} {}_pF_q(a_1+1,\vec{a};\vec{b};z) &= B^+_{a_1}\,{}_pF_q(a_1,\vec{a};\vec{b};z) = \frac{1}{a_1}\left(\theta + a_1\right){}_pF_q(a_1,\vec{a};\vec{b};z) , \\ {}_pF_q(\vec{a};b_1-1,\vec{b};z) &= H^-_{b_1}\,{}_pF_q(\vec{a};b_1,\vec{b};z) = \frac{1}{b_1-1}\left(\theta + b_1 - 1\right){}_pF_q(\vec{a};b_1,\vec{b};z) , \end{aligned} \tag{7} $$

which directly follow from the series representation of Eq. (4). The operators $B^+_{a_1}$ ($H^-_{b_1}$) are the step-up (step-down) operators for upper (lower) parameters of hypergeometric functions.

2.2 Non-exceptional values of parameters

In Ref. [12], it was shown that, for given step-up (step-down) operators, inverse step-down (step-up) operators, which are uniquely defined modulo Eq. (6), can be constructed. This type of operators were explicitly constructed for the hypergeometric function ${}_{p+1}F_p$ by Takayama in Ref. [22]. For completeness, we reproduce

[6] A full set of contiguous relations for an arbitrary hypergeometric function ${}_pF_q$ was considered by Rainville [21]. For ${}_pF_q$, there are $2(p+q)$ contiguous functions.

his result here:

$$
\begin{aligned}
{}_{p+1}F_p(a_i-1,\vec{a};\vec{b};z) &= B^-_{a_i}\,{}_{p+1}F_p(a_i,\vec{a};\vec{b};z)\ ,\\
B^-_{a_i} &= -\frac{a_i}{c_i}\left[t_i(\theta)-z\prod_{j\neq i}(\theta+a_j)\right]_{a_i\to a_i-1}\ ,\\
c_i &= -a_i\prod_{j=1}^{p}(b_j-1-a_i)\ ,\\
t_i(x) &= \frac{x\prod_{j=1}^{p}(x+b_j-1)-c_i}{x+a_i}\ , \qquad (8)\\
{}_{p+1}F_p(\vec{a};b_i+1,\vec{b};z) &= H^+_{b_i}\,{}_{p+1}F_p(\vec{a};b_1,\vec{b};z)\ ,\\
H^+_{a_i} &= \frac{b_i-1}{d_i}\left[\frac{\theta}{z}\prod_{j\neq i}(\theta+b_j-1)-s_i(\theta)\right]_{b_i\to b_i+1}\ ,\\
d_i &= \prod_{j=1}^{p+1}(1+a_j-b_i)\ ,\\
s_i(x) &= \frac{\prod_{j=1}^{p+1}(x+a_j)-d_i}{x+b_i-1}\ , \qquad (9)
\end{aligned}
$$

where $|_{a\to a+1}$ means substitution of a by $a+1$. The functions $t_i(x)$ and $s_i(x)$ are polynomials in x, so that Eqs. (8) and (9) are polynomials in the derivative θ. Let us introduce the symmetric polynomial $P^{(p)}_j(\{r_k\})$ as follows:

$$
\prod_{k=1}^{p}(z+r_k)=\sum_{j=0}^{p}P^{(p)}_{p-j}(\{r_k\})z^j=\sum_{j=0}^{p}P^{(p)}_{j}(\{r_k\})z^{p-j}\ , \qquad (10)
$$

so that

$$
\begin{aligned}
&P^{(p)}_0(\{r_k\})=1\ ,\\
&P^{(p)}_j(\{r_k\})=\sum_{i_1,\cdots,i_r=1}^{p}\ \prod_{i_1<\cdots<i_j} r_{i_1}\cdots r_{i_j}\ , \qquad j=1,\cdots,p\ . \qquad (11)
\end{aligned}
$$

For example, we have $P^{(p)}_1(\{r_k\})=\sum_{j=1}^{p}r_j$ and $P^{(p)}_p(\{r_k\})=\prod_{j=1}^{p}r_j$. Then, we may write

$$
t_i(x)=\sum_{j=0}^{p}P^{(p)}_{p-j}(\{b_r-1\})\frac{x^{j+1}-(-a_i)^{j+1}}{x+a_i}=\sum_{j=0}^{p}P^{(p)}_{p-j}(\{b_r-1\})\sum_{k=0}^{j}x^{j-k}(-a_i)^k\ . \qquad (12)
$$

A similar consideration is valid also for the last relation in Eq. (9):

$$
s_i(x)=\sum_{j=0}^{p+1}P^{(p+1)}_{p+1-j}(\{a_r\})\frac{x^j-(1-b_i)^j}{x-(1-b_i)}=\sum_{j=0}^{p}P^{(p+1)}_{p-j}(\{a_r\})\sum_{k=0}^{j}x^{j-k}(1-b_i)^k \qquad (13)
$$

The differential reduction has the form of a product of several differential step-up and step-down operators, $H^{\pm}_{b_k}$ and $B^{\pm}_{a_k}$, respectively:[7]

$$F(\vec{a}+\vec{m};\vec{b}+\vec{n};z) = \left(H^{\pm}_{\{a\}}\right)^{\sum_i m_i}\left(B^{\pm}_{\{b\}}\right)^{\sum_j n_j} F(\vec{a};\vec{b};z)\;, \tag{14}$$

so that the maximal power of θ in this expression is equal to $r \equiv \sum_i m_i + \sum_j n_j$. In a symbolic form, this may be written as

$$R(a_i,b_j,z)F(\vec{a}+\vec{m};\vec{b}+\vec{n};z) = \left[S_1(a_i,b_j,z)\theta^r + \cdots + S_{r+1}(a_i,b_j,z)\right] F(\vec{a};\vec{b};z)\;, \tag{15}$$

where R and $\{S_j\}$ are some polynomials. Since the hypergeometric function ${}_{p+1}F_p(\vec{a};\vec{b};z)$ satisfies the following differential equation of order $p+1$ [see Eq. (6)]:

$$\begin{aligned}&(1-z)\theta^{p+1}\,{}_{p+1}F_p(\vec{a};\vec{b};z)\\ &= \left\{\sum_{r=1}^{p}\left[zP^{(p+1)}_{p+1-r}(\{a_j\}) - P^{(p)}_{p+1-r}(\{b_j-1\})\right]\theta^r + z\prod_{k=1}^{p+1}a_k\right\}{}_{p+1}F_p(\vec{a};\vec{b};z)\;,\end{aligned} \tag{16}$$

it is possible to express all terms containing higher powers of the operator θ, θ^k with $k \geq p+1$, as a linear combination of rational functions of z depending parametrically on a and b multiplied by lower powers of θ, θ^j with $j \leq p$. In this way, any function ${}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z)$, where $\vec{m}$ and $\vec{k}$ are sets of integers, is expressible in terms of the basic function and its first p derivatives as

$$\begin{aligned}S(a_i,b_j,z)\,{}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z) &= \left\{R_1(a_i,b_j,z)\theta^p + R_2(a_i,b_j,z)\theta^{p-1}\right.\\ &\left. + \cdots + R_p(a_i,b_j,z)\theta + R_{p+1}(a_i,b_j,z)\right\}{}_{p+1}F_p(\vec{a};\vec{b};z)\;,\end{aligned} \tag{17}$$

where S and R_i are polynomials in the parameters $\{a_i\}$ and $\{b_j\}$ and the argument z.

From Eq. (8) it follows that, if one of the upper parameters a_j is equal to unity, then the application of the step-down operator $B^-_{a_j}$ to the hypergeometric function ${}_{p+1}F_p$ produces unity, $B^-_1\,{}_{p+1}F_p(1,\vec{a};\vec{b};z) = 1$. Taking into account the explicit form of the step-down operator B^-_1,

$$B^-_1 = \frac{1}{\prod_{k=1}^{p}(b_k-1)}\left[\prod_{j=1}^{p}(b_j-1) + \sum_{j=1}^{p}P^{(p)}_{p-j}(\{b_k-1\})\theta^j - z\prod_{j=1}^{p}(\theta+a_j)\right]\;, \tag{18}$$

we obtain the differential identity

$$\begin{aligned}&\left\{\prod_{j=1}^{p}(b_j-1) - z\prod_{j=1}^{p}a_j + (1-z)\theta^p\right\}{}_{p+1}F_p(1,\vec{a};\vec{b};z)\\ &+\left\{\sum_{j=1}^{p-1}\left[P^{(p)}_{p-j}(\{b_k-1\}) - zP^{(p)}_{p-j}(\{a_k\})\right]\theta^j\right\}{}_{p+1}F_p(1,\vec{a};\vec{b};z) = \prod_{k=1}^{p}(b_k-1)\;.\end{aligned} \tag{19}$$

[7] Due to the relation $\theta\,{}_pF_q(\vec{a};\vec{b};z) = z\prod_{i=1}^{p}a_i/(\prod_{j=1}^{q}b_j)\,{}_pF_q(\vec{a}+1;\vec{b}+1;z)$, not all step-up operators are independent. In fact, $(\prod_{j=1}^{p}H^+_{b_j})(\prod_{k=1}^{p+1}B^+_{a_k})F(\vec{a};\vec{b};z) = (\prod_{k=1}^{p+1}B^+_{a_k})(\prod_{j=1}^{p}H^+_{b_j})F(\vec{a};\vec{b};z) = \prod_{j=1}^{p}b_j/(\prod_{k=1}^{p+1}a_k)(d/dz)F(\vec{a};\vec{b};z)$.

The case where two or more upper parameters are equal to unity, e.g. $a_1 = a_2 = 1$, does not generate any new identities. As a consequence, if there is a subset $\vec{l}$ of positive integers in the set of upper parameters, the reduction procedure has the modified form:

$$\tilde{P}(a_i, b_j, z)\, {}_{p+1}F_p(\vec{l}, \vec{a} + \vec{m}; \vec{b} + \vec{k}; z) = \tilde{R}_1(a_i, b_j, z) + \Big\{ \tilde{R}_2(a_i, b_j, z)\theta^{p-1} + \cdots + \tilde{R}_p(a_i, b_j, z)\theta + \tilde{R}_{p+1}(a_i, b_j, z) \Big\}\, {}_{p+1}F_p(\vec{1}, \vec{a}; \vec{b}; z)\ . \tag{20}$$

Let us write explicit expressions for the inverse operators for several hypergeometric functions. For the Gauss hypergeometric function ${}_2F_1$, we have:

$$ {}_2F_1\left(\begin{matrix} a_1 - 1, a_2 \\ b_1 \end{matrix} \middle| z \right) = \frac{1}{b_1 - a_1} \left[(1-z)\theta + b_1 - a_1 - a_2 z \right] {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| z \right)\ , \tag{21}$$

$$ {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 + 1 \end{matrix} \middle| z \right) = \frac{b_1}{(b_1 - a_1)(b_1 - a_2)} \left[(1-z)\frac{d}{dz} + b_1 - a_1 - a_2 \right] {}_2F_1\left(\begin{matrix} a_1, a_2 \\ b_1 \end{matrix} \middle| z \right)\ .$$

For the hypergeometric function ${}_3F_2$, the inverse differential operators read:

$$\begin{aligned} & {}_3F_2\left(\begin{matrix} a_1 - 1, a_2, a_3 \\ b_1, b_2 \end{matrix} \middle| z \right) (b_1 - a_1)(b_2 - a_1) \\ & = \left\{ (1-z)\theta^2 + \left[(b_1 + b_2 - 1 - a_1) - z(a_2 + a_3) \right] \theta + (b_1 - a_1)(b_2 - a_1) - z a_2 a_3 \right\} \\ & \quad \times {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \middle| z \right)\ , \\ & {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1 + 1, b_2 \end{matrix} \middle| z \right) (a_1 - b_1)(a_2 - b_1)(a_3 - b_1) \\ & = b_1 \left\{ \frac{1-z}{z}\theta^2 + \left[\frac{b_2 - 1}{z} - (a_1 + a_2 + a_3 - b_1) \right] \theta \right. \\ & \quad \left. - \frac{1}{b_1} \left[a_1 a_2 a_3 - (a_1 - b_1)(a_2 - b_1)(a_3 - b_1) \right] \right\} {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \middle| z \right)\ . \end{aligned} \tag{22}$$

For the hypergeometric function ${}_4F_3$, the differential operators read:

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix}a_1-1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix}\middle|z\right)(b_1-a_1)(b_2-a_1)(b_3-a_1)\\
&=\left\{(1-z)\theta^3+\left[(b_1+b_2+b_3-2-a_1)-z(a_2+a_3+a_4)\right]\theta^2\right.\\
&\quad+\left[b_1b_2+b_1b_3+b_2b_3+a_1^2+(a_1+1)\left(1-\sum_{j=1}^{3}b_j\right)-z(a_2a_3+a_2a_4+a_3a_4)\right]\theta\\
&\quad\left.+(b_1-a_1)(b_2-a_1)(b_3-a_1)-za_2a_3a_4\right\}{}_4F_3\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix}\middle|z\right)\ ,\\
&{}_4F_3\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1+1,b_2,b_3\end{matrix}\middle|z\right)(a_1-b_1)(a_2-b_1)(a_3-b_1)(a_4-b_1)\\
&=b_1\left\{\frac{1-z}{z}\theta^3+\left[\frac{b_2+b_3-2}{z}-(a_1+a_2+a_3+a_4-b_1)\right]\theta^2\right.\\
&\quad+\left[\frac{(b_2-1)(b_3-1)}{z}-(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)+b_1\left(\sum_{j=1}^{4}a_j-b_1\right)\right]\theta\\
&\quad\left.-\frac{1}{b_1}\left[a_1a_2a_3a_4-(a_1-b_1)(a_2-b_1)(a_3-b_1)(a_4-b_1)\right]\right\}{}_4F_3\left(\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix}\middle|z\right)\ . \qquad (23)
\end{aligned}
$$

2.3 Differential-reduction algorithms for special values of parameters

A special consideration is necessary when $b_i = a_i + 1$. In this case, the expressions in Eqs. (8) and (9) are equal to zero. Let us start with the case when all a_i are different. In this case, the following relation should be applied (see Eq. (15) in Chapter 5 of Ref. [21], Chapter 7.2 in Ref. [23], and Ref. [24]):

$$
\begin{aligned}
(a-b)\,{}_pF_q\left(\begin{matrix}a,b,\cdots\\ a+1,b+1,\cdots\end{matrix}\middle|z\right) &= a\,{}_{p-1}F_{q-1}\left(\begin{matrix}b,\cdots\\ b+1,\cdots\end{matrix}\middle|z\right)\\
&\quad - b\,{}_{p-1}F_{q-1}\left(\begin{matrix}a,\cdots\\ a+1,\cdots\end{matrix}\middle|z\right)\ . \qquad (24)
\end{aligned}
$$

Repeating this procedure several times, we are able to split any original hypergeometric function with several parameters having unit difference into a set of hypergeometric functions with only one kind of parameters having unit difference (for the particular cases, see Eqs. (7.2.3.21)–(7.2.3.23) in Ref. [23]):

$$
\begin{aligned}
&{}_pF_q\left(\begin{matrix}\{a_1\}_{r_1},\{a_2\}_{r_2},\cdots,\{a_m\}_{r_m},c_1,\cdots,c_k\\ \{1+a_1\}_{r_1},\{1+a_2\}_{r_2},\cdots,\{1+a_m\}_{r_m},b_1,\cdots,b_l\end{matrix}\middle|z\right)\\
&\to\sum_{i=1}^{m}{}_{p-R+r_i}F_{q-R+r_i}\left(\begin{matrix}\{a_j\}_{r_i},c_1,\cdots,c_k\\ \{1+a_j\}_{r_i},b_1,\cdots,b_l\end{matrix}\middle|z\right)\ , \qquad (25)
\end{aligned}
$$

where $\{a_j\}_{r_i}$ denotes r_j repetitions of a_j in the argument list,

$$
R=\sum_{j=1}^{m}r_j\ ,\qquad c_i\neq a_i\ ,\qquad b_i\neq 1+a_i\ ,\qquad c_j\neq 1+b_j\ . \qquad (26)
$$

For a special set of parameters, Eqs. (7.2.3.21) and (7.2.3.23) in Ref. [23] are useful:

$$
{}_pF_q\left(\left.\begin{matrix}\{a\}_{p-2},\rho,\sigma\\ \{b\}_{q-2},\rho+n,\sigma+1\end{matrix}\right|z\right)=\frac{(\rho)_n}{(\rho-\sigma)_n}{}_{p-1}F_{q-1}\left(\left.\begin{matrix}\{a\}_{p-2},\sigma\\ \{b\}_{q-2},\sigma+1\end{matrix}\right|z\right)
$$

$$
-\frac{(\rho)_n\sigma}{(\rho-\sigma)_n}\Sigma_{k=1}^{n}\frac{(\rho-\sigma-1)_k}{(\rho)_k}{}_{p-1}F_{q-1}\left(\left.\begin{matrix}\{a\}_{p-2},\rho\\ \{b\}_{q-2},\rho+k\end{matrix}\right|z\right)\ , \tag{27}
$$

$$
{}_pF_q\left(\left.\begin{matrix}\{a\}_{p-n},\sigma_1,\cdots,\sigma_n\\ \{b\}_{q-n},\sigma_1+m_1,\cdots,\sigma_n+m_n\end{matrix}\right|z\right)=\prod_{j=1}^{n}\frac{(\sigma_j)_{m_j}}{(m_j-1)!}\Sigma_{k=1}^{n}\Sigma_{j_1=0}^{m_1-1}\cdots\Sigma_{j_n=0}^{m_n-1}\frac{1}{\sigma_k+j_k}
$$

$$
\times\prod_{l=1}^{n}\frac{(1-m_l)_{j_l}}{j_l!}\prod_{i=1;i\neq k}^{n}\frac{1}{\sigma_i+j_i-\sigma_k-j_k}{}_{p-n+1}F_{q-n+1}\left(\left.\begin{matrix}\{a\}_{p-n},\sigma_k+j_k\\ \{b\}_{q-n},\sigma_k+j_k+1\end{matrix}\right|z\right)\ . \tag{28}
$$

In Eq. (28), m_n are integers, all σ_i are different, and, if $\sigma_i-\sigma_k=N$ with $N=1,2,\cdots$, then $m_k<N$.

