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# MHV amplitude for $3 \to 3$ gluon scattering in Regge limit

J. Bartels,[1] L. N. Lipatov,[1,2] and A. Prygarin[1]

[1] *II. Institute of Theoretical Physics, Hamburg University, Germany*
[2] *St. Petersburg Nuclear Physics Institute, Russia*

We calculate corrections to the BDS formula for the six-particle planar MHV amplitude for the gluon transition $3 \to 3$ in the multi-Regge kinematics for the physical region, in which the Regge pole ansatz is not valid. The remainder function at two loops is obtained by an analytic continuation of the expression derived by Goncharov, Spradlin, Vergu and Volovich to the kinematic region described by the Mandelstam singularity exchange in the crossing channel. It contains both the imaginary and real contributions being in agreement with the BFKL predictions. The real part of the three loop expression is found from a dispersion-like all-loop formula for the remainder function in the multi-Regge kinematics derived by one of the authors. We also make a prediction for the all-loop real part of the remainder function multiplied by the BDS phase, which can be accessible through calculations in the regime of the strong coupling constant.

## INTRODUCTION

In recent years we witnessed a significant progress in understanding the structure of the scattering amplitudes in the supersymmetric theories. The pioneering paper of Parke and Taylor [1] on the MHV amplitudes eventually led to a formulation of a simple all-loop expression for multi-leg amplitudes in $\mathcal{N} = 4$ SYM by Anastasiou, Bern, Dixon and Kosower (ABDK) [2] and then by Bern, Dixon and Smirnov (BDS) [3]. However it was shown by two of the authors of this study in collaboration with Sabio Vera [4] that the BDS ansatz is violated at two loops starting from six external gluons, confirming a conclusion derived by Alday and Maldacena [5] that the BDS formula is to be violated at large number of external gluons. It was argued by two of the authors [4] that this violation is related to the fact that the BDS amplitude is not compatible with the Steinmann relations [6], imposing the absence of simultaneous singularities in the overlapping channels. Moreover, the BDS ansatz in some channels does not contain the contributions of the so-called Mandelstam cuts, which are the moving Regge singularities in the complex momenta plane. We call these channels the Mandelstam channels. The analytic properties of the BDS amplitude in the Regge kinematics were also investigated by Brower, Nastase, Schnitzer and C.-I Tan [7, 8].

The BDS amplitude differs from the full MHV amplitude by a factor [9] being a function of the dual conformal invariants according to the analysis of Drummond, Henn, Korchemsky and Sokatchev [10]. This function is commonly referred to as the remainder function $R_n^{(l)}$ for the $n$ external legs at $l$ loops. The leading logarithmic term of $R_6^{(2)}$ for the Mandelstam channels of the $2 \to 4$ and $3 \to 3$ scattering amplitudes in the Regge kinematics was explicitly calculated by the authors of ref. [11] using a solution to the color octet Balitsky-Fadin-Kuraev-Lipatov (BFKL) ([12]) equation, which is a special case of the Schrödinger equation for the open integrable Heisenberg spin chain [13].

It was suggested that in general kinematics $R_n^{(l)}$ can be obtained from the expectation value of the light-like polygonal Wilson loops [9]. In particular, the remainder function for the six-gluon MHV amplitude at two loops was calculated by Drummond, Henn, Korchemsky and Sokatchev [14] and then it was expressed in terms of the generalized polylogarithms by Del Duca, Duhr and Smirnov [16, 17]. Their lengthy expression for $R_6^{(2)}$ was greatly simplified by Goncharov, Spradlin, Vergu and Volovich (GSVV) [18] and written in terms of only classical polylogarithms. The GSVV expression was analytically continued by two of us [19] to the Mandelstam channel of the $2 \to 4$ scattering amplitude considered in ref. [11] and showed a full agreement within leading logarithmic accuracy. The leading logarithmic term and the real part of the next-to-leading term of the remainder function at three loops were found in ref. [20]. The analytic continuation at the strong coupling was performed by one of the authors with collaborators [21].

