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$\rho^0 - \gamma$ mixing in the neutral channel pion form factor $F_\pi^{(e)}(s)$ and its role in comparing e^+e^- with τ spectral functions

Fred Jegerlehner [a,b,*], Robert Szafron [c]

[a] *Humboldt-Universität zu Berlin, Institut für Physik, Newtonstrasse 15, D-12489 Berlin, Germany*
[b] *Deutsches Elektronen-Synchrotron (DESY), Platanenallee 6, D-15738 Zeuthen, Germany*
[c] *Institute of Physics, University of Silesia, ul. Uniwersytecka 4, PL-40007 Katowice, Poland*

Abstract

We study the effect of $\rho^0 - \gamma$ mixing in $e^+e^- \to \pi^+\pi^-$ and its relevance for the comparison of the square modulus of the pion from-factor $|F_\pi^{(e)}(s)|^2$, as measured in e^+e^- annihilation experiments, and $|F_\pi^{(\tau)}(s)|^2$ the corresponding quantity obtained after accounting for known isospin breaking effects by an isospin rotation from the τ-decay spectra. After correcting the τ data for the missing $\rho - \gamma$ mixing contribution, besides the other known isospin symmetry violating corrections, the $\pi\pi$ I=1 part of the hadronic vacuum polarization contribution to the muon $g-2$ are fully compatible between τ based and e^+e^- based evaluations. τ data thus confirm result obtained with e^+e^- data. Our evaluation of the leading order vacuum polarization contribution, based on all e^+e^- data including more recent BaBar and KLOE data, yields $a_\mu^{\rm had,LO}[e] = 690.75(4.72) \times 10^{-10}$ (e^+e^- based), while including τ data we find $a_\mu^{\rm had,LO}[e,\tau] = 690.96(4.65) \times 10^{-10}$ ($e^+e^- + \tau$ based). This backs the $\sim 3\,\sigma$ deviation between $a_\mu^{\rm experiment}$ and $a_\mu^{\rm theory}$. For the τ di-pion branching fraction we find $B_{\pi\pi^0}^{\rm CVC} = 25.20 \pm 0.0.17 \pm 0.28$ from e^+e^-+CVC, while $B_{\pi\pi^0} = 25.34 \pm 0.0.06 \pm 0.08$ is evaluated directly from the τ spectra.

Key words: $\gamma - \rho$ mixing, ρ-meson properties, e^+e^--annihilation, τ-decay pion form factor, muon anomalous magnetic moment.
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1. Introduction

Isovector data for the pion form factor obtained from hadronic τ-decay spectra can be compared with the mixed isovector-isoscalar data measured in the e^+e^- channel by means of theory input [1]. In particular, we need some model in order to be able to disentangle $\rho-\omega$ mixing as well as other isospin breaking (IB) effects. The general problem in confronting measured quantities like $|F_\pi^{(\tau)}(s)[I=1]|^2$ and $|F_\pi^{(e)}(s)|^2 = |F_\pi^{(e)}(s)[I=1] + F_\pi^{(e)}(s)[I=0]|^2$ is the fact that the latter object is subject to quantum interference between the two

* Corresponding author.
Email addresses: fjeger@physik.hu-berlin.de (Fred Jegerlehner), rszafron@us.edu.pl (Robert Szafron).
URL: www-com.physik.hu-berlin.de/~fjeger/ (Fred Jegerlehner).

amplitudes and in general may not be well approximated by $|F_\pi^{(e)}(s)[I=1]|^2 + |F_\pi^{(e)}(s)[I=0]|^2$. Without a specific model for the complex amplitudes one cannot get the precise relationship.

Commonly, pion form factors measured in the neutral channel in $e^+e^- \to \pi^+\pi^-$ and in the charged channel in $\tau^- \to \nu_\tau \pi^-\pi^0$ decay (or its charge conjugate) are parametrized by an extended Gounaris-Sakurai (GS) formula

$$F_\pi(s) = \frac{\mathrm{BW}^{\mathrm{GS}}_{\rho(770)}(s) \cdot \left(1 + \delta \frac{s}{M_\omega^2} \mathrm{BW}_\omega(s)\right) + \beta\, \mathrm{BW}^{\mathrm{GS}}_{\rho(1450)}(s) + \gamma\, \mathrm{BW}^{\mathrm{GS}}_{\rho(1700)}(s)}{1+\beta+\gamma} \ , \tag{1}$$

which results as a sum of mixing isovector states, each described by a Breit-Wigner (BW) type of amplitude. The pion form factor is related to the corresponding cross section by [1]

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3} \frac{\beta_\pi^3}{s} |F_\pi^{(e)}(s)|^2 = \frac{4\pi\alpha^2}{s} v_0(s), \tag{2}$$

for point-like pions $F_\pi(s) \equiv 1$, where β_π is the pion velocity in the c.m. frame: $\beta_\pi = \sqrt{1-4m_\pi^2/s}$. The spectral function $v_i(s)$ is related to the form factor by

$$v_i(s) = \frac{\beta_i(s)}{12\pi} |F_\pi^{(i)}(s)|^2 \ ; \ \ (i=0,-) \leftrightarrow (e,\tau) \ , \tag{3}$$

for the neutral (0) e^+e^-- and charged (-) τ-channel. The spectral function $v_-(s)$ can be measured very precisely in τ-decay:

$$\frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}s}(\tau^- \to \nu_\tau \pi^-\pi^0) = \frac{6|V_{ud}|^2\, S_{\mathrm{EW}}}{m_\tau^2} \frac{B_e}{B_{\pi\pi}} \left(1 - \frac{s}{m_\tau^2}\right)\left(1 + \frac{2s}{m_\tau^2}\right) v_-(s) \ , \tag{4}$$

with $m_\tau = (1776.84 \pm 0.17)$ MeV the τ mass, $|V_{ud}| = 0.9418 \pm 0.00019$ the CKM matrix element, $B_e = (17.818 \pm 0.032)\,\%$ the electron branching fraction, $B_{\pi\pi} = (25.51 \pm 0.09)\,\%$ the di-pion branching fraction and $S_{\mathrm{EW}} = 1.0235 \pm 0.0003$ the short distance electroweak correction.

Note that a single standard Breit-Wigner resonance yields

$$|F_\pi(s)|^2 = \frac{36}{\alpha^2} \frac{\Gamma(\rho \to e^+e^-)}{\beta_\pi^3 \Gamma(\rho \to \pi^+\pi^-)} \frac{s}{M_\rho^2} \frac{s\Gamma_\rho^2}{(s-M_\rho^2)^2 + M_\rho^2\Gamma_\rho^2} \ .$$

Denoting the $\gamma - \rho$ transition coupling by eM_ρ^2/g_ρ the branching fraction at resonance reads

$$R_\rho \doteq \frac{\Gamma(\rho \to e^+e^-)}{\Gamma(\rho \to \pi^+\pi^-)} = \frac{\alpha^2}{36} \left(\frac{g_\rho}{g_{\rho\pi\pi}}\right)^2 \left(\frac{M_\rho}{\Gamma_\rho}\right)^2 \beta_\rho^3 \ ,$$

with $\beta_\rho \doteq \beta_\pi(s = M_\rho^2)$. In the case of complete ρ dominance $g_\rho = g_{\rho\pi\pi}$ [2].

The GS formula (1) also describes the charged isovector channel provided $\delta = 0$, since there is no charged version of the ω. In the neutral channel the GS formula does not fully include $\rho^0 - \gamma$ mixing , which is known since the early 1960's, when the ρ had been discovered. A direct consequence of $\rho^0 - \gamma$ mixing is the vector meson dominance (VMD) model characterized by an effective Lagrangian [4]

$$\mathcal{L}_{\gamma\rho} = -\frac{e\, M_\rho^2}{g_\rho} \rho_\mu A^\mu \ . \tag{5}$$

[1] QED corrections to $e^+e^- \to \pi^+\pi^-$ have been summarized in [2,3]. They will not be considered in the following and we assume them to be taken into account in the extraction of the form factor from the experiments.

[2] At resonance the single BW pion form factor is given by

$$|F_\pi(M_\rho^2)|^2 = \frac{36}{\alpha^2} \frac{\Gamma_{ee}}{\beta_\rho^3 \Gamma_{\pi\pi}} \ ,$$

and for PDG values of the parameters yields $|F_\pi(M_\rho^2)|^2 \approx 39$ a reasonable value (see below).

However, this form does not preserve electromagnetic gauge invariance and the photon would acquire a mass unless we add a photon mass counterterm to the Lagrangian which is fine tuned appropriately. The pion form factor here takes the form

$$F_\pi(s) = -\frac{M_\rho^2}{s - M_\rho^2}\,\frac{g_{\rho\pi\pi}}{g_\rho} \tag{6}$$

and the condition of electromagnetic current conservation $F_\pi(0) = 1$ is satisfied only if $g_{\rho\pi\pi} = g_\rho$, which is called universality condition and corresponds to complete ρ dominance. In fact electromagnetic gauge invariance can be implemented by writing the effective VMD Lagrangian in the form [5]

$$\mathcal{L}_{\gamma\rho} = \frac{e}{2\,g_\rho}\,\rho_{\mu\nu}F^{\mu\nu} \quad , \tag{7}$$

in terms of the field strength tensors. As it satisfies gauge invariance, the form factor calculated here reads

$$F_\pi(s) = 1 - \frac{s}{s - M_\rho^2}\,\frac{g_{\rho\pi\pi}}{g_\rho} \tag{8}$$

and satisfies the current conservation condition $F_\pi(0) = 1$ in any case, irrespective of the universality constraint $g_{\rho\pi\pi} = g_\rho$ (for a recent discussion also see [6]). Obviously, this simple model is not able to describe the pion form factor measured in $e^+e^- \to \pi^+\pi^-$ at low energies, unless we take into account energy dependent finite widths effects of the ρ as it is done in the GS model [7] [3]. The energy dependence of the ρ-width has to reflect the off-shell $\rho^* \to \pi\pi$ process. So we have to model effectively a "rho-pion-photon" system, discarding the ω and its mixing with the ρ, which is well understood and will be taken into consideration in a second step. Our focus here is to work out the difference in the relation between the charged channel and the neutral channel, which results from the $\rho^0 - \gamma$ mixing . The latter like $\rho^0 - \omega$ mixing, has no counterpart in the charged channel. The purpose of this study is to understand better the discrepancy between τ and e^+e^- di-pion spectra, which has been clearly established in [16] under the assumption that all possible IB corrections were accounted for. More recent data form Belle [17] and KLOE [18,19], and applying improved IB corrections, confirmed a significant discrepancy [20]. Although the new $\pi^+\pi^-$ spectrum from BaBar [21], measured via the radiative return mechanism, is closer to the corresponding spectra obtained from τ-decays, a discrepancy persists.

