hep-lat 1008.0710

DESY 10-051
SFB/CPP-10-69

arXiv:1008.0710v1 [hep-lat] 4 Aug 2010

# Introduction to Non-perturbative Heavy Quark Effective Theory

R. Sommer

*NIC, DESY, Platanenallee 6, 15738 Zeuthen, Germany*

*Lectures at the Summer School on*
*"Modern perspectives in lattice QCD"*
*Les Houches, August 3–28, 2009*

# Preface

My lectures on the effective field theory for heavy quarks, an expansion around the static limit, concentrate on the motivation and formulation of HQET, its renormalization and discretization. This provides the basis for understanding that and how this effective theory can be formulated fully non-perturbatively in the QCD coupling, while by the very nature of an effective field theory, it is perturbative in the expansion parameter $1/m$. After the couplings in the effective theory have been determined, the result at a certain order in $1/m$ is unique up to higher order terms in $1/m$. In particular the continuum limit of the lattice regularized theory exists and leaves no trace of how it was regularized. In other words, the theory yields an asymptotic expansion of the QCD observables in $1/m$ – as usual in a quantum field theory modified by powers of logarithms. None of these properties has been shown rigorously (e.g. to all orders in perturbation theory) but perturbative computations and recently also non-perturbative lattice results give strong support to this "standard wisdom".

A subtle issue is that a theoretically consistent formulation of the theory is only possible through a non-perturbative matching of its parameters with QCD at finite values of $1/m$ (Sect. 4.4). As a consequence one finds immediately that the splitting of a result for a certain observable into, for example, lowest order and first order is ambiguous. Depending on how the matching between effective theory and QCD is done, a first order contribution may vanish and appear instead in the lowest order. For example, the often cited phenomenological HQET parameters $\bar{\Lambda}$ and $\lambda_1$ lack a unique non-perturbative definition. But this does not affect the precision of the asymptotic expansion in $1/m$. The final result for an observable is correct up to order $(1/m)^{n+1}$ if the theory was treated including $(1/m)^n$ terms.

Clearly, the weakest point of HQET is that it intrinsically is an expansion. In practise, carrying it out non-perturbatively beyond the order $1/m$ will be very difficult. In this context two observations are relevant. First, the expansion parameter for HQET applied to B-physics is $\Lambda_{\rm QCD}/m_{\rm b} \sim 1/(r_0 m_{\rm b}) = 1/10$ and indeed recent computations of $1/m_{\rm b}$ corrections showed them to be very small. Second, since HQET yields the asymptotic expansion of QCD, it becomes more and more accurate the larger the mass is. It can therefore be used to constrain the large mass behavior of QCD computations done at finite, varying, quark masses. At some point, computers and computational strategies will be sufficient to simulate with lattice spacings which are small enough for a relativistic b-quark. One would then like to understand the full mass-behavior of observables and a combination of HQET and relativistic QCD will again be most useful. Already now, there is a strategy (de Divitiis *et al.*, 2003*a*, de Divitiis *et al.*, 2003*b*, Guazzini *et al.*, 2008), which is related to the one discussed in Sect. 5.3 and which, in its final version combines HQET and QCD in such a manner. For a short review of this aspect I refer to (Tantalo, 2008).

## Acknowledgements

I am thankful for the nice collaboration in the team of organizers, with the director of the school and the staff of the school. It is also a pleasure to thank the members of the LGT discussion seminar at Humboldt-University and DESY, in particular Hubert Simma and Ulli Wolff, for their valuable suggestions on a first version of these lecture notes. I am grateful for a fruitful collaboration with Benoit Blossier, Michele Della Morte, Patrick Fritzsch, Nicolas Garron, Jochen Heitger, Georg von Hippel, Tereza Mendes, Mauro Papinutto and Hubert Simma on several of the subjects of these lectures and thank Nicolas Garron for providing me with tables and figures. Most of all I would like to thank Dorothy for her patience with me spending much time on these lectures and the school.

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# 1
# Introduction

## 1.1 Conventions

Our conventions for gauge fields, lattice derivatives etc. are summarized in the appendix.

## 1.2 The rôle of HQET

This school focuses on lattice gauge theories. How does heavy quark effective theory (HQET) fit into it? The first part of the answer is that HQET is expected to provide the true asymptotic expansion of quantities in powers (accompanied by logarithms) of $1/m$, the mass of the heavy quark, with all other scales held fixed. The accessible quantities are energies, matrix elements and Euclidean correlation functions with a single heavy (valence) quark, while all other quarks are light. A full understanding of QCD should contain this kinematical region.

The second part of the answer has to do with the challenge we are facing when we perform a Monte Carlo (MC) evaluation of the QCD path integral. This becomes apparent by considering the scales which are relevant for QCD. For low energy QCD and flavor physics excluding the top-quark, they range from

$$m_\pi \approx 140\,\mathrm{MeV} \text{ over } m_\mathrm{D} = 2\,\mathrm{GeV} \text{ to } m_\mathrm{B} = 5\,\mathrm{GeV}.$$

In addition, the ultraviolet cutoff of $\Lambda_\mathrm{UV} = a^{-1}$ of the discretized theory has to be large compared to all physical energy scales if the theory discretized with a lattice spacing $a$ is to be an approximation to a continuum. Finally, the linear extent of space time has to be restricted to a finite value $L$ in a numerical treatment: there is an infrared cutoff $L^{-1}$. Together the following constraints have to be satisfied.

$$\Lambda_\mathrm{IR} \;=\; L^{-1} \ll\; m_\pi\,,\; \ldots\;, m_\mathrm{D}\,, m_\mathrm{B} \;\ll a^{-1} \;=\; \Lambda_\mathrm{UV} \tag{1.1}$$

The infrared and the ultraviolet effects are systematic errors which have to be controlled. Infrared effects behave as (Lüscher, 1986) $\mathrm{O}(\mathrm{e}^{-Lm_\pi})$ and are known from chiral perturbation theory (Colangelo *et al.*, 2005) to be at the percent level when $L \gtrsim 4/m_\pi \approx 6\,\mathrm{fm}$, while the UV, discretization, errors are $\mathrm{O}((a\,m_\mathrm{quark})^2)$ in $\mathrm{O}(a)$-improved theories.[1] With a charm quark mass of around 1 GeV we have a requirement of $a \lesssim 1/(2m_\mathrm{c})\;\ldots\;1/(4m_\mathrm{c}) \approx 0.1\;\ldots\;0.05\,\mathrm{fm}$ (Kurth and Sommer, 2002) and thus

$$L/a \approx 60\;\ldots\;120\,. \tag{1.2}$$

[1] See Peter Weisz' lectures for the general discussion of discretization errors and improvement of lattice gauge theories.

Including b-quarks would increase the already rather intimidating estimate of $L/a$ by a factor 4. It is thus mandatory to resort to an effective theory where degrees of freedom with energy scales around the b-quark mass and higher are summarized in the coefficients of terms in the effective Lagrangian. A *precise* treatment of this theory has become very relevant because the search for physics beyond the Standard Model in the impressive first generation of B-physics flavor experiments has been unsuccessful so far. New physics contributions are very small and even higher precision is needed both in experiment and in theory to possibly reveal them. HQET is a very important ingredient in this effort.

Before we focus on our topic let us note that a factor two or so in $L/a$ may be saved by working at somewhat higher pion mass and extrapolating with chiral perturbation theory, see M. Golterman's lectures.

## 1.3 On continuum HQET

### 1.3.1 Idea

We consider hadrons with a single very heavy quark, e.g. a B-meson. Physical intuition tells us that these will be similar to a hydrogen atom with the analogy

| | | | | |
|---|---|---|---|---|
| hydrogen atom | : | heavy proton | + | light electron |
| B-meson | : | heavy b-quark | + | light anti-quark |
| b-baryons | : | heavy b-quark | + | two light quarks |

and so on.

When we take the limit $m = m_{\rm b} \to \infty$ ("static") the b-quark is at rest in the *rest-frame of the b-hadron* (B, $\Lambda_{\rm b}$, ...). In this situation, we should be able to find an effective Lagrangian describing the dynamics of the light quarks and glue with the heavy quark just representing a color source. Corrections in $1/m_{\rm b}$ should be systematically included in a series expansion in that variable. The Lagrangian is then expected to be given as a series in $D_k/m$ where the covariant derivatives act on the heavy quark field and correspond to its spatial momenta in the rest-frame of the heavy hadron.

Before proceeding to a heuristic derivation of the effective field theory, let us note some general properties of what we are actually seeking, comparing to other familiar effective field theories. In contrast to the low energy effective field theory for electroweak interactions, where the heavy particles (W- and Z-boson, top quark) are removed completely from the Lagrangian we here want to consider processes with b-quarks in initial and/or final states. The b-quark field is thus contained in the Lagrangian and we have to find its relevant modes to be kept.[2]

Another important effective field theory to compare to is the chiral effective theory, covered here by Maarten Golterman. Main differences are that this is a fully relativistic theory with loops of the (pseudo-) Goldstone bosons and that the interaction of the fields in the effective Lagrangian disappears for zero momentum. The theory can therefore be evaluated perturbatively. It is also called chiral perturbation theory. In

[2]However, when one carries out the expansion to include $1/m_{\rm b}^2$ terms, also a whole set of terms generated by b-quark loops in QCD which do not contain the b-quark field in the effective theory have to be taken into account. An example are 4-fermion operators made of the light quarks, just as they appear when one "integrates out" the W and Z-bosons in the Standard Model.

contrast, the b-quarks in HQET still interact non-perturbatively with the light quarks and gluons. This effective field theory therefore needs a lattice implementation in order to come to predictions beyond those that can be read off from its symmetries.

### 1.3.2 Derivation of the form of the effective field theory: FTW trafo

*Strategy*

Our strategy is to carry out the following steps, which we discuss in more detail below.

- We start from a Euclidean action.
- We identify the dominant degrees of freedom for the kinematical situation we are interested in: the "large" components of the b-quark field for the quark and the "small" components for the anti-quark.
- We decouple large components and small components, order by order in $D_k/m$ [ $\overline{\psi}_{\rm h} D_k/m \ \psi_{\rm h} \ll \overline{\psi}_{\rm h}\psi_{\rm h}$ ]. This assumes smooth gauge (and other) fields. It is thus essentially a classical derivation. The decoupling is achieved by a sequence of Fouldy Wouthuysen-Tani (FTW) transformations (see e.g. (Itzykson and Zuber, 1980)), following essentially (Körner and Thompson, 1991).
- The irrelevant modes are dropped from the theory (often it is said they are integrated out). Their effects are *not* expected to change the form of the local Lagrangian, but just to renormalize its parameters. Still it could be that local terms allowed by the symmetries happen to vanish in the classical theory. Thus the symmetries have to be considered and all terms of the proper dimension compatible with the symmetries have to be taken into account.
- At tree level the values of the parameters in the effective Lagrangian are given by the FTW transformation. In general (i.e. for any value of the QCD coupling) they have to be determined by matching to QCD: one expands QCD correlation functions in $1/m_{\rm b}$ and compares to HQET. This part of the strategy will be discussed in detail in later sections.

*Identifying the degrees of freedom*

We consider the *free* propagator of a Dirac-fermion in Euclidean space, in the time / space-momentum representation[3]:

$$\begin{aligned} S(x_0;\mathbf{k}) &= \int \mathrm{d}^3\mathbf{x}\, \mathrm{e}^{-i\mathbf{kx}} \langle \psi(x)\overline{\psi}(0)\rangle = \int \frac{\mathrm{d}k_0}{(2\pi)} \mathrm{e}^{ik_0x_0}\ [ik_\mu\gamma_\mu + m]^{-1} \\ &= S_+(x_0;\mathbf{k}) + S_-(x_0;\mathbf{k})\,, \end{aligned} \tag{1.3}$$

with

$$\begin{aligned} S_+(x_0;\mathbf{p}) &= \theta(x_0)\frac{m}{E(\mathbf{p})}\mathrm{e}^{-E(\mathbf{p})x_0}P_+(u)\,, \qquad P_+(u) = \frac{1-iu_\mu\gamma_\mu}{2}\,,\ u_\mu = p_\mu/m\,, \\ S_-(x_0;\mathbf{p}) &= \theta(-x_0)\frac{m}{E(\mathbf{p})}\mathrm{e}^{E(\mathbf{p})x_0}P_-(u)\,, \qquad P_-(u) = \frac{1+iu_\mu\gamma_\mu}{2}\,, \end{aligned} \tag{1.4}$$

[3]The expectation value $\langle . \rangle$ refers to the Euclidean path integral, here with the free Dirac action. We suggest to verify these formulae as an exercise.

where $p_\mu$ is the on-shell momentum, i.e.

$$p_0 = iE(\mathbf{p}) = i\sqrt{m^2 + \mathbf{p}^2}\,. \tag{1.5}$$

Here $S_+(x_0;\mathbf{p})$ describes the propagation of a quark from time $t = 0$ to $t = x_0$ and $S_-(x_0;\mathbf{p})$ describes the propagation of an anti-quark from $t = -x_0$ to $t = 0$. Since the Euclidean 4-velocity vector $u$ satisfies $u^2 = u_\mu u_\mu = -1$, the matrices $P \in \{P_+, P_-\}$ are projection operators,

$$[P(u)]^2 = P(u)\,,\ P_+(u)P_-(u) = 0\,,\ P_+(u) + P_-(u) = 1\,. \tag{1.6}$$

They allow us to project onto the on-shell components of a quark with velocity $\mathbf{u}$.

The "large" field components corresponding to the quark are given by the projection

$$\psi_{\mathrm{h},u}(x) = P(u)\psi(x)\,,\ \overline{\psi}_{\mathrm{h},u}(x) = \overline{\psi}(x)P(u) \tag{1.7}$$

and the "small" ones, the anti-quark field, are

$$\psi_{\bar{\mathrm{h}},u}(x) = P(-u)\psi(x)\,,\ \overline{\psi}_{\bar{\mathrm{h}},u}(x) = \overline{\psi}(x)P(-u)\,, \tag{1.8}$$

such that for free quarks

$$\int \mathrm{d}^3\mathbf{x}\,\mathrm{e}^{-i\mathbf{px}}\langle\psi_{\mathrm{h},u}(x)\overline{\psi}_{\mathrm{h},u}(0)\rangle = S_+(x_0;\mathbf{p}) \tag{1.9}$$

and similarly for the anti-quark.[4]

For a b-hadron with velocity $\mathbf{u}$, the fields $\psi_{\mathrm{h},u}(x)$, $\overline{\psi}_{\mathrm{h},u}(x)$ are expected to be the relevant ones with the other field-components giving subdominant contributions in the path integral representation of correlation functions (or scattering amplitudes in Minkowski space), while for a $\overline{\mathrm{b}}$-hadron $\psi_{\bar{\mathrm{h}},u}(x)$, $\overline{\psi}_{\bar{\mathrm{h}},u}(x)$ are expected to dominate.

*In the presence of a gauge field*

When a gauge field is present, we therefore expect an effective Lagrangian for the b-hadrons in terms of $\psi_{\mathrm{h},u}, \overline{\psi}_{\mathrm{h},u}$ plus a term for the anti-quark. When we rewrite the Dirac Lagrangian in terms of these fields,

$$\begin{aligned}\mathscr{L} &= \overline{\psi}(m + \mathcal{D})\psi \\ &= \overline{\psi}_{\mathrm{h},u}(m + \mathcal{D}_\parallel)\psi_{\mathrm{h},u} + \overline{\psi}_{\bar{\mathrm{h}},u}(m + \mathcal{D}_\parallel)\psi_{\bar{\mathrm{h}},u} + \overline{\psi}_{\mathrm{h},u}\mathcal{D}_\perp\psi_{\bar{\mathrm{h}},u} + \overline{\psi}_{\bar{\mathrm{h}},u}\mathcal{D}_\perp\psi_{\mathrm{h},u}\,,\end{aligned} \tag{1.10}$$

there are mixed contributions which involve

$$\mathcal{D}_\perp = \gamma_\mu D_\mu^\perp\,,\quad D_\mu^\perp = (\delta_{\mu\nu} + u_\mu u_\nu)\,D_\nu\,, \tag{1.11}$$

where the derivative is projected orthogonal to $u_\mu$. Analogously we have

$$\mathcal{D}_\parallel = \gamma_\mu D_\mu^\parallel\,,\quad D_\mu^\parallel = -u_\mu\,D_\nu u_\nu\,. \tag{1.12}$$

From our general consideration of the kinematical situation that we want to describe, $D_\mu^\perp$ acting on the heavy quark field is to be considered small (compared to $m$). In

[4]The terms "large" and "small" components are commonly used when discussing the non-relativistic limit of the Dirac equation for bound states, see e.g. (Itzykson and Zuber, 1980).

contrast, $D_\mu^{\|}$ applied to the field will yield approximately $p_\mu = u_\mu m$. We therefore carry out an expansion with

$$\begin{aligned} \mathcal{D}_\| \psi &= \mathrm{O}(m)\, \psi \,, \\ \mathcal{D}_\perp \psi &= \mathrm{O}(1)\, \psi \end{aligned} \tag{1.13}$$

and all other fields, such as $F_{\mu\nu}$, treated as order one. This is often called the power counting scheme.

*FTW trafo and Lagrangian at zero velocity*

Having identified the expansion, we perform a field rotation (FTW transformation) to decouple large and small components order by order in $1/m$. First we consider the special case of zero velocity,

$$\begin{aligned} u_k = 0: \quad & \mathcal{D}_\| = D_0\gamma_0 \,, \quad \mathcal{D}_\perp = D_k\gamma_k \,, \\ & P(u) = P_+ = \frac{1+\gamma_0}{2} \,, \quad P(-u) = P_- = \frac{1-\gamma_0}{2} \,. \end{aligned} \tag{1.14}$$

The FTW transformation is

$$\begin{aligned} \psi \to \psi' &= \mathrm{e}^S \psi \,, \quad S = \tfrac{1}{2m} D_k\gamma_k = -S^\dagger \,, \\ \overline{\psi} \to \overline{\psi}' &= \overline{\psi}\mathrm{e}^{-\overleftarrow{S}} = \overline{\psi}\mathrm{e}^{-\overleftarrow{D}_k\gamma_k/(2m)} \,. \end{aligned} \tag{1.15}$$

Its Jacobian is one. The Lagrangian written in terms of the transformed fields,

$$\mathscr{L} = \overline{\psi}'(\mathcal{D}' + m)\psi' \,, \tag{1.16}$$

yields a Dirac operator (note that $S$ acts to the right everywhere)

$$\mathcal{D}' + m = \mathrm{e}^{-S}(\mathcal{D} + m)\mathrm{e}^{-S} \,. \tag{1.17}$$

Expanding $\mathrm{e}^{-S} = 1 - S + \frac{1}{2}S^2 - \ldots$ in $S = \mathrm{O}(1/m)$ yields

$$\mathcal{D}' + m = \underbrace{\mathcal{D} + m}_{\mathrm{O}(m)} + \underbrace{\{-S, \mathcal{D} + m\}}_{\mathrm{O}(1)} + \tfrac{1}{2}\underbrace{\{-S, \{-S, \mathcal{D} + m\}\}}_{\mathrm{O}(1/m)} + \ldots \tag{1.18}$$

In the evaluation of the different terms we count all fields and derivatives of fields (e.g. $F_{\mu\nu}$) as order one except for $D_0$ *acting onto the heavy quark field*. We work out the expansion up to order $1/m$. A little algebra yields

$$\mathcal{D} + m + \{-S, \mathcal{D} + m\} = D_0\gamma_0 - \frac{1}{2m}[\gamma_k\gamma_0 F_{k0} + \frac{1}{i}\sigma_{kl}\, F_{kl} + 2D_k D_k] \tag{1.19}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, $F_{kl} = [D_k, D_l]$ and

$$\frac{1}{2}\{-S, \underbrace{\{-S, \mathcal{D} + m\}}_{-D_k\gamma_k + \mathrm{O}(1/m)}\} = \frac{1}{4m}[\frac{1}{i}\sigma_{kl}\, F_{kl} + 2D_k D_k] \,, \tag{1.20}$$

such that

$$\mathcal{D}' = D_0\gamma_0 - \frac{1}{2m}[\underbrace{\gamma_k\gamma_0 F_{k0}}_{\text{off-diagonal}} + \frac{1}{2i}\sigma_{kl}\,F_{kl} + D_k D_k] + \mathrm{O}(1/m^2)\,. \tag{1.21}$$

In the static part, $D_0\gamma_0$, the large and small components are decoupled, but one of the $1/m$ terms, $\gamma_k\gamma_0 F_{k0}$, is off-diagonal with respect to this split. We therefore seek a second transformation $\psi'' = \mathrm{e}^{S'}\psi'$ to cancel also that term, namely we want

$$\{-S', \mathcal{D}' + m\} = \frac{1}{2m}\gamma_k\gamma_0 F_{k0} + \mathrm{O}(1/m^2)\,. \tag{1.22}$$

The simple choice $S' = \frac{1}{4m^2}\gamma_0\gamma_k F_{k0}$ does the job. Now we have the classical HQET Lagrangian

$$\mathscr{L} = \mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} + \tfrac{1}{2m}\mathscr{L}_{\mathrm{h}}^{(1)} \;+\; \mathscr{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} + \tfrac{1}{2m}\mathscr{L}_{\bar{\mathrm{h}}}^{(1)} + \mathrm{O}(\tfrac{1}{m^2}) \tag{1.23}$$

$$\mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(m + D_0)\psi_{\mathrm{h}}\,, \quad P_+\psi_{\mathrm{h}} = \psi_{\mathrm{h}}\,, \quad \overline{\psi}_{\mathrm{h}}P_+ = \overline{\psi}_{\mathrm{h}}\,, \quad P_\pm = \tfrac{1\pm\gamma_0}{2} \tag{1.24}$$

$$\mathscr{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} = \overline{\psi}_{\bar{\mathrm{h}}}(m - D_0)\psi_{\bar{\mathrm{h}}}\,, \quad P_-\psi_{\bar{\mathrm{h}}} = \psi_{\bar{\mathrm{h}}}\,, \quad \overline{\psi}_{\bar{\mathrm{h}}}P_- = \overline{\psi}_{\bar{\mathrm{h}}}\,, \tag{1.25}$$

$$\mathscr{L}_{\mathrm{h}}^{(1)} = -(\mathcal{O}_{\mathrm{kin}} + \mathcal{O}_{\mathrm{spin}})\,, \quad \mathscr{L}_{\bar{\mathrm{h}}}^{(1)} = -(\bar{\mathcal{O}}_{\mathrm{kin}} + \bar{\mathcal{O}}_{\mathrm{spin}})\,, \tag{1.26}$$

correct up to terms of order $1/m^2$. We introduced

$$\mathcal{O}_{\mathrm{kin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\,\mathbf{D}^2\,\psi_{\mathrm{h}}(x)\,,\ \mathcal{O}_{\mathrm{spin}}(x) = \overline{\psi}_{\mathrm{h}}(x)\,\boldsymbol{\sigma}\cdot\mathbf{B}(x)\,\psi_{\mathrm{h}}(x)\,, \tag{1.27}$$

$$\bar{\mathcal{O}}_{\mathrm{kin}}(x) = \overline{\psi}_{\bar{\mathrm{h}}}(x)\,\mathbf{D}^2\,\psi_{\bar{\mathrm{h}}}(x)\,,\ \bar{\mathcal{O}}_{\mathrm{spin}}(x) = \overline{\psi}_{\bar{\mathrm{h}}}(x)\,\boldsymbol{\sigma}\cdot\mathbf{B}(x)\,\psi_{\bar{\mathrm{h}}}(x)\,, \tag{1.28}$$

$$\sigma_k = \tfrac{1}{2}\epsilon_{ijk}\sigma_{ij}\,, \quad B_k = i\tfrac{1}{2}\epsilon_{ijk}F_{ij}\,, \tag{1.29}$$

and the heavy quark fields are the transformed ones, i.e. we renamed $\psi''_{\mathrm{h}} \to \psi_{\mathrm{h}}$ etc.

Depending on the process/correlation function, just the heavy quark part or just the heavy anti-quark part of the Lagrangian will contribute, but there are also processes such as $\mathrm{B} - \bar{\mathrm{B}}$ oscillations where both are needed.

It is worth summarizing some issues that arose in this formal derivation.

- Assuming $D_k = \mathrm{O}(1)$ means that this is a classical derivation: in the quantum field theory path integral we integrate over rough fields, i.e. there are arbitrarily large derivatives.
  As emphasized before we therefore take this as a classical Lagrangian. Its renormalization will be discussed later, guided by dimensional counting.
- The derivation is perturbative in $1/m$, order by order. This is all that we want. In this way we expect to obtain the *asymptotic* expansion in powers of $1/m$.
- We note that there are alternative ways to derive the form of the Lagrangian. One may integrate out the components $\overline{\psi}_{\bar{\mathrm{h}}}, \psi_{\bar{\mathrm{h}}}$ in a path integral and then perform a formal expansion of the resulting non-local action for the remaining fields in terms of a series of local operators (Mannel *et al.*, 1992). Another option is to perform a hopping parameter expansion of the Wilson-Dirac lattice propagator. The leading term gives the propagator of the static action; see exercise 1.1.

*FTW transformation and Lagrangian at finite velocity*

At finite velocity the transformation is given again by eq. (1.15) but with $S = D_\mu^\perp \gamma_\mu/(2m)$. For the lowest order (static) approximation, just the anti-commutator

$$\{\mathcal{D}^\perp, \mathcal{D}^\parallel\} = \tfrac{1}{2}\{D_\mu^\perp, D_\nu^\parallel\} 2\delta_{\mu\nu} + \tfrac{1}{2}[D_\mu^\perp, D_\nu^\parallel][\gamma_\mu, \gamma_\nu]$$

is needed. Since $D_\mu^\perp D_\mu^\parallel = 0 = D_\mu^\parallel D_\mu^\perp$ and the second term just involves a commutator of derivatives, we see that $\{\mathcal{D}^\perp, \mathcal{D}^\parallel\} = \mathrm{O}(1)$. Consequently we find

$$\begin{aligned}\mathscr{L} &= \overline{\psi}_{\mathrm{h},u}(m + \mathcal{D}^\parallel)\psi_{\mathrm{h},u} + \overline{\psi}_{\bar{\mathrm{h}},u}(m + \mathcal{D}^\parallel)\psi_{\bar{\mathrm{h}},u} + \mathrm{O}(1/m) \\ &= \overline{\psi}_{\mathrm{h},u}(m - iu_\mu D_\mu)\psi_{\mathrm{h},u} + \overline{\psi}_{\bar{\mathrm{h}},u}(m + iu_\mu D_\mu)\psi_{\bar{\mathrm{h}},u} + \mathrm{O}(1/m) \end{aligned} \tag{1.30}$$

with the projected fields eq. (1.7) and eq. (1.8). [5]

Let us add a few *comments on the finite velocity theory*, since we will not discuss it further.

- O(4) (or Lorentz) invariance is broken. One therefore has to expect a different renormalization of $D_0$ and $D_k$ (or as is usually said, a renormalization of $\mathbf{u}$ (Christensen *et al.*, 2000, Mandula and Ogilvie, 1998)).
- The operator $-iD_k u_k$ is unbounded from below. Since it enters the Hamiltonian the theory seems to contain states with arbitrarily large negative energies. Resulting problems in the Euclidean formulation of the theory have been discussed in the literature (Aglietti *et al.*, 1992, Aglietti, 1994), but a compelling formulation of the theory seems not to have been found. There are also no modern applications of the finite velocity theory on the lattice. We will therefore concentrate entirely on zero velocity HQET from now on.