Let us return to the last expression in Eq. (25) and rewrite it as

$$
{}_{p+1}F_p\left(\left.\begin{matrix}\{a+m\}_r,a_1+k_1,\cdots,a_{p+1-r}+k_{p+1-r}\\ \{1+a+m\}_q,b_1+l_1,\cdots,b_{p-q}+l_{p-q}\end{matrix}\right|z\right)\ , \tag{29}
$$

where m, $\{k_r\}$, and $\{l_j\}$ are integers and a, $\{a_k\}$, and $\{b_j\}$ are parameters of the basis function. Using the reduction procedure described in Sec. 2.2, we may convert this function as

$$
{}_{p+1}F_p\left(\left.\begin{matrix}\{a+m\}_r,\{a_j+k_j\}_{p+1-r}\\ \{1+a+m\}_q,\{b_k+l_k\}_{p-q}\end{matrix}\right|z\right)\to{}_{p+1}F_p\left(\left.\begin{matrix}\{a+m\}_r,\{a_j+m\}_{p+1-r}\\ \{1+a+m\}_q,\{b_k+m\}_{p-q}\end{matrix}\right|z\right)\ , \tag{30}
$$

and then apply the differential relation (see Eq. (7.2.3.47) in Ref. [23])[8]

$$
{}_pF_q\left(\left.\begin{matrix}\{a_i+m\}_p\\ \{b_k+m\}_q\end{matrix}\right|z\right)=\frac{\prod_{k=1}^{q}\{(b_k)_m\}}{\prod_{j=1}^{p}\{(a_j)_m\}}\left(\frac{d}{dz}\right)^m{}_pF_q\left(\left.\begin{matrix}\{a_i\}_p\\ \{b_k\}_q\end{matrix}\right|z\right)\ , \tag{31}
$$

where the derivative $d/(dz)$ could be rewritten in terms of θ, with the help of

$$
\left(\frac{d}{dz}\right)^m=\left(\frac{1}{z}\right)^m\theta(\theta-1)\cdots(\theta-m+1)\ . \tag{32}
$$

If $m\geq p$, Eq. (31) can be converted to a differential identity of order $p-1$. Putting everything together, we obtain

$$
P(a_k,b_j,z){}_{p+1}F_p\left(\left.\begin{matrix}\{a+m\}_r,\{a_j+k_j\}_{p+1-r}\\ \{1+a+m\}_q,\{b_k+l_k\}_{p-q}\end{matrix}\right|z\right)
$$

$$
=\left[Q_1(a_k,b_j,z)\theta^p+\cdots+Q_{p+1}(a_k,b_j,z)\right]{}_{p+1}F_p\left(\left.\begin{matrix}\{a\}_r,a_1,\cdots,a_{p-r}\\ \{1+a\}_q,b_1,\cdots,b_{p-q-1}\end{matrix}\right|z\right)\ . \tag{33}
$$

[8] Another useful relation is Eq. (7.2.3.50) in Ref. [23]:

$$
\left(\frac{d}{dz}\right)^n\left[z^{\sigma+n-1}{}_pF_q\left(\left.\begin{matrix}\{a_i\}_{p-1},\sigma\\ \{b_k\}_{q-1}\end{matrix}\right|z\right)\right]=(\sigma)_n z^{\sigma-1}{}_pF_q\left(\left.\begin{matrix}\{a_i\}_{p-1},\sigma+n\\ \{b_k\}_q\end{matrix}\right|z\right)\ .
$$

We remark that Eq. (33) can be further simplified by using Eq. (7), which can be written as

$$\theta^k {}_{p+1}F_p\left(\left.\begin{matrix}A,\vec{a}\\1+A,\vec{b}\end{matrix}\right|z\right) = (-A)^k {}_{p+1}F_p\left(\left.\begin{matrix}A,\vec{a}\\1+A,\vec{b}\end{matrix}\right|z\right) - \sum_{j=0}^{k-1}(-A)^{k-j}\theta^j {}_pF_{p-1}\left(\left.\begin{matrix}\vec{a}\\\vec{b}\end{matrix}\right|z\right) \ . \quad (34)$$

Recursive application of this expression allows us to reduce higher powers of derivatives in Eq. (33) as

$$\left(\frac{\theta}{A}\right)^q {}_{p+1}F_p\left(\left.\begin{matrix}\{A\}_r,\vec{a}\\\{1+A\}_r,\vec{b}\end{matrix}\right|z\right) = \sum_{j=0}^{q}(-1)^{(j+q)}\binom{q}{j} {}_{p+1-j}F_{p-j}\left(\left.\begin{matrix}\{A\}_{r-j},\vec{a}\\\{1+A\}_{r-j},\vec{b}\end{matrix}\right|z\right) \ , \quad (35)$$

where $q \leq r$.

For a particular set of parameters, a further simplification of Eq. (33) can be achieved. For example, for $a = 1$, we have

$${}_pF_q\left(\left.\begin{matrix}1,\{a_i\}_{p-1}\\2,\{b_k\}_{q-1}\end{matrix}\right|z\right) = \frac{1}{z}\frac{\prod_{l=1}^{q-1}(b_l-1)}{\prod_{j=1}^{p-1}(a_j-1)}\left[{}_{p-1}F_{q-1}\left(\left.\begin{matrix}\{a_i-1\}_{p-1}\\\{b_k-1\}_{q-1}\end{matrix}\right|z\right)-1\right] \ , \quad (36)$$

where $a_j, b_k \neq 1$.

Further useful relations for particular values of parameters of the basis function were derived in Ref. [24], namely

$${}_{p+1}F_p\left(\left.\begin{matrix}\{a\}_p,b\\\{1+a\}_p\end{matrix}\right|z\right) = -\frac{(-a)^p}{\Gamma(p)}\int_0^1 dt\frac{t^{a-1}}{(1-zt)^b}\ln^{p-1}t \ , \quad (37)$$

and, for $b = 1$,

$${}_{p+1}F_p\left(\left.\begin{matrix}1,a,\cdots,a\\a+1,\cdots,a+1\end{matrix}\right|z\right) = a^p\Phi(z,p,a) \ , \quad (38)$$

where $\Phi(z,p,a)$ is the Lerch function defined as [9]

$$\Phi(z,p,a) = \sum_{k=0}^{\infty}\frac{z^k}{(a+k)^p} = \frac{1}{\Gamma(p)}\int_0^{\infty}\frac{e^{-(a-1)t}}{e^t-z}t^{p-1}dt \ , \quad (39)$$

so that

$$\mathrm{Li}_n(z) = z\Phi(z,n,1) \ . \quad (40)$$

2.4 Criteria of reducibility of hypergeometric functions

In this section, we formulate the criteria of reducibility of the hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$, i.e. we state under which conditions the hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ and its derivatives are expressible in terms of hypergeometric functions of lower order and/or with lower derivatives.

We call the result derived by Karlsson [25] the first criterion of reducibility of the hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ to hypergeometric functions of lower order,

$$\begin{aligned}&{}_pF_q\left(\left.\begin{matrix}b_1+m_1,\cdots,b_n+m_n,a_{n+1},\cdots,a_p\\b_1,\cdots,b_n,b_{n+1},\cdots,b_q\end{matrix}\right|z\right)\\ &= \sum_{j_1=0}^{m_1}\cdots\sum_{j_n=0}^{m_n}A(j_1,\cdots j_n)z^{J_n}\,{}_{p-n}F_{q-n}\left(\left.\begin{matrix}a_{n+1}+J_n,\cdots a_p+J_n\\b_{n+1}+J_n,\cdots,b_q+J_n\end{matrix}\right|z\right) \ , \quad (41)\end{aligned}$$

where m_j are positive integers, $J_n = j_1 + \cdots + j_n$, and

$$A(j_1, \cdots j_n) = \binom{m_1}{j_1} \cdots \binom{m_n}{j_n} \frac{(b_2+m_2)_{J_1}(b_3+m_3)_{J_2} \cdots (b_n+m_n)_{J_{n-1}}(a_{n+1})_{J_n} \cdots (a_p)_{J_n}}{(b_1)_{J_1}(b_2)_{J_2} \cdots (b_n)_{J_n}(b_{n+1})_{J_n} \cdots (b_q)_{J_n}} . \tag{42}$$

In particular, we have

$${}_pF_q\left(\begin{matrix} b_1+m_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots b_q \end{matrix}\middle| z\right) = \sum_{j=0}^{m_1} z^j \binom{m_1}{j} \frac{(a_2)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} {}_{p-1}F_{q-1}\left(\begin{matrix} a_2+j, \cdots a_p+j \\ b_2+j, \cdots b_q+j \end{matrix}\middle| z\right). \tag{43}$$

In words, the first criterion of reducibility of the hypergeometric function reads:
Criterion I
The hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ which has pairs of parameters satisfying $a_i = b_i + m_i$ with m_i being positive integers is expressible in terms of functions of lower order according to Eq. (41).

Equations (24) and (25) yield the second criterion of reducibility of the hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ to functions of lower order. In its explicit form, it was derived in Ref. [24] (see Eqs. (18)–(20) in Ref. [24]). Assuming that (i) $\{a_1, \cdots a_n\}$ are different and (ii) if $a_r - a_i = N$ with $N = 1, 2, \ldots$, then $m_i < N$, we have (see also Eqs. (7.2.3.21) and (7.2.3.23) in Ref. [23])

$$\begin{aligned} &{}_pF_q\left(\begin{matrix} a_1, \cdots a_n, a_{n+1} \cdots a_p \\ a_1+1+m_1, \cdots, a_n+1+m_n, b_{n+1}, \cdots, b_q \end{matrix}\middle| z\right) \prod_{r=1}^{n} \frac{1}{(a_r)_{m_r+1}} \\ &= \sum_{i=1}^{n} \sum_{j=0}^{m_i} \frac{(-m_i)_j}{j!(a_i+j)m_i!} \left(\prod_{r=1, r\neq i}^{n} \frac{1}{(a_r - a_i - j)_{m_r+1}} \right) \\ &\quad \times {}_{p-n+1}F_{q-n+1}\left(\begin{matrix} a_i+j, a_{n+1}, \cdots, a_p \\ a_i+1+j, b_{n+1}, \cdots, b_q \end{matrix}\middle| z\right) , \end{aligned} \tag{44}$$

so that we may formulate it as:
Criterion II
The hypergeometric function ${}_pF_q(\vec{a};\vec{b};z)$ which has two or more pairs of parameters satisfying $b_i = a_i + m_i + 1$ with m_i being positive integers is expressible in terms of functions of lower order if additional conditions on the parameters a_i (see Eq. (44)) are satisfied.

Equations (34) and (35) are considered as the third criterion of reducibility:
Criterion III
The result of the differential reduction of a hypergeometric function of the type ${}_{p+1}F_p(\vec{A}+\vec{m}, \vec{a}+\vec{k}; \vec{1}+\vec{A}+\vec{m}, \vec{b}+\vec{l}; z)$, where $\vec{m}$, $\vec{k}$, and $\vec{l}$ are sets of integers, is expressible in terms of the function ${}_{p+1}F_p(\vec{A}, \vec{a}; \vec{1}+\vec{A}, \vec{b}; z)$ and hypergeometric functions of lower order and their derivatives.

Equation (19) constitutes the fourth criterion of reducibility:
Criterion IV
If one of the upper parameters of a hypergeometric function is an integer,

the result of the differential reduction of this hypergeometric function has one less derivative and is described by Eq. (20).

Criteria I–IV of reducibility of hypergeometric functions are much simpler than the ordinary criterion of reducibility of Feynman diagrams [26].

2.5 All-order ε expansions of hypergeometric functions

Recently, several theorems on the structure of the coefficients of all-order ε expansions of hypergeometric functions about integer and/or rational values of their parameters have been proven [16–19,27–30]. For a recent review, see Ref. [30].

In this paper, we mainly consider hypergeometric functions of the type of Eq. (109). There still does not exist a rigorous mathematical proof regarding the structure of the coefficients of the all-order ε expansions of hypergeometric functions of this type beyond the Gauss hypergeometric functions ${}_2F_1$ [16,19]. Through functions of weight 4, the coefficients of the ε expansions were constructed in Ref. [20,31], as discussed in details in Appendix A), which is sufficient for next-to-next-to-leading-order (two-loop) calculations. Recently, more coefficients of the ε expansions of the Clausen functions ${}_3F_2$ have been evaluated in Ref. [32].

2.6 Reduction at $z = 1$ and construction of ε expansion

The value $z = 1$ is a particular case of a "hidden" variable. It is evident that the application of Eq. (6) to the r.h.s. of Eq. (17) gives rise to the generation of factors $1/(1-z)^k$, so that the direct limit $z \to 1$ cannot be taken. Let us recall that the hypergeometric series defined by Eq. (4) converges [9] for $|z| = 1$, when $\mathrm{Re}\sum_{j=1}^{p} b_j - \mathrm{Re}\sum_{j=1}^{p+1} a_j > 0$, so that, if the l.h.s. of Eq. (17) converges, the r.h.s. of the equation does also exist.

The main idea is to convert the original hypergeometric function ${}_{p+1}F_p(\vec{a};\vec{b};z)$ to a function of argument $1-z$. However, beyond type ${}_2F_1$, the analytical continuation of the hypergeometric function ${}_{p+1}F_p(\vec{A};\vec{B};z)\Big|_{z\to 1-z}$ is not expressible in terms of functions of the same type, but has a more complicated structure [33]. Nevertheless, we can perform the analytical continuation $z \to 1-z$ of the coefficients of the ε expansions of hypergeometric functions entering the r.h.s. of Eq. (17). It was shown in Ref. [34] that, under the transformation $z \to 1-z$, hyperlogarithms are expressible again in terms of hyperlogarithms. In this way, if the coefficients of the ε expansion are expressible in terms of hyperlogarithms, there is an opportunity to find the limit $z \to 1-z$ of the differential reduction, but only in fixed orders of the ε expansions and only for such values of parameters of the hypergeometric functions for which the analytical structures of the coefficients are known.

An alternative approach to evaluate hypergeometric functions at $z = 1$ was discussed in Refs. [22,27].

3 Application to Feynman diagrams

As an illustration of the differential-reduction algorithm, let us consider the diagrams shown in Figs. 1 and 2.[9] In general, Feynman diagrams suffer from irreducible numerators. Using the Davydychev-Tarasov algorithm [37], any tensor integral may be represented in terms of scalar integrals with shifted space-time dimensions and arbitrary (positive) powers of propagators. In our analysis of the structures of the coefficients of the ε expansions, we distinguish between two cases corresponding to even value $n = 2m - 2\varepsilon$ and odd value $n = 2m - 1 - 2\varepsilon$ of space-time dimension, where m is an integer.

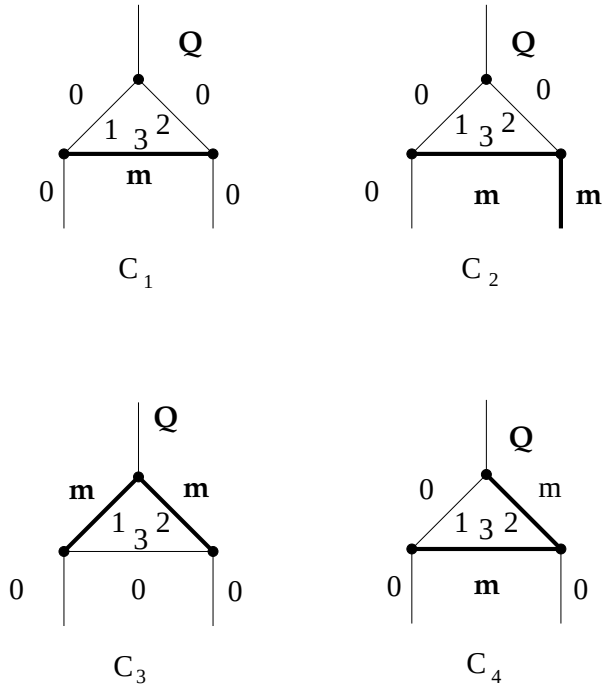

Figure 1: One-loop vertex diagrams expressible in terms of generalized hypergeometric functions. Bold and thin lines correspond to massive and massless propagators, respectively.

Since all diagrams shown in Fig. 2 contain massless subloops, we present for completeness the result for the q-loop massless sunset-type propagator with $q+1$ massless lines. It is given by

$$J_0(p^2,\sigma_1,\sigma_2,\cdots,\sigma_{q+1}) = \int \frac{d^n(k_1 k_2 \cdots k_q)}{[k_1^2]^{\sigma_1}[k_2^2]^{\sigma_2}\cdots[k_q^2]^{\sigma_q}[(k_1+k_2+\cdots+k_q+p)^2]^{\sigma_{q+1}}}$$
$$= \left\{\prod_{k=1}^{q+1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}\frac{\left[i^{1-n}\pi^{n/2}\right]^q\Gamma\left(\sigma-\frac{n}{2}q\right)}{\Gamma\left(\frac{n}{2}(q+1)-\sigma\right)}(p^2)^{\frac{n}{2}q-\sigma}\ , \quad (45)$$

[9] A few examples of the differential-reduction algorithm applied to the reduction of Feynman diagrams were presented in Refs. [35,36].

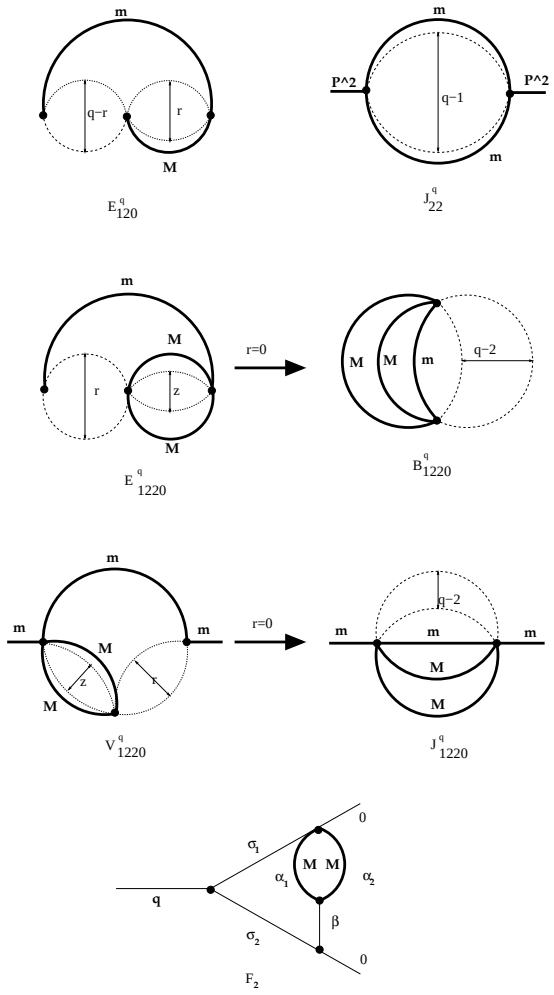

Figure 2: Diagrams considered in the paper. Bold and thin lines correspond to massive and massless propagators, respectively.

where

$$\sigma = \sum_{k=1}^{s} \sigma_k, \tag{46}$$

with $s = q + 1$. Some of the massless lines can be dressed by insertions of massless chains. In this case, the respective σ_k can be represented as $\sigma_k = r_k - R_k \frac{n}{2}$, where r_k and R_k and integers.

3.1 One-loop vertex: particular cases

We wish to remind the reader that any one-loop vertex diagram with arbitrary masses, external momenta, and powers of propagators can be reduced by recurrence relations, derived with the help of the integration-by-parts technique [15], to a vertex master integral plus propagator master integrals and bubble integrals (with all powers of propagators being equal to unity), or, in the case of zero Gram and/or

Cayley determinants, to a linear combination of propagator and bubble integrals. In terms of hypergeometric functions, vanishing Gram and/or Cayley determinants correspond to the situation where the hypergeometric function describing the original vertex diagram can be reduced to a Gauss hypergeometric function with the following sets of parameters [29,38,39]:

$$ {}_2F_1\left(\left.\begin{matrix}1, I_1 - \frac{n}{2}, \\ I_2 + \frac{n}{2}\end{matrix}\right| z\right) \ , \quad {}_2F_1\left(\left.\begin{matrix}1, I_1 - \frac{n}{2}, \\ I_2 + \frac{3}{2}\end{matrix}\right| y\right) \ , \tag{47} $$

where $\{I_a\}$ are arbitrary integers. In the case of non-zero Gram and/or Cayley determinants, the one-loop vertex master integral is expressible as a linear combination of Gauss hypergeometric functions and Appell functions F_1 [40].

For the diagrams shown in Fig. 1, both the Gram and Cayley determinants are non-zero. The hypergeometric representation of the corresponding master integral was published in Ref. [30].

