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Integrable spin chains and scattering amplitudes

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Abstract

In this review we show that the multi-particle scattering amplitudes in $\mathcal{N}=4$ SYM at large N_c and in the multi-Regge kinematics for some physical regions have the high energy behavior appearing from the contribution of the Mandelstam cuts in the complex angular momentum plane of the corresponding t-channel partial waves. These Mandelstam cuts or Regge cuts are resulting from gluon composite states in the adjoint representation of the gauge group $SU(N_c)$. In the leading logarithmic approximation (LLA) their contribution to the six point amplitude is in full agreement with the known two-loop result. The Hamiltonian for the Mandelstam states constructed from n gluons in LLA coincides with the local Hamiltonian of an integrable open spin chain. We construct the corresponding wave functions using the integrals of motion and the Baxter-Sklyanin approach.

This article is an invited review for a special issue of Journal of Physics A devoted to Scattering Amplitudes in Gauge Theories.

1 Introduction

At high energies $s \gg -t$ in QCD the elastic scattering amplitude for the process $AB \to A'B'$ in the leading logarithmic approximation (LLA)

$$\alpha_s \ln s \sim 1\,,\ \alpha_s \ll 1 \tag{1}$$

has the Regge form [1]

$$A_{2\to 2} = 2\,g\delta_{\lambda_A\lambda_{A'}}T^c_{AA'}\frac{s^{1+\omega(t)}}{t}\,g\,T^c_{BB'}\,\delta_{\lambda_B\lambda_{B'}}\,,\ t=-\mathbf{q}^2\,. \tag{2}$$

Here T^c are the generators of the gauge group $SU(N_c)$, λ_r are the particle helicities and $j(t)=1+\omega(t)$ is the gluon Regge trajectory for the space-time dimension $D=4-2\epsilon$

$$\omega(-\mathbf{q}^2) = -\frac{\alpha_s N_c}{(2\pi)^2}\,(2\pi\mu)^{2\epsilon}\int d^{2-2\epsilon}\mathbf{k}\,\frac{\mathbf{q}^2}{\mathbf{k}^2(\mathbf{q}-\mathbf{k})^2} \approx -\,a\left(\ln\frac{\mathbf{q}^2}{\mu^2}-\frac{1}{\epsilon}\right)\,. \tag{3}$$

In the framework of the dimensional regularization the parameter μ is the renormalization point for the 't Hooft coupling constant

$$a = \frac{\alpha_s\,N_c}{2\pi}\,, \tag{4}$$

where $\gamma=-\psi(1)$ is the Euler constant and $\psi(x)=(\ln\Gamma(x))'$. The gluon trajectory $j(t)$ was calculated also in the next-to-leading approximation in QCD [2] and in the SUSY gauge models [3].

In LLA gluons with momenta k_r (r=1,..,n) are produced in the multi-Regge kinematics

$$s=(p_A+p_B)^2 \gg s_r=(k_r+k_{r-1})^2 \gg -t_r=\mathbf{q}_r^2\,,\ k_r=q_{r+1}-q_r\,, \tag{5}$$

where the amplitude has the factorized form (see also section 2)

$$A_{2\to 2+n}=2\,s\,\delta_{\lambda_A\lambda_{A'}}g\,T^{c_1}_{AA'}\frac{s_1^{\omega(-\vec{q}_1^{\,2})}}{\vec{q}_1^{\,2}}gC_\mu(q_2,q_1)e^*_\mu(k_1)T^{d_1}_{c_2c_1}\frac{s_2^{\omega(-\vec{q}_2^{\,2})}}{\vec{q}_2^{\,2}}...\frac{s_{n+1}^{\omega(-\vec{q}_{n+1}^{\,2})}}{\vec{q}_{n+1}^{\,2}}\,g\,T^{c_{n+1}}_{BB'}\,\delta_{\lambda\lambda_{B'}}\,. \tag{6}$$

Here $C_\mu(q_2,q_1)$ is the effective Reggeon-Reggeon-gluon vertex. In the case when the polarization vector $e_\mu(k_1)$ describes the gluon with a positive helicity in its c.m. system with the particle A' one can obtain [4]

$$C\equiv C_\mu(q_2,q_1)\,e^*_\mu(k_1)=\sqrt{2}\,\frac{q_2^*q_1}{k_1}\,, \tag{7}$$

where the complex notation $q=q_x+iq_y$ for the two-dimensional transverse vector $\mathbf{q}$ was used.

The elastic scattering amplitude with vacuum quantum numbers in the t-channel is calculated in terms of the production amplitude $A_{2\to 2+n}$ with the use of the s-channel unitarity [1]. In this approach the Pomeron appears as a composite state of two Reggeized gluons. It is convenient to present the gluon transverse coordinates in the complex form together with their canonically conjugated momenta [4, 5]

$$\rho_k=x_k+iy_k\,,\ \rho_k^*=x_k-iy_k\,,\ p_k=i\frac{\partial}{\partial\rho_k}\,,\ p_k^*=i\frac{\partial}{\partial\rho_k^*}\,. \tag{8}$$

In this case the homogeneous Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for the Pomeron wave function can be written as follows [1]

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2)=H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\,,\ \Delta=-\frac{\alpha_sN_c}{2\pi}\min E\,, \tag{9}$$

where Δ is the Pomeron intercept entering in the asymptotic expression for the total cross-section $\sigma_t\sim s^\Delta$. The BFKL Hamiltonian has a rather simple operator representation [5]

$$H_{12}=\ln|p_1p_2|^2+\frac{1}{p_1p_2^*}(\ln|\rho_{12}|^2)\,p_1p_2^*+\frac{1}{p_1^*p_2}(\ln|\rho_{12}|^2)\,p_1^*p_2-4\psi(1) \tag{10}$$

with $\rho_{12}=\rho_1-\rho_2$. The kinetic energy is proportional to the sum of two gluon Regge trajectories $\omega(-|p_i|^2)$ $(i=1,2)$. The potential energy $\sim\ln|\rho_{12}|^2$ is obtained by the Fourier transformation from the product of two gluon production vertices C_μ. This Hamiltonian is invariant under the Möbius transformation [6]

$$\rho_k\to\frac{a\rho_k+b}{c\rho_k+d}\,, \tag{11}$$

where a,b,c and d are complex parameters. The eigenvalues of the corresponding Casimir operators are expressed in terms of the conformal weights

$$m=\frac{1}{2}+i\nu+\frac{n}{2}\,,\ \widetilde{m}=\frac{1}{2}+i\nu-\frac{n}{2}\,, \tag{12}$$

where ν and n are respectively real and integer numbers for the principal series of unitary representations of the Möbius group $SL(2,C)$. The eigenvalues of H_{12} depend on these parameters and can be written in the holomorphically separable form [8]

$$E_{m,\widetilde{m}}=\epsilon_m+\epsilon_{\widetilde{m}}\,,\ \epsilon_m=\psi(m)+\psi(1-m)-2\psi(1)\,, \tag{13}$$

where $\psi(x)=(\ln\Gamma(x))'$.

The Pomeron intercept in LLA is positive

$$\Delta = 4\,\frac{\alpha_s}{\pi} N_c \ln 2 > 0 \tag{14}$$

and therefore the Froissart bound $\sigma_t < c \ln^2 s$ for the total cross-section is violated [1]. To restore the broken s-channel unitarity one should take into account the contributions of diagrams corresponding to the t-channel exchange of an arbitrary number of reggeized gluons in the t-channel. The wave function of the colorless state constructed from n reggeized gluons can be obtained in LLA as a solution of the Bartels-Kwiecinski-Praszalowicz (BKP) equation [7]

$$E\,\Psi = H^{(0)}\,\Psi\,,\;\; \Delta = -\frac{\alpha_s N_c}{4\pi}\,\min E. \tag{15}$$

In the $N_c \to \infty$ limit the color structure is simplified and the corresponding Hamiltonian has the property of the holomorphic separability [8]

$$H^{(0)} = \sum_{k=1}^{n} H_{k,k+1} = h^{(0)} + h^{(0)*}\,,\;\; [h^{(0)}, h^{(0)*}] = 0\,. \tag{16}$$

It is a consequence of the similar property for the pair BFKL hamiltonian H_{12} (10) and the energy $E_{m,\widetilde{m}}$ (13).

The holomorphic Hamiltonian in the multi-color QCD can be written as follows (cf. (10))

$$h^{(0)} = \sum_k h^{(0)}_{k,k+1}\,,\;\; h^{(0)}_{12} = \ln(p_1 p_2) + \frac{1}{p_1}\,(\ln \rho_{12})\,p_1 + \frac{1}{p_2}\,(\ln \rho_{12})\,p_2 - 2\psi(1)\,, \tag{17}$$

where $\psi(x) = (\ln \Gamma(x))'$. As a result, the wave function Ψ has the holomorphic factorization [8]

$$\Psi = \sum_{r,\widetilde{r}} a_{r,\widetilde{r}}\,\Psi^r(\rho_1, ..., \rho_n)\,\Psi^{\widetilde{r}}(\rho_1^*, ..., \rho_n^*)\,, \tag{18}$$

which in the case of two-dimensional conformal field theories is a consequence of the infinite dimensional Virasoro group. Moreover, the holomorphic hamiltonian $h^{(0)}$ is invariant under the duality transformation [9]

$$p_i \to \rho_{i,i+1} \to p_{i+1}\,, \tag{19}$$

combined with its transposition.