In the present paper we consider $3 \to 3$ gluon scattering amplitude in the Mandelstam channels. This amplitude is generally not related to the $2 \to 4$ amplitude considered in the previous studies [19, 20] and brings new information about the analytic structure of the six-gluon MHV amplitude. We perform the analytic continuation of the GSVV expression for the two-loop remainder function to the Mandelstam channel of the $3 \to 3$ amplitude and simplify it in the Regge limit. The result is similar to that of the $2 \to 4$ case and differs by the overall sign and the presence of the real contribution. The obtained real contribution confirms general all-loop dispersion relations for the real and imaginary parts of the remainder function derived by one of the authors [22]. These dispersion relations are used to calculate also the leading logarithmic terms and the real part of the next-to-leading contribution at three loops. Another important result of the present study is the prediction, in the region under consideration, of the real constant part of the remainder

function multiplied by the phase present in the BDS amplitude. This prediction is valid for an arbitrary value of the coupling constant and can be accessible through the strong coupling calculations.

## ANALYTIC CONTINUATION

We consider a special case of the six-gluon planar MHV scattering amplitude for three gluon scattering ($3 \to 3$ amplitude) illustrated in Fig. 1. The energy invariants

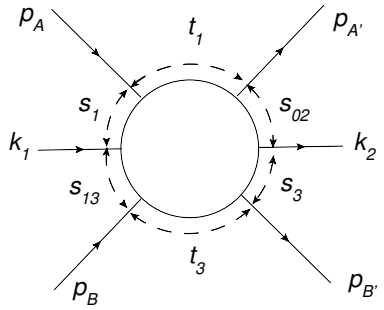


FIG. 1: The $3 \to 3$ gluon scattering amplitude.

are defined by $s_{13} = (p_B + k_1)^2, s_{02} = (p_{A'} + k_2)^2, s = (p_B + k_1 + p_A)^2, t_2^{'} = (p_A - p_{A'} - k_2)^2, s_1 = (k_1 + p_A)^2, s_3 = (p_{B'} + k_2)^2, t_2 = (p_A - p_{A'} + k_1)^2, t_1 = (p_A - p_{A'})^2$ and $t_3 = (p_B - p_{B'})^2$. The dual conformal cross ratios are expressed in terms of the energy invariants as follows

$$u_1 = \frac{s_{13} s_{02}}{s\ t_2^{'}},\ u_2 = \frac{s_1 s_3}{s\ t_2},\ u_3 = \frac{t_1 t_3}{t_2 t_2^{'}}. \tag{1}$$

In the multi-Regge kinematics for the direct channel, where all invariants are negative

$$-s \gg -s_1, -s_3, -t_2^{'} \gg -t_1, -t_2, -t_3 > 0, \tag{2}$$

the remainder function $R_6^{(l)}$ is zero, while in the physical region of the Mandelstam channel depicted in Fig. 2, where

$$s_1, s_3, s_{13}, s_{02} < 0 \quad \text{and} \quad s, t_2^{'} > 0, \tag{3}$$

it contains a non-vanishing contribution.

This situation was thoroughly discussed in refs. [4, 11] as well as in ref. [22]. The physical reason for the violation of the BDS ansatz in this region is the fact that the BDS formula does not have correct analytic properties, in particular, it does not account properly for the Mandelstam (Regge) cuts. In the Mandelstam channel (3) in the multi-Regge kinematics the dual conformal cross ratios (1) possess a non-zero phase

$$u_1 \to |u_1| e^{i2\pi},\ u_2 \to |u_2| e^{i\pi},\ u_3 \to |u_3| e^{i\pi}. \tag{4}$$

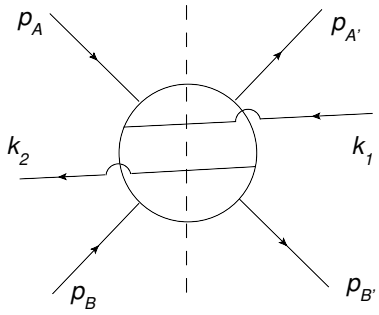


FIG. 2: The $3 \to 3$ gluon scattering amplitude in the Mandelstam channel given by $s_1, s_3, s_{13}, s_{02} < 0$ and $s, t_2^{'} > 0$.