2. A $\rho - \gamma$ mixing model and related self-energy effects

As already said, the VMD ansatz has to be replaced by a more realistic model which must take into account
– the finite ρ-width, related to its decay $\rho \to \pi^+\pi^-$,
– the $\rho - \gamma$ mixing, which leads to non-diagonal propagation of the $\rho - \gamma$ system, and
– the $\rho - \omega$ mixing, which we will consider in a second step.
This has to be implemented in an appropriate effective field theory (EFT). In a first step we consider the interaction of the ρ with the pions together with their electromagnetic interaction, assuming the pions to by point-like (scalar QED). As suggested long ago by Sakurai [22], the ρ may be treated as a massive gauge boson. The effective Lagrangian thus reads

$$\mathcal{L} = \mathcal{L}_{\gamma\rho} + \mathcal{L}_\pi \;\; ; \;\; \mathcal{L}_\pi = D_\mu\pi^+ D^{+\mu}\pi^- - m_\pi^2\pi^+\pi^- \;\; ; \;\; D_\mu = \partial_\mu - \mathrm{i}\,e\,A_\mu - \mathrm{i}\,g_{\rho\pi\pi}\rho_\mu \;\; . \tag{9}$$

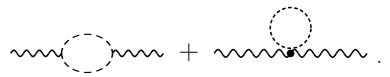
The corresponding Feynman rules in momentum space are

[3] Other models have been reviewed and investigated recently with emphasis on $\rho - \omega$ mixing in Ref. [8]. Frequently used descriptions of the low energy $\pi\pi$ form factor include the ChPT-based Guerrero-Pich formulation [9], the Leutwyler-Colangelo approach [10], the resonance Lagrangian approach [11] (see e.g. [12]) or the related Hidden Local Symmetry (HLS) model as applied in [13,14], and the phenomenological Kühn-Santamaria (KS) model [15]. As we will see our model is closely related to the GS model, and we adopt the latter for comparisons and fits.

$$
\begin{aligned}
A^\mu\pi\pi \;&\hat{=}\; -\mathrm{i}\,e\,(p+p')^\mu \quad ; \quad \rho^\mu\pi\pi \;\hat{=}\; -\mathrm{i}\,g_{\rho\pi\pi}\,(p+p')^\mu \\
A^\mu A^\nu\pi\pi \;&\hat{=}\; 2\,\mathrm{i}\,e^2\,g^{\mu\nu} \quad ; \quad \rho^\mu\rho^\nu\pi\pi \;\hat{=}\; 2\,\mathrm{i}\,g_{\rho\pi\pi}^2\,g^{\mu\nu} \\
A^\mu\rho^\nu\pi\pi \;&\hat{=}\; 2\,\mathrm{i}\,e\,g_{\rho\pi\pi}\,g^{\mu\nu} \quad ; \quad A^\mu\rho^\nu \;\hat{=}\; -\mathrm{i}\,e/g_\rho\,(p^2\,g^{\mu\nu}-p^\mu p^\nu)\ .
\end{aligned}
$$

The model should be understood as a simplified version of the better justified effective resonance Lagrangian approach [11], which extends the chiral structure of low energy QCD (chiral perturbation theory) to include spin 1 resonances in a consistent way. A variant is the HLS model, which in the same context has been applied to investigate the (ρ,ω,ϕ) mixing effects in [13]. Actually in [13] too, $V-\gamma$ ($V=\rho,\omega,\phi$) mixing amplitudes have been included (more on that below). The main difference to the GS model is that we take our EFT Lagrangian serious in the sense that we include all relevant contributions to $e^+e^-\to\pi^+\pi^-$, while in the GS model some of the contributions have been neglected. In fact the GS model is incomplete in the sense of a quantum field theory.

In sQED the contribution of a pion loop to the photon vacuum polarization is given diagrammatically by

$-\mathrm{i}\,\Pi_{\gamma\gamma}^{\mu\nu\,(\pi)}(q) =$  .

and one then obtains the bare $\gamma-\rho$ transverse self-energy functions

$$
\Pi_{\gamma\gamma}=\frac{e^2}{48\pi^2}\,f(q^2)\ ,\quad \Pi_{\gamma\rho}=\frac{e g_{\rho\pi\pi}}{48\pi^2}\,f(q^2)\quad\text{and}\quad \Pi_{\rho\rho}=\frac{g_{\rho\pi\pi}^2}{48\pi^2}\,f(q^2)\ , \tag{10}
$$

where

$$
f(q^2)\equiv q^2\,h(q^2)=\left(B_0(m_\pi,m_\pi;q^2)\,(q^2-4\,m_\pi^2)-4\,A_0(m_\pi)-4\,m_\pi^2+\frac{2}{3}q^2\right)\ , \tag{11}
$$

in terms of the standard scalar one-loop integrals $A_0(m)$ and $B_0(m,m;s)$ [23]. Explicitly [4], in the $\overline{\rm MS}$ scheme (μ the $\overline{\rm MS}$ renormalization scale)

$$
h(q^2)\equiv f(q^2)/q^2=2/3+2\,(1-y)-2\,(1-y)^2\,G(y)+\ln\frac{\mu^2}{m_\pi^2}\ , \tag{12}
$$

where $y=4m_\pi^2/s$ and $G(y)=\frac{1}{2\beta_\pi}\,(\ln\frac{1+\beta_\pi}{1-\beta_\pi}-\mathrm{i}\,\pi)$, for $q^2>4\,m_\pi^2$. Note that all components of the $(\gamma,\,\rho)$ 2×2 matrix propagator are proportional to the same function $f(q^2)$. The renormalization conditions are such that the matrix is diagonal and of residue unity at the photon pole $q^2=0$ and at the ρ resonance $s=M_\rho^2$, hence the renormalized self-energies read (see e.g. [24])

$$
\Pi_{\gamma\gamma}^{\rm ren}(q^2)=\Pi_{\gamma\gamma}(q^2)-q^2\,\Pi'_{\gamma\gamma}(0)\doteq q^2\,\Pi'^{\rm ren}_{\gamma\gamma}(q^2) \tag{13}
$$

$$
\Pi_{\gamma\rho}^{\rm ren}(q^2)=\Pi_{\gamma\rho}(q^2)-\frac{q^2}{M_\rho^2}\,{\rm Re}\,\Pi_{\gamma\rho}(M_\rho^2) \tag{14}
$$

$$
\Pi_{\rho\rho}^{\rm ren}(q^2)=\Pi_{\rho\rho}(q^2)-{\rm Re}\,\Pi_{\rho\rho}(M_\rho^2)-(q^2-M_\rho^2)\,{\rm Re}\,\frac{{\rm d}\Pi_{\rho\rho}}{{\rm d}s}(M_\rho^2) \tag{15}
$$

where $\Pi_{\gamma\gamma}(0)=\Pi_{\gamma\rho}(0)=\Pi_{\rho\rho}(0)=0$ and $\Pi'_{\gamma\gamma}(q^2)=\Pi_{\gamma\gamma}(q^2)/q^2$, has been used. Note, that the tree level mixing term in the Lagrangian contributes to the bare $\gamma\rho$ self-energy as $\Pi_{\gamma\rho}^{(0)}=q^2(e/g_\rho)$, which does not

[4] The standard Gounaris-Sakurai parametrization differs from our sQED model and utilizes $h(q^2)=-2\,(1-y)^2\,G(y)$ which for $q^2\to0$ behaves as $h(q^2)\to2y=8m_\pi^2/q^2$ i.e. $q^2h(q^2)\to8m_\pi^2$, which in $D_{\gamma\gamma}$ represents a non-vanishing photon mass. While the constant terms in (12) drop out by renormalization, the $+2(1-y)$ term is required by electromagnetic gauge invariance and in fact renders $h(q^2)\to$ const. regular in the static limit.

affect the renormalized self energies, however. In particular, $\delta\Pi^{\rm ren}_{\gamma\rho} = q^2\,\frac{e}{g_\rho} - \frac{q^2}{M_\rho^2}\,M_\rho^2\frac{e}{g_\rho} = 0$. The ρ wave function renormalization reads $Z_\rho = 1/(1+\frac{{\rm d}\Pi_{\rho\rho}}{{\rm d}s}(s=M_\rho^2))$ with

$$\frac{{\rm d}\Pi_{\rho\rho}}{{\rm d}s}(s=M_\rho^2) = \frac{g^2_{\rho\pi\pi}}{48\,\pi^2}\,\left\{8/3 - \beta_\rho^2\,\left[1 + (3-\beta_\rho^2)\,\frac{1}{2\,\beta_\rho}\ln\left(\frac{1+\beta_\rho}{1-\beta_\rho}\right)\right]\right\}\quad . \tag{16}$$

Numerically, $Z_\rho \simeq 1.1289$ at $\mu = m_\pi$.

It is crucial to observe that vacuum polarization effects affect mass renormalization of the ρ as well as γ - ρ mixing, in spite of the fact that photon vacuum polarization has to be subtracted in the definition of F_π. In other words, in sQED we would still have $F_\pi(s) = 1$ in (2) while vacuum polarization is absorbed into a running fine structure constant

$$\alpha \to \alpha(s) = \frac{\alpha}{1 + \Pi^{'{\rm ren}}_{\gamma\gamma}(s)}\quad , \tag{17}$$

which mean that in calculating $F_\pi(s)$ we have to multiply the result by $1 + \Pi^{'{\rm ren}}_{\gamma\gamma}(s)$.