Further, there are integrals of motion q_r commuting among themselves and with $h^{(0)}$ [5, 10]:

$$q_r^{(0)} = \sum_{k_1<k_2<...<k_r} \rho_{k_1k_2}\rho_{k_2k_3}...\rho_{k_2k_3}...\rho_{k_rk_1}\,p_{k_1}p_{k_2}...p_{k_r}\,,\;\; [q_r, h] = 0\,. \tag{20}$$

The integrability of the BFKL dynamics in LLA and its relation with the Baxter XXX-model was established in ref. [10]. This remarkable property is related also to the fact that h coincides with the local Hamiltonian of an integrable Heisenberg spin model [11] (see also [12]). Eigenvalues and eigenfunctions of this hamiltonian were constructed in refs. [13, 14] in the framework of the Baxter-Sklyanin approach [15].

In the next-to-leading approximation the integral kernel for the BFKL equation was calculated in Refs. [3, 16]. In QCD the eigenvalue of the kernel contains the Kronecker symbols $\delta_{n,0}$ and $\delta_{n,2}$ but in $\mathcal{N}=4$ SYM it is an analytic function of the conformal spin and has the property of the maximal transcendentality [3, 17]. This extended $\mathcal{N}=4$ supersymmetric theory appears in the framework of the AdS/CFT correspondence [18, 19, 20]. It is important, that the eigenvalues of one-loop anomalous dimension matrix for twist-2 operators in $\mathcal{N}=4$ SYM have the maximal transcendentality property and are proportional to the expression $\psi(1)-\psi(j-1)$, which is related to the integrability of evolution equations for quasi-partonic operators in this model [21]. The integrability persists also for some operators in QCD [22]. The maximal transcendentality principle suggested in ref. [17] gave a possibility to extract the universal anomalous dimension up to three loops in $\mathcal{N}=4$ SYM [23, 24] from the corresponding QCD results [25]. The integrability of the $\mathcal{N}=4$ model was demonstrated for anomalous dimensions of other operators in higher loops and at large coupling constants [26, 27, 28]. In particular, the asymptotic Bethe ansatz allowed to calculate the anomalous dimensions in four loops [29]. This result is in an agreement with the next-to-leading BFKL predictions after taking into account the wrapping effects [30]. The maximal transcendentality was helpful for finding a closed integral equation for the cusp anomalous dimension in this model [31, 32] with the use of the 4-loop result [33]. The thermodinamic Bethe ansatz and the approach based on the Y-systems allows to calculate the spectrum of anomalous dimensions in $\mathcal{N}=4$ SYM for an arbitrary coupling constant [34].

In recent years a new line of investigations has been started, which also shows remarkable properties of $\mathcal{N}=4$ SYM: the study of scattering amplitudes. A few years ago, Bern, Dixon and Smirnov (BDS) suggested a simple ansatz for the gluon scattering amplitudes in this model [35]. This ansatz was verified for the elastic amplitude in the strong coupling regime using the AdS/CFT correspondence [36]. But the BDS hypothesis does not agree in this regime with the calculation of the multi-particle amplitude [37], leading to the conclusion that a non-vanishing remainder function, $R^{(n)}$, has to exist which provides the necessary corrections to the BDS amplitudes. The property of the conformal invariance of the BDS amplitudes in the momentum space was discussed in ref. [38], and the relation with the Wilson loop approach was suggested in ref. [39] generalizing the results of the strong coupling calculations of ref. [36]. However, in ref. [40] it was found that the BDS amplitudes A_n for $n \geq 6$ in the multi-Regge kinematics do not have correct analytic properties compatible with the Steinmann relations [41]. It is a consequence of the fact, that these amplitudes do not contain the Mandelstam cuts [40]. The cut contribution was obtained from the BFKL-like equation for the amplitude with the t-channel exchange of two reggeized gluons in the adjoint representation of the gauge group [40]. This equation was solved in LLA and the two-loop expression for the 6-point scattering amplitude in the multi-Regge kinematics was derived [42]. The two-loop correction to the remainder function was calculated numerically for some values of external momenta in an agreement with expectations based on the Wilson loop approach [43]. In a recent paper [44], by solving the set of Y-equations, also the strong coupling limit of the remainder function has been studied.

The existence of the Mandelstam cut contribution, found first for the 6-point amplitude, generalizes to multiparticle amplitudes with $n > 6$. As an example, the $2 \to 2n$ amplitude will contain Mandelstam-cut contributions composed of n reggeized t-channel gluons. These n gluon t-channel states can be expressed in terms of solutions of the BKP-like equation in the adjoint representation. Most remarkable, in LLA the corresponding Hamiltonian is integrable: it coincides with the local Hamiltonian of an integrable open Heisenberg spin chain [45].

In this review we present a summary of the Mandelstam cut contributions to the inelastic scattering amplitudes in $N=4$ SYM and their properties of integrability. We first review the analytic structure of n-point amplitudes in the multi-Regge kinematics and describe the main features of the Mandelstam cut contributions. In the subsequent section we compare our results in LLA with the exact two-loop calculations of Goncharov, Spradlin, Vergu and Volovich (GSVV) [46] and consider a relation with collinear kinematics. The rest of the review is devoted to the integrability of the BKP Hamiltonian in the adjoint representation.

2 The analytic structure of scattering amplitudes in the Regge limit

Let us begin with a brief summary of the analytic properties of scattering amplitudes in the multi-Regge limit. It is well-known that the $2 \to 3$ amplitude in the multi-Regge kinematics with the exchanged reggeons having definite signatures $\tau_i = \pm 1$ in the crossing channels t_1 and t_2 can be written as a sum of two terms

$$\frac{M_{2\to 3}^{pole}}{\Gamma(t_1)\Gamma(t_2)} = |s_1|^{\omega_{12}}|s|^{\omega_2}\xi_{12}\xi_2\,\kappa_{12}^{\omega_2}c_1^{12} + |s_2|^{\omega_{21}}|s|^{\omega_1}\xi_{21}\xi_1\,\kappa_{12}^{\omega_1}c_2^{12}\,,\ \kappa_{12} = \mathbf{k}_a^2 = \frac{s_1 s_2}{s}\,, \tag{21}$$

where

$$\xi_1 = e^{-i\pi\omega_1} - \tau_1\,,\ \xi_2 = e^{-i\pi\omega_2} - \tau_2\,,\ \xi_{12} = e^{-i\pi\omega_{12}} + \tau_1\tau_2\,,\ \xi_{21} = e^{-i\pi\omega_{21}} + \tau_1\tau_2\,. \tag{22}$$

$\Gamma(t_i)$ are the residue functions of the exchanged Regge poles, and $\mathbf{k}_a$ is the transverse momentum of the produced particle. In (21) we assumed that s, s_1, s_2 and κ_{12} are measured in some characteristic mass μ^2.

The gluon Regge trajectories in $N = 4$ SYM can be written as (see (3))

$$\omega_i = \omega(-\mathbf{q}_i^2) = -\frac{\gamma_K}{4}\,\ln\frac{\mathbf{q}_i^2}{\lambda^2}\,,\ \gamma_K \approx 4a\,,\ a = \frac{g^2\,N_c}{8\pi^2}\,,\ \omega_{12} = \omega_1 - \omega_2\,, \tag{23}$$

where $t_i = -\mathbf{q}_i^2$, γ_K is the cusp anomalous dimension and $\lambda^2 \simeq \mu^2\exp(1/\epsilon)$ for $D = 4 - 2\epsilon$ with $\epsilon \to -0$. The parameter λ^2 can be considered as an effective mass of gluon. The real coefficients c_1^{12}, c_2^{12} in $\mathcal{N} = 4$ SYM are obtained from the BDS amplitude and given below [40]

$$c_1^{12} = |\Gamma_{12}|\,\frac{\sin\pi(\omega_1 - \omega_a)}{\sin\pi\omega_{12}}\,,\ c_2^{12} = |\Gamma_{12}|\,\frac{\sin\pi(\omega_2 - \omega_a)}{\sin\pi\omega_{21}}\,, \tag{24}$$

where the Reggeon-Reggeon-gluon vertex Γ_{12} in the physical region $s, s_1, s_2 > 0$ is

$$\Gamma_{12}(\ln\kappa_{12} - i\pi) = |\Gamma_{12}|\,\exp(i\pi\,\omega_a)\,,\ \omega_a = \frac{\gamma_K}{8}\,\ln\frac{\mathbf{k}_a^2\lambda^2}{\mathbf{q}_1^2\mathbf{q}_2^2}\,, \tag{25}$$

$$\ln|\Gamma_{12}| = \frac{\gamma_K}{4}\,\left(-\frac{1}{4}\ln^2\frac{\mathbf{k}_a^2}{\lambda^2} - \frac{1}{4}\ln^2\frac{\mathbf{q}_1^2}{\mathbf{q}_2^2} + \frac{1}{2}\ln\frac{\mathbf{q}_1^2\mathbf{q}_2^2}{\lambda^4}\ln\frac{\mathbf{k}_a^2}{\mu^2} + \frac{5}{4}\zeta_2\right)\,. \tag{26}$$

The amplitude representation in (21) is compatible with the Steinmann relations [41] forbidding the simultaneous singularities in the overlapping channels s_1 and s_2. The expression in (21) can be rewritten in the factorized form

$$\frac{M_{2\to 3}^{\tau_1\tau_2}}{\Gamma(t_1)\Gamma(t_2)} = |s_1|^{\omega_1}\xi_1\,V^{\tau_1\tau_2}\,|s_2|^{\omega_2}\xi_2\,,\ V^{\tau_1\tau_2} = \frac{\xi_{12}}{\xi_1}\,c_1^{12} + \frac{\xi_{21}}{\xi_2}\,c_2^{12}\,. \tag{27}$$

The BDS-amplitude which holds in the planar approximation can also be written as a sum of two terms:

$$\frac{M_{2\to 3}^{BDS}}{\Gamma(t_1)\Gamma(t_2)} = (-s_1)^{\omega_{12}}(-s\kappa_{12})^{\omega_2}c_1^{12} + (-s_2)^{\omega_{21}}(-s\kappa_{12})^{\omega_1}c_2^{12}\,, \tag{28}$$

where we put the normalization point μ^2 in the Regge factors equal to unity.

