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# A complete algebraic reduction of one-loop tensor Feynman integrals

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## Abstract

We set up a new, flexible approach for the tensor reduction of one-loop Feynman integrals. The 5-point tensor integrals up to rank $R = 5$ are expressed by 4-point tensor integrals of rank $R-1$, such that the appearance of the inverse 5-point Gram determinant is avoided. The 4-point tensor coefficients are represented in terms of 4-point integrals, defined in $d$ dimensions, $4-2\varepsilon \leq d \leq 4-2\varepsilon+2(R-1)$, with higher powers of the propagators. They can be further reduced to expressions which stay free of the inverse 4-point Gram determinants but contain higher-dimensional 4-point integrals with only the first power of scalar propagators, plus 3-point tensor coefficients. A direct evaluation of the higher dimensional 4-point functions would avoid the appearance of inverse powers of the Gram determinants completely. The simplest approach, however, is to apply here dimensional recurrence relations in order to reduce them to the familiar 2- to 4-point functions in generic dimension $d = 4-2\varepsilon$, introducing thereby coefficients with inverse 4-point Gram determinants up to power $R$ for tensors of rank $R$. For small or vanishing Gram determinants – where this reduction is not applicable – we use analytic expansions in positive powers of the Gram determinants. Improving the convergence of the expansions substantially with Padé approximants we close up to the evaluation of the 4-point tensor coefficients for larger Gram determinants. Finally, some relations are discussed which may be useful for analytic simplifications of Feynman diagrams.

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# Contents


**1 Introduction** **4**

**2 Tensor integrals in terms of integrals in shifted dimensions** **6**

**3 An efficient reduction of 5-point tensor integrals** **10**

**4 Reduction of 5-point tensor coefficients** **14**
4.1 Scalar and vector integrals . . . . . 14
4.2 The recursion formulae . . . . . 15
4.3 The tensor integral of rank $R = 2$ . . . . . 16
4.4 Reduction of integrals with rank $R = 3$ . . . . . 17
4.5 Reduction of integrals with rank $R = 4$ . . . . . 19
4.6 Reduction of integrals with rank $R = 5$ . . . . . 21

**5 Calculation of higher-dimensional 4-point functions** **23**
5.1 Difference quotients with $1/()_4$ . . . . . 24
5.2 Reduction of the difference quotients . . . . . 26
5.3 Expansion of $I_4^{[d+]^L}$ for small Gram determinants . . . . . 30

**6 Symmetrized recurrence relations** **34**

**7 Analytic simplifications for contractions of tensors with chords** **37**

**8 Conclusions** **40**

**A Recursion relations for the reduction of higher-dimensional 4-point functions** **41**
A.1 Reduction of 4-point integrals . . . . . 41
A.2 Reduction of 3-point integrals . . . . . 42
A.3 Reduction of 2-point integrals . . . . . 42

**B Divergent parts of higher-dimensional integrals** **46**

**C A numerical example** **47**

**D Notations and algebraic relations** **54**

**References** **55**

# List of Figures

2.1 Momenta flow of the $n$-point function. . . . . 6
C.1 (a) A six-point topology; (b) a four-point topology derived from (a). . . . . 49
C.2 The tensor coefficient $D_{1111}(x)$. . . . . 50

# List of Tables

C.1 Numerical values for the tensor coefficient $D_{1111}$. . . . . 52
C.2 Numerical values for the tensor coefficient $D_{111}$. . . . . 53

# 1 Introduction

The evaluation of tensorial Feynman integrals with $n$ external legs is an important technical ingredient of perturbative quantum field theoretical calculations with Feynman diagrams. They are needed in particular for the fast and efficient numerical evaluation of next-to-leading order contributions at high energy colliders. Special attention is concentrated these days on experiments performed at the LHC; for a snapshot on related activities see [1]. Of course, there is an unlimited variety of other reasons to use tensor reductions of Feynman integrals with quite diverse requirements in detail, and a unique all-purpose, final approach does not exist. For a recent overview see [2] and references therein.

The first systematic approach to reduce tensor components to a basis of scalar 1-point to 4-point integrals in generic dimension $d = 4 - 2\varepsilon$ for $n \leq 4$ is the Passarino-Veltman reduction [3], obtained by solving a system of linear equations. In fact, this is a unique basis.

We use, with some sophistication, Davydychev's approach [4], where $n$-point tensor coefficients are represented in terms of scalar Feynman integrals. For tensors of rank $R$ they are defined in space-time dimensions up to $4 - 2\varepsilon + 2R$, with an additional modification: propagators may appear with higher powers. These integrals are complicated objects, and an important step towards their evaluation is the application of dimensional recurrence relations, derived for $L$-loop functions in [5]. They have been systematically worked out for $L = 1$ in [6], and in a subsequent article [7] the evaluation of scalar integrals in $d$ dimensions with powers 1 of the scalar propagators is advocated. Alternatively, the straightforward derivation of representations in the generic dimension $4 - 2\varepsilon$, or finally just in four dimensions, by means of recurrence relations introduces coefficients containing inverse Gram determinants, which may become small in some kinematical domains and thus raise numerical problems. For $n \leq 4$ this problem was not very severe [8]. Serious problems arise, however, for $n$-point Feynman integrals with $n \geq 5$. In that case the choice of the tensor basis is not unique and the freedom may be used to completely avoid the appearance of inverse Gram determinants for $n \geq 5$ [9, 10, 11, 12, 13, 14, 15]. On the contrary, for $n < 5$ one has to find explicit methods to stabilize the numerics for vanishing or small Gram determinants.

In view of these facts, one may wonder if the approach of Davydychev can be worked out with an optimization of the handling of exceptional (small or vanishing) Gram determinants. This is what has been achieved in the present work.

For non-exceptional kinematics the $g^{\mu\nu}$ tensor, considering 5-point functions, is redundant and may be expressed in terms of 4-momenta. A nicely compact algebraic result is obtained due to this ansatz after applying *symmetrized* dimensional recurrences , but inverse Gram determinants of 5-point as well as of 4-point functions are introduced. In an earlier attempt [16, 17, 18], tensor ranks until $R = 3$ were presented *without* inverse Gram determinants of the 5-point function using the *algebra of the signed minors* [19], but a generalization to higher ranks was not evident. In the present work we first apply a particular recursion, which was obtained in [21]. It reduces the tensor rank from $R$ to $R - 1$ and the further reduction can be arranged in a systematic manner without introducing inverse 5-point Gram determinants. The 4-point tensor coefficients, which are in fact, due to [4], higher-dimensional 4-point functions with higher powers of the scalar propagators, were reduced by lengthy algebraic calculations to higher-dimensional 4-point functions with powers 1 of the scalar propgators plus tensor coefficients of 3-point functions. The latter higher-dimensional 4-point functions are tensor coefficients of the $g^{\mu_1\mu_2} \ldots g^{\mu_{2l-1}\mu_{2l}}$ terms of 4-point tensors. At this stage inverse 4-point Gram determinants are avoided completely. The recursions of all remaining 3-point functions may be performed simply á la [6]. This is the way the numerics for this article was performed. Nevertheless, the same approach as above can also be applied to the 3-point functions, leaving only higher dimensional 3-point functions with powers 1 of the scalar propagators, thereby avoiding inverse Gram determinants of the 3-point function.

Finally, one has to calculate the higher dimensional 4-point functions with powers 1 of the scalar propagators. This may be done by direct evaluation, see e.g. [22], or by further reduction to simpler integrals, which in general, however, introduces inverse powers of 4-point Gram determinants. For small Gram determinants we therefore derive a relatively simple *analytic expansion* in positive powers of the Gram determinant. Such an infinite series was, to our knowledge, first proposed in equation (36) of [7], but was not numerically applied so far. In fact, this expansion applies for higher-dimensional integrals with powers 1 of the scalar propagators only and would not be appropriate for a representation of tensor coefficients in their original form.

Another approach to the problem of small Gram determinants was chosen in sect. 5.4 of [23], where relations between different 3- and 4-point tensor coefficients are exploited and the full set is calculated with increasing iterations. The $n^{th}$ iteration requires all 3-point coefficients of rank $n$. In contrary to that our expansion is concerned only with the subset of the $g^{\mu_1\mu_2}\cdots g^{\mu_{2l-1}\mu_{2l}}$ coefficients of the 4-point tensors, which are approximated only by the corresponding subset of the 3-point tensor coefficients. Indeed, our series expansion of the higher dimensional 4-point functions is useful only since these are embedded in expressions which are already free of inverse Gram determinants .

In [7] it was shown that the integrals under consideration can be expressed in terms of multiple hypergeometric functions. One can apply their series expansion or, alternatively, one could use their representations in terms of 1-dimensional integrals also given in [7]. These, in general, present the integrals in rather different domains of phase space, including the case of small Gram determinants of the 4-point function. Thus our approach offers a variety of options to adjust to the given kinematical situation. For the time being we only use our series expansion applying Padé approximants. This turns out to be very efficient and allows to obtain high precision for the numerical values of the tensor coefficients of the 4-point function. In an example, we demonstrate that the combination of representations for non-exceptional kinematics with this expansion covers the complete phase-space from medium to vanishing Gram determinants.

In view of the importance of stable numerics for tensor reductions, it would be welcome to have one or more complete opensource programs for this task, including the treatment of small Gram determinants. To our knowledge, none is presently available. Following the approach of this article, a C++ program is under development to close this gap [24].

The article is organized as follows. In sect. 2, some definitions and basic formulae are recalled. Sect. 3 describes a compact analytical tensor reduction of 5-point functions with non-exceptional kinematics. In sect. 4 the 5-point functions up to rank $R = 5$ are reduced to 4-point tensor coefficients in terms of 4-point integrals in higher dimensions and with higher powers of the scalar propagators. In sect. 5, we reduce these 4-point integrals in several steps. In sect. 5.2 the integrals are reduced to 4-point integrals in higher dimensions with powers 1 of the scalar propagators plus 3-point tensor coefficients. The results are given in eqns. (5.10), (5.15), (5.19) and (5.21). Indeed, these eqns. are the central point of our approach since they allow to proceed further in different directions. One might e.g. apply the general method of calculating higher dimensional integrals of [7]. With app. A, alternatively a reduction to the Passarino-Veltman basis is straightforward. In subsect. 5.3, we recall how to expand the higher-dimensional integrals with powers 1 of the scalar propagators for the case of vanishing or small Gram determinants, the result being given in (5.46). The symmetrized recursion relations, useful for the 5-point functions with non-exceptional Gram determinants are presented in sect. 6. Additionally, in sect. 7 we give some relations which may be useful for an analytic simplification of Feynman diagrams. We end with conclusions in sect. 8. Appendix A contains a list of dimensional recurrences and app. B collects divergent parts of higher-dimensional integrals. A numerical example is discussed in app. C. Appendix D contains notations and some relevant algebraic relations.

Figure 2.1: Momenta flow of the $n$-point function.

# 2 Tensor integrals in terms of integrals in shifted dimensions

A tensorial Feynman integral with $n$ external legs is shown in fig. 2.1 and is defined as

$$I^{\mu_1\cdots\mu_R}_{n,\{\nu_j\}} = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\ \frac{\prod_{r=1}^R k^{\mu_r}}{\prod_{j=1}^n c_j^{\nu_j}}, \tag{2.1}$$

with denominators $c_j$, having *indices* $\nu_j$ and *chords* $q_j$,

$$c_j = (k-q_j)^2 - m_j^2 + i\varepsilon. \tag{2.2}$$

Here, we use the generic dimension $d = 4-2\varepsilon$ and $\mu = 1$. Reducing the tensors to 1- to 4-point scalar functions $I^d_n$, in general their expansions in terms of $\varepsilon$ is needed. The first expansion terms can be expressed in terms of Euler dilogarithmic (or simpler) functions [26, 27, 28, 25, 29].

The six-point tensor integrals may be expressed in terms of five-point tensor functions [6, 17, 30, 23]:

$$I_6^{\mu_1\ldots\mu_{R-1}\mu} = -\sum_{s=1}^6 I_5^{\mu_1\ldots\mu_{R-1},s}\bar{Q}_s^\mu, \tag{2.3}$$

where the auxiliary vectors $\bar{Q}_s$ are

$$\bar{Q}_s^\mu = \sum_{i=1}^6 q_i^\mu \frac{\binom{0s}{0i}_6}{\binom{0}{0}_6} \quad , \quad s = 1\ldots6. \tag{2.4}$$

A similar formula exists also for five-point tensor integrals [21]:

$$I_5^{\mu_1\ldots\mu_{R-1}\mu} = I_5^{\mu_1\ldots\mu_{R-1}}Q_0^\mu - \sum_{s=1}^5 I_4^{\mu_1\ldots\mu_{R-1},s}Q_s^\mu. \tag{2.5}$$

The auxiliary vectors here are:

$$Q_s^\mu = \sum_{i=1}^5 q_i^\mu \frac{\binom{s}{i}_5}{()_5}, \quad s = 0,\ldots,5. \tag{2.6}$$

For later use, we introduce also

$$\begin{aligned} Q_s^{t,\mu} &= \sum_{i=1}^{5} q_i^\mu \frac{\binom{st}{it}_5}{\binom{t}{t}_5} \\ &= Q_s^\mu - \frac{\binom{s}{t}_5}{\binom{t}{t}_5} Q_t^\mu, \end{aligned} \tag{2.7}$$

$$\begin{aligned} Q_s^{tu,\mu} &= \sum_{i=1}^{5} q_i^\mu \frac{\binom{stu}{itu}_5}{\binom{tu}{tu}_5} \\ &= Q_s^{u,\mu} - \frac{\binom{su}{tu}_5}{\binom{tu}{tu}_5} Q_t^{u,\mu}, \end{aligned} \tag{2.8}$$

and

$$g^{\mu\nu} = 2 \sum_{i,j=1}^{5} \frac{\binom{i}{j}_5}{()_5} q_i^\mu q_j^\nu. \tag{2.9}$$

In fact, (2.3) is essentially the same formula as (2.5), except that $()_5$ is replaced by $\binom{0}{0}_6$ etc. and $\binom{00}{0i}_6 = 0$. With the definition

$$Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \tag{2.10}$$

the *modified Cayley determinant* of a topology with internal lines $1 \cdots n$ becomes

$$()_n \equiv \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & Y_{11} & Y_{12} & \dots & Y_{1n} \\ 1 & Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}. \tag{2.11}$$

One chord may be chosen arbitrarily to vanish, $q_n = 0$, and then this object is the Gram determinant: [1]

$$()_n|_{q_n=0} = -\det G_{n-1}, \tag{2.12}$$

$$G_{n-1,ik} = 2q_i q_k, \;\; i,k = 1,\dots n-1. \tag{2.13}$$

The Gram determinant is independent of the internal masses. The *signed minors* [19] are denoted as follows:

$$\begin{pmatrix} j_1 & j_2 & \cdots j_m \\ k_1 & k_2 & \cdots k_m \end{pmatrix}_n. \tag{2.14}$$

They are determinants, labeled by those rows $j_1, j_2, \cdots j_m$ and columns $k_1, k_2, \cdots k_m$ which have been excluded from the definition of the Gram determinant $()_n$, with sign

$$\text{sign} \begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_n = (-1)^{j_1+j_2+\cdots+j_m+k_1+k_2+\cdots+k_m} \cdot S(j_1, j_2 \cdots j_m) \cdot S(k_1, k_2, \cdots k_m). \tag{2.15}$$

[1] Usually we will use indices $s,t,\cdots = 1,\cdots,n$ for labelling internal lines, and indices $i,j,\cdots = 1,\cdots,n-1$ for labelling the (non-vanishing) chords.

Here $S(j_1, j_2 \cdots j_m)$ gives the sign of permutations needed to place the indices in increasing order. [2] We have e.g.

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{12} & Y_{22} & \dots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_n . \tag{2.16}$$

Applying Davydychev's method [4], one expresses the tensor integrals $I_n^{\mu_1 \cdots \mu_R}$ by scalar Feynman integrals $I^{(d)}_{n,i^\cdots}$ in higher dimensions $d$ and with higher indices $\nu_i$. We reproduce here integrals with rank $R \leq 5$:

$$\begin{aligned} I_n^\mu &= \int \frac{d^d k}{i\pi^{d/2}} k^\mu \prod_{j=1}^n c_j^{-1} \\ &= -\sum_{i=1}^n q_i^\mu \, I_{n,i}^{[d+]}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} I_n^{\mu\,\nu} &= \int \frac{d^d k}{i\pi^{d/2}} k^\mu \, k^\nu \prod_{j=1}^n c_j^{-1} \\ &= \sum_{i,j=1}^n q_i^\mu \, q_j^\nu \, n_{ij} \, I_{n,ij}^{[d+]^2} - \frac{1}{2} g^{\mu\nu} \, I_n^{[d+]}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} I_n^{\mu\,\nu\,\lambda} &= \int \frac{d^d k}{i\pi^{d/2}} k^\mu \, k^\nu \, k^\lambda \prod_{j=1}^n c_j^{-1} \\ &= -\sum_{i,j,k=1}^n q_i^\mu \, q_j^\nu \, q_k^\lambda \, n_{ijk} \, I_{n,ijk}^{[d+]^3} + \frac{1}{2} \sum_{i=1}^n g^{[\mu\nu} q_i^{\lambda]} I_{n,i}^{[d+]^2}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} I_n^{\mu\,\nu\,\lambda\,\rho} &= \int \frac{d^d k}{i\pi^{d/2}} k^\mu \, k^\nu \, k^\lambda \, k^\rho \prod_{j=1}^n c_j^{-1} \\ &= \sum_{i,j,k,l=1}^n q_i^\mu \, q_j^\nu \, q_k^\lambda \, q_l^\rho \, n_{ijkl} \, I_{n,ijkl}^{[d+]^4} - \frac{1}{2} \sum_{i,j=1}^n g^{[\mu\nu} q_i^\lambda \, q_j^{\rho]} \, n_{ij} I_{n,ij}^{[d+]^3} + \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]} I_n^{[d+]^2}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} I_n^{\mu\,\nu\,\lambda\,\rho\,\sigma} &= \int \frac{d^d k}{i\pi^{d/2}} k^\mu \, k^\nu \, k^\lambda \, k^\rho \, k^\sigma \prod_{j=1}^n c_j^{-1} \\ &= -\sum_{i,j,k,l,m=1}^n q_i^\mu \, q_j^\nu \, q_k^\lambda \, q_l^\rho \, q_m^\sigma n_{ijklm} \, I_{n,ijklm}^{[d+]^5} + \frac{1}{2} \sum_{i,j,k=1}^n g^{[\mu\nu} q_i^\lambda \, q_j^\rho \, q_k^{\sigma]} \, n_{ijk} I_{n,ijk}^{[d+]^4} \\ &\quad - \frac{1}{4} \sum_{i=1}^n g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} I_{n,i}^{[d+]^3}. \end{aligned} \tag{2.21}$$

[2] The definitions are related to similar ones used in the literature, see app. D.

The following symmetrized tensors are used:

$$
\begin{aligned}
g^{[\mu\nu}q_i^{\lambda]} &= g^{\mu\nu}\,q_i^{\lambda} + g^{\mu\lambda}\,q_i^{\nu} + g^{\nu\lambda}\,q_i^{\mu}, && (2.22)\\
g^{[\mu\nu}q_i^{\lambda}q_j^{\rho]} &= g^{\mu\nu}\,q_i^{\lambda}\,q_j^{\rho} + g^{\mu\lambda}\,q_i^{\nu}\,q_j^{\rho} + g^{\nu\lambda}\,q_i^{\mu}\,q_j^{\rho} + g^{\mu\rho}\,q_i^{\nu}\,q_j^{\lambda} + g^{\nu\rho}\,q_i^{\mu}\,q_j^{\lambda} + g^{\lambda\rho}\,q_i^{\mu}\,q_j^{\nu}, && (2.23)\\
g^{[\mu\nu}g^{\lambda\rho]} &= g^{\mu\nu}\,g^{\lambda\rho} + g^{\mu\lambda}\,g^{\nu\rho} + g^{\mu\rho}\,g^{\nu\lambda}, && (2.24)\\
g^{[\mu\nu}q_i^{\lambda}q_j^{\rho}q_k^{\sigma]} &= g^{\mu\nu}q_i^{\lambda}q_j^{\rho}q_k^{\sigma} + g^{\mu\lambda}q_i^{\nu}q_j^{\rho}q_k^{\sigma} + g^{\mu\rho}q_i^{\nu}q_j^{\lambda}q_k^{\sigma} + g^{\mu\sigma}q_i^{\nu}q_j^{\lambda}q_k^{\rho} + g^{\nu\lambda}q_i^{\mu}q_j^{\rho}q_k^{\sigma}\\
&\quad + g^{\nu\rho}q_i^{\mu}q_j^{\lambda}q_k^{\sigma} + g^{\nu\sigma}q_i^{\mu}q_j^{\lambda}q_k^{\rho} + g^{\lambda\rho}q_i^{\mu}q_j^{\nu}q_k^{\sigma} + g^{\lambda\sigma}q_i^{\mu}q_j^{\nu}q_k^{\rho} + g^{\rho\sigma}q_i^{\mu}q_j^{\nu}q_k^{\lambda}, && (2.25)\\
g^{[\mu\nu}g^{\lambda\rho}q_i^{\sigma]} &= g^{[\mu\nu}g^{\lambda\rho]}q_i^{\sigma} + g^{[\mu\nu}g^{\lambda\sigma]}q_i^{\rho} + g^{[\mu\nu}g^{\rho\sigma]}q_i^{\lambda} + g^{[\mu\sigma}g^{\lambda\rho]}q_i^{\nu} + g^{[\nu\sigma}g^{\lambda\rho]}q_i^{\mu}. && (2.26)
\end{aligned}
$$

The scalar integrals are:

$$
I^{[d+]^l,stu\cdots}_{p,ijk\cdots} = \int \frac{d^{[d+]^l}k}{i\pi^{[d+]^l/2}} \prod_{r=1}^{n} \frac{1}{c_r^{1+\delta_{ri}+\delta_{rj}+\delta_{rk}+\cdots-\delta_{rs}-\delta_{rt}-\delta_{ru}-\cdots}}, \tag{2.27}
$$

where $[d+]^l = 4-2\varepsilon+2l$. The index $p$ is the number of propagators of the $p$-point function. Note that equal lower and upper indices cancel. The coefficients $n_{ij}, n_{ijk}$ and $n_{ijkl}$ etc. in (2.18) to (2.21) were introduced in [17]. They stand for the product of factorials of the number of equal indices: e.g. $n_{iiii} = 4!, n_{ijii} = 3!, n_{iijj} = 2!2!, n_{ijkk} = 2!, n_{ijkl} = 1!$; the indices $i, j, k, l$ are assumed here to be different from each other. The following relations are of particular relevance for the successive application of recurrence relations to reduce higher-dimensional integrals:

$$
\begin{aligned}
n_{ij} &= \nu_{ij},\\
n_{ijk} &= \nu_{ij}\nu_{ijk},\\
n_{ijkl} &= \nu_{ij}\nu_{ijk}\nu_{ijkl},\\
n_{ijklm} &= \nu_{ij}\nu_{ijk}\nu_{ijkl}\nu_{ijklm},
\end{aligned} \tag{2.28}
$$

and:

$$
\begin{aligned}
\nu_{ij} &= 1+\delta_{ij},\\
\nu_{ijk} &= 1+\delta_{ik}+\delta_{jk},\\
\nu_{ijkl} &= 1+\delta_{il}+\delta_{jl}+\delta_{kl}.\\
\nu_{ijklm} &= 1+\delta_{im}+\delta_{jm}+\delta_{km}+\delta_{lm}.
\end{aligned} \tag{2.29}
$$

In a second step, one may choose to express the higher-dimensional scalar integrals in terms of the generic scalar integrals The algorithm is based on recurrence relations with shifts of dimension $d \geq 4-2\varepsilon$ and indices $\nu_s \geq 1$,

$$
()_n\,\nu_s\left(\mathbf{s}^+ I_n^{(d+2)}\right) = -\binom{s}{0}_n I_n^{(d)} + \sum_{t=1}^{n}\binom{s}{t}_n \left(\mathbf{t}^- I_n^{(d)}\right), \tag{2.30}
$$

or with a shift of dimension $d$:

$$
()_n\,(d-\sum_{s=1}^{n}\nu_s+1) I_n^{(d+2)} = \binom{0}{0}_n I_n^{(d)} - \sum_{t=1}^{n}\binom{0}{t}_n \left(\mathbf{t}^- I_n^{(d)}\right). \tag{2.31}
$$

These relations hold for arbitrary index sets $\{\nu_s\}$. The integrals $\mathbf{s}^+ I_n^{(d)}$ and $\mathbf{t}^- I_n^{(d)}$ are obtained from $I_n^{(d)}$ by replacing $\nu_s \to (\nu_s+1)$ and $\nu_t \to (\nu_t-1)$, respectively. For more explicit expressions see app. A .