Using this phase structure one can perform an analytic continuation of the remainder function to our kinematic region. The two-loop remainder function for the six-gluon MHV amplitude in terms of the classical polylogarithms was calculated by Goncharov, Spradlin, Vergu and Volovich [18]. They found that in the variables

$$x_i^{\pm} = u_i x^{\pm}, \qquad x^{\pm} = \frac{u_1 + u_2 + u_3 - 1 \pm \sqrt{\Delta}}{2 u_1 u_2 u_3}, \tag{5}$$

where $\Delta = (u_1 + u_2 + u_3 - 1)^2 - 4 u_1 u_2 u_3$, the remainder function $R_6^{(2)}$ can be written in a rather compact way

$$R_6^{(2)}(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \operatorname{Li}_4(1 - 1/u_i) \right)$$
$$- \frac{1}{8} \left( \sum_{i=1}^{3} \operatorname{Li}_2(1 - 1/u_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72}. \tag{6}$$

The functions $L_4(x^+, x^-)$ and $J$ are defined by

$$L_4(x^+, x^-) = \frac{1}{8!!} \log(x^+ x^-)^4$$
$$+ \sum_{m=0}^{3} \frac{(-1)^m}{(2m)!!} \log(x^+ x^-)^m (\ell_{4-m}(x^+) + \ell_{4-m}(x^-)) \tag{7}$$

and

$$\ell_n(x) = \frac{1}{2} \left( \operatorname{Li}_n(x) - (-1)^n \operatorname{Li}_n(1/x) \right), \tag{8}$$
$$J = \sum_{i=1}^{3} (\ell_1(x_i^+) - \ell_1(x_i^-)).$$

In this paper we perform the analytic continuation of (6) in phases of $u_i$ given by (4) to the Mandelstam channel of the $3 \to 3$ scattering amplitude in the multi-Regge limit. We find that it is not so much different from the remainder function in the Mandelstam channel of the $2 \to 4$ gluon scattering amplitude calculated by the authors in ref. [19, 20]. This fact can be explained by the

Regge factorization as discussed below. The main difference between the two is that in the $3 \to 3$ case we get a non-vanishing real contribution in the next-to-leading logarithmic approximation (NLLA) as predicted by one of the authors [22]. At two loops we keep only the leading order (LLA) and constant terms (NLLA) in the logarithm of the energy $\ln(u_1 - 1)$ [24], where

$$u_1 - 1 \simeq \frac{(\mathbf{q}_1 + \mathbf{q}_3 - \mathbf{q}_2)^2}{|t_2'|}. \tag{9}$$

The logarithmic contributions in $\ln(u_1 - 1)$ originate from the discontinuities in $s$- and $t_2'$-channels illustrated in Fig. 2 and Fig. 3 respectively. Each order of the perturbation theory brings a power of $\ln(u_1 - 1)$ in the multi-Regge kinematics, so that at two loops one expects at most the first power of $\ln(u_1 - 1)$ (we start with an imaginary constant at one loop) and at three loops there appears a term proportional to $\ln^2(u_1 - 1)$.

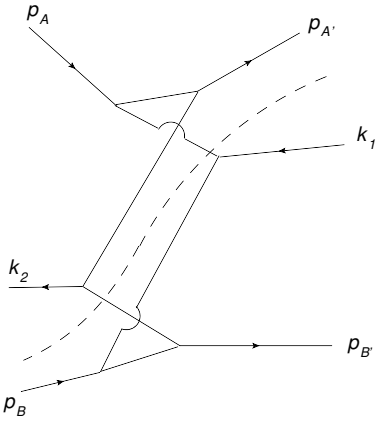


FIG. 3: The $3 \to 3$ gluon scattering amplitude with discontinuity in $t_2'$ channel.