For the $2 \to 4$ amplitude in the multi-Regge kinematics the situation is more complicated. In agreement with the Steinmann relations, the Regge-pole scattering amplitude can, again, be written as a sum of 5 terms as illustrated in Fig. 1.

For the signatured amplitude, this representation is equivalent to the factorized form:

$$\frac{M_{2\to 4}^{\tau_1\tau_2\tau_3}}{\Gamma(t_1)\Gamma(t_3)} = |s_1|^{\omega_1}\xi_1\,V^{\tau_1\tau_2}\,|s_2|^{\omega_2}\xi_2\,V^{\tau_2\tau_3}\,|s_3|^{\omega_3}\xi_3\,, \tag{29}$$

where $V^{\tau_2\tau_3}$ is obtained from $V^{\tau_1\tau_2}$ (27) with the corresponding substitutions

$$V^{\tau_2\tau_3} = \frac{\xi_{23}}{\xi_2}\,c_1^{23} + \frac{\xi_{32}}{\xi_3}\,c_2^{23}\,. \tag{30}$$

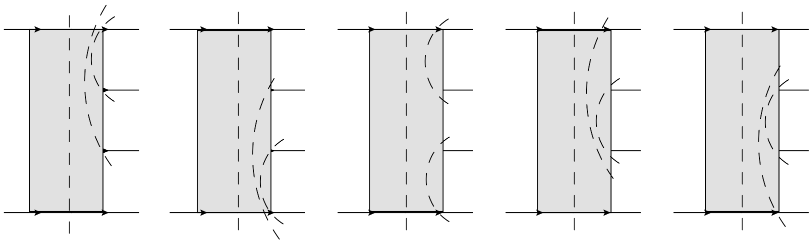

Figure 1: The analytic representation of the $2 \to 4$ scattering amplitude. The dashed lines denote possible energy discontinuities.

For the second produced gluon with the transverse momentum k_b the coefficients c^{23} and phase ω_b read

$$c_1^{23} = |\Gamma_{23}| \frac{\sin \pi(\omega_2 - \omega_b)}{\sin \pi\omega_{23}}, \; c_2^{23} = |\Gamma_{23}| \frac{\sin \pi(\omega_3 - \omega_b)}{\sin \pi\omega_{32}}, \; \omega_b = \frac{\gamma_K}{8} \ln \frac{\mathbf{k}_b^2 \lambda^2}{\mathbf{q}_2^2 \mathbf{q}_3^2}, \; \mathbf{k}_b^2 = \left| \frac{s_2 s_3}{s_{123}} \right| . \tag{31}$$

In the planar approximation one expects that the scattering amplitude, $M_{2\to4}^{pole}$, in accordance with the Steinmann relations [41], has the form [40]:

$$\begin{aligned} \frac{M_{2\to4}^{pole}}{\Gamma(t_1)\Gamma(t_3)} &= (-s_1)^{\omega_{12}} (-s_{012}\kappa_{12})^{\omega_{23}} (-s\kappa_{12}\kappa_{23})^{\omega_3} c_1^{12} c_1^{23} \\ &+(-s_3)^{\omega_{32}} (-s_{123}\kappa_{23})^{\omega_{21}} (-s\kappa_{12}\kappa_{23})^{\omega_1} c_2^{12} c_2^{23} + (-s\kappa_{12}\kappa_{23})^{\omega_2} (-s_1)^{\omega_{12}} (-s_3)^{\omega_{32}} c_1^{12} c_2^{23} \\ &+(-s_2)^{\omega_{21}} (-s_{012}\kappa_{12})^{\omega_{13}} (-s\kappa_{12}\kappa_{23})^{\omega_3} \frac{\sin \pi\omega_1}{\sin \pi\omega_2} \frac{\sin \pi\omega_{23}}{\sin \pi\omega_{13}} c_2^{12} c_1^{23} \\ &+ (-s_2)^{\omega_{23}} (-s_{123}\kappa_{23})^{\omega_{31}} (-s\kappa_{12}\kappa_{23})^{\omega_1} \frac{\sin \pi\omega_3}{\sin \pi\omega_2} \frac{\sin \pi\omega_{21}}{\sin \pi\omega_{31}} c_2^{12} c_1^{23} . \end{aligned} \tag{32}$$

A closer look at the last two terms shows that a model with Regge-poles only exhibits unphysical poles, indicating that a pure Regge model maybe incompatible with the correct analytic structure of multiparticle amplitudes in the multi-Regge kinematics. In fact, the LLA analysis of N=4 SYM gauge theory has shown that, in addition to the gluon Regge pole, there exists also a Mandelstam cut in the complex angular momentum plane, which removes this inconsistency. The Mandelstam

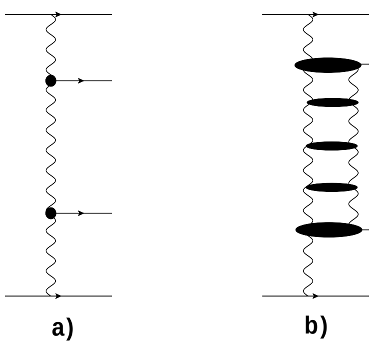

Figure 2: Contributions of **a**) the Regge poles and **b**) the Mandelstam cut to the $2 \to 4$ scattering amplitude in the t_2-channel. The wavy lines represent reggeized gluons. The Mandelstam cut appears as a bound state of two reggeized gluons.

cut appears in the angular momentum plane of the t_2 channel and describes a bound state of two or more reggeized gluons as depicted in Fig. 2. In the planar approximation, it shows up in the special physical kinematic regions where the invariants in the direct channels have the following signs $s, s_2 > 0$; $s_1, s_3, s_{012}, s_{123} < 0$ or $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$ [40], and it is not visible in the physical kinematic region where all energies are positive. In the following these special physical regions will be named "Mandelstam regions".

The $2 \to 4$ amplitude in the multi-Regge kinematics can be written as a sum of the Regge pole and Mandelstam cut contributions [40]

$$M_{2\to 4} = M_{2\to 4}^{pole} + M_{2\to 4}^{cut}\,, \tag{33}$$

where $M_{2\to 4}^{cut}$ is a generalization of two last terms in (32), and it is non-zero only in the two kinematic regions restricted by the inequalities $s, s_2 > 0$; $s_1, s_3, s_{012}, s_{123} < 0$ and $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$. There is some freedom in redistributing terms between the Regge pole $M_{2\to 4}^{pole}$ and the Mandelstam cut $M_{2\to 4}^{cut}$ contributions. Using this fact and representation (32) one can write in the region $s, s_2 > 0$; $s_1, s_3, s_{012}, s_{123} < 0$ [47]

$$\frac{M_{2\to 4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)} = e^{-i\pi\omega_2}\cos\pi\omega_{ab} \tag{34}$$

and

$$\frac{M_{2\to 4}^{cut}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)} = i\,e^{-i\pi\omega_2}\int_{-i\infty}^{i\infty}\frac{d\omega_{2'}}{2\pi i}\,f(\omega_{2'})\,e^{-i\pi\omega_{2'}}\,|s_2|^{\omega_{2'}}\,, \tag{35}$$

where ω_{ab} is obtained from (25) and (31) and reads

$$\omega_{ab} = \frac{\gamma_K}{8}\,\ln\frac{\mathbf{k}_a^2\mathbf{q}_3^2}{\mathbf{k}_b^2\mathbf{q}_1^2}. \tag{36}$$

In the other physical region, where $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$ (corresponding to the physical channel for the $3 \to 3$ transition) we find [47]

$$\frac{M_{2\to 4}^{pole}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)} = \cos\pi\omega_{ab} \tag{37}$$

and

$$\frac{M_{2\to 4}^{cut}}{|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}|\Gamma_{12}||\Gamma_{23}|\,\Gamma(t_1)\Gamma(t_3)} = -i\int_{-i\infty}^{i\infty}\frac{d\omega_{2'}}{2\pi i}\,f(\omega_{2'})\,|s_2|^{\omega_{2'}}\,. \tag{38}$$

The function $f(\omega_{2'})$ is pure real and denotes the partial wave in the complex angular momentum plane.

The origin of this 'restricted' appearance of the Mandelstam cut contribution $M_{2\to 4}^{cut}$ can be traced back to Mandelstam's argument for the existence of the Regge cuts (Fig. 3): If we put for the reggeon momentum $k = \alpha p_A + \beta p_B + k_\perp$, it is easy to see that, in the planar (large-N_c) limit with all energies being positive, the integrations over α and β have singularities only in the upper half planes and lead to the absence of the Mandelstam cut contribution. However, if by pulling the produced particles to the left we 'twist' the reggeons (ladders) in the t_1 and t_3 channels (Fig. 4), there will be singularities on both sides of the real α and β axis, and the Mandelstam singularity remains. Note that, despite this 'twisting', the amplitude is still planar. Returning to the sum of the five contributions in Fig. 1, it can be shown that the Mandelstam cut contribution should be present only in the last two terms: taking into account the phase structure one finds that, in the physical kinematic region, where all energies are positive, the cut cancels in the sum of the two terms. This cancellation does not work, if we are in the 'mixed' regions $s, s_2 > 0$; $s_1, s_3, s_{012}, s_{123} < 0$ or $s, s_1, s_2, s_3 < 0$; $s_{012}, s_{123} > 0$.