### 1.3.3 Propagator and Symmetries

*The continuum propagator.*

We consider the static approximation at zero velocity and the latter always from now on. The static Dirac operator for the quark is just $D_0 + m$ so its Green function, $G_\mathrm{h}$, (the propagator) in a gauge field $A_\mu(x)$ then satisfies

$$(\partial_{x_0} + A_0(x) + m)G_\mathrm{h}(x, y) = \delta(x - y)\, P_+ \,. \tag{1.31}$$

The solution of this equation is simply

$$G_\mathrm{h}(x, y) = \theta(x_0 - y_0)\, \exp(-m\,(x_0 - y_0))\, \mathcal{P} \exp\left\{ - \int_{y_0}^{x_0} \mathrm{d}z_0 A_0(z_0, \mathbf{x}) \right\} \delta(\mathbf{x} - \mathbf{y})\, P_+ \,, \tag{1.32}$$

were $\mathcal{P}$ denotes path ordering (fields at the end of the integration path to the left). In the same way the propagator for the anti-quark is [6]

[5] One can also obtain this Lagrangian by performing a boost of the zero velocity theory (Horgan *et al.*, 2009). In the quoted reference also the next to leading order terms are found.

[6] Note $\mathcal{P} \exp\left\{ - \int_{y_0}^{x_0} \mathrm{d}z_0 A_0(z_0, \mathbf{x}) \right\} = \mathcal{P} \exp\left\{ - \int_{x_0}^{y_0} \mathrm{d}z_0 A_0(z_0, \mathbf{x}) \right\}^\dagger$.

$$G_{\bar{\mathrm{h}}}(x,y) = \theta(y_0 - x_0)\, \exp(-m\,(y_0 - x_0))\, \mathcal{P} \exp\left\{ - \int_{y_0}^{x_0} \mathrm{d}z_0 A_0(z_0, \mathbf{x}) \right\} \delta(\mathbf{x} - \mathbf{y})\, P_- \,,$$
$$(-\partial_{x_0} - A_0(x) + m) G_{\bar{\mathrm{h}}}(x,y) = \delta(x-y)\, P_- \,. \tag{1.33}$$

The mass appears in a trivial way, with an explicit factor $\exp(-m\,|x_0 - y_0|)$ for any gauge field $A_\mu$. This exponential decay is then present also after path integration over the gauge fields in any 2-point function with a heavy quark,

$$C_{\mathrm{h}}(x,y;m) = C_{\mathrm{h}}(x,y;0)\, \exp(-m\,(x_0 - y_0))\,. \tag{1.34}$$

An explicit example is

$$C_{\mathrm{h}}^{\mathrm{PP}}(x,y;m) = \langle \overline{\psi}_{\mathrm{l}}(x) \gamma_5 \psi_{\mathrm{h}}(x)\; \overline{\psi}_{\mathrm{h}}(y) \gamma_5 \psi_{\mathrm{l}}(y) \rangle \,, \tag{1.35}$$

with $\psi_{\mathrm{l}}(x)$ a light-quark fermion field. Eq. (1.34) means that $m$ shifts *all* energies in the sector of the Hilbert space with a single heavy quark (or anti-quark). We may remove $m$ from the effective Lagrangian and add it to the energies later. We only have to be careful that $m \geq 0$ in eq. (1.31), eq. (1.33) selects the forward/backward propagation. Therefore we set

$$\mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(D_0 + \epsilon)\psi_{\mathrm{h}} \,, \quad \mathscr{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} = \overline{\psi}_{\bar{\mathrm{h}}}(-D_0 + \epsilon)\psi_{\bar{\mathrm{h}}} \,, \quad E_{\mathrm{h}/\bar{\mathrm{h}}}^{\mathrm{QCD}} = E_{\mathrm{h}/\bar{\mathrm{h}}}^{\mathrm{stat}} + m \,, \tag{1.36}$$

where the limit $\epsilon \to 0_+$ is to be understood.

We note that after performing this shift of the energies, there is no difference in the Lagrangian of a charm or a b-quark if both are treated at the lowest order in this expansion. We turn to discussing this as well as other symmetries of the static theory.

*Symmetries*

1. Flavor

If there are $F$ heavy quarks, we just add a corresponding flavor index and use a notation

$$\psi_{\mathrm{h}} \to \psi_{\mathrm{h}} = (\psi_{\mathrm{h}1}, \ldots, \psi_{\mathrm{h}F})^T \,, \quad \overline{\psi}_{\mathrm{h}} \to \overline{\psi}_{\mathrm{h}} = (\overline{\psi}_{\mathrm{h}1}, \ldots, \overline{\psi}_{\mathrm{h}F}) \tag{1.37}$$
$$\mathscr{L}_{\mathrm{h}}^{\mathrm{stat}} = \overline{\psi}_{\mathrm{h}}(D_0 + \epsilon)\psi_{\mathrm{h}} \,. \tag{1.38}$$

Then we obviously have the symmetry

$$\psi_{\mathrm{h}}(x) \to V\, \psi_{\mathrm{h}}(x) \,, \quad \overline{\psi}_{\mathrm{h}}(x) \to \overline{\psi}_{\mathrm{h}}(x) V^\dagger \,, \qquad V \in \mathrm{SU}(F) \tag{1.39}$$

and the same for the anti-quarks. Note that this symmetry emerges in the large mass limit irrespective of how the limit is taken. For example we may take ($F = 2$ with the first heavy flavor identified with charm and the second with beauty)

$$m_{\mathrm{b}} - m_{\mathrm{c}} = c \times \Lambda_{\mathrm{QCD}} \,, \quad \text{or} \quad m_{\mathrm{b}}/m_{\mathrm{c}} = c' \,, \quad m_{\mathrm{b}} \to \infty \tag{1.40}$$

with either $c$ or $c'$ fixed when taking $m_{\mathrm{b}} \to \infty$.

2. Spin

We further note that for each field there are also the two spin components but the Lagrangian contains no spin-dependent interaction. The associated SU(2) rotations are generated by the spin matrices eq. (1.29) (remember that $\psi_{\rm h}$, $\overline{\psi}_{\rm h}$ are kept as 4-component fields with 2 components vanishing)

$$\sigma_k = \frac{1}{2}\epsilon_{ijk}\sigma_{ij} \equiv \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} , \tag{1.41}$$

where the symbol $\sigma_k$ is used at the same time for the Pauli matrices and the $4 \times 4$ matrix. We here are in the Dirac representation where

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \; P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \; P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{1.42}$$

The spin rotation is then

$$\psi_{\rm h}(x) \to {\rm e}^{i\alpha_k\sigma_k}\, \psi_{\rm h}(x) , \qquad \overline{\psi}_{\rm h}(x) \to \overline{\psi}_{\rm h}(x){\rm e}^{-i\alpha_k\sigma_k} , \tag{1.43}$$

with arbitrary real parameters $\alpha_k$. It acts on each flavor component of the field. Obviously, the symmetry is even bigger. We can take $V \in {\rm SU}(2F)$ in eq. (1.39). This plays a rôle in heavy meson ChPT (Wise, 1992, Grinstein *et al.*, 1992, Burdman and Donoghue, 1992).

3. Local Flavor-number

The static Lagrangian contains no space derivative. The transformation

$$\psi_{\rm h}(x) \to {\rm e}^{i\eta(\mathbf{x})}\, \psi_{\rm h}(x) , \qquad \overline{\psi}_{\rm h}(x) \to \overline{\psi}_{\rm h}(x){\rm e}^{-i\eta(\mathbf{x})} , \tag{1.44}$$

is therefore a symmetry for any local phase $\eta(\mathbf{x})$. For every point $\mathbf{x}$ there is a corresponding Noether charge

$$Q_{\rm h}(x) = \overline{\psi}_{\rm h}(x)\psi_{\rm h}(x) \, [= \overline{\psi}_{\rm h}(x)\gamma_0\psi_{\rm h}(x) \,] \tag{1.45}$$

which we call local quark number. It is conserved,

$$\partial_0 Q_{\rm h}(x) = 0 \; \forall x . \tag{1.46}$$

### 1.3.4 Renormalizability of the static theory

Our effective field theory is in the category of local field theories with a Lagrangian made up from local fields. In $d$ space-time dimensions, standard wisdom says that such theories are renormalizable if the mass-dimension of the fields in the Lagrangian does not exceed $d$. Ultraviolet divergences can then be absorbed by adding a complete set of (composite) local fields with mass dimension smaller or equal to $d$ to the Lagrangian.

According to this (unproven[7]) rule, the static theory is renormalizable. The possible counter-terms have to share the symmetries of the bare Lagrangian. They are easily

[7]Power counting as discussed by Peter Weisz at this school is not applicable here, since the propagator does not fall off with all momentum components.

found. From the kinetic term in the Lagrangian eq. (1.36) we see that the dimension of the fields is $[\psi_{\rm h}] = 3/2$. Only 2-fermion terms with up to one derivative are then possible. Space-derivatives are excluded by the local phase invariance eq. (1.44). We then have the total quantum Lagrangian

$$\mathscr{L}_{\rm h}(x) = c_1\mathcal{O}_1(x) + c_2\mathcal{O}_2(x) \tag{1.47}$$

$$\mathcal{O}_1(x) = \overline{\psi}_{\rm h}(x)\psi_{\rm h}(x)\,, \quad \mathcal{O}_2(x) = \overline{\psi}_{\rm h}(x)D_0\psi_{\rm h}(x)\,, \tag{1.48}$$

where the convention $c_2 = 1$ can be chosen since it only fixes the unphysical field normalization, and $c_1 = \delta m$ has mass dimension $[\delta m] = 1$ and corresponds to an additive mass renormalization. From dimensional analysis and neglecting for simplicity the masses of the light quarks, it can be written as $\delta m = (e_1 g_0^2 + e_2 g_0^4 + \ldots)\,\Lambda_{\rm cut}$ in terms of the bare gauge coupling $g_0$ and a cutoff $\Lambda_{\rm cut}$, which in lattice regularization is $\Lambda_{\rm cut} = 1/a$. For a static quark there is of course no chiral symmetry to forbid additive mass renormalization.

This is the complete static Lagrangian. After the standard QCD renormalization of coupling and light quark masses, all divergences can be absorbed in $\delta m$, i.e. an energy shift. Flavor symmetry tells us that with several heavy flavors, $\delta m$ is proportional to the unit matrix in flavor space. Energies of *any state* are then

$$E^{\rm QCD}_{\rm h/\bar{h}} = E^{\rm stat}_{\rm h/\bar{h}}\Big|_{\delta m=0} + \delta m + m = E^{\rm stat}_{\rm h/\bar{h}}\Big|_{\delta m=0} + m_{\rm bare}\,. \tag{1.49}$$

Here $m_{\rm bare}$ and $\delta m$ compensate the linear divergence (self energy) of the static theory, while $m$ is finite. Note that there is no symmetry which would suggest a natural way of splitting $m_{\rm bare}$ into $\delta m$ and $m$. This split is arbitrary and convention dependent. The quantity $\delta m$ is often called the residual mass.

A rigorous proof of *renormalizability* to all orders in perturbation theory has not been given but we note the following.

- Perturbative computations have confirmed the standard wisdom. These computations reach up to three loops in dimensional regularization (Chetyrkin and Grozin, 2003, Grozin *et al.*, 2008), while in various different lattice regularizations 1-loop computations have been carried out (Eichten and Hill, 1990*a*, Eichten and Hill, 1990*c*, Eichten and Hill, 1990*b*, Boucaud *et al.*, 1989, Boucaud *et al.*, 1993, Flynn *et al.*, 1991, Borrelli and Pittori, 1992, Kurth and Sommer, 2001, Kurth and Sommer, 2002, Della Morte *et al.*, 2005, Palombi, 2008, Guazzini *et al.*, 2007, Grimbach *et al.*, 2008, Palombi *et al.*, 2006, Blossier *et al.*, 2006)
- We will see non-perturbative results which again yield a rather strong confirmation.
- Nevertheless a proof of renormalizability would be very desirable.

### 1.3.5 Normalization of states, scaling of decay constants

For the discussion of the mass-dependence of matrix elements we have to think about the normalization of states. Standard, relativistic invariant, normalization of bosonic one-particle states is

$$\langle \mathbf{p}|\mathbf{p}'\rangle_{\rm rel} = (2\pi)^3\, 2E(\mathbf{p})\,\delta(\mathbf{p}-\mathbf{p}')\,. \tag{1.50}$$

The states have a mass-dimension $[\,|\mathbf{p}\rangle_{\rm rel}\,] = -1$. The factor $E(\mathbf{p})$ introduces a spurious mass-dependence. In the large mass limit, relativistic invariance plays no rôle and we should choose a mass-independent normalization instead. The standard convention for such a non-relativistic normalization is

$$\langle \mathbf{p}|\mathbf{p}'\rangle_{\rm NR} \equiv \langle \mathbf{p}|\mathbf{p}'\rangle = 2\,(2\pi)^3\,\delta(\mathbf{p}-\mathbf{p}') \tag{1.51}$$

with $[\,|\mathbf{p}\rangle\,] = -3/2$ and

$$|\mathbf{p}\rangle_{\rm rel} = \sqrt{E(\mathbf{p})}\,|\mathbf{p}\rangle\,. \tag{1.52}$$

Consider as an example where the normalization of states plays a role, the leptonic decay of a B-meson, $B^- \to \tau^-\bar\nu_\tau$. The transition amplitude $\mathcal{A}$ for this decay is given to a good approximation in terms of the effective weak Hamiltonian. It factorizes into a leptonic and a hadronic part as

$$\mathcal{A} \propto \langle \tau\;\bar\nu|\tau(x)\gamma_\mu(1-\gamma_5)\bar\nu_\tau(x)|0\rangle\,\langle 0|\bar u(x)\gamma_\mu(1-\gamma_5)b(x)|B^-\rangle\,. \tag{1.53}$$

Using parity and Lorentz invariance, the hadronic part is

$$\langle 0|\bar u(x)\gamma_\mu(1-\gamma_5)b(x)|B^-(\mathbf{p})\rangle = \langle 0|A_\mu(x)|B^-(\mathbf{p})\rangle = p_\mu f_{\rm B}{\rm e}^{ipx} \tag{1.54}$$

in terms of the flavored axial current

$$A_\mu(x) = \bar u(x)\gamma_\mu\gamma_5 b(x)\,. \tag{1.55}$$

There is a single hadronic parameter $f_{\rm B}$ (matrix element) parameterizing the bound state dynamics in this decay. We note that it is very relevant for the phenomenological analysis of the CKM matrix (Antonelli *et al.*, 2009).

We may now use HQET to find the asymptotic mass-dependence of $f_{\rm B}$ for large $m = m_{\rm b}$. Since to lowest order in $1/m$ the FTW transformation is trivial, the HQET current is just

$$A_0^{\rm HQET}(x) = A_0^{\rm stat}(x) + {\rm O}(1/m)\,,\;\; A_0^{\rm stat}(x) = \bar u(x)\gamma_0\gamma_5\psi_{\rm h}(x)\,. \tag{1.56}$$

The static current $A_0^{\rm stat}$ has no explicit mass dependence. In static approximation we then have

$$\langle 0|A_0^{\rm stat}(0)|B^-(\mathbf{p}=0)\rangle = \Phi^{\rm stat}\,, \tag{1.57}$$

with a mass-independent $\Phi^{\rm stat}$. Its relation to $f_{\rm B}$,

$$\Phi^{\rm stat} = m_{\rm B}^{-1/2}\,p_0\,f_{\rm B} = m_{\rm B}^{1/2} f_{\rm B}\,, \tag{1.58}$$

takes eq. (1.52) into account ($p_0 = E(\mathbf{0}) = m_{\rm B}$). We arrive at the prediction

$$f_{\rm B} = \frac{\Phi^{\rm stat}}{\sqrt{m_{\rm B}}} + {\rm O}(1/m_{\rm b})\,,\quad \frac{f_{\rm B}}{f_{\rm D}} = \frac{\sqrt{m_{\rm D}}}{\sqrt{m_{\rm B}}} + {\rm O}(1/m_{\rm c})\,. \tag{1.59}$$

The latter use of course assumes $\Lambda_{\rm QCD}/m_{\rm c} \ll 1$. We will see later that these predictions are modified by the renormalization of the effective theory.

### 1.3.6 HQET and phenomenology

Heavy quark spin/flavor symmetry is very useful to classify the spectrum in terms of a few non-perturbative parameters or predict relations between different masses, e.g.

$$m^2_{\mathrm{B}^*} - m^2_{\mathrm{B}} \approx m^2_{\mathrm{D}^*} - m^2_{\mathrm{D}}\,, \tag{1.60}$$

$$m_{\mathrm{B}'} - m_{\mathrm{B}} \approx m_{\mathrm{D}'} - m_{\mathrm{D}}\,, \tag{1.61}$$

where $m_{\mathrm{B}^*}$, $m_{\mathrm{D}^*}$ are the vector meson masses and with $m_{\mathrm{B}'}$, $m_{\mathrm{D}'}$ we indicate the first excitation in the pseudo-scalar sector. The first of these relations has been seen to be approximately realized in nature.

More detailed statements about semi-leptonic transitions $B \to Dl\nu$, $B^\star \to D^\star l\nu$ are possible. In the heavy quark limit for both beauty and charm these are described by a single form factor, the Isgur Wise function, instead of several (Isgur and Wise, 1989, Isgur and Wise, 1990). These topics and many others are discussed in many reviews, e.g. (Neubert, 1994). We here concentrate on lattice HQET and where HQET helps to understand lattice results for states with a b-quark.

**Exercise 1.1** Static quarks from the hopping parameter expansion

Consider a Wilson quark propagator in a gauge background field. Evaluate the leading non-vanishing term in the hopping parameter expansion (with non-zero time-separation). Check that it is the continuum HQET propagator (restricted to the lattice points) up to an energy shift. Even if this is a nice piece of confirmation, note that one here takes the limit $\kappa \to 0$ corresponding to $ma \to \infty$, while the true limit for relating QCD observables $\Phi^{\mathrm{QCD}}$ to those of HQET is

$$\Phi^{\mathrm{HQET}} \sim \lim_{m\to\infty} \lim_{a\to 0} \Phi^{\mathrm{QCD}},$$

in that order!

# 2
# Lattice formulation

We start with the static approximation. The $1/m$ terms will be added after a discussion of the renormalization of the static theory.

## 2.1 Lattice action

For a static quark there is no chiral symmetry. Since we want to avoid doublers, we discretize à la Wilson (with $r = 1$). The continuum $D_0\,\psi_{\rm h}(x)$ is transcribed to the lattice as

$$D_0\gamma_0 \to \tfrac{1}{2}\{(\nabla_0 + \nabla_0^*)\gamma_0 - a\nabla_0^*\nabla_0\}\,, \tag{2.1}$$

and with $P_+\psi_{\rm h} = \psi_{\rm h}\,,\ P_-\psi_{\bar{\rm h}} = \psi_{\bar{\rm h}}$, we have the lattice identities

$$D_0\,\psi_{\rm h}(x) = \nabla_0^*\psi_{\rm h}(x)\,, \quad D_0\,\psi_{\bar{\rm h}}(x) = \nabla_0\psi_{\bar{\rm h}}(x)\,. \tag{2.2}$$

For later convenience we insert a specific normalization factor, defining the static lattice Lagrangians

$$\mathscr{L}_{\rm h} = \frac{1}{1+a\delta m}\overline{\psi}_{\rm h}(x)[\nabla_0^* + \delta m]\psi_{\rm h}(x)\,, \tag{2.3}$$

$$\mathscr{L}_{\bar{\rm h}} = \frac{1}{1+a\delta m}\overline{\psi}_{\bar{\rm h}}(x)[-\nabla_0 + \delta m]\psi_{\bar{\rm h}}(x)\,. \tag{2.4}$$

The following points are worth noting.

- Formally, this is just a one-dimensional Wilson fermion replicated for all space points $\mathbf{x}$, see also exercise 1.1.
- As a consequence there are no doubler modes.
- The construction of a positive hermitian transfer matrix for Wilson fermions (Lüscher, 1977, Montvay and Münster, 1994) can just be taken over.
- The choice of the backward derivative for the quark and the forward derivative for the anti-quark is selected by the Wilson term. We will see that this selects forward/backward propagation and an $\epsilon$-prescription as in eq. (1.36) is not needed.
- The form of this Lagrangian was first written down by Eichten and Hill (Eichten and Hill, 1990*a*).
- The lattice action preserves all the continuum heavy quark symmetries discussed in the previous section.

## 2.2 Propagator

From the Lagrangian eq. (2.3) we have the defining equation for the propagator

$$\frac{1}{1+a\,\delta m}(\nabla_0^* + \delta m)G_{\rm h}(x,y) = \delta(x-y)P_+ \equiv a^{-4}\prod_\mu \delta_{\frac{x_\mu}{a}\frac{y_\mu}{a}}P_+\,. \tag{2.5}$$

Obviously $G_{\rm h}(x,y)$ is proportional to $\delta(\mathbf{x}-\mathbf{y})$. Writing $G_{\rm h}(x,y) = g(n_0,k_0;\mathbf{x})\delta(\mathbf{x}-\mathbf{y})P_+$ with $x_0 = an_0\,,\ y_0 = ak_0$, the above equation yields a simple recursion for $g(n_0+1,k_0;\mathbf{x})$ in terms of $g(n_0,k_0;\mathbf{x})$ which is solved by

$$g(n_0,k_0;\mathbf{x}) = \theta(n_0-k_0)(1+a\delta m)^{-(n_0-k_0)}\mathcal{P}(y,x;0)^\dagger\,, \tag{2.6}$$

$$\mathcal{P}(x,x;0) = 1\,,\quad \mathcal{P}(x,y+a\hat{0};0) = \mathcal{P}(x,y;0)U(y,0)\,, \tag{2.7}$$

where

$$\theta(n_0-k_0) = \begin{cases} 0 & n_0 < k_0 \\ 1 & n_0 \geq k_0\,. \end{cases} \tag{2.8}$$

The static propagator reads

$$G_{\rm h}(x,y) = \theta(x_0-y_0)\;\delta(\mathbf{x}-\mathbf{y})\;\exp\big(-\widehat{\delta m}\,(x_0-y_0)\big)\;\mathcal{P}(y,x;0)^\dagger\;P_+\,, \tag{2.9}$$

$$\widehat{\delta m} = \tfrac{1}{a}\ln(1+a\delta m)\,. \tag{2.10}$$

The object $\mathcal{P}(x,y;0)$ parallel transports fields in the fundamental representation from $y$ to $x$ along a time-like path. Note that the derivation fixes $\theta(0)=1$ for the lattice $\theta$-function. As in the continuum, the mass counter term $\delta m$ just yields an energy shift; now, on the lattice, the shift is

$$E_{\rm h/\bar{h}}^{\rm QCD} = E_{\rm h/\bar{h}}^{\rm stat}\Big|_{\delta m=0} + m_{\rm bare}\,,\quad m_{\rm bare} = \widehat{\delta m} + m\,. \tag{2.11}$$

It is valid for all energies of states with a single heavy quark or anti-quark. As in the continuum the split between $\delta m$ and the finite $m$ is convention dependent.

In complete analogy the anti-quark propagator is given by

$$G_{\bar{\rm h}}(x,y) = \theta(y_0-x_0)\;\delta(\mathbf{x}-\mathbf{y})\;\exp\big(-\widehat{\delta m}\,(y_0-x_0)\big)\;\mathcal{P}(x,y;0)\;P_-\,. \tag{2.12}$$

## 2.3 Symmetries

All HQET symmetries are preserved on the lattice, in particular the U($2F$) spin-flavor symmetry and the local flavor-number conservation. The symmetry transformations can literally be carried over from the continuum, e.g. eq. (1.44). One just replaces the continuum fields by the lattice ones.

Note that these HQET symmetries are defined in terms of transformations of the heavy quark fields while the light quark fields do not change (unlike e.g. standard parity). Integrating out just the quark fields in the path integral while leaving the integral over the gauge fields, they thus yield identities for the integrand or one may say for "correlation functions in any fixed gauge background field".

## 2.4 Symanzik analysis of cutoff effects

According to the — by now well tested[1] — Symanzik conjecture, the cutoff effects of a lattice theory can be described in terms of an effective *continuum* theory (Symanzik, 1983*a*, Symanzik, 1983*b*, Lüscher *et al.*, 1996). Once the terms in Symanzik's effective Lagrangian are known, the cutoff effects can be canceled by adding terms of the same form to the lattice action, resulting in an improved action.

For a static quark, Symanzik's effective action is (Kurth and Sommer, 2001)

$$S_{\text{eff}} = S_0 + aS_1 + \dots\,, \quad S_i = \int \mathrm{d}^4x\, \mathscr{L}_{\mathrm{i}}(x) \tag{2.13}$$

where $\mathscr{L}_0(x) = \mathscr{L}_{\mathrm{h}}^{\text{stat}}(x)$ is the continuum static Lagrangian of eq. (1.47) and

$$\mathscr{L}_1(x) = \sum_{i=3}^{5} c_i\, \mathcal{O}_{\mathrm{i}}(x)\,, \tag{2.14}$$

is given in terms of local fields with mass dimension $[\mathcal{O}_{\mathrm{i}}(x)] = 5$. Their coefficients $c_i$ are functions of the bare gauge coupling. Assuming for simplicity mass-degenerate light quarks with a mass $m_{\mathrm{l}}$, the set of possible dimension five fields, which share the symmetries of the lattice theory, is

$$\mathcal{O}_3 = \overline{\psi}_{\mathrm{h}} D_0 D_0 \psi_{\mathrm{h}}\,, \quad \mathcal{O}_4 = m_{\mathrm{l}}\, \overline{\psi}_{\mathrm{h}} D_0 \psi_{\mathrm{h}}\,, \quad \mathcal{O}_5 = m_{\mathrm{l}}^2\, \overline{\psi}_{\mathrm{h}} \psi_{\mathrm{h}}\,. \tag{2.15}$$

Note that $P_+ \sigma_{0j} P_+ = 0$ means there is no term $\overline{\psi}_{\mathrm{h}} \sigma_{0j} F_{0j} \psi_{\mathrm{h}}$, and $\overline{\psi}_{\mathrm{h}} D_j D_j \psi_{\mathrm{h}}$ can't occur because it violates the local phase invariance eq. (1.44). Finally $\overline{\psi}_{\mathrm{h}} \sigma_{jk} F_{jk} \psi_{\mathrm{h}}$ is not invariant under the spin rotations eq. (1.43).

Furthermore, we are only interested in on-shell correlation functions and energies. For this class of observables $\mathcal{O}_3$, $\mathcal{O}_4$ do not contribute (Lüscher and Weisz, 1985, Lüscher *et al.*, 1996) because they vanish by the equation of motion[2],

$$D_0 \psi_{\mathrm{h}} = 0\,. \tag{2.16}$$

The only remaining term, $\mathcal{O}_5$, induces a redefinition of the mass counter-term $\delta m$ which therefore depends explicitly on the light quark mass.

We note that for almost all applications, $\delta m$ is explicitly canceled in the relation between physical observables and one thus has automatic on-shell $\mathrm{O}(a)$ improvement for the static action. No parameter has to be tuned to guarantee this property. Still, the improvement of matrix elements and correlation functions requires to also consider composite fields in the effective theory.

[1] See Peter Weisz' lectures for a theoretical discussion and chapter I of (Sommer, 2006) for an overview of tests. Finally (Balog *et al.*, 2009*b*, Balog *et al.*, 2009*a*) represents the most advanced understanding of the subject.