- Diagram C_1 with arbitrary powers of propagators is expressible in terms of two hypergeometric functions ${}_3F_2$ as (see also Eq. (3.44) in Ref. [41])

$$ \begin{aligned} \frac{C_1}{i^{1-n}\pi^{\frac{n}{2}}} &= (-m^2)^{\frac{n}{2}-j_{123}} \\ &\left\{ \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{12}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma(j_3)} {}_3F_2\left(\left.\begin{matrix} j_{123}-\frac{n}{2}, j_1, j_2 \\ \frac{n}{2}, 1+j_{12}-\frac{n}{2}\end{matrix}\right| -\frac{Q^2}{m^2}\right) \right. \\ &+ \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-j_{12}} \frac{\Gamma\left(\frac{n}{2}-j_1\right)\Gamma\left(\frac{n}{2}-j_2\right)\Gamma\left(j_{12}-\frac{n}{2}\right)}{\Gamma\left(n-j_{12}\right)\Gamma(j_1)\Gamma(j_2)} \\ &\left. \times {}_3F_2\left(\left.\begin{matrix} j_3, \frac{n}{2}-j_1, \frac{n}{2}-j_2 \\ n-j_{12}, \frac{n}{2}-j_{12}+1\end{matrix}\right| -\frac{Q^2}{m^2}\right)\right\} . \end{aligned} \tag{48} $$

Here and in the following, we use the short-hand notations $j_{ab} = j_a + j_b$ and $j_{abc} = j_a + j_b + j_c$ for compactness. In accordance with the differential-reduction algorithm, each of the two ${}_3F_2$ functions in Eq. (48) is expressible in terms of a ${}_2F_1$ function with one unit upper parameter, namely

$$ {}_2F_1\left(\left.\begin{matrix}1, 1 \\ I_1 + \frac{n}{2}\end{matrix}\right| z\right) \ , \quad {}_2F_1\left(\left.\begin{matrix}1, I_1 + \frac{n}{2} \\ I_2 + n\end{matrix}\right| z\right) \ , \tag{49} $$

respectively, plus rational functions of z. Standard approaches yield one vertex master integral plus massive bubble and massless propagator integrals. The latter two types of integrals are expressible in terms of products of Gamma functions and correspond to the rational functions in our approach.

- For diagram C_2 with arbitrary powers of propagators, the result is

$$\frac{C_2}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}} \left\{ \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{12}\right)\Gamma\left(n-j_{13}-2j_2\right)}{\Gamma\left(n-j_{123}\right)\Gamma\left(\frac{n}{2}-j_2\right)\Gamma(j_3)} {}_3F_2\left(\begin{matrix} j_{123}-\frac{n}{2}, j_1, j_2 \\ 1+j_{12}-\frac{n}{2}, 1+j_{13}+2j_2-n \end{matrix}\middle| \frac{Q^2}{m^2}\right) + \left(-\frac{Q^2}{m^2}\right)^{\frac{n}{2}-j_{12}} \frac{\Gamma\left(\frac{n}{2}-j_1\right)\Gamma\left(j_{12}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_{23}\right)}{\Gamma\left(n-j_{123}\right)\Gamma(j_1)\Gamma(j_2)} \times {}_3F_2\left(\begin{matrix} j_3, \frac{n}{2}-j_2, \frac{n}{2}-j_1 \\ \frac{n}{2}-j_{12}+1, 1+j_{23}-\frac{n}{2} \end{matrix}\middle| \frac{Q^2}{m^2}\right) \right\}. \tag{50}$$

  Similarly to the previous case, each of the two ${}_3F_2$ functions in Eq. (50) is expressible in terms of a ${}_2F_1$ function with one unit upper parameter, namely

$${}_2F_1\left(\begin{matrix} 1, 1 \\ I_1 - n \end{matrix}\middle| z\right), \quad {}_2F_1\left(\begin{matrix} 1, I_1+\frac{n}{2}, \\ I_2-\frac{n}{2} \end{matrix}\middle| z\right), \tag{51}$$

  respectively, plus rational functions. Standard approaches yield one vertex master integral plus massive bubble and massless propagator integrals.

- Diagram C_3 with arbitrary powers of propagators is expressible in terms of one ${}_4F_3$ function as

$$\frac{C_3}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}} \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_3\right)}{\Gamma\left(j_{12}\right)\Gamma\left(\frac{n}{2}\right)} {}_4F_3\left(\begin{matrix} j_{123}-\frac{n}{2}, j_1, j_2, \frac{n}{2}-j_3 \\ \frac{n}{2}, \frac{j_{12}}{2}, \frac{j_{12}+1}{2} \end{matrix}\middle| \frac{Q^2}{4m^2}\right). \tag{52}$$

  In accordance with the differential-reduction algorithm, this function may be written in terms of a ${}_3F_2$ function with one unit upper parameter and its first derivative,

$$\{1, \theta\} \times {}_3F_2\left(\begin{matrix} 1, I_1-\frac{n}{2}, \frac{n}{2}-1+I_2 \\ \frac{n}{2}+I_2, \frac{1}{2}+I_3 \end{matrix}\middle| z\right), \tag{53}$$

  and a rational function. Standard approaches yield one vertex and one propagator master integral plus massive bubble integrals.

- Diagram C_4 with arbitrary powers of propagators is expressible in terms of one ${}_3F_2$ function as

$$\frac{C_4}{i^{1-n}\pi^{\frac{n}{2}}} = (-m^2)^{\frac{n}{2}-j_{123}} \frac{\Gamma\left(j_{123}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}-j_1\right)}{\Gamma\left(j_{23}\right)\Gamma\left(\frac{n}{2}\right)} {}_3F_2\left(\begin{matrix} j_{123}-\frac{n}{2}, j_1, j_2 \\ \frac{n}{2}, j_{23} \end{matrix}\middle| \frac{Q^2}{m^2}\right). \tag{54}$$

  In accordance with the differential-reduction algorithm, this function may be written in terms of a ${}_3F_2$ function with one unit upper parameter and its first derivative,

$$\{1, \theta\} \times {}_3F_2\left(\begin{matrix} 1, I_1, I_2-\frac{n}{2} \\ I_1+1, \frac{n}{2}+I_2 \end{matrix}\middle| z\right), \tag{55}$$

  and a rational function. Standard approaches yield one vertex and one propagator master integral plus massive bubble integrals.

3.2 E^q_{120}

Let us consider the q-loop bubble diagram E^q_{120} depicted in Fig. 2 with two massive lines of different masses and $q-r$ plus r massless propagators,

$$E^q_{120}(m^2,M^2,\alpha,\beta,\sigma_1,\cdots,\sigma_{q-r},\rho_1,\cdots,\rho_r) = \int \frac{d^n(k_1\cdots k_q)}{[k_1^2]^{\sigma_1}\cdots[k_{q-r-1}^2]^{\sigma_{q-r-1}}[(k_1+k_2+k_{q-r-1}+k_q)^2]^{\sigma_{q-r}}[k_{q-1}^2-M^2]^{\alpha}[k_q^2-m^2]^{\beta}} \times \frac{1}{[k_{q-r}^2]^{\rho_1}\cdots[(k_{q-r}+\cdots k_{q-1}+k_q)^2)]^{\rho_r}} \; . \tag{56}$$

The special case $q=r$ was analyzed in Ref. [35]. The first non-trivial diagram of this type corresponds to a three-loop bubble ($q=3$) with $r=1$. The Mellin-Barnes representation of Eq. (56) reads:

$$E^q_{120}(m^2,M^2,\alpha,\beta,\sigma_1,\cdots,\sigma_{q-r},\rho_1,\cdots,\rho_r) = \frac{(-M^2)^{\frac{n}{2}r-\alpha-\rho}(-m^2)^{\frac{n}{2}(q-r)-\sigma-\beta}}{\left[i^{1-n}\pi^{\frac{n}{2}}\right]^{-q}\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)}$$
$$\left\{\prod_{j=1}^{q-r}\frac{\Gamma(\frac{n}{2}-\sigma_j)}{\Gamma(\sigma_j)}\right\}\left\{\prod_{k=1}^{r}\frac{\Gamma(\frac{n}{2}-\rho_k)}{\Gamma(\rho_k)}\right\}\frac{\Gamma\left(\sigma-\frac{n}{2}(q-r-1)\right)}{\Gamma\left(\frac{n}{2}(q-r)-\sigma\right)}\int dt\left(\frac{m^2}{M^2}\right)^t$$
$$\times\frac{\Gamma(-t)\Gamma\left(\rho-\frac{n}{2}(r-1)+t\right)\Gamma\left(\alpha+\rho-\frac{n}{2}r+t\right)\Gamma\left(\frac{n}{2}(q-r)-\sigma+t\right)\Gamma\left(\sigma+\beta-\frac{n}{2}(q-r)-t\right)}{\Gamma\left(\frac{n}{2}+t\right)} \; , \tag{57}$$

where σ is defined by Eq. (46) with $s=q-r$ and

$$\rho = \sum_{k=1}^{r}\rho_k \; . \tag{58}$$

Closing the contour of integration in Eq. (57) on the left, we obtain in the notation of Ref. [38]:

$$E^q_{120}(m^2,M^2,\alpha,\beta,\sigma_1,\cdots,\sigma_{q-r},\rho_1,\cdots,\rho_r) = \left[i^{1-n}\pi^{\frac{n}{2}}\right]^q\frac{(-M^2)^{\frac{n}{2}q-\alpha-\beta-\sigma-\rho}}{\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)}$$
$$\left\{\prod_{j=1}^{q-r}\frac{\Gamma(\frac{n}{2}-\sigma_j)}{\Gamma(\sigma_j)}\right\}\left\{\prod_{k=1}^{r}\frac{\Gamma(\frac{n}{2}-\rho_k)}{\Gamma(\rho_k)}\right\}\frac{\Gamma\left(\sigma-\frac{n}{2}(q-r-1)\right)}{\Gamma\left(\frac{n}{2}(q-r)-\sigma\right)}$$
$$\times\left\{\frac{\Gamma(\beta)\Gamma\left(\frac{n}{2}(q-r)-\sigma-\beta\right)\Gamma\left(\rho+\beta+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\alpha+\rho+\beta+\sigma-\frac{n}{2}q\right)}{\Gamma\left(\sigma+\beta-\frac{n}{2}(q-r-1)\right)}\right.$$
$$\left.\times\,{}_3F_2\left(\left.\begin{matrix}\beta,\beta+\sigma+\rho-\frac{n}{2}(q-1),\alpha+\beta+\sigma+\rho-\frac{n}{2}q\\ \sigma+\beta-\frac{n}{2}(q-r-1),1+\sigma+\beta-\frac{n}{2}(q-r)\end{matrix}\right|\frac{m^2}{M^2}\right)\right.$$
$$+\left(\frac{m^2}{M^2}\right)^{\frac{n}{2}(q-r)-\sigma-\beta}\frac{\Gamma\left(\frac{n}{2}(q-r)-\sigma\right)\Gamma\left(\sigma+\beta-\frac{n}{2}(q-r)\right)\Gamma\left(\alpha+\rho-\frac{n}{2}r\right)\Gamma\left(\rho-\frac{n}{2}(r-1)\right)}{\Gamma\left(\frac{n}{2}\right)}$$
$$\left.\times\,{}_3F_2\left(\left.\begin{matrix}\rho-\frac{n}{2}(r-1),\alpha+\rho-\frac{n}{2}r,\frac{n}{2}(q-r)-\sigma\\ \frac{n}{2},1+\frac{n}{2}(q-r)-\sigma-\beta,\end{matrix}\right|\frac{m^2}{M^2}\right)\right\} \; . \tag{59}$$

Let us discuss the differential reduction of the two hypergeometric functions in Eq. (59) assuming that β, σ_j ($\sigma \geq 2$), and ρ_k are integers. We distinguish between two cases: $r = 1$ ($q \geq 3$) and $r \geq 2$ ($q \geq 4$). For $r = 1$, both hypergeometric functions in Eq. (59) are reducible to ${}_2F_1$ functions with one unit upper parameter, namely

$$ {}_2F_1\left(\left.\begin{matrix}1, I_1 - \frac{n}{2}q \\ I_2 - \frac{n}{2}(q-2)\end{matrix}\right| z\right) \ , \quad {}_2F_1\left(\left.\begin{matrix}1, I_1 - \frac{n}{2}, \\ I_2 + \frac{n}{2}\end{matrix}\right| z\right) \ , \tag{60} $$

respectively. Standard approaches yield one three-loop bubble master integral and integrals expressible in terms of Gamma functions. For $r \geq 2$, the first hypergeometric function is expressible in terms of a ${}_3F_2$ function with one unit upper parameter, and the second one is reducible to a ${}_2F_1$ function with both upper parameters containing non-zero ε parts, namely

$$ (1,\theta)\times{}_3F_2\left(\left.\begin{matrix}1, I_1 - \frac{n}{2}(q-1), I_2 - \frac{n}{2}q \\ I_3 - \frac{n}{2}(q-r-1), I_4 - \frac{n}{2}(q-r)\end{matrix}\right| z\right) \ , \quad (1,\theta)\times{}_2F_1\left(\left.\begin{matrix} I_1 - \frac{n}{2}(r-1), I_2 - \frac{n}{2}r \\ I_3 + \frac{n}{2}\end{matrix}\right| z\right) \ , \tag{61} $$

respectively. For $q = 3$ and $r = 1$, we present the explicit form of the master integral, with the powers of propagator being all equal to unity, in dimension $n = 4 - 2\varepsilon$. It reads:

$$ E^3_{120}(m^2, M^2, 1, 1, 1, 1, 1) = \left[i\pi^{2-\varepsilon}\right]^3 (M^2)^{1-3\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma\left(1+2\varepsilon\right)\Gamma\left(1+\varepsilon\right)}{2\varepsilon^3(1-2\varepsilon)(1-\varepsilon)^2} \times $$
$$ \left\{ \frac{(1-\varepsilon)}{3(1-3\varepsilon)} \frac{\Gamma(1+3\varepsilon)}{\Gamma(1+\varepsilon)} \ {}_2F_1\left(\left.\begin{matrix}1, -1+3\varepsilon \\ 1+\varepsilon\end{matrix}\right| \frac{m^2}{M^2}\right) + \left(\frac{m^2}{M^2}\right)^{1-2\varepsilon} \frac{\Gamma(1+\varepsilon)}{\Gamma(1-\varepsilon)} \ {}_2F_1\left(\left.\begin{matrix}1, \varepsilon \\ 2-\varepsilon\end{matrix}\right| \frac{m^2}{M^2}\right) \right\} . \tag{62} $$

The two Gauss hypergeometric functions in Eq. (62) can be reduced to the basis functions considered in Refs. [16,20,35] by the following relations:

$$ (1-2\varepsilon) \ {}_2F_1\left(\left.\begin{matrix}1, \varepsilon \\ 2-\varepsilon\end{matrix}\right| z\right) = 1 - \varepsilon - \varepsilon(1-z) \ {}_2F_1\left(\left.\begin{matrix}1, 1+\varepsilon \\ 2-\varepsilon\end{matrix}\right| z\right) \ , $$
$$ (1-2\varepsilon) \ {}_2F_1\left(\left.\begin{matrix}1, -1+3\varepsilon \\ 1+\varepsilon\end{matrix}\right| z\right) = 1 - 2\varepsilon - z(1-3\varepsilon) $$
$$ + \frac{3\varepsilon(1-3\varepsilon)z(1-z)}{1+\varepsilon} \ {}_2F_1\left(\left.\begin{matrix}1, 1+3\varepsilon \\ 2+\varepsilon\end{matrix}\right| z\right) \ . \tag{63} $$

For illustration, we present here the first few coefficients of the ε expansion of Eq. (62):

$$\begin{aligned}
\frac{E^3_{120}(m^2,M^2,1,1,1,1,1)}{\left[i\pi^{2-\varepsilon}\right]^3\Gamma^3(1+\varepsilon)} &= (M^2)^{1-3\varepsilon}\left(\frac{1+2z}{6\varepsilon^3}+\frac{3+z\left[5-3\ln z\right)]}{3\varepsilon^2}\right.\\
&+\frac{1}{\varepsilon}\left\{\frac{25+34z}{6}-(1-z)\left[\mathrm{Li}_2\left(z\right)+\ln(z)\ln(1-z)\right]+z\ln z(\ln z-5)+\zeta_2\right\}\\
&+(1-z)\left[2\mathrm{S}_{1,2}(z)+2\left(\mathrm{Li}_2\left(z\right)-\zeta_2\right)\ln(1-z)+\mathrm{Li}_2\left(z\right)\ln z\right]\\
&+(1-z)\left\{\ln(1-z)\ln z\left[\ln z+\ln(1-z)\right]-5\left[\mathrm{Li}_2\left(z\right)+\ln(z)\ln(1-z)\right]\right\}\\
&+\frac{45+49z}{3}-17z\ln z+5z\ln^2 z+2\left(3-z\ln z\right)\zeta_2-\frac{5-2z}{3}\zeta_3-\frac{2}{3}z\ln^3 z\\
&+\varepsilon(1-z)\left\{\mathrm{Li}_2\left(z\right)\left[2\ln(1-z)\left(5-\ln(1-z)-\ln z\right)-\ln z(\ln z-5)-4\zeta_2-17\right]\right.\\
&\qquad+\mathrm{Li}_3\left(z\right)\ln z-\mathrm{Li}_4\left(z\right)+2\mathrm{S}_{1,2}(z)\left[5-2\ln(1-z)-\ln z\right]-4\mathrm{S}_{1,3}(z)\\
&\qquad+2\ln(1-z)\zeta_2\left[\ln(1-z)-5-\ln z\right]+4\ln(1-z)\zeta_3\\
&\qquad+\ln z\ln(1-z)\left[\ln(1-z)+\ln z\right]\left[5-\frac{2}{3}(\ln(1-z)+\ln z)\right]\\
&\qquad\left.+\ln z\ln(1-z)\left[\frac{1}{3}\ln z\ln(1-z)-17\right]\right\}\\
&+\varepsilon\left\{z\ln z\left(\frac{1}{3}\ln^2 z(\ln z-10)+17\ln z+2\zeta_2(\ln z-5)+2\zeta_3-49\right)\right.\\
&\qquad\left.\left.+\frac{301}{6}+43z+25\zeta_2-\frac{10}{3}(3-z)\zeta_3+\frac{1}{2}\zeta_4(23-14z)\right\}+\mathcal{O}(\varepsilon)\right), \quad (64)
\end{aligned}$$

where

$$z=\frac{m^2}{M^2}. \quad (65)$$

To check Eq. (64), we evaluate the first few coefficients of the ε expansion of the original diagram in the large-mass limit [42] using the program packages developed in Refs. [43,44] to find agreement.