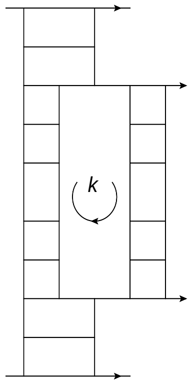

Figure 3: The diagrammatic structure of the Mandelstam cut.

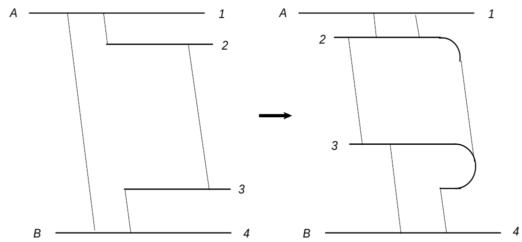

Figure 4: Twisting a planar $2 \to 4$ amplitude.

Turning now to the BDS amplitudes, this Mandelstam cut contribution is missing and must therefore be contained in the remainder function, $R_{2\to4}$. It is believed that the full MHV amplitude in the planar (large-N_c) approximation can be written in the factorized form:

$$M_{2\to4} = M_{2\to4}^{BDS}\, R_{2\to4}. \tag{39}$$

Indeed, in [42] it was shown that in the region $s, s_2 > 0$; $s_1, s_3, s_{012}, s_{123} < 0$ the correct form of the LLA $2 \to 4$ scattering amplitude is

$$M_{2\to4}^{LLA} = M_{2\to4}^{BDS}\,(1 + i\Delta_{2\to4}^{LLA}), \tag{40}$$

where $M_{2\to4}^{BDS}$ is the BDS amplitude [35] and

$$\Delta_{2\to4}^{LLA} = \frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\int_{-\infty}^{\infty}\frac{d\nu}{\nu^2+\frac{n^2}{4}}\left(\frac{q_3^* k_a^*}{k_b^* q_1^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3 k_a}{k_b q_1}\right)^{i\nu+\frac{n}{2}}\left(s_2^{\omega(\nu,n)}-1\right). \tag{41}$$

Here k_a, k_b are the complex transverse components of the produced gluon momenta, q_1, q_2, q_3 are the momenta of reggeons in the corresponding crossing channels, and

$$\omega(\nu, n) = 4a\,\Re\left(2\psi(1) - \psi(1 + i\nu + \frac{n}{2}) - \psi(1 + i\nu - \frac{n}{2})\right) \tag{42}$$

is the eigenvalue of the BFKL Hamiltonian in the adjoint representation. The correction $\Delta_{2\to4}^{LLA}$ is Möbius invariant in the momentum space, and it is important to note that it can be written in terms of the four-dimensional anharmonic ratios [42] in an accordance with the results of refs. [38]. Thus it can be viewed as part of the remainder function, $R_{2\to4}$, which is expected to depend only on the three anharmonic ratios u_i, $i = 1, 2, 3$. In section 3 we will come back for a closer look at this expression.

As we have already mentioned in the introduction, this Mandelstam cut structure of six-point amplitudes in the multi-Regge kinematics can be generalized to a larger number of external legs,

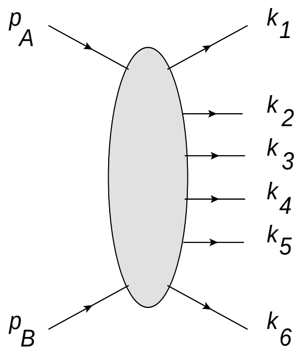

Figure 5: The $2 \to 6$ scattering amplitude.

$n > 6$. This generalization exhibits the remarkable feature of integrability [45]. As an example, let us consider the eight-point amplitude $M_{2\to 6}$ in multi-Regge kinematics (Fig. 5). Again, the amplitude can be written as a sum of terms which are compatible with the Steinmann relations. The number of terms is already 42 [1] and will not be discussed here in further detail. We only mention that, for the fully-signatured amplitude, the Regge-pole contribution can also be rewritten in the factorized representation (cf.(29)):

$$\frac{M_{2\to 6}^{\tau_1\tau_2\tau_3\tau_4}}{\Gamma(t_1)\Gamma(t_4)} = |s_1|^{\omega_1}\xi_1\, V^{\tau_1\tau_2}\, |s_2|^{\omega_2}\xi_2\, V^{\tau_2\tau_3}\, |s_3|^{\omega_3}\xi_3\, V^{\tau_3\tau_4}\, |s_4|^{\omega_4}\xi_4\,. \tag{43}$$

As it was already the case for the $2 \to 4$ scattering amplitude, there exist Mandelstam cut contributions which appear only in special physical regions. The most interesting one is illustrated in Fig. 6. This

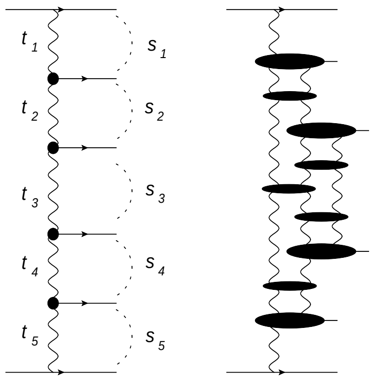

Figure 6: The Regge pole and two- and three-gluon cuts in the $2 \to 6$ amplitude.

Regge-cut piece belongs to singularities in the angular momentum variables j_2, j_3, and j_4 of the t_2, t_3, and t_4 channels, resp. Using the Mandelstam argument given above, it is easy to see that it appears, for example, in the physical region $s_3 > 0$, $s_{2345} > 0$, $s > 0$, $s_1 < 0$, $s_2 < 0$, $s_4 < 0$, $s_5 < 0$ [45]. This region is obtained by 'double twisting' of the planar amplitude and is further illustrated in Fig. 7.

[1] for the $2 \to n$ amplitude, the number of terms, N_n, obeys the recursion relation $N_n = \sum_{k=1}^{n-1} N_k N_{n-k}$ with $N_1 = N_2 = 1$.

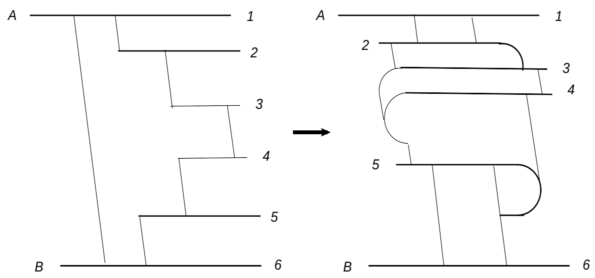

Figure 7: The 'double twisting' of the $2 \to 6$ amplitude.

The detailed form of this contribution will be given in a forthcoming paper. Here we only mention that the singularities in j_2 and j_4 are described by the BFKL Hamiltonian of the two reggeon state in the adjoint representation, whereas the j_3 channel is governed by the spectrum of the BKP Hamiltonian of three reggeized gluons, projected on the adjoint representation. In section 3 we will give a more detailed discussion: in particular, it will be shown that this Hamiltonian is integrable and belongs to an open spin chain.

3 BFKL approach and MHV amplitudes

In this section we discuss six-particle amplitudes in the multi-Regge kinematics in some of the Mandelstam regions. We compare results obtained in the BFKL approach with those calculated using Wilson Loop/Scattering Amplitude duality at two loops. In particular, the analysis of the two-loop result allows to obtain the impact factor for the Mandelstam cut contribution beyond the LLA. We also discuss briefly the collinear limit and write an explicit analytic form of the all-loop remainder function in the Double Leading Logarithmic Approximation (DLLA). The section consists of two parts devoted to $2 \to 4$ and $3 \to 3$ amplitudes.

3.1 $2 \to 4$ amplitude

The six-particle scattering amplitude corresponds to two physical processes, namely to $2 \to 4$ and $3 \to 3$ scattering. Firstly we discuss the $2 \to 4$ planar MHV amplitude illustrated in Fig. 8 and review the main result of the BFKL analysis applied to this case. The corresponding Mandelstam variables are defined as $s = (p_A + p_B)^2,\ s_1 = (p_{A'} + k_1)^2,\ s_2 = (k_1 + k_2)^2,\ s_3 = (p_{B'} + k_2)^2,\ s_{012} = (p_{A'} + k_1 + k_2)^2,\ s_{123} = (p_{B'} + k_1 + k_2)^2,\ t_1 = (p_A - p_{A'})^2,\ t_2 = (p_A - p_{A'} - k_1)^2,\ t_3 = (p_B - p_{B'})^2$ and the dual conformal cross ratios are given by

$$u_1 = \frac{s s_2}{s_{012}\, s_{123}},\ u_2 = \frac{s_1 t_3}{s_{012}\, t_2},\ u_3 = \frac{s_3 t_1}{s_{123} t_2}. \tag{44}$$

The multi-Regge kinematics, where

$$-s \gg -s_{012}, -s_{123} \gg -s_1, -s_2, -s_3 \gg -t_1, -t_2, -t_3 > 0 \tag{45}$$

implies

$$1 - u_1 \to +0,\ \ u_2 \to +0,\ \ u_3 \to +0,\ \ \frac{u_2}{1-u_1} \simeq \mathcal{O}(1),\ \ \frac{u_3}{1-u_1} \simeq \mathcal{O}(1), \tag{46}$$

which suggests that in this kinematics the convenient variables for the remainder function are $1 - u_1$ and the reduced cross ratios defined by

$$\tilde{u}_2 = \frac{u_2}{1-u_1},\ \ \tilde{u}_3 = \frac{u_3}{1-u_1}. \tag{47}$$

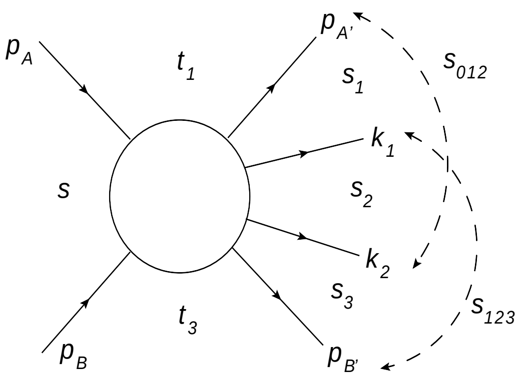

Figure 8: The $2 \to 4$ gluon scattering amplitude.