[2] The equations of motion follow just from a change of variable in the path integral. Contact terms are re-absorbed into the free coefficients $c_i$. We refer to Peter Weisz' lectures or (Lüscher *et al.*, 1996) for a more detailed discussion.

**Exercise 2.1** The static quark anti-quark potential.

A (time-local) field

$$O(t,\mathbf{x},\mathbf{y}) = \overline{\psi}_{\mathrm{h}}(x)\,\mathcal{P}(x,y)\gamma_5\,\psi_{\bar{\mathrm{h}}}(y)\,,\quad x_0 = y_0 = t$$

with $\mathcal{P}(x,y)$ being a parallel transporter from $y$ to $x$ in $x_0 = t$ plane, can be used to annihilate a quark-anti-quark pair at a separation $\mathbf{x}-\mathbf{y}$, while

$$\bar{O}(t,\mathbf{x},\mathbf{y}) = -\overline{\psi}_{\bar{\mathrm{h}}}(y)\,\mathcal{P}(y,x)\gamma_5\,\psi_{\mathrm{h}}(x)\,,\quad x_0 = y_0 = t \tag{2.17}$$

will create a quark-anti-quark pair at a separation $\mathbf{x}-\mathbf{y}$.

Show that for $t > 0$

$$\langle\,\bar{O}(t,\mathbf{x},\mathbf{y})\;O(0,\mathbf{x},\mathbf{y})\,\rangle = \text{const.}\;\mathrm{e}^{-2t\,\widehat{\delta m}}W(t,\mathbf{x}-\mathbf{y}) \tag{2.18}$$

where $W$ is the Wilson loop introduced in the lectures of P. Hernandez. Since the energy levels of HQET are finite (after inclusion of a suitable $\delta m$), one can conclude that

$$V_{\mathrm{R}}(\mathbf{x}-\mathbf{y}) = -\lim_{t\to\infty}\partial_t \ln(W(t,\mathbf{x}-\mathbf{y})) + 2\widehat{\delta m} \tag{2.19}$$

is a finite quantity: the divergent constant in the bare potential is absorbed by $\widehat{\delta m}$, i.e. by a renormalization of the heavy quark mass.

Furthermore, from the $\mathrm{O}(a)$ improvement of HQET, one concludes (Necco and Sommer, 2002)

$$V_{\mathrm{R}}(\mathbf{x}-\mathbf{y}) = V_{\mathrm{R}}^{\mathrm{cont}}(r = |\mathbf{x}-\mathbf{y}|) + \mathrm{O}(a^2) \tag{2.20}$$

if the action for the light fields is $\mathrm{O}(a)$ improved.

### 2.4.1 Renormalized and improved axial current.

We now also have to specify the discretization of the light quark field $\psi$. We will generically think of a standard $\mathrm{O}(a)$-improved Wilson discretization (Sheikholeslami and Wohlert, 1985, Lüscher *et al.*, 1996) but occasionally mention changes which occur when one has an action with exact chiral symmetry (Neuberger, 1998, Hasenfratz *et al.*, 1998, Lüscher, 1998)[3] or a Wilson regularization with a twisted mass term(Frezzotti *et al.*, 2001*a*, Frezzotti *et al.*, 2001*b*, Frezzotti and Rossi, 2004). As an example we study the time component of the axial current. In Symanzik's effective theory it is represented by

$$(A_0^{\mathrm{stat}})_{\mathrm{eff}} = A_0^{\mathrm{stat}} + a\sum_{k=1}^{4}\omega_k(\delta A_0^{\mathrm{stat}})_k\,,\quad A_0^{\mathrm{stat}} = \overline{\psi}\gamma_0\gamma_5\psi_{\mathrm{h}} \tag{2.21}$$

with some coefficients $\omega_k$. Here the flavor index of the field $\overline{\psi}$ is suppressed. It is considered to have some fixed but arbitrary value for our discussion, except where we indicate this explicitly. A basis for the dimension four fields $\{(\delta A_0^{\mathrm{stat}})_k\}$ is

[3] When the chiral symmetry realization of domain wall fermions (Shamir, 1993) is good enough, these fermions can of course also be considered to have an in practice exact chiral symmetry.

$$
\begin{aligned}
(\delta A_0^{\rm stat})_1 &= \overline{\psi}\overleftarrow{D}_j\gamma_j\gamma_5\psi_{\rm h}\,, \quad (\delta A_0^{\rm stat})_2 = \overline{\psi}\gamma_5 D_0\psi_{\rm h}\,,\\
(\delta A_0^{\rm stat})_3 &= \overline{\psi}\overleftarrow{D}_0\gamma_5\psi_{\rm h}\,, \quad (\delta A_0^{\rm stat})_4 = m_{\rm l}\,\overline{\psi}\gamma_0\gamma_5\psi_{\rm h}\,.
\end{aligned}
\tag{2.22}
$$

From eq. (2.16) we see that $k = 2$ does not contribute, while the equation of motion for $\overline{\psi}$ relates $(\delta A_0^{\rm stat})_3$, $(\delta A_0^{\rm stat})_4$ and $(\delta A_0^{\rm stat})_1$. We choose to remain with $k = 1$ (and in principle $k = 4$), but for simplicity assume[4] $a\,m_{\rm l} \ll 1$; we can then drop $(\delta A_0^{\rm stat})_4$. So for on-shell quantities the effective theory representation is

$$
(A_0^{\rm stat})_{\rm eff} = A_0^{\rm stat} + a\tilde{\omega}_1(\delta A_0^{\rm stat})_1\,. \tag{2.23}
$$

In order to achieve a cancellation of the $\mathrm{O}(a)$ lattice spacing effects, we add a corresponding combination of correction terms to the axial current in the lattice theory and write the improved and renormalized current in the form

$$
(A_{\rm R}^{\rm stat})_0 = Z_{\rm A}^{\rm stat}(g_0, a\mu)\,(A_{\rm I}^{\rm stat})_0\,, \tag{2.24}
$$

$$
(A_{\rm I}^{\rm stat})_0 = A_0^{\rm stat} + ac_{\rm A}^{\rm stat}(g_0)\,\overline{\psi}\gamma_j\gamma_5\tfrac{1}{2}(\overleftarrow{\nabla}_j + \overleftarrow{\nabla}_j^*)\psi_{\rm h}\,, \tag{2.25}
$$

with a mass-independent renormalization constant $Z_{\rm A}^{\rm stat}$ and a dimensionless improvement coefficient, $c_{\rm A}^{\rm stat}$, depending again on $g_0$ but not on the light quark mass.

The improvement coefficients can be determined such that for this (time component of the) *improved axial current* we have the representation

$$
(A_0^{\rm stat})_{\rm eff} = \overline{\psi}\gamma_0\gamma_5\psi_{\rm h} + \mathrm{O}(a^2)\,, \tag{2.26}
$$

in the Symanzik effective theory. In other words $\tilde{\omega}_1$ is then $\mathrm{O}(a)$ and cutoff effects are $\mathrm{O}(a^2)$.

The symmetries of the static theory are strong enough to improve all components of the flavor currents in terms of just $c_{\rm A}^{\rm stat}$ and to renormalize them by $Z_{\rm A}^{\rm stat}$. Let us discuss how this works.

## 2.5 The full set of flavor currents

The previous discussion literally carries over to the time component of the vector current,

$$
V_0^{\rm stat} = \overline{\psi}\gamma_0\psi_{\rm h}\,. \tag{2.27}
$$

Its improved and renormalized lattice version may be chosen as

$$
(V_{\rm R}^{\rm stat})_0 = Z_{\rm V}^{\rm stat}\,(V_{\rm I}^{\rm stat})_0 \tag{2.28}
$$

$$
(V_{\rm I}^{\rm stat})_0 = \overline{\psi}\gamma_0\psi_{\rm h} + ac_{\rm V}^{\rm stat}\overline{\psi}\gamma_j\tfrac{1}{2}(\overleftarrow{\nabla}_j + \overleftarrow{\nabla}_j^*)\psi_{\rm h}\,. \tag{2.29}
$$

[4] If the light quark action has an exact chiral symmetry or the light quarks are discretized with a twisted mass term at full twist, this restriction is unnecessary, since the term is excluded by the symmetry. Note that $(\delta A_0^{\rm stat})_1$ is, however, not forbidden by chiral symmetry and $c_{\rm A}^{\rm stat}$ is necessary for $\mathrm{O}(a)$-improvement in any case.

The chiral symmetry of the continuum limit can be used to relate $Z_{\rm V}^{\rm stat}$, $c_{\rm V}^{\rm stat}$ to $Z_{\rm A}^{\rm stat}$, $c_{\rm A}^{\rm stat}$ in the following way. We assume $N_{\rm f} \geq 2$ massless light quarks. Then the infinitesimal transformation

$$\delta_{\rm A}^a \psi(x) = \tfrac{1}{2}\tau^a \gamma_5 \psi(x), \qquad \delta_{\rm A}^a \overline{\psi}(x) = \overline{\psi}(x)\gamma_5 \tfrac{1}{2}\tau^a \ , \tag{2.30}$$

with the Pauli matrices $\tau^a$ acting on two of the flavor components of the light quark fields $\psi$, $\overline{\psi}$, is a (non-anomalous) symmetry of the theory. Identifying $V_0^{\rm stat} = \overline{\psi}_1 \gamma_0 \psi_{\rm h}$, where $\overline{\psi}_1$ is the first flavor component of $\overline{\psi}$, the vector current transforms as $\delta_{\rm A}^3 V_0^{\rm stat} = -\frac{1}{2} A_0^{\rm stat}$. The same property can then be required for the renormalized and improved lattice fields,

$$\delta_{\rm A}^3 (V_{\rm R}^{\rm stat})_0 = -\tfrac{1}{2}(A_{\rm R}^{\rm stat})_0 + {\rm O}(a^2)\,. \tag{2.31}$$

This condition can be implemented in the form of Ward identities relating different correlation functions, in particular in the Schrödinger functional . We refer to A. Vladikas' lectures and (Lüscher, 1998) for the principle; practical implementations have been studied in (Hashimoto *et al.*, 2002, Palombi, 2008). Such Ward identities determine $Z_{\rm V}^{\rm stat}$, $c_{\rm V}^{\rm stat}$ in terms of $Z_{\rm A}^{\rm stat}$, $c_{\rm A}^{\rm stat}$.

Furthermore by a finite spin-symmetry transformation (with $\sigma_k$ of eq. (1.41))

$$\psi_{\rm h} \to \psi'_{\rm h} = {\rm e}^{-i\pi\sigma_k/2}\psi_{\rm h} = -i\sigma_k \psi_{\rm h}\,, \qquad \overline{\psi}'_{\rm h} = \overline{\psi}_{\rm h} i\sigma_k\,, \tag{2.32}$$

we have

$$V_0^{\rm stat} \to \left[V_0^{\rm stat}\right]' = A_k^{\rm stat} \equiv \overline{\psi}\gamma_k\gamma_5\psi_{\rm h}\,, \quad \left[A_0^{\rm stat}\right]' = V_k^{\rm stat} \equiv \overline{\psi}\gamma_k\psi_{\rm h}\,, \tag{2.33}$$

and we can require the same for the correction terms,

$$\left[\delta V_0^{\rm stat}\right]' = \delta A_k^{\rm stat}\,, \quad \left[\delta A_0^{\rm stat}\right]' = \delta V_k^{\rm stat}\,. \tag{2.34}$$

We leave it as an exercise to determine the form of $\delta A_k^{\rm stat}$, $\delta V_k^{\rm stat}$. The discussed transformations are valid for the bare lattice fields at any lattice spacing. Thus renormalization and improvement of the spatial components is given completely in terms of the time-components once we define the renormalized fields to transform in the same way as the bare fields. A last property to note before writing down the renormalized and improved fields is that we have

$$Z_{\rm V}^{\rm stat}(g_0, a\mu) = Z_{\rm V/A}^{\rm stat}(g_0)\, Z_{\rm A}^{\rm stat}(g_0, a\mu) \tag{2.35}$$

with a $\mu$-independent function $Z_{\rm V/A}^{\rm stat}(g_0)$ and up to ${\rm O}(a^2)$, as soon as we require eq. (2.31). [5]

[5] A formal argument is as follows. Rewrite eq. (2.31) in terms of the bare operators, $Z_{\rm V}^{\rm stat}(g_0, a\mu)\delta_{\rm A}^3 V_0^{\rm stat} = -\frac{1}{2} Z_{\rm A}^{\rm stat}(g_0, a\mu) A_0^{\rm stat} + {\rm O}(a^2)$. Since the bare, regularized, operators $V_0^{\rm stat}, A_0^{\rm stat}$ carry no $\mu$-dependence, we see that $Z_{\rm V}^{\rm stat}(g_0, a\mu)/Z_{\rm A}^{\rm stat}(g_0, a\mu)$ is a function of $g_0$ only, apart from ${\rm O}(a^2)$ cutoff effects. To make the argument more rigorous one should rewrite the equation in the form of correlation functions which represent a Ward identity equivalent to eq. (2.31).

Let us disregard the $\mathrm{O}(a)$ improvement terms for simplicity. We can then summarize what we have learnt about the renormalization of the static-light bilinears as

$$(A_{\mathrm{R}}^{\mathrm{stat}})_0 = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, a\mu)\, A_0^{\mathrm{stat}}\,, \tag{2.36}$$

$$(V_{\mathrm{R}}^{\mathrm{stat}})_0 = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, a\mu)\, Z_{\mathrm{V/A}}^{\mathrm{stat}}(g_0)\, V_0^{\mathrm{stat}}\,, \tag{2.37}$$

$$(V_{\mathrm{R}}^{\mathrm{stat}})_k = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, a\mu)\, V_k^{\mathrm{stat}}\,, \tag{2.38}$$

$$(A_{\mathrm{R}}^{\mathrm{stat}})_k = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, a\mu)\, Z_{\mathrm{V/A}}^{\mathrm{stat}}(g_0)\, A_k^{\mathrm{stat}}\,, \tag{2.39}$$

where $Z_{\mathrm{V/A}}^{\mathrm{stat}}(g_0)$ can be determined from a chiral Ward identity (Hashimoto *et al.*, 2002, Palombi, 2008). Note that we denote the flavor currents in HQET in complete analogy to QCD. Still they do not form 4-vectors, as 4-dimensional rotation invariance is broken in HQET. For example $(A_{\mathrm{R}}^{\mathrm{stat}})_0$ cannot be rotated into $(A_{\mathrm{R}}^{\mathrm{stat}})_k$ by a 90 degree lattice rotation.

The only bilinears which are missing here are scalar, pseudo-scalar densities (and the tensor). These are equivalent to $A_0^{\mathrm{stat}}$ and $V_0^{\mathrm{stat}}$ in static approximation, for example

$$\overline{\psi}\gamma_5\psi_{\mathrm{h}} = \overline{\psi}\gamma_5\gamma_0\psi_{\mathrm{h}} = -A_0^{\mathrm{stat}}\,, \quad \overline{\psi}\psi_{\mathrm{h}} = \overline{\psi}\gamma_0\psi_{\mathrm{h}} = V_0^{\mathrm{stat}}\,. \tag{2.40}$$

At this stage it is therefore unnecessary to introduce renormalized scalar and pseudo-scalar densities.

We have so far written down expressions for the relevant renormalized heavy-light quark bilinears. The $Z$-factors can be chosen such that correlation functions of these fields have a continuum limit (with $\delta m$, gauge coupling and light quark masses properly determined). Beyond this requirement, however, also the finite parts need to be fixed by renormalization conditions. We have fixed some of them such that the renormalized fields satisfy chiral symmetry and heavy quark spin symmetry. Only one finite part (in $Z_{\mathrm{A}}^{\mathrm{stat}}$) then remains free. Preserving these symmetries by the renormalization is natural, but not absolutely required; e.g. eq. (2.33) could be violated in terms of the renormalized fields. As long as one just remains inside the effective field theory these ambiguities are not fixed. The proper conditions for the finite parts, valid for HQET as an effective theory of QCD, have to be determined from QCD with finite heavy quark masses. We will return to this later.

We may, however, already note that for renormalization group invariant fields, these ambiguities are not present. The renormalization group invariants are thus very appropriate. Still, relating the bare lattice fields to the renormalization group invariant ones is a non-trivial task in practice(Lüscher *et al.*, 1991, Capitani *et al.*, 1999). We will briefly discuss how it can be done (and has been done) for the static-light bilinears (Kurth and Sommer, 2001,Heitger *et al.*, 2003,Della Morte *et al.*, 2007*b*). For this and other purposes we need the Schrödinger functional . In the following we just give a simplified review of it and describe how static quarks are incorporated. Some more details are discussed by Peter Weisz.

## 2.6 HQET and Schrödinger Functional

The Schrödinger functional (Symanzik, 1981,Lüscher *et al.*, 1992,Sint, 1994,Sint, 1995) can just be seen as QCD in a finite Euclidean space-time of size $T \times L^3$, with spe-

cific boundary conditions. It is useful as a renormalizable probe of QCD, providing a definition of correlation functions which are accessible at all distances, short or long: gauge invariance is manifest and even at short distances (large momenta) cutoff effects can be kept small. It will help us to perform the non-perturbative renormalization of HQET and its matching to QCD. In all these applications it is advantageous to have a variety of kinematics at ones disposal. One element is to have access to finite but small momenta of the quarks (think of the free theory, a relevant starting point for the short distance regime).

To this end, the spatial boundary conditions were chosen to be $\psi(x + L\hat{k}) = \mathrm{e}^{i\theta_k}\psi(x)$ , $\overline{\psi}(x + L\hat{k}) = \mathrm{e}^{-i\theta_k}\overline{\psi}(x)$ in (Sint and Sommer, 1996), which allows momenta

$$p_k = \frac{2\pi l_k}{L} + \frac{\theta_k}{L}\,,\ l_k \in \mathbb{Z}\,, \tag{2.41}$$

in particular small ones when $l_k = 0$. Performing a variable transformation $\psi(x) \to \mathrm{e}^{i\theta_k x_k/L}\psi(x)$, $\overline{\psi}(x) \to \mathrm{e}^{-i\theta_k x_k/L}\psi(x)$, for $0 \le x_k \le L - a$, we see that this boundary condition is equivalent to periodic boundary conditions (without a phase) for the new fields, while the spatial covariant derivatives contain an additional phase, for example

$$\nabla_k\psi(x) = \frac{1}{a}\big[\mathrm{e}^{i\theta_k a/L}U(x,\mu)\psi(x + a\hat{k}) - \psi(x)\big]\,, \tag{2.42}$$

see also Sect. A.1. The phase $\theta_k a/L$ can be seen as a constant abelian gauge potential and the above variable transformation as a gauge transformation. Of course, the angles $\theta_k$ which we will set all equal from now on ($\theta_k = \theta$), are not specific to the Schrödinger functional ; they just have first been used in this context.

The standard Schrödinger functional boundary conditions in time are (Sint, 1994, Sint, 1995)

$$P_+\psi(x)|_{x_0=0} = 0\,, \quad P_-\psi(x)|_{x_0=T} = 0\,, \tag{2.43}$$

and

$$\overline{\psi}(x)P_-|_{x_0=0} = 0\,, \quad \overline{\psi}(x)P_+|_{x_0=T} = 0\,. \tag{2.44}$$

The gauge fields are taken periodic in space and the space components of the continuum gauge fields are set to zero at $x_0 = 0$ and $x_0 = T$ (on the lattice the boundary links $U(x,k)$ are set to unity).[6]

For the static quark the components projected by $P_-$ vanish anyway, so there is just

$$P_+\psi_{\mathrm{h}}(x)|_{x_0=0} = 0, \quad \overline{\psi}_{\mathrm{h}}(x)P_+|_{x_0=T} = 0. \tag{2.45}$$

Defining

$$\psi_{\mathrm{h}}(x) = 0 \quad \text{if } x_0 < 0 \text{ or } x_0 \ge T, \tag{2.46}$$

the lattice action for the static quark with Schrödinger functional boundary conditions can be written as

[6] In (Lüscher *et al.*, 1992, Lüscher *et al.*, 1994) the definition of a renormalized coupling uses more general boundary conditions for the gauge fields, but these are not needed here.

$$S_{\rm h} = \frac{1}{1+a\delta m} a^4 \sum_x \overline{\psi}_{\rm h}(x)[\nabla_0^* + \delta m]\psi_{\rm h}(x) \tag{2.47}$$

as before. In general the improvement of the Schrödinger functional requires to add boundary terms to the action as a straightforward generalization of Symanzik improvement. These terms are dimension four composite fields located on or at the boundaries, summed over space (Lüscher *et al.*, 1992, Lüscher *et al.*, 1996). Since they are not so important here and are also known sufficiently well, we do not discuss them. We just note that no boundary improvement terms involving static fields are needed (Kurth and Sommer, 2001), since the dimension four fields vanish either due to the equation of motion or the heavy quark symmetries.

We take the same periodicity in space as for relativistic quarks,

$$\psi_{\rm h}(x + L\hat{k}) = \psi_{\rm h}(x)\,, \quad \overline{\psi}_{\rm h}(x + L\hat{k}) = \overline{\psi}_{\rm h}(x)\,. \tag{2.48}$$

In the static theory this has no effect, since quarks at different $\mathbf{x}$ are not coupled, but it plays a rôle at order $1/m$ where $\theta$ is a useful kinematical variable.

An important feature of the Schrödinger functional is that one can form gauge invariant correlation functions of boundary quark fields. In particular, one can project those quark fields to small spatial momentum, e.g. $\mathbf{p} = 1/L \times (\theta, \theta, \theta)$ for the quarks and $-\mathbf{p}$ for the anti-quarks. For the precise definition of the boundary quark fields we refer to (Lüscher *et al.*, 1996) or for an alternative view we refer to (Lüscher, 2006). The details are here not so important. We only need to know that these boundary fields, i.e. fermion fields localized at the boundaries, exist. Those at $x_0 = 0$ are denoted by

$$\zeta_{\rm l}(\mathbf{x})\,,\ \overline{\zeta}_{\rm l}(\mathbf{x})\,,\ \zeta_{\bar{\rm h}}(\mathbf{x})\,,\ \overline{\zeta}_{\rm h}(\mathbf{x})\,,$$

and those at $x_0 = T$ by

$$\zeta_{\rm l}{}'(\mathbf{x})\,,\ \overline{\zeta}_{\rm l}{}'(\mathbf{x})\,,\ \zeta_{\rm h}{}'(\mathbf{x})\,,\ \overline{\zeta}_{\bar{\rm h}}{}'(\mathbf{x})\,.$$

### 2.6.1 Renormalization

These boundary quark fields are multiplicatively renormalized with factors $Z_\zeta\,, Z_{\zeta_{\rm h}}$, such that $(\zeta_{\rm l}(\mathbf{x}))_{\rm R} = Z_\zeta \zeta_{\rm l}(\mathbf{x})$ etc.

To illustrate a first use of the Schrödinger functional and the boundary fields we introduce three correlation functions

$$f_{\rm A}^{\rm stat}(x_0,\theta) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}} \left\langle (A_{\rm I}^{\rm stat})_0(x)\, \overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right\rangle : \tag{2.49}$$

$$f_1^{\rm stat}(\theta) = -\frac{a^{12}}{2L^6}\sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z}} \left\langle \overline{\zeta}_{\rm l}{}'(\mathbf{u})\gamma_5\zeta_{\rm h}{}'(\mathbf{v})\, \overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\rm l}(\mathbf{z})\right\rangle : \tag{2.50}$$

$$f_1^{\rm hh}(x_3,\theta) = -\frac{a^8}{2L^2}\sum_{x_1,x_2,\mathbf{y},\mathbf{z}} \langle \overline{\zeta}_{\bar{\rm h}}{}'(\mathbf{x})\gamma_5\zeta_{\rm h}{}'(\mathbf{0})\, \overline{\zeta}_{\rm h}(\mathbf{y})\gamma_5\zeta_{\bar{\rm h}}(\mathbf{z})\rangle : \tag{2.51}$$

In the graphs, double lines are static quark propagators. Note that the sum in eq. (2.51) runs on $x_1$ and $x_2$ and therefore yields an $x_3$-dependent correlation function. We further point out that $\sum_{\mathbf{y}}$ etc. project the boundary quark fields onto zero (space) momentum, but together with the abelian gauge field, this is equivalent to a physical momentum $p_k = \theta/L$. For example the time-decay of a free mass-less quark propagator projected this way contains an energy $E(\theta/L,\theta/L,\theta/L) = \sqrt{3}\theta/L$, cf. eq. (1.4).

The above functions are renormalized as

$$\left[f_{\rm A}^{\rm stat}\right]_{\rm R} = Z_{\rm A}^{\rm stat} Z_{\zeta_{\rm h}} Z_\zeta\, f_{\rm A}^{\rm stat}\,, \quad \left[f_1^{\rm stat}\right]_{\rm R} = Z_{\zeta_{\rm h}}^2 Z_\zeta^2\, f_1^{\rm stat}\,, \quad \left[f_1^{\rm hh}\right]_{\rm R} = Z_{\zeta_{\rm h}}^4\, f_1^{\rm hh}\,. \tag{2.52}$$

We remind the reader that an additional renormalization is the mass counter-term of the static action.

The ratio

$$\left[\frac{f_{\rm A}^{\rm stat}(T/2,\theta)}{\sqrt{f_1^{\rm stat}(\theta)}}\right]_{\rm R} = Z_{\rm A}^{\rm stat}(g_0,a\mu)\,\frac{f_{\rm A}^{\rm stat}(T/2,\theta)}{\sqrt{f_1^{\rm stat}(\theta)}} \tag{2.53}$$

renormalizes in a simple way and also needs no knowledge of $\delta m$, since it cancels out due to eq. (2.9). It is hence an attractive possibility to *define* the renormalization constant $Z_{\rm A}^{\rm stat}$ through this ratio. Explicitly we may choose

$$Z_{\rm A}^{\rm stat}(g_0,a\mu) \equiv \frac{\sqrt{f_1^{\rm stat}(\theta)}}{f_{\rm A}^{\rm stat}(L/2,\theta)}\left[\frac{f_{\rm A}^{\rm stat}(L/2,\theta)}{\sqrt{f_1^{\rm stat}(\theta)}}\right]_{g_0=0} \qquad \mu = 1/L\,, \quad T = L\,, \quad \theta = \tfrac{1}{2}\,, \tag{2.54}$$

which defines the finite part of $Z_{\rm A}^{\rm stat}$ in a so-called Schrödinger functional scheme. As usual the factor $\left[f_{\rm A}^{\rm stat}(L/2,\theta)/\sqrt{f_1^{\rm stat}(\theta)}\right]_{g_0=0}$ is inserted to ensure $Z_{\rm A}^{\rm stat} = 1 + {\rm O}(g_0^2)$. The name Schrödinger functional scheme just refers to the fact that the renormalization factor is defined in terms of correlation functions with Schrödinger functional boundary conditions. While $Z_{\rm A}^{\rm stat}$ refers to a specific regularization, the renormalization scheme is independent of that and can in principle be applied in a continuum

regularization. Many similar Schrödinger functional schemes can be defined (e.g. (Heitger *et al.*, 2003)), but by the choice $T = L, \theta = 0.5$ we have made eq. (2.54) unique. It is implied that the light quark masses are set to zero. We will soon come back to the $\mu$-dependence of the renormalized current and its relation to the RGI current. First let us show some numerical results which provide a non-perturbative test of the renormalizability of the static theory.

## 2.7 Numerical test of the renormalizability

The above listed renormalization structure of the Schrödinger functional correlation functions is just deduced from a simple dimensional analysis. A number of 1-loop calculations of the correlation functions defined above as well as of others (Kurth and Sommer, 2001, Kurth and Sommer, 2002, Della Morte *et al.*, 2005, Palombi, 2008) confirm the structure eq. (2.52) and more generally the renormalizability of the theory (by local counter-terms).