# 3 An efficient reduction of 5-point tensor integrals

The reduction of 5-point tensor integrals to 4-point tensor integrals at non-exceptional momenta may be performed by iterative application of (2.5). This was exemplified in [21], and an opensource Fortran code `olotic` [31] is available. In this sect., we derive a very compact, explicit representation of the tensor coefficients for 5-point functions in a minimal basis, chosen to be free of the metric tensor. This will rely on an exploitation of (2.9) and (3.9), a specifically useful relation of the *algebra of the signed minors*, and applying the *symmetrized* dimensional recurrences of sect. 6

We investigate the 5-point tensor integrals step by step. For the tensor of rank $R = 1$ we get from (2.5)

$$I_5^\mu = I_5 \cdot Q_0^\mu - \sum_{s=1}^{5} I_4^s \cdot Q_s^\mu, \tag{3.1}$$

and $I_5$ may be taken from (4.2).

Similarly for the tensor of rank 2,

$$I_5^{\mu\nu} = I_5^\mu \cdot Q_0^\nu - \sum_{s=1}^{5} I_4^{\mu,s} \cdot Q_s^\nu, \tag{3.2}$$

with four-point integrals from (2.17) and (A.6),

$$I_4^{\mu,s} = -\sum_{i=1}^{5} q_i^\mu I_{4,i}^{[d+],s}, \tag{3.3}$$

$$I_{4,i}^{[d+],s} = -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} I_4^s + \sum_{t=1}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{st}, \tag{3.4}$$

such that we can write the tensor of rank $R = 2$ as

$$I_5^{\mu\nu} = I_5^\mu \cdot Q_0^\nu - \sum_{s=1}^{5} \left\{ Q_0^{s,\mu} I_4^s - \sum_{t=1}^{5} Q_t^{s,\mu} I_3^{st} \right\} Q_s^\nu. \tag{3.5}$$

Compared to (2.18), this representation and the following ones are free of the metric tensor. Further, the compactness relies on the use of the auxiliary vectors $Q_s^\mu, Q_t^{s,\nu}$ instead of the chords $q_i^\mu$.

The tensor of rank $R = 3$ deserves a bit more effort,

$$I_5^{\mu\nu\lambda} = I_5^{\mu\nu} \cdot Q_0^\lambda - \sum_{s=1}^{5} I_4^{\mu\nu,s} \cdot Q_s^\lambda. \tag{3.6}$$

The corresponding 4-point function reads now due to (2.19) and with (A.5)

$$I_4^{\mu\nu,s} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \nu_{ij} I_{4,ij}^{[d+]^2,s} - \frac{1}{2} g^{\mu\nu} I_4^{[d+],s}, \tag{3.7}$$

$$\nu_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} + \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_4^{[d+],s} + \sum_{t=1}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st}. \tag{3.8}$$

Observe that for $i, j = s$ the integrals $I_{4,i}^{[d+],s}$ and $I_{4,ij}^{[d+]^2,s}$ vanish (due to vanishing signed minors) such that indeed a formal summation over all five values of $i, j$ is possible.

Now we use identity (D.3),

$$\begin{aligned}\frac{\binom{is}{js}_5}{\binom{s}{s}_5} &= \frac{\binom{i}{j}_5}{()_5} - \frac{\binom{s}{i}_5\binom{s}{j}_5}{()_5\binom{s}{s}_5} \\ &= \frac{\binom{i}{j}_5}{()_5} - \frac{()_5}{\binom{s}{s}_5} Q_s^i Q_s^j, \end{aligned} \tag{3.9}$$

where the $Q_s^i$ are the vector components of $Q_s^\mu$, see (2.6).

Performing summation over $i, j$ in (3.7), the first term on the right hand side of (3.9) yields $\frac{1}{2} g^{\mu\nu} I_4^{[d+],s}$ (see (2.9)) and thus cancels against the last term in (3.7). Thus we can write the 4-point tensor of rank $R = 2$ as

$$I_4^{\mu\nu,s} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu J_{4,ij}^s, \tag{3.10}$$

$$J_{4,ij}^s = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} I_{4,i}^{[d+],s} - \frac{\binom{s}{i}_5\binom{s}{j}_5}{()_5\binom{s}{s}_5} I_4^{[d+],s} + \sum_{t=1}^{5} \frac{\binom{ts}{js}_5}{\binom{s}{s}_5} I_{3,i}^{[d+],st}, \tag{3.11}$$

where the metric tensor has again disappeared compared to (2.18) and instead $q_i^\mu q_j^\nu$ contribute for $i, j = s$. A compact notation can now be used with (2.7) and (2.8):

$$I_4^{\mu\nu,s} = Q_0^{s,\mu} Q_0^{s,\nu} I_4^s - \frac{()_5}{\binom{s}{s}_5} Q_s^\mu Q_s^\nu I_4^{[d+],s} + \sum_{i=1}^{5} q_i^\mu q_j^\nu R_{3,ij}^{[d+]^2,s}, \tag{3.12}$$

with $R_{3,ij}^{[d+]^2,s}$ the scratched version of (6.2). Inserting (3.12) in (3.6) yields the compact expression for $I_5^{\mu\nu\lambda}$, free of the metric tensor.

Next, for the tensor of rank $R = 4$ of the 5-point function,

$$I_5^{\mu\nu\lambda\rho} = I_5^{\mu\nu\lambda} \cdot Q_0^\rho - \sum_{s=1}^{5} I_4^{\mu\nu\lambda,s} \cdot Q_s^\rho, \tag{3.13}$$

we need the 4-point function of rank $R = 3$, according to (2.19):

$$I_4^{\mu\nu\lambda,s} = -\sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I_{4,ijk}^{[d+]^3,s} + \frac{1}{2} \sum_{i=1}^{5} g^{[\mu\nu} q_i^{\lambda]} I_{4,i}^{[d+]^2,s}, \tag{3.14}$$

$$n_{ijk} I_{4,ijk}^{[d+]^3,s} = -\frac{\binom{0s}{is}_5\binom{0s}{js}_5\binom{0s}{ks}_5}{\binom{s}{s}_5^3} I_4^s + \left\{ \frac{\binom{is}{js}_5}{\binom{s}{s}_5} I_{4,k}^{[d+]^2,s} + (j \leftrightarrow k) + (i \leftrightarrow k) \right\} + R_{3,ijk}^{[d+]^3,s}. \tag{3.15}$$

Here we have used for $I_{4,ijk}^{[d+]^3,s}$, instead of (A.4), the symmetrized form (6.8) with $R_{3,ijk}^{[d+]^3,s}$ being the scratched version of (6.9). Applying again (3.9) we obtain the analogue of (3.12),

$$I_4^{\mu\nu\lambda,s} = \sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda J_{4,ijk}^s, \tag{3.16}$$

$$J_{4,ijk}^s = \frac{\binom{0s}{is}_5\binom{0s}{js}_5\binom{0s}{ks}_5}{\binom{s}{s}_5^3} I_4^s + \frac{1}{()_5} \left\{ \frac{\binom{s}{i}_5\binom{s}{j}_5}{\binom{s}{s}_5} I_{4,k}^{[d+]^2,s} + (j \leftrightarrow k) + (i \leftrightarrow k) \right\} - R_{3,ijk}^{[d+]^3,s}. \tag{3.17}$$

For the following it also pays to introduce

$$\begin{aligned} J_4^{\mu,s} &= \sum_{i=1}^{5} q_i^\mu I_{4,i}^{[d+]^2,s} \\ &= -Q_0^{s,\mu} I_4^{[d+]} + \sum_{t=1}^{5} Q_t^{s,\mu} I_3^{[d+],st}, \end{aligned} \tag{3.18}$$

so that finally

$$I_4^{\mu\nu\lambda,s} = Q_0^{s,\mu} Q_0^{s,\nu} Q_0^{s,\lambda} I_4^s + \frac{()_5}{\binom{s}{s}_5} Q_s^{[\mu} Q_s^\nu J_4^{\lambda,s]} - \sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda R_{3,ijk}^{[d+]^3,s}. \tag{3.19}$$

with $R_{3,ijk}^{[d+]^3,s}$ the scratched version of (6.9). This finishes the determination of $I_5^{\mu\nu\lambda\rho}$.

Finally, for the tensor of rank $R = 5$ of the 5-point function,

$$I_5^{\mu\nu\lambda\rho\sigma} = I_5^{\mu\nu\lambda\rho} \cdot Q_0^\sigma - \sum_{s=1}^{5} I_4^{\mu\nu\lambda\rho,s} \cdot Q_s^\sigma, \tag{3.20}$$

we need the scratched tensor of rank $R = 4$ of the 4-point function. The corresponding symmetrized tensor coefficients are taken from (6.11) and (6.11) by scratching. We begin with the term $I_4^{[d+]^2}$ from (6.11). Using again (3.9) we have

$$\begin{aligned} & \frac{\binom{is}{ls}_5}{\binom{s}{s}_5}\frac{\binom{js}{ks}_5}{\binom{s}{s}_5} + \frac{\binom{js}{ls}_5}{\binom{s}{s}_5}\frac{\binom{is}{ks}_5}{\binom{s}{s}_5} + \frac{\binom{ks}{ls}_5}{\binom{s}{s}_5}\frac{\binom{is}{js}_5}{\binom{s}{s}_5} = \frac{\binom{i}{l}_5}{()_5}\frac{\binom{j}{k}_5}{()_5} + \frac{\binom{j}{l}_5}{()_5}\frac{\binom{i}{k}_5}{()_5} + \frac{\binom{k}{l}_5}{()_5}\frac{\binom{i}{j}_5}{()_5} \\ - \quad & \frac{()_5}{\binom{s}{s}_5}\left\{ \frac{\binom{i}{l}_5}{()_5} Q_s^j Q_s^k + \frac{\binom{j}{k}_5}{()_5} Q_s^i Q_s^l + \frac{\binom{j}{l}_5}{()_5} Q_s^i Q_s^k + \frac{\binom{i}{k}_5}{()_5} Q_s^j Q_s^l + \frac{\binom{k}{l}_5}{()_5} Q_s^i Q_s^j + \frac{\binom{i}{j}_5}{()_5} Q_s^k Q_s^l \right\} \\ + \quad & 3\frac{()_5^2}{\binom{s}{s}_5^2} Q_s^i Q_s^j Q_s^k Q_s^l. \end{aligned} \tag{3.21}$$

The first term on the right hand side of (3.21) yields after summation over $i, j, k, l$

$$\sum_{ijkl=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho \left\{ \frac{\binom{i}{l}_5}{()_5}\frac{\binom{j}{k}_5}{()_5} + \frac{\binom{j}{l}_5}{()_5}\frac{\binom{i}{k}_5}{()_5} + \frac{\binom{k}{l}_5}{()_5}\frac{\binom{i}{j}_5}{()_5} \right\} = \frac{1}{4} g^{[\mu\nu} g^{\lambda\rho]}. \tag{3.22}$$

The same contribution comes directly from (2.20). Finally, the second term of (2.20) contributes with the term $\sim I_4^{[d+]^2,s}$ of (6.11) and again the first term on the right hand side of (3.9)

$$-\frac{1}{2}\sum_{i,j=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^{\rho]} \frac{\binom{i}{j}_5}{()_5} \cdot I_4^{[d+]^2,s} = -\frac{1}{2} g^{[\mu\nu} g^{\lambda\rho]} \cdot I_4^{[d+]^2,s}, \tag{3.23}$$

i.e. the terms $g^{[\mu\nu} g^{\lambda\rho]} \cdot I_4^{[d+]^2,s}$ cancel. The second term on the right hand side of (3.21) yields after

summation over $i,j$

$$
\begin{aligned}
& -\frac{()_5}{\binom{s}{s}_5}\sum_{i,j,k,l=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho \\
& \left\{\frac{\binom{i}{l}_5}{()_5}Q_s^j Q_s^k + \frac{\binom{j}{k}_5}{()_5}Q_s^i Q_s^l + \frac{\binom{j}{l}_5}{()_5}Q_s^i Q_s^k + \frac{\binom{i}{k}_5}{()_5}Q_s^j Q_s^l + \frac{\binom{k}{l}_5}{()_5}Q_s^i Q_s^j + \frac{\binom{i}{j}_5}{()_5}Q_s^k Q_s^l\right\}\cdot I_4^{[d+]^2,s} \\
= & -\frac{1}{2}\frac{()_5}{\binom{s}{s}_5} g^{[\mu\nu}Q_s^\lambda Q_s^{\rho]}\cdot I_4^{[d+]^2,s}. \qquad (3.24)
\end{aligned}
$$

A contribution of this type also comes from the second term of (2.20) with the term $\sim I_4^{[d+]^2,s}$ of (6.11), but now the second part on the right hand side of (3.9) is

$$
\frac{1}{2}\frac{()_5}{\binom{s}{s}_5} g^{[\mu\nu}Q_s^\lambda Q_s^{\rho]}\cdot I_4^{[d+]^2,s}, \qquad (3.25)
$$

which means that also these terms cancel. Finally there remains the last term in (3.21), which does not cancel but contains no $g^{\mu\nu}$

$$
3\frac{()_5^2}{\binom{s}{s}_5^2} Q_s^\mu Q_s^\nu Q_s^\lambda Q_s^\rho \cdot I_4^{[d+]^2,s}. \qquad (3.26)
$$

The contributions from (6.11) of the type $I_{4,i}^{[d+]^2,s}$ - after cancelling the first term on the right hand side of (6.11) - can also be written in a compact manner,

$$
\frac{()_5}{\binom{s}{s}_5} Q_s^{[\mu} Q_s^\nu J_4^{\lambda,s} Q_0^{s,\rho]}, \qquad (3.27)
$$

where the symmetrization in the tensor indices is understood. Introducing like in (3.18)

$$
\begin{aligned}
J_3^{\mu,st} &= \sum_{i=1}^{5} q_i^\mu I_{3,i}^{[d+]^2,st} \\
&= -Q_0^{st,\mu} I_3^{[d+],st} + \sum_{u=1}^{5} Q_u^{st,\mu} I_2^{[d+],stu}, \qquad (3.28)
\end{aligned}
$$

we can finally write

$$
\begin{aligned}
I_4^{\mu\nu\lambda\rho,s} \quad & = Q_0^{s,\mu}Q_0^{s,\nu}Q_0^{s,\lambda}Q_0^{s,\rho} I_4^s + 3\frac{()_5^2}{\binom{s}{s}_5^2} Q_s^\mu Q_s^\nu Q_s^\lambda Q_s^\rho \cdot I_4^{[d+]^2,s} + \frac{()_5}{\binom{s}{s}_5} Q_s^{[\mu} Q_s^\nu J_4^{\lambda,s} Q_0^{s,\rho]} \\
& -\sum_{\substack{t=1\\ t\neq s}}^{5}\left\{ Q_0^{st,\mu}Q_0^{st,\nu}Q_0^{st,\lambda} I_3^{st} + \frac{1}{\binom{st}{st}_5}\left[\binom{t}{t}_5 Q_s^{[\mu} Q_s^{t,\nu} J_3^{\lambda,st]} + \binom{s}{s}_5 Q_t^{[\mu} Q_t^{s,\nu} J_3^{\lambda,st]}\right]\right\} Q_t^{s,\rho} \\
& -Q_0^{s,\rho}\sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda R_{3,ijk}^{[d+]^3,s} + \sum_{\substack{t=1\\ t\neq s}}^{5} Q_t^{s,\rho}\sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda R_{2,ijk}^{[d+]^3,st}. \qquad (3.29)
\end{aligned}
$$

The $J_4^{\lambda,s}$ defined in (3.18) occurs again, the $R_{3,ijk}^{[d+]^3,s}$ is given in (6.9), and $R_{2,ijk}^{[d+]^3,st}$ in (6.13). With the expression for $I_4^{\mu\nu\lambda\rho,s}$, free of the metric tensor, we complete the rewriting of $I_5^{\mu\nu\lambda\rho\sigma}$ with (3.20).

It is remarkable that all the coefficients of the $g^{\mu\nu}$ terms of the four-point functions in (2.18)-(2.20) completely cancel in a way that the remaining tensor coefficients are much simpler than the original ones. This is achieved due to the symmetrization of the recurrence relations given in sect. 6 and would have been seen less easily with the "standard" recursions of app. A. The new representations for the tensors may be useful in several respects. First of all we have here an extremely compact notation, due to the use of auxiliary vectors $Q_s^\mu$, which is not evident at the outset. Further, the representations may be used for a completely independent programming and thus for stringent numerical cross checks. The latter one is an important aspect because there are not too many opportunities for that in case of the 5-point and 6-point functions. Finally, the auxiliary vectors $Q_s^\mu$ have some specific properties so that they may be used for simplifying manipulations with physical amplitudes, see sect. 7.

# 4 Reduction of 5-point tensor coefficients

The purpose of this sect. is to express the 5-point tensor coefficients in terms of 4-point tensor coefficients, which will be evaluated in Sec. 5 in such a way that also the case of inverse sub-Gram determinants can be dealt with in an elegant manner. The difference to the former sect. is the fact that we avoid all inverse Gram determinants, $1/()_5$ as well as $1/()_4$. In this case we have to keep the $g^{\mu\nu}$-terms.

## 4.1 Scalar and vector integrals

For the *scalar* 5-point function $I_5$, we use the recurrence relation (2.31):

$$(d-4)\,()_5\, I_5^{[d+]} = \binom{0}{0}_5 I_5 - \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s. \tag{4.1}$$

The integral $I_5^{[d+]}$ is finite for $d=4$, and we get in this limit:

$$I_5 \equiv E = \frac{1}{\binom{0}{0}_5} \sum_{s=1}^{5} \binom{0}{s}_5 I_4^s, \tag{4.2}$$

i.e. the scalar 5-point function is expressed in the limit $d \to 4$ in terms of scalar 4-point functions, which are obtained by scratching in the five terms of the sum the $s^{th}$ scalar propagator, respectively. This was already derived in [19], see Eq. (6.1) there. See also [20].

The tensor $n$-point integral of rank $R=1$ in (2.17) can be expressed by integrals $I_{n,i}^{[d+]}$, and we obtain quite similarly

$$I_n^\mu = \sum_{i=1}^{n} q_i^\mu E_i, \tag{4.3}$$

$$\begin{aligned} E_i &\equiv -I_{n,i}^{[d+]} \\ &= (d+1-n)\frac{\binom{0}{i}_n}{\binom{0}{0}_n} I_n^{[d+]} - \frac{1}{\binom{0}{0}_n} \sum_{s=1}^{n} \binom{0i}{0s}_n I_{n-1}^s, \end{aligned} \tag{4.4}$$

where again for $n = 5$ in the limit $d \to 4$ the scalar integral $I_5^{[d+]}$ disappears:

$$E_i = \sum_{s=1}^{5} E_i^s, \tag{4.5}$$

$$E_i^s = -\frac{\binom{0i}{0s}_5}{\binom{0}{0}_5} I_4^s. \tag{4.6}$$

## 4.2 The recursion formulae

For the general case, we use as a starting point (2.5). In order to solve this eqn. recursively, we multiply it with $\binom{0}{0}_5$ [3] and use the identity

$$\binom{0}{0}_5 \binom{s}{i}_5 = \binom{0s}{0i}_5 ()_5 + \binom{0}{i}_5 \binom{s}{0}_5. \tag{4.7}$$

The first term on the right-hand-side can cancel already a Gram determinant $()_5$, and the second one transforms a vector $Q_s^\mu$ into a vector $Q_0^\mu$. As a result, we get from (2.5) the general form

$$\binom{0}{0}_5 I_5^{\mu_1 \cdots \mu_{R-1} \mu} = T^{\mu_1 \cdots \mu_{R-1}} Q_0^\mu - \sum_{s=1}^{5} I_4^{\mu_1 \cdots \mu_{R-1}, s} \bar{Q}_s^{0,\mu}, \tag{4.8}$$

with:

$$T^{\mu_1 \cdots \mu_{R-1}} = \binom{0}{0}_5 I_5^{\mu_1 \cdots \mu_{R-1}} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu_1 \cdots \mu_{R-1}, s}, \tag{4.9}$$

and

$$\bar{Q}_s^{0,\mu} = \sum_{i=1}^{5} q_i^\mu \binom{0s}{0i}_n, \quad s = 1, \ldots, 5. \tag{4.10}$$

The barred vectors are free of the inverse Gram determinant $()_5$. Evidently, the reduction of $I_4^{\mu_1 \cdots \mu_{R-1}, s}$ is also free of $()_5$, and we have to care only about the product $T^{\mu_1 \cdots \mu_{R-1}} Q_0^\mu$.

The following observation will prove to be useful: $T^{\mu_1 \cdots \mu_{R-1}}$ contains general tensor structures as given in (2.17)–(2.21) with chords $q_i$ and the metric tensor. In fact, when calculating the 5-point tensor recursively, we keep at this stage the 4-point tensor as given there. With the 5-point tensor of rank $R = 1$, given in (4.4) above, the recursion is started.

In order to cancel $1/()_5$, in each recursive step a term $\binom{s}{i}_5$ will be generated and summed over with the corresponding chord $q_i$. We will apply to such terms the identity

$$\binom{s}{i}_5 \frac{\binom{0}{j}_5}{()_5} = -\binom{0i}{sj}_5 + \binom{s}{0}_5 \frac{\binom{i}{j}_5}{()_5}. \tag{4.11}$$

The ratio $\binom{0}{j}_5 / ()_5$ comes from $Q_0^\mu$, see (2.6) . In the first term of the right-hand-side of (4.11) the Gram determinant $()_5$ has cancelled and the second term yields a $g^{\mu\nu}$ contribution according to (2.9). The metric tensors in the original $T^{\mu_1 \cdots \mu_{R-1}}$ remain unchanged. From the following examples the scheme will become more evident.

[3] Throughout the present work we assume $\binom{0}{0}_5 \neq 0$. In case of vanishing and/or small $\binom{0}{0}_5$ see the discussion in Sec. 2.2. of [6]).

## 4.3 The tensor integral of rank $R = 2$

Equation (4.8) reads for the tensor of rank $R = 2$:

$$\binom{0}{0}_5 I_5^{\mu\nu} = \left[\binom{0}{0}_5 I_5^{\mu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu,s}\right] Q_0^{\nu} - \sum_{s=1}^{5} I_4^{\mu,s} \bar{Q}_s^{0,\nu}. \tag{4.12}$$

The square bracket, a special case of (4.9) for $R = 2$, will be rewritten now. We use (2.17) and (4.6) for $d = 4$ and insert the reduction (A.6) with $l = 1$:

$$\begin{aligned} T^{\mu} &= \sum_{s=1}^{5} T^{\mu,s}, & (4.13)\\ T^{\mu,s} &= \sum_{i=1}^{5} q_i^{\mu} \left\{ \binom{0}{0}_5 E_i^s + \binom{s}{0}_5 I_{4,i}^{[d+],s} \right\} \\ &= \sum_{i=1}^{5} q_i^{\mu} \left\{ -\binom{0s}{0i}_5 I_4^s + \binom{s}{0}_5 \left[ -\binom{0s}{is}_5 I_4^s + \sum_{t=1,t\neq s}^{5} \binom{ts}{is}_5 I_3^{st} \right] \frac{1}{\binom{s}{s}_5} \right\}. & (4.14) \end{aligned}$$

Using further

$$\binom{s}{0}_5 \binom{0s}{is}_5 = \binom{s}{i}_5 \binom{0s}{0s}_5 - \binom{s}{s}_5 \binom{0s}{0i}_5, \tag{4.15}$$

we see the cancellation of $E_i^s$. Additionally, it is

$$\binom{s}{0}_5 \binom{ts}{is}_5 = \binom{s}{i}_5 \binom{ts}{0s}_5 - \binom{s}{s}_5 \binom{ts}{0i}_5. \tag{4.16}$$

Here the $\binom{s}{s}_5$ term cancels and the remaining factor $\binom{ts}{0i}_5$ is antisymmetric in $s,t$, yielding a vanishing contribution after summation over $s,t$. With (A.7), reintroducing $I_4^{[d+],s}$, we obtain

$$\begin{aligned} T^{\mu,s} &= \sum_{i=1}^{5} q_i^{\mu} T_i^s, & (4.17)\\ T_i^s &= -\binom{s}{i}_5 I_4^{[d+],s}. & (4.18) \end{aligned}$$

Here we observe the first occurrence of a term $\binom{s}{i}_5$, as mentioned in sect. 4.2. Using (4.11) and the notation

$$I_4^{\mu\,\nu} = \sum_{i,j=1}^{5} q_i^{\mu}\, q_j^{\nu} E_{ij} + g^{\mu\nu} E_{00}, \tag{4.19}$$

we finally get, taking into account (2.17) for $n = 4$, the expressions for the tensor coefficients:

$$\begin{aligned} E_{00} &\equiv \sum_{s=1}^{5} E_{00}^s \\ &= -\sum_{s=1}^{5} \frac{1}{2} \frac{1}{\binom{0}{0}_5} \binom{s}{0}_5 I_4^{[d+],s}, & (4.20)\\ E_{ij} &\equiv \sum_{s=1}^{5} E_{ij}^s \\ &= \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[ \binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s} \right]. & (4.21) \end{aligned}$$

The functions $I_4^{[d+],s}, I_{4,i}^{[d+],s}$ will be further treated in sect. 5. A comparison shows that the tensor coefficients $E_{ij}$ given in Eqs. (3.10)–(3.12) and (A.22) of [17] are much more involved.