The remainder function after the analytic continuation (4) to the Mandelstam channel of $3 \to 3$ amplitude in the multi-Regge is given by

$$\begin{aligned} & R_6^{(2)}\left(|u_1|e^{i2\pi}, \frac{e^{i\pi}|1-u_1|}{|1+w|^2}, \frac{e^{i\pi}|1-u_1||w|^2}{|1+w|^2}\right) \qquad (10) \\ & \simeq -\frac{i\pi}{2}\ln(u_1-1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 \\ & +\frac{\pi^2}{2}\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 - \frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2 \\ & +\frac{i\pi}{3}\ln^3|1+w|^2 - i\pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) \\ & +i2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right). \end{aligned}$$

The complex variable $w$ is expressed in terms of the reduced cross ratios

$$\tilde{u}_2 = \frac{|u_2|}{|1-u_1|},\ \tilde{u}_3 = \frac{|u_3|}{|1-u_1|} \tag{11}$$

through

$$w = \frac{B^+}{\tilde{u}_2},\ w^* = \frac{B^-}{\tilde{u}_2} \tag{12}$$

for $B^\pm$ defined by [20]

$$B^\pm = \frac{1-\tilde{u}_2-\tilde{u}_3 \pm \sqrt{(1-\tilde{u}_2-\tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3}}{2}. \tag{13}$$

In the course of the analytic continuation we obtain terms of the order $\ln^2(u_1 - 1)$ and $\ln^3(u_1 - 1)$. These higher order terms in the logarithm of the energy all cancel in the final result. The remainder function $R_6^{(2)}$ in (10) is symmetric under substitution $w \to 1/w$ and vanishes in the limit $|w| \to 0$ or $|w| \to \infty$.

The expression in (10) was obtained by the analytic continuation for the $3 \to 3$ amplitude (see (4)) of the remainder function of Goncharov et al. in (6) and then simplified in the Regge limit

$$|u_1| \to 1^+,\ |u_2| \to 0^+,\ |u_3| \to 0^+,\ \tilde{u}_{2,3} \sim \mathcal{O}(1). \tag{14}$$

However it is also possible to get the same expression from the remainder function for the $2 \to 4$ amplitude with the use of its cyclic symmetry [22]. The remainder function for the $2 \to 4$ case was found by two of the authors [19, 20] analytically continuing (6) in

$$u_1 \to |u_1|e^{-i2\pi},\ u_2 \to u_2,\ u_3 \to u_3 \tag{15}$$

and then simplifying in the Regge limit

$$|u_1| \to 1^-,\ |u_2| \to 0^+,\ |u_3| \to 0^+,\ \tilde{u}_{2,3} \sim \mathcal{O}(1). \tag{16}$$

Another interesting physical region of the Mandelstam channel is depicted in Fig. 4. It can be obtained from Fig. 3 by twisting the lower part of figure. In this region $t_2', s, s_1, s_3 < 0$ and $s_{13}, s_{02} > 0$, so that the corresponding analytic continuation is given by

$$u_1 \to |u_1|e^{-i2\pi},\ u_2 \to u_2,\ u_3 \to u_3. \tag{17}$$

Thus this case is trivially related to the $2 \to 4$ amplitude considered in refs. [4, 11, 19, 20], in particular, here we also have $|u_1| < 1$. The similarity between the two cases is expected from the Regge factorization of the scattering amplitudes. In the rest of the present paper we focus on the non-trivial region of the $3 \to 3$ amplitude shown in Fig. 3.

The remainder function for the Mandelstam channel of the $3 \to 3$ case in (10) differs from the remainder function for the corresponding channel of the $2 \to 4$ case (see (22) of ref. [20]) only by the overall sign and the presence of the real term subleading in the logarithm of the energy $\ln(u_1 - 1)$. The origin of this difference is best understood from a general expression for remainder function both for $3 \to 3$ and $2 \to 4$ cases derived by one of the authors [22].

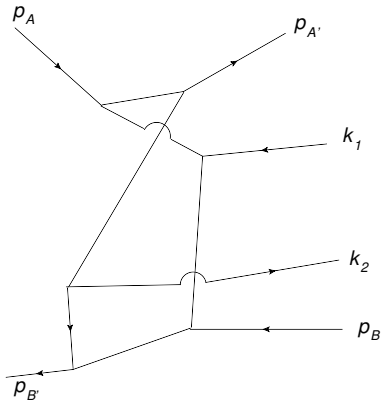


FIG. 4: The $3 \to 3$ gluon scattering amplitude for the Mandelstam channel, where $t_2', s, s_1, s_3 < 0$ and $s_{13}, s_{02} > 0$.