A convenient representation of $\Pi^{\rm ren}_{\rho\rho}$ is given by

$$\Pi^{\rm ren}_{\rho\rho}(s) = \frac{\Gamma_\rho}{\pi M_\rho\,\beta_\rho^3}\,\left\{s\,\left(h(s) - {\rm Re}\,h(M_\rho^2)\right) - (s-M_\rho^2)\,M_\rho^2\,\left.{\rm Re}\,\frac{{\rm d}h}{{\rm d}s}\right|_{s=M_\rho^2}\right\} \tag{18}$$

with

$$s\,\frac{{\rm d}h}{{\rm d}s}(s) = 3\,y - 1 - 3\,y\,(1-y)\,G(y)\quad . \tag{19}$$

In particular [5]:

$$\Pi^{\rm ren}_{\rho\rho}(0) = \frac{M_\rho\Gamma_\rho}{\pi\,\beta_\rho^3}\,M_\rho^2\,\left.\frac{{\rm d}h}{{\rm d}s}\right|_{s=M_\rho^2} = \frac{M_\rho\Gamma_\rho}{\pi\,\beta_\rho^3}\,\left(3y_\rho - 1 - 3\,y_\rho\,\beta_\rho^2\,G(y_\rho)\right)\quad ;\quad y_\rho = 4m_\pi^2/M_\rho^2\quad . \tag{20}$$

Without mixing, pion production mediated by the ρ resonance, yields the GS type pion form factor, normalized to $F_\pi(0) = 1$,

$$F_\pi^{\rm GS}(s) = \frac{-M_\rho^2 + \Pi^{\rm ren}_{\rho\rho}(0)}{s - M_\rho^2 + \Pi^{\rm ren}_{\rho\rho}(s)}\quad . \tag{21}$$

The renormalized mixing self-energy may be written in a form

$$\Pi^{\rm ren}_{\gamma\rho}(s) = \frac{e g_{\rho\pi\pi}}{48\pi^2}\,\left\{s\,\left(h(s) - {\rm Re}\,h(M_\rho^2)\right)\right\}\quad . \tag{22}$$

Note that while the inverse propagator matrix is diagonal at the two propagator poles, off the poles it is not diagonal. This is the main effect we are going to discuss now [6].

The propagators are obtained by inverting the symmetric 2×2 self energy matrix

[5] In contrast to sQED, in the standard GS formula

$$h(s) = -2(1-y)^2\,G(y)\quad ;\quad s\,\frac{{\rm d}h}{{\rm d}s}(s) = y - 1 - 3\,y\,(1-y)\,G(y)\quad ,$$

which is singular for $s\to 0$. The first term in (18) in this case yields a finite contribution $sh(s)\to 8m_\pi^2$ and thus

$$\Pi^{\rm ren}_{\rho\rho}(0) = \frac{M_\rho\Gamma_\rho}{\pi\,\beta_\rho^3}\,\left(\frac{8m_\pi^2}{M_\rho^2} + M_\rho^2\,\left.\frac{{\rm d}h}{{\rm d}s}\right|_{s=M_\rho^2}\right) = \frac{M_\rho\Gamma_\rho}{\pi\,\beta_\rho^3}\,\left(3y_\rho - 1 - 3\,y_\rho\,\beta_\rho^2\,G(y_\rho)\right)\quad ,$$

i.e., $\Pi^{\rm ren}_{\rho\rho}(0) \equiv -d\,\Gamma_\rho\,M_\rho$ is actually not modified, in spite of lacking manifest gauge invariance.

[6] These effects are very similar to $Z^0-\gamma$ mixing [25] which has been investigated theoretically as well as experimentally at LEP with high precision (see Refs. [26,27] and references therein). Typically, these effects at low energy or near the Z pole are

$$\hat{D}^{-1} = \begin{pmatrix} q^2 + \Pi_{\gamma\gamma}(q^2) & \Pi_{\gamma\rho}(q^2) \\ \Pi_{\gamma\rho}(q^2) & q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2) \end{pmatrix} \tag{23}$$

with the result:

$$\begin{aligned} D_{\gamma\gamma} &= \frac{1}{q^2 + \Pi_{\gamma\gamma}(q^2) - \frac{\Pi^2_{\gamma\rho}(q^2)}{q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2)}} \simeq \frac{1}{q^2 + \Pi_{\gamma\gamma}(q^2)} \\ D_{\gamma\rho} &= \frac{-\Pi_{\gamma\rho}(q^2)}{(q^2 + \Pi_{\gamma\gamma}(q^2))(q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2)) - \Pi^2_{\gamma\rho}(q^2)} \simeq \frac{-\Pi_{\gamma\rho}(q^2)}{(q^2 + \Pi_{\gamma\gamma}(q^2))(q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2))} \\ D_{\rho\rho} &= \frac{1}{q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2) - \frac{\Pi^2_{\gamma\rho}(q^2)}{q^2 + \Pi_{\gamma\gamma}(q^2)}} \simeq \frac{1}{q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2)} \ . \end{aligned} \tag{24}$$

These expressions sum correctly all the irreducible self-energy bubbles [7]. The approximations indicated are the one-loop results. The extra terms are higher order contributions and are particularly relevant near the resonance, characterized by the location s_P of the pole of the propagator, which is given by the zero of the inverse propagator:

$$s_P - m^2_{\rho^0} - \Pi_{\rho^0\rho^0}(s_P) - \frac{\Pi^2_{\gamma\rho^0}(s_P)}{s_P - \Pi_{\gamma\gamma}(s_P)} = 0 \ , \tag{25}$$

with $s_P = \tilde{M}^2_{\rho^0}$ complex. The usual (no mixing) considerations in determining the physical mass and width of a resonance remain true if we denote the self-energies by Π_V ($V = \rho^0, \rho^\pm$) with

$$\Pi_{\rho^\pm}(p^2, \cdots) = \Pi_{\rho^+\rho^-}(p^2, \cdots)$$

and

$$\Pi_{\rho^0}(p^2, \cdots) = \Pi_{\rho^0\rho^0}(p^2, \cdots) + \frac{\Pi^2_{\gamma\rho^0}(p^2, \cdots)}{p^2 - \Pi_{\gamma\gamma}(p^2, \cdots)} \ .$$

Thus the location of the pole may be written as

$$\tilde{M}^2 - m^2 + \Pi(\tilde{M}^2, m^2, \cdots) = 0, \tag{26}$$

for both the $\rho^\pm$ and the ρ^0, where

$$\tilde{M}^2_\rho \equiv \left(q^2\right)_{\text{pole}} = M_\rho^2 - \mathrm{i}\, M_\rho\, \Gamma_\rho$$

is characterized by mass and width of the ρ [8]. Note that the imaginary part of the self-energy function is energy dependent, which implies an energy dependent width, of course with the correct phase-space behavior of $\rho \to \pi\pi$ decay.

How do off-diagonal elements of the $\gamma - \rho$ propagator affect the line-shape of the ρ? We assume we know the ρ mass M_ρ and the ρ width Γ_ρ for the unmixed ρ as it is seen e.g. in the isovector τ decay spectra, i.e. we compare the result with a charged $\rho^\pm$ assuming equal mass and width. We therefore compare result first with the Belle data [17]. Of course our model does not fit the data, because a more sophisticated extended

expected to be small because of the smallness of the electromagnetic fine structure constant, and at the Z resonance, because of the large mass and very small width of the Z^0 boson. In case of the ρ witch's mass lies not very far above the hadronic $\pi\pi$-threshold (which is very low by the fact that pions are quasi Nambu-Goldstone bosons) and due to the relatively large (hadronic) width we expect corresponding mixing effects to be much more relevant. In the charged channel, in principle, there is $W^\pm - \rho^\pm$ mixing, with some effective $W^+\rho^-$ + h.c. coupling term. However, this produces a negligible effect because the W propagator pole is far away from the ρ propagator pole and from the energy range of interest. In fact the mixing matrix, diagonalized at the ρ-pole, remains essentially diagonal in the whole range of interest (< 2 GeV).

[7] It is of curse well known that this Dyson summation is crucial for a proper description of the particle/resonance structure in particular near the poles, where naive perturbation theory in any case breaks down.

[8] For γ - Z^0 mixing in the electroweak Standard Model explicit results up to two loops have been worked out in [28].

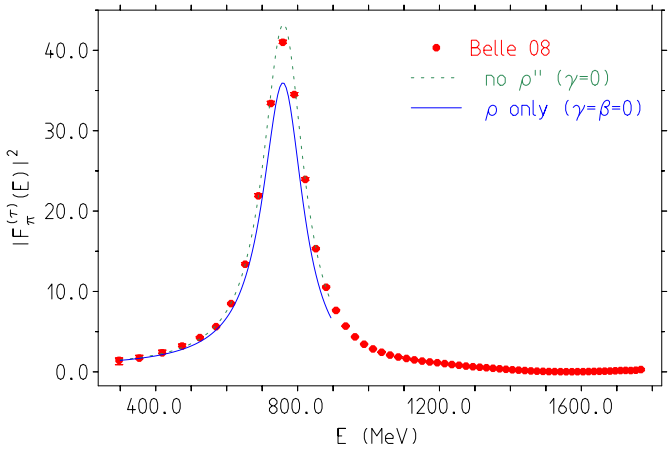


Fig. 1. GS fits of the Belle data and the effects of including higher states ρ' and ρ'' at fixed M_ρ and Γ_ρ. Doubts are in order whether the higher resonances really affect the ρ resonance in the way suggested by the commonly adopted GS parametrization.

Gounaris-Sakurai model (1) has been used to extract the ρ parameters. If we switch off the contributions from ρ' and ρ'' by setting $\gamma = 0$ and $\beta = \gamma = 0$ we observe a substantial change in $F_\pi(s)$ as illustrated in Fig. 1. Note that in an EFT one would expect the heavier states to decouple, while in the GS type modeling the low energy tail is normalized away by the $1 + \beta + \gamma$ normalization factor. In field theory in place of this normalization a factor $s/M_{\rm res}^2$ would imply automatic decoupling. But that is not the way mass and width of the ρ are determined usually. Evidently, in the GS model, in the ρ-region the higher resonances serve as a continuum background without which good fits in general are not possible. So if we stick with our simplified model we cannot expect to get a good representation of the data without corresponding extensions. On the other hand, the simplified model allows us to work out more clearly the effect of γ - ρ mixing. Above we have diagonalized the mixing propagator matrix at the poles, this allows to make precise the meaning of mass and width of the heavy unstable state. This is achieved if renormalized mixing self-energy is given by

$$\Pi_{\gamma\rho\ ren}(q^2) = \Pi_{\gamma\rho}(q^2) - \Pi_{\gamma\rho}(0) - \frac{q^2}{M_\rho^2}\left(Re\Pi_{\gamma\rho}(M_\rho^2) - \Pi_{\gamma\rho}(0)\right) \ . \tag{27}$$

This can be achieved by two subsequent transformations of the bare fields:

i) Infinitesimal (perturbative) rotation

$$\begin{pmatrix} A_b \\ \rho_b \end{pmatrix} = \begin{pmatrix} 1 & -\Delta_0 \\ \Delta_0 & 1 \end{pmatrix} \begin{pmatrix} A' \\ \rho' \end{pmatrix}$$

diagonalizing the mass matrix at one-loop (n+1-loop) order given that the mass matrix has been diagonalized at tree (n-loop) level.

ii) Upper diagonal matrix wave function renormalization inducing a kinetic mixing term (this cannot be done by an orthogonal transformation)

$$\begin{pmatrix} A' \\ \rho' \end{pmatrix} = \begin{pmatrix} \sqrt{Z_\gamma} & -\Delta_\rho \\ 0 & \sqrt{Z_\rho} \end{pmatrix} \begin{pmatrix} A_r \\ \rho_r \end{pmatrix}$$

which allows to normalize the residues to one for the γ- and ρ-propagator, respectively, and to shift to zero the mixing propagator at the ρ-pole.