In the Regge limit they can be expressed through s_2 and the transverse momenta

$$1 - u_1 \simeq \frac{(\mathbf{k}_1 + \mathbf{k}_2)^2}{s_2}, \quad \tilde{u}_2 \simeq \frac{\mathbf{k}_1^2 \, \mathbf{q}_3^2}{(\mathbf{k}_1 + \mathbf{k}_2)^2 \, \mathbf{q}_2^2}, \quad \tilde{u}_3 \simeq \frac{\mathbf{k}_2^2 \, \mathbf{q}_1^2}{(\mathbf{k}_1 + \mathbf{k}_2)^2 \, \mathbf{q}_2^2}. \tag{48}$$

Note that $\tilde{u}_2$ and $\tilde{u}_3$ are rational functions of cross ratios in four dimensions, but in the Regge limit they are simple cross ratios in the two-dimensional transverse space as one can see from (48). The region of possible values of $\tilde{u}_2$ and $\tilde{u}_3$ that correspond to physical momenta is depicted in Fig. 9 as a semi-infinite strip [48, 49].

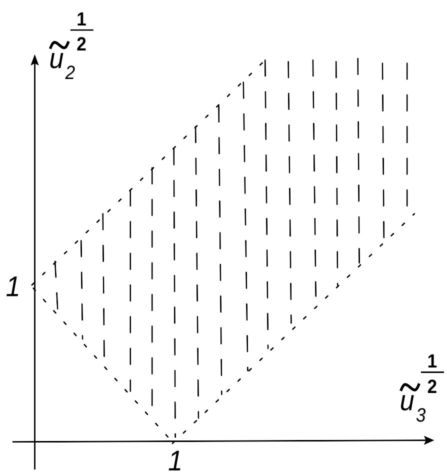

Figure 9: The physical values of $\sqrt{\tilde{u}_2}$ and $\sqrt{\tilde{u}_3}$ lie in the shaded semi-infinite strip.

In the "Euclidean" kinematics, where all invariants are negative and thus all u_i are positive, the remainder function vanishes asymptotically as follows from the analysis presented in refs. [40, 42]. However, these studies also show that this is not the case in a slightly different physical region, where one or more dual conformal cross ratios possess a phase. This happens when some energy invariants change the sign and here we consider one of such regions of the $2 \to 4$ scattering amplitude having

$$u_1 = |u_1| e^{-i2\pi}, \; u_2 \text{ and } u_3 \text{ are fixed and positive.} \tag{49}$$

It corresponds to the physical region mentioned before (the Mandelstam region), where

$$s\,, s_2 > 0; \quad s_1, \; s_3, \; s_{012}, \; s_{123} < 0 \tag{50}$$

as illustrated in Figs. 4 and 10. The t-variables are all negative in the physical regions under consideration for the $2 \to 4$ scattering amplitude in the Regge kinematics. It is worth emphasizing that the scattering amplitude in Fig. 10 is still planar, but the produced particles have reversed momenta k_1 and k_2 with a negative energy components.

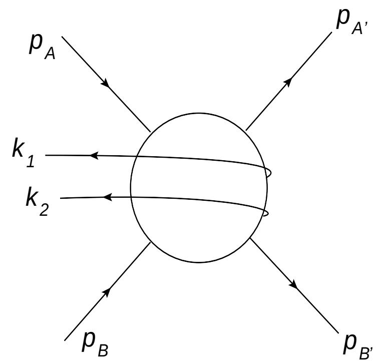

Figure 10: The Mandelstam channel of the $2 \to 4$ gluon planar scattering amplitude.

In the Mandelstam channel the remainder function grows with energy s_2 and was first calculated using the BFKL approach by two of the authors in collaboration with A. Sabio Vera in ref. [42]. The BFKL approach, based on the analyticity and unitarity was developed more than thirty years ago [1]. In this approach one sums the contributions from the Feynman diagrams, which are enhanced by the logarithms of the energy ($1 - u_1 \simeq (\mathbf{k}_1 + \mathbf{k}_2)^2/s_2$ in our case). The Leading Logarithmic Approximation (LLA) allows to write an integral representation of the remainder function R to any order of the parameter $g^2 \ln s_2$. As it was already discussed in the previous section (see (40)), the amplitude in this Mandelstam channel is given by [42]

$$M_{2\to4} = M_{2\to4}^{BDS}\, R_{2\to4} = M_{2\to4}^{BDS}\,(1 + i\Delta_{2\to4}), \tag{51}$$

where $M_{2\to4}^{BDS}$ is the BDS expression [35] and the correction $\Delta_{2\to4}$ was calculated in all orders with a leading logarithmic accuracy using the solution to the BFKL equation in the adjoint representation. The all-order LLA expression for $\Delta_{2\to4}$ was given in (41)

$$\Delta_{2\to4}^{LLA} \simeq \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}}\, (w^*)^{i\nu - \frac{n}{2}}\, (w)^{i\nu + \frac{n}{2}} \left((1-u_1)^{-\omega(\nu,n)} - 1 \right). \tag{52}$$

Here k_1, k_2 are complex transverse components of the gluon momenta, q_1, q_2, q_3 are the corresponding momenta of reggeons in the crossing channels. It is convenient to define holomorphic and antiholomorphic variables in the transverse space as

$$w = \frac{q_3 k_1}{k_2 q_1}, \quad w^* = \frac{q_3^* k_1^*}{k_2^* q_1^*} \tag{53}$$

related to the reduced cross ratios of (47) by

$$|w|^2 = \frac{\tilde{u}_2}{\tilde{u}_3} = \frac{u_2}{u_3}, \quad w = |w| e^{i(\phi_2 - \phi_3)}, \quad \cos(\phi_2 - \phi_3) = \frac{1 - \tilde{u}_2 - \tilde{u}_3}{2\sqrt{\tilde{u}_2 \tilde{u}_3}} = \frac{1 - u_1 - u_2 - u_3}{2\sqrt{u_2 u_3}}. \tag{54}$$

The energy behavior of the remainder function is determined by the Mandelstam cut intercept

$$\omega(\nu, n) = -a E_{\nu,n}, \tag{55}$$

where a and $E_{\nu,n}$ are the perturbation theory parameter and the eigenvalue of the BFKL Hamiltonian in the adjoint representation given by

$$a = \frac{\alpha_s N_c}{2\pi}, \quad E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2 + \frac{n^2}{4}} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1). \tag{56}$$

Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ and $\gamma = -\psi(1)$ is the Euler constant. The two loop LLA expression for the remainder function in the BFKL approach was first found from (51) and (52) in ref. [42] and it reads

$$R^{(2)\,LLA}_{2\to 4} = \frac{i\pi}{2}\ln(1-u_1)\ln\tilde{u}_2\ln\tilde{u}_3 = \frac{i\pi}{2}\ln(1-u_1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2. \tag{57}$$

The remainder function in (57) is pure imaginary and symmetric under $w \to 1/w$ transformation, which corresponds to the target-projectile symmetry $p_A \leftrightarrow p_B$, $p_{A'} \leftrightarrow p_{B'}$ and $k_1 \leftrightarrow k_2$ in accordance with (52).

This result was shown by Schabinger [50] to agree numerically with the analytic continuation of the expression for the two-loop remainder function found by Drummond, Henn, Korchemsky and Sokatchev [51] from Wilson Loop/Scattering Amplitude duality. A rather complicated expression of ref. [51] was largely simplified by Del Duca, Duhr and Smirnov [52, 53] and then by Goncharov, Spradlin, Vergu and Volovich (GSVV) [46]. The prediction in (57) was analytically confirmed by two of the authors [48] performing the analytic continuation of the GSVV expression for the remainder function at two loops. The analytic continuation allowed also to extract the next-to-leading logarithmic (NLLA) contribution, not yet available from the BFKL approach

$$\begin{aligned} R^{(2)\,NLLA}_{2\to 4} &= \frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2 - \frac{i\pi}{3}\ln^3|1+w|^2 + i\pi\ln|w|^2\left(\mathrm{Li}_2(-w) + \mathrm{Li}_2(-w^*)\right) \\ &-i2\pi\left(\mathrm{Li}_3(-w) + \mathrm{Li}_3(-w^*)\right). \end{aligned} \tag{58}$$

The NLLA remainder function in (57) is also pure imaginary and symmetric under $w \to 1/w$ transformation. Both of the contributions are pure imaginary due to a cancellation of the real part coming from the Mandelstam cut, Regge pole and a phase present in the BDS amplitude as was shown by one of the authors [47]. Starting at three loops the cancellation does not happen anymore and the real part gives a non-vanishing contribution at the next-to-leading level. The analysis of ref. [47] based on analyticity and other general properties of the scattering amplitudes resulted in a formulation of the dispersion-like relation for the real and imaginary parts of the remainder function in the Regge kinematics in this Mandelstam region

$$R_{2\to 4}\, e^{i\pi\,\delta} = \cos\pi\omega_{ab} + i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}\, f(\omega)\, e^{-i\pi\omega}\,(1-u_1)^{-\omega}\,, \tag{59}$$

where the first term in RHS corresponds to the contribution of the Regge pole (see (34)). This term as well as the phase δ in LHS of (59) are obtained directly from the BDS formula (see also (36))

$$\delta = \frac{\gamma_K}{8}\,\ln(\tilde{u}_2\tilde{u}_3) = \frac{\gamma_K}{8}\ln\frac{|w|^2}{|1+w|^4}\,,\ \omega_{ab} = \frac{\gamma_K}{8}\,\ln\frac{\tilde{u}_2}{\tilde{u}_3} = \frac{\gamma_K}{8}\ln|w|^2. \tag{60}$$