3.3 J^q_{22}

Let us consider the q-loop sunset diagram J^q_{22} in Fig. 2 with two massive lines of the same mass m and $q-2$ massless subloops, which is one of the most frequently studied Feynman diagrams. It is defined as

$$\begin{aligned}
&J^q_{22}(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) =\\
&\int\frac{d^n(k_1\cdots k_q)}{[k_1^2]^{\sigma_1}\cdots[k_{q-1}^2]^{\sigma_{q-1}}[k_q^2-m^2]^{\alpha_1}[(k_1+k_2+\cdots+k_q+p)^2-m^2]^{\alpha_2}}\,. \quad (66)
\end{aligned}$$

The Mellin-Barnes representation of Eq. (66) reads:[10]

$$J_{22}^q(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \frac{(-m^2)^{\frac{n}{2}-\alpha_{1,2}}(p^2)^{\frac{n}{2}(q-1)-\sigma}}{\left[i^{1-n}\pi^{n/2}\right]^{-q}\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\{\prod_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$$
$$\times \int dt \left(-\frac{p^2}{m^2}\right)^t \frac{\Gamma(\alpha_1+t)\Gamma(\alpha_2+t)\Gamma(\alpha_{1,2}-\frac{n}{2}+t)\Gamma(\frac{n}{2}+t)\Gamma(\sigma-\frac{n}{2}(q-1)-t)}{\Gamma(\alpha_{1,2}+2t)\Gamma(\frac{n}{2}q-\sigma+t)} , \quad (67)$$

where σ is defined by Eq. (46) with $s = q-1$ and

$$\alpha_{1,2} = \alpha_1 + \alpha_2 . \quad (68)$$

Closing the contour of integration in Eq. (67) on the left, we obtain in the notation of Ref. [38]:

$$J_{22}^q(m^2,p^2,\alpha_1,\alpha_2,\sigma_1,\cdots,\sigma_{q-1}) = \left[i^{1-n}\pi^{n/2}\right]^q \frac{(-m^2)^{\frac{n}{2}q-\alpha_{1,2}-\sigma}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\{\prod_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$$
$$\times \frac{\Gamma\left(\alpha_1+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\alpha_2+\sigma-\frac{n}{2}(q-1)\right)\Gamma\left(\sigma-\frac{n}{2}(q-2)\right)\Gamma\left(\alpha_{1,2}+\sigma-\frac{n}{2}q\right)}{\Gamma\left(\alpha_{1,2}+2\sigma-n(q-1)\right)\Gamma\left(\frac{n}{2}\right)}$$
$$\times \ {}_4F_3\left(\begin{matrix}\alpha_1+\sigma-\frac{n}{2}(q-1), \alpha_2+\sigma-\frac{n}{2}(q-1), \sigma-\frac{n}{2}(q-2), \alpha_{1,2}+\sigma-\frac{n}{2}q \\ \frac{n}{2}, \frac{1}{2}(\alpha_{1,2}-n(q-1))+\sigma, \frac{1}{2}(1+\alpha_{1,2}-n(q-1))+\sigma,\end{matrix}\middle| \frac{p^2}{4m^2}\right) . \quad (69)$$

For $q = 1$ (see Footnote 10), the ${}_4F_3$ function in Eq. (69) is reduced to a ${}_3F_2$ function, in agreement with Ref. [38]. In the two-loop case ($q = 2$), the hypergeometric representation of this diagram was derived in Refs. [45,46].

Let us analyze the reduction of the hypergeometric function in Eq. (69) assuming that all parameters, α_1, α_2, and σ_k, are integer. For $q = 1$ ($\sigma_k = 0$), we obtain a ${}_3F_2$ function with integer differences between upper and lower parameters, which, according to *Criterion 1*, is reducible to a ${}_2F_1$ function with one integer upper parameter, so that we have a basis hypergeometric function plus a rational function. (For the one-loop propagator, there are one master integral and bubble integrals.) For $q = 2$, we get a ${}_4F_3$ function with integer parameter differences and one integer upper parameter, so that it is reducible to a ${}_3F_2$ function with one integer upper parameter and its first derivative,

$$(1,\theta) \times \ {}_3F_2\left(\begin{matrix}1, I_1-\frac{n}{2}, I_2-n \\ I_3+\frac{n}{2}, I_4+\frac{1}{2}-\frac{n}{2}\end{matrix}\middle| z\right) , \quad (70)$$

plus a rational function. In this case, there are two nontrivial master integrals of the same topology and bubble integrals, in accordance with the results of Ref. [47]. For $q \geq 3$, we have a ${}_4F_3$ function with integer differences of parameters, which, according to *Criterion 1*, is reducible to a ${}_3F_2$ function and its first two derivatives,

$$(1,\theta,\theta^2) \times \ {}_3F_2\left(\begin{matrix}I_1-\frac{n}{2}(q-1), I_2-\frac{n}{2}(q-2), I_3-\frac{n}{2}q \\ \frac{n}{2}, I_4+\frac{1}{2}-\frac{n}{2}(q-1)\end{matrix}\middle| z\right) . \quad (71)$$

[10] In the one-loop case ($q = 1$), the factor $\left\{\prod_{k=1}^{q-1}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)}\right\}$ is equal to unity, and the representation of Eq. (67) agrees with the one in Ref. [38].

3.4 E^q_{1220} and B^q_{1220}

Let us consider the q-loop bubble diagram E^q_{1220} in Fig. 2 with two massive lines of different masses and s plus r massless propagators,

$$E^q_{1220}(m^2,M^2,\alpha_1,\alpha_2,\beta,\sigma_1,\cdots,\sigma_s,\rho_1,\cdots,\rho_r) = \int \frac{d^n(k_1\cdots k_q)}{[k_1^2]^{\sigma_1}\cdots[k_s^2]^{\sigma_s}[k_{s+1}^2-M^2]^{\alpha_1}[(k_1+k_2+\cdots+k_{s+1}+k_q)^2-M^2]^{\alpha_2}} \times \frac{1}{[k_{s+2}^2]^{\rho_1}\cdots[k_{q-1}^2]^{\rho_{r-1}}[(k_{s+2}+\cdots k_{q-1}+k_q)^2)]^{\rho_r}(k_q^2-m^2)^{\beta}} , \quad (72)$$

where, by construction, $s \geq 0$ and $r \geq 2$ and, as a consequence, $q = s + r + 1 \geq 3$. The Mellin-Barnes representation of Eq. (72) reads:

$$E^q_{1220}(m^2,M^2,\alpha_1,\alpha_2,\beta,\sigma_1,\cdots,\sigma_s,\rho_1,\cdots,\rho_r) = \frac{(-M^2)^{\frac{n}{2}-\alpha_{1,2}}(-m^2)^{\frac{n}{2}(q-1)-\sigma-\beta-\rho}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)} \times \left[i^{1-n}\pi^{\frac{n}{2}}\right]^q \left\{\prod_{j=1}^{s}\frac{\Gamma(\frac{n}{2}-\sigma_j)}{\Gamma(\sigma_j)}\right\}\left\{\prod_{k=1}^{r}\frac{\Gamma(\frac{n}{2}-\rho_k)}{\Gamma(\rho_k)}\right\}\frac{\Gamma\left(\rho-\frac{n}{2}(r-1)\right)}{\Gamma\left(\frac{n}{2}r-\rho\right)} \times \int dt \left(\frac{m^2}{M^2}\right)^t \frac{\Gamma(\alpha_1+t)\Gamma(\alpha_2+t)\Gamma(\alpha_{1,2}-\frac{n}{2}+t)\Gamma(\frac{n}{2}+t)}{\Gamma(\alpha_{1,2}+2t)} \times \frac{\Gamma\left(\sigma-\frac{n}{2}s-t\right)\Gamma\left(\sigma+\beta+\rho-\frac{n}{2}(q-1)-t\right)\Gamma\left(\frac{n}{2}(q-1)-\sigma-\rho+t\right)}{\Gamma\left(\frac{n}{2}(s+1)-\sigma+t\right)} , \quad (73)$$

where σ, ρ, and $\alpha_{1,2}$ are defined in Eqs. (46), (58), and (68), respectively.

Diagram B^q_{1220} in Fig. 2 emerges from E^q_{1220} with $r = 1$ and $s = q - 2$ in the smooth limit $\rho = \rho_1 \to 0$. Then, Eq. (73) simplifies due to $\frac{\Gamma\left(\frac{n}{2}(q-1)-\sigma-\rho+s\right)}{\Gamma\left(\frac{n}{2}(s+1)-\sigma+s\right)} = 1$.

Closing the contour of integration in Eq. (73) on the left, we obtain in the

notation of Ref. [38]:

$$E^q_{1220}(m^2,M^2,\alpha_1,\alpha_2,\beta,\sigma_1,\cdots,\sigma_z,\rho_1,\cdots,\rho_r) = \left[i^{1-n}\pi^{\frac{n}{2}}\right]^q \frac{(-M^2)^{\frac{n}{2}q-\alpha_{1,2}-\sigma-\beta-\rho}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)}$$
$$\left\{\prod_{j=1}^{s}\frac{\Gamma(\frac{n}{2}-\sigma_j)}{\Gamma(\sigma_j)}\right\}\left\{\prod_{k=1}^{r}\frac{\Gamma(\frac{n}{2}-\rho_k)}{\Gamma(\rho_k)}\right\}\frac{\Gamma\left(\rho-\frac{n}{2}(r-1)\right)}{\Gamma\left(\frac{n}{2}r-\rho\right)}$$
$$\times\left\{\left(\frac{m^2}{M^2}\right)^{\frac{n}{2}r-\beta-\rho}\frac{\Gamma\left(\sigma-\frac{n}{2}(s-1)\right)\Gamma\left(\frac{n}{2}r-\rho\right)\Gamma\left(\rho+\beta-\frac{n}{2}r\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha_{1,2}+2\sigma-ns\right)}\right.$$
$$\times\Gamma\left(a_{1,\sigma}-\tfrac{n}{2}s\right)\Gamma\left(a_{2,\sigma}-\tfrac{n}{2}s\right)\Gamma\left(a_{1,2,\sigma}-\tfrac{n}{2}(s+1)\right)$$
$${}_5F_4\left(\left.\begin{matrix}a_{1,\sigma}-\frac{n}{2}s, a_{2,\sigma}-\frac{n}{2}s, a_{1,2,\sigma}-\frac{n}{2}(s+1), \sigma-\frac{n}{2}(s-1), \frac{n}{2}r-\rho \\ \frac{n}{2}, \frac{\alpha_{1,2}-ns}{2}+\sigma, \frac{\alpha_{1,2}+1-ns}{2}+\sigma, \frac{n}{2}r-\rho+1-\beta\end{matrix}\right| \frac{m^2}{4M^2}\right)$$
$$+\frac{\Gamma(\beta)\Gamma\left(\frac{n}{2}r-a_{\beta,\rho}\right)\Gamma\left(a_{\sigma,\beta,\rho}-\frac{n}{2}(q-2)\right)}{\Gamma\left(\alpha_{1,2}+2\sigma+2\beta+2\rho-n(q-1)\right)\Gamma\left(a_{\beta,\rho}-\frac{n}{2}(r-1)\right)}$$
$$\times\Gamma\left(a_{1,\sigma,\beta,\rho}-\tfrac{n}{2}(q-1)\right)\Gamma\left(a_{2,\sigma,\beta,\rho}-\tfrac{n}{2}(q-1)\right)\Gamma\left(a_{1,2,\sigma,\beta,\rho}-\tfrac{n}{2}q\right)$$
$${}_5F_4\left(\left.\begin{matrix}a_{1,\sigma,\beta,\rho}-\frac{n}{2}(q-1), a_{2,\sigma,\beta,\rho}-\frac{n}{2}(q-1), a_{1,2,\sigma,\beta,\rho}-\frac{n}{2}q, a_{\sigma,\beta,\rho}-\frac{n}{2}(q-2), \beta \\ 1+a_{\beta,\rho}-\frac{n}{2}r, a_{\sigma,\beta,\rho}+\frac{\alpha_{1,2}-n(q-1)}{2}, a_{\sigma,\beta,\rho}+\frac{\alpha_{1,2}+1-n(q-1)}{2}, a_{\beta,\rho}-\frac{n}{2}(r-1)\end{matrix}\right| \frac{m^2}{4M^2}\right)\Bigg\}\,, \tag{74}$$

where we have introduced the short-hand notations

$$\begin{aligned} a_{i,\sigma} &= \alpha_1+\sigma\,, \qquad a_{\beta,\rho} = \beta+\rho\,,\\ a_{1,2,\sigma} &= \alpha_1+\alpha_2+\sigma\,, \qquad a_{\sigma,\beta,\rho} = \sigma+\beta+\rho\,,\\ a_{1,2,\sigma,\beta} &= \alpha_1+\alpha_2+\sigma+\beta\,, \qquad a_{i,\sigma,\beta,\rho} = \alpha_i+\sigma+\beta+\rho\,,\\ a_{1,2,\sigma,\beta,\rho} &= \alpha_1+\alpha_2+\sigma+\beta+\rho\,, \qquad i=1,2\,. \end{aligned} \tag{75}$$

Let us analyze the result of the differential reduction of the two hypergeometric functions in Eq. (74) assuming that all parameters, $\alpha_1, \alpha_2, \beta, \sigma_k, \rho_j$, are integer. We distinguish between the three cases: $s=0$, $s=1$, and $s\geq 2$. For $s=0$ ($\sigma_k=0$, $r=q-1$), both hypergeometric functions in Eq. (74) are reducible to ${}_2F_1$ functions with one unit upper parameter,

$${}_2F_1\left(\left.\begin{matrix}1, I_1-\frac{n}{2}\\ I_2+\frac{1}{2}\end{matrix}\right| z\right)\,, \qquad {}_2F_1\left(\left.\begin{matrix}1, I_1-\frac{n}{2}q,\\ \frac{1}{2}+I_2-\frac{n}{2}(q-1)\end{matrix}\right| z\right)\,. \tag{76}$$

Standard approaches yield one master integral. For $s=1$ ($r=q-2$), each of the two hypergeometric functions is reducible to a ${}_3F_2$ function with one unit upper parameter,

$$(1,\theta)\times{}_3F_2\left(\left.\begin{matrix}1, I_1-n, I_2-\frac{n}{2}\\ I_3+\frac{n}{2}, I_4+\frac{1}{2}-\frac{n}{2}\end{matrix}\right| z\right)\,, \qquad (1,\theta)\times{}_3F_2\left(\left.\begin{matrix}1, I_1-\frac{n}{2}(q-1), I_2-\frac{n}{2}q\\ I_3-\frac{n}{2}(q-3), I_4+\frac{1}{2}-\frac{n}{2}(q-1)\end{matrix}\right| z\right)\,. \tag{77}$$

For $s\geq 2$, the first hypergeometric function is reducible to a ${}_3F_2$ function and its first two derivatives, while the second one is expressible in terms of a ${}_4F_3$ function

and its first two derivatives, namely

$$(1,\theta,\theta^2)\times {}_3F_2\left(\left.\begin{matrix} I_1-\frac{n}{2}s, I_2-\frac{n}{2}(s+1), I_3-\frac{n}{2}(s-1) \\ I_4+\frac{n}{2}, I_5+\frac{1}{2}-\frac{n}{2}s \end{matrix}\right| z\right) ,$$
$$(1,\theta,\theta^2)\times {}_4F_3\left(\left.\begin{matrix} 1, I_1-\frac{n}{2}q, I_2-\frac{n}{2}(q-1), I_3-\frac{n}{2}(q-2) \\ \frac{1}{2}+I_4-\frac{n}{2}(q-1), I_5-\frac{n}{2}r, I_6-\frac{n}{2}(r-1) \end{matrix}\right| z\right) , \qquad (78)$$

respectively. Notice that, according to *Criterion 4*, a third derivative does not appear because one of the upper parameters is integer.

We now present an explicit result for the three-loop case in which there is no massless propagator inside the massive loop, $q=3$, $s=0$, $\sigma_k=\sigma=0$, $r=2$, and $n=4-2\varepsilon$. In this case, there is one master integral with $\alpha_1=\alpha_2=\beta=\rho_1=\rho_2=1$ ($\alpha_{1,2}=\rho=2$),

$$E^3_{1220}(m^2,M^2,1,1,1,1,1) = \left[i\pi^{2-\varepsilon}\right]^3 (M^2)^{1-3\varepsilon}$$
$$\frac{\Gamma(1-\varepsilon)\Gamma^2(1+\varepsilon)\Gamma(1+2\varepsilon)}{2\varepsilon^3(1-\varepsilon)(1-2\varepsilon)}\left\{\left(\frac{m^2}{M^2}\right)^{1-2\varepsilon} {}_2F_1\left(\left.\begin{matrix} 1,\varepsilon \\ \frac{3}{2} \end{matrix}\right| \frac{m^2}{4M^2}\right)\right.$$
$$\left.+\frac{2}{3(1-3\varepsilon)}\frac{\Gamma(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma(1+4\varepsilon)\Gamma(1+\varepsilon)} {}_2F_1\left(\left.\begin{matrix} 1,-1+3\varepsilon \\ \frac{1}{2}+2\varepsilon \end{matrix}\right| \frac{m^2}{4M^2}\right)\right\} . \qquad (79)$$

To reduce the Gauss hypergeometric functions in Eq. (79) to sets of functions studied in Refs. [16,20,35], we apply the following relations:

$$(1-2\varepsilon)\ {}_2F_1\left(\left.\begin{matrix} 1,\varepsilon \\ \frac{3}{2} \end{matrix}\right| z\right) = 1-2\varepsilon(1-z)\ {}_2F_1\left(\left.\begin{matrix} 1,1+\varepsilon \\ \frac{3}{2} \end{matrix}\right| z\right) ,$$
$${}_2F_1\left(\left.\begin{matrix} 1,-1+3\varepsilon \\ \frac{1}{2}+2\varepsilon \end{matrix}\right| z\right) = 1-2z\frac{(1-3\varepsilon)}{1-2\varepsilon}+\frac{12\varepsilon(1-3\varepsilon)z(1-z)}{(1-2\varepsilon)(1+4\varepsilon)}\ {}_2F_1\left(\left.\begin{matrix} 1,1+3\varepsilon \\ \frac{3}{2}+2\varepsilon \end{matrix}\right| z\right) . \qquad (80)$$

The first few coefficients of the ε expansion of Eq. (79) are given by

$$\frac{E^3_{1220}(m^2,M^2,1,1,1,1,1)}{\Gamma^3(1+\varepsilon)\left[i\pi^{2-\varepsilon}\right]^3} = (M^2)^{1-3\varepsilon}\left(\frac{(1+z)}{3\varepsilon^3}+\frac{6+z(5-3\ln z)}{3\varepsilon^2}\right.$$
$$+\frac{1}{3\varepsilon}\left\{25+z(17-15\ln z+3\ln^2 z+3\zeta_2)\right\}$$
$$+2\frac{(4-z)}{\varepsilon}\frac{(1-y)}{(1+y)}\left[\mathrm{Li}_2(1-y)+\frac{1}{4}\ln^2 y+\frac{1}{2}\ln z\ln y\right]$$
$$+\frac{1}{3}(90+49z)+\frac{1}{3}\zeta_3(8-7z)-z\left[17\ln z-5\ln^2 z+\frac{2}{3}\ln^3 z\right]+\zeta_2 z(5-2\ln z)$$
$$+\frac{1-y}{1+y}(4-z)\left\{4\mathrm{S}_{1,2}(y)+2\mathrm{S}_{1,2}\left(y^2\right)-4\mathrm{S}_{1,2}(-y)-6\mathrm{Li}_3(y)-2\mathrm{Li}_3(-y)\right.$$
$$+\mathrm{Li}_2(-y)\left[4\ln(1-y)-2\ln z\right]+\mathrm{Li}_2(1-y)\left[10-8\ln(1-y)-4\ln(1+y)\right]$$
$$+\ln(1-y)\left[2\zeta_2-4\ln y\ln(1-y)-3\ln^2 y\right]+\ln^2 y\left[\frac{5}{2}-\ln(1+y)+\frac{2}{3}\ln y\right]$$
$$\left.\left.-\zeta_2(\ln z-4\ln y)-\zeta_3+\ln z\ln y\left[5+\frac{1}{2}\ln y-2\ln(1+y)-\ln z\right]\right\}+\mathcal{O}(\varepsilon)\right) , \qquad (81)$$

where z is defined in Eq. (65) and the variable y is defined as

$$y = \frac{1 - \sqrt{\frac{z}{z-4}}}{1 + \sqrt{\frac{z}{z-4}}}, \qquad z = -\frac{(1-y)^2}{y}, \qquad 4 - z = \frac{(1+y)^2}{y}. \tag{82}$$