Thus the relationship between the bare and the renormalized (LSZ) fields is (expanded to linear order)

$$\begin{aligned} A_b &= \sqrt{Z_\gamma} A_r - \ (\Delta_\rho + \Delta_0)\ \rho_r \\ \rho_b &= \sqrt{Z_\rho}\rho_r + \Delta_0\ A_r \ , \end{aligned} \tag{28}$$

generalizing the usual multiplicative field renormalization represented by the first term for both fields. The counter-terms Δ_0 and Δ_ρ are determined by the condition (27)

$$\Delta_0 = \frac{\Pi_{\gamma\rho}(0)}{M_\rho^2}$$
$$\Delta_\rho = \frac{Re\Pi_{\gamma\rho}(M_\rho^2) - \Pi_{\gamma\rho}(0)}{M_\rho^2} \quad . \tag{29}$$

For our model $\Delta_0 = 0$ and $\Delta_\rho = e/g_\rho$ to leading order. The field transformations of course induce mixing counter terms at the vertices, which are absorbed into the definition of the physical couplings. In principle, this non-symmetric transformation only affects the bookkeeping such that the propagator pole structure becomes obvious. It does not change the value of the functional integral i.e. the mixing counter terms cancel in the interior of Feynman diagrams, unless the photon and/or the rho are involved as external fields (states).

As a consequence of the diagonalization the physical ρ acquires a direct coupling to the electron: starting as usual from the bare Lagrangian

$$\mathcal{L}_{\rm QED} = \bar{\psi}_e \gamma^\mu (\partial_\mu - \mathrm{i}\, e_b\, A_{b\mu})\, \psi_e \tag{30}$$

we obtain

$$\mathcal{L}_{\rm QED} = \bar{\psi}_e \gamma^\mu (\partial_\mu - \mathrm{i}\, e\, A_\mu + \mathrm{i}\, g_{\rho ee} \rho_\mu)\, \psi_e \tag{31}$$

with $g_{\rho ee} = e\,(\Delta_\rho + \Delta_0)$, where in our case $\Delta_0 = 0$.

The $e^+e^- \to \pi^+\pi^-$ matrix element in sQED is given by

$$\mathcal{M} = -\mathrm{i}\, e^2\, \bar{v} \gamma^\mu u\, (p_1 - p_2)_\mu\, F_\pi(q^2) \tag{32}$$

with $F_\pi(q^2) = 1$. In our extended VMD model we have the four terms shown in Fig. 2 and thus

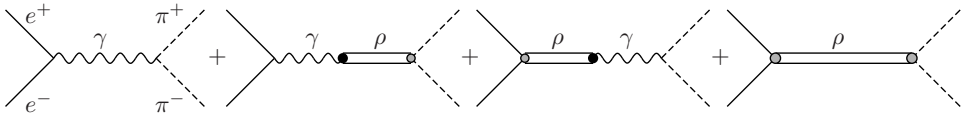


Fig. 2. Diagrams contributing to the process $e^+e^- \to \pi^+\pi^-$.

$$F_\pi(s) \propto e^2\, D_{\gamma\gamma} + e g_{\rho\pi\pi}\, D_{\gamma\rho} - g_{\rho ee} e D_{\rho\gamma} - g_{\rho ee} g_{\rho\pi\pi}\, D_{\rho\rho}\,,$$

where the first term properly normalized must be unity [9]. Thus

$$F_\pi(s) = \left[e^2\, D_{\gamma\gamma} + e\,(g_{\rho\pi\pi} - g_{\rho ee})\, D_{\gamma\rho} - g_{\rho ee} g_{\rho\pi\pi}\, D_{\rho\rho}\right] / \left[e^2\, D_{\gamma\gamma}\right] \quad . \tag{33}$$

Note the sign of the induced coupling $g_{\rho ee}$ in (31), which leads to the signs as given in (33). Typical couplings read $g_{\rho\pi\pi\,\rm bare} = 5.8935$, $g_{\rho\pi\pi\,\rm ren} = 6.1559$, $g_{\rho ee} = 0.018149$ and $x = g_{\rho\pi\pi}/g_\rho = 1.15128$.

Real parts and moduli of the individual terms normalized to the sQED photon exchange term are displayed in Fig. 3.

An improved theory of the pion form factor has been developed in [10]. One of the key ingredients in this approach is the strong interaction phase shift $\delta_1^1(s)$ of $\pi\pi$ (re)scattering in the final state. In Fig. 4 we compare the phase of $F_\pi(s)$ in our model with the one obtained by solving the Roy equation with $\pi\pi$-scattering data as input. We notice that the agreement is surprisingly good up to about 1 GeV. It is not difficult to replace our phase by the more precise exact one.

We note that the precise s-dependence of the effective ρ-width is obtained by evaluating the imaginary part of the ρ self-energy:

[9] Note that the conserved vector current (CVC) condition $F_\pi(0) = 1$ in our model is given and saturated by the sQED term $D_{\gamma\gamma}$ alone, while in the GS model $F_\pi(0) = 1$ is imposed by force on the term $D_{\rho\rho}$, the only one present in the GS case.

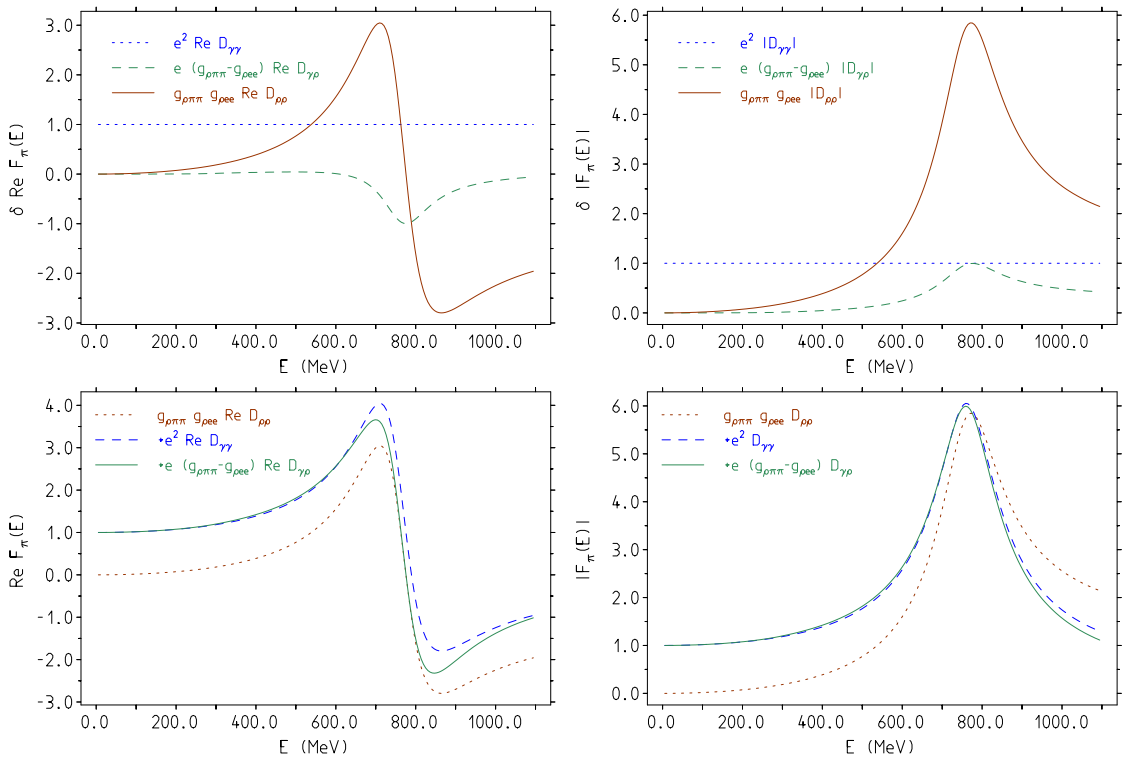


Fig. 3. The real parts and moduli of the three terms of (33), individual and added up.

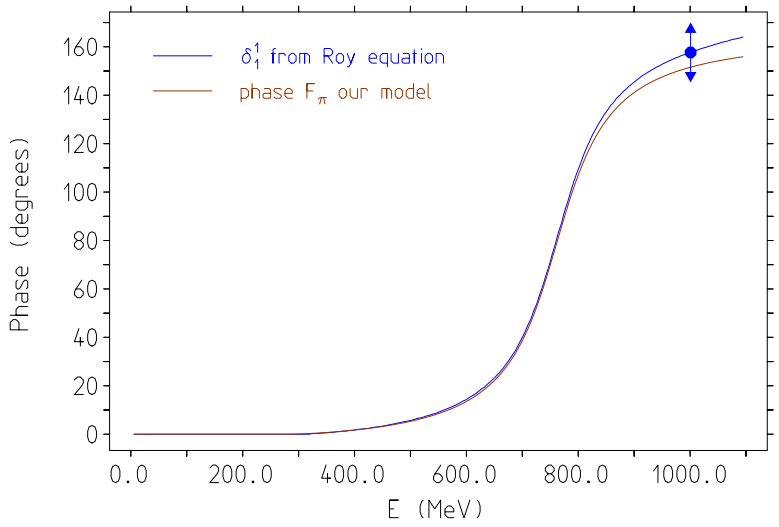


Fig. 4. The phase of $F_\pi(E)$ as a function of the c.m. energy E. We compare the result of the elaborate Roy equation analysis of Ref. [10] with the one due to the sQED pion-loop. The solution of the Roy equation depends on the normalization at a high energy point (typically 1 GeV). In our calculation we could adjust it by varying the coupling $g_{\rho\pi\pi}$.