For the $2 \to 4$ amplitude in region defined by (15) and (16) it is given by

$$R_6 e^{i\pi\delta} = \cos\pi\omega_{ab} + i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i} f(\omega) e^{-i\pi\omega}|1-u_1|^{-\omega}, \quad (18)$$

and for the $3 \to 3$ amplitude in region defined by (4) and (14) it reads

$$R_6 e^{-i\pi\delta} = \cos\pi\omega_{ab} - i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i} f(\omega)|1-u_1|^{-\omega}, \quad (19)$$

where $R_6$ are the remainder functions for corresponding process. The phases $\delta$ and $\omega_{ab}$ are defined by [22]

$$\delta = \frac{\gamma_K}{8}\ln\tilde{u}_2\tilde{u}_3 = \frac{\gamma_K}{8}\ln\frac{|w|^2}{|1+w|^4}, \quad (20)$$
$$\omega_{ab} = \frac{\gamma_K}{8}\ln\frac{\tilde{u}_3}{\tilde{u}_2} = \frac{\gamma_K}{8}\ln|w|^2,$$

where $\gamma_K$ is the cusp anomalous dimension $\gamma_K \simeq 4a$ for $a = g^2 N_c/8/\pi^2$ and $\omega$ is related to the angular momentum in $t_2$-channel [25].

The all-loop expressions in (18) and (19) have a meaning of dispersion relations, which establish a connection between real and imaginary parts of the scattering amplitude. It is worth emphasizing that the integral term in (18) and (19) is formally divergent at one loop and should be understood in the sense of the principal value prescription (cf. [11]). It cancels the one-loop contribution from the BDS phase $\delta$, so that $R_6^{(1)}$ is zero as expected.

A few words to be said about the structure of (18) and (19). As it was discussed in refs. [4, 22] assuming the Regge pole factorization the six-gluon amplitude can be written as five contributions compatible with the Steinmann relations. Values of four out of five relative coefficients are fixed by the BDS amplitude in the four physical regions. Using the Weis factorization property [23] one can fix the whole Regge pole structure of the six-gluon amplitude, however the resulting expression has unpleasant analytic properties. Namely, it includes some singularities incompatible with the perturbation theory. It was argued by one of the authors [22] that these dangerous terms can be absorbed in the Mandelstam cut contribution because they have the same phase structure. The resulting expressions have correct analytic properties and can be written in the form of (18) and (19). The contributions of the Regge pole ($\cos\pi\omega_{ab}$) and the Mandelstam cut (the integral over $\omega$) are functions of only dual conformal cross ratios. The factors $e^{\pm i\pi\delta}$ for the corresponding physical regions accounts for a phase already present in the BDS amplitude. They are extracted from the BDS amplitude to make it self-consistent. For more details the reader is referred to section 2 of ref. [22]. The "dispersion" relations in (18) and (19) are correct in the Regge kinematics for any number of loops and thus allow us to make predictions also for a strong coupling regime as discussed below.

Substituting the expansion of the remainder function

$$R_6 \simeq 1 + a^2 R_6^{(2)} \quad (21)$$

and the leading logarithmic (LLA) approximation of $R_6^{(2)}$ for $2 \to 4$ amplitude calculated in ref. [11] we find that the subleading in $\ln(1-u_1)$ real term in $R_6^{(2)}$ fully cancels with the contributions from $\delta$ and $\omega_{ab}$ [22]. The physical meaning of this is that contributions from Regge poles (the cosine term depending on $\omega_{ab}$ ) and Mandelstam cuts with the factor $e^{-i\pi\omega}$ in the integral of (18) are related to each other due to the analyticity of the amplitude.