The second terms in RHS of (59) stands for the contribution of the Mandelstam cut (see (35)). The coefficient $\gamma_K \simeq 4a$ is the cusp anomalous dimension known to an arbitrary order of the perturbation theory. The only unknown piece in (59) is the real function $f(\omega)$, which contains the Mandelstam cut in ω, depends only on the transverse particle momenta and has no energy dependence. In the leading logarithmic approximation $f(\omega)$ is given by

$$f^{LLA}(\omega) = \frac{a}{2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} d\nu \frac{1}{\omega - \omega(\nu,n)}\frac{(-1)^n}{\nu^2+\frac{n^2}{4}}\,(w^*)^{i\nu-\frac{n}{2}}\,(w)^{i\nu+\frac{n}{2}}\,, \tag{61}$$

where $\omega(\nu, n)$ is defined in (55).

The dispersion-like relation in (59) was used [49] for calculating the three loop contributions to $R^{(3)}_{2\to 4}$ (leading imaginary and the sub-leading real terms) in the multi-Regge kinematics

$$R^{(3)\,LLA}_{2\to 4} = i\Delta^{(3)}_{2\to 4}/a^3 = \frac{i\pi}{4}\ln^2(1-u_1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \tag{62}$$
$$-\frac{1}{4}\ln^2|w|^2\ln|1+w|^2 + \frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\Big)$$

and

$$\Re\left(R^{(3)\,NLLA}_{2\to 4}\right) = \frac{\pi^2}{4}\ln(1-u_1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \tag{63}$$
$$-\frac{1}{2}\ln^2|w|^2\ln|1+w|^2 - \ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) + 2\mathrm{Li}_3(-w) + 2\mathrm{Li}_3(-w^*)\Big).$$

As in the two loop case, both (62) and (63) are symmetric under $w \to 1/w$ transformation, which is obvious from (52) and corresponds to the target-projectile symmetry of the scattering amplitude. The corrections, subleading in the logarithm of the energy, are not captured by (52) and require some knowledge of the next-to-leading impact factor and the eigenvalue of the BFKL Kernel in the adjoint representation.

While the latter is still to be found from the next-to-leading BFKL equation constructed by Fadin and Fiore [54, 55], the correction to the impact factor was obtained in ref. [49] extracting it from (58). The integrals in (52) and (59) come as a convolution of the propagator of the BFKL state G_{BFKL}

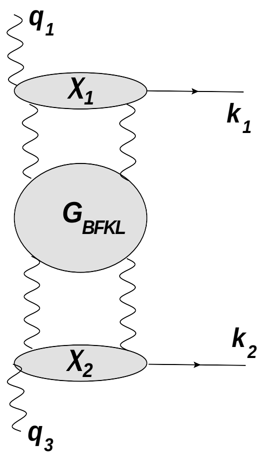

Figure 11: A graphic representation of the expression in (52). Two impact factors χ_1 and χ_2 are convoluted with the propagator of the BFKL state G_{BFKL}.

and two impact factors χ_1 and χ_2 as shown in Fig. 11. The leading logarithmic impact factor χ_i^{LLA} was calculated by two of the authors directly from the Feynman diagrams in ref. [42]

$$\chi_1^{LLA} = \frac{1}{2}\frac{1}{\left(i\nu+\frac{n}{2}\right)}\left(-\frac{q_1}{k_1}\right)^{-i\nu-\frac{n}{2}}\left(-\frac{q_1^*}{k_1^*}\right)^{-i\nu+\frac{n}{2}}, \chi_2^{LLA} = -\frac{1}{2}\frac{1}{\left(i\nu-\frac{n}{2}\right)}\left(\frac{q_3^*}{k_2^*}\right)^{i\nu-\frac{n}{2}}\left(\frac{q_3}{k_2}\right)^{i\nu+\frac{n}{2}}, \tag{64}$$

while the NLO impact factor was extracted from (58) and read [49]

$$\chi_1^{NLO} = \frac{a}{2}\left(E^2_{\nu,n} - \frac{1}{4}\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2}\right)\chi_1^{LLA}, \tag{65}$$

where $E_{\nu,n}$ is defined in (56). The NLO correction to χ_2 has a similar form found in ref. [49]. An important feature of χ_i^{NLO} in (65) is the fact that, in contrast to the leading order, it has lost the property of holomorphic separability: we cannot write the factor in front of χ_1^{LLA} in (65) as a sum of terms, which depends only on either $i\nu + n/2$ or $-i\nu + n/2$. It is worth emphasizing that the NLO impact factors χ_i^{NLO} are factorized in the product of the Born impact factors in (64) and a term expressed through the eigenvalue $E_{\nu,n}$ of the BFKL equation in the LLA. The form of the NLO impact factor in the ν, n representation resembles the three-loop remainder function in the LLA, emphasizing the intimate relation between the two. Indeed, it is easy to see that expanding the integrand of (52) to the third order in a one gets the $E_{\nu,n}^2$ term.

In the general case the integral in (52) is not easy to calculate, but one can consider a more restrictive kinematics, where it can be found explicitly at any order of the coupling a. One of such possibilities is the so-called collinear kinematics, when two adjacent particles become collinear, e.g. if in Fig. 8 the momenta p_B and $p_{B'}$ coincide. In the limit $t_3 \to 0$ the remainder function vanishes at two loops and beyond, in both the direct channel of Fig. 8 and in the Mandelstam channel of Fig. 10. The multi-Regge limit followed by the collinear limit in terms of the dual conformal cross ratios (compare to the Regge kinematics in (46)) reads

$$1-u_1 \to +0, \;\; u_2 \to +0, \;\; u_3 \to +0, \;\; \frac{u_2}{1-u_1} = \tilde{u}_2 \to +0, \;\; \frac{u_3}{1-u_1} = \tilde{u}_3 \simeq 1, \tag{66}$$

which in terms of w and w^* implies (see (54))

$$1-u_1 \to +0, \;\; |w| \to +0, \;\; \cos(\phi_2 - \phi_3) \simeq \mathcal{O}(1). \tag{67}$$

For $|w| \to 0$ the main contribution to the LLA remainder function given by (52) comes from poles at $\nu = \pm in/2$ for the conformal spin $n = 1$. In this case one can drop the ψ functions in (56) and perform integration of (52) at any order of the coupling constant in the Double Leading Logarithmic Approximation (DLLA), where one sums contributions from the powers of $a \ln|w| \ln(1-u_1)$ resulting in [56]

$$R_{2\to4}^{DLLA} = 1 + i2\pi a \cos(\phi_2 - \phi_3)\, |w| \left(1 - I_0\left(2\sqrt{a \ln|w| \ln(1-u_1)}\right)\right), \tag{68}$$

where $I_n(z)$ is the modified Bessel function. The Double Leading Logarithmic Approximation is analogous to the summation of the contributions of powers of $g^2 \ln s \ln Q^2$ in Deep Inelastic Scattering (DIS), where the BFKL and the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations overlap. Then the real part of the next-to-leading corrections to the remainder function in DLLA was calculated [49] using the dispersion relation (59) and (68).

3.2 $3 \to 3$ scattering amplitude

Beyond the $2 \to 4$ scattering the six-particle scattering amplitude describes also the $3 \to 3$ scattering illustrated in Fig. 12. The analysis of the $3 \to 3$ amplitude is similar to that of the $2 \to 4$, but there are some rather interesting features we want to emphasize. Firstly we start with the definition of the kinematic invariants $s_{13} = (p_B + k_1)^2, s_{02} = (p_{A'} + k_2)^2, s = (p_B + k_1 + p_A)^2, t_2^{'} = (p_A - p_{A'} - k_2)^2, s_1 = (k_1 + p_A)^2, s_3 = (p_{B'} + k_2)^2, t_2 = (p_A - p_{A'} + k_1)^2, t_1 = (p_A - p_{A'})^2$ and $t_3 = (p_B - p_{B'})^2$. The dual conformal cross ratios are expressed in terms of these invariants as follows

$$u_1 = \frac{s_{13} s_{02}}{s\, t_2^{'}}, \;\; u_2 = \frac{t_1 t_3}{t_2 t_2^{'}}, \;\; u_3 = \frac{s_1 s_3}{s\, t_2}. \tag{69}$$

In the multi-Regge kinematics for the direct channel in Fig. 12, where all invariants are negative

$$-s \gg -s_1, -s_3, -t_2^{'} \gg -t_1, -t_2, -t_3 > 0, \tag{70}$$

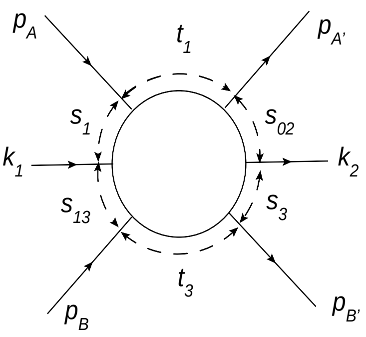

Figure 12: The $3 \to 3$ gluon scattering amplitude.