$$\mathrm{Im}\,\Pi_{\rho\rho} = \frac{g^2_{\rho\pi\pi}}{48\,\pi}\,\beta_\pi^3\,s \equiv M_\rho\,\Gamma_\rho(s)\ , \tag{34}$$

which yields

$$\Gamma_\rho(s)/M_\rho = \frac{g^2_{\rho\pi\pi}}{48\,\pi}\,\beta_\pi^3\,\frac{s}{M_\rho^2}\ ;\ \ \Gamma_\rho/M_\rho = \frac{g^2_{\rho\pi\pi}}{48\,\pi}\,\beta_\rho^3\ . \tag{35}$$

In our model, in the given approximation, the on ρ-mass-shell form factor reads

$$F_\pi(M_\rho^2) = 1 - \mathrm{i}\,\frac{g_{\rho ee}g_{\rho\pi\pi}}{e^2}\,\frac{M_\rho}{\Gamma_\rho}\ , \tag{36}$$

and the square modulus may be written as

$$|F_\pi(M_\rho^2)|^2 = 1 + \frac{36}{\alpha^2}\frac{\Gamma_{ee}}{\beta_\rho^3\,\Gamma_\rho}\ , \tag{37}$$

with

$$\Gamma_{\rho ee} = \frac{1}{3}\,\frac{g^2_{\rho ee}}{4\pi}\,M_\rho\ \ \text{or}\ \ \ g_{\rho ee} = \sqrt{12\pi\,\Gamma_{\rho ee}/M_\rho}\ . \tag{38}$$

It is interesting to note that the GS formula (21) does not involve $\Gamma_{\rho ee}$ in any direct way, since the normalization is fixed by applying an overall factor $1 + d\,\Gamma_\rho/M_\rho \equiv 1 - \Pi^{\rm ren}_{\rho\rho}(0)/M_\rho^2$ to enforce $F_\pi(0) = 1$. The leptonic width is then given by

$$\Gamma^{\rm GS}_{\rho ee} = \frac{2\alpha^2\,\beta_\rho^3 M_\rho^2}{9\,\Gamma_\rho}\,\left(1 + d\,\Gamma_\rho/M_\rho\right)^2\ . \tag{39}$$

In the CMD-2 fit $1 + d\,\Gamma_\rho/M_\rho \simeq 1.089$.

The result for $|F_\pi(s)|^2$ (using mass and width as before) is displayed in Fig. 5. We compare the results obtained when ρ - γ mixing is properly taken into account with the one obtained by ignoring mixing and with a GS fit with just the ρ taken into account. At first look, the results agree fairly well but do not fit the Belle data as expected if we do not include the higher resonances.

A detailed comparison, in terms of the ratio

$$r_{\rho\gamma}(s) \equiv \frac{|F_\pi(s)|^2}{|F_\pi(s)|^2_{D_{\gamma\rho}=0}}\ , \tag{40}$$

shown in Fig. 6, however reveals substantial differences and proves the relevance of the mixing. We also plotted the same ratio for the I=1 part of the GS fit, which exhibits a similar behavior as the true $F_\pi(s)$. This is not really surprising as one fits the same data just in a different way, i.e. with different parameters for the ρ. This mixing affects, however, the relationship to the τ channel, which does not exhibit this effect. Of course at higher energies, not to far above the ρ, it is not known whether the simple EFT model can be trusted. Note that dropping the $\Pi^2_{\gamma\rho}$ terms (approximation indicated in (24)) in the Dyson resummed propagators does not affect the result. We have checked that $\omega - \gamma$, $\phi - \gamma$ or $\Upsilon(4S) - \gamma$ mixing effects are tiny away from the resonances and thus should not affect the interpretation of radiative return spectra as measured at KLOE and BaBar.

We have to compare the model with the $I = 1$ part of the e^+e^--data. To this end we may take the CMD-2 fit of the CMD-2 data [29] and set the mixing parameter $\delta = 0$ as illustrated in Fig. 7. In this way we obtain the isovector part of the square of the pion form factor $|F_\pi^{(e)}[I = 1](s)|^2$.

In order to compare $F_\pi^{(\tau)}(s)$ extracted from τ-decay spectra with $F_\pi^{(e)}(s)$ measured in e^+e^--annihilation we have to apply isospin breaking corrections as investigated in [30] and [31] (see also [16,32,20]):

– Mass shift: the Cottingham formula, which allows us for a rather precise calculation of the electromagnetic pion mass shift $\delta m_\pi = m_{\pi^\pm} - m_{\pi^0} \simeq 4.6$ MeV, suggest the relation $\Delta m_\pi^2 \simeq \Delta M_\rho^2$, which yields a shift between changed an neutral ρ by $\delta M_\rho = M_{\rho^\pm} - M_{\rho^0} \simeq \frac{1}{2}\,\frac{\Delta m_\pi^2}{M_{\rho^0}} = 0.814$ MeV.

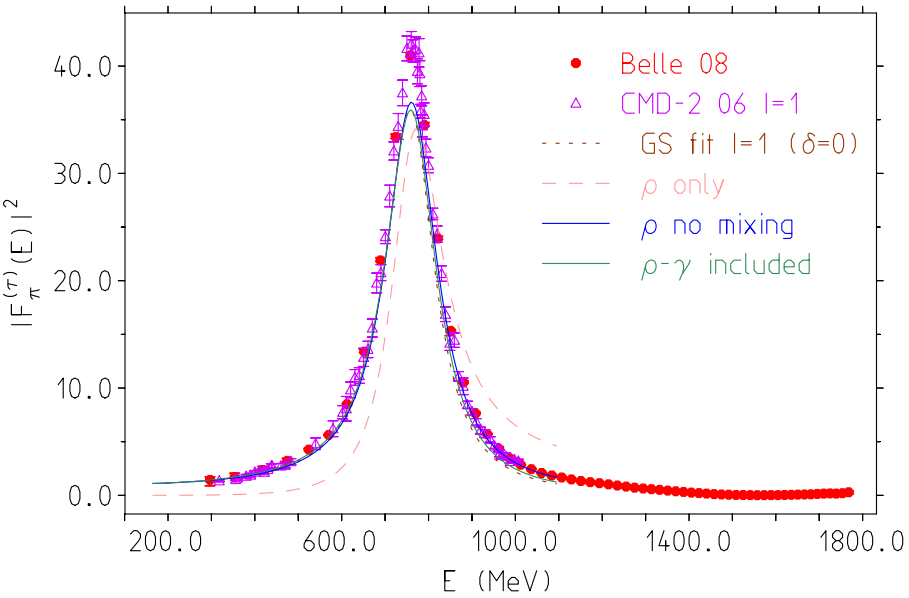


Fig. 5. Effect of γ - ρ mixing based on the simple EFT model (9). Parameters: $M_\rho = 775.5$ MeV, $\Gamma_\rho = 143.85$ MeV, $\mathcal{B}[(\rho \to ee)/(\rho \to \pi\pi)] = 4.67 \times 10^{-5}$, $e = 0.302822$, $g_{\rho\pi\pi} = 5.92$, $g_{\rho ee} = 0.01826$. The crucial point is the difference between "ρ no mixing" [1st plus 3rd term of (33)] and "$\rho - \gamma$ mixing included" [all terms of (33)]. The interference with the mixing term lowers the form factor above the $\sqrt{s} \sim M_\rho$, an effect not present in the charged (pure $I = 1$) channel. For comparison the GS fit with switched off ρ' and ρ'' and the "ρ only" [3rd term of (33) only] are shown.

.

- Width shift: The kinematic shift form pion and rho mass differences is $\delta\Gamma_\rho = \Gamma_{\rho^\pm} - \Gamma_{\rho^0} = \frac{g^2_{\rho\pi\pi}}{48\pi}\,(\beta^3_{\rho^\pm}\,M_{\rho^\pm} - \beta^3_{\rho^0}\,M_{\rho^0}) = 1.300$ MeV.
- Pion velocities are $\beta_{\pi,0} = \sqrt{1 - \frac{(2m_{\pi^\pm})^2}{s}}$ and $\beta_{\pi,\pm} = \sqrt{1 - \frac{(m_{\pi^\pm}+m_{\pi^0})^2}{s} - \frac{(m_{\pi^\pm})-m^2_{\pi^0}}{s}}$ for neutral and charged channel. respectively. Their on-resonance values read $\beta_{\rho^\pm} = \beta_{\pi,\pm}(s = M^2_{\rho^\pm})$ and $\beta_{\rho^0} = \beta_{\pi,0}(s = M^2_{\rho^0})$.
- In the charged channel (τ-decay) the appropriate phase-space for the $\pi^\pm\pi^0$ system, replacing the $\pi^+\pi^-$ one, has to be considered. For the energy dependent width one has $\Gamma_{\rho^\pm}(s) = \Gamma_{\rho^\pm}\,\frac{\beta^3_{\pi,\pm}}{\beta^3_{\rho^\pm}}\,\frac{s}{M^2_{\rho^\pm}}$.

We have made use of the fact that the strong coupling factor $\frac{g^2_{\rho\pi\pi}}{48\pi} = \frac{\Gamma_{\rho^0}}{M_{\rho^0}\,\beta^3_{\rho^0}} = \frac{\Gamma_{\rho^\pm}}{M_{\rho^\pm}\,\beta^3_{\rho^\pm}}$ is charge independent. Note that all of these corrections represent corrections in $F_0(s)/F_-(s)$ the ratio between neutral and charged channel F_π's.

- Electromagnetic corrections $G_{\rm EM}(s)$ as calculated in [30,31]. Specifically, we will apply the correction given by [30], since the ones given in [31] differ quite a lot for reasons we have not yet understood. It does not affect the main conclusion of our analysis.