## 4.4 Reduction of integrals with rank $R = 3$

For the tensor integral of rank $R = 3$, Eq. (4.8) reads:

$$\binom{0}{0}_5 I_5^{\mu\nu\lambda} = \left[\binom{0}{0}_5 I_5^{\mu\nu} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu,s}\right] Q_0^\lambda - \sum_{s=1}^{5} I_4^{\mu\nu,s} \bar{Q}_s^{0,\lambda}. \tag{4.22}$$

Investigating the square bracket, i.e. the tensor (4.9) for $R = 3$, we see that the corresponding $g^{\mu\nu}$ term vanishes. Indeed, from (4.20) and (2.18) we have:

$$\binom{0}{0}_5 E_{00}^s + \frac{1}{2}\binom{s}{0}_5 I_4^{[d+],s} = 0. \tag{4.23}$$

This is interesting in view of our general scheme, which was described in Sec. 4.2: Since there is no vector $q_i$ in this contribution, no $\binom{s}{i}_5$ is produced, and if we assume that no inverse Gram $()_5$ should occur in this case, the contribution must vanish.

Further, from (4.21), (2.18) and (A.5) we obtain

$$T^{\mu\nu} = \sum_{s=1}^{5} T^{\mu\nu,s}, \tag{4.24}$$

$$T^{\mu\nu,s} = \sum_{i,j=1}^{5} q_i^\mu q_j^\nu T_{ij}^s, \tag{4.25}$$

and

$$\begin{aligned} T_{ij}^s &= \binom{0}{0}_5 E_{ij}^s - \binom{s}{0}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \\ &= \binom{s}{s}_5 \left[\binom{0i}{sj}_5 I_4^{[d+],s} + \binom{0s}{0j}_5 I_{4,i}^{[d+],s}\right] \\ &\quad - \binom{s}{0}_5 \left[-\binom{0s}{js}_5 I_{4,i}^{[d+],s} + \binom{is}{js}_5 I_4^{[d+],s} + \sum_{t=1,t\neq s,i}^{5} \binom{ts}{js}_5 I_{3,i}^{[d+],st}\right] \frac{1}{\binom{s}{s}_5}. \end{aligned} \tag{4.26}$$

With (4.15) and

$$\binom{s}{0}_5 \binom{is}{js}_5 = \binom{s}{i}_5 \binom{0s}{js}_5 + \binom{s}{s}_5 \binom{0i}{sj}_5, \tag{4.27}$$

we see that the complete term $E_{ij}^s$ cancels. As above we use again (4.16) and with the same arguments as before we see that only $\binom{s}{i}_5$-type terms remain such that (4.11) can be used again to cancel the Gram determinant.

Before collecting all contributions, we would like to point out that, after the above manipulations, the expressions are in general not explicitly symmetric in their indices, although the original integral *is* symmetric in $\mu, \nu, \lambda$. Consequently, our result must also be symmetric in the indices $i, j, k$, however, after summation over $s$ and $t$. For an explicit example see also the discussion after (6.1). If there is no

explicit symmetry before summation over $s$ and $t$ it may be useful to symmetrize the result. With this in mind, collecting all contributions, we have

$$T^s_{ij} = \quad \Bigg\{ -\left[ \binom{s}{i}_5 \binom{0s}{js}_5 + \binom{s}{j}_5 \binom{0s}{is}_5 \right] I_4^{[d+],s} + \sum_{t=1,t\neq s,i}^{5} \left[ \binom{s}{i}_5 \binom{ts}{js}_5 + \binom{s}{j}_5 \binom{ts}{is}_5 \right] \frac{d-2}{2} I_3^{[d+],st} \Bigg\} \frac{1}{\binom{s}{s}_5}. \tag{4.28}$$

To obtain this result, the vector integral $I_4^\mu$, represented by tensor coefficients $I_{4,i}^{[d+],s}$, and the vector integral $I_3^\mu$, represented by tensor coefficients $I_{3,i}^{[d+],st}$ in (4.26), have been reduced to scalar 2-,3-, and 4-point integrals in generic dimension $d$ by means of (A.6) and (A.10). The 2-point functions cancel here. Further we need the identity

$$\binom{s}{s}_5 \binom{0st}{0st}_5 = \binom{0s}{0s}_5 \binom{st}{st}_5 - \binom{ts}{0s}_5^2, \tag{4.29}$$

and in order to get rid of the vector indices in the 2-point functions, we need the relation

$$\left[ \binom{ts}{0s}_5 \binom{ust}{jst}_5 - \binom{ts}{js}_5 \binom{ust}{0st}_5 \right] \binom{s}{s}_5 = \left[ \binom{ts}{0s}_5 \binom{us}{js}_5 - \binom{ts}{js}_5 \binom{us}{0s}_5 \right] \binom{st}{st}_5, \tag{4.30}$$

which shows that after cancellation of $\binom{st}{st}_5$, Eq. (4.30) is antisymmetric in $t$ and $u$ such that it can be effectively considered to vanish after summation over $t$ and $u$. This allows finally to introduce $I_3^{[d+],st}$ according to (A.11) into (4.28).

There is a further subtlety concerning (4.28). The ultraviolet (UV) divergency

$$I_{3,\mathrm{UV}}^{[d+],st} = -\frac{1}{2\varepsilon}, \tag{4.31}$$

when combined with $\frac{d-2}{2} = 1-\varepsilon$, yields a constant finite contribution $\frac{1}{2}$.[4] Since, however,

$$\sum_{t=1}^{5} \binom{ts}{is}_5 = 0, \tag{4.32}$$

this term does *not* contribute and we can put $d = 4$. In that case (4.28) reads

$$T^s_{ij} = \binom{s}{i}_5 I_{4,j}^{[d+]^2,s} + \binom{s}{j}_5 I_{4,i}^{[d+]^2,s}, \tag{4.33}$$

to be compared with (4.18). According to our general scheme, each $q_i$ generates a factor $\binom{s}{i}_5$, the further factor being a higher-dimensional integral with index (indices) being the same as in the remaining chords. In fact, (4.18) is a vector coefficient so that no additional index is available and thus the higher-dimensional integral cannot carry an index.

We just mention that, due to (4.32) and (A.6), also the integral $I_{4,i}^{[d+]^2,s}$ is UV and infrared (IR-) finite. Applying (4.11) in (4.22), we obtain products $\binom{s}{0}_5 I_{4,i}^{[d+]^2,s}$, for which we can write, using (4.15), (4.16), (4.27) and (A.7) and setting $d \to d+2$:

$$\binom{s}{0}_5 I_{4,i}^{[d+]^2,s} = \binom{0s}{0i}_5 I_4^{[d+],s} - \binom{s}{i}_5 (d-1) I_4^{[d+]^2,s}. \tag{4.34}$$

[4] See the discussion after (A.11).

Collecting all the contributions, our final result for the tensor of rank $R = 3$ can be written as follows:

$$I_4^{\mu\,\nu\,\lambda} = \sum_{i,j,k=1}^{5} q_i^\mu\, q_j^\nu\, q_k^\lambda E_{ijk} + \sum_{k=1}^{5} g^{[\mu\nu} q_k^{\lambda]} E_{00k}, \tag{4.35}$$

with

$$\begin{aligned} E_{00j} &\equiv \sum_{s=1}^{5} E_{00j}^s \\ &= \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left[ \frac{1}{2} \binom{0s}{0j}_5 I_4^{[d+],s} - \frac{d-1}{3} \binom{s}{j}_5 I_4^{[d+]^2,s} \right], \end{aligned} \tag{4.36}$$

$$\begin{aligned} E_{ijk} &\equiv \sum_{s=1}^{5} E_{ijk}^s \\ &= -\sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[ \binom{0j}{sk}_5 I_{4,i}^{[d+]^2,s} + (i \leftrightarrow j) \right] + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^2,s} \right\}. \end{aligned} \tag{4.37}$$

In (4.36), we can put $d = 4$ because, similar to the discussion by means of (4.32), it is

$$\sum_{s=1}^{5} \binom{s}{j}_5 = 0. \tag{4.38}$$

Another possibility to argue uses that, due to (A.7), $l = 2$, the $d - 1$ cancels.

The rank $R = 3$ tensors were also treated in [17]. We proved there successfully the cancellation of $1/()_5$, although the corresponding formulae were quite a bit longer than here: see Eqs. (3.41)–(3.42), (3.30)–(3.33), (3.40) in [17]. For the rank $R > 3$, however, the tensor reduction would become really awkward with the older approach.

## 4.5 Reduction of integrals with rank $R = 4$

For the tensor integral of rank $R = 4$, Eq. (4.8) reads:

$$\binom{0}{0}_5 I_5^{\mu\nu\lambda\rho} = \left[ \binom{0}{0}_5 I_5^{\mu\nu\lambda} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu\lambda,s} \right] Q_0^\rho - \sum_{s=1}^{5} I_4^{\mu\nu\lambda,s} \bar{Q}_s^{0,\rho}. \tag{4.39}$$

Here $I_5^{\mu\nu\lambda}$ is given in (4.35) to (4.37), $I_4^{\mu\nu\lambda,s}$ in (2.19), taken at $n = 4$. In a similar manner we decompose the square bracket in (4.39):

$$T_5^{\mu\,\nu\,\lambda} = \sum_{s=1}^{5} T_5^{\mu\,\nu\,\lambda,s}, \tag{4.40}$$

$$T_5^{\mu\,\nu\,\lambda,s} = \sum_{i,j,k=1}^{5} q_i^\mu\, q_j^\nu\, q_k^\lambda T_{ijk}^s + \sum_{i=1}^{5} g^{[\mu\nu} q_i^{\lambda]} T_{00i}^s, \tag{4.41}$$

according to which:

$$\begin{aligned} T_{00i}^s &= \left[ \binom{0}{0}_5 E_{00i}^s - \frac{1}{2} \binom{s}{0}_5 I_{4,i}^{[d+]^2,s} \right] \\ &= \frac{1}{2} \binom{s}{i}_5 I_4^{[d+]^2,s}. \end{aligned} \tag{4.42}$$

Obviously, the tensor coefficient $E^s_{00k}$ has been completely eliminated - as observed before in (4.14). As in (4.35), we now write

$$I_4^{\mu\nu\lambda\rho} = \sum_{i,j,k,l=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho E_{ijkl} + \sum_{i,j=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^{\rho]} E_{00ij} + g^{[\mu\nu} g^{\lambda\rho]} E_{0000}. \tag{4.43}$$

Proceeding as before, using (4.11), from (4.42) we obtain:

$$E_{0000} = \sum_{s=1}^{5} E^s_{0000}, \tag{4.44}$$

$$E^s_{0000} = \frac{1}{4} \frac{\binom{s}{0}_5}{\binom{0}{0}_5} I_4^{[d+]^2,s}. \tag{4.45}$$

We remark that since (4.42) is summed over $s$, all (constant UV-) divergent contributions from $I_4^{[d+]^2,s}$ can be dropped; see also the discussion at the end of sect. 4.4.

In the next step we calculate $T^s_{ijk}$:

$$\begin{aligned}
T^s_{ijk} =& \binom{0}{0}_5 E^s_{ijk} + \binom{s}{0}_5 \nu_{ij}\nu_{ijk} I^{[d+]^3,s}_{4,ijk} \\
=& \frac{1}{\binom{s}{s}_5} \left\{ \binom{s}{s}_5 \left[ -\binom{0i}{sk}_5 I^{[d+]^2,s}_{4,j} - \binom{0j}{sk}_5 I^{[d+]^2,s}_{4,i} - \binom{0s}{0k}_5 \nu_{ij} I^{[d+]^2,s}_{4,ij} \right] \right. \\
&+ \left. \binom{s}{0}_5 \left[ \binom{is}{ks}_5 I^{[d+]^2,s}_{4,j} + \binom{js}{ks}_5 I^{[d+]^2,s}_{4,i} - \binom{0s}{ks}_5 \nu_{ij} I^{[d+]^2,s}_{4,ij} + \sum_{t=1}^{5} \binom{ts}{ks}_5 \nu_{ij} I^{[d+]^2,st}_{3,ij} \right] \right\} \\
=& \left\{ \binom{s}{i}_5 \binom{0s}{ks}_5 I^{[d+]^2,s}_{4,j} + \binom{s}{j}_5 \binom{0s}{ks}_5 I^{[d+]^2,s}_{4,i} - \binom{s}{k}_5 \binom{0s}{0s}_5 \nu_{ij} I^{[d+]^2,s}_{4,ij} \right. \\
&+ \left. \binom{s}{k}_5 \sum_{t=1}^{5} \binom{ts}{0s}_5 \nu_{ij} I^{[d+]^2,st}_{3,ij} \right\} \frac{1}{\binom{s}{s}_5},
\end{aligned} \tag{4.46}$$

where again (4.15), (4.16) and (4.27) have been applied and $\nu_{ijk} I^{[d+]^3,s}_{4,ijk}$ has been replaced by means of (A.4). Again we observe that the complete tensor of lower rank (here $E^s_{ijk}$) cancels. After further lengthy manipulations and subsequent symmetrization, the following analogue of (4.18) and (4.33) can be verified:

$$T^s_{ijk} = -\left\{ \binom{s}{i}_5 \nu_{jk} I^{[d+]^3,s}_{4,jk} + \binom{s}{j}_5 \nu_{ik} I^{[d+]^3,s}_{4,ik} + \binom{s}{k}_5 \nu_{ij} I^{[d+]^3,s}_{4,ij} \right\}. \tag{4.47}$$

Using again (4.11), we can immediately write down the pure spatial components:

$$\begin{aligned}
E_{ijkl} &\equiv \sum_{s=1}^{5} E^s_{ijkl} \\
&= \sum_{s=1}^{5} \frac{1}{\binom{0}{0}_5} \left\{ \left[ \binom{0k}{sl}_5 \nu_{ij} I^{[d+]^3,s}_{4,ij} + (i \leftrightarrow k) + (j \leftrightarrow k) \right] + \binom{0s}{0l}_5 n_{ijk} I^{[d+]^3,s}_{4,ijk} \right\}.
\end{aligned} \tag{4.48}$$

For the mixed terms $E^s_{00ij}$, i.e. those containing the metric tensor, we have contributions from different origins. From $T^s_{00k}$ (see (4.42)) we get

$$-\frac{1}{2} \sum_{k,l=1}^{5} g^{[\mu\nu} q_k^{\lambda]} q_l^\rho \binom{0k}{sl}_5 I_4^{[d+]^2,s}. \tag{4.49}$$

From (4.47), we get

$$
\begin{aligned}
& -\sum_{i,j,k,l=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho \binom{s}{0}_5 \left\{ \frac{\binom{i}{l}_5}{()_5} \nu_{jk} I_{4,jk}^{[d+]^3,s} + \frac{\binom{j}{l}_5}{()_5} \nu_{ik} I_{4,ik}^{[d+]^3,s} + \frac{\binom{k}{l}_5}{()_5} \nu_{ij} I_{4,ij}^{[d+]^3,s} \right\}. \\
= \quad & -\frac{1}{2}\binom{s}{0}_5 \left\{ g^{\mu\rho} \sum_{j,k=1}^{5} q_j^\nu q_k^\lambda \nu_{jk} I_{4,jk}^{[d+]^3,s} + g^{\nu\rho} \sum_{i,k=1}^{5} q_i^\mu q_k^\lambda \nu_{ik} I_{4,ik}^{[d+]^3,s} + g^{\lambda\rho} \sum_{i,j=1}^{5} q_i^\mu q_j^\nu \nu_{ij} I_{4,ij}^{[d+]^3,s} \right\}.
\end{aligned}
\tag{4.50}
$$

Finally, there is a contribution from the second term of (2.19):

$$
-\frac{1}{2}\sum_{i,l=1}^{5} (\, g^{\mu\nu} q_i^\lambda + g^{\mu\lambda} q_i^\nu + g^{\nu\lambda} q_i^\mu \,) q_l^\rho \binom{0s}{0l}_5 I_{4,i}^{[d+]^2,s}. \tag{4.51}
$$

Collecting these contributions without symmetrization we have:

$$
E_{00ij} = \sum_{s=1}^{5} E_{00ij}^s, \tag{4.52}
$$

$$
E_{00ij}^s = -\frac{1}{4\binom{0}{0}_5} \left\{ \binom{0i}{sj}_5 I_4^{[d+]^2,s} + \binom{0s}{0j}_5 I_{4,i}^{[d+]^2,s} + \binom{s}{0}_5 \nu_{ij} I_{4,ij}^{[d+]^3,s} \right\}. \tag{4.53}
$$

A general comment is in order at this place: The only UV divergent term in (4.53) is $I_4^{[d+]^2,s}$, which comes from (4.42). We see, however, that due to (4.38) this (constant) term does not contribute when $T_{00k}^s$ is summed over $s$. Thus, the UV divergent part can be dropped in $I_4^{[d+]^2,s}$ and as a consequence it does also not appear in (4.53). This, after all, is only an expression of the fact that the original tensor integral under consideration is finite.

## 4.6 Reduction of integrals with rank $R = 5$

For the tensor integral of rank $R = 5$, Eq. (4.8) reads:

$$
\binom{0}{0}_5 I_5^{\mu\nu\lambda\rho\sigma} = \left[ \binom{0}{0}_5 I_5^{\mu\nu\lambda\rho} - \sum_{s=1}^{5} \binom{s}{0}_5 I_4^{\mu\nu\lambda\rho,s} \right] Q_0^\sigma - \sum_{s=1}^{5} I_4^{\mu\nu\lambda\rho,s} \bar{Q}_s^{0,\sigma}. \tag{4.54}
$$

Writing the square bracket in (4.54) as

$$
T_5^{\mu\,\nu\,\lambda\,\rho} = \sum_{s=1}^{5} T_5^{\mu\,\nu\,\lambda\,\rho,s}, \tag{4.55}
$$

$$
T_5^{\mu\,\nu\,\lambda\,\rho,s} = \sum_{i,j,k,l=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho T_{ijkl}^s + \sum_{i,j=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^{\rho]} T_{00ij}^s + g^{[\mu\nu} g^{\lambda\rho]} T_{0000}^s, \tag{4.56}
$$

we, first of all, observe that the $g^{[\mu\nu} g^{\lambda\rho]}$ term vanishes. Indeed, from (4.45) and (2.21) we have:

$$
T_{0000}^s = \binom{0}{0}_5 E_{0000}^s - \frac{1}{4}\binom{s}{0}_5 I_4^{[d+]^2,s} = 0. \tag{4.57}
$$

Again we have a situation like in (4.23): There is no vector $q_i$, and no $\binom{s}{i}_5$ is produced. Thus, no inverse Gram determinant appears since this term vanishes. The next term, $T^s_{00ij}$, is calculated similarly as is scetched in sect. 4.4 with a result generalizing (4.42):

$$\begin{aligned} T^s_{00ij} &= \left[\binom{0}{0}_5 E^s_{00ij} + \frac{1}{2}\binom{s}{0}_5 \nu_{ij} I^{[d+]^3,s}_{4,ij}\right] \\ &= -\frac{d}{8}\left\{\binom{s}{i}_5 I^{[d+]^3,s}_{4,j} + \binom{s}{j}_5 I^{[d+]^3,s}_{4,i}\right\}. \end{aligned} \tag{4.58}$$

In contrary to the discussion at the end of sect. 4.5, summing (4.58) over $s$, the UV divergence of the integrals $I^{[d+]^3,s}_{4,i}$ does not drop out since $I^{[d+]^3,s}_{4,i} = 0$ for $s = i$. The corresponding divergence cancels in this case against a divergence coming from the last term of (2.20).

In analogy to (4.18), (4.33) and (4.47) we also have

$$T^s_{ijkl} = \binom{s}{i}_5 n_{jkl} I^{[d+]^4,s}_{4,jkl} + \binom{s}{j}_5 n_{ikl} I^{[d+]^4,s}_{4,ikl} + \binom{s}{k}_5 n_{ijl} I^{[d+]^4,s}_{4,ijl} + \binom{s}{l}_5 n_{ijk} I^{[d+]^4,s}_{4,ijk}. \tag{4.59}$$

It is interesting to note that a second chain of tensor coefficients has developed for the square bracket tensor $T^{\mu_1\cdots\mu_{R-1}}$ (4.9) which follows the same rule when proceeding to higher ranks, namely (4.42) and (4.58) to be compared with the chain (4.18), (4.33), (4.47) and (4.59).

The complete tensor of rank $R = 5$ (2.21) now reads

$$I_5^{\mu\,\nu\,\lambda\rho\sigma} = \sum_{s=1}^{5} I_5^{\mu\,\nu\,\lambda\rho\sigma,s}, \tag{4.60}$$

$$I_5^{\mu\,\nu\,\lambda\rho\sigma,s} = \sum_{i,j,k,l,m=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma E^s_{ijklm} + \sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^\rho q_k^{\sigma]} E^s_{00ijk} + \sum_{i=1}^{5} g^{[\mu\nu} g^{\lambda\rho} q_i^{\sigma]} E^s_{0000i}. \tag{4.61}$$

Using (4.10) and (4.11), we obtain for the pure spatial part

$$E^s_{ijklm} = -\frac{1}{\binom{0}{0}_5}\left\{\left[\binom{0l}{sm}_5 n_{ijk} I^{[d+]^4,s}_{4,ijk} + (i \leftrightarrow l) + (j \leftrightarrow l) + (k \leftrightarrow l)\right] + \binom{0s}{0m}_5 n_{ijkl} I^{[d+]^4,s}_{4,ijkl}\right\}. \tag{4.62}$$

Next we consider again the mixed terms and begin with $E_{00ijk}$. From (4.58) we have

$$\frac{1}{2}\sum_{i,j,k=1}^{5} g^{[\mu\nu} q_i^\lambda q_j^{\rho]} q_k^\sigma \left[\binom{0i}{sk}_5 I^{[d+]^3,s}_{4,j} + \binom{0j}{sk}_5 I^{[d+]^3,s}_{4,i}\right]. \tag{4.63}$$

From (4.59) we get:

$$\begin{aligned} &\sum_{i,j,k,l.m=1}^{5} q_i^\mu q_j^\nu q_k^\lambda q_l^\rho q_m^\sigma \binom{s}{0}_5 \left\{\frac{\binom{i}{m}_5}{()_5} n_{jkl} I^{[d+]^4,s}_{4,jkl} + \frac{\binom{j}{m}_5}{()_5} n_{ikl} I^{[d+]^4,s}_{4,ikl}\right. \\ &\qquad\left. + \frac{\binom{k}{m}_5}{()_5} n_{ijl} I^{[d+]^4,s}_{4,ijl} + \frac{\binom{l}{m}_5}{()_5} n_{ijk} I^{[d+]^4,s}_{4,ijk}\right\} \\ = \;& \frac{1}{2}\binom{s}{0}_5 \left\{g^{\mu\sigma} \sum_{j,k,l=1}^{5} q_j^\nu q_k^\lambda q_l^\rho n_{jkl} I^{[d+]^4,s}_{4,jkl} + g^{\nu\sigma} \sum_{i,k,l=1}^{5} q_i^\mu q_k^\lambda q_l^\rho n_{ikl} I^{[d+]^4,s}_{4,ikl}\right. \\ &\qquad\left. + g^{\lambda\sigma} \sum_{i,j,l=1}^{5} q_i^\mu q_j^\nu q_l^\rho n_{ijl} I^{[d+]^4,s}_{4,ijl} + g^{\rho\sigma} \sum_{i,j,k=1}^{5} q_i^\mu q_j^\nu q_k^\lambda n_{ijk} I^{[d+]^4,s}_{4,ijk}\right\}. \end{aligned} \tag{4.64}$$

There is a 4-point contribution from the second term of (2.20):

$$\frac{1}{2}\sum_{i,j,k=1}^{5} g^{[\mu\nu}q_i^{\lambda}\,q_j^{\rho]}q_k^{\sigma}\binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^3,s}. \tag{4.65}$$

In (4.63) and (4.65) we have the tensor structure $g^{[\mu\nu}q_i^{\lambda}q_j^{\rho]}q_k^{\sigma}$, and in (4.61) the structure $g^{[\mu\nu}q_i^{\lambda}q_j^{\rho}q_k^{\sigma]}$. In order to identify them, we can make use of the fact that the tensor $I_5^{\mu\nu\lambda\rho\sigma}$ is symmetric in all indices due to which we have only to count the number of terms in the structures to be compared: e.g. (2.25) contains ten terms, while (2.23) contains six terms. Thus, replacing (2.23), multiplied by $q_k^{\sigma}$, by (2.25) we have to introduce a factor taking care of the ratio of the numbers of terms in each of them. Similarly this applies for (4.64). In this way we obtain

$$E^s_{00ijk} = \frac{1}{2\binom{0}{0}_5}\left\{\frac{3}{5}\left[\frac{d}{4}\binom{0i}{sk}_5 I_{4,j}^{[d+]^3,s} + \frac{d}{4}\binom{0j}{sk}_5 I_{4,i}^{[d+]^3,s} + \binom{0s}{0k}_5 \nu_{ij} I_{4,ij}^{[d+]^3,s}\right] + \frac{2}{5}\binom{s}{0}_5 n_{ijk} I_{4,ijk}^{[d+]^4,s}\right\}. \tag{4.66}$$

This concludes the $E_{00ijk}$ and finally we have to collect the contributions to $E_{0000i}$. They come from the last term of (2.20) and (4.58) with the result

$$E^s_{0000i} = -\frac{1}{4\binom{0}{0}_5}\frac{1}{5}\left\{\binom{0s}{0i}_5 I_4^{[d+]^2,s} + d\binom{s}{0}_5 I_{4,i}^{[d+]^3,s}\right\}. \tag{4.67}$$

We will not proceed here further, but by now it may be evident to the reader how to treat tensors of higher rank. One remark, however, is in order: identifying the $I_{4,i\ldots}^{[d+]^l}$ in (2.17) - (2.21) as 4-point tensor coefficients, the above tensor coefficients of the 5-point functions should be equivalent to (6.17) -(6.21) of [23]. Nevertheless, working with higher dimensional 4-point functions and in particular using the *algebra of the signed minors* appears advantageous to us.