The $\mathcal{O}(\varepsilon)$ term is too long to be presented here, but is available from Ref. [48]. We mention here only that, in accordance with the expansion of the Gauss hypergeometric function constructed in Refs. [20,35], this term is expressible just in terms of Nielsen polylogarithms. To check Eq. (81), we evaluate the first few coefficients of the ε expansion of the original diagram in the large-mass limit [42] using the program packages developed in Refs. [43,44] to find agreement. In order to construct the series expansion about $1 - y$, which serves as the small parameter of the large-mass expansion, we employ the trick described in Ref. [49]. In fact, the variable $1 - y$ can be written as a series in the small parameter z as

$$1 - y = i\sqrt{z}\left[\sqrt{1 - \frac{z}{4}} - i\sqrt{\frac{z}{4}}\right] = i\sqrt{z}\left[1 - i\frac{\sqrt{z}}{2} - \frac{z}{8}\sum_{k=0}^{\infty}\frac{(2k)!}{(k!)^2}\frac{1}{k+1}\left(\frac{z}{16}\right)^k\right]. \tag{83}$$

For completeness, we also present the explicit result for diagram B^q_{1220} in Fig. 2:[11]

$$\begin{aligned}
&B^q_{1220}(m^2, M^2, \alpha_1, \alpha_2, \beta, \sigma_1, \cdots, \sigma_{q-2}) \\
&= \frac{(-M^2)^{\frac{n}{2}q - \alpha_{1,2} - \sigma - \beta}\Gamma\left(\frac{n}{2} - \beta\right)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)\left[i^{1-n}\pi^{\frac{n}{2}}\right]^{-q}}\left\{\prod_{k=1}^{q-2}\frac{\Gamma(\frac{n}{2} - \sigma_k)}{\Gamma(\sigma_k)}\right\} \\
&\times\left\{\frac{\Gamma\left(a_{1,\sigma,\beta} - \frac{n}{2}(q-1)\right)\Gamma\left(a_{2,\sigma,\beta} - \frac{n}{2}(q-1)\right)\Gamma\left(a_{1,2,\beta,\sigma} - \frac{n}{2}q\right)\Gamma\left(a_{\sigma,\beta} - \frac{n}{2}(q-2)\right)}{\Gamma\left(\alpha_{1,2} + 2\beta + 2\sigma - n(q-1)\right)}\right. \\
&\quad {}_4F_3\left(\left.\begin{matrix} a_{1,\beta,\sigma} - \frac{n}{2}(q-1), a_{2,\beta,\sigma} - \frac{n}{2}(q-1), \alpha_{1,2} + \beta + \sigma - \frac{n}{2}q, a_{\sigma,\beta} - \frac{n}{2}(q-2) \\ \frac{1}{2}(\alpha_{1,2} - n(q-1)) + \sigma + \beta, \frac{1}{2}(\alpha_{1,2} + 1 - n(q-1)) + \sigma + \beta, 1 + \beta - \frac{n}{2} \end{matrix}\right| \frac{m^2}{4M^2}\right) \\
&\quad + \left(\frac{m^2}{M^2}\right)^{\frac{n}{2} - \beta}\frac{\Gamma\left(a_{1,\sigma} - \frac{n}{2}(q-2)\right)\Gamma\left(a_{2,\sigma} - \frac{n}{2}(q-2)\right)\Gamma\left(a_{1,2,\sigma} - \frac{n}{2}(q-1)\right)}{\Gamma\left(\frac{n}{2} - \beta\right)\Gamma\left(\alpha_{1,2} + 2\sigma - n(q-2)\right)} \\
&\quad \times\Gamma\left(\sigma - \tfrac{n}{2}(q-3)\right)\Gamma\left(\beta - \tfrac{n}{2}\right) \\
&\quad \left.{}_4F_3\left(\left.\begin{matrix} a_{1,\sigma} - \frac{n}{2}(q-2), a_{2,\sigma} - \frac{n}{2}(q-2), a_{1,2,\sigma} - \frac{n}{2}(q-1), \sigma - \frac{n}{2}(q-3) \\ \frac{1}{2}(\alpha_{1,2} - n(q-2)) + \sigma, \frac{1}{2}(\alpha_{1,2} + 1 - n(q-2)) + \sigma, 1 - \beta + \frac{n}{2} \end{matrix}\right| \frac{m^2}{4M^2}\right)\right\},
\end{aligned} \tag{84}$$

where $q \geq 2$. For $q = 2$, the hypergeometric representation was derived in Ref. [51].[12] The result of the differential reduction of the hypergeometric functions in Eq. (84) assuming that all parameters, α_1, α_2, β, and σ_k, are integer may be derived for $q = 2$, $q = 3$, and $q \geq 4$ by directly substituting $r = 1$ and $s = q - 2$ in Eqs. (76)–(78), respectively. We only point out here that, for $q = 3$, the second hypergeometric

[11] For the four-loop case ($q = 4$) with equal masses $m = M$, the Mellin-Barnes representation of this diagram was published in Ref. [50]. In order to recover the result of Ref. [50] from Eq. (73), it is sufficient to redefine the variable of integration t to be $t = N/2 - \alpha_{12} - s$.

[12] In this case, we have $\sigma = 0$ and $\left\{\prod_{k=1}^{q-2}\frac{\Gamma(\frac{n}{2} - \sigma_k)}{\Gamma(\sigma_k)}\right\} = 1$.

function in Eq. (77) is reducible to a ${}_2F_1$ function and its first derivative, so that the differential reduction yields

$$(1,\theta)\times\ {}_2F_1\left(\left.\begin{matrix} I_1-n, I_2-\frac{3n}{2} \\ \frac{1}{2}+I_3-n \end{matrix}\right| z\right)\ , \quad (1,\theta)\times\ {}_3F_2\left(\left.\begin{matrix} 1, I_1-n, I_2-\frac{n}{2} \\ \frac{1}{2}+I_3-\frac{n}{2}, I_4+\frac{n}{2} \end{matrix}\right| z\right)\ , \quad (85)$$

and that, for $q \geq 4$, the second hypergeometric function in Eq. (78) is reducible to a ${}_3F_2$ function and its first two derivatives, so that the differential reduction yields

$$\begin{aligned}
&(1,\theta,\theta^2)\times {}_3F_2\left(\left.\begin{matrix} I_1-\frac{n}{2}q, I_2-\frac{n}{2}(q-1), I_3-\frac{n}{2}(q-2) \\ \frac{1}{2}+I_4-\frac{n}{2}(q-1), I_5-\frac{n}{2} \end{matrix}\right| z\right)\ , \\
&(1,\theta,\theta^2)\times {}_3F_2\left(\left.\begin{matrix} I_1-\frac{n}{2}(q-2), I_2-\frac{n}{2}(q-1), I_3-\frac{n}{2}(q-3) \\ I_4+\frac{n}{2}, I_5+\frac{1}{2}-\frac{n}{2}(q-2) \end{matrix}\right| z\right)\ . \qquad (86)
\end{aligned}$$

It is interesting to note that, in the single-scale case $m = M$ [52], there is only one master integral [43]. This is because the hypergeometric functions ${}_3F_2$ and ${}_2F_1$ with special values of parameters and argument $z = 1/4$ are expressible as products of Gamma functions. For details, see Eqs. (4.36) and (4.42) in Ref. [29]. A diagrammatic interpretation of similar identities was presented in Ref. [53]. On the other hand, this is a manifestation of the existence of functional relations between Feynman diagrams [54].

We now present the explicit result for the three-loop case $q = 3$, $n = 4 - 2\varepsilon$. In this case, the first master integral corresponds to $\alpha_1 = \alpha_2 = \beta = \sigma_1 = 1$, while we choose $\alpha_1 = \beta = \sigma_1 = 1$ and $\alpha_2 = 2$ for the second one,

$$\begin{aligned}
&B^3_{1220}(m^2, M^2, 1, 1, 1, 1) = (M^2)^{2-3\varepsilon}\frac{\Gamma^3(1+\varepsilon)(i\pi^{2-\varepsilon})^3}{\varepsilon^3(1-\varepsilon)^2(1-2\varepsilon)} \\
&\times\left\{\frac{2(1-\varepsilon)(1-4\varepsilon)\Gamma^2\left(1+2\varepsilon\right)\Gamma\left(1+3\varepsilon\right)\Gamma(1-\varepsilon)}{3(1-3\varepsilon)(2-3\varepsilon)\Gamma(1+4\varepsilon)\Gamma^2(1+\varepsilon)}\ {}_2F_1\left(\left.\begin{matrix} -1+2\varepsilon, -2+3\varepsilon \\ -\frac{1}{2}+2\varepsilon \end{matrix}\right| \frac{m^2}{4M^2}\right)\right. \\
&\qquad \left. + \left(\frac{m^2}{M^2}\right)^{1-\varepsilon}\ {}_3F_2\left(\left.\begin{matrix} 1, \varepsilon, -1+2\varepsilon \\ 2-\varepsilon, \frac{1}{2}+\varepsilon \end{matrix}\right| \frac{m^2}{4M^2}\right)\right\}, \\
&B^3_{1220}(m^2, M^2, 1, 2, 1, 1) = (M^2)^{1-3\varepsilon}\frac{\Gamma^3(1+\varepsilon)(i\pi^{2-\varepsilon})^3}{2\varepsilon^3(1-\varepsilon)^2} \\
&\times\left\{\frac{2(1-\varepsilon)(1-4\varepsilon)\Gamma^2\left(1+2\varepsilon\right)\Gamma\left(1+3\varepsilon\right)\Gamma(1-\varepsilon)}{3(1-2\varepsilon)(1-3\varepsilon)\Gamma(1+4\varepsilon)\Gamma^2(1+\varepsilon)}\ {}_2F_1\left(\left.\begin{matrix} -1+2\varepsilon, -1+3\varepsilon \\ -\frac{1}{2}+2\varepsilon \end{matrix}\right| \frac{m^2}{4M^2}\right)\right. \\
&\qquad \left. + \left(\frac{m^2}{M^2}\right)^{1-\varepsilon}\ {}_3F_2\left(\left.\begin{matrix} 1, \varepsilon, 2\varepsilon \\ 2-\varepsilon, \frac{1}{2}+\varepsilon \end{matrix}\right| \frac{m^2}{4M^2}\right)\right\}. \qquad (87)
\end{aligned}$$

In order to express all hypergeometric functions entering Eq. (87) in terms of functions which were analyzed in Ref. [20] (see Appendix A for details), we apply the

following set of relations:

$$(1-4\varepsilon)\ {}_2F_1\left(\left.\begin{matrix}-1+2\varepsilon,-2+3\varepsilon\\ -\frac{1}{2}+2\varepsilon\end{matrix}\right|z\right)=-\frac{24z(1-z)(1+2z)\varepsilon^2}{(1+4\varepsilon)}\ {}_2F_1\left(\left.\begin{matrix}1+2\varepsilon,1+3\varepsilon\\ \frac{3}{2}+2\varepsilon\end{matrix}\right|z\right)$$
$$+\left[1-4\varepsilon-4z+8\varepsilon z^2+14\varepsilon z\right]\ {}_2F_1\left(\left.\begin{matrix}2\varepsilon,3\varepsilon\\ \frac{1}{2}+2\varepsilon\end{matrix}\right|z\right)\ ,$$
$$(1-4\varepsilon)\ {}_2F_1\left(\left.\begin{matrix}-1+2\varepsilon,-1+3\varepsilon\\ -\frac{1}{2}+2\varepsilon\end{matrix}\right|z\right)=-\frac{24z(1-z)\varepsilon^2}{1+4\varepsilon}\ {}_2F_1\left(\left.\begin{matrix}1+2\varepsilon,1+3\varepsilon\\ \frac{3}{2}+2\varepsilon\end{matrix}\right|z\right)$$
$$+\left[1-4\varepsilon-2z+10z\varepsilon)\right]\ {}_2F_1\left(\left.\begin{matrix}2\varepsilon,3\varepsilon\\ \frac{1}{2}+f\varepsilon\end{matrix}\right|z\right)\ ,$$
$$\frac{2z(1-2\varepsilon)(1-3\varepsilon)}{1-\varepsilon}\ {}_3F_2\left(\left.\begin{matrix}1,\varepsilon,2\varepsilon\\ 2-\varepsilon,\frac{1}{2}+\varepsilon\end{matrix}\right|z\right)=1-2\varepsilon$$
$$+\left[2\varepsilon(1-4z)-(1-2z)\right]\ {}_3F_2\left(\left.\begin{matrix}1,\varepsilon,2\varepsilon\\ 1-\varepsilon,\frac{1}{2}+\varepsilon\end{matrix}\right|z\right)+\frac{8z(1-z)\varepsilon^2}{(1-\varepsilon)(1+2\varepsilon)}\ {}_3F_2\left(\left.\begin{matrix}2,1+\varepsilon,1+2\varepsilon\\ 2-\varepsilon,\frac{3}{2}+\varepsilon\end{matrix}\right|z\right)\ ,$$
$$z(1-3\varepsilon)(2-3\varepsilon)\ {}_3F_2\left(\left.\begin{matrix}1,\varepsilon,-1+2\varepsilon\\ 2-\varepsilon,\frac{1}{2}+\varepsilon\end{matrix}\right|z\right)=\frac{1}{2}(1-\varepsilon)(1-2\varepsilon+2z\varepsilon)$$
$$+\frac{4\varepsilon^2z(1-z)(1+2z)}{(1+2\varepsilon)}\ {}_3F_2\left(\left.\begin{matrix}2,1+\varepsilon,1+2\varepsilon\\ 2-\varepsilon,\frac{3}{2}+\varepsilon\end{matrix}\right|z\right)$$
$$-\frac{(1-\varepsilon)}{2}\left[1-4z-2\varepsilon(1-8z-2z^2)\right]\ {}_3F_2\left(\left.\begin{matrix}1,\varepsilon,2\varepsilon\\ 1-\varepsilon,\frac{1}{2}+\varepsilon\end{matrix}\right|z\right)\ . \qquad (88)$$

Here, the following relation was used:

$$z(b-a_2)(b-a_3)(1+b-a_3)(f-a_3)\ {}_3F_2\left(\left.\begin{matrix}1,a_2,a_3-1\\ b+1,f\end{matrix}\right|z\right)$$
$$=-b(1-a_3)(1-z)\left[a_3-f+(a_2-b)z\right]\theta\ {}_3F_2\left(\left.\begin{matrix}1,a_2,a_3\\ b,f\end{matrix}\right|z\right)$$
$$+b(1-a_3)\Bigg\{a_2(a_2-b)z^2+(a_3-f)(1-f)-b(1+a_2+2a_3-b)z$$
$$-\left[f(2a_2+a_3-2b)+a_2(a_2-1)-(a_2+a_3)^2\right]z\Bigg\}\ {}_3F_2\left(\left.\begin{matrix}1,a_2,a_3\\ b,f\end{matrix}\right|z\right)$$
$$-b(f-1)\left[(1-a_3)(f-a_3)-(b-1)(b-a_2)z\right]\ . \qquad (89)$$

The first coefficients of the ε expansions of Eq. (87) read:

$$\frac{B^3_{1220}(m^2,M^2,1,1,1,1)}{\Gamma^3(1+\varepsilon)(i\pi^{2-\varepsilon})^3} = \left(M^2\right)^{2-3\varepsilon}\left(\frac{1+2z}{3\varepsilon^3}+\frac{1}{\varepsilon^2}\left\{\frac{7}{6}+\frac{8}{3}z-\frac{1}{12}z^2-z\ln z\right\}\right.$$
$$+\frac{1}{\varepsilon}\left\{\frac{25}{12}+\frac{20}{3}z-\frac{5}{8}z^2+\frac{1}{4}z\ln z\left[z+2\ln z-16\right]\right\}$$
$$+\frac{8}{3}\zeta_3(1-z)-\frac{5}{24}+\frac{35}{3}z-\frac{145}{48}z^2-\frac{1}{6}z\ln^3 z+\frac{(16-z)}{8}z\ln^2 z-\frac{(80-15z)}{8}z\ln z$$
$$-4(1-z)\left[\mathrm{S}_{1,2}(1-y)+\ln y\mathrm{Li}_2\left(1-y\right)\right]-(1-z)\ln^2 y\left[\ln z+\frac{2}{3}\ln y\right]$$
$$\left.-(8+2z-z^2)\frac{(1-y)}{(1+y)}\left[\mathrm{Li}_2\left(1-y\right)+\frac{1}{2}\ln z\ln y+\frac{1}{4}\ln^2 y\right]+\mathcal{O}(\varepsilon)\right),$$
$$\frac{B^3_{1220}(m^2,M^2,1,2,1,1)}{\Gamma^3(1+\varepsilon)(i\pi^{2-\varepsilon})^3} = \left(M^2\right)^{1-3\varepsilon}\left(\frac{1+z}{3\varepsilon^3}+\frac{1}{6\varepsilon^2}\left\{4+5z-3z\ln z\right\}\right.$$
$$+\frac{1}{3\varepsilon}\left\{1+4z\right\}-\frac{1}{4\varepsilon}z\ln z\left\{4-\ln z\right\}-\frac{10}{3}+\frac{5}{6}z-\frac{1}{2}z\ln z\left[1-\ln z+\frac{1}{6}\ln^2 z\right]$$
$$+(4-z)\frac{(1-y)}{(1+y)}\left[2\mathrm{Li}_2\left(1-y\right)+\frac{1}{2}\ln^2 y+\ln z\ln y\right]$$
$$\left.+2(z-2)\left[\mathrm{S}_{1,2}(1-y)+\ln y\mathrm{Li}_2\left(1-y\right)+\frac{1}{6}\ln^3 y+\frac{1}{4}\ln z\ln^2 y-\frac{2}{3}\zeta_3\right]+\mathcal{O}(\varepsilon)\right), \quad (90)$$

where y and z are defined in Eqs. (82) and (65), respectively. The $\mathcal{O}(\varepsilon)$ term is too long to be presented here, but is available from Ref. [48]. To check Eq. (90), we evaluate the first few coefficients of the ε expansion of the original diagram in the large-mass limit [42] using the program packages developed in Refs. [43,44] to find agreement.

3.5 V^q_{1220} and J^q_{1220}

Let us consider the q-loop on-shell propagator diagram V^q_{1220} in Fig. 2 with s plus r massless propagators,

$$V^q_{1220}(m^2,M^2,\alpha_1,\alpha_2,\beta,\sigma_1,\cdots,\sigma_s,\rho_1,\cdots,\rho_r)$$
$$=\int\frac{d^n(k_1\cdots k_q)}{[k_1^2]^{\sigma_1}\cdots[k_s^2]^{\sigma_s}[k_{s+1}^2-M^2]^{\alpha_1}[(k_1+k_2+\cdots+k_{s+1}+k_q)^2-M^2]^{\alpha_2}}$$
$$\times\left.\frac{1}{[k_{s+2}^2]^{\rho_1}\cdots[k_{q-1}^2]^{\rho_{r-1}}[(k_{s+2}+\cdots+k_{q-1}+k_q)^2)]^{\rho_r}((k_q-p)^2-m^2)^{\beta}}\right|_{p^2=m^2}, \quad (91)$$

where $q = s + r + 1$. The Mellin-Barnes representation of Eq. (91) reads:

$$V^q_{1220}(m^2, M^2, \alpha_1, \alpha_2, \beta, \sigma_1, \cdots, \sigma_s, \rho_1, \cdots, \rho_r) = \frac{(-M^2)^{\frac{n}{2}-\alpha_{1,2}}(-m^2)^{\frac{n}{2}(q-1)-\sigma-\beta-\rho}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)}$$
$$\times \left[i^{1-n}\pi^{\frac{n}{2}}\right]^q \left\{\prod_{j=1}^{s}\frac{\Gamma(\frac{n}{2}-\sigma_j)}{\Gamma(\sigma_j)}\right\}\left\{\prod_{k=1}^{r}\frac{\Gamma(\frac{n}{2}-\rho_k)}{\Gamma(\rho_k)}\right\}\frac{\Gamma\left(\rho-\frac{n}{2}(r-1)\right)}{\Gamma\left(\frac{n}{2}r-\rho\right)}$$
$$\times \int dt \left(\frac{m^2}{M^2}\right)^t \frac{\Gamma(\alpha_1+t)\Gamma(\alpha_2+t)\Gamma(\alpha_{1,2}-\frac{n}{2}+t)\Gamma(\frac{n}{2}+t)}{\Gamma(\alpha_{1,2}+2t)}$$
$$\times \frac{\Gamma\left(\sigma-\frac{n}{2}s-t\right)\Gamma\left(\sigma+\beta+\rho-\frac{n}{2}(q-1)-t\right)\Gamma\left(n(q-1)-2\sigma-\beta-2\rho+2t\right)}{\Gamma\left(\frac{n}{2}(s+1)-\sigma+t\right)\Gamma\left(\frac{n}{2}q-\sigma-\beta-\rho+t\right)}, \quad (92)$$

where σ, ρ, and $\alpha_{1,2}$ are defined in Eqs. (46), (58), and (68), respectively.