In total a correction[10]

$$v_0(s) = R_{\rm IB}(s)\,v_-(s) \;\;;\;\; R_{\rm IB}(s) = \frac{1}{G_{\rm EM}(s)}\,\frac{\beta^3_0(s)}{\beta^3_-(s)}\left|\frac{F_0(s)}{F_-(s)}\right|^2 \tag{41}$$

[10] If we would not include the $\rho - \gamma$ mixing in $F_0(s)$ the correction formula would read

$$v_0(s) = r_{\rho\gamma}(s)\,R_{\rm IB}(s)\,v_-(s) \;\; .$$

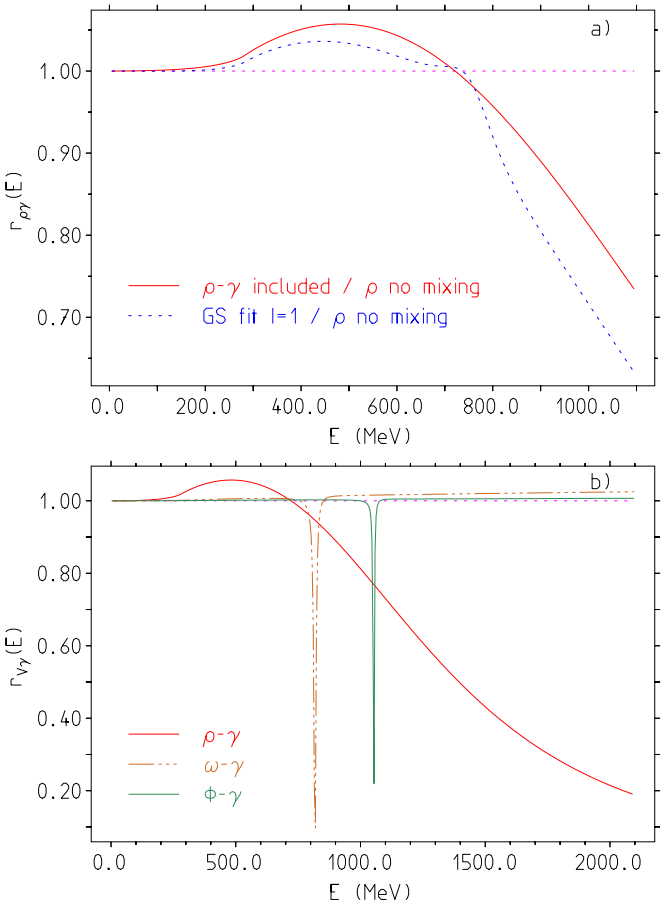


Fig. 6. a) Ratio of the full $|F_\pi(E)|^2$ in units of the same quantity omitting the mixing term together with a standard GS fit with PDG parameters. b) The same mechanism scaled up by the branching fraction $\Gamma_V/\Gamma(V \to \pi\pi)$ for $V = \omega$ and ϕ. In the $\pi\pi$ channel the effects for resonaces $V \neq \rho$ are tiny if not very close to resonance.

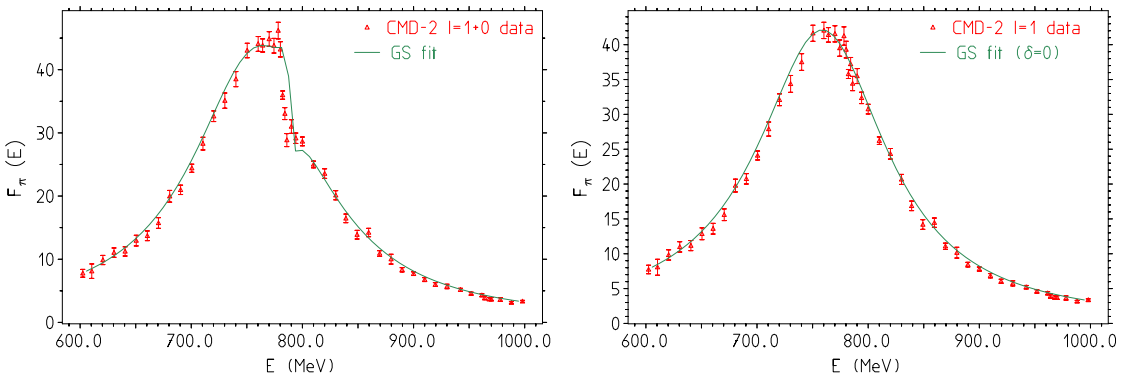


Fig. 7. CMD-2 data for $|F_\pi|^2$ in $\rho - \omega$ region together with Gounaris-Sakurai fit. Left before subtraction right after subtraction of the ω.

has to be applied in the relation between the spectral functions. Final state radiation correction FSR(s) and vacuum polarization effects we have been subtracted from all e^+e^--data.

In Fig. 8 we illustrate the consequence of $\rho - \gamma$ mixing. After applying the correction (for our set of parameters, which is not far from standard GS fit parameters) the consistency of τ and e^+e^- data is

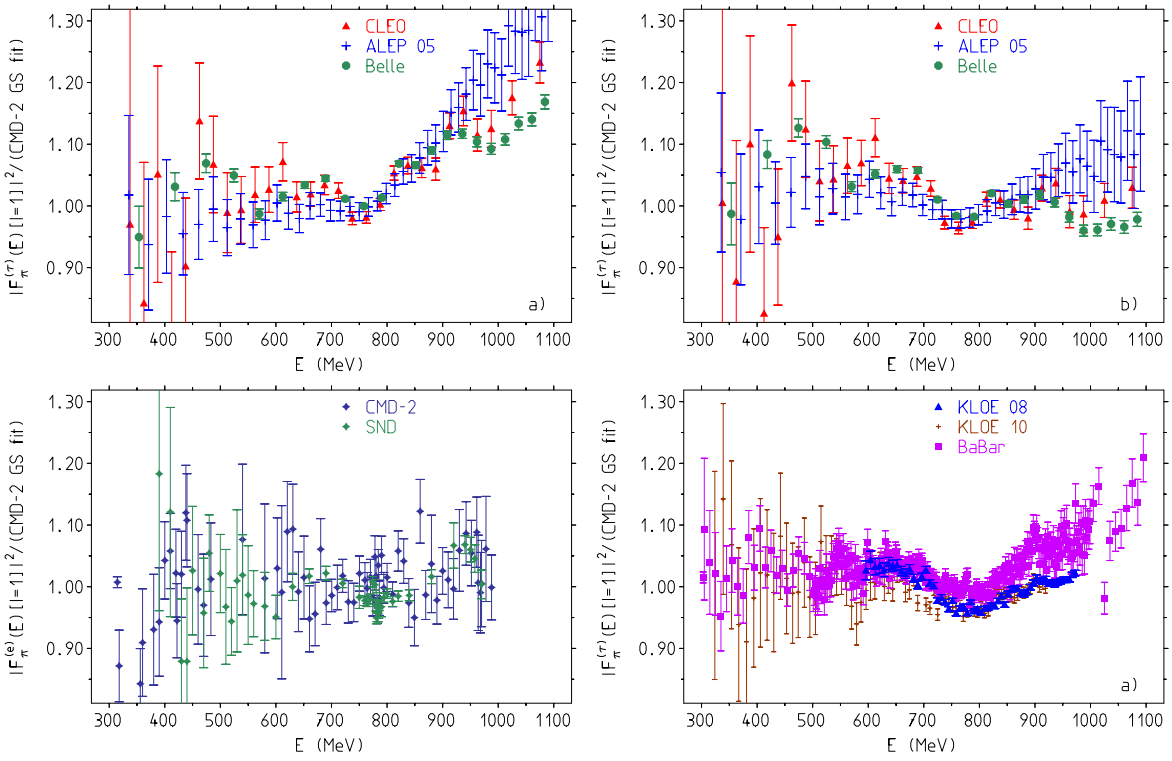


Fig. 8. $|F_\pi(E)|^2$ ratio τ vs. e^+e^- I=1 (CMD-2 GS fit): a) uncorrected for $\rho-\gamma$ mixing, and b) the same after correcting for it. The correction factor is given by the solid curve of Fig. 6 a). Lower panel: e^+e^- energy scan data [left] and e^+e^- radiative return data [right]. The GS fit chosen as a reference is represented by the full line in the right Fig. 7. The I=1 part for the e^+e^- sets is obtained by subtracting the difference of the two curves shown in Fig. 7. The choice of the particular reference is of course ambiguous.

dramatically improved. However, substantial differences of different measurements remain as a problem.

How does the new correction affect the evaluation of the hadronic contribution to the anomalous magnetic moment of the muon? To lowest order in terms of e^+e^--data, represented by $R(s)$, we have

$$a_\mu^{\rm had,LO}(\pi\pi) = \frac{\alpha^2}{3\pi^2}\int\limits_{4m_\pi^2}^{\infty} {\rm d}s\, R^{(0)}_{\pi\pi}(s)\frac{K(s)}{s}\ , \tag{42}$$

with the well-known kernel $K(s)$ and

$$R^{(0)}_{\pi\pi}(s) = (3s\sigma_{\pi\pi})/4\pi\alpha^2(s)) = 3v_0(s)\ .$$

Note that the $\rho-\gamma$ interference is included in the measured e^+e^--data, and so is its contribution to $a_\mu^{\rm had}$. In fact $a_\mu^{\rm had}$ is intrinsic an e^+e^--based "observable" (neutral current channel). If we want to use the τ spectral function [11] isospin breaking corrections to the τ data must be applied: traditionally $v_-(s) \to v_0(s) = R_{\rm IB}(s)\, v_-(s)$. For our comparison we switch off the I=0 ($\rho-\omega$ mixing part) part from the e^+e^- data, which we do not include in $R_{\rm IB}$. In addition we have to account for the missing $\rho-\gamma$ mixing in the τ spectra. The results for the I=1 part of $a_\mu^{\rm had}$ is given in the Tab. 1. Remarkably, the contributions to $a_\mu^{\rm had}$ from the e^+e^--data on the one hand and from the τ data on the other hand agree surprisingly well after including the mixing. The $\rho-\gamma$ mixing correction has been applied for the ρ part only (ρ' and ρ'' subtracted and added back with the help of the GS fit from Belle). We do not attempt here a complete analysis of all τ

[11] In the exact $SU(2)$ (isospin) limit we would have the CVC relation $v_-(s) = v_0(s)$ and the τ data would give the same contribution as the e^+e^- data.

Table 1
Isovector (I=1) contribution to $a_\mu^{\rm had} \times 10^{10}$ from the range [0.592 - 0.975] GeV from selected experiments. First entry: results from τ-data after standard isospin breaking (IB) corrections. Second entry: results from τ-data after applying in addition the $\rho-\gamma$ mixing corrections $r_{\rho\gamma}(s)$, with fitted values for M_ρ, Γ_ρ and $\Gamma_{\rho ee}$ [$M_\rho = 775.65$ MeV, $\Gamma_\rho = 149.99$ MeV, $\mathcal{B}[(\rho \to ee)/(\rho \to \pi\pi)] = 4.10 \times 10^{-5}$]. Third entry: as second one, however using PDG values for the ρ parameters [$M_\rho = 775.5$ MeV, $\Gamma_\rho = 143.85$ MeV, $\mathcal{B}[(\rho \to ee)/(\rho \to \pi\pi)] = 4.67 \times 10^{-5}$] (for illustration of sensitivity to the choice of parameters). For the $\rho-\omega$ mixing we subtracted 2.67×10^{-10}. Errors are statistical, systematic, isospin breaking and $\rho-\gamma$ mixing, assuming a 10% uncertainty for the latter. Final state radiation is not included.