# 5 Calculation of higher-dimensional 4-point functions

In the foregoing sects. the 5-point tensor coefficients $I_{5,i\ldots}^{[d+]^{R-r}}$ have been rewritten in terms of 4-point tensor coefficients. The factor $1/()_5$ has been completely avoided. In detail we have:

- Tensors with $R=2$:
  The tensor coefficients $E_{00}, E_{ij}$ are expressed by $I_4^{[d+],s}, I_{4,i}^{[d+],s}$.
- Tensors with $R=3$:
  The tensor coefficients $E_{00k}, E_{ijk}$ are expressed by $I_4^{[d+],s}, I_4^{[d+]^2,s}, I_{4,i}^{[d+]^2,s}, I_{4,ij}^{[d+]^2,s}$.
- Tensors with $R=4$:
  The tensor coefficients $E_{0000}, E_{00ij}, E_{ijkl}$ are expressed by $I_4^{[d+]^2,s}, I_{4,i}^{[d+]^2,s}, I_{4,ij}^{[d+]^3,s}, I_{4,ijk}^{[d+]^3,s}$.
- Tensors with $R=5$:
  The tensor coefficients $E_{0000i}, E_{00ijk}, E_{ijklm}$ are expressed by $I_4^{[d+]^2,s}$, $I_{4,i}^{[d+]^3,s}$, $I_{4,ij}^{[d+]^3,s}$, $I_{4,ijk}^{[d+]^4,s}$, $I_{4,ijkl}^{[d+]^4,s}$.

It is our goal to find a representation of these integrals which is suited for the most problematic cases occurring in practical calculations, namely for vanishing sub-Gram determinants $()_4 = \binom{s}{s}_5$. In the numerics we will make use of opensource programs for the calculation of few *master integrals*, chosen here to be the scalar 1-point to 4-point functions in generic dimension $d = 4 - \varepsilon$, in standard notation the integrals $A_0$, $B_0$, $C_0$ and $D_0$. They are available from e.g. the LoopTools/FF package [25, 28] or from the QCDloop/FF package [29, 28]. For this purpose, we have to reduce dimension and indices of the above integrals. This may be done by recurrence relations (2.30) and (2.31), given in detail in app. A . In each recursion step an inverse power of $()_4$ is generated, which causes numerical problems for small $()_4$ although the original integrals $I^{[d+]^l,s}_{4,i\cdots}$ are finite and well-behaved there.

We proceed in two steps. In subsect. 5.1, an intermediate step, we manage to write the integrals in the form:

$$I^{[d+]^l,s}_{4,ij\cdots k} \sim \frac{\binom{0s}{ks}_5}{\binom{s}{s}_5}\left[I^{[d+]^{l-1},s}_{4,ij\cdots} - Z^{[d+]^{l-1},s}_{4,ij\cdots}\right] + R_{4,ij\cdots k}. \tag{5.1}$$

Here $Z^{[d+]^{l-1},s}_{4,ij\cdots}$ is constructed such that in the limit $()_4 \to 0$ it has the same value as $I^{[d+]^{l-1},s}_{4,ij\cdots}$, i.e. the first term, the difference quotient $\left[I^{[d+]^{l-1},s}_{4,ij\cdots} - Z^{[d+]^{l-1},s}_{4,ij\cdots}\right] / \binom{s}{s}_5$, stays finite in this limit. Further, dimension and indices are reduced. The remainder $R_{4,ij\cdots k}$ does no contain an inverse $()_4$. In a second step, in subsect. 5.2, we will eliminate the inverse $()_4$ in the difference quotient.

## 5.1 Difference quotients with $1/()_4$

In this subsect. we derive optimized, compact expressions, where the appearance of possible singular $1/()_4$-terms is reduced as much as possible. We will treat the singular behaviour using the fact that the integrals are exactly known in the limit $()_n \to 0$. In fact, if $()_n = 0$, due to (2.31) the $n$-point integrals degenerate to integrals with scratched propagators:

$$\lim_{()_n \to 0} I^{(d)}_{n,i\cdots} = \sum_{t=1}^{n} \frac{\binom{t}{0}_n}{\binom{0}{0}_n}\ \mathbf{t}^- I^{(d)}_{n,i\cdots}. \tag{5.2}$$

Accordingly we define objects which converge in the limit $()_4 \to 0$ to the corresponding tensor coefficients, taken in that limit:

$$Z^{(d),s}_4 = \sum_{t=1}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I^{(d),st}_3, \tag{5.3}$$

$$Z^{(d),s}_{4,i} = \sum_{t=1,t\neq i}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I^{(d),st}_{3,i} + \frac{\binom{is}{0s}_5}{\binom{0s}{0s}_5} I^{(d),s}_4, \tag{5.4}$$

$$\nu_{ij} Z^{(d),s}_{4,ij} = \sum_{t=1,t\neq i,j}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} \nu_{ij} I^{(d),st}_{3,ij} + \frac{\binom{is}{0s}_5}{\binom{0s}{0s}_5} I^{(d),s}_{4,j} + \frac{\binom{js}{0s}_5}{\binom{0s}{0s}_5} I^{(d),s}_{4,i}, \tag{5.5}$$

$$\begin{aligned} \nu_{ij}\nu_{ijk} Z^{(d),s}_{4,ijk} = & \sum_{t=1,t\neq i,j,k}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} \nu_{ij}\nu_{ijk} I^{(d),st}_{3,ijk} \\ & + \frac{\binom{ks}{0s}_5}{\binom{0s}{0s}_5} \nu_{ij} I^{(d),s}_{4,ij} + \frac{\binom{js}{0s}_5}{\binom{0s}{0s}_5} \nu_{ik} I^{(d),s}_{4,ik} + \frac{\binom{is}{0s}_5}{\binom{0s}{0s}_5} \nu_{jk} I^{(d),s}_{4,jk}. \end{aligned} \tag{5.6}$$

Eq. (A.7) reads in this notation

$$
\begin{aligned}
I_4^{(d+2),s} &= \binom{0s}{0s}_5 \frac{1}{\binom{s}{s}_5} \left[ I_4^{(d),s} - \sum_{t=1}^{5} \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} I_3^{(d),st} \right] \frac{1}{d-3} \\
&\equiv \binom{0s}{0s}_5 \frac{1}{\binom{s}{s}_5} \left[ I_4^{(d),s} - Z_4^{(d),s} \right] \frac{1}{d-3},
\end{aligned}
\tag{5.7}
$$

such that indeed for $\binom{s}{s}_5 \to 0$ the $I_4^{(d+2),s}$ remains finite. We need this relation with $d = 4-2\varepsilon$ and $d = [d+] = 6-2\varepsilon$ for tensors of rank $R = 2,3$ and rank $R = 3,4,5$, respectively.

The next integral is $I_{4,i}^{(d),s}$. The recursion of integrals with one index, $I_{4,i}^{(d),s}$, is (A.6). To rewrite (A.6) in a similar manner as (5.7), we evaluate the right hand side of (A.6), replacing $I_4$ by $Z_4$ :

$$
\begin{aligned}
\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} Z_4^{(d),s} - \sum_{t=1}^{5} \frac{\binom{ts}{is}_5}{\binom{s}{s}_5} I_3^{(d),st} &= \sum_{t=1}^{5} \frac{1}{\binom{s}{s}_5} \left[ \binom{0s}{is}_5 \frac{\binom{ts}{0s}_5}{\binom{0s}{0s}_5} - \binom{ts}{is}_5 \right] I_3^{(d),st} \\
&= -\frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{(d),st},
\end{aligned}
\tag{5.8}
$$

where the latter eqn. is due to

$$
\binom{0s}{is}_5 \binom{ts}{0s}_5 - \binom{0s}{0s}_5 \binom{ts}{is}_5 = -\binom{s}{s}_5 \binom{0st}{0si}_5 .
\tag{5.9}
$$

The factor $1/\binom{s}{s}_5$ has cancelled and we obtain the analogue to (5.7):

$$
\begin{aligned}
I_{4,i}^{(d+2),s} &= -\frac{\binom{0s}{is}_5}{\binom{s}{s}_5} \left[ I_4^{(d),s} - Z_4^{(d),s} \right] + \frac{1}{\binom{0s}{0s}_5} \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{(d),st} \\
&= \frac{1}{\binom{0s}{0s}_5} \left[ -\binom{0s}{is}_5 (d-3) I_4^{(d+2),s} + \sum_{t=1}^{5} \binom{0st}{0si}_5 I_3^{(d),st} \right].
\end{aligned}
\tag{5.10}
$$

In the first line of (5.10) we have introduced a difference quotient which can in the next step be replaced, due to (5.7), by $I_4^{(d+2),s}$, i.e. an integral of the same dimension as the original integral on the left hand side. In fact the second line of (5.10) is already our final result for this type of tensor coefficient. We need this result for the 5-point tensors of rank $R = 2$ with $d = 4-2\varepsilon$ (generic dimension), for $R = 3,4$ with $d = [d+]$, and for $R = 5$ with $d = [d+]^2$. Eqn. (5.10) demonstrates our principle as described above.

In the tensor integrals of higher rank more complicated difference quotients appear and will be dealt with in the next subsect. 5.2. The procedure of calculation is the same as before. In order to obtain, e.g., $v_{ij} I_{4,ij}^{[d+]^2,s}$, we calculate the right hand side of (A.5) (for $l = 2$), replacing $I_{4,i}^{[d+]}$ by $Z_{4,i}^{[d+]}$. Using again relation (5.9) we find

$$
v_{ij} I_{4,ij}^{[d+]^2,s} = -\frac{\binom{0s}{js}_5}{\binom{s}{s}_5} \left[ I_{4,i}^{[d+],s} - Z_{4,i}^{[d+],s} \right] + \frac{1}{\binom{0s}{0s}_5} \left[ \binom{0si}{0sj}_5 I_4^{[d+],s} + \sum_{t=1,t\neq i}^{5} \binom{0st}{0sj}_5 I_{3,i}^{[d+],st} \right].
\tag{5.11}
$$

Next we obtain from (A.4)

$$\begin{aligned}\nu_{ij}\nu_{ijk}I^{[d+]^3,s}_{4,ijk} = \quad & -\frac{\binom{0s}{ks}_5}{\binom{s}{s}_5}\nu_{ij}\left[I^{[d+]^2,s}_{4,ij} - Z^{[d+]^2,s}_{4,ij}\right] \\ & + \frac{1}{\binom{0s}{0s}_5}\left[\binom{0si}{0sk}_5 I^{[d+]^2,s}_{4,j} + \binom{0sj}{0sk}_5 I^{[d+]^2,s}_{4,i} + \sum_{t=1,t\neq i,j}^{5}\binom{0st}{0sk}_5 \nu_{ij} I^{[d+]^2,st}_{3,ij}\right],\end{aligned} \tag{5.12}$$

and with (A.3)

$$\begin{aligned}n_{ijkl}I^{[d+]^4,s}_{4,ijkl} = & -\frac{\binom{0s}{ls}_5}{\binom{s}{s}_5}\nu_{ij}\nu_{ijk}\left[I^{[d+]^3,s}_{4,ijk} - Z^{[d+]^3,s}_{4,ijk}\right] + \frac{1}{\binom{0s}{0s}_5}\left[\binom{0sk}{0sl}_5 \nu_{ij} I^{[d+]^3,s}_{4,ij} + \right. \\ & \left. \binom{0sj}{0sl}_5 \nu_{ik} I^{[d+]^3,s}_{4,ik} + \binom{0si}{0sl}_5 \nu_{jk} I^{[d+]^3,s}_{4,jk} + \sum_{t=1,t\neq i,j,k}^{5}\binom{0st}{0sl}_5 \nu_{ij}\nu_{ijk} I^{[d+]^3,st}_{3,ijk}\right].\end{aligned} \tag{5.13}$$

We now have collected all contributions to higher-dimensional integrals with an $\binom{s}{s}_5$ in the denominator in such a way that also the numerator vanishes for $\binom{s}{s}_5 = 0$, see (5.2). These results are only a rewriting of the recurrence relations, but they make the finiteness of the integrals at $\binom{s}{s}_5 = 0$ manifest. They will be a starting point to find a final representation, which is truly optimal for kinematical points around $()_4 = 0$. In the second line of (5.10) we observe that there are no explicit inverse Gram determinants anymore. In the following we will show that this also holds for integrals with any number of indices.

## 5.2 Reduction of the difference quotients

In app. A we reproduce a list of the recurrence relations needed for the evaluation of the 5-point functions. In fact, since all tensor coefficients of the 5-point functions have been reduced to higher-dimensional 4-point functions, we need only the recursions for the latter. When applying these formulae to 5-point functions, we have to identify $()_4 = \binom{s}{s}_5$ and $I^{[d+]}_4 = I^{[d+],s}_5$, etc. In the present sect. we will drop the index $s$ in the Gram determinant and in the upper indices of the integrals.

We now discuss the higher-dimensional 4-point functions needed for the different tensor ranks of the 5-point functions. For the tensor of rank $R = 2$ (4.21) we need $I^{[d+]}_4$ and $I^{[d+]}_{4,i}$ given in (5.7) and (5.10). In the spirit of our approach they are already in the final form. For the tensor of rank $R = 3$, (4.36) and (4.37), we further need $I^{[d+]^2}_4$, $I^{[d+]^2}_{4,i}$ and $\nu_{ij} I^{[d+]^2}_{4,ij}$. These are given in (5.7), (5.10) and (5.11). In fact, the first two are already in the final form, while in the last one a new difference quotient appears.

Our general approach to cancel Gram determinants is, first of all, to use the recurrence relations "backward", i.e. to express a 4-point function of dimension $d$ by one of dimension $d+2$, multiplied by a Gram determinant, plus a sum over 3-point functions. The factorized Gram determinant can be cancelled and for the collected sum over 3-point functions the algebra of Cayley determinants allows to combine them such that again the Gram determinant factorizes and can be cancelled.

With the notation $() \equiv ()_4$ we obtain

$$\frac{\binom{0}{0}}{()}\left[I^{[d+]}_{4,i} - Z^{[d+]}_{4,i}\right] = -(d-2)\left[\frac{\binom{0}{i}}{\binom{0}{0}}(d-1)I^{[d+]^2}_4 - \frac{1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i} I^{[d+],t}_3\right], \tag{5.14}$$

and from (5.11)

$$\begin{aligned} \nu_{ij} I_{4,ij}^{[d+]^2} = \quad & \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} (d-2)(d-1) I_4^{[d+]^2} + \frac{\binom{0i}{0j}}{\binom{0}{0}} I_4^{[d+]} \\ & - \frac{\binom{0}{j}}{\binom{0}{0}} \frac{d-2}{\binom{0}{0}} \sum_{t=1}^{4} \binom{0t}{0i} I_3^{[d+],t} \quad + \frac{1}{\binom{0}{0}} \sum_{t=1}^{4} \binom{0t}{0j} I_{3,i}^{[d+],t}. \end{aligned} \tag{5.15}$$

This is the form we wanted to obtain. The higher-dimensional 3-point functions can be calculated by means of the recurrence relations given in app. A.2, reducing them to scalar functions in generic dimension. If $()$ is not small, the same applies for the higher-dimensional 4-point functions, in particular (A.7); otherwise we will use (5.3) by setting up an expansion in the small Gram determinant. This will be done in subsect. 5.3.

It is worth mentioning that we can deal with 3-point functions like $I_{3,i}^{(d),t}$ in (5.15) in the same manner as we dealt with the 4-point functions:

$$I_{3,i}^{[d+],t} = - \frac{\binom{0t}{it}}{\binom{0t}{0t}} (d-2) I_3^{[d+],t} + \frac{1}{\binom{0t}{0t}} \sum_{u=1}^{4} \binom{0tu}{0ti} I_2^{tu}, \tag{5.16}$$

to be compared with (5.10). This allows to handle the 3-point functions in case $\binom{t}{t} = 0$, for which case (A.11) does not work - expanding in small $\binom{t}{t}$ if needed by the use of (5.2). The $\binom{0t}{0t}$, however, vanishes for an infrared 3-point function - thus we have to assume here that $\binom{t}{t}$ and $\binom{0t}{0t}$ don't vanish simultaneously in order to be able to apply at least one of them; this is also implicitly assumed for the case of the 4-point functions.

Exploiting this approach systematically, it can be achieved in general that the indices are carried, like in (5.15) and (5.16), only by the Cayley determinants, multiplied by scalar integrals in higher dimension. This property might become useful for further analytical evaluation of the original Feynman diagrams, in performing partial sums over indices explicitly where needed. We point out that due to the powers of $d$ in front of higher-dimensional integrals we have to take into account finite rational contributions arising from the divergencies of the integrals; see app. B for a list of examples.

For the tensors of rank $R = 4$(see (4.45), (4.48) and (4.53)) we further need $I_{4,ij}^{[d+]^3}$ and $\nu_{ij}\nu_{ijk} I_{4,ijk}^{[d+]^3}$. The tensor with two indices was treated in (5.15) and we have only to shift the dimension: $d \to d+2$.

Much more involved is now the calculation of $I_{4,ijk}^{[d+]^3}$. The crucial point of our approach is to obtain here an expression for the following difference quotient with the envisaged properties:

$$\begin{aligned} \frac{\binom{0}{0}}{()} \nu_{ij} \left[ I_{4,ij}^{[d+]^2} - Z_{4,ij}^{[d+]^2} \right] = \quad & d \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} (d-3)(d+1) I_4^{[d+]^3} + (d-3) \frac{1}{\binom{0}{0}} \binom{0i}{0j} I_4^{[d+]^2} + 2\nu_{ij} I_{4,ij}^{[d+]^3} \\ & + \frac{1}{()} \left\{ d \frac{\binom{0}{i}}{\binom{0}{0}} \frac{\binom{0}{j}}{\binom{0}{0}} (d-3) \sum_{t=1}^{4} \binom{t}{0} I_3^{[d+]^2,t} + \frac{1}{\binom{0}{0}} \binom{0i}{0j} \sum_{t=1}^{4} \binom{t}{0} I_3^{[d+],t} \right. \\ & - (d-2) \frac{\binom{0}{j}}{\binom{0}{0}} \sum_{t=1}^{4} \binom{0t}{0i} I_3^{[d+],t} + \sum_{t=1}^{4} \binom{0t}{0j} I_{3,i}^{[d+],t} \\ & \left. - \sum_{t=1}^{4} \binom{t}{0} \nu_{ij} I_{3,ij}^{[d+]^2,t} - \sum_{t=1}^{4} \binom{t}{i} I_{3,j}^{[d+]^2,t} - \sum_{t=1}^{4} \binom{t}{j} I_{3,i}^{[d+]^2,t} \right\}. \end{aligned} \tag{5.17}$$

By construction, the 4-point functions have no explicit inverse Gram determinant anymore. It is interesting that the formerly calculated $I_{4,ij}^{[d+]^3}$ (see (5.15)) enters here as a whole. The remaining task is

now to show that in the sum of 3-point functions in (5.17) a Gram determinant $()$ factorizes and thus cancels its overall factor $1/()$. Indeed this is so. After a tremendeous amount of cancellations one gets the result:

$$\begin{aligned}\frac{\binom{0}{0}}{()}\nu_{ij}\left[I_{4,ij}^{[d+]^2}-Z_{4,ij}^{[d+]^2}\right] &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}(d-1)d(d+1)I_4^{[d+]^3}+(d-1)\frac{1}{\binom{0}{0}}\binom{0i}{0j}I_4^{[d+]^2}\\ &-\frac{(d-1)d}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+]^2,t}+\frac{d-1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I_{3,i}^{[d+]^2,t}. \quad (5.18)\end{aligned}$$

The $\nu_{ij}I_{4,ij}^{[d+]^3}$ from (5.15) ($d\to d+2$) has now been explicitly inserted since it has the same structure as the final result. Adding all contributions, using (5.10), we finally have

$$\begin{aligned}\nu_{ij}\nu_{ijk}I_{4,ijk}^{[d+]^3} = &-\frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}(d-1)d(d+1)I_4^{[d+]^3}-\frac{\binom{0i}{0j}\binom{0}{k}+\binom{0i}{0k}\binom{0}{j}+\binom{0j}{0k}\binom{0}{i}}{\binom{0}{0}^2}(d-1)I_4^{[d+]^2}\\ &+\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{(d-1)d}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I_3^{[d+]^2,t}-\frac{\binom{0}{k}}{\binom{0}{0}}\frac{d-1}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I_{3,i}^{[d+]^2,t}\\ &+\sum_{t=1}^{4}\frac{\binom{0i}{0k}\binom{0t}{0j}+\binom{0j}{0k}\binom{0t}{0i}}{\binom{0}{0}^2}I_3^{[d+],t}+\frac{1}{\binom{0}{0}}\sum_{t=1,t\neq i,j}^{4}\binom{0t}{0k}\nu_{ij}I_{3,ij}^{[d+]^2,t}. \quad (5.19)\end{aligned}$$

It is obvious where the various contributions come from: Those being proportional to $-\binom{0}{k}/\binom{0}{0}$ come from (5.18), while all the others come from the second part of (5.12). We could indeed list these contributions separately without inserting the second part explicitly. However, in this way the coefficient of $I_4^{[d+]^2}$ gets contributions from both terms which combine to make the resulting coefficient explicitly symmetric in all indices. The symmetry of the 3-point functions is not so easily seen. Neverheless, numerically, it might be faster not to combine these terms but just to retain what had been derived.

Also here, as discussed in (5.16), we can replace the tensor- 3-point functions:

$$\begin{aligned}\nu_{ij}I_{3,ij}^{[d+]^2,t} = &\frac{\binom{0t}{it}}{\binom{0t}{0t}}\frac{\binom{0t}{jt}}{\binom{0t}{0t}}(d-1)d\;I_3^{[d+]^2,t}+\frac{1}{\binom{0t}{0t}}\binom{0ti}{0tj}I_3^{[d+],t}\\ &-\frac{\binom{0t}{jt}}{\binom{0t}{0t}}\frac{d-1}{\binom{0t}{0t}}\sum_{u=1}^{4}\binom{0tu}{0ti}I_2^{[d+],tu}+\frac{1}{\binom{0t}{0t}}\sum_{u=1}^{4}\binom{0tu}{0tj}I_{2,i}^{[d+],tu}. \quad (5.20)\end{aligned}$$

At this point, we observe a simple rule of how to obtain (5.19): replace in (5.15) $d\to d+2$ and multiply with $-(d-1)\frac{\binom{0}{k}}{\binom{0}{0}}$, where $(d-1)$ is to be chosen such that all factors $(d+i)$ increase by steps of 1 (see also (5.15) ). In this manner we increase simulaneously the dimension and the number of indices. Then one has to add the second part of (5.12).