Diagram J^q_{1220} in Fig. 2 emerges from V^q_{1220} with $r = 1$ and $s = q-2$ in the smooth limit $\rho = \rho_1 \to 0$. For the special case $q = 3$, the Mellin-Barnes representation has recently been presented in Ref. [55].

Closing the contour of integration in Eq. (92) on the left, we obtain:

$$V^q_{1220}(m^2, M^2, \alpha_1, \alpha_2, \beta, \sigma_1, \cdots, \sigma_s, \rho_1, \cdots, \rho_r) = \frac{\left(i^{1-n}\pi^{\frac{n}{2}}\right)^q (-M^2)^{\frac{n}{2}q-\alpha_{1,2,\sigma,\beta,\rho}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)}$$
$$\times \left\{\prod_{j=1}^{s}\frac{\Gamma(\frac{n}{2}-\sigma_j)}{\Gamma(\sigma_j)}\right\}\left\{\prod_{k=1}^{r}\frac{\Gamma(\frac{n}{2}-\rho_k)}{\Gamma(\rho_k)}\right\}\frac{\Gamma\left(\rho-\frac{n}{2}(r-1)\right)}{\Gamma\left(\frac{n}{2}r-\rho\right)}$$
$$\times \left\{\left(\frac{m^2}{M^2}\right)^{\frac{n}{2}r-a_{\beta,\rho}}\frac{\Gamma\left(\sigma-\frac{n}{2}(s-1)\right)\Gamma\left(nr-2\rho-\beta\right)\Gamma\left(\rho+\beta-\frac{n}{2}r\right)}{\Gamma\left(\alpha_{1,2}+2\sigma-ns\right)\Gamma\left(\frac{n}{2}(r+1)-\rho-\beta\right)}\right.$$
$$\times \Gamma\left(a_{1,\sigma}-\tfrac{n}{2}s\right)\Gamma\left(a_{2,\sigma}-\tfrac{n}{2}s\right)\Gamma\left(a_{1,2,\sigma}-\tfrac{n}{2}(s+1)\right)$$
$$\times \ {}_6F_5\left(\left.\begin{matrix}a_{1,\sigma}-\frac{n}{2}s, a_{2,\sigma}-\frac{n}{2}s, a_{1,2,\sigma}-\frac{n}{2}(s+1), \sigma-\frac{n}{2}(s-1), \frac{nr-\beta}{2}-\rho, \frac{nr-\beta+1}{2}-\rho \\ \frac{n}{2}, \frac{\alpha_{1,2}-ns}{2}+\sigma, \frac{\alpha_{1,2}+1-ns}{2}+\sigma, \frac{n}{2}(r+1)-\beta-\rho, \frac{n}{2}r-\beta-\rho+1\end{matrix}\right| \frac{m^2}{M^2}\right)$$
$$+ \frac{\Gamma\left(\frac{n}{2}r-a_{\beta,\rho}\right)\Gamma\left(a_{\sigma,\beta,\rho}-\frac{n}{2}(q-2)\right)\Gamma(\beta)}{\Gamma\left(\alpha_{1,2}+2\sigma+2\beta+2\rho-n(q-1)\right)\Gamma\left(a_{\beta,\rho}-\frac{n}{2}(r-1)\right)}$$
$$\times \Gamma\left(a_{1,\sigma,\beta,\rho}-\tfrac{n}{2}(q-1)\right)\Gamma\left(a_{2,\sigma,\beta,\rho}-\tfrac{n}{2}(q-1)\right)\Gamma\left(a_{1,2,\sigma,\beta,\rho}-\tfrac{n}{2}q\right)$$
$$\times \left.{}_6F_5\left(\left.\begin{matrix}a_{1,\sigma,\beta,\rho}-\frac{n}{2}(q-1), a_{2,\sigma,\beta,\rho}-\frac{n}{2}(q-1), a_{1,2,\sigma,\beta,\rho}-\frac{n}{2}q, a_{\sigma,\beta,\rho}-\frac{n}{2}(q-2), \frac{\beta}{2}, \frac{\beta+1}{2} \\ \frac{n}{2}, 1+a_{\beta,\rho}-\frac{n}{2}r, \frac{\alpha_{1,2}-n(q-1)}{2}+a_{\sigma,\beta,\rho}, \frac{\alpha_{1,2}+1-n(q-1)}{2}+a_{\sigma,\beta,\rho}, a_{\beta,\rho}-\frac{n}{2}(r-1)\end{matrix}\right| \frac{m^2}{M^2}\right)\right\}, \quad (93)$$

where $a_{1,\sigma}$, $a_{2,\sigma}$, $a_{\beta,\rho}$, $a_{1,2,\sigma}$, $a_{\sigma,\beta,\rho}$, $a_{1,\sigma,\beta,\rho}$, and $a_{2,\sigma,\beta,\rho}$ are defined in Eq. (75). In the special case in which there is no massless propagator inside the massive loop, $s = 0$ ($r = q-1$, $\sigma = 0$, $\prod_{k=1}^{q-2}\frac{\Gamma(\frac{n}{2}-\sigma_k)}{\Gamma(\sigma_k)} = 1$), the ${}_6F_5$ functions in Eq. (93) are reduced to ${}_5F_4$ functions. For $q = 2$, $s = 0$, and $r = 1$, the hypergeometric representation was given in Ref. [56].

Let us analyze the result of the differential reduction of the hypergeometric functions in Eq. (93) assuming that all parameters, α_1, α_2, β, σ_k, and ρ_k, are

integer. For $r \neq 0$, we distinguish between the three cases $s = 0$, $s = 1$, and $s \geq 2$. For $s = 0$ ($r = q-1$, $\sigma = 0$), each of the two hypergeometric functions in Eq. (93) is reducible to a ${}_3F_2$ function with one unit upper parameter and its first derivative,

$$(1,\theta) \times \ {}_3F_2\left(\left.\begin{matrix} 1, I_1 - \frac{n}{2}, I_2 + \frac{1}{2} + \frac{n}{2}(q-1) \\ I_3 + \frac{1}{2}, I_4 + \frac{n}{2}q \end{matrix}\right| z\right) \ ,$$
$$(1,\theta) \times \ {}_3F_2\left(\left.\begin{matrix} 1, I_1 + \frac{1}{2}, I_2 - \frac{n}{2}q \\ I_3 + \frac{n}{2}, \frac{1}{2} + I_4 - \frac{n}{2}(q-1) \end{matrix}\right| z\right) \ . \tag{94}$$

For $s = 1$ ($r = q-2$), each of the two hypergeometric functions in Eq. (93) is reducible to a ${}_4F_3$ function with one unit upper parameter and its first two derivatives,

$$(1,\theta,\theta^2) \times \ {}_4F_3\left(\left.\begin{matrix} 1, I_1 - n, I_2 - \frac{n}{2}, I_3 + \frac{1}{2} + \frac{n}{2}(q-2) \\ I_4 + \frac{n}{2}, I_5 + \frac{1}{2} - \frac{n}{2}, I_6 + \frac{n}{2}(q-1) \end{matrix}\right| z\right) \ ,$$
$$(1,\theta,\theta^2) \times \ {}_4F_3\left(\left.\begin{matrix} 1, I_1 + \frac{1}{2}, I_2 - \frac{n}{2}q, I_3 - \frac{n}{2}(q-1) \\ I_4 + \frac{n}{2}, \frac{1}{2} + I_5 - \frac{n}{2}(q-1), I_6 - \frac{n}{2}(q-3) \end{matrix}\right| z\right) \ . \tag{95}$$

For $s \geq 2$, one of the hypergeometric functions in Eq. (93) is reducible to a ${}_4F_3$ function with all upper parameters having non-zero ε parts, whereas the second one is expressible in terms of a ${}_5F_4$ function with one unit upper parameter,

$$(1,\theta,\theta^2,\theta^3) \times \ {}_4F_3\left(\left.\begin{matrix} I_1 - \frac{n}{2}(s-1), I_2 - \frac{n}{2}s, I_3 - \frac{n}{2}(s+1), I_4 + \frac{1}{2} + \frac{n}{2}r \\ I_5 + \frac{n}{2}, I_6 + \frac{1}{2} - \frac{n}{2}s, I_7 + \frac{n}{2}(r+1) \end{matrix}\right| z\right) \ ,$$
$$(1,\theta,\theta^2,\theta^3) \times \ {}_5F_4\left(\left.\begin{matrix} 1, I_1 + \frac{1}{2}, I_2 - \frac{n}{2}q, I_3 - \frac{n}{2}(q-1), I_4 - \frac{n}{2}(q-2) \\ I_5 + \frac{n}{2}, \frac{1}{2} + I_6 - \frac{n}{2}(q-1), I_7 - \frac{n}{2}r, I_8 - \frac{n}{2}(r-1) \end{matrix}\right| z\right) \ . \tag{96}$$

For completeness, we also present an explicit result for diagram J^q_{1220} in Fig. 2:

$$\begin{aligned}
&J^q_{1220}(m^2, M^2, \alpha_1, \alpha_2, \beta, \sigma_1, \cdots, \sigma_{q-2}) \\
&= \frac{\left[i^{1-n}\pi^{\frac{n}{2}}\right]^q \Gamma\left(\beta - \frac{n}{2}\right) (-M^2)^{\frac{n}{2}q - \alpha_{1,2,\sigma,\beta}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)\Gamma\left(\frac{n}{2}\right)} \left\{ \prod_{k=1}^{q-2} \frac{\Gamma(\frac{n}{2} - \sigma_k)}{\Gamma(\sigma_k)} \right\} \\
&\quad \times \left\{ \left(\frac{m^2}{M^2}\right)^{\frac{n}{2}-\beta} \frac{\Gamma\left(\sigma - \frac{n}{2}(q-3)\right) \Gamma\left(a_{1,\sigma} - \frac{n}{2}(q-2)\right) \Gamma\left(a_{2,\sigma} - \frac{n}{2}(q-2)\right) \Gamma\left(a_{1,2,\sigma} - \frac{n}{2}(q-1)\right)}{\Gamma\left(\alpha_{1,2} + 2\sigma - n(q-2)\right)} \right. \\
&\quad \times \ {}_6F_5\left(\left.\begin{matrix} a_{1,\sigma} - \frac{n}{2}(q-2), a_{2,\sigma} - \frac{n}{2}(q-2), a_{1,2,\sigma} - \frac{n}{2}(q-1), \sigma - \frac{n}{2}(q-3), \frac{n-\beta}{2}, \frac{n-\beta+1}{2} \\ \frac{n}{2}, n-\beta, \frac{n}{2}+1-\beta, \frac{\alpha_{1,2}-n(q-2)}{2}+\sigma, \frac{\alpha_{1,2}+1-n(q-2)}{2}+\sigma \end{matrix}\right| \frac{m^2}{M^2}\right) \\
&\quad + \frac{\Gamma\left(\frac{n}{2}-\beta\right) \Gamma\left(a_{\sigma,\beta} - \frac{n}{2}(q-2)\right) \Gamma\left(a_{1,\sigma,\beta} - \frac{n}{2}(q-1)\right) \Gamma\left(a_{2,\sigma,\beta} - \frac{n}{2}(q-1)\right) \Gamma\left(\alpha_{1,2,\sigma,\beta} - \frac{n}{2}q\right)}{\Gamma\left(\beta - \frac{n}{2}\right) \Gamma\left(\alpha_{1,2} + 2\sigma + 2\beta - n(q-1)\right)} \\
&\quad \left. \times \ {}_6F_5\left(\left.\begin{matrix} a_{1,\sigma,\beta} - \frac{n}{2}(q-1), a_{2,\sigma,\beta} - \frac{n}{2}(q-1), a_{1,2,\sigma,\beta} - \frac{n}{2}q, a_{\sigma,\beta} - \frac{n}{2}(q-2), \frac{\beta}{2}, \frac{\beta+1}{2} \\ \frac{n}{2}, \beta, 1+\beta-\frac{n}{2}, \frac{\alpha_{1,2}-n(q-1)}{2}+\sigma+\beta, \frac{\alpha_{1,2}+1-n(q-1)}{2}+\sigma+\beta \end{matrix}\right| \frac{m^2}{M^2}\right) \right\} .
\end{aligned} \tag{97}$$

In this case, the result of the differential reduction assuming that all parameters, α_1, α_2, β, and σ_k, are integer can be derived for $q = 2$, $q = 3$, and $q \geq 4$ by directly

substituting $r = 1$ and $s = q - 2$ in Eqs. (94)–(96), respectively. We only point out here that, for $q = 3$, the second hypergeometric function in Eq. (95) is reducible to a ${}_3F_2$ function and its first two derivatives, so that the differential reduction yields

$$
\begin{aligned}
&(1, \theta, \theta^2) \times \; {}_4F_3 \left(\left. \begin{matrix} 1, I_1 - n, I_2 - \frac{n}{2}, I_3 + \frac{1}{2} + \frac{n}{2} \\ I_4 + \frac{n}{2}, I_5 + \frac{1}{2} - \frac{n}{2}, I_6 + n \end{matrix} \right| z \right) \; , \\
&(1, \theta, \theta^2) \times \; {}_3F_2 \left(\left. \begin{matrix} I_1 + \frac{1}{2}, I_2 - n, I_3 - \frac{3n}{2} \\ I_4 + \frac{n}{2}, \frac{1}{2} + I_5 - n \end{matrix} \right| z \right) \; ,
\end{aligned}
\tag{98}
$$

and that, for $q \geq 4$, the ${}_5F_4$ hypergeometric functions in Eq. (96) is reducible to a ${}_4F_3$ function with all upper parameters having non-zero ε parts:

$$
\begin{aligned}
&(1, \theta, \theta^2, \theta^3) \times \; {}_4F_3 \left(\left. \begin{matrix} I_1 - \frac{n}{2}(q-3), I_2 - \frac{n}{2}(q-2), I_3 - \frac{n}{2}(q-1), I_4 + \frac{1}{2} + \frac{n}{2} \\ I_5 + \frac{n}{2}, I_6 + \frac{1}{2} - \frac{n}{2}(q-2), I_7 + n \end{matrix} \right| z \right) \; , \\
&(1, \theta, \theta^2, \theta^3) \times \; {}_4F_3 \left(\left. \begin{matrix} I_1 + \frac{1}{2}, I_2 - \frac{n}{2}q, I_3 - \frac{n}{2}(q-1), I_4 - \frac{n}{2}(q-2) \\ I_5 + \frac{n}{2}, \frac{1}{2} + I_6 - \frac{n}{2}(q-1), I_7 - \frac{n}{2} \end{matrix} \right| z \right) \; .
\end{aligned}
\tag{99}
$$

In particular, for $q = 2$, in accordance with Ref. [57], there are two nontrivial master integrals plus bubble integrals. For $q = 3$, in accordance with Ref. [55], there are three nontrivial master integrals plus bubble integrals.

3.6 Two-loop vertex

Let us consider the diagram F_2 shown in Fig. 2

$$
\begin{aligned}
&F_2(M^2, p^2, \alpha_1, \alpha_2, \sigma_1, \sigma_2,) = \\
&\int \frac{d^n(k_1 k_2)}{[(k_1 - p_1)^2]^{\sigma_1} [(k_1 - p_2)^2]^{\sigma_2} [k_1^2]^{\beta} [k_2^2 - M^2]^{\alpha_1} [(k_1 - k_2)^2 - M^2]^{\alpha_2}} \Bigg|_{p_1^2 = p_2^2 = 0} \\
&= \left[i^{1-n} \pi^{\frac{n}{2}} \right]^2 \frac{(-M^2)^{\frac{n}{2} - \alpha_{12}} (p^2)^{\frac{n}{2} - \sigma_{12} - \beta}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\sigma_1) \Gamma(\sigma_2)} \\
&\quad \times \int dt \left(-\frac{p^2}{M^2} \right)^t \frac{\Gamma(-t) \Gamma(\alpha_1 + t) \Gamma(\alpha_2 + t) \Gamma(\alpha_{12} - \frac{n}{2} + t)}{\Gamma\left(\alpha_{12} + 2t\right)} \\
&\quad \frac{\Gamma\left(\frac{n}{2} - \sigma_1 - \beta + t\right) \Gamma\left(\frac{n}{2} - \sigma_2 - \beta + t\right) \Gamma\left(\sigma_{12} + \beta - \frac{n}{2} - t\right)}{\Gamma(n - \sigma_{12} - \beta + t)} \; .
\end{aligned}
\tag{100}
$$

where σ is defined in Eq. (46) with $s = 2$ and α_{12} in Eq. (68). Closing the contour of integration on the left we obtain in the notation of Ref. [38]:

$$F_2(M^2,p^2,\alpha_1,\alpha_2,\sigma_1,\sigma_2,) = \left[i^{1-n}\pi^{\frac{n}{2}}\right]^2 \frac{(-M^2)^{\frac{n}{2}-\alpha_{12}}(p^2)^{\frac{n}{2}-\sigma_{12}-\beta}\Gamma\left(\frac{n}{2}-\sigma_{12}-\beta\right)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}$$
$$\times\Bigg\{\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma\left(\frac{n}{2}-\sigma_1-\beta\right)\Gamma\left(\frac{n}{2}-\sigma_2-\beta\right)\Gamma\left(\sigma_{12}+\beta-\frac{n}{2}\right)\Gamma\left(\alpha_{12}-\frac{n}{2}\right)}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\alpha_{12})\Gamma\left(n-\sigma_{12}-\beta\right)\Gamma\left(\frac{n}{2}-\sigma_{12}-\beta\right)}$$
$$\times\ {}_5F_4\left(\left.\begin{matrix}\alpha_1,\alpha_2,\alpha_{12}-\frac{n}{2},\frac{n}{2}-\sigma_1-\beta,\frac{n}{2}-\sigma_2-\beta\\ n-\sigma_{12}-\beta,1+\frac{n}{2}-\sigma_{12}-\beta,\frac{\alpha_{12}}{2},\frac{\alpha_{12}+1}{2}\end{matrix}\right|-\frac{p^2}{4M^2}\right)$$
$$+\left(-\frac{p^2}{M^2}\right)^{\beta+\sigma_{12}-\frac{n}{2}}\frac{\Gamma\left(\alpha_1+\sigma_{12}+\beta-\frac{n}{2}\right)\Gamma\left(\alpha_2+\sigma_{12}+\beta-\frac{n}{2}\right)\Gamma\left(\sigma_{12}+\alpha_{12}+\beta-n\right)}{\Gamma(\alpha_{12}+2\sigma_{12}+2\beta-n)\Gamma\left(\frac{n}{2}\right)}$$
$$\times\ {}_5F_4\left(\left.\begin{matrix}\sigma_1,\sigma_2,\alpha_1+\sigma_{12}+\beta-\frac{n}{2},\alpha_2+\sigma_{12}+\beta-\frac{n}{2},\alpha_{12}+\sigma_{12}+\beta-n\\ \frac{n}{2},1+\sigma_{12}+\beta-\frac{n}{2},\sigma_{12}+\beta+\frac{\alpha_{12}-n}{2},\sigma_{12}+\beta+\frac{\alpha_{12}+1-n}{2}\end{matrix}\right|-\frac{p^2}{4M^2}\right)\Bigg\} \quad (101)$$

By differential reduction, the two hypergeometric functions in Eq. (101) may be reduced to

$$(1,\theta)\times\ {}_3F_2\left(\left.\begin{matrix}1,I_1-\frac{n}{2},\frac{n}{2}+I_2\\ n+I_3,\frac{1}{2}+I_4\end{matrix}\right|-\frac{p^2}{4M^2}\right)\ ,\quad (1,\theta)\times\ {}_3F_2\left(\left.\begin{matrix}1,1,I_1-n\\ I_2+\frac{n}{2},I_3+\frac{1}{2}-\frac{n}{2}\end{matrix}\right|-\frac{p^2}{4M^2}\right) \quad (102)$$

respectively, where $\{I_a\}$ is a set of integers. In accordance with Ref. [20], the first few coefficients of the ε expansion of this diagram are expressible in terms of Remiddi-Vermaseren functions, i.e. multiple polylogarithms of the square roots of unity, of argument

$$\tilde{y} = \frac{1-\sqrt{\frac{z}{z+4}}}{1+\sqrt{\frac{z}{z+4}}}, \quad (103)$$

where $z = p^2/M^2$ (see also the discussion in Ref. [58]). The on-shell value $z = 1$ corresponds to Remiddi-Vermaseren functions of argument $(3-\sqrt{5})/2 = [(\sqrt{5}-1)/2]^2$, in agreement with results of Ref. [59]. It is interesting to note that these constants are generated not only in Higgs-boson decay [59], but also in orthopositronium decay [60].