Data	standard IB corrections	incl. $\rho-\gamma$ mixing	same using GS fit parameters for the ρ
ALEPH 1997 [33]	390.75(2.69)(1.97)(1.45)	385.63(2.65)(1.94)(1.43)(0.50)	–
ALEPH 2005 [34]	388.74(4.05)(2.10)(1.45)	383.54(4.00)(2.07)(1.43)(0.50)	385.05(4.01)(2.08)(1.44)(0.36)
OPAL 1999 [35]	380.25(7.36)(5.13)(1.45)	375.39(7.27)(5.06)(1.43)(0.50)	–
CLEO 2000 [36]	391.59(4.16)(6.81)(1.45)	386.61(4.11)(6.72)(1.43)(0.50)	388.10(4.12)(6.75)(1.44)(0.36)
BELLE 2008 [17]	394.67(0.53)(3.66)(1.45)	389.62(0.53)(3.66)(1.43)(0.50)	391.06(0.52)(3.63)(1.44)(0.36)
average	391.06(1.42)(1.47)(1.45)	385.96(1.40)(1.45)(1.43)(0.50)	
CMD-2 2006 [29]		386.34(2.26)(2.65)	
SND 2006 [37]		383.99(1.40)(4.99)	
KLOE 2008 [18]		380.24(0.34)(3.27)	
KLOE 2010 [19]		377.35(0.71)(3.50)	
BABAR 2009 [21]		389.35(0.37)(2.00)	
average		385.12(0.87)(2.18)	
all e^+e^- data		385.21(0.18)(1.54)	
$e^+e^- + \tau$		385.42 (0.53)(1.21)	

data. The $\rho-\gamma$ mixing contribution, which implies a moderate positive interference below the ρ resonance and a stronger negative one above the ρ resonance, shift the τ data to lie perfectly within the ballpark of the e^+e^- data [12].

Another important quantity which can be directly measured is the branching fraction $B^{\rm CVC}_{\pi\pi^0} = \Gamma(\tau \to \nu_\tau \pi\pi^0)/\Gamma_\tau$. This "$\tau$-observable" can be evaluated in terms of the I=1 part of the $e^+e^- \to \pi^+\pi^-$ cross section, after taking into account the IB correction $v_0(s) \to v_-(s) = v_0(s)/R_{\rm IB}(s)$,

$$B^{\rm CVC}_{\pi\pi^0} = \frac{2 S_{\rm EW} B_e |V_{ud}|^2}{m_\tau^2} \int\limits_{4m_\pi^2}^{m_\tau^2} {\rm d}s \, R^{(0)}_{\pi^+\pi^-}(s) \left(1 - \frac{2}{m_\tau^2}\right)^2 \left(1 + \frac{2s}{m_\tau^2}\right) \frac{1}{R_{\rm IB}(s)} \ , \tag{43}$$

where here we also have to "undo" the $\rho-\gamma$ mixing which is absent in the charged isovector channel. Results are given in Tab. 2. The shift by the $\rho-\gamma$ mixing is +0.62 % to which we assign an error of 10 % [13]. For the direct τ branching fractions the first error is statistical the second systematic. For e^+e^-+CVC the first error is experimental the second error includes uncertainties of the IB correction +0.06 from the new mixing effect. Remaining problems seem to be experimental, there are significant differences in the spectral functions from different experiments (see Fig. 8).

[12] Note that the moderate shifts of rho masses and widths [32] which diminish somewhat the discrepancy are included in the IB corrections, as detailed above.

[13] Averages given in [20,38] are 25.42 ± 0.10 % for τ and 24.78 ± 0.28 % for e^+e^- + CVC. Adding the new correction we get 25.40 ± 0.28 ± 0.06 % for the latter, in perfect agreement with the τ result. The BaBar di-pion spectral function, not included in the previous numbers, yields 25.15 ± 0.18 ± 0.22 % or 25.77 ± 0.18 ± 0.28 including the new correction. Results differ slightly from ours because we apply slightly different IB corrections.

Table 2
Calculated branching fractions in % from selected experiments. Experimental data completed down to threshold and up to m_τ by corresponding world averages where necessary. CMD-2 is a combination of 2003 and 2006 data. The experimental world average of direct branching fractions is $B^{\rm CVC}_{\pi\pi^0} = 25.51 \pm 0.09$ % [39].

τ experiments	$B_{\pi\pi^0}$[%]	e^+e^- experiments	$B^{\rm CVC}_{\pi\pi^0}$[%]
ALEPH 1997	$25.27 \pm 0.17 \pm 0.13$	CMD-2 2006	$25.40 \pm 0.21 \pm 0.28$
ALEPH 2005	$25.40 \pm 0.10 \pm 0.09$	SND 2006	$25.09 \pm 0.30 \pm 0.28$
OPAL 1999	$25.17 \pm 0.17 \pm 0.29$	KLOE 2008	$24.82 \pm 0.29 \pm 0.28$
CLEO 2000	$25.28 \pm 0.12 \pm 0.42$	KLOE 2010	$24.65 \pm 0.29 \pm 0.28$
Belle 2008	$25.40 \pm 0.01 \pm 0.39$	BaBar 2009	$25.45 \pm 0.18 \pm 0.28$
combined	$25.34 \pm 0.06 \pm 0.08$	combined	$25.20 \pm 0.17 \pm 0.28$

3. $\rho - \omega$ mixing

In order to include the I=0 contribution form $\omega \to \pi^+\pi^-$ we need to consider the corresponding symmetric (γ, ρ, ω) 3×3 matrix propagator, with new entries $\Pi_{\gamma\omega}(q^2)$, $\Pi_{\rho\omega}(q^2)$ and $q^2 - M_\omega^2 + \Pi_{\omega\omega}(q^2)$, supplementing the inverse propagator matrix (23) by a 3rd row/column. Treating all off-diagonal elements as perturbations (after diagonalization) to linear order the new elements in the propagator read:

$$
\begin{aligned}
D_{\gamma\omega} &\simeq \frac{-\Pi_{\gamma\omega}(q^2)}{(q^2 + \Pi_{\gamma\gamma}(q^2))\,(q^2 - M_\omega^2 + \Pi_{\omega\omega}(q^2))} \\
D_{\rho\omega} &\simeq \frac{-\Pi_{\rho\omega}(q^2)}{(q^2 - M_\rho^2 + \Pi_{\rho\rho}(q^2))\,(q^2 - M_\omega^2 + \Pi_{\omega\omega}(q^2))} \\
D_{\omega\omega} &\simeq \frac{1}{q^2 - M_\omega^2 + \Pi_{\omega\omega}(q^2)}\ .
\end{aligned}
\tag{44}
$$

The self-energies again are the renormalized ones and in the two pion channel $e^+e^- \to \pi^+\pi^-$ given up to different coupling factors by the same self-energy functions as in the $\gamma - \rho$ sector. Thus, the bare self-energy functions read

$$
\Pi_{\gamma\omega} = \frac{e g_{\omega\pi\pi}}{48\pi^2}\, f(q^2)\ ,\quad \Pi_{\rho\omega} = \frac{g_{\rho\pi\pi} g_{\omega\pi\pi}}{48\pi^2}\, f(q^2) \quad \text{and} \quad \Pi_{\omega\omega} = \frac{g^2_{\omega\pi\pi}}{48\pi^2}\, f(q^2)\ ,
\tag{45}
$$

and they are renormalized analogous to (14,15) subtracted at the ω mass shell. The $\rho - \omega$ mixing term is special here because if we diagonalize it on the ρ mass shell the matrix is no longer diagonal at the ω-resonance, where

$$
\Pi^{\rm ren}_{\rho\omega}(q^2) = \Pi_{\rho\omega}(q^2) - \frac{q^2}{M_\rho^2}\,{\rm Re}\,\Pi_{\rho\omega}(M_\rho^2) \stackrel{q^2=M_\omega^2}{\to} \Pi_{\rho\omega}(M_\omega^2) - \frac{M_\omega^2}{M_\rho^2}\,{\rm Re}\,\Pi_{\rho\omega}(M_\rho^2) \neq 0\ ,
\tag{46}
$$

and which yields the leading I=0 contribution to the pion form factor [14]. The ω induced terms contribute to the pion form factor

$$
\Delta F_\pi^{(\omega)}(s) = \left[e\ (g_{\omega\pi\pi} - g_{\omega ee})\ D_{\gamma\omega} - (g_{\rho ee} g_{\omega\pi\pi} + g_{\omega ee} g_{\rho\pi\pi})\, D_{\rho\omega}\right] / \left[e^2\, D_{\gamma\gamma}\right]\ ,
\tag{47}
$$

which adds to (33). The direct $e^+e^- \to \omega \to \pi\pi$ term given by $-g_{\omega\pi\pi} g_{\omega ee}\, D_{\omega\omega}$ by convention is taken into account as part of the complete ω-resonance contribution.

So far we have extended our effective Lagrangian by including direct $\rho - \omega$, $\gamma - \omega$, $\omega\pi\pi$ and ωee vertices only, such that at the one-loop level only the previous pion loops show up. Missing are $\omega\pi^+\pi^-\pi^0$ and $\omega\pi^0\gamma$

[14] Typically, $\Pi^{\rm ren}_{\gamma\rho}(M_\omega^2) = \frac{e g_{\rho\pi\pi}}{48\pi^2}\, M_\omega^2 \left(h(M_\omega^2) - {\rm Re}\, h(M_\rho^2)\right)$ and $D_{\gamma\rho}(M_\omega^2) = -\frac{e g_{\rho\pi\pi}}{48\pi^2} \frac{\left(h(M_\omega^2) - {\rm Re}\, h(M_\rho^2)\right)}{M_\omega^2 - M_\rho^2 + {\rm i}\, M_\rho\, \Gamma_\rho}$. Similarly, $D_{\rho\rho}(M_\omega^2) = \frac{1}{M_\omega^2 - M_\rho^2 + {\rm i}\, M_\rho \Gamma_\rho}$ taking $\Gamma_\rho(M_\omega^2) \sim \Gamma_\rho$.