Also non-perturbative tests exist. A stringent and precise one (Della Morte *et al.*, 2005) is based on the ratios

$$\xi_{\mathrm{A}}(\theta,\theta') = \frac{f_{\mathrm{A}}^{\mathrm{stat}}(T/2,\theta)}{f_{\mathrm{A}}^{\mathrm{stat}}(T/2,\theta')}\,, \quad \xi_1(\theta,\theta') = \frac{f_1^{\mathrm{stat}}(\theta)}{f_1^{\mathrm{stat}}(\theta')}\,, \quad h(d/L,\theta) = \frac{f_1^{\mathrm{hh}}(d,\theta)}{f_1^{\mathrm{hh}}(L/2,\theta)}\,. \tag{2.55}$$

The additional dependence on $L$ and the lattice resolution $a/L$ of these ratios is not indicated explicitly. With eq. (2.52), we see that all renormalization factors cancel in these ratios. They should have a finite limit $a/L \to 0$, approached asymptotically with a rate $(a/L)^2$. This is tested in Fig. 2.1, where $L$ is kept fixed in units of the reference length scale $r_0$(Sommer, 1994) to $L/r_0 = 1.436$. This choice corresponds to about $L \approx 0.7\,\mathrm{fm}$. The same continuum limit has to be reached for different lattice discretizations. Also this universality is tested in the graphs, where four different choices of the covariant derivative $D_0$ in the static action are used. All actions defined by the different choices of $D_0$ have the symmetries discussed earlier.

## 2.8 Scale dependence of the axial current and the RGI current

Let us first recapitulate the scale dependence in perturbation theory. At one-loop order one has

$$(A_{\mathrm{R}}^{\mathrm{stat}})_0(x) = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0,\mu a) A_0^{\mathrm{stat}}(x)\,, \tag{2.56}$$

$$Z_{\mathrm{A}}^{\mathrm{stat}}(g_0,\mu a) = 1 + g_0^2\,[B_0 - \gamma_0 \ln(a\,\mu)] + \ldots\,, \quad \gamma_0 = -\frac{1}{4\pi^2}\,. \tag{2.57}$$

In the lattice minimal subtraction scheme the $Z$–factors are polynomials in $\ln(a\mu)$ without constant part; thus $B_0 = 0$. Instead, when the renormalization scheme is defined by eq. (2.54) a one-loop computation of $f_{\mathrm{A}}^{\mathrm{stat}}$, $f_1^{\mathrm{stat}}$ yields(Kurth and Sommer, 2001) [7] $B_0 = -0.08458$. As usual there is the renormalization group equation (RGE) (remember $g_0^2 = \bar{g}^2 + \mathrm{O}(\bar{g}^4)$)

[7] As usual in perturbation theory, terms of order $(a\mu)^n\,, n \geq 1$ are dropped.

**Fig. 2.1** Lattice spacing dependence of various ratios of correlation functions for which Z-factors cancel. Different symbols correspond to different actions. Computation and figure from (Della Morte *et al.*, 2005).

$$\mu\frac{\partial}{\partial\mu}(A_{\rm R}^{\rm stat})_0 = \gamma(\bar g)(A_{\rm R}^{\rm stat})_0\,, \quad \gamma(\bar g) = -\bar g^2\left\{\gamma_0 + \bar g^2\gamma_1 + \ldots\right\}\,. \tag{2.58}$$

Combining it with the RGE for the coupling eq. (A.29) it is easily integrated to (see eq. (A.32) for the definition of the beta-function coefficients $b_i$)

$$(A_{\rm R}^{\rm stat})_0(\mu) = (A^{\rm RGI})_0\ \exp\left\{\int^{\bar g(\mu)} {\rm d}x\frac{\gamma(x)}{\beta(x)}\right\} \tag{2.59}$$

$$\equiv (A^{\rm RGI})_0\ \left[\,2b_0\bar g^2\,\right]^{\gamma_0/2b_0}\exp\left\{\int_0^{\bar g}{\rm d}x\left[\,\frac{\gamma(x)}{\beta(x)} - \frac{\gamma_0}{b_0x}\,\right]\right\} \tag{2.60}$$

where eq. (2.60) provides the definition of the lax notation for the second factor in eq. (2.59). The integration "constant" is the renormalization group invariant field. It can also be written as

$$(A^{\rm RGI})_0 = \lim_{\mu\to\infty}\left[\,2b_0\bar g^2(\mu)\,\right]^{-\gamma_0/2b_0}(A_{\rm R}^{\rm stat})_0(\mu)\ ,$$

since the last factor in eq. (2.60) converges to one as $\mu\to\infty$. Using also that $\gamma_0, b_0$ are independent of the renormalization scheme, as well as $O_S(\mu) = O_{S'}(\mu)(1+{\rm O}(\bar g^2(\mu))$ (valid for any operator $O$ and standard schemes $S, S'$), this representation also shows

that the renormalization group invariant operator $(A^{\rm RGI})_0$ is independent of scale and scheme.[8]

Let us now go beyond perturbation theory and start from a non-perturbative definition of the renormalized current, such as eq. (2.54), together with a non-perturbative definition of a renormalized coupling (Lüscher *et al.*, 1991,Lüscher *et al.*, 1992,Lüscher *et al.*, 1994). With the step scaling method discussed in more detail by Peter Weisz, one can then determine the change

$$(A_{\rm R}^{\rm stat})_0(\mu) = \sigma_{\rm A}^{\rm stat}(\bar{g}^2(2\mu))\,(A_{\rm R}^{\rm stat})_0(2\mu)\,, \quad \mu = 1/L \tag{2.61}$$

of the renormalized field $(A_{\rm R}^{\rm stat})_0(\mu)$ when the renormalization scale $\mu$ is changed by a factor of two. The so-called step scaling function $\sigma_{\rm A}^{\rm stat}$ is parameterized in terms of the running coupling $\bar{g}(\mu)$. Its argument is $\mu = 1/L$ in terms of the linear extent, $L = T$, of a Schrödinger functional .

Instead of the scale dependence of $(A_{\rm R}^{\rm stat})_0(\mu)$ we will often discuss a generic matrix element

$$\Phi(\mu) = \langle\alpha|\widehat{(A_{\rm R}^{\rm stat})_0}(\mu)|\beta\rangle \tag{2.62}$$

of the associated operator $\widehat{A_0^{\rm stat}}$ in Hilbert space.

In a non-perturbative calculation, the continuum $\sigma_{\rm A}^{\rm stat}$ is obtained through a numerical extrapolation

$$\sigma_{\rm A}^{\rm stat}(u) = \lim_{a/L\to 0} \Sigma_{\rm A}^{\rm stat}(u, a/L) \tag{2.63}$$

of the lattice step scaling functions

$$\Sigma_{\rm A}^{\rm stat}(u, a/L) = \left.\frac{Z_{\rm A}^{\rm stat}(g_0, a/2L)}{Z_{\rm A}^{\rm stat}(g_0, a/L)}\right|_{\bar{g}^2(1/L)=u} \tag{2.64}$$

obtained directly from simulations. Here, $\bar{g}^2(1/L)$ is kept fixed to remain at constant $L$ while $L/a$ is varied in the continuum extrapolation.

The $\mu$-dependence of $\Phi$ can then be constructed iteratively via

$$u_0 = \bar{g}^2(1/L_0)\,, \quad L_n = 2^n\,L_0\,, \;: \Phi(1/L_{n+1}) = \sigma_{\rm A}^{\rm stat}(u_n)\,\Phi(1/L_n)$$
$$u_{n+1} = \sigma(u_n)\,,$$

where the step scaling function $\sigma$ of the running coupling enters. The length scale $L_0$ is chosen deep in the perturbative domain, typically $L_0 \approx 1/100\,\mathrm{GeV}$ and therefore the $\mu$-dependence can be completed perturbatively to infinite $\mu$, i.e. to the RGI using eq. (2.60).

For $N_{\rm f} = 2$ the analysis has been done for $\mu \approx 300\,\mathrm{MeV}...80\,\mathrm{GeV}$. After it was verified that the steps at smallest $L$ ($L \le L_2$) are accurately described by perturbation theory (see Fig. 2.2), the two-loop anomalous dimension was used in eq. (2.60) with

[8] Of course, a trivial definition dependence due to the choice of pre-factors in eq. (2.60) is present. Unfortunately there is no uniform choice for those in the literature.

**Fig. 2.2** Relation $\Phi(\mu)/\Phi_{\rm RGI}$ between RGI and matrix element at finite $\mu$ in a Schrödinger functional scheme and for $N_{\rm f} = 2$. The $\Lambda$-parameter in the SF-scheme is around 100 MeV. Everything was computed non-perturbatively from continuum extrapolated step scaling functions (Della Morte *et al.*, 2007*b*).

$\mu = 1/L_p$ to connect to the RGI current. The result (Kurth and Sommer, 2001, Heitger *et al.*, 2003, Della Morte *et al.*, 2007*b*) is conveniently written as

$$Z^{\rm stat}_{\rm A,RGI}(g_0) = \frac{\Phi_{\rm RGI}}{\Phi(\mu)} \times Z^{\rm stat}_{\rm A}(g_0, a\mu)\Big|_{\mu=1/(2L_{\rm max})}, \tag{2.65}$$

where only the second factor depends on the lattice action, and $\bar{g}^2(1/L_{\rm max}) = u_{\rm max}$ is a convenient value covered by the non-perturbative results for the above recursion.

We show the result for the first factor in Fig. 2.2 for a series of $\mu$ with $N_{\rm f} = 2$ dynamical quarks. The different points in the graph correspond to different $n$ in the recursion. Note that the two-loop running becomes accurate only at rather small $L$. There is an about 5% difference in $\frac{\Phi_{\rm RGI}}{\Phi(\mu)}$ between a two-loop result and the non-perturbative one at the smallest $\mu$.

The details of this calculation and strategy is not that important for the following. We have mainly discussed it since
– first the RGI matrix elements play a prominent role in HQET in static approximation and it is relevant to understand that they can be obtained completely non-perturbatively and
– second we also want to later emphasize the difference between the here used – by now more classic – renormalization of the static theory and the strategy discussed in Sect. 5.3.

Let us further note that the strategy above has been extended to four-fermion operators relevant for $B - \bar{B}$ oscillations in (Palombi *et al.*, 2006, Dimopoulos *et al.*,

**Fig. 2.3** Relation $\hat{c} = \Phi_{\rm RGI}/\Phi(\mu)$ of the RGI matrix element $\Phi_{\rm RGI}$ and the matrix element at finite $\mu$ of the "$VA+AV$" four–fermion operator in a Schrödinger functional scheme. It was computed non-perturbatively from continuum extrapolated step scaling functions (Dimopoulos *et al.*, 2008).

2008). Since more than one operator is involved and twisted mass QCD is used in order to avoid mixing with operators of wrong chirality, the strategy and computation are somewhat more involved. In particular two different four-fermion operators contribute at leading order in $1/m$. As an example, we just show in Fig. 2.3 the result for the four–fermion operator which dominates in the physical process.

## 2.9 Eigen-states of the Hamiltonian

The eigenstates of the static Hamiltonian can be diagonalized simultaneously with the local heavy flavor number operator (remember $Q_{\rm h}(x) = \overline{\psi}_{\rm h}(x)\psi_{\rm h}(x)$ ). We consider a finite volume with periodic boundary conditions. Since the theory is translation invariant, there is a $k \times L^3/a^3$ – fold degeneracy of states with a single heavy quark [9], where $k$ arises from degeneracies on top of the translation invariance discussed here. For the lowest energy level one can choose a basis of eigenstates of the Hamiltonian as

$$|\tilde{B}(\mathbf{x})\rangle\,, \quad \langle\tilde{B}(\mathbf{x})|\tilde{B}(\mathbf{y})\rangle = 2\delta(\mathbf{x}-\mathbf{y})\,, \quad \hat{Q}_{\rm h}(\mathbf{y})|\tilde{B}(\mathbf{x})\rangle = \delta(\mathbf{x}-\mathbf{y}) \tag{2.66}$$

or their Fourier transformed

$$|B(\mathbf{p})\rangle = a^3\sum_{\mathbf{x}} {\rm e}^{-i\mathbf{p}\mathbf{x}}|\tilde{B}(\mathbf{x})\rangle\,, \quad \langle\tilde{B}(\mathbf{p}')|\tilde{B}(\mathbf{p})\rangle = 2(2\pi)^3\delta(\mathbf{p}-\mathbf{p}') \tag{2.67}$$

$$\delta(\mathbf{p}-\mathbf{p}') = (L/(2\pi))^3\prod_i \delta_{l_i l_i'}\,, \quad k_i = \frac{2\pi l_i}{L}\,, l_i \in \mathbb{Z}. \tag{2.68}$$

(Here we set $\theta = 0$.) We usually work with the zero momentum eigenstate, denoted for short by $|B\rangle = |B(\mathbf{p}=0)\rangle$, as this is related to an eigenstate of the finite mass QCD Hamiltonian, which in finite volume has normalization $\langle B|B\rangle = 2L^3$.

[9] In certain types of quark smearing, this has to be properly taken into account(Christ *et al.*, 2007).

# 3

# Mass dependence at leading order in 1/m: Matching

We now discuss the "matching" of HQET to QCD using the example of a simple correlation function. As mentioned before, the issue is to fix the finite parts of renormalization constants such that the effective theory describes the underlying theory QCD. Throughout this section we remain in static approximation. Matching including $1/m$ terms will be discussed in the next section.

## 3.1 A correlation function in QCD

We start from a simple QCD correlation function, which we write down in the lattice regularization,

$$C^{\rm QCD}_{\rm AA,R}(x_0) = Z_{\rm A}^2\, a^3 \sum_{\mathbf{x}} \Big\langle A_0(x) A_0^\dagger(0) \Big\rangle_{\rm QCD} \tag{3.1}$$

with the bare heavy-light axial current in QCD, $A_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi_{\rm b}$, and $A_\mu^\dagger = \overline{\psi}_{\rm b}\gamma_\mu\gamma_5\psi$. The current is formed with the relativistic b-quark field $\psi_{\rm b}$. In QCD, the renormalization factor, $Z_{\rm A}(g_0)$, is fixed by chiral Ward identities (Bochicchio *et al.*, 1985, Lüscher *et al.*, 1997). It therefore does not depend on a renormalization scale.

One reson to consider this correlation function is that at large time the B-meson state dominates its spectral representation via

$$\begin{aligned} C^{\rm QCD}_{\rm AA,R}(x_0) &= Z_{\rm A}^2 a^3 \sum_{\mathbf{x}} \langle 0|A_0^\dagger(x)|B\rangle \frac{1}{2L^3} \langle B|A_0(0)|0\rangle {\rm e}^{-x_0 m_{\rm B}} \left[1 + {\rm O}({\rm e}^{-x_0\Delta})\right] \\ &= Z_{\rm A}^2 \tfrac{1}{2} \langle 0|A_0^\dagger(0)|B\rangle \langle B|A_0(0)|0\rangle {\rm e}^{-x_0 m_{\rm B}} \left[1 + {\rm O}({\rm e}^{-x_0\Delta})\right] \end{aligned} \tag{3.2}$$

and the B-meson mass and its decay constant can be obtained from[1]

$$\Gamma^{\rm QCD}_{\rm AA}(x_0) = -\widetilde{\partial}_0 \, \ln(C^{\rm QCD}_{\rm AA}(x_0)) \, = m_{\rm B} + {\rm O}({\rm e}^{-x_0\Delta}) \tag{3.3}$$

$$\begin{aligned} \left[\Phi^{\rm QCD}\right]^2 &\equiv f_{\rm B}^2\, m_{\rm B} \\ &= \left|\langle B|Z_{\rm A}A_0|0\rangle\right|^2 = 2 \lim_{x_0\to\infty} \exp(x_0\, \Gamma^{\rm QCD}_{\rm AA}(x_0)) C^{\rm QCD}_{\rm AA}(x_0)\,. \end{aligned} \tag{3.4}$$

Note that we use the normalization eq. (1.52) (in finite volume $\langle B|B\rangle = 2L^3$) for the zero momentum state $|B\rangle$. The gap $\Delta$ is the energy difference between the second

[1] It is technically of advantage to consider so-called smeared-smeared and local-smeared correlation functions, but this is irrelevant in the present discussion.

energy level and the first energy level in the zero momentum (flavored) sector of the Hilbert space of the finite volume lattice theory.

## 3.2 The correlation function in static approximation

In the static approximation we replace $Z_{\rm A} A_0 \to Z_{\rm A}^{\rm stat}(g_0, \mu a) A_0^{\rm stat}$ and define

$$C_{\rm AA,R}^{\rm stat}(x_0) = (Z_{\rm A}^{\rm stat})^2\, C_{\rm AA}^{\rm stat}(x_0) = (Z_{\rm A}^{\rm stat})^2\, a^3 \sum_{\mathbf{x}} \Big\langle A_0^{\rm stat}(x) (A_0^{\rm stat})^\dagger(0) \Big\rangle_{\rm stat} \tag{3.5}$$

$$\Gamma_{\rm AA}^{\rm stat}(x_0) = -\widetilde{\partial}_0\, \ln(C_{\rm AA}^{\rm stat}(x_0))\,, \tag{3.6}$$

$$\begin{aligned} \big[\Phi(\mu)\big]^2 &\equiv \big|\langle B| Z_{\rm A}^{\rm stat} A_0^{\rm stat} |0\rangle_{\rm stat}\big|^2 \\ &= 2 \lim_{x_0\to\infty} \exp(x_0\, \Gamma_{\rm AA}^{\rm stat}(x_0)) (Z_{\rm A}^{\rm stat})^2\, C_{\rm AA}^{\rm stat}(x_0)\,. \end{aligned} \tag{3.7}$$

The $\mu$-dependence of $\Phi$ results from the renormalization of the current in the effective theory,

$$Z_{\rm A}^{\rm stat}(g_0, \mu a) = 1 + g_0^2\, [B_0 - \gamma_0 \ln(a\,\mu)] + {\rm O}(g_0^4)\,. \tag{3.8}$$

Different renormalization schemes have different constants $B_0$. Alternatively one uses the renormalization group invariant operator $(A^{\rm RGI})_0$. We come to that shortly.

## 3.3 Matching

The correlation function $C_{\rm AA}^{\rm QCD}$ and the matrix element $\Phi^{\rm QCD}$, eq. (3.4), are independent of any renormalization scale, due to the chiral symmetry of QCD in the massless limit. But of course they depend on the mass of the b-quark.

In the effective theory we first renormalize in an arbitrary scheme, which we do not need to specify for the following, resulting in a scale-dependent $\Phi(\mu)$. The two quantities are then related through the matching equation (without explicit superscripts "QCD" we refer to HQET quantities, here static),

$$\Phi^{\rm QCD}(m) = \widetilde{C}_{\rm match}(m, \mu) \times \Phi(\mu) + {\rm O}(1/m)\,. \tag{3.9}$$

Somewhat symbolically the same equation could be written for the current instead of its matrix element; we write "symbolically" since the two currents belong to theories with different field contents. However, thinking in terms of the currents, it is clear that eq. (3.9) can be thought of as a change of renormalization scheme in the effective theory, where the new renormalization scale is $m = m_{\rm b}$ and the finite part is exactly fixed by eq. (3.9). In fact, since at tree level we have constructed the effective theory such that $\Phi = \Phi^{\rm QCD}$, the tree-level value for $\widetilde{C}_{\rm match}$ is one and we have a perturbative expansion

$$\widetilde{C}_{\rm match}(m, \mu) = 1 + c_1(m/\mu)\bar{g}^2(\mu) + \ldots \tag{3.10}$$

The finite renormalization factor $\widetilde{C}_{\rm match}$ may be determined such that eq. (3.9) holds for some particular matrix element of the current and will then be valid *for all matrix elements* or correlation functions. Some aspects of the above equation still need explanation. The $\mu$-dependence is not present on the left-hand-side and this should be made explicit also on the right-hand-side; further one may wonder which definition of the quark mass and coupling constant one is to choose.

### 3.3.1 One-loop

Before coming to these issues, it is illustrative to write down explicitly what eq. (3.9) looks like at 1-loop order. Ignore for now how we renormalized the current in eq. (2.54) and use instead lattice minimal subtraction,

$$Z_{\rm A}^{\rm stat}(g_0, \mu a) = 1 - \gamma_0 \ln(a\,\mu)\, g_0^2 + \dots\,, \tag{3.11}$$

in the static theory.

Instead of the decay constant, use as an observable a perturbatively accessible quantity. We take[2]

$$\Phi^{\rm QCD} = Y_{\rm R}^{\rm QCD}(\theta, m_{\rm R}, L) \equiv \lim_{a\to 0} Z_{\rm A}(g_0)\, \frac{f_{\rm A}(L/2, \theta, m_{\rm R})}{\sqrt{f_1(\theta, m_{\rm R})}}\,, \tag{3.12}$$

where for our one-loop discussion we do not need to specify the normalization condition for the renormalized heavy quark mass $\overline{m} = m_{\rm R}$ and coupling $\bar{g} = g_{\rm R}$. The one-loop expansion of these functions has been computed (Kurth and Sommer, 2002), and the result can be summarized as

$$\begin{aligned}\Phi = Y_{\rm R}^{\rm stat}(\theta, \mu, L) &= \lim_{a\to 0} Z_{\rm A}^{\rm stat}(g_0, \mu a)\, \frac{f_{\rm A}^{\rm stat}(L/2, \theta)}{\sqrt{f_1^{\rm stat}(\theta)}} \\ &= A(\theta)[1 - \gamma_0 \ln(\mu L)\, g_{\rm R}^2] + D(\theta) g_{\rm R}^2 + {\rm O}(g_{\rm R}^4)\end{aligned} \tag{3.13}$$

in static approximation and

$$\Phi^{\rm QCD} = A(\theta)[1 + (D' - \gamma_0 \ln(m_{\rm R} L)) g_{\rm R}^2] + D(\theta) g_{\rm R}^2 + {\rm O}(1/(m_{\rm R} L)) + {\rm O}(g_{\rm R}^4)$$

in QCD. From these expressions we can read off

$$c_1(m_{\rm R}/\mu) = \gamma_0\, \ln(\mu/m_{\rm R}) + D'\,. \tag{3.14}$$

Furthermore, the fact that the same functions $A(\theta)$, $D(\theta)$ appear in the static theory and in QCD is a (partial) confirmation that the static approximation is the effective theory for QCD. In particular the logarithmic L-dependence in QCD matches the one in the static theory. With eq. (3.14), the matching of QCD and static theory holds for all $\theta$, and also for other matrix elements of $A_0$.

### 3.3.2 Renormalization group invariants

Having seen how QCD and effective theory match at one-loop order, we now proceed to a general discussion of eq. (3.9), beyond one-loop. Obviously, the $\mu$-dependence in eq. (3.9) is artificial, since we have a scale-independent quantity in QCD. Only the

[2] The correlation functions $f_{\rm A}$, $f_1$ are the relativistic versions of $f_{\rm A}^{\rm stat}$, $f_1^{\rm stat}$.

mass-dependence is for real. We may then choose any value for $\mu$. For convenience we set all renormalization scales equal to the mass itself [3],

$$\mu = m_\star = \overline{m}(m_\star)\,,\ g_\star = \bar{g}(m_\star)\,, \tag{3.15}$$

where $\overline{m}(\mu), \bar{g}(\mu)$ are running mass and coupling in an unspecified massless renormalization scheme.[4] This simplifies the matching function to

$$\widetilde{C}_{\rm match}(m_\star, m_\star) = C_{\rm match}(g_\star) = 1 + c_1(1)\, g_\star^2 + \dots\,. \tag{3.16}$$

Further we want to eliminate the dependence on the renormalization scheme for $\overline{m}, \bar{g}, (A_{\rm R}^{\rm stat})_0$. As a first step we change from $\Phi(\mu)$ to the RGI matrix element

$$\Phi_{\rm RGI} = \exp\left\{-\int^{\bar{g}(\mu)} {\rm d}x\, \frac{\gamma(x)}{\beta(x)}\right\}\, \Phi(\mu)\,, \tag{3.17}$$

and arrive at the form

$$\Phi^{\rm QCD} = C_{\rm match}(g_\star) \times \Phi(\mu) = C_{\rm match}(g_\star) \exp\left\{\int^{g_\star} {\rm d}x \frac{\gamma(x)}{\beta(x)}\right\} \Phi_{\rm RGI} \tag{3.18}$$

$$\equiv \exp\left\{\int^{g_\star} {\rm d}x\, \frac{\gamma_{\rm match}(x)}{\beta(x)}\right\} \Phi_{\rm RGI}\,. \tag{3.19}$$

Everywhere terms of order $1/m$ are dropped, since we are working to static order. Eq. (3.19) defines $\gamma_{\rm match}$, which describes the physical mass dependence via,

$$\frac{m_\star}{\Phi^{\rm QCD}} \frac{\partial \Phi^{\rm QCD}}{\partial m_\star} = \gamma_{\rm match}(g_\star)\,, \tag{3.20}$$

but it still depends on the chosen renormalization scheme through the choice of $\overline{m}$ (the scheme, not the scale). We eliminate also this scheme dependence by switching to the RGI mass, $M$, and the $\Lambda$-parameter,

$$\frac{\Lambda}{\mu} = \exp\left\{-\int^{\bar{g}(\mu)} {\rm d}x\, \frac{1}{\beta(x)}\right\}\,, \tag{3.21}$$

$$\frac{M}{\overline{m}(\mu)} = \exp\left\{-\int^{\bar{g}(\mu)} {\rm d}x\, \frac{\tau(x)}{\beta(x)}\right\}\,. \tag{3.22}$$

Exact expressions, defining the constant parts in these equation, are given in the appendix.

[3] Note that $m_\star$ is implicitly defined through $m_\star = \overline{m}(m_\star)$.

[4] In a massless renormalization scheme, the renormalization factors do not depend on the masses. Consequently the renormalization group functions do not depend on the masses.

Just based on dimensional analysis, we expect a relation

$$\Phi^{\rm QCD} = C_{\rm PS}(M/\Lambda) \times \Phi_{\rm RGI} \tag{3.23}$$

to hold. Indeed, remembering eq. (3.15), $\mu = m_\star = \overline{m}$, we can combine eq. (3.21) and eq. (3.22) to

$$\frac{\Lambda}{M} = \exp\left\{ -\int^{g_\star(M/\Lambda)} {\rm d}x\, \frac{1-\tau(x)}{\beta(x)} \right\}, \tag{3.24}$$

from which $g_\star$ can be determined for any value of $M/\Lambda$; we write $g_\star = g_\star(M/\Lambda)$. It follows that

$$M \frac{\partial g_\star(m_\star(M/\Lambda))}{\partial M} = \frac{\beta(g_\star)}{1-\tau(g_\star)}, \tag{3.25}$$

and the matching function is

$$C_{\rm PS}(M/\Lambda) = \exp\left\{ \int^{g_\star(M/\Lambda)} {\rm d}x\, \frac{\gamma_{\rm match}(x)}{\beta(x)} \right\}. \tag{3.26}$$

We note that the dependence on $M$ is described by a function[5]

$$\frac{M}{\Phi}\frac{\partial \Phi}{\partial M}\bigg|_\Lambda = \frac{M}{C_{\rm PS}}\frac{\partial C_{\rm PS}}{\partial M}\bigg|_\Lambda = \frac{\gamma_{\rm match}(g_\star)}{1-\tau(g_\star)}, \quad g_\star = g_\star(M/\Lambda). \tag{3.31}$$

With

$$\gamma_{\rm match}(g_\star) \overset{g_\star\to 0}{\sim} -\gamma_0 g_\star^2 - \gamma_1^{\rm match} g_\star^4 + \ldots, \qquad \beta(\bar g) \overset{\bar g\to 0}{\sim} -b_0 \bar g^3 + \ldots \tag{3.32}$$

we can now give the leading large mass behavior

$$C_{\rm PS} \overset{M\to\infty}{\sim} (2b_0 g_\star^2)^{-\gamma_0/2b_0} \sim [\ln(M/\Lambda)]^{\gamma_0/2b_0}. \tag{3.33}$$

Functions such as $C_{\rm PS}$ convert from the static RGI matrix elements to the QCD matrix element; we call them conversion functions.