Finally, for the tensor of rank $R=5$ we need $I_{4,ijkl}^{[d+]^4}$. Due to the above rule we need not again perform a complicated calculation, rather we apply the rule as proceeding from (5.15) to (5.19), increasing simultaneously dimension and number of indices: we have to shift in (5.19) $d\to d+2$,

muliply with $-d\frac{\binom{0}{l}}{\binom{0}{0}}$ and add the second part of (5.13). We obtain:

$$
\begin{aligned}
v_{ij}v_{ijk}v_{ijkl}I^{[d+]^4}_{4,ijkl} &= \frac{\binom{0}{i}}{\binom{0}{0}}\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}}d(d+1)(d+2)(d+3)I^{[d+]^4}_4 \\
&+\frac{\binom{0i}{0j}\binom{0}{k}\binom{0}{l}+\binom{0i}{0k}\binom{0}{j}\binom{0}{l}+\binom{0j}{0k}\binom{0}{i}\binom{0}{l}+\binom{0i}{0l}\binom{0}{j}\binom{0}{k}+\binom{0j}{0l}\binom{0}{i}\binom{0}{k}+\binom{0k}{0l}\binom{0}{i}\binom{0}{j}}{\binom{0}{0}^3}d(d+1)I^{[d+]^3}_4 \\
&+\frac{\binom{0i}{0l}\binom{0j}{0k}+\binom{0j}{0l}\binom{0i}{0k}+\binom{0k}{0l}\binom{0i}{0j}}{\binom{0}{0}^2}I^{[d+]^2}_4 \\
&-\frac{\binom{0}{j}}{\binom{0}{0}}\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}}\frac{d(d+1)(d+2)}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0i}I^{[d+]^3,t}_3+\frac{\binom{0}{k}}{\binom{0}{0}}\frac{\binom{0}{l}}{\binom{0}{0}}\frac{d(d+1)}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0j}I^{[d+]^3,t}_{3,i} \\
&-\frac{d}{\binom{0}{0}^3}\sum_{t=1}^{4}\left[\binom{0i}{0k}\binom{0t}{0j}+\binom{0j}{0k}\binom{0t}{0i}\right]\binom{0}{l}I^{[d+]^2,t}_3 \\
&-\frac{d}{\binom{0}{0}^3}\sum_{t=1}^{4}\left[\binom{0j}{0l}\binom{0t}{0i}\binom{0}{k}+\binom{0i}{0l}\binom{0t}{0j}\binom{0}{k}+\binom{0k}{0l}\binom{0t}{0i}\binom{0}{j}\right]I^{[d+]^2,t}_3 \\
&+\frac{1}{\binom{0}{0}^2}\sum_{t=1}^{4}\left[\binom{0j}{0l}\binom{0t}{0k}I^{[d+]^2,t}_{3,i}+\binom{0i}{0l}\binom{0t}{0k}I^{[d+]^2,t}_{3,j}+\binom{0k}{0l}\binom{0t}{0j}I^{[d+]^2,t}_{3,i}\right] \\
&-\frac{\binom{0}{l}}{\binom{0}{0}}\frac{d}{\binom{0}{0}}\sum_{t=1}^{4}\binom{0t}{0k}v_{ij}I^{[d+]^3,t}_{3,ij}+\frac{1}{\binom{0}{0}}\sum_{t=1,t\neq i,j}^{4}\binom{0t}{0l}v_{ij}v_{ijk}I^{[d+]^3,t}_{3,ijk}.
\end{aligned}
\tag{5.21}
$$

Again we, first of all, mention that the 3-point function $I^{[d+]^3,t}_{3,ijk}$ appearing here can as well be calculated like (5.16) and (5.20), essentially by taking over the 4-point result, shifting $d \to d+1$:

$$
\begin{aligned}
v_{ij}v_{ijk}I^{[d+]^3,t}_{3,ijk} = \quad & -\frac{\binom{0t}{it}}{\binom{0t}{0t}}\frac{\binom{0t}{jt}}{\binom{0t}{0t}}\frac{\binom{0t}{kt}}{\binom{0t}{0t}}d(d+1)(d+2)I^{[d+]^3,t}_3-\frac{\binom{0ti}{0tj}\binom{0t}{kt}+\binom{0ti}{0tk}\binom{0t}{jt}+\binom{0tj}{0tk}\binom{0t}{it}}{\binom{0t}{0t}^2}dI^{[d+]^2,t}_3 \\
&+\frac{\binom{0t}{jt}}{\binom{0t}{0t}}\frac{\binom{0t}{kt}}{\binom{0t}{0t}}\frac{d(d+1)}{\binom{0t}{0t}}\sum_{u=1}^{4}\binom{0tu}{0ti}I^{[d+]^2,tu}_2-\frac{\binom{0t}{kt}}{\binom{0t}{0t}}\frac{d}{\binom{0t}{0t}}\sum_{u=1}^{4}\binom{0tu}{0tj}I^{[d+]^2,tu}_{2,i} \\
&+\sum_{u=1}^{4}\frac{\binom{0ti}{0tk}\binom{0tu}{0tj}+\binom{0tj}{0tk}\binom{0tu}{0ti}}{\binom{0t}{0t}^2}I^{[d+],tu}_2+\frac{1}{\binom{0t}{0t}}\sum_{u=1,t\neq i,j}^{4}\binom{0tu}{0tk}v_{ij}I^{[d+]^2,tu}_{2,ij}.
\end{aligned}
\tag{5.22}
$$

In fact, having a closer look at our 4-point tensor coefficients, we observe that apart from the higher dimensional 4-point functions, all other terms are 3-point tensor coefficients, occasionally of higher tensor rank. Eq. 5.21 has been obtained by an educated guess. Indeed, a step by step derivation would have been extremely tedious if one would have had the courage at all to try the calculation. Of course one needs a verification by numerical checks: among others we found for non-exceptional Gram determinants an agreement with LoopTools of typically more than ten decimals; see sect. C for some details. Concerning these non-exceptional Gram determinants, we just mention that we evaluate (5.15), (5.19) and (5.21) by means of the recurrence relations of app. A. For further details see Sec. 6.

## 5.3 Expansion of $I_4^{[d+]^L}$ for small Gram determinants

The tensor coefficients in (2.17) to (2.20), in particular (5.15), (5.19) and (5.21), have been expressed in terms of 4-point functions in higher dimensions: $I_4^{[d+]}, I_4^{[d+]^2}, I_4^{[d+]^3}$ and $I_4^{[d+]^4}$. In our approach only these integrals can cause problems for small Gram determinants and therefore, finding a special approach for their calculation, will finalize the problem of calculating the 4-poin tensor coefficients. We start from the fact that for exactly vanishing Gram determinants (5.3) yields a finite value for 4-point functions of any dimension and, taking into account higher orders, we set up an infinite series in terms of powers of the Gram determinant:[5]

$$I_4^{[d+]^L} = \sum_{j=0}^{\infty} r^j \;\; I_{4,j}^L, \quad L = 1, \cdots, 4, \tag{5.23}$$

where the coefficients $I_{4,j}^L$ have to be determined and

$$r = \frac{()}{\binom{0}{0}}. \tag{5.24}$$

We use the observation that the recurrence relation with shift of dimension (2.31) (see also A.7) for the scalar higher-dimensional 4-point functions can be written as follows:

$$I_4^{[d+]^l} \;\; = \;\; Z_4^{[d+]^l} + \frac{()}{\binom{0}{0}} \left[(2l+1) - 2\varepsilon\right] I_4^{[d+]^{(l+1)}}, \quad l = 1, \cdots \tag{5.25}$$

where we re-wrote (5.3) as follows:

$$Z_4^{[d+]^l} = \frac{1}{\binom{0}{0}} \sum_{t=1}^{4} \binom{t}{0} I_3^{[d+]^l, t}, \quad l = 1, \cdots \tag{5.26}$$

From now on we assume $l > 0$, so that the integrals are infrared finite and the leading singularity in $\varepsilon$ is at most of the order $1/\varepsilon$. Integrals in generic dimension ($l = 0$) will be discussed at the end of the section. We treat the finite and divergent parts separately:

$$I_4^{[d+]^l} \;\; = \;\; F_4^{[d+]^l} + \frac{D_4^{[d+]^l}}{\varepsilon} + \mathcal{O}(\varepsilon^2), \tag{5.27}$$

$$Z_4^{[d+]^l} \;\; = \;\; Z_{4F}^l + \frac{Z_{4D}^l}{\varepsilon} + \mathcal{O}(\varepsilon^2). \tag{5.28}$$

The first few iterations of (5.25) are

$$\begin{aligned}
I_4^{[d+]^l} \;\; &= \;\; Z_4^{[d+]^l} + \frac{()}{\binom{0}{0}} c_{l+1} I_4^{[d+]^{(l+1)}} && (5.29)\\
&= \;\; Z_4^{[d+]^l} + \frac{()}{\binom{0}{0}} c_{l+1} \left\{ Z_4^{[d+]^{l+1}} + \frac{()}{\binom{0}{0}} c_{l+2} I_4^{[d+]^{(l+2)}} \right\} \\
&= \;\; Z_4^{[d+]^l} + \frac{()}{\binom{0}{0}} c_{l+1} Z_4^{[d+]^{l+1}} + \frac{()^2}{\binom{0}{0}^2} c_{l+1} c_{l+2} \left\{ Z_4^{[d+]^{l+2}} + \frac{()}{\binom{0}{0}} c_{l+3} I_4^{[d+]^{(l+3)}} \right\} \\
&= \;\; \cdots \;\; = \;\; Z_4^{[d+]^l} + \sum_{i=1}^{\infty} \frac{()^i}{\binom{0}{0}^i} \left[ \prod_{j=1}^{i} c_{l+j} \right] \;\; Z_4^{[d+]^{l+i}}, && (5.30)
\end{aligned}$$

[5]The series is not a Taylor series because the expansion coefficients are not the derivatives of $I_4^{[d+]^L}$ at $() = 0$.

with

$$c_{l+j} \;=\; 2(l+j)-1-2\varepsilon. \tag{5.31}$$

The 4-point functions are expressed in terms of an infinite power series in $()/\binom{0}{0}$ with higher-dimensional 3-point functions in the expansion coeffients.

For the finite and divergent part of (5.27) we get to lowest order

$$F_4^{[d+]^l} \;=\; Z_{4F}^l + \frac{()}{\binom{0}{0}}\left[(2l+1)F_4^{[d+]^{(l+1)}} - 2D_4^{[d+]^{(l+1)}}\right], \tag{5.32}$$

$$D_4^{[d+]^l} \;=\; Z_{4D}^l + \frac{()}{\binom{0}{0}}\left[(2l+1)D_4^{[d+]^{(l+1)}}\right]. \tag{5.33}$$

Now the second part in (5.32) is proportional to $()$ and can be considered as a correction term for small Gram determinants, where also a proper approximation for $F_4^{[d+]^{(l+1)}}$ has to be chosen. There are two simple choices. One has just to set in (5.32) $F_4^{[d+]^{(l+1)}} = Z_{4F}^{l+1}$ and neglect the unknown higher order terms, or one selects a close kinematical point with $()=0$ and uses $F_4^{[d+]^{(l+1)}} = Z_{4F}^{l+1}|_{()=0}$. We come back to these two alternatives later.

To calculate higher order corrections, we perform now iterations. Defining the correction term as

$$\delta Z_{4F,i}^l = \frac{()}{\binom{0}{0}}\left[(2l+1)Z_{4F,i}^{(l+1)} - 2\,D_4^{[d+]^{(l+1)}}\right], \quad i=0,1,2\cdots, \tag{5.34}$$

the iterative scheme then reads:

$$Z_{4F,i}^l = Z_{4F}^l + \delta Z_{4F,(i-1)}^l, \quad i=1,2,\cdots \tag{5.35}$$

The index $i$ counts the highest power of $()_4$ and the series $Z_{4F,i}^l$ is supposed to converge for growing $i$ towards $F_4^{[d+]^l}$. As a condition of applicability of the iteration we can obviously use

$$\frac{\delta Z_{4F,i}^l}{Z_{4F}^l} \sim \frac{()}{\binom{0}{0}} \times \text{scale} \ll 1, \tag{5.36}$$

where the scale has dimension of a squared mass. It is worth to perform the first few steps in the iteration explicitly:

$$\begin{aligned} Z_{4F,i}^l &= Z_{4F}^l + \frac{()}{\binom{0}{0}}\left[(2l+1)Z_{4F,(i-1)}^{(l+1)} - 2D_4^{[d+]^{(l+1)}}\right] \qquad (5.37)\\ &= Z_{4F}^l + \frac{()}{\binom{0}{0}}(2l+1)\left\{Z_{4F}^{(l+1)} + \frac{()}{\binom{0}{0}}\left[(2l+3)Z_{4F,(i-1)}^{(l+2)} - 2D_4^{[d+]^{(l+2)}}\right]\right\} - 2\frac{()}{\binom{0}{0}}D_4^{[d+]^{(l+1)}}\\ &= F_4^{[d+]^l} + \mathcal{O}(r^i). \end{aligned}$$

Performing $i$ steps in the iteration, we have

$$Z_{4F,i}^L \;=\; \sum_{j=0}^{i-1} a_j^L r^j Z_{4F}^{(L+j)} + a_i^L r^i Z_{4F,0}^{(L+i)} - 2\sum_{j=0}^{i-1} a_j^L r^{j+1} D_4^{[d+]^{(L+j+1)}}, \tag{5.38}$$

where $r$ is given in (5.24) and

$$a_j^L = 2^j \frac{\Gamma(L+j+\frac{1}{2})}{\Gamma(L+\frac{1}{2})}. \tag{5.39}$$

The $\Gamma(z)$ is the Euler Gamma function. In (5.38) we have to define yet $Z_{4F,0}^{(L+i)}$. Strictly speaking for $Z_{4F,0}^{(L+i)} = F_4^{[d+]^{L+i}}$ we have $Z_{4F,i}^{(l)} = F_4^{[d+]^l}$ $(i = 1, 2\cdots)$ but in order to evaluate (5.38) we have to choose an appropriate approximation.

Taking $i \to \infty$, however, and assuming convergence of the series, we have

$$F_4^{[d+]^L} = \sum_{j=0}^{\infty} a_j^L r^j Z_{4F}^{(L+j)} - 2 \sum_{j=0}^{\infty} a_j^L r^{j+1} D_4^{[d+]^{(L+j+1)}}, \tag{5.40}$$

i.e. in this limit the term with $Z_{4F,0}^{(L+i)}$ drops out. The choice of an approximation for $Z_{4F,0}^{(L+i)}$ thus can influence only the first few partial sums of (5.40). In fact, as will be seen in Sec. C in an example, the convergence is quite good for moderate $r$ and after a few steps the result is not very much dependent on the approximant of $Z_{4F,0}^{(L+i)}$.

It remains to deal with the last term in (5.38), i.e. $D_4^{[d+]^{(L+j+1)}}$. As in (5.38), we define a partial sum

$$Z_{4D,i}^L = \sum_{j=0}^{i} a_j^L r^j Z_{4D}^{(L+j)}, \tag{5.41}$$

and get corresponding to (5.40)

$$D_4^{[d+]^L} = \sum_{j=0}^{\infty} a_j^L r^j Z_{4D}^{(L+j)}. \tag{5.42}$$

In order to avoid in (5.38) higher order terms coming from $D_4^{[d+]^{(L+j+1)}}$, i.e. higher than those contained in the finite part, we re-write

$$\sum_{j=0}^{i-1} a_j^L r^{j+1} D_4^{[d+]^{(L+j+1)}} = \sum_{j=0}^{i-1} a_j^L r^{j+1} Z_{4D,i-1-j}^{(L+j+1)} + \mathscr{O}(r^{(i+1)}). \tag{5.43}$$

E.g. for $j = i-1$ on the right hand side of (5.43) there contributes a term $r^i Z_{4D,0}^{(L+i)}$, where $i$ is the highest power of $r$ which occurs also in the finite part of (5.38). Some algebra yields

$$\sum_{j=0}^{i-1} a_j^L r^{j+1} Z_{4D,i-1-j}^{(L+j+1)} = \frac{1}{2} \sum_{j=0}^{i} a_j^L b_j^L r^j Z_{4D}^{(L+j)}, \tag{5.44}$$

with coefficients

$$\begin{aligned} b_j^L &= \sum_{k=L+1}^{L+j} \frac{2}{2k-1} \\ &= \psi(L+j+\frac{1}{2}) - \psi(L+\frac{1}{2}), \end{aligned} \tag{5.45}$$

such that as final expression in terms of an infinite sum, we can write the solution of (5.23):

$$I_4^{[d+]^L} = \sum_{j=0}^{\infty} a_j^L r^j \left[ Z_4^{(L+j)} - b_j^L Z_{4D}^{(L+j)} \right], \quad L = 0, \cdots 4. \tag{5.46}$$

The $\psi(z)$ is the logarithmic derivative of the Gamma function (digamma function).

It is interesting to note that (5.46) is also valid for $L = 0$. Possible additional infrared divergent terms are then contained for $j = 0$ in $Z_4^{(0)}$. Comparing (5.46) with (5.23), we see that our goal is achieved.

Having now obtained the compact expression (5.46), it remains to discuss how to calculate the $Z_4^l$. Quite naturally one first calculates the needed $I_3^{[d+]^l,t}$ by means of recurrence (A.11) and sums over $t$ according to (5.26). This is possible since in general for $()_4 = 0$, $\binom{t}{t}_4 \neq 0$ for $t = 0, \cdots, 4$. Thus in order to evaluate (5.46), what is needed at the end, are the $I_3^t$, $t = 1, \cdots, 4$ , $I_2^{t,u}$, $t, u = 1, \cdots, 4, t \neq u$ and 4 1-point funcions for the kinematical point under consideration, i.e. we need 14 master integrals. Applying (A.11) in order to get the finite parts of $I_3^{[d+]^l,t}$, one also has to calculate the divergent parts of the higher-dimensional 2- and 3-point functions $I_2^{[d+]^{l-1},tu}$ and $I_3^{[d+]^{l-1},t}$. These have been discussed in app. B, but only for low values of $l$. They are needed for quite large $l$ ($l \sim 10$ and larger), and for larger $l$ the analytic expressions blow up considerably. Further, for $l > 6$ the analytic cancellation of the occurring Gram determinants is hard to perform. So, we preferred to work numerically. Amazingly, the situation is different for the 2-point functions. Without any problem we can produce with Mathematica any higher divergences with recurrence (A.12), cancelling thereby the $\binom{tu}{tu}$ Gram determinants. For details see the discussion in sect. A.3. To remain numerically as accurate as possible we use these analytic expressions when calculating the divergences of the 3-point functions numerically by recursion, starting with $D_3^{[d+]}(t) = -\frac{1}{2}$, see (B.4). As mentioned above, this has to be done for $l = 1, \cdots, l_{max} - 1$. As a result, in a practical calculation one has the objects

$$Z_{4F,i}^L, \quad i = 0, \ldots, l_{max} - L, \tag{5.47}$$

which are a sequence of approximations for the finite part of the integrals $I_4^{[d+]^L}$, i.e. $F_4^{[d+]^L}$. In app. C we give an example for their numerical evaluation.

At this point we stress that formulae (5.15), (5.19) and (5.21) are free of indexed inegrals $I_{4,i\cdots}^{[d+]^l}$ and thus enable a new access to the calculation of tensor 4-point functions. Relation (5.46) was already obained in [7], see (36) there: for $n$-point functions ($n = 1, 2, 3, 4$) of arbitrary dimension *with generic indices* $\nu_i = 1$ this series was derived in similar manner as above and a general scheme was developed of how to find analyic continuations to kinematical domains, where this series does not converge. E.g., if $\binom{0}{0}_4$ is small, $r$ in (5.24) is large. For this case [7] contains the description of how to modify the procedure of solving the recursion such that the expansion parameter is small, i.e. how to obtain from the recursion relation a series in $\frac{1}{r}$, see eqn. (23) ibid. Beyond that, from this series the $d$-dimensional $n$-point functions are obtained iteratively in terms of multiple hypergeometric series with ratios of different signed minors as arguments. For the 4-point function, e.g., the generalized hypergeometric functions ${}_2F_1$, Appell function $F_1$ and the Lauricella-Saran funcion $F_S$ appear, see e.g. (98) of [7]. Transformation formulae of the generalized hypergeometric functions allow to extend their applicability to different domains of the phase space. In particular in our situation we have to deal with integrals of dimension $d \geq 6 - 2\varepsilon$. The hypergeometric functions in [7] are also expressed in terms of 1-dimensional integrals and inspection shows that for the large dimensions these are particularly well suited for numerical evaluation (see eqns. (78) and (96)). We leave this for further study.

Another attempt to perform the described series of approximations was undertaken in [32]; see Eq. (5) there. A specific example was studied, namely forward light-by-light scattering through a massless fermion loop. The approach was then not further followed.

# 6 Symmetrized recurrence relations

So far in the former sect. we were concerned with the evaluation of the 4-point tensor coefficients for small Gram determinants. If, however, the Gram determinant is not small there are other ways of doing the reduction. In fact the "standard" Passarino-Veltman [3] reduction is one possibility. This is, however, not a unique procedure.

While in [21] a systematic application of recursion relations of type (2.5) was performed for all tensor $n$-point functions, here we take a different point of view, namely to arrange the tensor coefficients in (2.17) to (2.20) for the 4-point functions in such a way that a possible analytic simplification of the tensor as a whole is achieved.

We begin with $I_4^{\mu\nu}$ defined in (2.18), containing as most complicated object $I_{4,ij}^{[d+]^2}$. This integral, represented also by (5.15), may be reduced by (2.30),

$$\nu_{ij} I_{4,ij}^{[d+]^2} = \frac{\binom{0}{i}\binom{0}{j}}{()^2} I_4 + \frac{\binom{i}{j}}{()} I_4^{[d+]} + R_{3,ij}^{[d+]^2}, \tag{6.1}$$

with

$$\begin{aligned} R_{3,ij}^{[d+]^2} &= -\frac{\binom{0}{j}}{()} \sum_{t=1}^{4} \frac{\binom{t}{i}}{()} I_3^t + \sum_{t=1}^{4} \frac{\binom{t}{j}}{()} I_{3,i}^{[d+],t} \\ &= -\sum_{t=1}^{4} \frac{\binom{0}{i}\binom{t}{j} + \binom{0}{j}\binom{t}{i} - \binom{t}{i}\binom{t}{j}\frac{\binom{t}{0}}{\binom{t}{t}}}{()^2} I_3^t + \frac{1}{()} \sum_{t,u=1}^{4} \frac{\binom{t}{j}}{\binom{t}{t}} \binom{ut}{it} I_2^{tu}. \end{aligned} \tag{6.2}$$

The first observation of interest here is the symmetry in the indices $i, j$. Only the last term is not obviously symmetric. As was mentioned earlier, the symmetry is in general seen only after summation over $s,t$, which we can exemplify here. We use the relation

$$\binom{t}{j}\binom{ut}{it} = \binom{t}{i}\binom{ut}{jt} + \binom{t}{t}\binom{tu}{ji}. \tag{6.3}$$

Inserting the left hand side into (6.2), the first term on the right hand side has just exchanged indices $i, j$. In the second contribution $\binom{t}{t}$ cancels due to which the sum over $s,t$ vanishes since $\binom{tu}{ji}$ is antisymmetric in $t,u$.

In (6.1) there remains as higher-dimensional integral $I_4^{[d+]}$, which is evaluated according to (A.7). Often it is as well used as "master integral" since it is UV and IR finite. Having a look at (2.18) we see that the second amplitude of the rank $R = 2$ tensor is also just $I_4^{[d+]}$. This allows the following way of writing for this tensor. Similarly as in [21] we introduce [6]

$$\begin{aligned} G^{\mu\nu} &= g^{\mu\nu} - 2 \sum_{i,j=1}^{4} q_i^\mu q_j^\nu \frac{\binom{i}{j}}{()} \\ &= \frac{8 v^\mu v^\nu}{()}, \end{aligned} \tag{6.4}$$

[6] $G^{\mu\nu}$ here differs from the definition (24) in [21] by a factor of 2.

with

$$v^\mu = \varepsilon^{\mu\lambda\rho\sigma}(q_1-q_4)_\lambda(q_2-q_4)_\rho(q_3-q_4)_\sigma, \tag{6.5}$$

and $v^2 = \frac{1}{8}()$. This allows to drop $I_4^{[d+]}$ in (6.1) and to replace $g^{\mu\nu}$ in (2.18) by $G^{\mu\nu}$:

$$\begin{aligned} I_4^{\mu\nu} &= \sum_{i,j=1}^4 q_i^\mu q_j^\nu \left[ \frac{\binom{0}{i}\binom{0}{j}}{()^2} I_4 - \sum_{t=1}^4 \frac{\binom{0}{i}\binom{t}{j}+\binom{0}{j}\binom{t}{i}-\binom{t}{i}\binom{t}{j}\frac{\binom{t}{0}}{\binom{t}{t}}}{()^2} I_3^t + \frac{1}{()}\sum_{t,u=1}^4 \frac{\binom{t}{j}}{\binom{t}{t}}\binom{ut}{it} I_2^{tu}\right] \\ &\quad - \frac{1}{2}G^{\mu\nu} I_4^{[d+]}. \end{aligned} \tag{6.6}$$

This representation may become advantageous in the analytic evaluation of diagrams since $G^{\mu\nu}$, contracted with a (proper difference of) chord(s), vanishes. Remember that any external momentum may be written as a sum of chords. For the tensor of rank $R=2$ this corresponds to (20) of [21].

We will derive here the corresponding relations for the higher tensors as well.