4 Master integrals in differential reduction and IBP technique

In this section, we compare the differential-reduction and IPB techniques with respect to the numbers of master integrals which they produce. For this purpose, we return to the Feynman diagrams depicted in Figs. 1 and 2. They all have the following hypergeometric structure:

$$\Phi(n,\vec{j};z) = \sum_{a=1}^{k} z^{l_a} C_{l_a}(n,\vec{j})\,{}_{p+1}F_p(\vec{A}_a;\vec{B}_a;\kappa z)\ , \quad (104)$$

Table 1: Highest powers of the differential operator θ generated by the differential reduction of the Feynman diagrams shown in Fig. 2.

h	E^q_{120}	s	E^q_{1220}	V^q_{1220}	q	J^q_{22}	B^q_{1220}	J^q_{1220}
1	1	0	1	2	1	1	–	–
≥ 2	2	1	2	3	2	2	1	2
		≥ 2	3	4	3	3	2	3
					≥ 4	3	3	4

where $\vec{j}$ is the set of the powers of the propagators of the Feynman diagram, n is dimension of space-time, k is the number of hypergeometric functions, $\{l_a, A_a, B_a\}$ are linear combinations of $\vec{j}$ and n with rational coefficients, κ is a rational number (being $\kappa = 1, 1/4$ in the considered cases), and C_{l_a} are products of Γ functions with arguments only depending on n and $\vec{j}$. Being a sum of holonomic functions, $\Phi(\vec{j}; \vec{z})$ is also holonomic. Thus, the number of basis elements on the r.h.s. of Eq. (104) is equal to the number of master-integrals $\{\Phi_k(\vec{z})\}$ that may be derived from the l.h.s. of Eq. (104) by applying the integration-by-parts technique, which may be written symbolically as

$$\Phi(n, \vec{j}; z) = \sum_{k=1}^{h} B_k(n, \vec{j}; z) \Phi_k(n; z) \; . \tag{105}$$

Here, it is understood that diagrams that are expressible in terms of Gamma functions [61] are not counted. The number of basis elements in the framework of differential reduction is defined to be the highest power of the differential operator θ in

$$ {}_{p+1}F_p(\vec{A}; \vec{B}; z) = \sum_{l=0}^{v} P_l(z) \theta^l \, {}_{s+1}F_s(\vec{A} - \vec{I}_1; \vec{B} - \vec{I}_2; z) \; , \tag{106}$$

where $\vec{I}_1, \vec{I}_2$ are sets of integers, $P_l(z)$ are rational functions, and the differential-reduction algorithm for exceptional values of parameters of hypergeometric function, constructed in Sections (2.3) and (2.4) is employed, in which case one has $v \leq p$ in general. Our analysis demonstrates that there is a very simple relation between the number h of nontrivial master integrals follows from IBP, which are not expressible in terms of Gamma functions, and the maximal power v of θ generated by the differential reduction in Eq. (107), namely

$$h = v + 1 \; . \tag{107}$$

This relation does not depend on the number k of hypergeometric functions entering Eq. (104). In Table 1, these highest powers are collected for the Feynman diagrams shown in Fig. 2.

5 Discussion and conclusion

The idea that any Feynman diagram can be associated with a generalized Horn-type hypergeometric function or a linear combination of such functions was born at the end of the 60s [7] and was applied during the last twenty years [38,46,62]. The Mellin-Barnes representation (1) of a Feynman diagram is universal. Under certain conditions, it can be converted to a linear combination of Horn-type hypergeometric functions (2). The latter possess very interesting properties. In fact, the systems of differential equations which they satisfy are sufficient for (i) the differential reduction of the original functions to restricted sets of basis functions, whose number follows directly from the system of differential equations; (ii) the construction of the all-order ε expansions of the basis hypergeometric functions in form of iterated integrals [16–19]. To our knowledge, the first property has never been discussed in the context of a reduction procedure of Feynman diagrams.

The aim of the present paper was to show how the differential-reduction algorithm can be applied to reduce Feynman diagrams without having to exploit the integration-by-parts technique [15] or its dimensional generalization [37]. In our analysis, we considered a phenomenologically interesting class of Feynman diagrams, namely those shown in Figs. 1 and 2, which are expressible in terms of generalized hypergeometric functions ${}_{p+1}F_p$ of one variable and cannot be treated with currently available program packages.

As first steps in our analysis, Takayama's differential-reduction algorithm [22] was extended to the case of *exceptional* values of parameters in Section 2.3, and the basis of differential reduction was written explicitly in Section 2.4. In particular, this algorithm allowed us to complete the proofs of the theorems regarding the construction of all-order ε expansions of hypergeometric functions presented in Refs. [16–19]. For non-exceptional values of parameters, this algorithm is implemented in the Mathematica based program `HYPERDIRE` to be described in a in a separate communication [63].

The differential-reduction formalism via the construction of step-up and step-down operators can be applied to the reduction of any Feynman diagram. Its advantages reside in the simplicity and universality of the construction of this type of operators, the full control over the analytical structure of the Feynman diagrams via the hypergeometric representations, the simplicity of the criterion of reducibility of hypergeometric functions to simpler functions, and the existence of a few, recently developed algorithms [16–20,27] for analytically constructing the coefficients of the ε expansions of hypergeometric functions.

The main criteria of reducibility of hypergeometric functions, namely integer difference between upper and lower parameters and integer value of one of the upper parameters, are also valid for Horn-type hypergeometric functions of more than one variables. Moreover, the first type of reduction can be applied directly to Mellin-Barnes representations of Feynman diagrams.

This approach becomes incomplete when some of the variables of a Horn-type hypergeometric function belong to its surface of singularities; for hypergeometric functions of one variable, this corresponds to $z = 1$. In this case, the extended Ore algebra technique [64], the telescoping approach [65], or the Laporta algorithm [66]

can be employed to obtain exact analytical results. In addition to these algorithms, there is another approach based on the analytical structure of the coefficients of the ε expansions of hypergeometric functions: instead of looking for exact recurrence relations, the limit $z \to 1$ can be taken for the coefficients of the Laurent expansions. In particular, this procedure can be applied to any hypergeometric function whose ε expansion has coefficients that are expressible in terms of multiple polylogarithms. This algorithm may be applied to evaluate Feynman diagrams, if just the first few coefficients of their ε expansions are required.

As an illustration of the usefulness of our differential-reduction procedure, we considered a few examples, namely the Feynman diagrams shown in Figs. 1 and 2 with arbitrary powers of propagators and space-time dimension, and explained how to express them in terms of hypergeometric functions and to construct their ε expansions to higher orders.

An interesting observation we made here is that, up to products of one-loop bubbles, the number of master integrals constructed through the integration-by-parts technique is equal to the highest power of the differential operator θ generated by the differential reduction of hypergeometric functions plus one, independently of the number of hypergeometric functions occuring in the expressions for the Feynman diagrams.

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A Sets of basis functions for the Laurent expansions of hypergeometric functions

A.1 All parameters are integer

Using the algorithm described in Ref. [18], the derivatives of a basis hypergeometric function with integer parameters may be expressed in terms of the $\rho_{p+k}^{(p-a)}(z)$ functions defined in that reference, as[13]

$$\theta^j {}_pF_{p-1}(\vec{a}\varepsilon; 1+\vec{b}\varepsilon; z) = \delta_{j0} + \varepsilon^p \left(\prod_{r=1}^{p} a_r\right) \sum_{k=0}^{\infty} \varepsilon^k \rho_{p+k}^{(p-1-j)}(z) \;, \qquad j = 0, \cdots, p-1 \;. \tag{108}$$

[13] In Eq. (2.11a) of Ref. [18], the last parameter should have the value $p-2$ rather than $p-1$.

The iterative solution of the $\rho_{p+k}^{(p-1-j)}(z)$ functions is given by Eqs. (2.13a)–(2.13b) in Ref. [18], and their explicit forms in terms of generalized polylogarithms are collected in Section 3 therein.

A.2 One lower parameter is half-integer

In Refs. [20,29], the ε expansions of the functions

$$ {}_{P+1}F_P\left(\left.\begin{array}{c}\{1+a_i\varepsilon\}^K, \{2+d_i\varepsilon\}^{P-K+1}\\ \frac{3}{2}+f\varepsilon, \{1+e_j\varepsilon\}^R, \{2+c_i\varepsilon\}^{P-R-1}\end{array}\right| z\right) , \tag{109} $$

with $K-R \geq 2$, were constructed up to functions of weight 4 and, in the case of $z=1/4$, up to weight 5.[14] For these parameters, after some trivial factorization, the coefficients of the ε expansions contain functions of only one definite weight. In this section, we present explicit relations between the basis functions of the differential reduction, defined by Eq. (17), and the functions of Eq. (109). One of the auxiliary relations intensively used in the following reads:

$$ {}_pF_{p-1}\left(\left.\begin{array}{c}a_1, a_2, \cdots, a_p\\ b_1, b_2, \cdots, b_{p-1}\end{array}\right| z\right) = 1 + z\frac{\prod_{j=1}^p a_j}{\prod_{k=1}^{p-1} b_k}\ {}_{p+1}F_p\left(\left.\begin{array}{c}1, 1+a_1, 1+a_2, \cdots, 1+a_p\\ 2, 1+b_1, 1+b_2, \cdots, 1+b_{p-1}\end{array}\right| z\right) . \tag{110} $$

A.2.1 ${}_2F_1$

A detailed discussion of the ${}_2F_1$ function was presented in Refs. [16,19,35].

A.2.2 ${}_3F_2$

For the hypergeometric function

$$ {}_3F_2\left(\left.\begin{array}{c}I_1+a_1\varepsilon, I_2+a_2\varepsilon, I_3+a_3\varepsilon\\ I_4+\frac{1}{2}+f\varepsilon, I_5+c_1\varepsilon\end{array}\right| z\right) , \tag{111} $$

where $\{I_k\}$ are arbitrary integers and all $a_i \neq 0$, the basis of the differential reduction is

$$ \{1, \theta, \theta^2\} \times\ {}_3F_2\left(\left.\begin{array}{c}a_1\varepsilon, a_2\varepsilon, a_3\varepsilon\\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon\end{array}\right| z\right) . \tag{112} $$

For $P=2$, there are four functions of the type of Eq. (109):

$$ {}_4F_3\left(\left.\begin{array}{c}1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2\end{array}\right| z\right) , \quad {}_3F_2\left(\left.\begin{array}{c}1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{array}\right| z\right) , $$
$$ {}_3F_2\left(\left.\begin{array}{c}1+a_1\varepsilon, 1+a_2\varepsilon, 2+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{array}\right| z\right) , \quad {}_3F_2\left(\left.\begin{array}{c}1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 1+c_1\varepsilon\end{array}\right| z\right) . \tag{113} $$

[14] The series representations of these hypergeometric functions are proportional to $\sum_{j=0}^{\infty} \frac{z^j}{j^{K-R-1}}$.

The functions of Eqs. (112) and (113) are related as

$$
{}_3F_2\left(\left.\begin{matrix}a_1\varepsilon, a_2\varepsilon, a_3\varepsilon\\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon\end{matrix}\right| z\right) = 1 + \frac{2a_1a_2a_3\varepsilon^3}{(1+2f\varepsilon)(1+c_1\varepsilon)}\ {}_4F_3\left(\left.\begin{matrix}1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2\end{matrix}\right| z\right) ,
$$

$$
\theta\ {}_3F_2\left(\left.\begin{matrix}a_1\varepsilon, a_2\varepsilon, a_3\varepsilon\\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon\end{matrix}\right| z\right) = 2z\frac{a_1a_2a_3\varepsilon^3}{(1+2f\varepsilon)(1+c_1\varepsilon)}\ {}_3F_2\left(\left.\begin{matrix}1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{matrix}\right| z\right) . \tag{114}
$$

The second derivative in Eq. (112) may be obtained from any of the following relations:

$$
{}_3F_2\left(\left.\begin{matrix}1+a_1, 1+a_2, 1+a_3\\ 1+f, c_1\end{matrix}\right| z\right) = \frac{1}{z}\frac{f\theta(\theta+c_1-1)}{a_1a_2a_3}\ {}_3F_2\left(\left.\begin{matrix}a_1, a_2, a_3\\ f, c_1\end{matrix}\right| z\right) ,
$$

$$
{}_3F_2\left(\left.\begin{matrix}1+a_1, 1+a_2, 2+a_3\\ 1+f, 1+c_1\end{matrix}\right| z\right) = \frac{1}{z}\frac{c_1f\theta(\theta+a_3)}{a_1a_2a_3(1+a_3)}\ {}_3F_2\left(\left.\begin{matrix}a_1, a_2, a_3\\ f, c_1\end{matrix}\right| z\right) , \tag{115}
$$

where

$$
a_i \to a_i\varepsilon, \qquad f \to \frac{1}{2}+f\varepsilon, \qquad c_1 \to 1+c_1\varepsilon, \tag{116}
$$

are to be substituted. In particular, it can be expressed as a linear combination of the second, third, and fourth functions in Eq. (113) as

$$
\begin{aligned}
&\frac{(\gamma_1+\gamma_2)}{z}\frac{fc_1}{a_1a_2a_3}\theta^2\ {}_3F_2\left(\left.\begin{matrix}a_1, a_2, a_3\\ c_1, f\end{matrix}\right| z\right)\\
&= \gamma_1c_1\ {}_3F_2\left(\left.\begin{matrix}1+a_1, 1+a_2, 1+a_3\\ c_1, 1+f\end{matrix}\right| z\right) + \gamma_2(1+a_3)\ {}_3F_2\left(\left.\begin{matrix}1+a_1, 1+a_2, 2+a_3\\ 1+c_1, 1+f\end{matrix}\right| z\right)\\
&- [\gamma_1(c_1-1)+\gamma_2a_3]\ {}_3F_2\left(\left.\begin{matrix}1+a_1, 1+a_2, 1+a_3\\ 1+c_1, 1+f\end{matrix}\right| z\right) ,
\end{aligned} \tag{117}
$$

where γ_1 and γ_2 are arbitrary numbers. Putting $\gamma_2 = -\gamma_1$, we obtain a linear relation between the hypergeometric functions on the r.h.s. of Eq. (117). We checked that the ε expansions of the hypergeometric functions constructed in Ref. [20] and collected in Ref. [48] satisfy this relation.

In the case $a_1 = 1$, the basis of the differential reduction is

$$
\{1, \theta\} \times\ {}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{matrix}\right| z\right) . \tag{118}
$$

The first derivative in Eq. (118) can be obtained from any of the two functions

$$
{}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 1+c_1\varepsilon\end{matrix}\right| z\right) , \qquad {}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 2+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{matrix}\right| z\right) , \tag{119}
$$

by the single application of the step-up and step-down operators defined in Eq. (7). The result is

$$
\begin{aligned}
&(\gamma_1+\gamma_2)\theta\ {}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{matrix}\right| z\right) =\\
&\gamma_1(1+a_3\varepsilon)\ {}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 2+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{matrix}\right| z\right) + \gamma_2(1+c_1\varepsilon)\ {}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 1+c_1\varepsilon\end{matrix}\right| z\right)\\
&- [\gamma_1(1+a_3\varepsilon)+\gamma_2(1+c_1\varepsilon)]\ {}_3F_2\left(\left.\begin{matrix}1, 1+a_2\varepsilon, 1+a_3\varepsilon\\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon\end{matrix}\right| z\right) ,
\end{aligned} \tag{120}
$$

where γ_1 and γ_2 are arbitrary numbers.