[5] This is seen from

$$\frac{M}{\Phi}\frac{\partial \Phi}{\partial M} = \underbrace{\frac{M}{m_\star}\frac{\partial m_\star}{\partial M}}_{\frac{1}{1-\tau(g_\star)}} \underbrace{\frac{m_\star}{\Phi}\frac{\partial \Phi}{\partial m_\star}}_{\gamma_{\rm match}(g_\star)} = \frac{\gamma_{\rm match}(g_\star)}{1-\tau(g_\star)}, \tag{3.27}$$

where we used

$$m_\star = M\, \exp\left\{ \int^{g_\star} {\rm d}x \frac{\tau(x)}{\beta(x)} \right\} \tag{3.28}$$

$$\frac{\partial m_\star}{\partial M} = \frac{m_\star}{M} + \frac{\tau(g_\star)}{\beta(g_\star)}\frac{\partial g_\star}{\partial M} m_\star = \frac{m_\star}{M} + \frac{\tau(g_\star)}{\beta(g_\star)}\beta(g_\star)\frac{\partial m_\star}{\partial M}, \tag{3.29}$$

which shows that

$$\frac{M}{m_\star}\frac{\partial m_\star}{\partial M} = \frac{1}{1-\tau(g_\star)}. \tag{3.30}$$

**Fig. 3.1** $C_{\rm PS}$ estimated in perturbation theory. For B-physics we have $\Lambda_{\overline{\rm MS}}/M_{\rm b} \approx 0.04$. Figure from (Heitger *et al.*, 2004).

An interesting application is the asymptotics of the decay constant of a heavy-light pseudo-scalar (e.g. B):[6]

$$F_{\rm PS} \stackrel{M\to\infty}{\sim} \frac{[\ln(M/\Lambda)]^{\gamma_0/2b_0}}{\sqrt{m_{\rm PS}}}\Phi_{\rm RGI} \times [1+{\rm O}([\ln(M/\Lambda)]^{-1}]\,. \tag{3.34}$$

At leading order in $1/m$ the conversion function $C_{\rm PS}$ contains the full (logarithmic) mass-dependence. The non-perturbative effective theory matrix elements, $\Phi_{\rm RGI}$, are mass independent numbers. Conversion functions such as $C_{\rm PS}$ are universal for all (low energy) matrix elements of their associated operator. For example

$$C^{\rm QCD}_{\rm AA,R}(x_0) \stackrel{x_0\gg 1/m}{\sim} [C_{\rm PS}(\tfrac{M}{\Lambda_{\overline{\rm MS}}})\, Z^{\rm stat}_{\rm A,RGI}]^2 \underbrace{\langle A_0^{\rm stat}(x)^\dagger A_0^{\rm stat}(0)\rangle}_{C^{\rm stat}_{\rm AA}(x_0)\ \text{(bare)}} + {\rm O}(\tfrac{1}{m})\,, \tag{3.35}$$

is a straight forward generalization of eq. (3.9).

Analogous expressions for the conversion functions are valid for the time component of the axial current replaced by other composite fields, for example the space components of the vector current. Based on the work of (Broadhurst and Grozin, 1991, Shifman and Voloshin, 1987, Politzer and Wise, 1988) and recent efforts their perturbative expansion is known including the 3-loop anomalous dimension $\gamma_{\rm match}$ obtained from the 3-loop anomalous dimension $\gamma$ (Chetyrkin and Grozin, 2003) in the $\overline{\rm MS}$-scheme and the 2-loop matching function $C_{\rm match}$ (Ji and Musolf, 1991, Broadhurst and Grozin, 1995, Gimenez, 1992).

Figure 3.1 seems to indicate that the remaining ${\rm O}(\bar{g}^6(m_{\rm b}))$ errors in $C_{\rm PS}$ are relatively small. However, as discussed in more detail in App. A.2, such a conclusion is premature. By now *ratios* of conversion functions for different currents are known to even one more order in perturbation theory (Bekavac *et al.*, 2010). We show an example in the first column of Fig. 3.2, where the x-axis is approximately proportional to $g^2_\star(M/\Lambda)$ and for B-physics one needs $1/\ln(\Lambda_{\overline{\rm MS}}/M_{\rm b}) \approx 0.3$. For a quark mass around

[6] Note the slow, logarithmic, decrease of the corrections in eq. (3.34). We will see below, in the discussion of Figs. 3.1,3.2, that the perturbative evaluation of $C_{\rm PS}(M_{\rm b}/\Lambda)$ is somewhat problematic.

**Fig. 3.2** The ratio $C_{\rm PS}/C_{\rm V}$, evaluated in the first column as described here. In columns two and three the expansion in $g_\star$ is generalized to an expansion in $\bar{g}(m_\star/s)$, see App. A.2.2. The last column contains the conventionally used $\hat{C}^{\rm PS}_{\rm match}(m_{\rm Q},m_{\rm Q},m_{\rm Q})/\hat{C}^{\rm V}_{\rm match}(m_{\rm Q},m_{\rm Q},m_{\rm Q})$, see App. A.2. For B-physics we have $\Lambda_{\overline{\rm MS}}/M_{\rm b}\approx 0.04$ and $1/\ln(\Lambda_{\overline{\rm MS}}/M_{\rm b})\approx 0.3$. The loop order changes from one-loop (long-dashes) up to 4-loop (full line) anomalous dimension.

the mass of the b-quark and lower, the higher order contributions in perturbation theory do not decrease significantly and perturbation theory is not trustworthy. It seems impossible to estimate a realistic error of the perturbative expansion. Only for somewhat higher masses the expansion looks reasonable.

Moreover, using the freedom to choose the scale $\mu$ in eq. (3.10), the $l$'th order coefficients (as far as they are known) can be brought down in magnitude below about $(4\pi)^{-l}$, which means there is a fast decrease of terms in the perturbative series once $\alpha(\mu)\lesssim 1/3$. This is shown in columns two and three of the figure. Unfortunately, the required scale $\mu$ is around a factor 4 or more below the mass of the quark. For the b-quark, $\alpha$ is rather large at that scale and the series is again unreliable. Only for even larger masses, say $m_\star > 15\,\mathrm{GeV}$, the asymptotic convergence of the series is noticeably better after adjusting the scale. More details are found in App. A.2. Unfortunately we see no way out of the conclusion that for B-physics with a trustworthy error budget aiming at the few percent level, one needs a non-perturbative matching, *even in the static approximation*.

We return to the full set of heavy-light flavor currents of Sect. 2.5. The bare fields satisfy the symmetry relations eq. (2.33). The same is then true for the RGI fields in static approximation. It follows that *in static approximation* the effective currents are given by

$$A_0^{\rm HQET} = C_{\rm PS}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\, Z^{\rm stat}_{\rm A,RGI}(g_0)\, A_0^{\rm stat}\,, \tag{3.36}$$

$$V_k^{\rm HQET} = C_{\rm V}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\, Z^{\rm stat}_{\rm A,RGI}(g_0)\; V_k^{\rm stat}\,, \tag{3.37}$$

$$V_0^{\rm HQET} = C_{\rm PS}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\, Z^{\rm stat}_{\rm A,RGI}(g_0)\, Z^{\rm stat}_{\rm V/A}(g_0)\, V_0^{\rm stat}\,, \tag{3.38}$$

$$A_k^{\rm HQET} = C_{\rm V}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\, Z^{\rm stat}_{\rm A,RGI}(g_0) Z^{\rm stat}_{\rm V/A}(g_0)\, A_k^{\rm stat}\,. \tag{3.39}$$

The factor $Z^{\rm stat}_{\rm A,RGI}(g_0)$ is known as discussed in the previous lecture. Note that $Z^{\rm stat}_{\rm A,RGI}(g_0)$ is common to all (components of the) currents. Due to the HQET symmetries, there is one single anomalous dimension. A dependence on the different fields comes in only through matching, i.e. through the QCD matrix elements. In the above equations, chiral symmetry (of the continuum theory), eq. (2.30), has been used to relate conversion functions of axial and vector currents.

**Exercise 3.1** Pseudo-scalar and Scalar densities

Start from the PCAC, PCVC relations in QCD

$$\partial_\mu (A_{\rm R})_\mu = (\overline{m}_{\rm b}(\mu) + \overline{m}_{\rm l}(\mu)) P_{\rm R}(\mu)\,, \tag{3.40}$$

$$\partial_\mu (V_{\rm R})_\mu = (\overline{m}_{\rm b}(\mu) - \overline{m}_{\rm l}(\mu)) S_{\rm R}(\mu)\,. \tag{3.41}$$

Replace all quantities by their RGI's. Take the matrix elements between vacuum and a suitable B-meson state to show that

$$P^{\rm HQET} = -C_{\rm PS}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\, \frac{m_{\rm B}}{M_{\rm b}}\, Z^{\rm stat}_{\rm A,RGI}(g_0) A_0^{\rm stat}\,, \tag{3.42}$$

$$S^{\rm HQET} = C_{\rm V}(M_{\rm b}/\Lambda_{\overline{\rm MS}})\, \frac{m_{\rm B}}{M_{\rm b}}\, Z^{\rm stat}_{\rm V/A}(g_0)\, Z^{\rm stat}_{\rm A,RGI}(g_0)\, V_0^{\rm stat}\,, \tag{3.43}$$

is valid up to terms of order $1/m$. What happens if you choose a different matrix element?

### 3.3.3 Applications

As an application, we can now modify the scaling law for the decay constant to include renormalization and matching effects

$$\frac{f_{\rm B}\sqrt{m_{\rm B}}}{C_{\rm PS}(M_{\rm b}/\Lambda_{\overline{\rm MS}})} = \Phi_{\rm RGI} + {\rm O}(1/m) \tag{3.44}$$

$$\frac{f_{\rm B}}{f_{\rm D}} \approx \frac{\sqrt{m_{\rm D}}\, C_{\rm PS}(M_{\rm b}/\Lambda_{\overline{\rm MS}})}{\sqrt{m_{\rm B}}\, C_{\rm PS}(M_{\rm c}/\Lambda_{\overline{\rm MS}})}\,, \tag{3.45}$$

where the latter equation is maybe stretching the applicability domain of HQET.

Despite the discussion above, let us assume that the conversion functions $C$ are known with reasonably small errors from perturbation theory. In this case, the knowledge of the leading term in expansions such as eq. (3.44) is very useful to constrain the large mass behavior of QCD observables, computed on the lattice with unphysical quark masses $m_{\rm h} < m_{\rm b}$, typically $m_{\rm h} \approx m_{\rm charm}$. (Such a calculation is done with a relativistic (Wilson, tmQCD, ...) formulation, extrapolating $am_{\rm h} \to 0$ at fixed $m_{\rm h}$.) As illustrated in Fig. 3.3, one can then, with a reasonable smoothness assumption, interpolate to the physical point.

**Fig. 3.3** Example of an interpolation between a static result and results with $m_{\rm h} < m_{\rm b}$. The function $C_{\rm PS}$ is estimated at three-loop order. Continuum extrapolations are done before the interpolation (Blossier *et al.*, 2010*c*). The point at $1/r_0 m_{\rm PS} = 0$ is given by $r_0^{3/2}\Phi_{\rm RGI}$. This quenched computation is done for validating and demonstrating the applicability of HQET.

Given the unclear precision of the perturbative predictions, the above interpolation method has to be taken with care. The inherent perturbative error remains to be estimated.

The relation between the RGI fields and the bare fields has also been obtained for the two parity violating $\Delta B = 2$ four fermion operators (Palombi *et al.*, 2006,Palombi *et al.*, 2007) for $N_{\rm f} = 0$ and $N_{\rm f} = 2$ (Dimopoulos *et al.*, 2008). Their matrix elements, evaluated in twisted mass QCD will give the standard model B-parameter for B-$\overline{\rm B}$ mixing.

We now turn to the natural question whether one can directly compute the $1/m$ corrections in HQET, which will lead us again to the necessity of performing a non-perturbative matching between HQET and QCD.

**Exercise 3.2** Anomalous dimension $\gamma_{\rm match}$

Show that

$$\gamma_{\rm match} = -\gamma_0 g_\star^2 - [\gamma_1 + 2b_0 c_1(1)] g_\star^4 + \dots . \tag{3.46}$$

where $c_1$ is the 1-loop matching coefficient in the same scheme as $\gamma_1$.

# 4

# Renormalization and matching at order 1/m

## 4.1 Including $1/m$ corrections

We here work directly in lattice regularization. The continuum formulae are completely analogous. The expressions for $\mathcal{O}_{\rm kin}, \mathcal{O}_{\rm spin}$ are discretized in a straight forward way,

$$D_k D_k \to \nabla^*_k \nabla_k \,, \quad F_{kl} \to \widehat{F}_{kl} \tag{4.1}$$

with the latter given by the clover leaf representation, defined e.g. in (Lüscher *et al.*, 1996). Of course other discretizations of these composite fields are possible.

Apart from the terms in the classical Lagrangian, renormalization can in principle introduce new local fields compatible with the symmetries (but not necessarily the heavy quark symmetries which are broken by $\mathcal{O}_{\rm spin}, \mathcal{O}_{\rm kin}$) and with dimension $d_{\rm op} \leq 5$. Also the field equations can be used to eliminate terms. With these rules one finds that no new terms are needed and it suffices to treat the coefficients of $\mathcal{O}_{\rm spin}, \mathcal{O}_{\rm kin}$ as free parameters which depend on the bare coupling of the theory and on $m$.

The $1/m$ Lagrangian then reads

$$\mathscr{L}^{(1)}_{\rm h}(x) = -(\omega_{\rm kin}\, \mathcal{O}_{\rm kin}(x) + \omega_{\rm spin}\, \mathcal{O}_{\rm spin}(x)) \,. \tag{4.2}$$

Since these terms are composite fields of dimension five, the theory defined with a path integral weight ($\mathscr{L}_{\rm light}$ collects all contributions of QCD with the heavy quark(s) dropped)

$$W_{\rm NRQCD} \propto \exp(-a^4 \sum_x [\mathscr{L}_{\rm light}(x) + \mathscr{L}^{\rm stat}_{\rm h}(x) + \mathscr{L}^{(1)}_{\rm h}(x)]) \tag{4.3}$$

is *not* renormalizable. In perturbation theory, new divergences will occur at each order in the loop expansion, which necessitate to introduce new counter-terms. The continuum limit of the lattice theory will then not exist(Thacker and Lepage, 1991). Since the effective theory is "only" supposed to reproduce the $1/m$ expansion of the observables order by order in $1/m$, we instead expand the weight $W$ in $1/m$, counting $\omega_{\rm kin} = {\rm O}(1/m) = \omega_{\rm spin}$,

$$W_{\rm NRQCD} \to W_{\rm HQET} \equiv \exp(-a^4 \sum_x [\mathscr{L}_{\rm light}(x) + \mathscr{L}^{\rm stat}_{\rm h}(x)]) \left\{ 1 - a^4 \sum_x \mathscr{L}^{(1)}_{\rm h}(x) \right\} .$$

This rule is part of the definition of HQET, just like the same step is part of Symanzik's effective theory discussed by Peter Weisz.

Let us remark here on the difference to chiral perturbation theory. In chiral perturbation theory one computes the asymptotic expansion in powers of $p^2$. Each term in the expansion requires a finite number of counter terms, since there are only a finite number of (pion) loops. The theory is thus renormalizable order by order in the expansion. In NRQCD and HQET one expands in $1/m$. At each order of the expansion an arbitrary number of loops remain, coming from the gluons and light quarks. In fact, we are even interested in more than an arbitrary number of loops: in non-perturbative results in $\alpha$.

NRQCD can then only be formulated with a cutoff and results depend on how the cutoff is introduced and on it's value. On the lattice, the cutoff is identified with the one present for the other fields, $\Lambda_{\rm cut} \sim 1/a$. Instead of taking a continuum limit, one then relies on physics results not depending on the lattice spacing (the cutoff) within a window (Thacker and Lepage, 1991)

$$1/m \ll a \ll \Lambda_{\rm QCD}\,. \qquad \text{[in NRQCD]} \tag{4.4}$$

In HQET the discussion is rather simple, since the static theory is (believed to be) renormalizable; we will come to the renormalization of the insertion of $\mathscr{L}_{\rm h}^{(1)}$ shortly.

Up to and including $\mathrm{O}(1/m)$, expectation values in HQET are therefore defined as

$$\begin{aligned}\langle\mathcal{O}\rangle &= \langle\mathcal{O}\rangle_{\rm stat} + \omega_{\rm kin}a^4\sum_x\langle\mathcal{O}\mathcal{O}_{\rm kin}(x)\rangle_{\rm stat} + \omega_{\rm spin}a^4\sum_x\langle\mathcal{O}\mathcal{O}_{\rm spin}(x)\rangle_{\rm stat}\\ &\equiv \langle\mathcal{O}\rangle_{\rm stat} + \omega_{\rm kin}\langle\mathcal{O}\rangle_{\rm kin} + \omega_{\rm spin}\langle\mathcal{O}\rangle_{\rm spin}\,,\end{aligned} \tag{4.5}$$

where the path integral average

$$\langle\mathcal{O}\rangle_{\rm stat} = \frac{1}{\mathcal{Z}}\int_{\rm fields}\mathcal{O}\exp(-a^4\sum_x[\mathscr{L}_{\rm light}(x) + \mathscr{L}_{\rm h}^{\rm stat}(x)]) \tag{4.6}$$

is taken with respect to the lowest order action. The integral extends over all fields and the normalization $\mathcal{Z}$ is fixed by $\langle 1\rangle_{\rm stat} = 1$.[1]

In order to compute matrix elements or correlation functions in the effective theory, we also need the effective composite fields. At the classical level they can again be obtained from the Fouldy-Wouthuysen rotation. In the quantum theory one adds all local fields with the proper quantum numbers and dimensions. For example the effective axial current (time component) is given by

$$A_0^{\rm HQET}(x) = Z_{\rm A}^{\rm HQET}\,[A_0^{\rm stat}(x) + \sum_{i=1}^{2} c_{\rm A}^{(i)}A_0^{(i)}(x)]\,, \tag{4.7}$$

$$A_0^{(1)}(x) = \overline{\psi}(x)\frac{1}{2}\gamma_5\gamma_i(\nabla_i^{\rm S} - \overleftarrow{\nabla}_i^{\rm S})\psi_{\rm h}(x)\,, \tag{4.8}$$

$$A_0^{(2)}(x) = -\widetilde{\partial}_i\,A_i^{\rm stat}\,, \tag{4.9}$$

where all derivatives are symmetric,

[1] A straight expansion gives e.g. $\omega_{\rm kin}a^4\sum_x\langle\mathcal{O}[\mathcal{O}_{\rm kin}(x) - \langle\mathcal{O}_{\rm kin}(x)\rangle_{\rm stat}\rangle_{\rm stat}$, but this just corresponds to an irrelevant shift of $\mathcal{O}_{\rm kin}(x)$ etc. by a constant.

$$\widetilde{\partial}_i = \tfrac{1}{2}(\partial_i + \partial_i^*)\,, \quad \overleftarrow{\nabla}_i^{\rm S} = \tfrac{1}{2}(\overleftarrow{\nabla}_i + \overleftarrow{\nabla}_i^*)\,, \quad \nabla_i^{\rm S} = \tfrac{1}{2}(\nabla_i + \nabla_i^*)\,, \tag{4.10}$$

and we recall $A_i^{\rm stat}(x) = \overline{\psi}(x)\gamma_i\gamma_5\psi_{\rm h}(x)$. One arrives at these currents, writing down all dimension four operators with the right flavor structure and transformation under spatial lattice rotations and parity. The equations of motion of the light and static quarks are used to eliminate terms but heavy quark symmetries (spin and local flavor) can't be used since they are broken at order $1/m$.[2]

For completeness let us write down the other HQET currents:

$$\begin{aligned} A_k^{\rm HQET}(x) &= Z_{\mathbf{A}}^{\rm HQET}\,[A_k^{\rm stat}(x) + \sum_{i=3}^{6} c_{\rm A}^{(i)} A_k^{(i)}(x)]\,, \\ A_k^{(3)}(x) &= \overline{\psi}(x)\frac{1}{2}\gamma_k\gamma_5\gamma_i(\nabla_i^{\rm S} - \overleftarrow{\nabla}_i^{\rm S})\psi_{\rm h}(x)\,, \quad A_k^{(4)}(x) = \overline{\psi}(x)\frac{1}{2}(\nabla_k^{\rm S} - \overleftarrow{\nabla}_k^{\rm S})\gamma_5\psi_{\rm h}(x)\,, \\ A_k^{(5)}(x) &= \widetilde{\partial}_i\,\left(\overline{\psi}(x)\gamma_k\gamma_5\gamma_i\psi_{\rm h}(x)\right)\,, \quad A_k^{(6)}(x) = \widetilde{\partial_k}\,A_0^{\rm stat}\,. \end{aligned} \tag{4.11}$$

The vector current components are just obtained by dropping $\gamma_5$ in these expressions and changing $c_{\rm A}^{(i)} \to c_{\rm V}^{(i)}$. The classical values of the coefficients are $c_{\rm A}^{(1)} = c_{\rm A}^{(2)} = c_{\rm A}^{(3)} = c_{\rm A}^{(5)} = -\frac{1}{2m}$, while $c_{\rm A}^{(4)} = c_{\rm A}^{(6)} = 0$. We note that with periodic boundary conditions in space we have

$$a^3\sum_{\mathbf{x}} A_0^{(1)}(x) = a^3\sum_{\mathbf{x}} \overline{\psi}(x)\overleftarrow{\nabla}_i^{\rm S}\gamma_i\gamma_5\psi_{\rm h}(x)\,, \quad a^3\sum_{\mathbf{x}} A_0^{(2)}(x) = 0\,, \tag{4.12}$$

which for instance may be used in the determination of the B decay constant.

Before entering into details of the renormalization, we show some examples how the $1/m$-expansion works.

## 4.2 $1/m$-expansion of correlation functions and matrix elements

For now we assume that the coefficients

$$\begin{aligned} {\rm O}(1)\ &:\ \delta m\,,\ Z_{\rm A}^{\rm HQET}\,, \\ {\rm O}(1/m)\ &:\ \omega_{\rm kin}\,,\ \omega_{\rm spin}\,,\ c_{\rm A}^{(1)}\,, \end{aligned} \tag{4.13}$$

are known as a function of the bare coupling $g_0$ and the quark mass $m$. Their non-perturbative determination will be discussed later.

The rules of the $1/m$-expansion are illustrated on the example of $C_{\rm AA,R}^{\rm QCD}(x_0)$, eq. (3.1). One uses eq. (4.5) and the HQET representation of the composite field eq. (4.7). Then the expectation value is expanded consistently in $1/m$, counting powers of $1/m$ as in eq. (4.13). At order $1/m$, terms proportional to $\omega_{\rm kin} \times c_{\rm A}^{(1)}$ etc. are to be dropped. As a last step, we have to take the energy shift between HQET and

[2] An operator $\frac{m_l}{m}A_0^{\rm stat}$ is included as a corresponding mass-dependence of $Z_{\rm A}^{\rm HQET}$. In practice, since $\frac{m_l}{m_{\rm b}} \lll 1$, and this term appears only at one-loop order, this dependence on the light quark mass can be neglected.

QCD into account. Therefore correlation functions with a time separation $x_0$ obtain an extra factor $\exp(-x_0\, m)$, where the scheme dependence of $m$ is compensated by a corresponding one in $\delta m$. Dropping all terms $\mathrm{O}(1/m^2)$ without further notice, one arrives at the expansion

$$\begin{aligned} C_{\mathrm{AA}}^{\mathrm{QCD}}(x_0) &= \mathrm{e}^{-mx_0}(Z_{\mathrm{A}}^{\mathrm{HQET}})^2 \Big[C_{\mathrm{AA}}^{\mathrm{stat}}(x_0) + c_{\mathrm{A}}^{(1)}\, C_{\delta\mathrm{AA}}^{\mathrm{stat}}(x_0) \\ &\qquad + \omega_{\mathrm{kin}}\, C_{\mathrm{AA}}^{\mathrm{kin}}(x_0) + \omega_{\mathrm{spin}}\, C_{\mathrm{AA}}^{\mathrm{spin}}(x_0)\Big] \qquad (4.14) \\ &\equiv \mathrm{e}^{-mx_0}(Z_{\mathrm{A}}^{\mathrm{HQET}})^2\, C_{\mathrm{AA}}^{\mathrm{stat}}(x_0)\Big[1 + c_{\mathrm{A}}^{(1)}\, R_{\delta A}^{\mathrm{stat}}(x_0) \\ &\qquad + \omega_{\mathrm{kin}}\, R_{\mathrm{AA}}^{\mathrm{kin}}(x_0) + \omega_{\mathrm{spin}}\, R_{\mathrm{AA}}^{\mathrm{spin}}(x_0)\Big] \qquad (4.15) \end{aligned}$$

with (remember the definitions in eq. (4.5))

$$\begin{aligned} C_{\delta\mathrm{AA}}^{\mathrm{stat}}(x_0) &= a^3 \sum_{\mathbf{x}} \langle A_0^{\mathrm{stat}}(x)(A_0^{(1)}(0))^\dagger\rangle_{\mathrm{stat}} \;+\; a^3 \sum_{\mathbf{x}} \langle A_0^{(1)}(x)(A_0^{\mathrm{stat}}(0))^\dagger\rangle_{\mathrm{stat}}\,, \\ C_{\mathrm{AA}}^{\mathrm{kin}}(x_0) &= a^3 \sum_{\mathbf{x}} \langle A_0^{\mathrm{stat}}(x)(A_0^{\mathrm{stat}}(0))^\dagger\rangle_{\mathrm{kin}} \\ C_{\mathrm{AA}}^{\mathrm{spin}}(x_0) &= a^3 \sum_{\mathbf{x}} \langle A_0^{\mathrm{stat}}(x)(A_0^{\mathrm{stat}}(0))^\dagger\rangle_{\mathrm{spin}}\,. \end{aligned}$$

The contribution of $A_0^{(2)}$ vanishes due to eq. (4.12). It is now a straight forward exercise to obtain the expansion of the B-meson mass[3]

$$\begin{aligned} m_{\mathrm{B}} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0 \ln C_{\mathrm{AA}}^{\mathrm{QCD}}(x_0) \qquad (4.16) \\ &= m_{\mathrm{bare}} - \lim_{x_0\to\infty} \widetilde{\partial}_0 \big[\, \ln C_{\mathrm{AA}}^{\mathrm{stat}}(x_0) + c_{\mathrm{A}}^{(1)}\, R_{\delta A}^{\mathrm{stat}}(x_0) + \\ &\qquad + \omega_{\mathrm{kin}}\, R_{\mathrm{AA}}^{\mathrm{kin}}(x_0) + \omega_{\mathrm{spin}}\, R_{\mathrm{AA}}^{\mathrm{spin}}(x_0)\,\big]_{\delta m=0} \qquad (4.17) \\ &= m_{\mathrm{bare}} + E^{\mathrm{stat}} + \omega_{\mathrm{kin}} E^{\mathrm{kin}} + \omega_{\mathrm{spin}} E^{\mathrm{spin}}\,, \qquad (4.18) \\ E^{\mathrm{stat}} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0\, \ln C_{\mathrm{AA}}^{\mathrm{stat}}(x_0)\Big|_{\delta m=0}\,, \qquad (4.19) \\ E^{\mathrm{kin}} &= -\lim_{x_0\to\infty} \widetilde{\partial}_0\, R_{\mathrm{AA}}^{\mathrm{kin}}(x_0)\,, \quad E^{\mathrm{spin}} = -\lim_{x_0\to\infty} \widetilde{\partial}_0\, R_{\mathrm{AA}}^{\mathrm{spin}}(x_0)\,. \qquad (4.20) \end{aligned}$$

Again we have made the dependence on $\delta m$ explicit through $m_{\mathrm{bare}} = m_{\mathrm{b}} + \widehat{\delta m}$ and then quantities in the theory with $\delta m = 0$ appear. Note that the ratios $R_{\mathrm{AA}}^{x}$ (and therefore $E^{\mathrm{kin}}, E^{\mathrm{spin}}$) do not depend on $\delta m$; the quantities $E^{\mathrm{kin}}, E^{\mathrm{spin}}$ have mass dimension two and we have already anticipated eq. (4.25).