This cancellation does not happen for the $3 \to 3$ amplitude, where the contribution from the Mandelstam cut is pure imaginary (no factor $e^{-i\pi\omega}$ in the integrand). The contribution from Regge poles $\cos\pi\omega_{ab}$ is pure real, but the phase of BDS amplitude $\delta$ mixes between real and imaginary parts of the remainder function. Thus the $3 \to 3$ remainder function does have a real part at two loops in region (4). Indeed, expanding (19) to the second order in $a$ we obtain

$$a^2 R_6^{(2)} - \frac{\pi^2\delta^2}{2} = -\frac{\pi^2\omega_{ab}^2}{2} \quad (22)$$
$$-i\frac{a^2}{2}\frac{\partial^2}{\partial a^2}\left(\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i} f(\omega)(u_1-1)^{-\omega}\right)|_{a=0}.$$

The integral term in (22) gives a pure imaginary contribution and thus the real part of the $3 \to 3$ remainder function reads

$$\Re\left(R_6^{(2)}\right) = \frac{\pi^2\delta^2}{2a^2} - \frac{\pi^2\omega_{ab}^2}{2a^2} = \frac{\pi^2}{2}\ln\tilde{u}_3\ln\tilde{u}_2 \quad (23)$$
$$= \frac{\pi^2}{2}\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2,$$

in full agreement with (10).

From (18) and (19) we deduce that the remainder function of the $3 \to 3$ amplitude can be obtained from that of the $2 \to 4$ amplitude by a simple transformation

$$\ln(1-u_1) \to \ln(u_1-1) - i\pi \tag{24}$$

together with the subsequent complex conjugation. This is related to the fact, that the Mandelstam cut contribution is constructed from impact factors and the corresponding Green functions, which are the same for $2 \to 4$ and $3 \to 3$ amplitudes. It is easy to see that the transformation (24) is true for (22) of ref. [20] and (10).

## $3 \to 3$ REMAINDER FUNCTION AT THREE LOOPS

In this section we find the remainder function for the $3 \to 3$ scattering amplitude at three loops with logarithmic accuracy $R_6^{(3)\,LLA}$ as well as the real part of the next-to-leading logarithmic term $\Re\left(R_6^{(3)\,NLLA}\right)$. In accordance to the general analytic properties of the scattering amplitudes the leading term in the leading logarithmic approximation (LLA) is pure imaginary. The integral all-order representation for the LLA part of the remainder function was found in ref. [11] using the solution to the octet BFKL equation. It was shown that the LLA term of the remainder function at arbitrary numbers of loops in the Mandelstam channel of the $3 \to 3$ amplitude differs from that of the $2 \to 4$ amplitude only by the overall sign. The explicit expression for $R_6^{(3)\,LLA}$ in the Mandelstam channel of the $2 \to 4$ amplitude was found in ref. [20]. Using this result we readily calculate the leading logarithmic contribution to the remainder function of the Mandelstam channel of the $3 \to 3$ amplitude at three loops

$$\begin{aligned} R_6^{(3)\,LLA} = -\frac{i\pi}{4}\ln(u_1-1)^2 \Big(\ln|w|^2 \ln^2|1+w|^2 \\ -\frac{2}{3}\ln^3|1+w|^2 + \frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) \\ -\frac{1}{4}\ln^2|w|^2\ln|1+w|^2 - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\Big). \end{aligned} \tag{25}$$

The real part of the next-to-leading contribution is found expanding (19) to the third order in $a$ and extracting the real terms

$$\Re\left(R_6^{(3)\,NLLA}\right) = \frac{i\pi\delta}{a} R_6^{(2)\,LLA}. \tag{26}$$

For an arbitrary number of loops $\ell \geq 3$ this reads

$$\Re\left(R_6^{(\ell)\,NLLA}\right) = \frac{i\pi\delta}{a} R_6^{(\ell-1)\,LLA}, \tag{27}$$

where

$$R_6^{(\ell)\,LLA} = \frac{-i}{\ell!}\frac{\partial^\ell}{\partial a^\ell}\left(\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i} f(\omega)(u_1-1)^{-\omega}\right)|_{a=0} \tag{28}$$

as follows from (19).