Proceeding to $I_4^{\mu\nu\lambda}$, (2.19), we ,first of all, need in the second term on the right hand side $I_{4,i}^{[d+]^2}$, ((A.6) with $l=2$):

$$I_{4,i}^{[d+]^2} = -\frac{\binom{0}{i}}{()} I_4^{[d+]} + \sum_{t=1}^4 \frac{\binom{t}{i}}{()} I_3^{[d+],t}, \tag{6.7}$$

where again $I_4^{[d+]}$ appears and $I_3^{[d+],t}$ is given in (A.11). From the recursions of subsect. A.1 we obtain

$$v_{ij}v_{ijk}I_{4,ijk}^{[d+]^3} = -\frac{\binom{0}{i}\binom{0}{j}\binom{0}{k}}{()^3} I_4 + \left\{ \frac{\binom{i}{j}}{()} I_{4,k}^{[d+]^2} + (j\leftrightarrow k) + (i\leftrightarrow k)\right\} + R_{3,ijk}^{[d+]^3}, \tag{6.8}$$

where we have introduced an abbreviation for the remaining 3- and 2-point functions:

$$\begin{aligned} R_{3,ijk}^{[d+]^3} = &\quad -\frac{\binom{0}{k}}{()} R_{3,ij}^{[d+]^2} - \sum_{t=1}^4 \frac{\binom{t}{k}}{()}\frac{\binom{0t}{jt}}{\binom{t}{t}} I_{3,i}^{[d+],t} - \sum_{t=1}^4 \frac{\binom{t}{i}\binom{t}{j}\binom{t}{k}}{()^2\binom{t}{t}} I_3^{[d+],t} + \sum_{t,u=1}^4 \frac{\binom{t}{k}\binom{ut}{jt}}{()\binom{t}{t}} I_{2,i}^{[d+],tu} \\ = &\quad \sum_{t=1}^4 \frac{\binom{0}{i}\binom{0}{j}\binom{t}{k}+\binom{0}{i}\binom{t}{j}\binom{0}{k}+\binom{t}{i}\binom{0}{j}\binom{0}{k}}{()^3} I_3^t \\ - &\quad \sum_{t=1}^4 \frac{\binom{0}{i}\binom{t}{j}\binom{t}{k}+\binom{t}{i}\binom{0}{j}\binom{t}{k}+\binom{t}{i}\binom{t}{j}\binom{0}{k}-\binom{t}{i}\binom{t}{j}\binom{t}{k}\frac{\binom{t}{0}}{\binom{t}{t}}}{()^3}\frac{\binom{t}{0}}{\binom{t}{t}} I_3^t - \sum_{t=1}^4 \frac{\binom{t}{i}\binom{t}{j}\binom{t}{k}}{()^2\binom{t}{t}} I_3^{[d+],t} \\ - &\quad \sum_{t,u=1}^4 \frac{\left[\binom{0}{j}\binom{t}{k}+\binom{0}{k}\binom{t}{j}-\binom{t}{j}\binom{t}{k}\frac{\binom{t}{0}}{\binom{t}{t}}\right]}{()^2}\frac{\binom{ut}{it}}{\binom{t}{t}} I_2^{tu} + \sum_{t,u=1}^4 \frac{\binom{t}{k}\binom{ut}{jt}}{()\binom{t}{t}} I_{2,i}^{[d+],tu}. \end{aligned} \tag{6.9}$$

In (6.8), the $\frac{\binom{i}{j}}{()} I_4^{[d+]}$ of (6.1) together with part of the last contribution in (A.9) has now been absorbed in $\frac{\binom{i}{j}}{()} I_{4,k}^{[d+]^2}$: in (A.4) only two terms of this type appear explicitely, i.e. this form has no obvious symmetry. Further, in (6.8) and (6.9) only the 4- and 3- point functions are explicitly symmetric in

the indices $i,j,k$. In order to demonstrate symmetry also for the 2-point functions one would have to reduce also $I_{2,i}^{[d+],tu}$, which is given in app. A.

We may now combine the results and simplify $I_4^{\mu\nu\lambda}$ correspondingly: The $I_{4,k}^{[d+]^2}$ in (6.8) can be combined with the second part of (2.19), i.e. it can be dropped in (6.8), and in (2.19) the $g^{\mu\nu}$ must then be replaced by $G^{\mu\nu}$. (6.8) has also been identified with a complete reduction of (5.19).

For $I_4^{\mu\nu\lambda\rho}$ (2.20), we start from (A.3):

$$\begin{aligned}
n_{ijkl}I_{4,ijkl}^{[d+]^4} &= \frac{\binom{0}{i}\binom{0}{j}\binom{0}{k}\binom{0}{l}}{()^4}I_4 - \frac{\binom{0}{l}}{()}\left\{\frac{\binom{i}{j}}{()}I_{4,k}^{[d+]^2} + (j\leftrightarrow k) + (i\leftrightarrow k) + R_{3,ijk}^{[d+]^3}\right\} \\
&+ \left\{\frac{\binom{i}{l}}{()}\frac{\binom{j}{k}}{()} + \frac{\binom{j}{l}}{()}\frac{\binom{i}{k}}{()} + \frac{\binom{k}{l}}{()}\frac{\binom{i}{j}}{()}\right\}I_4^{[d+]^2} \\
&- \left\{\frac{\binom{i}{l}}{()}\frac{\binom{0}{k}}{()}I_{4,j}^{[d+]^2} + \frac{\binom{j}{l}}{()}\frac{\binom{0}{k}}{()}I_{4,i}^{[d+]^2} + \frac{\binom{k}{l}}{()}\frac{\binom{0}{j}}{()}I_{4,i}^{[d+]^2}\right\} \\
&+ \frac{\binom{i}{l}}{()}\sum_{t=1}^{4}\frac{\binom{t}{k}}{()}I_{3,j}^{[d+]^2,t} + \frac{\binom{j}{l}}{()}\sum_{t=1}^{4}\frac{\binom{t}{k}}{()}I_{3,i}^{[d+]^2,t} + \frac{\binom{k}{l}}{()}\sum_{t=1}^{4}\frac{\binom{t}{j}}{()}I_{3,i}^{[d+]^2,t} + \sum_{t=1}^{4}\frac{\binom{t}{l}}{()}n_{ijk}I_{3,ijk}^{[d+]^3,t}.
\end{aligned} \tag{6.10}$$

As can be seen, the $n_{ijkl}I_{4,ijkl}^{[d+]^4}$ can be expressed in a form which mainly contains terms which also occur in

$$\nu_{ij}I_{4,ij}^{[d+]^3} = -\frac{\binom{0}{j}}{()}I_{4,i}^{[d+]^2} + \frac{\binom{i}{j}}{()}I_4^{[d+]^2} + \sum_{t=1}^{4}\frac{\binom{t}{j}}{()}I_{3,i}^{[d+]^2,t}, \tag{6.11}$$

see (A.5). The $n_{ijk}I_{3,ijk}^{[d+]^3,t}$ can be written similarly like (6.8):

$$\nu_{ij}\nu_{ijk}I_{3,ijk}^{[d+]^3,t} = -\frac{\binom{0t}{it}\binom{0t}{jt}\binom{0t}{kt}}{\binom{t}{t}^3}I_3^t + \left\{\left[\frac{\binom{i}{j}}{()} - \frac{\binom{t}{i}\binom{t}{j}}{()\binom{t}{t}}\right]I_{3,k}^{[d+]^2,t} + (j\leftrightarrow k) + (i\leftrightarrow k)\right\} + R_{2,ijk}^{[d+]^3,t}, \tag{6.12}$$

where $R_{2,ijk}^{[d+]^3,t}$ collects the remaining 2- and 1-point functions and is obtained from (6.9) as follows: all 3-point functions are replaced by 2-point functions ($3\to 2$) and the 2-point functions are replaced by 1-point functions ($2\to 1$). All summation indices $u$ must be replaced by $v$ and summation indices $t$ must be replaced by $u$. Finally in all determinants and integrals columns, lines and propagators $t$ must be scratched - like in (6.12), i.e.

$$R_{2,ijk}^{[d+]^3,t} = -\frac{\binom{0t}{kt}}{\binom{t}{t}}R_{2,ij}^{[d+]^2,t} - \sum_{u=1}^{4}\frac{\binom{ut}{kt}}{\binom{t}{t}}\frac{\binom{0tu}{jtu}}{\binom{tu}{tu}}I_{2,i}^{[d+],tu} - \sum_{u=1}^{4}\frac{\binom{ut}{it}\binom{ut}{jt}\binom{ut}{kt}}{\binom{t}{t}^2\binom{tu}{tu}}I_2^{[d+],tu} + \sum_{u,v=1}^{4}\frac{\binom{ut}{kt}\binom{vtu}{jtu}}{\binom{t}{t}\binom{tu}{tu}}I_{1,i}^{[d+],tuv}, \tag{6.13}$$

and

$$R_{2,ij}^{[d+]^2,t} = -\frac{\binom{0t}{jt}}{\binom{t}{t}}\sum_{u=1}^{4}\frac{\binom{ut}{it}}{\binom{t}{t}}I_2^{tu} + \sum_{u=1}^{4}\frac{\binom{ut}{jt}}{\binom{t}{t}}I_{2,i}^{[d+],tu}. \tag{6.14}$$

To indicate how the various tensor components may be combined to simplify the result, we only count here the number of contributions of a certain type. The second term in (2.20) contains, according to (2.23), 6 terms of type (6.11). Each type of the 3 terms in (6.11) is also contained in the other tensor components in (2.20):

- $\frac{\binom{0}{j}}{()} I_{4,i}^{[d+]^2}$ occurs 6 times in (6.11),
- $\frac{\binom{i}{j}}{()} I_{4}^{[d+]^2}$ occurs 3 times in (6.11) and occurs 3 times in the $3^{rd}$ term of (2.20),
- $\sum_{t=1}^{4} \frac{\binom{t}{j}}{()} I_{3,i}^{[d+]^2,t}$ occurs 3 times in (6.11) and occurs 3 times in (6.12).

Thus, rewriting (6.4) as

$$\sum_{i,j=1}^{4} q_i^\mu q_j^\nu \frac{\binom{i}{j}}{()} = \frac{1}{2}\left(g^{\mu\nu} - G^{\mu\nu}\right), \tag{6.15}$$

one can convince oneself that $g^{\mu\nu}$ cancels and it remains $G^{\mu\nu}$, which after contraction with a chord drops out.

As we have seen, the above treatment of tensor coefficients requires at least one step of iteration of the recursion relation, because of which it is applicable only for non-vanishing Gram determinants. For these, however, it is very useful when we consider 4-point functions obtained by scratching one line of a 5-point function. In this case it is indeed possible to perform a cancellation of terms, which above still have the factor $G^{\mu\nu}$. Also for this case, which we dealt with in sect. 3, the above presentation of the 4-point tensor coefficients can be applied - only scratching of one propagator has to be taken into account.

Beyond that the present approach results in reductions which make the symmetry in the indices $i, j, k, l$ more transparent and as a consequence yield certain blocks which can be calculated separately and combined to yield the complete tensor coefficients.

# 7 Analytic simplifications for contractions of tensors with chords

Before a numerical program for calculations is set up, it turns out to be advantageous to simplify Feynman diagrams analytically. A standard example is the following. If in the numerator of a Feynman integral a scalar product $q_i \cdot k$ of a chord and an integration momentum occurs, this product is usually expressed in terms of the difference of two scalar propagators which can be cancelled against propagators in the denominator. Already in [21] an alternative was indicated, making use of the fact that the contraction of a vector of the type (2.6) with a chord yields a simple expression:

$$q_i \cdot Q_0 = \sum_{j=1}^{n-1} q_i q_j \frac{\binom{0}{j}_n}{()_n} = -\frac{1}{2}\left(Y_{in} - Y_{nn}\right), \quad i = 1, \ldots, n-1, \tag{7.1}$$

and

$$q_i \cdot Q_s = \sum_{j=1}^{n-1} q_i q_j \frac{\binom{s}{j}_n}{()_n} = \frac{1}{2}\left(\delta_{is} - \delta_{ns}\right), \quad i = 1, \ldots, n-1, \quad s = 1, \ldots n. \tag{7.2}$$

In (7.1) and (7.2) $q_n = 0$ is assumed since only in this case

$$q_i \cdot q_j = \frac{1}{2}\left[Y_{ij} - Y_{in} - Y_{nj} + Y_{nn}\right], \tag{7.3}$$

which is needed for their derivations. Thus, if the reduction relation (2.5) for $I_5^{\mu_1 \cdots \mu_{R-1}\mu}$ is contracted with a $q_{i,\mu}$, we can advantageously apply (7.1) and (7.2) with $n = 5$.

In case one considers a process with 5 external legs, one can choose from the very beginning $q_5 = 0$ in the tensor integrals. If, however, the 5-point tensor is obtained by reducing a 6-point tensor, cases with $q_5 \neq 0$ will occur. In order to be able to apply (7.1) and (7.2), it is recommended to perform a shift of the integration momentum like $k \to k + q_5$, i.e. $q_i \to q_i - q_5$. Such a shift is not a problem at all, nevertheless it is interesting to see how this shift can be implemented in the formalism. The scalar integrals and the signed minors are invariant under the shift. We exemplify this for $I_n^{\mu\nu}$, writing

$$\begin{aligned} -\sum_{i,j=1}^{n} (q_i - q_n)^\mu (q_j - q_n)^\nu \nu_{ij} I_{n,ij}^{[d+]^2} = \quad & -\sum_{i,j=1}^{n} q_i^\mu q_j^\nu \nu_{ij} I_{n,ij}^{[d+]^2} + q_n^\mu \sum_{i,j=1}^{n} q_j^\nu \nu_{ij} I_{n,ij}^{[d+]^2} \\ & + q_n^\nu \sum_{i,i=1}^{n} q_i^\nu \nu_{ij} I_{n,ij}^{[d+]^2} - q_n^\mu q_n^\nu \sum_{i,j=1}^{n} \nu_{ij} I_{n,ij}^{[d+]^2}, \end{aligned} \tag{7.4}$$

with

$$\nu_{ij} I_{n,ij}^{[d+]^2} = -\frac{\binom{0}{j}_n}{()_n} I_{n,i}^{[d+]} + \sum_{t=1, t\neq i}^{n} \frac{\binom{t}{j}_n}{()_n} I_{n-1,i}^{[d+],t} + \frac{\binom{i}{j}_n}{()_n} I_n^{[d+]}. \tag{7.5}$$

The sums

$$\sum_{j}^{n} \nu_{ij} I_{n,ij}^{[d+]^2} = -I_{n,i}^{[d+]}, \tag{7.6}$$

$$\sum_{i=1}^{n} \nu_{ij} I_{n,ij}^{[d+]^2} = -I_{n,j}^{[d+]}, \tag{7.7}$$

$$\sum_{i,j=1}^{n} \nu_{ij} I_{n,ij}^{[d+]^2} = I_n \tag{7.8}$$

can be obtained by making use of

$$\sum_{i=1}^{n} \binom{0}{i}_n = ()_n, \tag{7.9}$$

$$\sum_{i=1}^{n} \binom{s}{i}_n = 0. \tag{7.10}$$

Thus one has

$$\sum_{i,j=1}^{n} q_i^\mu q_j^\nu \nu_{ij} I_{n,ij}^{[d+]^2} = \sum_{i,j=1}^{n-1} q'^\mu_i q'^\nu_i \nu_{ij} I_{n,ij}^{[d+]^2} - q_n^\mu \sum_{i=1}^{n} q_i^\nu I_{n,i}^{[d+]} - q_n^\nu \sum_{i=1}^{n} q_i^\mu I_{n,i}^{[d+]} - q_n^\mu q_n^\nu I_n, \tag{7.11}$$

using the abbreviation $q'^\mu_i = q_i^\mu - q_n^\mu$, and with

$$-\sum_{i=1}^{n} q_i^\mu I_{n,i}^{[d+]} = -\sum_{i=1}^{n-1} q'^\mu_i I_{n,i}^{[d+]} + q_n^\mu I_n, \tag{7.12}$$

derived in the same manner, with (2.17) and (2.18) the standard result of shifting the integration momentum is obtained. The point is that the extra contributions obtained by the shift contain only integrals which were needed already in the unshifted integral so that the shift does not require the calculation of any new integral, see (7.5).

Assume now again that we are dealing with the 5-point tensor and have $q_5 = 0$. The above trick to avoid an increase of the tensor rank can be applied to (4.8) as well: contracting with $q_{i,\mu}$ the first term yields the contribution $q_i \cdot Q_0$ given in (7.1), for the second term we have to find a formula for the scalar product $q_i \cdot \bar{Q}^0_s$. Indeed,

$$\begin{aligned} q_i \bar{Q}^0_s &= \sum_{j=1}^{4} q_i q_j \binom{0s}{0j}_5 \\ &= \frac{1}{2}\left[\binom{0}{0}_5 (\delta_{is} - \delta_{5s}) + \binom{s}{0}_5 (Y_{i5} - Y_{55})\right]. \end{aligned} \tag{7.13}$$

For the first term in (4.8) also another possibility exists, provided a contraction with a further vector is available. In such a case the first term on the right hand side of (4.11), which shows up explicitly in the tensor components $E_{ij\ldots}$, yields a double-sum like

$$\sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \binom{0i}{sj}_5 = \frac{1}{2} q_a \cdot q_b \binom{s}{0}_5 + \frac{1}{4}()_5 (Y_{b5} - Y_{55})(\delta_{as} - \delta_{5s}). \tag{7.14}$$

Further sums are obtained if the 4-point tensors are contracted. These are all of the type $\binom{is}{js}_5$, i.e. with line $s$ scratched. We just list a few of them:

$$\sum_{j=1}^{4} q_a \cdot q_j \binom{0s}{js}_5 = -\frac{1}{2}\left[\binom{s}{0}_5 (\delta_{as} - \delta_{5s}) + \binom{s}{s}_5 (Y_{a5} - Y_{55})\right], \tag{7.15}$$

$$\sum_{i,j=1}^{4} q_i \cdot q_j \binom{0s}{is}_5 \binom{0s}{js}_5 = \frac{1}{2}\binom{s}{s}_5 \left[\binom{0s}{0s}_5 + Y_{55}\binom{s}{s}_5 + 2\binom{s}{0}_5 \delta_{5s}\right], \tag{7.16}$$

$$\sum_{i,j=1}^{4} q_i \cdot q_j \binom{is}{js}_5 = \frac{3}{2}\binom{s}{s}_5, \tag{7.17}$$

$$\sum_{i,j=1}^{4} (q_a \cdot q_i)(q_b \cdot q_j) \binom{is}{js}_5 = \frac{1}{2} q_a \cdot q_b \binom{s}{s}_5 - \frac{1}{4}()_5 (\delta_{ab}\delta_{as} + \delta_{5s}), \tag{7.18}$$

and for 3-point functions

$$\begin{aligned} \sum_{j=1}^{4} q_a \cdot q_j \binom{ts}{js}_5 &= \frac{1}{2}\left\{\binom{s}{s}_5 [(1-\delta_{as})\delta_{at} - (1-\delta_{5s})\delta_{5t}]\right. \\ &\quad \left. - \binom{t}{a}_5 (1-\delta_{at})\delta_{as} + \binom{t}{5}_5 (1-\delta_{5t})\delta_{5s}\right\}, \end{aligned} \tag{7.19}$$

$$\sum_{i,j=1}^{4} q_i \cdot q_j \binom{ts}{is}_5 \binom{ts}{js}_5 = \frac{1}{2}\binom{s}{s}_5 \binom{st}{st}_5, \tag{7.20}$$

$$\sum_{i,j=1}^{4} q_i \cdot q_j \binom{ts}{is}_5 \binom{0s}{js}_5 = \frac{1}{2}\binom{s}{s}_5 \left\{\binom{0s}{ts}_5 - \binom{s}{s}_5 (1-\delta_{5s})\delta_{5t} + \binom{t}{5}_5 (1-\delta_{5t})\delta_{5s}\right\}, \tag{7.21}$$

$$\sum_{i,j=1}^{4} q_i \cdot q_j \binom{ist}{jst}_5 = \binom{st}{st}_5. \tag{7.22}$$

Even a quadrupel sum appears:

$$\sum_{i,j,k,l=1}^{4} (q_i \cdot q_j)(q_k \cdot q_l) \binom{0i}{sl}_5 \binom{ts}{js}_5 \binom{ts}{ks}_5 = \frac{1}{4} \binom{s}{0}_5 \binom{s}{s}_5 \binom{st}{st}_5 . \tag{7.23}$$

In fact, there are many more such sums. Our conclusion here is that in every scalar, which is obtained by contraction with chords the appearing sums can be evaluated analytically in order to yield compact expressions. This is due to the fact that the indices of the chords $i, j, c \ldots$ are carried by signed minors while the integrals don't necessarily carry indices anymore.

# 8 Conclusions

We have developed a new approach to reduce tensorial one-loop $n$-point Feynman integrals based on an algebraic method elaborated in earlier papers. The approach is worked out up to 6-point tensors with rank $R \leq 6$ and a rule is found how to extend the method to higher ranks. The first step was to reduce 5-point tensors up to rank 5 to 4-point tensor coefficients given in terms of higher-dimensional, indexed 4-point functions. The latter are expressed in terms of higher-dimensional integrals $I_4^{[d+]^L}$, $L = 1, \cdots, 4$, plus 3-point tensor coefficients in (5.10), (5.15), (5.19) and (5.21). So far no Gram determinants $()_4$ occur. Inverse powers of $()_4$ do occur if the integrals $I_4^{[d+]^L}$ are reduced to standard $A_0, B_0, C_0, D_0$ functions in generic dimension. For small $()_4$ this is avoided by using the expansion (5.46) in positive powers of $()_4$. Application of Padé-approximants based on the $\varepsilon$-algorithm to this expansion allows to calculate the $I_4^{[d+]^L}$ in a simple manner to such a precision that the complete phase space is covered with high numerical precision. A numerical opensource code of the formulae derived in this article is under development.

As a matter of fact, (5.46) is a special case of the general method developed in [7]. Apart from this special series expansion, the integrals $I_4^{[d+]^L}$ are expressed in [7] in terms of multiple hypergeometric functions ${}_2F_1$, Appell function $F_1$ and Lauricella-Saran function $F_S$. In this context, it is a crucial property of eqns. (5.10), (5.15), (5.19) and (5.21) to be free of integrals $I_{4,i\cdots}^{[d+]^L}$ with indices larger than one. Using the notion of special functions allows a variety of options to adjust to various kinematical situations, and it might be interesting to explore their potential for a further improvement of numerical programs.

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# A Recursion relations for the reduction of higher-dimensional 4-point functions

In this app. we quote explicitly the needed recurrence relations for the reduction of the 4-point functions. In [6] three recurrence relations are given: one reducing simultaneously index and dimension, one reducing only the dimension and another one reducing only the index. In the present work only the first two relations are used, i.e. (2.30) and (2.31). Special cases of (2.31) are (A.7), (A.11) and (A.12), i.e. these are the ones used to reduce the dimension of the scalar 4-, 3- and 2-point functions. All the other ones are special cases of (2.30) and reduce the tensor indices. Reducing 4-point functions, 3- and 2-point functions are generated, for which we also have to give the corresponding recurrence relations.