The expansion for the decay constant is

[3]It follows from the simple form of the static propagator that there is no dependence on $\delta m$ except for the explicitly shown energy shift $\widehat{\delta m}$.

$$f_{\rm B}\sqrt{m_{\rm B}} = \lim_{x_0\to\infty}\left\{2\exp(m_{\rm B}x_0)\,C_{\rm AA}^{\rm QCD}(x_0)\right\}^{1/2} \tag{4.21}$$

$$\begin{aligned} &= Z_{\rm A}^{\rm HQET}\,\Phi^{\rm stat}\lim_{x_0\to\infty}\left\{1+\tfrac{1}{2}x_0\left[\omega_{\rm kin}E^{\rm kin}+\omega_{\rm spin}E^{\rm spin}\right]\right.\\ &\quad\left.+\tfrac{1}{2}c_{\rm A}^{(1)}R_{\delta A}^{\rm stat}(x_0)+\tfrac{1}{2}\omega_{\rm kin}R_{\rm AA}^{\rm kin}(x_0)+\tfrac{1}{2}\omega_{\rm spin}R_{\rm AA}^{\rm spin}(x_0)\right\}, \end{aligned} \tag{4.22}$$

$$\Phi^{\rm stat} = \lim_{x_0\to\infty}\left\{2\exp(E^{\rm stat}x_0)\,C_{\rm AA}^{\rm stat}(x_0)\right\}^{1/2}.$$

Using the transfer matrix formalism (with normalization $\langle B|B\rangle = 2L^3$), one further observes that (do it as an exercise)

$$E^{\rm kin} = -\frac{1}{2L^3}\langle B|a^3\sum_{\mathbf{z}}\mathcal{O}_{\rm kin}(0,\mathbf{z})|B\rangle_{\rm stat} = -\frac{1}{2}\langle B|\mathcal{O}_{\rm kin}(0)|B\rangle_{\rm stat} \tag{4.23}$$

$$E^{\rm spin} = -\frac{1}{2}\langle B|\mathcal{O}_{\rm spin}(0)|B\rangle_{\rm stat}\,, \tag{4.24}$$

$$0 = \lim_{x_0\to\infty}\widetilde{\partial}_0 R_{\delta A}^{\rm stat}(x_0)\,. \tag{4.25}$$

As expected, only the parameters of the action are relevant in the expansion of hadron masses.

A correct split of the terms in eq. (4.18) and eq. (4.22) into leading order and next to leading order pieces which are separately renormalized and which hence *separately have a continuum limit* requires more thought on the renormalization of the $1/m$-expansion. We turn to this now.

## 4.3 Renormalization beyond leading order

For illustration we check the self consistency of eq. (4.14). The relevant question concerns renormalization: are the "free" parameters $\delta m \ldots c_{\rm A}^{(1)}$ sufficient to absorb all divergences on the r.h.s.? We consider the term $\propto C_{\rm AA}^{\rm kin}(x_0)$ since its renormalization displays all subtleties. As a first step we rewrite $\omega_{\rm kin}\mathcal{O}_{\rm kin} = \frac{1}{2m_{\rm R}}\big(\mathcal{O}_{\rm kin}\big)_{\rm R}$ in terms of a renormalized mass and the renormalized operator

$$\big(\mathcal{O}_{\rm kin}\big)_{\rm R}(z) = Z_{\mathcal{O}_{\rm kin}}\big(\mathcal{O}_{\rm kin}(z) + \frac{c_1}{a}\,\overline{\psi}_{\rm h}(z)D_0\psi_{\rm h}(z) + \frac{c_2}{a^2}\,\overline{\psi}_{\rm h}(z)\psi_{\rm h}(z)\big)\,. \tag{4.26}$$

The latter involves a subtraction of lower dimensional ones with dimensionless coefficients $c_i(g_0)$. The renormalization scheme for $m_{\rm R}$ is irrelevant, as any change of scheme can be compensated by $Z_{\mathcal{O}_{\rm kin}}, c_i$ whose finite parts need to be fixed by matching to QCD. We further expand

$$(Z_{\rm A}^{\rm HQET})^2 = (Z_{\rm A}^{\rm stat})^2 + 2Z_{\rm A}^{\rm stat}Z_{\rm A}^{(1/m)} + {\rm O}(1/m^2) \tag{4.27}$$

which we will discuss more below. With these rules we then have

$$\big(Z_{\rm A}^{\rm stat}\big)^2\,\omega_{\rm kin}C_{\rm AA}^{\rm kin}(x_0) = \tfrac{1}{2m_{\rm R}}a^7\sum_{\mathbf{x},\,z}G(x,z) + \text{ subtraction terms}\,, \tag{4.28}$$

where

$$G(x,z) = \left\langle [A_0^{\rm stat}]_{\rm R}(x)\,([A_0^{\rm stat}]_{\rm R}(0))^\dagger\,\big(\mathcal{O}_{\rm kin}\big)_{\rm R}(z)\right\rangle_{\rm stat}. \tag{4.29}$$

The subtraction terms are due to the lower dimensional operators with coefficients $c_1$ and $c_2$. Since we are interested in on-shell observables ($x_0 > 0$ in eq. (4.14)), we may use the equation of motion $D_0\psi_{\rm h}(z) = 0$ to see that the $c_1$-term does not contribute, while $\frac{c_2}{a^2}\overline{\psi}_{\rm h}(z)\psi_{\rm h}(z)$, is equivalent to a mass shift. In the full correlation function eq. (4.14) it hence contributes to $\delta m$ which becomes quadratically divergent when the $1/m$ terms are included.

While $G(x,z)$ is a renormalized correlation function for all physical separations, its integral over $z$ (or on the lattice the continuum limit of the sum over $z$) does not exist due to singularities at $z \to 0$ and as $z \to x$. These contact term singularities can be analyzed by the operator product expansion. We discuss them first in the continuum and regulate the short distance region by just integrating for $z^2 \geq r^2$ with some small $r$. The operator product expansion then yields

$$\begin{aligned}&\int_{z^2\geq r^2} {\rm d}^4 z\, G(x,z) \\ &\overset{r\to 0}{\sim} \left\langle [A_0^{\rm stat}]_{\rm R}(x)\,[d_1''\frac{1}{r}\,(A_0^{\rm stat}(0))^\dagger + d_2''(A_0^{(1)}(0))^\dagger + d_3''(A_0^{(2)}(0))^\dagger]\right\rangle_{\rm stat}\end{aligned} \tag{4.30}$$

up to terms which are finite as $r \to 0$. The coefficients $d_i''$ in the operator product expansion have a further logarithmic dependence on $r$.[4] For (the continuum version of) eq. (4.29) we need $r \to 0$. In this limit short distance divergences emerge which have to be subtracted by counter-terms. In the lattice regularization, short distance singularities are regulated by the lattice spacing $a$ and we have in full analogy

$$\begin{aligned}&\left\langle [A_0^{\rm stat}]_{\rm R}(x)\,\big[a^4\sum_z\,([A_0^{\rm stat}]_{\rm R}(0))^\dagger\,\big(\mathcal{O}_{\rm kin}\big)_{\rm R}(z)\big]\right\rangle_{\rm stat} \\ &\overset{a\to 0}{\sim} \left\langle [A_0^{\rm stat}]_{\rm R}(x)\,[d_1'\frac{1}{a}\,(A_0^{\rm stat}(0))^\dagger + d_2'(A_0^{(1)}(0))^\dagger + d_3'(A_0^{(2)}(0))^\dagger]\right\rangle_{\rm stat}\end{aligned} \tag{4.31}$$

up to terms which have a continuum limit $a \to 0$ and up to the singular terms originating from $z \approx x$ . The coefficients $d_i$ contain a logarithmic dependence on $a$. Treating the singular terms at $z \approx x$ in the same way and noting that the term with $A_0^{(2)}(0)$ vanishes upon summation over $\mathbf{x}$ we find

$$Z_{\rm A}^{\rm stat}\,[\,d_1\frac{1}{a}\,C_{\rm AA}^{\rm stat}(x_0) + d_2\,C_{\delta{\rm A}}^{\rm stat}(x_0)\,] \tag{4.32}$$

for the contact term singularities in eq. (4.28). These are absorbed in eq. (4.14) through counter-terms contained in $Z_{\rm A}^{\rm HQET}$ and $c_{\rm A}^{(1)}$,

$$2Z_{\rm A}^{(1/m)} = -\frac{d_1}{2am_{\rm R}} + \ldots\,, \quad c_{\rm A}^{(1)} = -\frac{d_2}{2m_{\rm R}Z_{\rm A}^{\rm stat}} + \ldots\,. \tag{4.33}$$

The change from $d_i'$ to $d_i$ is due to the use of the equation of motion above. This step is valid only up to contact terms, resulting in the shift $d' \to d$. The ellipses contain the physical, finite $1/m$ terms.

[4] We have written down the integrated version, since then a smaller number of operators can appear and we are ultimately interested in the integral.

We now comment further on the expansion eq. (4.27). Our discussion shows that the quadratic term $(Z_{\rm A}^{(1/m)})^2$ in eq. (4.27) *must be dropped*; otherwise an uncanceled $1/(a^2m^2)$ divergence remains. As we have seen there is no $1/(a^2m^2)$ in $C_{\rm AA}^{\rm kin}(x_0)$ and the other pieces in eq. (4.14) are less singular. This is just a manifestation of the general rule of an effective field theory that all quantities are to be expanded in $1/m$ whether they are divergent or not. With this rule the various HQET parameters can be determined such that they absorb all divergences. [5]

The lesson of our discussion is that counter-terms with the correct structure are automatically present because in the effective theory all the relevant local composite fields are included with free coefficients. These free parameters may thus be chosen such that the continuum limit of the HQET correlation functions exists. Finally, their finite parts are to be determined such that the effective theory yields the $1/m$ expansion of the QCD observables.

## 4.4 The need for non-perturbative conversion functions

An important step remains to be explained: the determination of the HQET parameters. As discussed in Sect. 3 at the leading order in $1/m$, this can be done with the help of perturbation theory for conversion functions such as $C_{\rm PS}$. However, as soon as a $1/m$ correction is to be included, the leading order conversion functions have to be known non-perturbatively. This general feature in the determination of power corrections in QCD is seen in the following way. Consider the error made in eq. (3.9), when the anomalous dimension has been computed at $l$ loops and $C_{\rm match}$ at $l-1$ loop order. The conversion function

$$C_{\rm PS} = \exp\left\{-\int^{g_\star} {\rm d}x \frac{\gamma_0 x^2 + \ldots + \gamma_{l-1}^{\rm match} x^{2l}}{\beta(x)}\right\} + \Delta(C_{\rm PS}) \tag{4.34}$$

is then known up to a relative *error*

$$\frac{\Delta(C_{\rm PS})}{C_{\rm PS}} \propto [\bar{g}^2(m)]^l \sim \left\{\frac{1}{2b_0 \ln(m/\Lambda_{\rm QCD})}\right\}^l \overset{m\to\infty}{\gg} \frac{\Lambda_{\rm QCD}}{m}\,. \tag{4.35}$$

As $m$ is made large, this perturbative error becomes dominant over the power correction one wants to determine. Taking a perturbative conversion function and adding power corrections to the leading order effective theory is thus a phenomenological approach, where one assumes that for example at the b-quark mass, the coefficient of the $[\bar{g}^2(m_{\rm b})]^l$ term (as well as higher order ones) is small, such that the $\Lambda/m_{\rm b}$ corrections

[5] It is convenient to avoid the multiplication of $1/m$ terms explicitly by a choice of observables, for example

$$\tilde{\Phi} = \ln(f_{\rm B}\sqrt{m_{\rm B}}) = \ln(Z_{\rm A}^{\rm HQET}) + \ln(\Phi^{\rm stat}) + \lim_{x_0\to\infty}\left\{\tfrac{1}{2}x_0\omega_{\rm kin}E^{\rm kin} + \tfrac{1}{2}\omega_{\rm kin}R_{\rm AA}^{\rm kin}(x_0) + \ldots\right\},$$

$$\ln(Z_{\rm A}^{\rm HQET}) = \ln(Z_{\rm A}^{\rm stat}) + \frac{Z_{\rm A}^{(1/m)}}{Z_{\rm A}^{\rm stat}} \equiv \ln(Z_{\rm A}^{\rm stat}) + [\ln(Z_{\rm A})]^{1/m}$$

In this convention all $1/m$-terms appear linearly.

dominate. In such a phenomenological determination of a power correction, its size depends on the order of perturbation theory considered. A theoretically consistent evaluation of power corrections requires a fully non-perturbative formulation of the theory including a non-perturbative matching to QCD. Note that the essential point of Eq. (4.35) is not the expected factorial growth of the coefficients of the perturbative expansion. Rather it is due to the truncation of perturbation theory as such. Of course a renormalon-like growth of the coefficients does not help.

The foregoing discussion is completely generic, applying to any regularization. When we define the theory on the lattice, there are in addition power divergences, e.g. in eq. (4.31). It is well known that they have to be subtracted non-perturbatively if one wants the continuum limit to exist.

## 4.5 Splitting leading order (LO) and next to leading order (NLO)

We just learned that the very definition of a NLO correction to $f_\mathrm{B}$ means to take eq. (4.22) with all coefficients $Z_\mathrm{A}^\mathrm{HQET} \dots c_\mathrm{A}^{(1)}$ determined non-perturbatively. We want to briefly explain that, as a consequence, the split between LO and NLO is not unique. This is fully analogous to the case of standard perturbation theory in $\alpha$, where the split between different orders depends on the renormalization scheme used, and on the experimental observable used to determine $\alpha$ in the first place.

Consider the lowest order. The only coefficient needed in eq. (4.22) is then $Z_\mathrm{A}^\mathrm{HQET} = C_\mathrm{PS} Z_\mathrm{A,RGI}^\mathrm{stat}$. It has to be fixed by matching some matrix element of $A_0^\mathrm{stat}$ to the matrix element of $A_0$ in QCD. For example one may choose $\langle B'|A_0^\dagger|0\rangle$, with $|B'\rangle$ denoting some other state such as an excited pseudo-scalar state. Or one may take a finite volume matrix element defined through the Schrödinger functional as we will do later. Since the matching involves the QCD matrix element, there are higher order in $1/m$ "pieces" in these equations. There is no reason for them to be independent of the particular matrix element. So from matching condition to matching condition, $C_\mathrm{PS} Z_\mathrm{A,RGI}^\mathrm{stat}$ determined at the leading order in $1/m$ differs by $\mathrm{O}(\Lambda_\mathrm{QCD}/m_\mathrm{b})$ terms.

The matrix element $f_\mathrm{B}$ in static approximation inherits this $\mathrm{O}(\Lambda_\mathrm{QCD}/m_\mathrm{b})$ ambiguity. These corrections are hence not unique. Fixing a matching condition, the leading order $f_\mathrm{B}$ as well as the one including the corrections can be computed and have a continuum limit. Their difference can be defined as the $1/m$ correction. However, what matters is not the ambiguous NLO term, but the fact that the uncertainty is reduced from $\mathrm{O}(\Lambda_\mathrm{QCD}/m_\mathrm{b})$ in the LO term to $\mathrm{O}(\Lambda_\mathrm{QCD}^2/m_\mathrm{b}^2)$ in the sum.

The following table illustrates the point explicitly.

| Observables | $\langle B\|A_0^\dagger\|0\rangle$ | $\langle B'\|A_0^\dagger\|0\rangle$ | $\langle B''\|A_0^\dagger\|0\rangle$ |
|---|---|---|---|
| matching condition | * | | |
| error in HQET result | 0 | $\mathrm{O}(\Lambda/m_\mathrm{b})$ | $\mathrm{O}(\Lambda/m_\mathrm{b})$ |
| matching condition | | * | |
| error in HQET result | $\mathrm{O}(\Lambda/m_\mathrm{b})$ | 0 | $\mathrm{O}(\Lambda/m_\mathrm{b})$ |
| matching condition | | | * |
| error in HQET result | $\mathrm{O}(\Lambda/m_\mathrm{b})$ | $\mathrm{O}(\Lambda/m_\mathrm{b})$ | 0 |

As a consequence, there is no strict meaning to the statement "*the* $1/m$ correction to $f_{\rm B}$ is 10%".

## 4.6 Mass formulae

Often cited mass formulae are

$$m_{\rm B}^{\rm av} \equiv \frac{1}{4}[m_{\rm B} + 3m_{\rm B^*}] = m_{\rm b} + \overline{\Lambda} + \frac{1}{2m_{\rm b}}\lambda_1 + {\rm O}(1/m_{\rm b}^2) \tag{4.36}$$

$$\Delta m_{\rm B} \equiv m_{\rm B^*} - m_{\rm B} = -\frac{2}{m_{\rm b}}\lambda_2 + {\rm O}(1/m_{\rm b}^2) \tag{4.37}$$

with (ignoring renormalization)

$$\lambda_1 = \langle B|\mathcal{O}_{\rm kin}|B\rangle\,, \quad \lambda_2 = \tfrac{1}{3}\langle B|\mathcal{O}_{\rm spin}|B\rangle\,. \tag{4.38}$$

The quantity $\overline{\Lambda}$ is termed "static binding energy". Also here, depending on how one formulates the matching condition which determines $m_{\rm b}$, one changes $\overline{\Lambda}$ by a term of order $\Lambda_{\rm QCD}$, e.g. one may define $\overline{\Lambda} = 0$. Similarly, the kinetic term $\lambda_1/(2m_{\rm b})$ has a non-perturbative matching scheme dependence of order $\Lambda_{\rm QCD}$ and thus $\lambda_1$ itself has a matching scheme dependence of order $m_{\rm b}$. The situation for $\overline{\Lambda}$ is similar to the gluon "condensate". The non-perturbative scheme dependence has the same size as the gluon "condensate" itself. In contrast, $\lambda_2$ is the leading term in the $1/m$ expansion and does not have such an ambiguity. We refer also to the more detailed discussion in (Sommer, 2006).

## 4.7 Non-perturbative determination of HQET parameters

We close our theoretical discussion of HQET by stating the correct procedure to determine the $N_{\rm HQET}$ parameters in the effective theory at a certain order in $1/m$. One requires

$$\Phi_i^{\rm QCD}(m) = \Phi_i^{\rm HQET}(m, a)\,, \quad i = 1 \ldots N_{\rm HQET}\,, \tag{4.39}$$

where the $m$-dependence on the r.h.s. is entirely inside the HQET parameters. On the l.h.s. the continuum limit in QCD is assumed to have been taken, but the r.h.s. refers to a given lattice spacing where it defines the bare parameters of the theory at that value of $a$. We emphasize that as this matching has to be invoked by numerical data, it is done at a given finite value of $1/m$. Carrying it out with just the static parameters defines the static approximation etc.

As simple as it is written down, it is non-trivial to implement eq. (4.39) in practice such that

1) the HQET expansion is accurate and one may thus truncate at a given order,
2) the numerical precision is sufficient,
3) lattice spacings are available for which large volume computations of physical matrix elements can be performed.

In the following section we explain how these criteria can be satisfied using Schrödinger functional correlation functions and a step scaling method. The first part will be a test of HQET on some selected correlation functions. This establishes how 1) and 2) can be met. We can then explain the complete strategy which also achieves 3).

## 4.8 Relation to RGI matrix elements and conversion functions

The matching equations eq. (4.39) provide a definition of all HQET parameters, in principle at any given order in the expansion. If considered at the static order, it also provides the renormalization of the static axial current, which we discussed at length in Sect. 3. The relation between the two ways of parametrizing the current in static approximation are

$$Z_{\mathrm{A}}^{\mathrm{HQET}} = Z_{\mathrm{A,RGI}}^{\mathrm{stat}}\, C_{\mathrm{PS}}(M/\Lambda) + \mathrm{O}(1/m, a)\,. \tag{4.40}$$

While eq. (4.39) is a matching equation determining directly the product $Z_{\mathrm{A,RGI}}^{\mathrm{stat}}\, C_{\mathrm{PS}}$, the r.h.s. separates the problem into a pure HQET problem, the determination of the RGI operator, and a pure QCD problem, the the "anomalous dimension" $\gamma_{\mathrm{match}}$, see eq. (3.20). Note that in this simple form, such a separation is only possible at the lowest order in $1/m$.

Since the breaking of spin symmetry is due to a single operator at order $1/m$, there is also an analogous representation of $\omega_{\mathrm{spin}}$. We refer the interested reader to (Guazzini *et al.*, 2007).

# 5
# Non-perturbative HQET

After our long discussion of the theoretical issues in the renormalization of HQET, we turn to a complete strategy for the non-perturbative implementation. To this end the three criteria in Sect. 4.7 have to be fulfilled. Establishing 1) is equivalent to testing HQET. We therefore start with such a test. Item 2) has to do with finding matching conditions sensitive to the $1/m$-suppressed contributions. For this purpose we then expand a little on correlation functions in the Schrödinger functional before coming to a full description of the matching strategy.

## 5.1 Non-perturbative tests of HQET

Although it is generally accepted that HQET is an effective theory of QCD, tests of this equivalence are rare and mostly based on phenomenological analysis of experimental results. A pure theory test can be performed if QCD including a heavy enough quark can be simulated on the lattice at lattice spacings which are small enough to be able to take the continuum limit. This has been done in the last few years (Heitger *et al.*, 2004, Della Morte *et al.*, 2008) and will be summarized below.

We start with the QCD side of such a test. Lattice spacings such that $am_{\rm b} \ll 1$ can be reached if one puts the theory in a finite volume, $L^3 \times T$ with $L, T$ not too large. We shall use $T = L$. For various practical reasons, Schrödinger functional boundary conditions are chosen. Equivalent boundary conditions are imposed in the effective

**Fig. 5.1** Testing eq. (5.6) through numerical simulations in the quenched approximation and for $L \approx 0.2\,\mathrm{fm}$ (Heitger *et al.*, 2004). The graph uses notation $Y_{\rm R}^{\rm QCD} \equiv Y_{\rm PS}$). The physical mass of the b-quark corresponds to $z \approx 6$. Two different orders of perturbation theory for $C_{\rm PS}$ are shown.

theory. As in Sect. 3.3 we consider the ratio $Y_{\rm R}^{\rm QCD}(\theta, m_{\rm R}, L)$ built from the correlation functions $f_{\rm A}$ and $f_1$.

It can be written as

$$Y_{\rm R}^{\rm QCD}(\theta, m_{\rm R}, L) = \frac{\langle\Omega(L)|A_0|B(L)\rangle}{||\,|\Omega(L)\rangle\,||\;||\,|B(L)\rangle\,||}, \tag{5.1}$$

$$|B(L)\rangle = \mathrm{e}^{-L\mathbb{H}/2}|\varphi_{\rm B}(L)\rangle\,,\ |\Omega(L)\rangle = \mathrm{e}^{-L\mathbb{H}/2}|\varphi_0(L)\rangle\,,$$

in terms of the boundary states $|\varphi_{\rm B}(L)\rangle\,,\ |\varphi_0(L)\rangle$. Expanded in energy eigenstates with energies $E_n \geq m_{\rm B}$ in the B-sector and energies $\tilde{E}_n$ in the vacuum sector, we have

$$|B(L)\rangle = \sum_n \mathrm{e}^{-LE_n/2}\langle n, B|\varphi_{\rm B}(L)\rangle\ |n, B\rangle \tag{5.2}$$

$$\sim \sum_{n\ |\ E_n - m_{\rm B} < k/L} \mathrm{e}^{-LE_n/2}\langle n, B|\varphi_{\rm B}(L)\rangle\ |n, B\rangle + \mathrm{O}(\mathrm{e}^{-k/2})\,, \tag{5.3}$$

$$|\Omega(L)\rangle = \sum_n \mathrm{e}^{-L\tilde{E}_n/2}\langle n, 0|\varphi_0(L)\rangle\ |n, 0\rangle\,, \tag{5.4}$$

$$\sim \sum_{n\ |\ \tilde{E}_n < k/L} \mathrm{e}^{-L\tilde{E}_n/2}\langle n, 0|\varphi_0(L)\rangle\ |n, 0\rangle + \mathrm{O}(\mathrm{e}^{-k/2})\,, \tag{5.5}$$

which shows that only energy eigenstates with $E_n - E_0 = \mathrm{O}(1/L)$ contribute significantly. For $z = LM_{\rm b} \gg 1$, HQET will thus describe the correlation functions and the ratio $Y_{\rm R}^{\rm QCD}$. We come to the conclusion that

$$Y_{\rm R}^{\rm QCD}(\theta, m_{\rm R}, L) = C_{\rm PS}(M_{\rm b}/\Lambda)\, X_{\rm RGI} + \mathrm{O}(1/z)\,, \quad z = M_{\rm b}L\,, \tag{5.6}$$

$$X_{\rm RGI} = \lim_{a\to 0} Z_{\rm A,RGI}^{\rm stat}(g_0)\frac{f_{\rm A}^{\rm stat}(L/2, \theta)}{\sqrt{f_1^{\rm stat}(\theta)}} \tag{5.7}$$

and similarly for other observables. Note that one could also just argue that the only relevant scales are $L, \Lambda, m_{\rm b}$. Therefore with $L \approx 1/\Lambda$ there is a $\Lambda/m_{\rm b} \sim 1/z$ expansion.

Of course relations such as eq. (5.6) are expected after the continuum limit of both sides has been taken separately. For the case of $Y_{\rm R}^{\rm QCD}$, this is done by the following steps:

- Fix a value $u_0$ for the renormalized coupling $\bar{g}^2(L)$ (in the Schrödinger functional scheme) at vanishing quark mass. In (Heitger *et al.*, 2004) $u_0$ was chosen such that $L \approx 0.2\,\mathrm{fm}$.
- For a given resolution $L/a$, determine the bare coupling from the condition $\bar{g}^2(L) = u_0$. This step is well known by now (Capitani *et al.*, 1999).
- Fix the bare quark mass $m_{\rm q}$ of the heavy quark such that $LM = z$ using the known renormalization factors $Z_{\rm M}, Z$ in $M = Z_{\rm M}Z\,(1 + ab_{\rm m}m_{\rm q})\,m_{\rm q}$, where $Z, Z_{\rm M}, b_{\rm m}$ are all known non-perturbatively (Guagnelli *et al.*, 2001, Della Morte *et al.*, 2007*a*).
- Evaluate $Y_{\rm R}^{\rm QCD}$ and repeat for better resolution $a/L$.
- Extrapolate to the continuum as shown in Fig. 5.1, left.