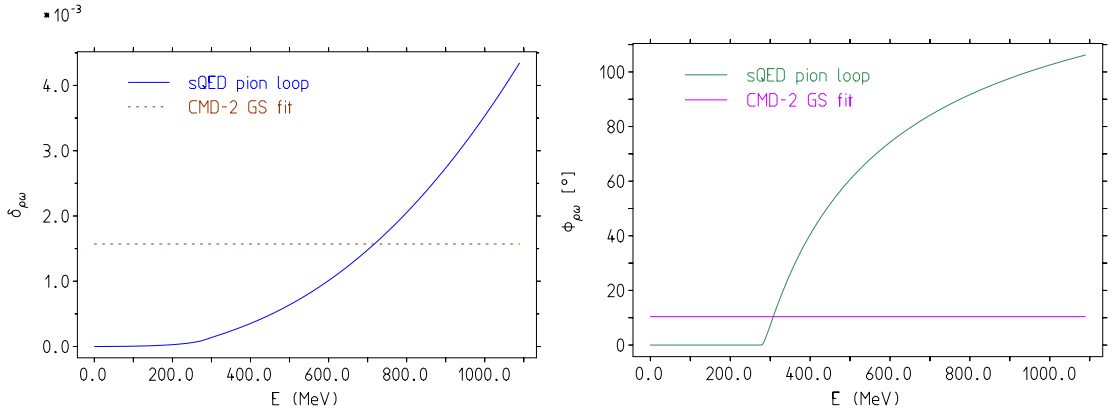


Fig. 9. Dynamical mixing parameter $\delta(E)$ obtained in our EFT, in contrast to the approximation by a constant. The latter seems justified by the narrow width of the ω.

effective vertices, which are necessary in order to obtain the correct full ω-width in place of the $\omega \to \pi\pi$ partial width only. Since the ω is very narrow we expect to obtain a good approximation if we use the proper full width in $\mathrm{Im}\,\Pi_{\omega\omega} = \mathrm{i}\, M_\omega \Gamma_\omega(s)$, namely,

$$\Gamma_\omega \to \Gamma_\omega(s) = \sum_X \Gamma(\omega \to X, s) = \frac{s}{M_\omega^2}\Gamma_\omega \left\{ \sum_X Br(\omega \to X) \frac{F_X(s)}{F_X(M_\omega^2)} \right\}, \tag{48}$$

where $Br(V \to X)$ denotes the branching fraction for the channel $X = 3\pi, \pi^0\gamma, 2\pi$ and $F_X(s)$ is the phase space function for the corresponding channel normalized such that $F_X(s) \to \text{const}$ for $s \to \infty$ [40].

If we include $\omega - \rho$ mixing in the usual way (see (1)) by writing

$$F_\pi(s) = \left[e^2\, D_{\gamma\gamma} + e\,(g_{\rho\pi\pi} - g_{\rho ee})\, D_{\gamma\rho} - g_{\rho ee} g_{\rho\pi\pi}\, D_{\rho\rho} \cdot \left(1 + \delta \frac{s}{M_\rho^2} BW_\omega(s) \right) \right] / \left[e^2\, D_{\gamma\gamma} \right] \quad . \tag{49}$$

with $\mathrm{BW}_\omega(s) = -M_\omega^2/((s - M_\omega^2) + \mathrm{i}\, M_\omega \Gamma_\omega(s))$ in our approach $\delta_{\mathrm{eff}}(s)$ is given by

$$\delta_{\mathrm{eff}}(s) = \frac{(g_{\rho ee} g_{\omega\pi\pi} + g_{\omega ee} g_{\rho\pi\pi})\, D_{\rho\omega} - e\,(g_{\omega\pi\pi} - g_{\omega ee})\, D_{\gamma\rho}}{(g_{\rho\pi\pi} g_{\rho ee})\, D_{\rho\rho} \cdot \mathrm{BW}_\omega(s)} \tag{50}$$

which is well approximated by

$$\begin{aligned} \delta_{\mathrm{dyn}} &= -\frac{(g_{\rho ee} g_{\omega\pi\pi} + g_{\omega ee} g_{\rho\pi\pi})}{g_{\rho\pi\pi} g_{\rho ee}} \frac{\Pi_{\rho\omega}^{\mathrm{ren}}(s)}{M_\omega^2} \\ &\overset{s \sim M_\omega^2}{\sim} -\frac{(g_{\rho ee} g_{\omega\pi\pi} + g_{\omega ee} g_{\rho\pi\pi})}{g_{\rho\pi\pi} g_{\rho ee}} \frac{g_{\rho\pi\pi} g_{\omega\pi\pi}}{48\pi^2} \left(h(M_\omega^2) - \mathrm{Re}\, h(M_\rho^2) \right) \quad . \end{aligned} \tag{51}$$

The second term $g_{\omega ee} g_{\rho\pi\pi} \sim 0.03$ is an order of magnitude larger than than the first one $g_{\rho ee} g_{\omega\pi\pi} \sim 0.003$ and thus is sensitive to $g_{\omega ee}$ once the $g_{\rho\pi\pi}$ has been fixed in the ρ-sector. In leading approximation $\delta \propto g_{\omega ee}/g_{\rho ee} \cdot g_{\rho\pi\pi} g_{\omega\pi\pi}$. The phase is actually fixed by the pion loop alone as we take couplings to be real (unitarity). We have $|\delta| = 1.945 \times 10^{-3}$ and $\phi_\delta = 90.49°$.

A complete EFT treatment of the $\rho - \omega$ mixing, as well as the proper inclusion of the higher ρ's, requires the extension of our model, e.g. in the HLS version as performed in [13,14]. This is beyond the scope of the present study. Nevertheless, the discussion of the $\rho - \omega$ mixing presented above illustrates the need for a reconsideration of the subject.

4. Summary and Conclusions

Our main point is to properly take into account the $\rho - \gamma$ mixing, which is responsible for the major part of the τ vs. e^+e^- discrepancy. The general message we have is that only a consequent application of the effective field theory approach can help to make progress in understanding low energy hadron data. Such attempts have been made recently within the HLS effective theory [13,14], where also a common consistent description (global HLS-model fit) of e^+e^-- and τ-data has been found within a (ρ, ω, ϕ) mixing scheme. We have some difficulties to understand details of these elaborate calculations, but Fig. 8 of of [13] includes a component which has to be attributed to the $\rho - \gamma$ mixing, what we have been discussing here (our Fig. 6). Of course the effects considered here can easily be incorporated in other approaches like the ones proposed in [9] or [10].

We have based our "modeling" of the pion from factor on the low energy effective field theory of ρ, $\pi\pi$ and γ, with the main assumption the pions to behave as point particles (sQED). We avoid some ad hoc elements, like imposing $F_\pi(0) = 1$ by hand, which is common practice when using GS like ansätze. Our result demonstrates two things: a) models should in any case be based on effective field theory, the "right" Lagrangian, to be sorted out by global fit strategies, b) obviously our simplest model has to be extended towards a full fledged resonance Lagrangians in order to be able to control the $\rho - \omega$ mixing and the energy range above the two kaon threshold. A proper treatment of the $\rho - \omega$ mixing requires an extension of the model to include the $\omega 3\pi$ and $\omega\pi\gamma$ couplings, which have anomalous parity, like $\pi^0\gamma\gamma$. Our study should be understood as a first step towards reconsidering the proper extraction of resonance parameters from experiments.

In spite of the fact that we have to make use of a model which has its limitations, the relevant $\rho - \gamma$ mixing effect (needed for example to correct the τ data to be applicable for the evaluation of $a_\mu^{\rm had}$) can be determined from the e^+e^--data solely. Hence, the τ data represent independent additional information, which can be used to improve evaluations of hadronic effects which initially are directly related to e^+e^--data.

With this additional insight the original idea promoted in [41] indeed can work at the level of present standards in precision. Nevertheless, we should keep in mind that photon radiation effects from the composite hadrons are not fully under control and corresponding uncertainties are not easy to specify beyond the few per mil level.

The model we use reproduces in some approximation the standard Gounaris-Sakurai model and improves it in several respects: as it should be we need no photon mass renormalization (which is intrinsically there in the GS model, as explained above), in addition to the s-dependence of the width (as incorporated properly in most versions of the GS model) we include the s-dependent $\rho - \gamma$ mixing, which leads to substantial modification of the GS model. The current conservation condition $F_\pi(0) = 1$ is realized in our approach in a natural way (just by gauge invariance and by standard electromagnetic charge renormalization) as for the gauge invariant VMD type (7) and not by hand as it is done in the GS model, which is of the VMD type (6).

What does it mean for the muon $g - 2$? It definitely shows that the τ-decay isovector form factor must be corrected also for $\rho - \gamma$ mixing interferences, which means that relevant corrections not accounted for so far must be applied. These corrections are model dependent to some extent, as we assume pions to be point-like. Our calculation shows that the bulk of the effect is real and a 10% uncertainty seems to be a reasonable guess. The effects which only depend on the ρ parameters mass, width and leptonic branching fraction and for reasonable values of these parameters brings into fair agreement τ-data based evaluations of $a_\mu^{\rm had}$ and the e^+e^--based ones. Thus phenomenologically, it reproduces rather precisely the pattern of the discrepancy between τ and e^+e^- extracted pion form factors (modulo differences which show up in the different measurements anyway). Our result strongly supports that the observed muon $g - 2$ discrepancy between theory and experiment is real and at the 3 σ level (see e.g. [42],[38]). The τ-data if properly corrected for isospin violating effects support this conclusion.

For the lowest order hadronic vacuum polarization (VP) contribution to a_μ we find

$$a_\mu^{\rm had,LO}[e,\tau] = 690.96(1.06)(4.63) \times 10^{-10} \qquad (e+\tau)$$

for the higher order vacuum polarization terms we find $-206.68(0.36)(1.56) \times 10^{-11}$, $103.89(0.16)(0.70) \times$

10^{-11} and $3.0(0.1) \times 10^{-11}$, which adds to

$$a_\mu^{\rm had,VP,HO} = -99.79(0.38)(0.86) \times 10^{-11}$$

(systematic errors of the first two essentially anti-correlated). The corresponding updated value for the muon $g-2$ is $a_\mu^{\rm the} = 116591797(60) \times 10^{-11}$ which deviates from the experimental value $a_\mu^{\rm exp} = 116592080(54)(33) \times 10^{-11}$ [43] by $a_\mu^{\rm exp} - a_\mu^{\rm the} = (283 \pm 87) \times 10^{-11}$ corresponding to $3.3\,\sigma$.

It is important to note that the $\rho-\gamma$ mixing effect discussed here is included when evaluating the electromagnetic current correlator, $F_\pi(s)$ and $a_\mu^{\rm had}$ from first principles in lattice QCD [44,45]. By comparison with the charged channel isovector correlator, the ratio $F_0(s)/F_-(s)$ could be measured within QCD, without reference to sQED or other hadronic models.

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