the remainder function $R^{(l)}_{3\to3}$ is zero, while in the physical region of the Mandelstam channel depicted in Fig. 13, where

$$s_1, s_3, s_{13}, s_{02} < 0 \quad \text{and} \quad s, t_2^{'} > 0, \tag{71}$$

it contains a non-vanishing contribution, growing with energy $t_2^{'}$. In the Mandelstam channel (71) in

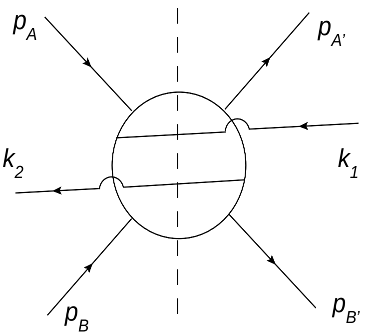

Figure 13: The $3 \to 3$ gluon scattering amplitude in the Mandelstam channel given by $s_1, s_3, s_{13}, s_{02} < 0$ and $s, t_2^{'} > 0$.

the multi-Regge kinematics the dual conformal cross ratios (69) possess a non-zero phase

$$u_1 \to |u_1|e^{i2\pi}, \ u_2 \to |u_2|e^{i\pi}, \ u_3 \to |u_3|e^{i\pi} \tag{72}$$

and the analytic continuation of the GSVV expression in multi-Regge kinematics gives

$$\begin{aligned} R^{(2)\,LLA+NLLA}_{3\to3} &= -\frac{i\pi}{2}\ln(u_1-1)\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2+\frac{\pi^2}{2}\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 \\ &-\frac{i\pi}{2}\ln|w|^2\ln^2|1+w|^2+\frac{i\pi}{3}\ln^3|1+w|^2-i\pi\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) \\ &+i2\pi\left(\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\right). \end{aligned} \tag{73}$$

As in the $2 \to 4$ case the remainder function has the target-projectile symmetry ($p_A \leftrightarrow p_B, p_{A'} \leftrightarrow p_{B'}$, $k_1 \leftrightarrow k_2$ or $|w| \to 1/|w|$), but in contrast to the $2 \to 4$ amplitude (73) has a real part $\frac{\pi^2}{2}\ln|1+w|^2\ln\left|1+\frac{1}{w}\right|^2$. This fact is in full agreement with the dispersion-like relation [47] for the $3 \to 3$ amplitude

$$R_{3\to3}e^{-i\pi\delta} = \cos\pi\omega_{ab} - i\int_{-i\infty}^{i\infty}\frac{d\omega}{2\pi i}f(\omega)|1-u_1|^{-\omega}, \tag{74}$$

which differs from (59) by signs of the phase on LHS and the integral on RHS, as well the absence of the phase $e^{-i\pi\omega}$ in the integrand. This phase mixes the contribution from the Regge poles and the Mandelstam cut in the dispersion relation (59), which leads to the full cancellation of the real part at two loops for the $2 \to 4$ amplitude. In the $3 \to 3$ case this cancellation does not happen anymore due the absence of the phase $e^{-i\pi\omega}$, and one obtains a real term in (73). The real part does not cancel out in both the $2 \to 4$ and $3 \to 3$ amplitudes at higher loops. The dispersion relation (74) was used by two of the authors [49] to find the LLA and the real part of the NLLA contributions to the remainder function of the $3 \to 3$ amplitude

$$
\begin{aligned}
R^{(3)\,LLA}_{3\to3} &= -\frac{i\pi}{4}\ln^2(u_1-1)\Big(\ln|w|^2\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 \\
&+\frac{1}{2}\ln|w|^2\left(\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\right) - \frac{1}{4}\ln^2|w|^2\ln|1+w|^2 - \mathrm{Li}_3(-w) - \mathrm{Li}_3(-w^*)\Big)
\end{aligned}
\tag{75}
$$

and

$$
\Re\left(R^{(3)\,NLLA}_{3\to3}\right) = -\frac{\pi^2}{4}\ln(u_1-1)\left(\ln^2|1+w|^2\ln\left|1+\frac{1}{w}\right|^2 + \ln|1+w|^2\ln^2\left|1+\frac{1}{w}\right|^2\right). \tag{76}
$$

As we have already mentioned in the $3 \to 3$ case there is no mixing between the real contributions coming from the Regge poles and Mandelstam cuts. This fact allows us to make a prediction valid at an arbitrary value of the coupling constant for the following expression [57]

$$
\Re\left(R_{3\to3}e^{-i\pi\delta}\right) = \cos\pi\omega_{ab}. \tag{77}
$$

Both δ and ω_{ab} are known and are given by (60) as functions of the cusp anomalous dimension and the anharmonic ratios. However there is a difficulty in understanding (77) at the strong coupling because of the rapid oscillations as $a \to \infty$.

Similarly to the $2 \to 4$ case one can also consider the collinear limit $t_3 \to 0$ ($|w| \to 0$) preceded by the Regge limit. In this kinematics it is possible to calculate explicitly the LLA and the real part of the NLLA $3 \to 3$ remainder function at an arbitrary number of loops [56].

In the next section we return to the discussion of section 2 and consider a composite state of an arbitrary number of reggeized gluons. These BKP states appear in the scattering amplitudes with 8 or more external legs.

4 Integrability of the n gluon Hamiltonian

Here we discuss composite states of n reggeized gluons in the adjoint representation at large N_c (cf. a similar approach for the simple case $n = 2$ in ref. [42]). One can write the homogeneous BKP equation for its wave function described by an amplitude with amputated propagators in the form [45]

$$
H\Psi = E\Psi\,,\ \Delta_n = -\frac{g^2N_c}{16\pi^2}E\,. \tag{78}
$$

Here H is a redefined hamiltonian obtained after a subtraction of the gluon Regge trajectory $\omega(t)$ containing infrared divergencies. Namely, the Regge trajectory of the composite state is [45]

$$
\omega_n(t) = a\left(\frac{1}{\epsilon} - \ln\frac{-t}{\mu^2}\right) + \Delta_n\,,\ a = \frac{g^2N_c}{8\pi^2}\,, \tag{79}
$$

where Δ_n is the infrared stable quantity expressed in terms of the energy E.

The hamiltonian H in the multi-color limit can be written in the holomorphically separable form (see [45])

$$H = h + h^* \,, \; h = \ln \frac{p_1\, p_n}{q^2} + \sum_{r=1}^{n-1} h^t_{r,r+1} \,, \; q = \sum_1^n p_r \,, \tag{80}$$

where the pair hamiltonian $h^t_{r,r+1}$ is transposed to the corresponding unamputated operator (17)

$$h^t_{r,r+1} = \ln(p_r p_{r+1}) + p_r \ln(\rho_{r,r+1}) \, \frac{1}{p_r} + p_{r+1} \ln(\rho_{r,r+1}) \, \frac{1}{p_{r+1}} + 2\gamma \,. \tag{81}$$

It is seen from (80) that the holomorphic hamiltonian for the composite state in the adjoint representation differs from the corresponding expression for the singlet case $h^{(0)}$ (17) after its transposition only by the substitution

$$h_{n,1} \to \ln \frac{p_1\, p_n}{q^2} \,, \tag{82}$$

which is related to the fact, that the planar Feynman diagrams have the topology of a strip and the infrared divergencies in the Regge trajectories of the particles 1 and n are not compensated by the contribution from the pair potential energy $V_{n,1}$.

It turns out, that the eigenvalues E do not depend on $|q|^2$ due to the scale invariance of H, as it will be demonstrated below. As a result, the t-dependence of $\omega_n(t)$ is the same as in the gluon Regge trajectory.

The normalization condition for the wave function in two-dimensional space can be written as follows

$$||\Psi||^2 = \int \prod_{r=1}^{n-1} d^2 p_r \, \Psi^* \prod_{s=1}^{n} |p_s|^{-2} \Psi \,, \; \sum_{s=1}^{n} p_s = q \,. \tag{83}$$

Using the duality transformation (see [9] and [45])

$$p_1 = z_{0,1} \,, \; p_r = z_{r-1,r} \,, \; q = z_{0,n} \,, \; \rho_{r,r+1} = i \frac{\partial}{\partial z_r} = i \partial_r \,, \tag{84}$$

the holomorphic hamiltonian can be presented as follows

$$h = \ln \frac{z_{0,1}\, z_{n-1,n}}{z^2_{0,n}} + \sum_{r=1}^{n-1} h^t_{r,r+1} \,. \tag{85}$$

Further, by regrouping its terms we can write the holomorphic hamiltonian in another form [45]

$$h = -2 \ln z_{0,n} + \ln(z^2_{0,1} \partial_1) + \ln(z^2_{n-1,n} \partial_{n-1}) + 2\gamma + \sum_{r=1}^{n-2} h'_{r,r+1} \,, \tag{86}$$

where the new pair hamiltonian is

$$h'_{r,r+1} = \ln(z^2_{r,r+1} \partial_r) + \ln(z^2_{r,r+1} \partial_{r+1}) - 2 \ln z_{r,r+1} + 2\gamma$$

$$= \ln(\partial_r) + \ln(\partial_{r+1}) + \frac{1}{\partial_r} \, \ln z_{r,r+1} \, \partial_r + \frac{1}{\partial_{r+1}} \, \ln z_{r,r+1} \, \partial_{r+1} + 2\gamma \,. \tag{87}$$

The operator $h'_{r,r+1}$ coincides in fact after the substitution $z_r \to \rho_r$ with the corresponding hamiltonian in the coordinate representation (17) acting on the wave function with non-amputated propagators.