Using $R_6^{(2)\,LLA}$ given by the first term on RHS of (10) we get

$$\begin{aligned} \Re\left(R_6^{(3)\,NLLA}\right) &= \frac{\pi^2}{4}\ln(u_1-1)\ln(\tilde{u}_2\tilde{u}_3)\ln\tilde{u}_2\ln\tilde{u}_3 \\ &= -\frac{\pi^2}{4}\ln(u_1-1)\left(\ln^2|1+w|^2\ln\left|1+\frac{1}{w}\right|^2\right. \\ &\left. +\ln|1+w|^2\ln^2\left|1+\frac{1}{w}\right|^2\right). \end{aligned} \tag{29}$$

The full remainder function at three loops in the Regge kinematics has also NLLA imaginary and next-to-next-to-leading logarithmic (NNLLA) contributions. To find them one needs to know the higher order corrections to the function $f(\omega)$ in (19), which are not available at the moment apart from the next-to-leading contribution to the impact factor calculated in ref. [20].

Both $R_6^{(3)\,LLA}$ and $R_6^{(3)\,NLLA}$ are symmetric under inversion $w \to 1/w$ and vanishing for $|w| \to 0$ or $|w| \to \infty$. The $w \to 1/w$ symmetry implies a target-projectile symmetry, where the amplitude is symmetric with respect to the transformation $p_A, p_{A'}, p_B, p_{B'}, k_1, k_2 \to p_B, p_{B'}, p_A, p_{A'}, k_2, k_1$.

As it was already mentioned the BDS phase $\delta$ mixes between real and imaginary parts of the remainder function. This means that by virtue of (19) the real part of the remainder function at an arbitrary number of loops $R_6^{(l)}$ is expressed through the BDS phase $\delta$ and the remainder function with a lower number of loops $R_6^{(l-1)}$, $R_6^{(l-2)}$ etc. Despite this fact we can make an all-loop prediction for a value of

$$\Re\left(R_6 e^{-i\pi\delta}\right) = \cos\pi\omega_{ab}, \tag{30}$$

where $\delta$ and $\omega_{ab}$ are given by (20). In the Regge limit $\Re\left(R_6 e^{-i\pi\delta}\right)$ gives the constant term, which is not accompanied by any logarithm of the energy $\ln(u_1-1)$. This all-loop result can be accessible through calculations in the strong coupling regime. However there is some difficulty in understanding (30) in this regime related to the fact that at large coupling constants the functions $\delta$ and $\omega_{ab}$ grow, and therefore the expression in (30) rapidly oscillates and does not have a definite limit. Note, that the expression in (30) is valid only for the Mandelstam channel of the $3 \to 3$ scattering amplitude in the multi-Regge kinematics. In the $2 \to 4$ case this simple structure is spoiled by the presence of the $e^{-i\pi\omega}$ factor in the integral in (18), although this factor disappears after the analytic continuation to the non-physical region $u_1 > 1$.

## DISCUSSION

In the present study we consider $3 \to 3$ planar gluon MHV amplitude in the multi-Regge kinematics. We per-

form the analytic continuation of the six-gluon remainder function at two loops found by Goncharov, Spradlin, Vergu and Volovich to the Mandelstam channel illustrated in Fig. 2 and then extract the logarithmic and constant terms in the Regge limit. We find that despite the fact that $2 \to 4$ and $3 \to 3$ amplitudes have a rather different structure, the corresponding remainder functions have a similar form as expected from the Regge factorization of scattering amplitudes. The only difference between them at two loops is the overall sign and the presence of the real term for the $3 \to 3$ remainder function. This result is in full agreement with a general all-loop dispersion relations (18) and (19) derived by one of the authors [22]. Using these dispersion relations we predict the leading term (25) and the subleading real term (29) of the $3 \to 3$ remainder function at three loops. We also make a prediction for all-loop expression of the real part of the $3 \to 3$ remainder function multiplied by the BDS phase (30). This relation can be accessible through calculations in the regime of the large coupling constant.

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[24] Note that in the $3 \to 3$ case $u_1 > 1$, in contrast to the $2 \to 4$ case, where $u_1 < 1$. For more details the reader is referred to refs. [4, 11].
[25] More details on the equations (18) and (19), including the rigorous definitions of $\omega$ and $f(\omega)$ are presented in ref. [20, 22]