**Fig. 5.2** Continuum extrapolation of $X_{\mathrm{RGI}}$ (Heitger *et al.*, 2004).

In the effective theory the same steps are followed. As a simplification, no quark mass needs to be fixed and the continuum extrapolation is much easier as illustrated in Fig. 5.2.

The comparison of the static result and the relativistic theory, Fig. 5.1, looks rather convincing,[1] but we note that the b-quark mass point is $1/z = 1/z_{\mathrm{b}} \approx 0.17$, where $1/z^2$ terms are not completely negligible. The displayed fit has a 8% contribution by the $1/z$ term and a 2% $1/z^2$ piece.

For a precision application (Sect. 5.3) it is thus safer to have $L \gtrsim 0.4\,\mathrm{fm}$ instead of the $L = 0.2\,\mathrm{fm}$ chosen in the first test, reducing $1/z^2$ by a factor four. For $L \approx 0.5\,\mathrm{fm}$ we show two different examples, Fig. 5.3, Fig. 5.4 which involve

$$k_1(\theta) = -\frac{a^{12}}{6L^6} \sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z},k} \left\langle \bar{\zeta}_{\mathrm{l}}{}'(\mathbf{u})\gamma_k \zeta'_{\mathrm{b}}(\mathbf{v})\, \bar{\zeta}_{\mathrm{b}}(\mathbf{y})\gamma_k \zeta_{\mathrm{l}}(\mathbf{z}) \right\rangle \tag{5.8}$$

in addition to the previously introduced correlation functions. The considered combinations are

$$R_1 = \tfrac{1}{4}\left(\ln\left(\frac{f_1(\theta_1)k_1(\theta_1)^3}{f_1(\theta_2)k_1(\theta_2)^3}\right)\right) \tag{5.9}$$

$$\widetilde{R}_1 = \frac{3}{4}\ln\left(\frac{f_1}{k_1}\right) . \tag{5.10}$$

Their HQET expansion contains no conversion functions at leading order and they are thus free of the associated perturbative uncertainty. While $R_1$ has a finite static limit, $\widetilde{R}_1$ vanishes as $z \to \infty$ due to the spin symmetry. The expected HQET behavior is confirmed with surprisingly small $1/z^2$ corrections for a charm quark. The quadratic fits in $1/z$ displayed in the figures are not constrained to pass through the separately displayed static limit.

[1] Note that the comparison Fig. 5.1 has to be taken with a grain of salt due to the perturbative uncertainty in $C_{\mathrm{PS}}$ discussed in Sect. 3.3.2.

**Fig. 5.3** The logarithmic ratio $R_1$ for different pairs $(\theta_1, \theta_2)$ with $N_\mathrm{f} = 2$ flavors and for $L = T \approx 0.5\,\mathrm{fm}$ (Della Morte *et al.*, 2008) with $N_\mathrm{f} = 2$. The value of $1/z$ for charm and bottom quarks are indicated by the vertical bands.

**Fig. 5.4** The logarithmic ratio $\widetilde{R}_1$ for different values $\theta_0$ with $N_\mathrm{f} = 2$ flavors and for $L = T \approx 0.5\,\mathrm{fm}$ (Della Morte *et al.*, 2008) with $N_\mathrm{f} = 2$.

## 5.2 HQET expansion of Schrödinger functional correlation functions

In complete analogy to the case of a manifold without boundary we can write down the expansions of the Schrödinger functional correlation functions to first order in $1/m$:

$$[f_{\rm A}]_{\rm R} = Z_{\rm A}^{\rm HQET} Z_{\zeta_{\rm h}} Z_\zeta {\rm e}^{-m_{\rm bare}x_0} \left\{ f_{\rm A}^{\rm stat} + c_{\rm A}^{(1)} f_{\delta{\rm A}}^{\rm stat} + \omega_{\rm kin} f_{\rm A}^{\rm kin} + \omega_{\rm spin} f_{\rm A}^{\rm spin} \right\} , \quad (5.11)$$

$$[f_1]_{\rm R} = Z_{\zeta_{\rm h}}^2 Z_\zeta^2 {\rm e}^{-m_{\rm bare}T} \left\{ f_1^{\rm stat} + \omega_{\rm kin} f_1^{\rm kin} + \omega_{\rm spin} f_1^{\rm spin} \right\} , \quad (5.12)$$

$$[k_1]_{\rm R} = Z_{\zeta_{\rm h}}^2 Z_\zeta^2 {\rm e}^{-m_{\rm bare}T} \left\{ f_1^{\rm stat} + \omega_{\rm kin} f_1^{\rm kin} - \tfrac{1}{3}\omega_{\rm spin} f_1^{\rm spin} \right\} . \quad (5.13)$$

Apart from

$$f_{\delta{\rm A}}^{\rm stat}(x_0,\theta) = -\frac{a^6}{2} \sum_{\mathbf{y},\mathbf{z}} \left\langle A_0^{(1)}(x)\, \bar\zeta_{\rm h}(\mathbf{y}) \gamma_5 \zeta_{\rm l}(\mathbf{z}) \right\rangle \quad (5.14)$$

the labeling of the different terms follows directly the one introduced in eq. (4.5). The relation between the $1/m$ terms in $f_1$ and $k_1$ is a simple consequence of the spin symmetry of the static action, valid at any lattice spacing. A further simplicity is that no $1/m$ boundary corrections are present. Potential such terms have dimension four. After using the equations of motion, only one candidate remains, which however does not contribute to any correlation function.[2]

## 5.3 Strategy for non-perturbative matching

After the tests of HQET described above, it is clear how one can non-perturbatively match HQET to QCD. Consider the action as well as $A_0$ (just at $\mathbf{p} = 0$) and denote the free parameters of the effective theory by $\omega_i$ , $i = 1 \ldots N_{\rm HQET}$. In static approximation we then have

$$\omega^{\rm stat} = (\, m_{\rm bare}^{\rm stat} \,,\, [\ln(Z_{\rm A})]^{\rm stat} \,)^t \,, \quad N_{\rm HQET} = 2 \quad (5.15)$$

and including the first order terms in $1/m$ together with the static ones, the HQET parameters are

$$\omega^{\rm HQET} = (\, m_{\rm bare} \,,\, \ln(Z_{\rm A}^{\rm HQET}) \,,\, c_{\rm A}^{(1)} \,,\, \omega_{\rm kin} \,,\, \omega_{\rm spin} \,)^t \quad N_{\rm HQET} = 5 \,. \quad (5.16)$$

The pure $1/m$ parameters may be defined as $\omega^{(1/m)} = \omega^{\rm HQET} - \omega^{\rm stat}$, with all of them, e.g. also $m_{\rm bare}^{(1/m)}$, non-zero. In fact our discussion of renormalization of the $1/m$ terms shows that $m_{\rm bare}^{(1/m)}$ diverges as $1/(a^2 m)$.

With suitable observables

$$\Phi_i(L_1, M, a) \,,\ i = 1 \ldots N_{\rm HQET} \,,$$

in a Schrödinger functional with $L = T = L_1 \approx 0.5\,{\rm fm}$, we then require matching[3]

$$\Phi_i(L_1, M, a) = \Phi_i^{\rm QCD}(L_1, M, 0) \,,\ i = 1 \ldots N_{\rm HQET} \,. \quad (5.17)$$

[2] In the notation of (Lüscher *et al.*, 1996) it reads $\bar\rho_{\rm h}(\mathbf{x})\gamma_k D_k \rho_{\rm h}(\mathbf{x})$ at $x_0 = 0$. Such a term does not contribute to any correlation function due to the form of the static propagator.

[3] Recall that observables without a superscript refer to HQET.

Note that the continuum limit is taken in QCD, while in HQET we want to extract the bare parameters of the theory from the matching equation and thus have a finite value of $a$. It is convenient to pick observables with HQET expansions linear in $\omega_i$,

$$\Phi(L,M,a) = \eta(L,a) + \phi(L,a)\,\omega(M,a)\,, \tag{5.18}$$

in terms of a $N_{\rm HQET} \times N_{\rm HQET}$ coefficient matrix $\phi$. A natural choice for the first two observables is

$$\Phi_1 = L\Gamma^{\rm P} \equiv -L\widetilde{\partial}_0 \ln(-f_{\rm A}(x_0))_{x_0=L/2} \stackrel{L\to\infty}{\sim} Lm_{\rm B} \tag{5.19}$$

$$\Phi_2 = \ln(Z_{\rm A}\frac{-f_{\rm A}}{\sqrt{f_1}}) \stackrel{L\to\infty}{\sim} L^{3/2} f_{\rm B}\sqrt{m_{\rm B}/2}\,, \tag{5.20}$$

since in static approximation these determine directly $\omega_1$ and $\omega_2$. We will introduce the other $\Phi_i$ later. The explicit form of $\eta, \phi$ is

$$\eta = \begin{pmatrix} \Gamma^{\rm stat} \\ \zeta_{\rm A} \\ \dots \end{pmatrix}\,, \quad \phi = \begin{pmatrix} L & 0 \dots \\ 0 & 1 \dots \\ \dots & \end{pmatrix} \tag{5.21}$$

with

$$\Gamma^{\rm stat} = -L\widetilde{\partial}_0 \ln(f_{\rm A}^{\rm stat}(x_0))_{x_0=L/2}\,, \quad \zeta_{\rm A} = \ln(\frac{-f_{\rm A}^{\rm stat}}{\sqrt{f_1^{\rm stat}}})\,. \tag{5.22}$$

In static approximation, the structure of the matrix $\phi$ is perfect: one observable determines one parameter. This is possible since there is no (non-trivial) mixing at that order.

Having specified the matching conditions, the HQET parameters $\omega_i(M,a)$ can be obtained from eqs.(5.17,5.18), but only for rather small lattice spacings since a reasonable suppression of lattice artifacts requires $L_1/a = {\rm O}(10)$ and thus $a = {\rm O}(0.05\,{\rm fm})$.

Larger lattice spacings as needed in large volume, can be reached by adding a step scaling strategy, illustrated in Fig. 5.5. Let us now go through the various steps of this strategy.

(1) Take the continuum limit

$$\Phi_i^{\rm QCD}(L_1,M,0) = \lim_{a/L_1\to 0} \Phi_i^{\rm QCD}(L_1,M,a)\,. \tag{5.23}$$

This is similar to the HQET tests and as we saw there, it requires $L_1/a = 20\dots40$, or $a = 0.025\,{\rm fm}\dots0.012\,{\rm fm}$.

(2a) Set the HQET observables equal to the QCD ones, eq. (5.17) and extract the parameters

$$\tilde{\omega}(M,a) \equiv \phi^{-1}(L_1,a)\,[\Phi(L_1,M,0) - \eta(L_1,a)] \tag{5.24}$$

$$= \begin{pmatrix} L_1^{-1}\Phi_1(L_1,M,0) - \Gamma^{\rm stat}(L_1,a) \\ \Phi_2(L_1,M,0) - \zeta_{\rm A}(L_1,a) \\ \dots \end{pmatrix}\,. \tag{5.25}$$

The only restriction here is $L_1/a \gg 1$, so one can use $L_1/a = 10\dots20$, which means $a = 0.05\,{\rm fm}\dots0.025\,{\rm fm}$.

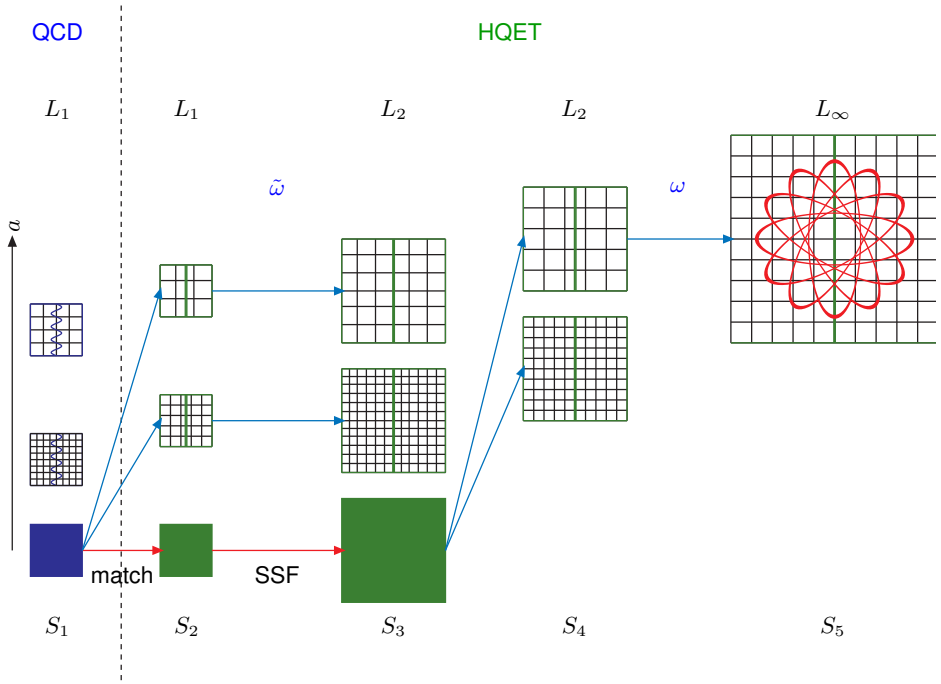


**Fig. 5.5** Strategy for non-perturbative HQET (Blossier *et al.*, 2010*b*). Note that in the realistic implementation(Blossier *et al.*, 2010*b*) finer resolutions are used.

(2b.) Insert $\tilde{\omega}$ into $\Phi(L_2, M, a)$:

$$\begin{aligned}\Phi(L_2,M,0) &= \lim_{a/L_2\to 0}\{\eta(L_2,a)+\phi(L_2,a)\,\tilde{\omega}(M,a)\} \qquad (5.26)\\ &= \lim_{a/L_2\to 0}\begin{pmatrix} L_2\Gamma^{\rm stat}(L_2,a)+\frac{L_2}{L_1}\Phi_1(L_1,M,0)-L_2\Gamma^{\rm stat}(L_1,a)\\ \zeta_{\rm A}(L_2,a)+\Phi_2(L_1,M,0)-\zeta_{\rm A}(L_1,a)\\ \cdots\end{pmatrix}\\ &= \lim_{a/L_2\to 0}\underbrace{\begin{pmatrix} L_2[\Gamma^{\rm stat}(L_2,a)-\Gamma^{\rm stat}(L_1,a)]\\ \zeta_{\rm A}(L_2,a)-\zeta_{\rm A}(L_1,a)\\ \cdots\end{pmatrix}}_{\text{finite HQET SSF's}}+\underbrace{\begin{pmatrix}\frac{L_2}{L_1}\Phi_1(L_1,M,0)\\ \Phi_2(L_1,M,0)\\ \cdots\end{pmatrix}}_{\text{QCD, mass dependence}}.\end{aligned}$$

In the last line we have identified pieces which are separately finite. This step can be done as long as the lattice spacing is common to the $n_2 = L_2/a$ and $n_1 = L_1/a$-lattices and

$$s = L_2/L_1 = n_2/n_1 \qquad (5.27)$$

is kept at a fixed, small, ratio.[4]

(3.) Repeat (2a.) for $L_1 \to L_2$:

$$\omega(M,a) \equiv \phi^{-1}(L_2,a)\,[\Phi(L_2,M,0) - \eta(L_2,a)]\,. \tag{5.28}$$

With the same resolutions $L_2/a = 10\ldots20$ one has now reached $a = 0.1\,\mathrm{fm}\ldots0.05\,\mathrm{fm}$.

(4.) Finally insert $\omega$ into the expansion of large volume observables, e.g.

$$m_\mathrm{B} = \omega_1 + E^\mathrm{stat}\,. \tag{5.29}$$

In the chosen example the result is the relation between the RGI b-quark mass and the B-meson mass $m_\mathrm{B}$. It is illustrative to put the different steps into one equation,

$$\begin{aligned} m_\mathrm{B} = & \qquad\qquad\qquad\qquad\qquad\qquad (5.30)\\ & \lim_{a\to0}[E^\mathrm{stat} - \Gamma^\mathrm{stat}(L_2,a)] \qquad a = 0.1\mathrm{fm}\ldots0.05\mathrm{fm} \qquad [S_4,S_5]\\ & + \lim_{a\to0}[\Gamma^\mathrm{stat}(L_2,a) - \Gamma^\mathrm{stat}(L_1,a)] \qquad a = 0.05\mathrm{fm}\ldots0.025\mathrm{fm} \qquad [S_2,S_3]\\ & + \frac{1}{L_1}\lim_{a\to0}\Phi_1(L_1,M_\mathrm{b},a) \qquad a = 0.025\,\mathrm{fm}\ldots0.012\,\mathrm{fm} \qquad [S_1]\,. \end{aligned}$$

We have indicated the lattices drawn in Fig. 5.5 and the typical lattice spacings of these lattices. The explicit expression for the decay constant in static approximation is even more simple; write it down as an exercise!

So far we have spelled out only those observables which are needed in the static approximation. The following heuristics helps to find observables suitable for the determination of the $1/m$-terms. Recall that $\theta \neq 0$ means $\frac{1}{2}(\nabla_j + \nabla_j^*) \sim i\theta/L$ (acting onto a quark field) when the gauge fields are weak, as is the case in small volume. Hence, expanding in $1/m$

$$\Phi_3(L,M,a) = \frac{f_\mathrm{A}(\theta_1)}{f_\mathrm{A}(\theta_2)} \sim \ldots + c_\mathrm{A}^{(1)}\,[\theta_2 - \theta_1]/L \tag{5.31}$$

for weakly coupled quarks. In the same way the combination (recall eq. (5.9))

$$\Phi_4(L,M,a) = R_1 = R_1^\mathrm{stat} + \omega_\mathrm{kin}\,R_1^\mathrm{kin}$$

has a sensitivity to $\omega_\mathrm{kin}$ of $R_1^\mathrm{kin} \propto\ \theta_1^2 - \theta_2^2$ while in the specific linear combination of $f_1$ and $k_1$ which form $R_1$ the parameter $\omega_\mathrm{spin}$ drops out. Finally the choice

$$\Phi_5(L,M,a) = \widetilde{R}_1 = \omega_\mathrm{spin} R_1^\mathrm{spin} \tag{5.32}$$

allows for a direct determination of $\omega_\mathrm{spin}$. These choices leave relatively many zeros in the matrix $\phi$, which has a block structure,

$$\phi = \begin{pmatrix} C & B \\ 0 & A \end{pmatrix}\,, \quad \phi^{-1} = \begin{pmatrix} C^{-1} & -C^{-1}BA^{-1} \\ 0 & A^{-1} \end{pmatrix}\,, \quad C = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{5.33}$$

The listed observables $\Phi_i$ have been shown to work in practice, i.e. in a numerical application (Blossier *et al.*, 2010*b*).

[4] A fixed ratio $s$ ensures that the cutoff effects are a smooth function of $a/L_i$.

## 5.4 Numerical computations in the effective theory

Before showing some results, we should briefly mention that it is not entirely straight forward to obtain precise numerical results in the effective theory. The reason is a generically rather strong growth of statistical errors as a function of the Euclidean time separation of the correlation functions. Two ideas help to overcome this problem. We sketch them here; more details are available in the cited literature.

### 5.4.1 The static action

Consider a typical two-point function, for example eq. (3.5). At large time it decays exponentially and so does the variance. Setting $\delta m = 0$ the decay of the signal is

$$C(x_0) \sim \mathrm{e}^{-E_{\mathrm{stat}}\, x_0}\,, \tag{5.34}$$

while the variance decays with an exponential rate given by the pion mass. Thus the noise-to-signal ratio for the B-meson correlation function behaves as

$$R_{\mathrm{NS}} \propto e^{[E_{\mathrm{stat}} - m_\pi/2]\, x_0}\,. \tag{5.35}$$

The self energy of a static quark is power divergent, in particular in perturbation theory

$$E^{\mathrm{stat}} \sim \left(\frac{1}{a} r^{(1)} + \mathrm{O}(a^0)\right)\, g_0^2 + \mathrm{O}(g_0^4)\,. \tag{5.36}$$

This divergence yields the leading behavior of eq. (5.35) for small $a$. It is potentially dangerous since we are interested in the continuum limit. The scale of the problem can be reduced considerably by the replacement

$$U(x,0) \;\to\; W_{\mathrm{HYPi}}(x,0)\,, \tag{5.37}$$

in the covariant derivative $\nabla_0^*$ in the static action. Here $W_{\mathrm{HYPi}}$ is a so-called HYP-smeared link. Table 5.1 shows how the self energy is reduced for two choices of $W_{\mathrm{HYPi}}$.

| $S_{\mathrm{h}}^{\mathrm{W}}$ | $r^{(1)}$ | $aE_{\mathrm{stat}}$ |
|---|---|---|
| $S_{\mathrm{h}}^{\mathrm{EH}}$ | 0.16845(2) | 0.68(9) |
| $S_{\mathrm{h}}^{\mathrm{HYP1}}$ | 0.04844(1) | 0.44(2) |
| $S_{\mathrm{h}}^{\mathrm{HYP2}}$ | 0.03523(1) | 0.41(1) |

**Table 5.1** One loop coefficients $r^{(1)}$, eq. (5.36) and non-perturbative values for $aE_{\mathrm{stat}}$ at $\beta = 6/g_0^2 = 6$ and a (quenched) light quark with the mass of the strange quark. "EH" refers to Eichten-Hill, i.e. $W(x,0) = U(x,0)$, while "HYP1,HYP2" are two versions of HYP-smearing (Hasenfratz and Knechtli, 2001, Della Morte *et al.*, 2005)

It is mandatory to check that such a change of action does not introduce large cutoff effects. This was done for single smearing in (Della Morte *et al.*, 2005): the points with smallest error bars in Fig. 2.1 are for these actions. We expect that large cutoff effects would however appear if smearing was repeated several times.

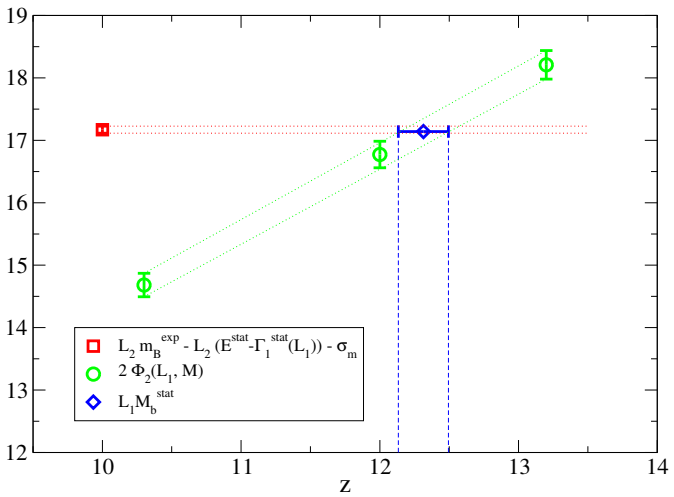


**Fig. 5.6** Numerical solution of the equation for $M_{\rm b}$ (Blossier *et al.*, 2010*b*) made dimensionless by multiplication with $L_2$. The figure uses a notation $\sigma_{\rm m} = \lim_{a\to 0} L_2\,[\Gamma^{\rm stat}(L_2,a) - \Gamma^{\rm stat}(L_1,a)]$ and $\Phi_2$ in the figure is $\Phi_1$ in our notation.

### 5.4.2 Generalized Eigenvalue Method

For the numerical evaluation of matrix elements such as $\Phi^{\rm stat}$, eq. (1.57), or of energy levels it is advisable to use an improvement over the straight forward formula eq. (3.7). The reason is as follows. Let us label the energies in the sector contributing to a given correlation function by $E_n$, $n = 1, 2, 3$. Then there are corrections to the desired ground state matrix element due to excited state contaminations of order $\mathrm{e}^{-x_0\Delta}$ and $\Delta = E_2 - E_1$. From an investigation of the spectrum in the B-meson sector one finds numerically $\Delta \approx 600\,\mathrm{MeV}$ and thus $\Delta\, x_0 \approx 3x_0/\mathrm{fm}$. The suppression of excited state contaminations is then not necessarily small enough for $x_0 \sim 1\mathrm{fm}$ but using eq. (3.7) beyond $x_0 \sim 1\mathrm{fm}$ is very difficult because statistical errors grow quite rapidly with $x_0$.

A considerable improvement is achieved if one considers the generalized eigenvalue problem (GEVP)(Michael and Teasdale, 1983, Lüscher and Wolff, 1990, Blossier *et al.*, 2009). It uses additional information in the form of a matrix correlation function formed from $N$ different interpolating fields on one time slice and the same interpolating fields on another time slice. When this matrix correlation function is analyzed in a specific way, described in (Blossier *et al.*, 2009), one can prove that a much larger gap, $\Delta = E_{N+1} - E_1$ appears for the dominating correction terms due to excited states. These then disappear much more quickly with growing time.

The GEVP is straight forwardly applicable to HQET, order by order in $1/m$. The precision of the numerical results that we show below is largely due to this method, together with the use of HYP1/2 actions.

## 5.5 Examples of results

We now discuss a few numerical results (Blossier *et al.*, 2010*b*, Blossier *et al.*, 2010*a*, Blossier *et al.*, 2010*c*) in order to give an indication of what can be done at present.

| | LO (static) | NLO (static + O($1/m$)) | | |
|---|---|---|---|---|
| | | $(\theta_1, \theta_2) = (0, 0.5)$ | $(\theta_1, \theta_2) = (0.5, 1)$ | $(\theta_1, \theta_2) = (0, 1)$ |
| $\theta_0 = 0$ | $17.1 \pm 0.2$ | $17.1 \pm 0.2$ | $17.1 \pm 0.2$ | $17.1 \pm 0.2$ |
| $\theta_0 = 0.5$ | $17.2 \pm 0.2$ | $17.2 \pm 0.2$ | $17.2 \pm 0.2$ | $17.1 \pm 0.2$ |
| $\theta_0 = 1$ | $17.2 \pm 0.2$ | $17.3 \pm 0.3$ | $17.3 \pm 0.3$ | $17.3 \pm 0.3$ |

**Table 5.2** Dimensionless b-quark mass, $r_0 M_\mathrm{b}$, obtained from the $B_\mathrm{s}$ meson mass, for different values of $\theta_i$.

**Fig. 5.7** Continuum extrapolations in HQET. Left: $\Phi^{\mathrm{HQET}} = f_{\mathrm{B_s}}\sqrt{m_{\mathrm{B_s}}}/C_{\mathrm{PS}}$ (diamonds) in HQET with $1/m$ corrections included(Blossier *et al.*, 2010*c*) and its static limit $\Phi_{\mathrm{RGI}}$ (circles). The value of $C_{\mathrm{PS}}$ does not depend on the lattice spacing. It renders the two quantities directly comparable. Right: pseudo scalar energy levels (Blossier *et al.*, 2010*a*). From bottom to top: 2s – 1s splitting static, 2s – 1s splitting static + $1/m$, 3s – 1s splitting static.

**Fig. 5.8** Static results together with results with $m_\mathrm{h} < m_\mathrm{b}$ and an HQET computation with $1/m$ corrections included. Continuum extrapolations are done before the interpolation (Blossier *et al.*, 2010*c*). $C_{\mathrm{PS}}$ is evaluated with the three-loop approximation of $\gamma_{\mathrm{match}}$.