## A.1 Reduction of 4-point integrals

To learn about practical applications of the recurrence relations it is useful to investigate the most complicated one under consideration in some detail. Recall the definition given after (2.27), $[d+]^l = 4+2l-2\varepsilon = d+2l$. With (2.30),

$$\nu_{ijkl} I^{[d+]^4}_{4,ijkl} = -\frac{\binom{0}{l}_4}{()_4} I^{[d+]^3}_{4,ijk} + \sum_{t=1,t\neq i,j,k}^{4} \frac{\binom{t}{l}_4}{()_4} I^{[d+]^3,t}_{3,ijk} + \frac{\binom{i}{l}_4}{()_4} I^{[d+]^3}_{4,jk} + \frac{\binom{j}{l}_4}{()_4} I^{[d+]^3}_{4,ik} + \frac{\binom{k}{l}_4}{()_4} I^{[d+]^3}_{4,ij}. \quad (A.1)$$

The integral contributes to the rank $R=5$ tensor coefficient $E_{ijklm}$, see (5.13). Strictly speaking (A.1) is in this form valid only if all indices $i,j,k$ are different. In case that some or all indices are equal there is no repetition of the same terms on the right hand side and the question is how to take into account this property in a general manner. Let us recall that in fact in the original integral (2.21) we have to deal with $n_{ijkl} I^{[d+]^4}_{4,ijkl}$, where $n_{ijkl} = \nu_{ij}\nu_{ijk}\nu_{ijkl}$. Thus it is recommended to multiply (A.1) with $\nu_{ij}\nu_{ijk}$. Let us further introduce for the sum of the last three terms in (A.1) the notation $[ijk]^{(l)}$ (with repetition) and $[ijk]^{(l)}_{red}$ (without repetition). Then we have the following useful relation:

$$\begin{aligned} \nu_{ij}\nu_{ijk}[ijk]^{(l)}_{red} &= [ijk]^{(l)} + \delta_{jk}\frac{\binom{i}{l}_4}{()_4} I^{[d+]^3}_{4,jk} + \delta_{ik}\frac{\binom{j}{l}_4}{()_4} I^{[d+]^3}_{4,ik} + \delta_{ij}\frac{\binom{k}{l}_4}{()_4} I^{[d+]^3}_{4,ij} \\ &= \nu_{jk}\frac{\binom{i}{l}_4}{()_4} I^{[d+]^3}_{4,jk} + \nu_{ik}\frac{\binom{j}{l}_4}{()_4} I^{[d+]^3}_{4,ik} + \nu_{ij}\frac{\binom{k}{l}_4}{()_4} I^{[d+]^3}_{4,ij}. \end{aligned} \quad (A.2)$$

Thus, with $n_{ijk} = \nu_{ij}\nu_{ijk}$ (A.1) reads

$$\begin{aligned} n_{ijkl} I^{[d+]^4}_{4,ijkl} = & -\frac{\binom{0}{l}_4}{()_4} n_{ijk} I^{[d+]^3}_{4,ijk} \\ & + \sum_{t=1,t\neq i,j,k}^{4} \frac{\binom{t}{l}_4}{()_4} n_{ijk} I^{[d+]^3,t}_{3,ijk} + \frac{\binom{i}{l}_4}{()_4} \nu_{jk} I^{[d+]^3}_{4,jk} + \frac{\binom{j}{l}_4}{()_4} \nu_{ik} I^{[d+]^3}_{4,ik} + \frac{\binom{k}{l}_4}{()_4} \nu_{ij} I^{[d+]^3}_{4,ij}. \end{aligned} \quad (A.3)$$

Correspondingly we have

$$n_{ijk} I^{[d+]^l}_{4,ijk} = -\frac{\binom{0}{k}_4}{()_4} \nu_{ij} I^{[d+]^{l-1}}_{4,ij} + \sum_{t=1,t\neq i,j}^{4} \frac{\binom{t}{k}_4}{()_4} \nu_{ij} I^{[d+]^{l-1},t}_{3,ij} + \frac{\binom{i}{k}_4}{()_4} I^{[d+]^{l-1}}_{4,j} + \frac{\binom{j}{k}_4}{()_4} I^{[d+]^{l-1}}_{4,i}, \quad (A.4)$$

$$\nu_{ij} I^{[d+]^l}_{4,ij} = -\frac{\binom{0}{j}_4}{()_4} I^{[d+]^{l-1}}_{4,i} + \sum_{t=1,t\neq i}^{4} \frac{\binom{t}{j}_4}{()_4} I^{[d+]^{l-1},t}_{3,i} + \frac{\binom{i}{j}_4}{()_4} I^{[d+]^{l-1}}_{4}, \quad (A.5)$$

and directly from (2.30) and (2.31) we obtain:

$$I_{4,i}^{[d+]^l} = -\frac{\binom{0}{i}_4}{()_4} I_4^{[d+]^{l-1}} + \sum_{t=1}^{4} \frac{\binom{t}{i}_4}{()_4} I_3^{[d+]^{l-1},t}, \tag{A.6}$$

$$I_4^{[d+]^l} = \left[ \frac{\binom{0}{0}_4}{()_4} I_4^{[d+]^{l-1}} - \sum_{t=1}^{4} \frac{\binom{t}{0}_4}{()_4} I_3^{[d+]^{l-1},t} \right] \frac{1}{d+2l-5}. \tag{A.7}$$

In (A.7) we can put $d=4$ for $l=1$ since $I_4^{[d+]}$ is UV and IR finite. Concerning the factors $n_{ijkl\ldots}$ and $\nu_{ijkl\ldots}$ we see that the recursions work as if there were no such factors at all: each recursion eliminates one of the factors $\nu_{ij\ldots}$. This property continues to be valid for 3- and 2-point functions.

## A.2 Reduction of 3-point integrals

$$n_{ijk} I_{3,ijk}^{[d+]^3,t} = -\frac{\binom{0t}{kt}_4}{\binom{t}{t}_4} \nu_{ij} I_{3,ij}^{[d+]^2,t} + \sum_{u=1,u\neq i,j}^{4} \frac{\binom{ut}{kt}_4}{\binom{t}{t}_4} \nu_{ij} I_{2,ij}^{[d+]^2,tu} + \frac{\binom{it}{kt}_4}{\binom{t}{t}_4} I_{3,j}^{[d+]^2,t} + \frac{\binom{jt}{kt}_4}{\binom{t}{t}_4} I_{3,i}^{[d+]^2,t}, \tag{A.8}$$

$$\nu_{ij} I_{3,ij}^{[d+]^l,t} = -\frac{\binom{0t}{jt}_4}{\binom{t}{t}_4} I_{3,i}^{[d+]^{l-1},t} + \sum_{u=1,u\neq t,i}^{4} \frac{\binom{ut}{jt}_4}{\binom{t}{t}_4} I_{2,i}^{[d+]^{l-1},tu} + \frac{\binom{it}{jt}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t}, \tag{A.9}$$

$$I_{3,i}^{[d+]^l,t} = -\frac{\binom{0t}{it}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} + \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{it}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu}, \tag{A.10}$$

$$I_3^{[d+]^l,t} = \left[ \frac{\binom{0t}{0t}_4}{\binom{t}{t}_4} I_3^{[d+]^{l-1},t} - \sum_{u=1,u\neq t}^{4} \frac{\binom{ut}{0t}_4}{\binom{t}{t}_4} I_2^{[d+]^{l-1},tu} \right] \frac{1}{d+2l-4}, \tag{A.11}$$

Of special interest is the case $l=1$: the $I_3^{[d+],t}$ is IR finite but UV infinite. The $I_3^t$ is in any case UV finite, however, if it is IR divergent then the coefficient $\binom{0t}{0t}_4$ is zero. Thus for the 3-point function we can put $d=4$. On the other hand, the UV divergence of $I_3^{[d+],t}$, coming from $I_{2,UV}^{tu} = 1/\varepsilon$, results in $I_{3,\mathrm{UV}}^{[d+],t} = -1/(2\varepsilon)$. Thus we have to keep the factor $1/(d-2)$ in (A.11) when multipying $I_2^{tu}$.

## A.3 Reduction of 2-point integrals

For the 2-point functions, surprisingly enough, a number of peculiarities occur. Let us begin with the scalar integrals in arbitrary dimension. The recurrence relation quite generally reads

$$I_2^{[d+]^l,tu} = \left\{ \frac{\binom{0tu}{0tu}_4}{\binom{tu}{tu}_4} I_2^{[d+]^{l-1},tu} - \sum_{v=1,v\neq t,u}^{4} \frac{\binom{vtu}{0tu}_4}{\binom{tu}{tu}_4} I_1^{[d+]^{l-1},tuv} \right\} \frac{1}{d+2l-3}. \tag{A.12}$$

The $\binom{tu}{tu}_4 = -2(q_i - q_{i'})^2 \equiv -2q^2,\ i,i' \neq t,u\ (i' \neq i)$ is independent of masses and is in particular the argument of the 2-point function, i.e. $I_2 = I_2(m_1, m_2, q^2)$. Quite often $q^2 = 0$, which has to be considered separately. This situation occurs, e.g., in our case of calculating the higher-dimensional 4-point functions where the corresponding 2-point functions are generated by the application of the recurrence relations for the 4-point functions. A more physical case, e.g., occurs if we consider radiation of a photon from an internal massive line. For $q^2 \neq 0$, nevertheless, we can apply (A.12). In

that case it is first of all worth to investigate the 1-point functions. These depend only on one mass $m$ and can be expressed for arbitrary dimension as

$$I_1^{[d+]^l} = (-1)^l\, I_1^d\, \frac{(2m^2)^l}{d(d+2)\cdots(d+2l-2)}, \tag{A.13}$$

with

$$I_1^d = -\frac{\Gamma(1-\frac{d}{2})}{(m^2)^{1-\frac{d}{2}}} \sim A_0(m^2) + \frac{m^2}{\varepsilon}. \tag{A.14}$$

The question now arises how to implement these expressions in recurrence (A.12). Here the following observation is useful: in (A.12), performing the sum over $v$, there are only contributions if all indices $t,u,v$ are different. In the $I_1$-functions the propagators with indices $t,u,v$ are scratched. Let us call the remaining propagator $w$. Since all indices are running from 1 to 4 and all are different, theire sum is 10. Thus $w = 10-t-u-v$ and the mass in $I_1$, (A.13), can be expressed as $m^2 = \frac{1}{2}Y_{10-t-u-v,10-t-u-v}$, where the $Y_{ij}$ is defined in (2.10). This property has in particular been used to calculate the divergent parts of the 2-point functions $I_2^{[d+]^l,tu}$ for $t,u = 1,\cdots,4$ as it was needed in Sec. 5.3.

Making use of

$$(m_i^2 - m_{i'}^2) I_2^{(d)}(m_i, m_{i'}, q^2 = 0) = I_1^{(d)}(m_i) - I_1^{(d)}(m_{i'}), \tag{A.15}$$

we obtain the following relation of special interest:

$$\begin{aligned} I_2^{[d+],tu} = \quad & \frac{1}{2}\left\{ \left[q^2 - 2(m_i^2+m_{i'}^2)\right] I_2^{tu}(q^2) + (m_i^2-m_{i'}^2)^2 \frac{I_2^{tu}(q^2) - I_2^{tu}(0)}{q^2} \right. \\ & \left. - \left[I_1^{tui} + I_1^{tui'}\right] \right\} \frac{1}{d-1}, \end{aligned} \tag{A.16}$$

where $q^2 = (q_i - q_{i'})^2$. The second term is obviously UV finite and vanishing for equal masses. Also for $q^2 = 0$ this expression makes sense, the difference quotient resulting in the $DB0$-function in LoopTools notation [25]. In general, however, and in particular for high dimensions one should start from (5.2) in case of $q^2 = 0$. For convenience we may drop indices $t,u$ in $I_2^{[d+],tu}$, i.e. we work in this case with 2-point functions only:

$$\begin{aligned} I_2^{[d+]^l}(m_i, m_{i'}, q^2 = 0) \quad & = Z_2^{[d+]^l}(m_i, m_{i'}, q^2 = 0) \\ & = \sum_{t=1}^{2} \frac{\binom{t}{0}_2}{\binom{0}{0}_2} I_1^{[d+]^l,t}(m_i, m_{i'}, q^2 = 0), \end{aligned} \tag{A.17}$$

and with $q^2 = 0$:

$$\frac{\binom{t}{0}_2}{\binom{0}{0}_2} = (-1)^t \frac{1}{m_i^2 - m_{i'}^2}, \quad m_i \neq m_{i'}, \tag{A.18}$$

such that

$$I_2^{[d+]^l}(m_i, m_{i'}, q^2 = 0) \;=\; \frac{1}{m_i^2 - m_{i'}^2} I_1^{[d+]^l}(m_i) + \frac{1}{m_{i'}^2 - m_i^2} I_1^{[d+]^l}(m_{i'}). \tag{A.19}$$

For the case $m_i = m_{i'}$ and $q^2 = 0$ the ratio $\binom{t}{0}_2 / \binom{0}{0}_2 = 0/0$ does not yield correct results [7] and we have to choose a different approach, based on relation (29) of [6] (see also [4], [5] ):

$$\sum_{j=1}^{n} \nu_j \mathbf{j}^+ I_n^{(d+2)} = -I_n^{(d)}. \tag{A.20}$$

With

$$I_2^{(d)}(m,m,q^2=0) = I_{1,1}^{(d)}, \tag{A.21}$$

the eqnn. (A.20) yields

$$\begin{aligned} I_{1,1}^{(d)} &= -I_1^{(d-2)} \\ &= \frac{d-2}{2m^2}\, I_1^{(d)}. \end{aligned} \tag{A.22}$$

The latter eqn. is obtained from (A.14), and finally

$$I_2^{(d)}(m,m,q^2=0) = \frac{d-2}{2m^2}\, I_1^{(d)}. \tag{A.23}$$

For the vector integral $I_2^{\mu}$ the recursion reads

$$I_{2,i}^{[d+]^l,tu} = -\frac{\binom{0tu}{itu}_4}{\binom{tu}{tu}_4} I_2^{[d+]^{l-1},tu} + \sum_{v=1,v\neq t,u}^{4} \frac{\binom{vtu}{itu}_4}{\binom{tu}{tu}_4} I_1^{[d+]^{l-1},tuv}, \tag{A.24}$$

and similarly to (A.16):

$$I_{2,i}^{[d+],tu} = -\frac{1}{2} I_2^{tu}(q^2) + \frac{1}{2}(m_i^2 - m_{i'}^2) \frac{I_2^{tu}(q^2) - I_2^{tu}(0)}{q^2}, \;\; i,i' \neq t,u, \tag{A.25}$$

which seems particularly useful for numerical evaluations even if $q^2 = 0$, yielding the function $DB0(0,m_1^2,m_2^2)$ of LoopTools. To clarify the notation we mention that the integral $I_{2,j}^{[d+],tu}$ occurs with the indices $j = i$ and $j = i'$ ($i \neq i'$), corresponding to the chords $q_i$ and $q_{i'}$ and the masses $m_i$ and $m_{i'}$ of propagators $c_i$ and $c_{i'}$, respectively [8].

For $m_i = m_{i'} = m$ and $q^2 = 0$ we have

$$I_{2,i}^{(d+2)}(m,m,q^2=0) = -\frac{1}{2} I_2^{(d)}(m,m,q^2=0) = -\frac{d-2}{4m^2}\, I_1^{(d)}(m). \tag{A.26}$$

Further we need the tensor coefficients

$$\nu_{ij} I_{2,ij}^{[d+]^l,tu} = -\frac{\binom{0tu}{jtu}_4}{\binom{tu}{tu}_4} I_{2,i}^{[d+]^{l-1},tu} + \sum_{v=1,v\neq t,u,i}^{4} \frac{\binom{vtu}{jtu}_4}{\binom{tu}{tu}_4} I_{1,i}^{[d+]^{l-1},tuv} + \frac{\binom{itu}{jtu}_4}{\binom{tu}{tu}_4} I_2^{[d+]^{l-1},tu}, \; l = 1,2. \tag{A.27}$$

[7] In fact the result depends on the order of taking the limits $m_i - m_{i'} \to 0$ or $q^2 \to 0$.

[8] For the comparison wih LoopTools. e.g., we put $q_2 = 0,\; m_{i'} = m_1^{LT}$ and $q_1 = -p, m_i = m_2^{LT}$.

For $q^2 \neq 0$ all the occurring integrals on the right-hand side of (A.27) are given above. For $q^2 = 0$ and $m_i \neq m_{i'}$ we now have

$$\begin{aligned} \nu_{ij} I^{[d+]^2}_{2,ij}(q^2=0) = & \quad \nu_{ij} Z^{[d+]^2}_{2,ij}(q^2=0) \\ = & \quad \frac{\binom{t}{0}_2}{\binom{0}{0}_2} \nu_{ij} I^{[d+]^2,t}_{1,ij} + \frac{\binom{i}{0}_2}{\binom{0}{0}_2} I^{[d+]^2}_{2,j}(q^2=0) + \frac{\binom{j}{0}_2}{\binom{0}{0}_2} I^{[d+]^2}_{2,i}(q^2=0), \;\; t \neq i,j. \end{aligned} \tag{A.28}$$

Note that for $i \neq j$ the first term does not exist and for $i = j$ we have $\nu_{ii} = 2$. Thus we have for $i \neq j$:

$$I^{[d+]^2}_{2,ij}(q^2=0) = \frac{\binom{i}{0}_2}{\binom{0}{0}_2} I^{[d+]^2}_{2,j}(q^2=0) + \frac{\binom{j}{0}_2}{\binom{0}{0}_2} I^{[d+]^2}_{2,i}(q^2=0), \tag{A.29}$$

and for $i = j$ we have

$$I^{[d+]^2}_{2,ii}(q^2=0) = \frac{\binom{i}{0}_2}{\binom{0}{0}_2} I^{[d+]^2}_{2,i}(q^2=0) + \frac{\binom{t}{0}_2}{\binom{0}{0}_2} \frac{1}{2} I^d_1(m_i), \quad t \neq i, \tag{A.30}$$

where due to (A.20)

$$I^{[d+]^2}_{1,11} = \frac{1}{2} I^d_1, \tag{A.31}$$

with $I^d_1$ given in (A.14). To evaluate (A.29) and (A.30) we need

$$I^{[d+]^2}_{2,i}(q^2=0) = -\frac{1}{2} I^{[d+]}_2(q^2=0) + \frac{1}{2}(m_i^2 - m_{i'}^2) \frac{\partial I^{[d+]}_2}{\partial q^2}(q^2=0) \tag{A.32}$$

with

$$\frac{\partial I^{[d+]}_2}{\partial q^2}(q^2=0) = \frac{1}{2} \left\{ I_2 - 2\left(m_i^2 + m_{i'}^2\right) \frac{\partial I_2}{\partial q^2} + \frac{1}{2}\left(m_i^2 - m_{i'}^2\right)^2 \frac{\partial^2 I_2}{\partial (q^2)^2} \right\} (q^2=0) \frac{1}{d-1}, \tag{A.33}$$

i.e. the second derivative $\frac{\partial^2 I_2}{\partial (q^2)^2}(q^2=0)$ is needed in addition to the function DB0 used also in LoopTools.

Last but not least we have to deal with $q^2 = 0$ and $m_i = m_{i'} = m$, where we avoid the appearance of a ratio $\frac{0}{0}$ as follows: (A.20) reads in this case

$$\begin{aligned} \sum_{j=1}^{2} \nu_{ij} I^{[d+]^2}_{2,ij} &= -I^{[d+]}_{2,i} \\ &= \frac{d-2}{4m^2} I^d_1(m), \end{aligned} \tag{A.34}$$

the latter eqn. being obtained from (A.26). Assuming $q = 0$, all integrals $I^{[d+]^2}_{2,ij}$ are equal and we have [7]

$$I^{[d+]^2}_{2,ij} = \frac{1}{3} \frac{d-2}{4m^2} I^d_1(m). \tag{A.35}$$

# B Divergent parts of higher-dimensional integrals

In our final results (5.15), (5.19) and (5.21) we obtained contributions of certain higher-dimensional integrals muliplied with polynomials in $d = 4 - 2\varepsilon$ such that the $1/\varepsilon$ parts of the UV divergent 4- and 3-point integrals combine with the $\varepsilon$-powers of the polynomials to yield finite contributions. In a numerical approach these contributions have to be explicitly calculated, and for that purpose we list the infinite parts of those integrals and of scalar 2-point functions appearing in the reductions. As described before, for the calculation of corrections for small Gram determinants, we need 3-point integrals of even higher dimensions, which are, however, too complicated to be listed – apart from being difficult to obtain. They have been calculated iteratively in the numerics, see sect. 5.3 for details.

The higher-dimensional integrals are in general UV divergent and we write them in the form

$$I_n^{(d)} = F_n^{(d)} + \frac{1}{\varepsilon} D_n^{(d)} + \mathscr{O}(\varepsilon^2), \quad d = [d+]^l. \tag{B.1}$$

The terms $D_n^{(d)}$ are obtained from the recurrence relations (see app. A), starting from lower dimensions. We mention that, whenever in the following list two of the occurring indices $i, j, k, t, u, v$ are equal, the corresponding $D's$ vanish.

For 1-point functions, it is

$$D_1^{[d+]^l,t,u,v} = (-1)^l \frac{1}{2 \cdot 4 \cdots 2(l+1)} Y^{l+1}_{10-t-u-v,10-t-u-v}. \tag{B.2}$$

For $i, j \neq t, u$ we have for 2-point functions:

$$\begin{aligned}
D_2^{t,u} &= 1, \\
D_2^{[d+],t,u} &= -\frac{1}{6}\left[Y_{ii} + Y_{jj} + Y_{ij}\right], \\
D_2^{[d+]^2,t,u} &= \frac{1}{120}\left[3Y_{ii}^2 + Y_{ii}Y_{jj} + 3Y_{jj}^2 + 3Y_{ij}\left(Y_{ii} + Y_{jj}\right) + 2Y_{ij}^2\right], \\
D_2^{[d+]^3,t,u} &= -\frac{1}{1680}\left[5Y_{ii}^3 + Y_{ii}^2 Y_{jj} + Y_{ii}Y_{jj}^2 + 5Y_{jj}^3 + Y_{ij}\left(5Y_{ii}^2 + 3Y_{ii}Y_{jj} + 5Y_{jj}^2\right)\right. \\
&\quad + \left. 4Y_{ij}^2\left(Y_{ii} + Y_{jj}\right) + 2Y_{ij}^3\right], \\
D_2^{[d+]^4,t,u} &= \frac{1}{120960}\left[35Y_{ii}^4 + 5Y_{ii}^3 Y_{jj} + 3Y_{ii}^2 Y_{jj}^2 + 5Y_{ii}Y_{jj}^3 + 35Y_{jj}^4\right. \\
&\quad + 5Y_{ij}\left(7Y_{ii}^3 + 3Y_{ii}^2 Y_{jj} + 3Y_{ii}Y_{jj}^2 + 7Y_{jj}^3\right) \\
&\quad + \left. 6Y_{ij}^2\left(5Y_{ii}^2 + 4Y_{ii}Y_{jj} + 5Y_{jj}^2\right) + 20Y_{ij}^3\left(Y_{ii} + Y_{jj}\right) + 8Y_{ij}^4\right], \\
D_{2,i}^{[d+]^2,t,u} &= \frac{1}{24}\left[3Y_{ii} + 2Y_{ij} + Y_{jj}\right].
\end{aligned} \tag{B.3}$$

For $i,j,k \neq t$ we have for 3- and 4-point functions:

$$
\begin{aligned}
D_3^{[d+],t} &= -\frac{1}{2}, \\
D_3^{[d+]^2,t} &= \frac{1}{24}\left[Y_{ii}+Y_{ij}+Y_{ik}+Y_{jj}+Y_{jk}+Y_{kk}\right], \\
D_3^{[d+]^3,t} &= -\frac{1}{720}\left\{3\left[Y_{ii}\left(Y_{ii}+Y_{ij}+Y_{ik}\right)+Y_{jj}\left(Y_{ji}+Y_{jj}+Y_{jk}\right)+Y_{kk}\left(Y_{ki}+Y_{kj}+Y_{kk}\right)\right]\right. \\
&\quad + \quad 2\left[Y_{ij}\left(Y_{ij}+Y_{jk}\right)+Y_{ik}\left(Y_{ij}+Y_{ik}\right)+Y_{jk}\left(Y_{ik}+Y_{jk}\right)\right] \\
&\quad + \quad \left.\left[Y_{ii}\left(Y_{jj}+Y_{jk}\right)+Y_{jj}\left(Y_{ik}+Y_{kk}\right)+Y_{kk}\left(Y_{ii}+Y_{ij}\right)\right]\right\}, \\
D_{3,i}^{[d+]^2,t} &= \frac{1}{6}, \\
D_{3,i}^{[d+]^3,t} &= -\frac{1}{120}\left\{3\ \ Y_{ii}+2(Y_{ij}+Y_{ik})+Y_{jj}+Y_{jk}+Y_{kk}\right\}, \quad j,k \neq i,t \qquad (j \neq k), \\
D_{3,ij}^{[d+]^3,t} &= -\frac{1}{24},
\end{aligned}
\tag{B.4}
$$

and

$$
\begin{aligned}
D_4^{[d+]} &= 0, \\
D_4^{[d+]^2} &= \frac{1}{6}, \\
D_4^{[d+]^3} &= -\frac{1}{120}\left[Y_{11}+Y_{12}+Y_{13}+Y_{14}+Y_{22}+Y_{23}+Y_{24}+Y_{33}+Y_{34}+Y_{44}\right], \\
D_4^{[d+]^4} &= \frac{1}{5040}\left\{3\left[Y_{11}(Y_{11}+Y_{12}+Y_{13}+Y_{14})+Y_{22}(Y_{12}+Y_{22}+Y_{23}+Y_{24})+\right.\right. \\
&\qquad Y_{33}(Y_{13}+Y_{23}+Y_{33}+Y_{34})+Y_{44}(Y_{14}+Y_{24}+Y_{34}+Y_{44})\left.\right]+ \\
&\qquad 2\left[Y_{12}(Y_{12}+Y_{13}+Y_{14})+Y_{13}(Y_{13}+Y_{14}+Y_{23})+Y_{14}(Y_{14}+Y_{24}+Y_{34})\ +\right. \\
&\qquad Y_{23}(Y_{12}+Y_{23}+Y_{24})+Y_{24}(Y_{12}+Y_{24}+Y_{34})+Y_{34}(Y_{13}+Y_{23}+Y_{34})\left.\right]+ \\
&\qquad Y_{11}(Y_{22}+Y_{23}+Y_{24}+Y_{33}+Y_{34})+Y_{22}(Y_{13}+Y_{14}+Y_{33}+Y_{34}+Y_{44})\ \ + \\
&\qquad Y_{33}(Y_{12}+Y_{14}+Y_{24}+Y_{44}) \qquad +Y_{44}(Y_{11}+Y_{12}+Y_{13}+Y_{23})+ \\
&\qquad \left.Y_{12}Y_{34}+Y_{13}Y_{24}+Y_{14}Y_{23}\right\}, \\
D_{4,i}^{[d+]^3} &= -\frac{1}{24}, \\
D_{4,ij}^{[d+]^3} &= 0, \\
D_{4,ijk}^{[d+]^4} &= 0.
\end{aligned}
\tag{B.5}
$$