A.2.3 ${}_4F_3$

Similarly, for the hypergeometric function

$$ {}_4F_3\left(\left.\begin{matrix} I_1+a_1\varepsilon, I_2+a_2\varepsilon, I_3+a_3\varepsilon, I_4+a_4\varepsilon \\ I_5+\frac{1}{2}+f\varepsilon, I_6+c_1\varepsilon, I_7+c_2\varepsilon \end{matrix}\right| z\right) , \tag{121} $$

where $\{I_k\}$ are arbitrary integers and all $a_i \neq 0$, the basis of the differential reduction is

$$ \{1,\theta,\theta^2,\theta^3\} \times \ {}_4F_3\left(\left.\begin{matrix} a_1\varepsilon, a_2\varepsilon, a_3\varepsilon, a_4\varepsilon \\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon \end{matrix}\right| z\right) . \tag{122} $$

There are seven functions of the type of Eq. (109), namely

$$ \begin{aligned} &{}_5F_4\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon, 1 \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon, 2 \end{matrix}\right| z\right) . \qquad {}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon \end{matrix}\right| z\right) , \\ &{}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 2+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon \end{matrix}\right| z\right) , \qquad {}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 1+c_2\varepsilon \end{matrix}\right| z\right) , \\ &{}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 2+a_3\varepsilon, 2+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon \end{matrix}\right| z\right) , \qquad {}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 2+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon \end{matrix}\right| z\right) , \\ &{}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon \end{matrix}\right| z\right) . \end{aligned} \tag{123} $$

The two bases defined by Eqs. (122) and (123) are related as

$$ \begin{aligned} &{}_4F_3\left(\left.\begin{matrix} a_1\varepsilon, a_2\varepsilon, a_3\varepsilon, a_4\varepsilon \\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon \end{matrix}\right| z\right) = \\ &1+\frac{2a_1a_2a_3a_4\varepsilon^4}{(1+2f\varepsilon)(1+c_1\varepsilon)(1+c_2\varepsilon)}\ {}_5F_4\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon, 1 \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon, 2 \end{matrix}\right| z\right) , \\ &\theta\ {}_4F_3\left(\left.\begin{matrix} a_1\varepsilon, a_2\varepsilon, a_3\varepsilon, a_4\varepsilon \\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon \end{matrix}\right| z\right) \\ &= 2z\frac{a_1a_2a_3a_4\varepsilon^4}{(1+2f\varepsilon)(1+c_1\varepsilon)(1+c_2\varepsilon)}\ {}_4F_3\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon \end{matrix}\right| z\right) . \end{aligned} \tag{124} $$

The second derivative in Eq. (122) may be obtained from any of the following relations:

$$ \begin{aligned} &{}_4F_3\left(\left.\begin{matrix} 1+a_1, 1+a_2, 1+a_3, 1+a_4 \\ 1+f, 1+c_1, c_2 \end{matrix}\right| z\right) = \frac{f}{z}\frac{c_1\theta(\theta+c_2-1)}{a_1a_2a_3a_4}\ {}_4F_3\left(\left.\begin{matrix} a_1, a_2, a_3, a_4 \\ f, c_1, c_2 \end{matrix}\right| z\right) , \\ &{}_4F_3\left(\left.\begin{matrix} 1+a_1, 1+a_2, 1+a_3, 2+a_4 \\ 1+f, 1+c_1, 1+c_2 \end{matrix}\right| z\right) = \frac{f}{z}\frac{c_1c_2\theta(\theta+a_4)}{a_1a_2a_3a_4(1+a_4)}\ {}_4F_3\left(\left.\begin{matrix} a_1, a_2, a_3, a_4 \\ f, c_1, c_2 \end{matrix}\right| z\right) , \end{aligned} \tag{125} $$

with the substitutions of Eq. (116). In particular, it can be expressed it as a linear combination of the second, third, and fourth hypergeometric functions in Eq. (123) as

$$\begin{aligned}
&\frac{(\gamma_1+\gamma_2)}{z}\frac{f c_1 c_2}{a_1 a_2 a_3 a_4}\theta^2\ {}_4F_3\left(\left.\begin{matrix}a_1,a_2,a_3,a_4\\ f,c_1,c_2\end{matrix}\right|z\right) = \gamma_1 c_2\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\ 1+f,1+c_1,c_2\end{matrix}\right|z\right)\\
&-\left[a_4\gamma_2+(c_2-1)\gamma_1\right]\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\ 1+f,1+c_1,1+c_2\end{matrix}\right|z\right)\\
&+\gamma_2(1+a_4)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4\\ 1+f,1+c_1,1+c_2\end{matrix}\right|z\right)\ ,
\end{aligned} \tag{126}$$

where γ_1 and γ_2 are arbitrary numbers. Putting $\gamma_2 = -\gamma_1$, we obtain a linear relation between the hypergeometric functions on the r.h.s. of Eq. (126). We checked that the ε expansions of the hypergeometric functions constructed in Ref. [20] satisfy this relation. The third derivative in Eq. (122) may be obtained from any of the following relations:

$$\begin{aligned}
{}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\ 1+f,c_1,c_2\end{matrix}\right|z\right) &= \frac{f}{z}\frac{\theta(\theta+c_1-1)(\theta+c_2-1)}{a_1a_2a_3a_4}\ {}_4F_3\left(\left.\begin{matrix}a_1,a_2,a_3,a_4\\ f,c_1,c_2\end{matrix}\right|z\right)\ ,\\
{}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,2+a_3,2+a_4\\ 1+f,1+c_1,1+c_2\end{matrix}\right|z\right) &= \frac{f}{z}\frac{c_1c_2\theta(\theta+a_3)(\theta+a_4)}{a_1a_2a_3a_4(1+a_3)(1+a_4)}\ {}_4F_3\left(\left.\begin{matrix}a_1,a_2,a_3,a_4\\ f,c_1,c_2\end{matrix}\right|z\right)\ ,\\
{}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4\\ 1+f,1+c_1,c_2\end{matrix}\right|z\right) &= \frac{f}{z}\frac{c_1\theta(\theta+a_4)(\theta+c_2-1)}{a_1a_2a_3a_4(1+a_4)}\ {}_4F_3\left(\left.\begin{matrix}a_1,a_2,a_3,a_4\\ f,c_1,c_2\end{matrix}\right|z\right)\ ,
\end{aligned} \tag{127}$$

with the substitutions of Eq. (116). As a consequence of Eq. (127), the following linear relations between hypergeometric functions are valid:

$$
\begin{aligned}
&c_1\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\1+f,c_1,c_2\end{matrix}\right|z\right)-(1+a_4)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4\\1+f,1+c_1,c_2\end{matrix}\right|z\right)\\
&=(c_1-a_4-1)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\1+f,1+c_1,c_2\end{matrix}\right|z\right)\ ,\\
&c_2\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4\\1+f,1+c_1,c_2\end{matrix}\right|z\right)-(1+a_3)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,2+a_3,2+a_4\\1+f,1+c_1,1+c_2\end{matrix}\right|z\right)\\
&=(c_2-a_3-1)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4\\1+f,1+c_1,1+c_2\end{matrix}\right|z\right)\ ,\\
&(\beta_1+\beta_2)\Bigg\{c_1c_2\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\1+f,c_1,c_2\end{matrix}\right|z\right)\\
&-(1+a_3)(1+a_4)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,2+a_3,2+a_4\\1+f,1+c_1,1+c_2\end{matrix}\right|z\right)\Bigg\}\\
&+(c_2-a_4-1)\left[\beta_1(c_2-a_3-1)-\beta_2(c_1-a_4-1)\right]\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\1+f,1+c_1,1+c_2\end{matrix}\right|z\right)\\
&-\beta_1(c_1+c_2-2-a_3-a_4)c_2\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4\\1+f,1+c_1,c_2\end{matrix}\right|z\right)\\
&-\beta_2(c_1+c_2-2-a_3-a_4)(1+a_4)\ {}_4F_3\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4\\1+f,1+c_1,1+c_2\end{matrix}\right|z\right)=0\ ,
\end{aligned}
\tag{128}
$$

where β_1 and β_2 are arbitrary numbers. We checked that the ε expansions of the hypergeometric functions ${}_4F_3$ constructed in Ref. [20] and collected in Ref. [48] satisfy all these relations.

In the case $a_1 = 1$, the basis of the differential reduction is

$$
\{1,\theta,\theta^2\}\times\ {}_4F_3\left(\left.\begin{matrix}1,1+a_2\varepsilon,1+a_3\varepsilon,1+a_4\varepsilon\\\frac{3}{2}+f\varepsilon,2+c_1\varepsilon,2+c_2\varepsilon\end{matrix}\right|z\right)\ . \tag{129}
$$

The first derivative in Eq. (129) can be related to any of the two functions,

$$
{}_4F_3\left(\left.\begin{matrix}1,1+a_2\varepsilon,1+a_3\varepsilon,2+a_4\varepsilon\\\frac{3}{2}+f\varepsilon,2+c_1\varepsilon,2+c_2\varepsilon\end{matrix}\right|z\right)\ ,\qquad {}_4F_3\left(\left.\begin{matrix}1,1+a_2\varepsilon,1+a_3\varepsilon,1+a_4\varepsilon\\\frac{3}{2}+f\varepsilon,2+c_1\varepsilon,1+c_2\varepsilon\end{matrix}\right|z\right)\ , \tag{130}
$$

by single application of the operators $B^{(+)}_{a_4}$ or $H^{(-)}_{c_2}$ given in Eq. (7). The second derivative in Eq. (129) is related to any of the following hypergeometric functions:

$$
\begin{aligned}
&{}_4F_3\left(\left.\begin{matrix}1+a_1\varepsilon,1+a_2\varepsilon,1+a_3\varepsilon,1+a_4\varepsilon\\\frac{3}{2}+f\varepsilon,1+c_1\varepsilon,1+c_2\varepsilon\end{matrix}\right|z\right)\ ,\quad {}_4F_3\left(\left.\begin{matrix}1+a_1\varepsilon,1+a_2\varepsilon,2+a_3\varepsilon,2+a_4\varepsilon\\\frac{3}{2}+f\varepsilon,2+c_1\varepsilon,2+c_2\varepsilon\end{matrix}\right|z\right)\ ,\\
&{}_4F_3\left(\left.\begin{matrix}1+a_1\varepsilon,1+a_2\varepsilon,1+a_3\varepsilon,2+a_4\varepsilon\\\frac{3}{2}+f\varepsilon,2+c_1\varepsilon,1+c_2\varepsilon\end{matrix}\right|z\right)\ ,
\end{aligned}
\tag{131}
$$

which are generated from the basis function of Eq. (129) via application of $H^{(-)}_{b_2}H^{(-)}_{b_3}$, $B^{(+)}_{a_3}B^{(+)}_{a_4}$, or $H^{(-)}_{b_3}B^{(+)}_{a_4}$.

A.2.4 ${}_5F_4$

Similarly, for the hypergeometric function

$$ {}_5F_4\left(\left.\begin{matrix} I_1+a_1\varepsilon, I_2+a_2\varepsilon, I_3+a_3\varepsilon, I_4+a_4\varepsilon, I_5+a_5\varepsilon \\ I_6+\frac{1}{2}+f\varepsilon, I_7+c_1\varepsilon, I_8+c_2\varepsilon, I_9+c_3\varepsilon \end{matrix}\right| z\right) , \tag{132} $$

where $\{I_k\}$ are arbitrary integers and all $a_i \neq 0$, the basis of the differential reduction is

$$ \{1,\theta,\theta^2,\theta^3,\theta^4\} \times {}_4F_3\left(\left.\begin{matrix} a_1\varepsilon, a_2\varepsilon, a_3\varepsilon, a_4\varepsilon, a_5\varepsilon \\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon, 1+c_3\varepsilon \end{matrix}\right| z\right) . \tag{133} $$

In this case, there are eleven functions of the type of Eq. (109). The two bases defined by Eqs. (133) and (109) are related as

$$ \begin{aligned} &{}_5F_4\left(\left.\begin{matrix} a_1\varepsilon, a_2\varepsilon, a_3\varepsilon, a_4\varepsilon, a_5\varepsilon \\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon, 1+c_3\varepsilon \end{matrix}\right| z\right) = 1 + \\ &\frac{2\varepsilon^5\left(\prod_{i=1}^5 a_i\right)}{(1+2f\varepsilon)\left[\prod_{k=1}^3(1+c_k\varepsilon)\right]}\ {}_6F_5\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon, 1+a_5\varepsilon, 1 \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2\varepsilon, 2+c_3\varepsilon, 2 \end{matrix}\right| z\right) , \\ &\theta\ {}_5F_4\left(\left.\begin{matrix} a_1\varepsilon, a_2\varepsilon, a_3\varepsilon, a_4\varepsilon, a_5\varepsilon \\ \frac{1}{2}+f\varepsilon, 1+c_1\varepsilon, 1+c_2\varepsilon, 1+c_3\varepsilon \end{matrix}\right| z\right) \\ &= \frac{2z\left(\prod_{i=1}^5 a_i\right)\varepsilon^5}{(1+2f\varepsilon)\left[\prod_{k=1}^3(1+c_k\varepsilon)\right]}\ {}_5F_4\left(\left.\begin{matrix} 1+a_1\varepsilon, 1+a_2\varepsilon, 1+a_3\varepsilon, 1+a_4\varepsilon, 1+a_5\varepsilon \\ \frac{3}{2}+f\varepsilon, 2+c_1\varepsilon, 2+c_2, 2+c_3\varepsilon\varepsilon \end{matrix}\right| z\right) \end{aligned} \tag{134} $$

The second derivative in Eq. (133) may be obtained as

$$ \begin{aligned} &\frac{(\gamma_1+\gamma_2)}{z}\frac{f\left(\prod_{k=1}^3 c_k\right)}{\left(\prod_{j=1}^5 a_j\right)}\theta^2\ {}_5F_4\left(\left.\begin{matrix} a_1, a_2, a_3, a_4, a_5 \\ f, c_1, c_2, c_3 \end{matrix}\right| z\right) \\ &= \gamma_1 c_3\ {}_5F_4\left(\left.\begin{matrix} 1+a_1, 1+a_2, 1+a_3, 1+a_4, 1+a_5 \\ 1+f, 1+c_1, 1+c_2, c_3 \end{matrix}\right| z\right) \\ &\quad + \gamma_2(1+a_5)\ {}_5F_4\left(\left.\begin{matrix} 1+a_1, 1+a_2, 1+a_3, 1+a_4, 2+a_5 \\ 1+f\varepsilon, 1+c_1, 1+c_2, 1+c_3 \end{matrix}\right| z\right) \\ &\quad - [\gamma_1(c_3-1)+\gamma_2 a_5]\ {}_5F_4\left(\left.\begin{matrix} 1+a_1, 1+a_2, 1+a_3, 1+a_4, 1+a_5 \\ 1+f\varepsilon, 1+c_1, 1+c_2, 1+c_3 \end{matrix}\right| z\right) , \end{aligned} \tag{135} $$

with the substitutions of Eq. (116), where γ_1 and γ_2 are arbitrary numbers. Putting $\gamma_2 = -\gamma_1$, we obtain a linear relation between the hypergeometric functions on the r.h.s. of Eq. (135). We checked that the ε expansions of the hypergeometric functions constructed in Ref. [20] satisfy this relation. The third derivative in Eq. (133) is

related to one of the following functions or their linear combination:

$$
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4,2+a_5\\1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right)=\frac{f}{z}\frac{\left(\prod_{k=1}^{3}c_k\right)\theta(\theta+a_4)(\theta+a_5)}{\left(\prod_{j=1}^{5}a_j\right)(1+a_4)(1+a_5)}F(z)\ ,
$$

$$
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,2+a_5\\1+f,1+c_1,1+c_2,c_3\end{matrix}\right|z\right)=\frac{f}{z}\frac{\left(\prod_{k=1}^{2}c_k\right)\theta(\theta+c_3-1)(\theta+a_5)}{\left(\prod_{j=1}^{5}a_j\right)(1+a_5)}F(z)\ ,
$$

$$
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\1+f,1+c_1,c_2,c_3\end{matrix}\right|z\right)=\frac{f}{z}\frac{c_1\theta(\theta+c_2-1)(\theta+c_3-1)}{\left(\prod_{j=1}^{5}a_j\right)}F(z)\ , \tag{136}
$$

where

$$
F(z)\equiv\ {}_5F_4\left(\left.\begin{matrix}a_1,a_2,a_3,a_4,a_5\\f,c_1,c_2,c_3\end{matrix}\right|z\right)\ , \tag{137}
$$

As a consequence of Eq. (136), the following linear relations between hypergeometric functions hold:

$$
\begin{aligned}
&(1+a_4-c_3)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,2+a_5\\1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right)\\
&=(1+a_4)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4,2+a_5\\1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right)\\
&\quad-c_3\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,2+a_5\\1+f,1+c_1,1+c_2,c_3\end{matrix}\right|z\right)\ ,\\
&(1+a_5-c_2)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\1+f,1+c_1,1+c_2,c_3\end{matrix}\right|z\right)\\
&=(1+a_5)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,2+a_5\\1+f,1+c_1,1+c_2,c_3\end{matrix}\right|z\right)\\
&\quad-c_2\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\1+f,1+c_1,c_2,c_3\end{matrix}\right|z\right)\ ,\\
&(a_5-c_3+1)\left[\beta_1\left(c_3-a_4-1\right)-\beta_2\left(c_2-a_5-1\right)\right]\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right)\\
&=(\beta_1+\beta_2)c_2c_3\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\1+f,1+c_1,c_2,c_3\end{matrix}\right|z\right)\\
&\quad-(\beta_1+\beta_2)(1+a_4)(1+a_5)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4,2+a_5\\1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right)\\
&\quad+\beta_1c_3\left(a_4+a_5-c_2-c_3+2\right)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\1+f,1+c_1,1+c_2,c_3\end{matrix}\right|z\right)\\
&\quad+\beta_2(1+a_5)\left(a_4+a_5-c_2-c_3+2\right)\ {}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,2+a_5\\1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right)\ .
\end{aligned}
\tag{138}
$$

The forth derivative in Eq. (133) is related to any of the following hypergeometric functions:

$$
\begin{aligned}
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,2+a_3,2+a_4,2+a_5\\ 1+f,1+c_1,1+c_2,1+c_3\end{matrix}\right|z\right) &= \frac{f}{z}\frac{c_1c_2c_3\theta(\theta+a_3)(\theta+a_4)(\theta+a_5)}{\left(\prod_{j=1}^5 a_j\right)(1+a_3)(1+a_4)(1+a_5)}F(z)\ ,\\
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,2+a_4,2+a_5\\ 1+f,1+c_1,1+c_2,c_3\end{matrix}\right|z\right) &= \frac{f}{z}\frac{c_1c_2\theta(\theta+c_3-1)(\theta+a_4)(\theta+a_5)}{\left(\prod_{j=1}^5 a_j\right)(1+a_4)(1+a_5)}F(z)\ ,\\
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,2+a_5\\ 1+f,1+c_1,c_2,c_3\end{matrix}\right|z\right) &= \frac{f}{z}\frac{c_1\theta(\theta+c_2-1)(\theta+c_3-1)(\theta+a_5)}{\left(\prod_{j=1}^5 a_j\right)(1+a_5)}F(z)\ ,\\
{}_5F_4\left(\left.\begin{matrix}1+a_1,1+a_2,1+a_3,1+a_4,1+a_5\\ 1+f,c_1,c_2,c_3\end{matrix}\right|z\right) &= \frac{f}{z}\frac{\theta(\theta+c_1-1)(\theta+c_2-1)(\theta+c_3-1)}{\left(\prod_{j=1}^5 a_j\right)}F(z)\ ,
\end{aligned}
\tag{139}
$$

with the substitutions of Eq. (116). The set of algebraic relations between the hypergeometric functions of Eq. (139) is to long to be presented here. We checked that the ε expansions of the hypergeometric functions ${}_5F_4$ constructed in Ref. [20] satisfy all these linear relations.

A.2.5 ${}_{p+1}F_p$, $p \geq 6$

For the hypergeometric function

$$
{}_{p+1}F_p\left(\left.\begin{matrix}\{1\}_{p+1},\\ \frac{3}{2},\{2\}_{p-1}\end{matrix}\right|z\right)\ , \tag{140}
$$

the following representation was derived in Refs. [67]:

$$
\begin{aligned}
\frac{z}{2}\ {}_{p+1}F_p\left(\left.\begin{matrix}\{1\}_{p+1},\\ \frac{3}{2},\{2\}_{p-1}\end{matrix}\right|z\right) &= \sum_{k=1}^{\infty}\frac{1}{\binom{2n}{n}}\frac{z^n}{n^p}\\
&= -\sum_{j=0}^{p-2}\frac{(-2)^j}{(a-2-j)!j!}(\ln z)^{a-2-j}\mathrm{Ls}_{j+2}^{(1)}\left(2\arcsin\frac{\sqrt{z}}{2}\right)
\end{aligned}
\tag{141}
$$

where $0 \leq z \leq 4$, $p \geq 2$, and $\mathrm{Ls}_a^{(j)}(z)$ denotes the generalized log-sine function[15] [69]. Later, it was shown in Ref. [29] that any generalized log-sine function of argument θ can be written in terms of Nielsen polylogarithms with argument[16] $y = \exp(\pm i\theta)$ (see Eqs. (2.18) and (A.21) in Ref. [29]).

[15] A computer program for the numerical evaluation of the generalized log-sine functions with high precision was developed in Ref. [68].

[16] In terms of the variable z in Eq. (141), we have

$$
y = \cos\theta \pm i\sin\theta\big|_{\theta=2\arcsin\frac{\sqrt{z}}{2}} = 1 - \frac{z}{2} \pm i\sqrt{z}\sqrt{1-\frac{z}{4}} \equiv \frac{1-\sqrt{\frac{z}{z-4}}}{1+\sqrt{\frac{z}{z-4}}}\ .
$$

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