The graphs and numbers are for the quenched approximation (the light quark is a strange quark) but these computations are also on the way for dynamical fermions. The statistics employed in the quenched approximation is rather modest: only 100 configurations were analyzed. One can easily use a larger number, even with dynamical fermions. We skip numerical details in the following discussion.

As a first step, one wants to fix the b-quark mass. This is done through eq. (5.30) and its $1/m$ corrections. Its graphical solution is illustrated in Fig. 5.6 where all plotted numbers originate from prior continuum extrapolations. The resulting mass of the b-quark is displayed in Table 5.2. Observe that it depends very little on the matching condition, i.e. the choice of $\theta_0, \theta_1, \theta_2$ and moreover the $1/m$ corrections are small.

Next we look at the lattice spacing dependence of the decay constant. For the results including $1/m$ corrections no significant dependence on $a$ is seen in Fig. 5.7 despite a good precision of about 2%. In static approximation, discretization errors are visible but small. Table 5.3 lists the $B_\mathrm{s}$ decay constant using $r_0 = 0.5\mathrm{fm}$ to convert to MeV for illustration. The actual number is affected by an unknown "quenching effect" and thus not so important. It is more relevant to observe the precision that can be reached with just 100 configurations and how the spread in the numbers in static approximation is reduced when the $1/m$ corrections are included.

| | LO (static) | NLO (static + O($1/m$)) | | |
|---|---|---|---|---|
| | | $(\theta_1, \theta_2) = (0, 0.5)$ | $(\theta_1, \theta_2) = (0.5, 1)$ | $(\theta_1, \theta_2) = (0, 1)$ |
| $\theta_0 = 0$ | $233 \pm 6$ | $220 \pm 9$ | $218 \pm 9$ | $218 \pm 9$ |
| $\theta_0 = 0.5$ | $229 \pm 7$ | $221 \pm 9$ | $219 \pm 8$ | $219 \pm 9$ |
| $\theta_0 = 1$ | $219 \pm 6$ | $223 \pm 9$ | $221 \pm 8$ | $222 \pm 8$ |

**Table 5.3** Pseudo-scalar heavy-light decay constant $f_{\mathrm{B_s}}$ in MeV, for different values of $\theta_i$.

Further, the comparison with results in the charm mass region, Fig. 5.8, seems to indicate that the $1/m$ expansion works very well even for charm quarks. This is a bit surprising and certainly requires further confirmation. Note also that this comparison makes use of the perturbatively evaluated $C_\mathrm{PS}$ whose intrinsic uncertainty due to perturbation theory is difficult to evaluate. Of course this uncertainty does neither affect the non-perturbatively computed static value at $1/(r_0 m_\mathrm{PS}) = 0$, nor $f_{\mathrm{B_s}}\sqrt{m_{\mathrm{B_s}}}$ computed with $1/m$ corrections at the mass of the b-quark, corresponding to $1/(r_0 m_\mathrm{PS}) \approx 0.07$. It only affects the comparison to the results for $1/(r_0 m_\mathrm{PS}) \gtrsim 0.15$ since only for the purpose of this comparison the logarithmic mass dependence described by $C_\mathrm{PS}$ has to be divided out.

Finally we show some results concerning the spectrum. The splitting between radial excitations in the pseudo-scalar sector is displayed in the right part of Fig. 5.7. As throughout in our results, the $1/m$-correction is rather small.

## 5.6 Perspectives

Meanwhile it has been established that HQET with non-perturbatively determined parameters is a precision tool. However, we are still at the beginning concerning appli-

cations. Results for the quantities shown here will be available for $N_{\mathrm{f}} = 2$ dynamical fermions rather soon. But there are many more applications which remain unexplored and open interesting avenues of research for the future.

# Appendix A

## A.1 Notation

### A.1.1 Index conventions

Lorentz indices $\mu, \nu, \ldots$ are taken from the middle of the Greek alphabet and run from 0 to 3. Latin indices $k, l, \ldots$ run from 1 to 3 and are used to label the components of spatial vectors. For the Dirac indices capital letters $A, B, \ldots$ from the beginning of the alphabet are taken. They run from 1 to 4. Color vectors in the fundamental representation of SU($N$) carry indices $\alpha, \beta, \ldots$ ranging from 1 to $N$, while for vectors in the adjoint representation, Latin indices $a, b, \ldots$ running from 1 to $N^2 - 1$ are employed.

Repeated indices are always summed over unless otherwise stated and scalar products are taken with Euclidean metric.

### A.1.2 Dirac matrices

In the *chiral representation* for the Dirac matrices, we have

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix}. \tag{A.1}$$

The $2 \times 2$ matrices $e_\mu$ are taken to be

$$e_0 = -1, \qquad e_k = -i\sigma_k, \tag{A.2}$$

with $\sigma_k$ the Pauli matrices. It is then easy to check that

$$\gamma_\mu^\dagger = \gamma_\mu, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{A.3}$$

Furthermore, if we define $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, we have

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.4}$$

In particular, $\gamma_5 = \gamma_5^\dagger$ and $\gamma_5^2 = 1$. The hermitian matrices

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right] \tag{A.5}$$

are explicitly given by ($\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$)

$$\sigma_{0k} = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \qquad \sigma_{ij} = -\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \equiv -\epsilon_{ijk}\sigma_k, \tag{A.6}$$

where $\epsilon_{ijk}$ is the totally anti-symmetric tensor with $\epsilon_{123} = 1$.

In the *Dirac representation* we have

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.7}$$

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{ij} = -\epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} = \sigma_k, \tag{A.8}$$

### A.1.3 Lattice conventions

Ordinary forward and backward lattice derivatives act on color singlet functions $f(x)$ and are defined through

$$\begin{aligned} \partial_\mu f(x) &= \frac{1}{a}\big[f(x + a\hat{\mu}) - f(x)\big], \\ \partial^*_\mu f(x) &= \frac{1}{a}\big[f(x) - f(x - a\hat{\mu})\big], \end{aligned} \tag{A.9}$$

where $\hat{\mu}$ denotes the unit vector in direction $\mu$. We also use the symmetric derivative

$$\widetilde{\partial}_\mu = \tfrac{1}{2}(\partial_\mu + \partial^*_\mu)\,. \tag{A.10}$$

The gauge covariant derivative operators, acting on a quark field $\psi(x)$, are given by

$$\nabla_\mu \psi(x) = \frac{1}{a}\big[\lambda_\mu U(x,\mu)\psi(x + a\hat{\mu}) - \psi(x)\big], \tag{A.11}$$

$$\nabla^*_\mu \psi(x) = \frac{1}{a}\big[\psi(x) - \lambda_\mu^{-1} U(x - a\hat{\mu},\mu)^{-1}\psi(x - a\hat{\mu})\big]\,, \tag{A.12}$$

with the constant phase factors

$$\lambda_\mu = \mathrm{e}^{ia\theta_\mu/L}, \qquad \theta_0 = 0, \quad -\pi < \theta_k \le \pi\,, \tag{A.13}$$

explained in Sect. 2.6. The left action of the lattice derivative operators is defined by

$$\overline{\psi}(x)\overleftarrow{\nabla}_\mu = \frac{1}{a}\left[\,\overline{\psi}(x + a\hat{\mu})U(x,\mu)^{-1}\lambda_\mu^{-1} - \overline{\psi}(x)\,\right], \tag{A.14}$$

$$\overline{\psi}(x)\overleftarrow{\nabla}^*_\mu = \frac{1}{a}\left[\,\overline{\psi}(x) - \overline{\psi}(x - a\hat{\mu})U(x - a\hat{\mu},\mu)\lambda_\mu\,\right]. \tag{A.15}$$

Our lattice version of $\delta$-functions are

$$\delta(x_\mu) = a^{-1}\delta_{x_\mu 0}\,, \quad \delta(\mathbf{x}) = \prod_{k=1}^{3}\delta(x_k)\,, \quad \delta(x) = \prod_{\mu=0}^{3}\delta(x_\mu) \tag{A.16}$$

and we use

$$\begin{aligned} &\theta(x_\mu) = 1 \text{ for } x_\mu \ge 0 \\ &\theta(x_\mu) = 0 \text{ otherwise} \end{aligned} \tag{A.17}$$

Fields in momentum space are introduced by the Fourier transformation

$$\tilde{f}(p) = a^4 \sum_x \mathrm{e}^{-ipx} f(x) \Leftrightarrow \begin{cases} f(x) = \frac{1}{L^3 T}\sum_p \mathrm{e}^{ipx}\tilde{f}(p) & \text{in a } T \times L^3 \text{ volume} \\ f(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{e}^{ipx}\tilde{f}(p) & \text{in infinite volume} \end{cases} \tag{A.18}$$

### A.1.4 Continuum gauge fields

An SU($N$) gauge potential in the continuum theory is a vector field $A_\mu(x)$ with values in the Lie algebra su($N$). It may thus be written as

$$A_\mu(x) = A^a_\mu(x) T^a \tag{A.19}$$

with real components $A^a_\mu(x)$ and

$$(T^a)^\dagger = -T^a, \quad \mathrm{tr}\,\{T^a T^b\} = -\tfrac{1}{2}\delta^{ab}. \tag{A.20}$$

The associated field tensor,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)], \tag{A.21}$$

may be decomposed similarly and the right and left action of the covariant derivative $D_\mu$ is defined by

$$D_\mu \psi(x) = (\partial_\mu + A_\mu)\psi(x), \tag{A.22}$$

$$\overline{\psi}(x)\overleftarrow{D}_\mu = \overline{\psi}(x)(\overleftarrow{\partial}_\mu - A_\mu). \tag{A.23}$$

We note that periodic boundary conditions up to a phase $\theta_\mu$ are equivalent to adding a constant abelian gauge field $i\theta_\mu/L$: in the above we replace $A_\mu \to A_\mu + i\theta_\mu/L$.

### A.1.5 Lattice action

Let us first assume that the theory is defined on an infinite lattice. A gauge field $U$ on the lattice is an assignment of a matrix $U(x,\mu) \in \mathrm{SU}(N)$ to every lattice point $x$ and direction $\mu = 0,1,2,3$. Quark and anti-quark fields, $\psi(x)$ and $\overline{\psi}(x)$, reside on the lattice sites and carry Dirac, colour and flavour indices. The (unimproved) lattice action is of the form

$$S[U,\overline{\psi},\psi\,] = S_{\mathrm{G}}[U] + S_{\mathrm{F}}[U,\overline{\psi},\psi\,], \tag{A.24}$$

where $S_{\mathrm{G}}$ denotes the usual Wilson plaquette action and $S_{\mathrm{F}}$ the Wilson quark action. Explicitly we have

$$S_{\mathrm{G}}[U] = \frac{1}{g_0^2}\sum_p \mathrm{tr}\,\{1 - U(p)\} = \frac{1}{g_0^2}\sum_x \sum_{\mu,\nu} P_{\mu\nu}(x)\,, \tag{A.25}$$

$$P_{\mu\nu}(x) = U(x,\mu)\,U(x+a\hat{\mu},\nu)\,U(x+a\hat{\nu},\mu)^{-1}\,U(x,\nu)^{-1} \tag{A.26}$$

with $g_0$ being the bare gauge coupling and $U(p)$ the parallel transporter around the plaquette $p$. The sum runs over all *oriented* plaquettes $p$ on the lattice, i.e. independently over $\mu,\nu$. The quark action,

$$S_{\mathrm{F}}[U,\overline{\psi},\psi\,] = a^4 \sum_x \overline{\psi}(x)(D_{\mathrm{W}} + m_0)\psi(x), \tag{A.27}$$

is defined in terms of the Wilson-Dirac operator

$$D_{\mathrm{W}} = \tfrac{1}{2}\left\{\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - a\nabla^*_\mu\nabla_\mu\right\}, \tag{A.28}$$

which involves the gauge covariant lattice derivatives $\nabla_\mu$ and $\nabla^*_\mu$, eq. (A.9), and the bare quark mass *matrix*, $m_0 = \mathrm{diag}(m_{0\mathrm{u}}, m_{0\mathrm{d}}, \ldots)$ .

### A.1.6 Renormalization group functions and invariants

Our RG functions are defined through

$$\mu\frac{\partial\bar{g}}{\partial\mu} = \beta(\bar{g}) \ , \tag{A.29}$$

$$\frac{\mu}{\overline{m}}\frac{\partial\overline{m}}{\partial\mu} = \tau(\bar{g}) \ , \tag{A.30}$$

$$\frac{\mu}{\Phi}\frac{\partial\Phi}{\partial\mu} = \gamma(\bar{g}) \tag{A.31}$$

in terms of running coupling and running quark mass as well as some matrix element $\Phi$ of a (multiplicatively renormalizable) composite field. They have asymptotic expansions

$$\beta(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3\left\{b_0+\bar{g}^2 b_1+\ldots\right\} \ , \tag{A.32}$$
$$b_0 = \tfrac{1}{(4\pi)^2}\big(11-\tfrac{2}{3}N_{\rm f}\big) \ , \quad b_1 = \tfrac{1}{(4\pi)^4}\big(102-\tfrac{38}{3}N_{\rm f}\big) \ ,$$

$$\tau(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^2\left\{d_0+\bar{g}^2 d_1+\ldots\right\} , \qquad d_0 = 8/(4\pi)^2 \ , \tag{A.33}$$

$$\gamma(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^2\left\{\gamma_0+\bar{g}^2\gamma_1+\ldots\right\} \tag{A.34}$$

The integration constants of the solutions to the RGEs define the RG invariants

$$\Lambda = \mu\left(b_0\bar{g}^2\right)^{-b_1/(2b_0^2)}\mathrm{e}^{-1/(2b_0\bar{g}^2)}\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\tfrac{1}{\beta(x)}+\tfrac{1}{b_0x^3}-\tfrac{b_1}{b_0^2x}\right]\right\} \ , \tag{A.35}$$

$$M = \overline{m}\,(2b_0\bar{g}^2)^{-d_0/2b_0}\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\frac{\tau(x)}{\beta(x)}-\frac{d_0}{b_0 g}\right]\right\} \ . \tag{A.36}$$

$$\Phi_{\rm RGI} = \Phi\left[\,2b_0\bar{g}^2\,\right]^{-\gamma_0/2b_0}\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\,\frac{\gamma(x)}{\beta(x)}-\frac{\gamma_0}{b_0x}\,\right]\right\} \tag{A.37}$$

where $\bar{g}\equiv\bar{g}(\mu)$ ... $\Phi\equiv\Phi(\mu)$. We will also us the shorthand notation

$$\frac{\Lambda}{\mu} = \varphi_g(\bar{g}) = \exp\left\{-\int^{\bar{g}}\mathrm{d}x\tfrac{1}{\beta(x)}\right\} , \tag{A.38}$$

$$\frac{M}{\overline{m}} = \varphi_m(\bar{g}) = \exp\left\{-\int^{\bar{g}}\mathrm{d}x\tfrac{\tau(x)}{\beta(x)}\right\} , \tag{A.39}$$

$$\frac{\Phi_{\rm RGI}}{\Phi} = \varphi_\Phi(\bar{g}) = \exp\left\{-\int^{\bar{g}}\mathrm{d}x\tfrac{\gamma(x)}{\beta(x)}\right\} , \tag{A.40}$$

with the constants exactly as defined above.

## A.2 Conversion functions and anomalous dimensions

Conversion functions and the anomalous dimensions $\gamma_{\rm match}$ are not part of the standard phenomenology literature. For completeness we give the explicit relations to the matching coefficients found directly in the literature and discuss the accuracy of their perturbative expansion.

### A.2.1 Matching coefficients and anomalous dimension

We here describe the result (Bekavac *et al.*, 2010) and its relation to the anomalous dimension. We denote a matrix element of some heavy-light quark bilinear $\overline{\psi}\Gamma\psi_{\rm h}$ in the effective theory by $\Phi(\mu)$. The Dirac structure $\Gamma$ is left implicit.[1]

All quantities are renormalized in the $\overline{\rm MS}$-scheme, with a scale $\mu_o$ for the QCD bilinear and a scale $\mu$ in HQET. Choosing the pole quark mass $m_{\rm Q}$,[2] the matrix element is then (without explicit superscripts "QCD" we refer to HQET quantities, in the static approximation),

$$\Phi^{\rm QCD}(m_{\rm Q},\mu_o;\mathcal{V}_{\rm kin}) = \widehat{C}_{\rm match}(m_{\rm Q},\mu_o,\mu)\times\Phi(\mu;\mathcal{V}_{\rm kin}) + {\rm O}(1/m)\,. \tag{A.41}$$

The kinematical variables entering the matrix element $\Phi$ are denoted by $\mathcal{V}_{\rm kin}$. For the (partially) conserved currents $V_\mu, A_\mu$ there is no $\mu_o$-dependence on the l.h.s. of eq. (A.41), $(\partial_{\mu_o}\,\Phi^{\rm QCD}(m_{\rm Q},\mu_o)=0)$, while in general we have

$$\frac{\mu}{\Phi^{\rm QCD}(m_{\rm Q},\mu_o)}\frac{\partial\Phi^{\rm QCD}(m_{\rm Q},\mu_o)}{\partial\mu_o} = \frac{\partial\ln(\Phi^{\rm QCD}(m_{\rm Q},\mu_o))}{\partial\ln(\mu_o)} \equiv \gamma_o(\bar{g}(\mu_o))\,. \tag{A.42}$$

We pass to the RGI matrix element in QCD via (O$(1/m)$ is dropped without notice)

$$\Phi^{\rm QCD}_{\rm RGI} = \exp\left\{-\int^{\bar{g}(\mu_o)}{\rm d}x\frac{\gamma_o(x)}{\beta(x)}\right\}\,\Phi^{\rm QCD}(m_{\rm Q},\mu_o;\mathcal{V}_{\rm kin}) \tag{A.43}$$

$$= \exp\left\{-\int^{\bar{g}(\mu_o)}{\rm d}x\frac{\gamma_o(x)}{\beta(x)}\right\}\,\widehat{C}_{\rm match}(m_{\rm Q},\mu_o,\mu)\times\Phi(\mu;\mathcal{V}_{\rm kin})\,. \tag{A.44}$$

(O$(1/m)$ is dropped without notice). It depends on the quark mass but not on a renormalization scale. The physical anomalous dimension is given by

$$\gamma_{\rm match}(g_\star) = \frac{{\rm d}\ln(m_{\rm Q})}{{\rm d}\ln(m_\star)}\frac{\partial\ln(\widehat{C}_{\rm match}(m_{\rm Q},\mu_o,\mu))}{\partial\ln(m_{\rm Q})}\,, \tag{A.45}$$

where the first factor is computed from the expansion (Gray *et al.*, 1990, Fleischer *et al.*, 1999, Melnikov and Ritbergen, 2000, Bekavac *et al.*, 2010)

$$m_{\rm Q} = m_\star\,[1+\sum_{l\geq1}k_l[\bar{a}(m_\star)]^l]\,,\,, \quad \bar{a}(\mu)=\frac{\bar{g}^2(\mu)}{4\pi^2} \tag{A.46}$$

$$k_1 = 4/3\,,\quad k_2 = -1.0414(N_{\rm f}-1)+13.4434\,,$$

$$k_3 = 0.6527(N_{\rm f}-1)^2 - 26.655(N_{\rm f}-1)+190.595\,.$$

[1] The notation $C_{\tilde{\Gamma}}$ of (Broadhurst and Grozin, 1995) translates to our $\Gamma$ as $\tilde{\Gamma}=(1,\,\gamma_0,\,\gamma_1,\,\gamma_0\gamma_1)\to\Gamma=(\gamma_5,\,\gamma_0\gamma_5,\,\gamma_k,\,\gamma_0\gamma_k)$ and (Bekavac *et al.*, 2010) uses the notation of (Broadhurst and Grozin, 1995) when one sets $v_\mu\gamma_\mu=\gamma_0$, $\gamma_\perp=\gamma_k$ as it is the case in the rest frame. We will also refer to the bilinears as $({\rm PS}, A_0, V_k, {\rm T})$. In comparison to (Bekavac *et al.*, 2010) we add a subscript Q to the pole quark mass and a bar to the running mass $(m\to m_{\rm Q},\ m(\mu)\to\overline{m}(\mu))$ for clarity.

[2] While in the complete, non-perturbative theory, the pole mass is ill-defined, in perturbation theory it exists order by order in the expansion. We use it here, because the formulae in the literature are written in terms of it. It will be eliminated in the final formulae.

The authors of Ref. (Bekavac *et al.*, 2010) set $\mu_o = \mu$. Building on (Ji and Musolf, 1991, Broadhurst and Grozin, 1995, Gimenez, 1992), they give the perturbative expansion

$$\widehat{C}_{\rm match}(m_{\rm Q},\mu,\mu) = 1 + \sum_{l\geq 1}\sum_{k=0}^{l} L_{lk}[\ln(m_{\rm Q}^2/\mu^2)]^k\,[\bar{a}(m_{\rm Q})]^l\,, \tag{A.47}$$

with coefficients $L_{lk}$ depending on the Dirac-structure, $\Gamma$.

Independence of the l.h.s. of eq. (A.41) of $\mu$ yields

$$\frac{\partial\ln(\widehat{C}_{\rm match}(m_{\rm Q},\mu_o,\mu))}{\partial\ln(\mu)} = -\frac{\partial\ln(\Phi(\mu))}{\partial\ln(\mu)} = -\gamma^{\rm stat}(\bar{g}(\mu))\,, \tag{A.48}$$

and with $\frac{\partial\ln(\widehat{C}_{\rm match}(m_{\rm Q},\mu_o,\mu))}{\partial\ln(\mu_o)} = \gamma_o(\bar{g}(\mu_o))$ we have

$$\begin{aligned}&\frac{\mathrm{d}\ln(\widehat{C}_{\rm match}(m_{\rm Q},m_{\rm Q},m_{\rm Q}))}{\mathrm{d}\ln(m_{\rm Q})}\\ &\quad= \left.\frac{\partial\ln((\widehat{C}_{\rm match}(m_{\rm Q},\mu_o,\mu))}{\partial\ln(m_{\rm Q})}\right|_{\mu_o=\mu=m_{\rm Q}} + \gamma_o(\bar{g}(m_{\rm Q})) - \gamma_{\rm stat}(\bar{g}(m_{\rm Q}))\,.\end{aligned} \tag{A.49}$$

From these equations $\gamma_{\rm match}(g_\star)$ can be determined up to three-loop order and the differences $\gamma^{\Gamma'}_{\rm match}(g_\star) - \gamma^{\Gamma}_{\rm match}(g_\star)$ up to four-loop order.

### A.2.2 Numerical results and the behavior of perturbation theory

Let us now look at the numerical size of the perturbative coefficients of the RG functions. The following table lists results for $N_{\rm f} = 3$. This is enough to understand the general picture since for smaller $N_{\rm f}$ the higher order coefficients are generically somewhat larger, but not by much.

| coefficient | $i=1$ | $i=2$ | $i=3$ | $i=4$ |
|---|---|---|---|---|
| $(4\pi)^i b_{i-1}$ | 0.71620 | 0.40529 | 0.32445 | 0.47367 |
| $(4\pi)^i d_{i-1}$ | 0.63662 | 0.76835 | 0.80114 | 0.90881 |
| $(4\pi)^i\,\gamma_{{\rm stat},i-1}$ | -0.31831 | -0.26613 | -0.25917 | |
| $(4\pi)^i\,\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1}$ | -0.31831 | -0.57010 | -0.94645 | |
| $(4\pi)^i\,\gamma^{\gamma_k}_{{\rm match},i-1}$ | -0.31831 | -0.87406 | -3.12585 | |
| $(4\pi)^i\,[\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1} - \gamma^{\gamma_k}_{{\rm match},i-1}]$ | 0 | 0.30396 | 2.17939 | 14.803 |
| $(4\pi)^i\,[\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1} - \gamma^{\gamma_5}_{{\rm match},i-1}]$ | | | | |

The normalization $(4\pi)^i$ has been inserted such that the series is well behaved for $\alpha \lesssim 1/3$ if the coefficients are order one. Indeed this is the magnitude of the coefficients in the first three rows which show as a comparison the beta-function, mass anomalous dimension and the anomalous dimension of the static-light bilinears (all in the $\overline{\rm MS}$–scheme). In contrast in the physical anomalous dimension of the vector current $\gamma^{\gamma_k}_{\rm match}$, the 3-loop coefficient is rather big and the difference $(4\pi)^i\,[\gamma^{\gamma_0\gamma_5}_{{\rm match},3} - \gamma^{\gamma_k}_{{\rm match},3}]$ is even

above ten. Perturbation theory is then useful only at rather small $\alpha$; in particular not really for the b-quark.

An attempt to improve the perturbative series is to re-expand $\gamma_{\rm match}$ in the coupling at a different scale, adjusting the scale to obtain smaller coefficients. In fact, since the effective theory is valid at energy scales below the mass of the quark, it is plausible that scales smaller than $m_\star$ are more suitable. So we choose a coupling $\hat{g}^2 = \bar{g}^2(s^{-1}m_\star) = \sigma(g_\star^2, s)$ and

$$\hat{\gamma}_{\rm match}(\hat{g}) = \gamma_{\rm match}([\sigma(\hat{g}^2, 1/s)]^{1/2})\,, \tag{A.50}$$

which is of course expanded order by order,

$$g_\star^2 = \sigma(\hat{g}^2, 1/s) = \hat{g}^2 - 2b_0 \ln(s)\, \hat{g}^4 + \dots\,. \tag{A.51}$$

The conversion functions are then expressed as

$$C_{\rm PS}(M/\Lambda) = \exp\left\{ \int^{\hat{g}} {\rm d}x \frac{\hat{\gamma}_{\rm match}(x)}{\beta(x)} \right\}. \tag{A.52}$$

The difference comes from truncating eq. (A.50) as a series in $\hat{g}^2$. The argument above suggests $s > 1$. The perturbative coefficients are listed in the following table for a few choices of $s$, for example the one which brings the two-loop coefficient $\gamma_1$ to zero.

| coefficient | $i=1$ | $i=2$ | $i=3$ | $i=4$ | $s$ |
|---|---|---|---|---|---|
| $(4\pi)^i\,\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1}$ | -0.31831 | -0.57010 | -0.94645 | | 1 |
| | -0.31831 | 0 | 0.39720 | | 3.4916 |
| $(4\pi)^i\,\gamma^{\gamma_k}_{{\rm match},i-1}$ | -0.31831 | -0.87406 | -3.12585 | | 1 |
| | -0.31831 | 0 | -0.231121 | | 6.8007 |
| $(4\pi)^i\,[\gamma^{\gamma_0\gamma_5}_{{\rm match},i-1} - \gamma^{\gamma_k}_{{\rm match},i-1}]$ | 0 | 0.30396 | 2.17939 | 14.803 | 1 |
| | 0 | 0.30396 | 0.972221 | 4.733 | 4 |
| | 0 | 0.30396 | -0.05414 | 1.82678 | 13 |
| | 0 | 0.30396 | -0.23495 | 1.85344 | 16 |

The higher order coefficients can indeed be reduced significantly but $s \gtrsim 4$ is required. For B-physics $\alpha(m_{\star{\rm b}}/s)$ is then not small and there is no really useful improvement for phenomenology, see Fig. 3.2. We emphasize, however, that with $s \approx 4$ the series is much better behaved for masses that are a factor two or more higher than the b-quark mass. The pattern visible in in the tables reflects itself in Fig. 3.2.

Let us finally mention that the same behavior is found for $\hat{C}_{\rm match}(m_{\rm Q}, m_{\rm Q}, m_{\rm Q})$ for all Dirac structures of the currents. Their perturbative expansion in a coupling $\bar{g}(m_{\rm Q}/s)$ is better behaved for $s \gtrsim 4$ than for $s = 1$.

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