# C A numerical example

In order to demonstrate the use of our small Gram determinant expansion, we reproduce the numerics for the topology shown in Fig. C.1 (b), which arises from the on-shell six-point topology of Fig. C.1 (a). The example is taken from [2], and first results obtained with our approach were reported in [33]. In LoopTools [25] conventions, the tensor coefficients $D_{ijl}$ are defined as follows:

$$
D_{\mu\nu\lambda} = \sum_{i,j,l=1}^{3} K_{i\mu}K_{j\nu}K_{l\lambda}D_{ijl} + \sum_{i=1}^{3}(g_{\mu\nu}K_{i\lambda}+g_{\nu\lambda}K_{i\mu}+g_{\lambda\mu}K_{i\nu})D_{00i}. \tag{C.1}
$$

For our conventions see (2.19). The external momenta are assumed to be incoming: $p_1 = p_{e^+}, p_2 = p_{\mu^-}, p_3 = p_{\bar{\nu}_\mu} + p_u, p_4 = p_{e^-} + p_{\bar{d}}$. The inverse propagators are here $c_j = [(k-q_j)^2 - m_j^2]$, and in LoopTools conventions $c_j = [(k+K_{j-1})^2 - m_j^2]$. The $K_i$ are the internal momenta, expressible by the $p_i$: $K_1 = p_1, K_2 = K_1 + p_2, K_3 = K_2 + p_3, K_4 = 0$. Then, with $p_i^2 = s_i, (p_i + p_j)^2 = s_{ij}$, we set $s_{12} = t_{\bar{e}\mu}, s_{23} = s_{\mu\bar{\nu}u}, s_3 = s_{\bar{\nu}u}, s_4 = t_{ed}$. The corresponding tensor integrals are, in LoopTools [25] notation:

$$\mathrm{D0i}(\mathrm{id}, 0, 0, s_{\bar{\nu}u}, t_{ed}, t_{\bar{e}\mu}, s_{\mu\bar{\nu}u}, 0, M_Z^2, 0, 0). \tag{C.2}$$

The Gram determinant is:

$$\begin{aligned} ()_4 \;=\; \Delta^{(3)} &= -\mathrm{Det}\left(2K_iK_j\right) \\ &= -2t_{\bar{e}\mu}[s_{\mu\bar{\nu}u}^2 + s_{\bar{\nu}u}t_{ed} - s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})], \end{aligned} \tag{C.3}$$

and it vanishes if

$$t_{ed} \to t_{ed,\mathrm{crit}} = \frac{s_{\mu\bar{\nu}u}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu})}{s_{\mu\bar{\nu}u} - s_{\bar{\nu}u}}. \tag{C.4}$$

In terms of a dimensionless scaling parameter $x$,

$$t_{ed} \;=\; (1+x)t_{ed,\mathrm{crit}}, \tag{C.5}$$

the Gram determinant becomes

$$()_4 = -2\,x\,s_{\mu\bar{\nu}u}t_{\bar{e}\mu}(s_{\mu\bar{\nu}u} - s_{\bar{\nu}u} + t_{\bar{e}\mu}). \tag{C.6}$$

Following [2], we have chosen

$$\begin{aligned} s_{\mu\bar{\nu}u} &= 2\times 10^4 \mathrm{GeV}^2, \\ s_{\bar{\nu}u} &= 1\times 10^4 \mathrm{GeV}^2, \\ t_{\bar{e}\mu} &= -4\times 10^4 \mathrm{GeV}^2, \end{aligned} \tag{C.7}$$

and get $t_{ed,\mathrm{crit}} = -6\times 10^4 \mathrm{GeV}^2$. For $x$=1, the Gram determinant becomes $()_4 = -4.8\times 10^{13}$ GeV$^3$.

We also need the modified Cayley determinant:

$$\begin{aligned} \binom{0}{0}_4 &= \begin{vmatrix} 2M_Z^2 & M_Z^2 & M_Z^2 - s_{\mu\bar{\nu}u} & M_Z^2 \\ M_Z^2 & 0 & -s_{\bar{\nu}u} & M_Z^2 \\ M_Z^2 - s_{\mu\bar{\nu}u} & -s_{\bar{\nu}u} & 0 & -t_{ed} \\ M_Z^2 & -t_{\bar{e}\mu} & -t_{ed} & 0 \end{vmatrix} \\ &= s_{\mu\bar{\nu}u}^2 t_{\bar{e}\mu}^2 + 2\,M_Z^2 t_{\bar{e}\mu}[-2s_{\bar{\nu}u}t_{ed} + s_{\mu\bar{\nu}u}(s_{\bar{\nu}u} + t_{ed} - t_{\bar{e}\mu})] \\ &\quad + M_Z^4\left[s_{\bar{\nu}u}^2 + (t_{ed} - t_{\bar{e}\mu})^2 - 2s_{\bar{\nu}u}(t_{ed} + t_{\bar{e}\mu})\right]. \end{aligned} \tag{C.8}$$

From (5.34) we see that a *small-Gram determinant expansion* will be applicable when the following dimensionless parameter becomes small:

$$R \;=\; \frac{()_4}{\binom{0}{0}_4} \times s, \tag{C.9}$$

Figure C.1: *(a) A six-point topology; (b) a four-point topology derived from (a).*

where $s$ is a typical scale of the process. We choose here $s = s_{\mu\bar{\nu}u}$. Obviously, due to (C.6), $x$ must also be small. Indeed, for e.g. $x = 0.01$ we have $R = -0.064$.[9]

In Fig. C.2 we show the tensor coefficient $D_{1111}(x)$, and in Tabs. C.1–C.2 the tensor coefficients $D_{1111}$ and $D_{111}$ are tabulated for the region of interest of $x$. Because we assume in our formulae $q_1 = 0$, and in LoopTools $K_4 = 0$ is assumed, one has to care about specific correspondences, in particular:

$$D_{1111} = n_{2222} \times I^{[d+]^4}_{4,2222}, \tag{C.10}$$

$$D_{111} = n_{222} \times I^{[d+]^3}_{4,222}. \tag{C.11}$$

The integral $I^{[d+]^4}_{4,2222}$ contributes also to the rank $R = 5$ tensor coefficients $E_{2222m}$, see (4.62), and $I^{[d+]^3}_{4,222}$ to the rank $R = 4$ tensor coefficients $E_{222l}$, see (4.48).

The numerics have been performed with Mathematica v.7.0, using LoopTools v.2.5 for the scalar 4-, 3- and 2-point functions in generic dimension. Of course the first step to be done is to verify (5.21), (5.19), and (5.15) by comparing their numerical evaluation with results from LoopTools v.2.5 [10]. This verification is naturally only possible for non-vanishing Gram determinants, i.e. in this case all (higher-dimensional) integrals have been calculated by means of the recurrence relations given in app. A. For $x \in (0.1, 1.0)$ we find in general an agreement to more than 10 decimals: in Tabs. C.1–C.2 these cases are marked by $I^{[d+]^4}_{4,2222}$ and $I^{[d+]^3}_{4,222}$, respectively. In the next step we have to evaluate (5.21), (5.19), and (5.15) for small Gram determinants. If $l = l_{max}$ is the highest available index for which the $Z^l_{4F}$ are calculated, we see that the upper limit $i = i_{max}$ in (5.38) can be at most $i_{max} = l_{max} - L$. Since the integral of highest dimension is $I^{[d+]^4}_4$, we take the value $L = 4$ as reference value.

Tabs. C.1–C.2 have been produced with a varying number of correction terms, specified by $i_{max} = l_{max} - 4$, dependent on the size of $()_4$, specified in terms of the parameter $x$ (see (C.6)). It is clear that for very small Gram determinants a few terms only have to be taken into account. Higher terms don't change anything beyond the shown accuracy.

Due to our discussion following (5.47) we have a certain choice for the value of $Z^{(L+i)}_{4F,0}$ for $i = i_{max}$, see (5.38). The choices are $()_4 = 0$ and $()_4 \neq 0$, taken at the kinematics under consideration. Due to (C.6) the two options are parametrized here by $x$, specified in (C.5). In the tables we mark the

[9] The dependence of $R(x)$ on $x$ is shown in Fig. 1 of [34].

[10] In fact in some cases we found a slightly better agreement with LoopTools v.2.5 than with LoopTools v.2.4.

(a)

Figure C.2: *The tensor coefficient $D_{1111}(x)$; the data points for a least square fit are shown. Input: grey (red) dots originate from the expansion at small x, black dots from the values at larger x.*

approximations correspondingly by 0 and $x$. We specify the expansions as [exp p,$i_{max}$], where $Z_{4F,0}^{(L+i_{max})}$ in (5.38) is calculated for $p = 0$ and $p = x$, respectively.

We have, however, still another option. With (5.47) we provide a sequence of *partial sums* $S^i$ of a series expansion for $F_4^{[d+]^L}$. This is exactly the input needed for the calculation of a Padé approximation for $F_4^{[d+]^L}$. We apply the $\varepsilon$-algorithm for sequence transformations [35, 36]. It is described in detail in [37]. The $\varepsilon$-algorithm allows an efficient calculation of elements of the so-called $\varepsilon$-table. The first column is zero, and the second one consists of the sequence $S^i = Z_{4F,i}^L$, the convergence of which shall be improved. From the first two columns, the others are determined iteratively:

$$\varepsilon_{-1}^{(i)} = 0, \tag{C.12}$$

$$\varepsilon_{0}^{(i)} = Z_{4F,i}^L, \quad i = 0, \cdots, l_{\max} - L, \tag{C.13}$$

$$\varepsilon_{k+1}^{(i)} = \varepsilon_{k-1}^{(i+1)} + \frac{1}{\varepsilon_k^{(i+1)} - \varepsilon_k^{(i)}}. \tag{C.14}$$

The $\varepsilon$-table and the Padé table are related:

$$\varepsilon_{2k}^{(i)} = [k + i/k], \tag{C.15}$$

where the symbol $[k+i/k]$ stands for the degrees $k+i$ of numerator and $k$ of denominator polynomials of the corresponding Padé approximant $[k+i/k]$. We took the choice $k = l_{max} - L$ and:

$$F_4^{[d+]^L} \sim \varepsilon_{2k}^{(0)} \equiv [k/k]_{F_4^{[d+]^L}}. \tag{C.16}$$

In the tables we present the Padé approximants together with the corresponding sums, for $x = 0$ as well as for $x \neq 0$: they are denoted by [pade 0,$i_{max}/2$] and [pade x,$i_{max}/2$], respectively. In general

the Padé approximants provide a remarkable improvement of precision compared to the sums such that we even close up to the values provided by the non-small Gram determinant representation. It is also remarkable that there is only very little difference between the values obtained by iterations starting from $x = 0$ or from $x \neq 0$. In fact the Padé starting with $x = 0$ is only slighly better than the one calculated with $x \neq 0$. The difference arises because the integrals $F_4^{[d+]^l}$ change much slower with $x$ than the approximants $Z^L_{4F,i}$ so that for a start at $x = 0$ the latter are already much closer to the final values than those starting at $x \neq 0$. Nevertheless, after a few iterations the difference has already almost disappeared. At this point we want to remind the reader that we approximate the integrals $I_4^{[d+]^l}$ and they must indeed be very precise since there are considerable cancellations with the remaining contributions to the tensor coefficients under consideration.

To discuss a few results given in Tabs. C.1 and C.2 we mention in particular the results for $x = 0.05$. In Tab. C.1 we can assume the values of [pade 0,10] to be accurate to 10 decimals. One reason is that even Padé approximants up to [pade 0,13] (based on 27 terms in (5.38), not shown here) remain stable in the $10^{th}$ and $11^{th}$ decimal. With LoopTools we have an agreement of 8 decimals - thus it seems that this point is the value where the representation in terms of non-small Gram determinants starts to loose precision. We also see that there is even agreement beween [pade 0,10] and [pade x,10] up to 9 decimals. Just for curiosity we have also calculated the values for $x = 0.1$. Here LoopTools and also our calculation with the larger Gram determinants work perfectly so that we can say to be in the domain of larger Gram deteteminants. Nevertheless [pade 0,13] yields a precision of 3 decimals - while [exp 0,26] is off by 2 orders of magnitude. It is clear that for lower $x$ we obtain good results with less terms in the expansion. The results in Tab. C.2 are of similar quality - in fact they are slightly better since the highest dimensional 4-point function is only $I_4^{[d+]^3}$.

Finally we show in Fig. C.2 the smooth behaviour of $D_{1111}(x)$. The smoothness is in striking contrast to the complexity of its precise calculation. Indeed, in the critical domain the function is almost constant and one has to spend quite some effort to calculate it with high enough precision.

| $x$ | | $\mathfrak{Re}\, D_{1111}$ | $\mathfrak{Im}\, D_{1111}$ |
|---|---|---|---|
| 0. | [exp 0,0] | 2.05969289730 E-10 | 1.55594910118 E-10 |
| $10^{-8}$ | [exp x,2] | 2.05969289342 E-10 | 1.55594909187 E-10 |
| | [exp 0,2] | 2.05969289349 E-10 | 1.55594909187 E-10 |
| $10^{-4}$ | [exp x,5] | 2.05965609497 E-10 | 1.55585605343 E-10 |
| | [exp 0,5] | 2.05965609495 E-10 | 1.55585605343 E-10 |
| 0.001 | [exp 0,6] | 2.05932484380 E-10 | 1.55501912433 E-10 |
| | [exp x,6] | 2.05932484381 E-10 | 1.55501912433 E-10 |
| | $I^{[d+]^4}_{4,2222}$ | 2.02292295240 E-10 | 1.54974785467 E-10 |
| | $D_{1111}$ | 2.01707671668 E-10 | 1.62587142251 E-10 |
| 0.005 | [exp 0,6] | 2.05786054801 E-10 | 1.55131031024 E-10 |
| | [pade 0,3] | 2.05785198947 E-10 | 1.55131031003 E-10 |
| | [exp x,6] | 2.05786364440 E-10 | 1.55131031024 E-10 |
| | [pade x,3] | 2.05785199805 E-10 | 1.55131030706 E-10 |
| | $I^{[d+]^4}_{4,2222}$ | 2.05778894114 E-10 | 1.55135794453 E-10 |
| | $D_{1111}$ | 2.05779811490 E-10 | 1.55136343923 E-10 |
| 0.01 | [exp 0,6] | 2.05703298143 E-10 | 1.54669910676 E-10 |
| | [pade 0,3] | 2.05600940065 E-10 | 1.54669907784 E-10 |
| | [exp 0,10] | 2.05600964693 E-10 | 1.54669910676 E-10 |
| | [pade 0,5] | 2.05600955381 E-10 | 1. 54669910676E-10 |
| | [exp x,10] | 2.05600963675 E-10 | 1.54669910676 E-10 |
| | [pade x,5] | 2.05600955381 E-10 | 1.54669910676 E-10 |
| | $I^{[d+]^4}_{4,2222}$ | 2.05600013702 E-10 | 1.54670651917 E-10 |
| | $D_{1111}$ | 2.05600239280 E-10 | 1.54670771210 E-10 |
| 0.05 | [exp 0,6] | 4.83822963052 E-09 | 1.51077429118 E-10 |
| | [pade 0,3] | 2.01518061131 E-10 | 1.50591643209 E-10 |
| | [exp 0,20] | 2.04218962072 E-10 | 1.51077424143 E-10 |
| | [pade 0,10] | 2.04122727654 E-10 | 1.51077424149 E-10 |
| | [exp x,20] | 2.04190274030 E-10 | 1.51077424143 E-10 |
| | [pade x,10] | 2.04122727971 E-10 | 1.51077423985 E-10 |
| | $I^{[d+]^4}_{4,2222}$ | 2.04122726387 E-10 | 1.51077422901 E-10 |
| | $D_{1111}$ | 2.04122726601 E-10 | 1.51077423320 E-10 |
| 0.1 | [exp 0,26] | 2.20215264409 E-08 | 1.46815247004 E-10 |
| | [pade 0,13] | 2.01749674352 E-10 | 1.46681287362 E-10 |
| | [exp x,26] | 2.08190721550 E-08 | 1.46815247004 E-10 |
| | [pade x,13] | 2.03995221326 E-10 | 1.46785977364 E-10 |
| | $I^{[d+]^4}_{4,2222}$ | 2.02269485177 E-10 | 1.46815247061 E-10 |
| | $D_{1111}$ | 2.02269485217 E-10 | 1.46815247051 E-10 |
| 1. | $I^{[d+]^4}_{4,2222}$ | 1.72115440143 E-10 | 9.74550747662 E-11 |
| | $D_{1111}$ | 1.72115440148 E-10 | 9.74550747662 E-11 |

Table C.1: *Numerical values for the tensor coefficient $D_{1111}$. Values marked by $D_{1111}$ are evaluated with LoopTools, the $I^{[d+]^4}_{4,2222}$ corresponds to* (5.21)*. The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Padé approximant $[n,n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.*

| $x$ | | $\mathfrak{Re}\, D_{111}$ | $\mathfrak{Im}\, D_{111}$ |
|---|---|---|---|
| 0 | [exp 0,0] | –3.15407250453 E-10 | –3.31837792634 E-10 |
| $10^{-8}$ | [exp x,1] | –3.15407250057 E-10 | –3.31837790700 E-10 |
| | [exp 0,1] | –3.15407250057 E-10 | –3.31837790700 E-10 |
| $10^{-4}$ | [exp x,4] | –3.15403282194 E-10 | –3.31818461838 E-10 |
| | [exp 0,4] | –3.15403282194 E-10 | –3.31818461838 E-10 |
| 0.001 | [exp x,6] | –3.15367545429 E-10 | –3.31644587150 E-10 |
| | [exp 0,6] | –3.15367545429 E-10 | –3.31644587150 E-10 |
| | $I^{[d+]^3}_{4,222}$ | –3.15372092999 E-10 | –3.31645245644 E-10 |
| | $D_{111}$ | –3.15372823537 E-10 | –3.31635736868 E-10 |
| 0.005 | [exp x,6] | –3.15208222856 E-10 | –3.30874035862 E-10 |
| | [pade x,3] | –3.15208230282 E-10 | –3.30874035931 E-10 |
| | [exp 0,6] | –3.15208224867 E-10 | –3.30874035862 E-10 |
| | [pade 0,3] | –3.15208230411 E-10 | –3.30874035867 E-10 |
| | $I^{[d+]^3}_{4,222}$ | –3.15208269791 E-10 | –3.30874006110 E-10 |
| | $D_{111}$ | –3.15208264077 E-10 | –3.30874002667 E-10 |
| 0.01 | [exp 0,6] | –3.15006665284 E-10 | –3.29915926110 E-10 |
| | [pade 0,3] | –3.15007977830 E-10 | –3.29915888075 E-10 |
| | [exp 0,10] | –3.15007991203 E-10 | –3.29915926110 E-10 |
| | [pade 0,5] | –3.15007991324 E-10 | –3.29915926110 E-10 |
| | [exp x,10] | –3.15007991217 E-10 | –3.29915926110 E-10 |
| | [pade x,5] | –3.15007991324 E-10 | –3.29915936110 E-10 |
| | $I^{[d+]^3}_{4,222}$ | –3.15008000292 E-10 | –3.29915916848 E-10 |
| | $D_{111}$ | –3.15008000292 E-10 | –3.29915915368 E-10 |
| 0.05 | [exp 0,6] | –1.34278470211 E-11 | –3.22448580722 E-10 |
| | [pade 0,3] | –3.13432516570 E-10 | –3.22580791799 E-10 |
| | [exp 0,20] | –3.13359445767 E-10 | –3.22448581032 E-10 |
| | [pade 0,10] | –3.13365675001 E-10 | –3.22448581024 E-10 |
| | [exp x,20] | –3.13361302214 E-10 | –3.22448581032 E-10 |
| | [pade x,10] | –3.13365674956 E-10 | –3.22448581051 E-10 |
| | $I^{[d+]^3}_{4,222}$ | –3.13365675084 E-10 | –3.22448581110 E-10 |
| | $D_{111}$ | –3.13365675070 E-10 | –3.22448581084 E-10 |
| 0.1 | [exp 0,26] | –2.49466252165 E-09 | –3.13582331984 E-10 |
| | [pade 0,13] | –3.11144777695 E-10 | –3.13599283949 E-10 |
| | [exp x,26] | –2.34010823441 E-09 | –3.135823319836 E-10 |
| | [pade x,13] | –3.10806582023 E-10 | –3.135870111996 E-10 |
| | $I^{[d+]^3}_{4,222}$ | –3.11226750699 E-10 | –3.13582331977 E-10 |
| | $D_{111}$ | –3.11226750695 E-10 | –3.13582331978 E-10 |
| 1. | $I^{[d+]^3}_{4,222}$ | –2.70193791372 E-10 | –2.10251973821 E-10 |
| | $D_{111}$ | –2.70193791373 E-10 | –2.10251973821 E-10 |

Table C.2: *Numerical values for the tensor coefficient $D_{111}$. Values marked by $D_{111}$ are evaluated with LoopTools, the $I^{[d+]^3}_{4,222}$ is defined in* (5.19)*. The labels [exp 0,2n] and [pade 0,n] denote iteration $2n$ and Padé approximant $[n,n]$ when the small Gram determinant expansion starts at $x = 0$, and [exp x,2n] and [pade x,n] are the corresponding numbers for an expansion starting at x.*

# D Notations and algebraic relations

In the following we quote some relations for the signed minors, which are of particular relevance for the present work (see [19] and also [17]). First of all we used the sums

$$\sum_{i=1}^{n} \binom{0}{i}_n = ()_n \tag{D.1}$$

and

$$\sum_{i=1}^{n} \binom{j}{i}_n = 0, \ (j \neq 0). \tag{D.2}$$

Next a kind of standard relation, which allows to disentangle signed minors:

$$()_n \binom{il}{jk}_n = \binom{i}{j}_n \binom{l}{k}_n - \binom{i}{k}_n \binom{l}{j}_n ; \ \ i,j,k,l = 0,\ldots,n \tag{D.3}$$

and finally, what we call **"master formula"** in the present context ( (A.13) of [19] )

$$\binom{s}{i}_n \binom{\tau s}{0s}_n = \binom{s}{0}_n \binom{\tau s}{is}_n + \binom{s}{s}_n \binom{\tau s}{0i}_n , \quad \tau = 0,1,\ldots 5. \tag{D.4}$$

The following correspondences have to be taken into account when comparing our notations to that of e.g. [23]:

$$\begin{aligned}
()_5 &= -\tilde{X}^{(4)}_{00}, && \text{(D.5)}\\
\binom{0}{0}_5 &= \det X^{(4)}, && \text{(D.6)}\\
\binom{0}{i}_5 &= \tilde{X}^{(4)}_{0i}, && \text{(D.7)}\\
\binom{i}{j}_5 &= -\tilde{Z}^{(4)}_{ij} \ = \ \tilde{\tilde{X}}^{(4)}_{(0i)(0j)}, && \text{(D.8)}\\
\binom{0i}{0j}_5 &= \tilde{X}^{(4)}_{ij}, && \text{(D.9)}\\
\binom{0i}{kj}_5 &= \tilde{\tilde{X}}^{(4)}_{(0i)(jk)} \ = \ \tilde{\tilde{X}}^{(4)}_{(jk)(0i)}. && \text{(D.10)}
\end{aligned}$$

The notation in terms of $D_{\ldots}$-functions for 4-point tensors with even number of indices reads

$$\begin{aligned}
I_n^{\mu\nu} &= \int \frac{d^dk}{i\pi^{d/2}} k^\mu k^\nu \prod_{j=1}^{n} c_j^{-1} \qquad = \cdots + g^{\mu\nu} D_{00}, && \text{(D.11)}\\
I_n^{\mu\nu\lambda\rho} &= \int \frac{d^dk}{i\pi^{d/2}} k^\mu k^\nu k^\lambda k^\rho \prod_{j=1}^{n} c_j^{-1} = \cdots + g^{[\mu\nu} g^{\lambda\rho]} D_{0000} \quad \text{etc.}, && \text{(D.12)}
\end{aligned}$$

i.e. the functions $D_{00\ldots}$ correspond in our notation to integrals in higher dimension $I_4^{[d+]^l}$ $(l = 1,2,\cdots)$ and our $Z_4^{[d+]^l}$-functions correspond to approximations of the $D_{00\ldots}$-functions. We write

$$Z_4^{[d+]^l} = Z_{4F}^l + \frac{Z_{4D}^l}{\varepsilon} + \mathscr{O}(\varepsilon^2) \tag{D.13}$$

and the series (5.46) thus uses only approximations for the $D_{00\ldots}$-functions.

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