It is important, that h (86) is invariant under the Möbius transformations

$$z_k \to \frac{a z_k + b}{c z_k + d} \tag{88}$$

and does not contain the derivatives ∂_0 and ∂_n. Therefore we can put

$$z_0 = 0\,, \;\; z_n = \infty\,, \tag{89}$$

which leads to the simplified expression for h

$$h \to h' = \ln(z_1^2 \partial_1) + \ln(\partial_{n-1}) + 2\gamma + \sum_{r=1}^{n-2} h'_{r,r+1}\,. \tag{90}$$

To return to initial variables in the final expression for the wave function one should perform the following substitution of z_k

$$z_k \to -\frac{z_k - z_0}{z_k - z_n} = \frac{\sum_{r=1}^{k} p_r}{q - \sum_{r=1}^{k} p_r}\,. \tag{91}$$

According to the above representation (85) for h, its transposed part h'^t can be obtained from h by the similarity transformation which can be written in terms of h' as follows

$$h'^t = z_1^{-1} \left(\prod_{r=1}^{n-2} z_{r,r+1} \right)^{-1} h' \; z_1 \left(\prod_{r=1}^{n-2} z_{r,r+1} \right)\,, \tag{92}$$

which is compatible with the following normalization condition for the wave function in the full two-dimensional space

$$||\Psi||_1^2 = \int \frac{d^2 z_{n-1}}{|z_1|^2} \prod_{r=1}^{n-2} \frac{d^2 z_r}{|z_{r,r+1}|^2}\, |\Psi|^2\,. \tag{93}$$

On the other hand, from the expression (90) for h' we obtain another relation for h'^t

$$h'^t = \left(\prod_{r=1}^{n-1} \partial_r \right) h' \left(\prod_{r=1}^{n-1} \partial_r \right)^{-1}\,, \tag{94}$$

corresponding to the second normalization condition for Ψ compatible with the hermicity properties of the total hamiltonian

$$||\Psi||_2^2 = \int \prod_{r=1}^{n-1} d^2 z_r\, \Psi^* \prod_{r=1}^{n-1} |\partial_r|^2\, \Psi\,. \tag{95}$$

By comparing two above relations between h' and h'^t one can conclude (cf. [10]), that the operator

$$A' = z_1 \prod_{s=1}^{n-2} z_{s,s+1} \prod_{r=1}^{n-1} \partial_r \tag{96}$$

commutes with the holomorphic hamiltonian

$$[A', h'] = 0\,. \tag{97}$$

4.1 Integrable open spin chain

Let us verify [45], that the holomorphic hamiltonian h' (90) also commutes with the differential operator $D(u)$ being the matrix element T_{22} of the monodromy matrix (cf. [10])

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = L_1(u) L_2(u) ... L_{n-1}(u)\,, \tag{98}$$

where the L-operator is defined by the relation

$$L_r(u) = \left(\begin{array}{cc} u + iz_r\partial_r & i\partial_r \\ -iz_r^2\partial_r & u - iz_r\partial_r \end{array} \right) . \tag{99}$$

To prove the commutativity of h' and $D(u)$ one can use the following relation [45]

$$[L_k(u)\, L_{k+1}(u), h'_{k,k+1}] = -i\, (L_k(u) - L_{k+1}(u)) \ , \tag{100}$$

valid in particular due to the Möbius symmetry of the pair hamiltonian

$$[\vec{M}_{k,k+1}, h'_{k,k+1}] = 0\, , \ \vec{M}_{k,k+1} = \vec{M}_k + \vec{M}_{k+1}. \tag{101}$$

and the fact that its eigenvalue is a linear combination of polygamma functions (see (13))

Relation (100) leads to the equality

$$[T(u), \sum_{r=1}^{n-2} h'_{r,r+1}] = iL_2(u)L_3(u)...L_{n-1}(u) - iL_1(u)L_2(u)...L_{n-2}(u)\, . \tag{102}$$

On the other hand, one can easily verify, that

$$[T_{22}(u), \ln(z_1^2\partial_1) + \ln \partial_{n-1}] = (0\, ,\ 1)\, [T(u), \ln(z_1^2\partial_1) + \ln \partial_{n-1}] \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$$

$$= -i\, (0\, ,\ 1)\, (L_2(u)L_3(u)...L_{n-1}(u) - L_1(u)L_2(u)...L_{n-2}(u)) \left(\begin{array}{c} 0 \\ 1 \end{array} \right) , \tag{103}$$

which proves that the differential operator $D(u) = T_{22}(u)$ is an integral of motion [45]

$$[D(u), h'] = 0\, . \tag{104}$$

Note, that, if instead of condition (89) we shall use the equivalent condition

$$z_0 = \infty\, , \ z_n = 0\, ,$$

the matrix element $A(u) = T_{11}(u)$ of the monodromy matrix will be an integral of motion.

Thus, our hamiltonian is the local hamiltonian for an open integrable Heisenberg spin model with the spins which are generators of the Möbius group.

With the use of the following decomposition of the L-operators

$$L_r(u) = \left(\begin{array}{cc} u & 0 \\ 0 & u \end{array} \right) + \left(\begin{array}{c} 1 \\ -z_r \end{array} \right) (z_r\, ,\ 1)\ i\partial_r \tag{105}$$

one can construct the matrix element $T_{22} = D(u)$ in an explicit way

$$D(u) = \sum_{k=0}^{n-1} u^{n-1-k}\, q'_k\, , \tag{106}$$

where

$$q'_k = - \sum_{0<r_1<r_2<...<r_k<n} z_{r_1} \prod_{s=1}^{k-1} z_{r_s, r_{s+1}} \prod_{t=1}^{k} i\partial_{r_t}\, . \tag{107}$$

Note, that one can parameterize the monodromy matrix in another form

$$T(u) = \left(\begin{array}{cc} j_0(u) + j_3(u) & j_+(u) \\ j_-(u) & j_0(u) - j_3(u) \end{array} \right) , \ j_\pm(u) = j_1(u) \pm ij_2(u)\, . \tag{108}$$

In this case the Yang-Baxter equations for the currents j_μ have the Lorentz-invariant representation [9]

$$[j_\mu(u), j_\nu(v)] = \frac{\epsilon_{\mu\nu\rho\sigma}}{2(u-v)}\, (j^\rho(u)j^\sigma(v) - j^\rho(v)j^\sigma(u))\ . \tag{109}$$

Here $\epsilon_{\mu\nu\rho\sigma}$ is the antisymmetric tensor in the four-dimensional Minkowski space and $\epsilon_{1230} = 1\, ,\ g_{\mu\nu} = (1, -1, -1, -1)$.

4.2 Composite states of two and three gluons

In the case $n = 2$ we have only one non-trivial integral of motion

$$q_1' = -iz_1\,\partial_1\,. \tag{110}$$

Taking into account the normalization condition for the eigenfunction in the two-dimensional space

$$||\Psi||^2 = \int \frac{d^2z_1}{|z_1|^2}\,|\Psi|^2\,, \tag{111}$$

we find the orthonormalized and complete set of eigenfunctions

$$\Psi^{(2)}_{m,\widetilde{m}} = z_1^{-\frac{1}{2}+m}\,(z_1^*)^{-\frac{1}{2}+\widetilde{m}}\,,\ m = \frac{1+n}{2} + i\nu\,,\ \widetilde{m} = \frac{1+n}{2} - i\nu\,, \tag{112}$$

satisfying the single-valuedness requirement. Note, that using the substitution (91) one can reproduce the wave functions of two gluon composite states in the momentum space (see [42]).

For the case $n = 3$ the operator $D(u)$ is given below

$$D_3(u) = u^2 - iu(z_1\,\partial_1 + z_2\,\partial_2) + z_1 z_{1,2}\,\partial_1\partial_2\,. \tag{113}$$

With taking into account the normalization condition

$$||\Psi||^2 = \int \frac{d^2z_1\,d^2z_2}{|z_1|^2|z_{1,2}|^2}\,|\Psi|^2\,, \tag{114}$$

one can search the holomorphic eigenfunction of this operator in the form

$$\Psi^{(3)}_m = z_2^{-\frac{1}{2}+m}\,f\left(\frac{z_2}{z_1}\right)\,. \tag{115}$$

The function $f(x)$ satisfies the equation

$$\left(x(1-x)\partial^2 + (\frac{1}{2}+m)(1-x)\partial + \lambda\right)\,f = 0\,,\ x = \frac{z_2}{z_1}\,, \tag{116}$$

where λ is the eigenvalue of the operator $z_1 z_{1,2}\partial_1\partial_2$. Two independent solutions of this equation can be expressed in terms of the hypergeometric function F

$$f_1(x) = F(a_1, a_2; 1 + a_1 + a_2; x)\,,\ f_2(x) = x^{a_1+a_2}\,F(-a_2, -a_1; 1 - a_1 - a_2; x)\,, \tag{117}$$

where the parameters a_1 and a_2 are obtained from the set of equations

$$a_1 + a_2 = -\frac{1}{2} + m\,,\ a_1 a_2 = -\lambda\,. \tag{118}$$

We have the similar solutions for the eigenvalue of the operator D^* in the antiholomorphic subspace. They can be obtained by the substitution

$$x \to x^*\,,\ a_1 \to \widetilde{a}_1\,,\ a_2 \to \widetilde{a}_2\,,\ m \to \widetilde{m} = \frac{1-n}{2} + i\nu\,. \tag{119}$$

To construct the wave function Ψ with the property of the single-valuedness in the two-dimensional subspaces $\vec{z}_1$ and $\vec{x}$ we should write a bilinear combination of the functions $f_i(x)$ and the corresponding functions in the anti-holomorphic subspace $\widetilde{f}_i(x^*)$.

One can write the integral representation for the wave function satisfying the above constraints

$$\Psi \sim z_2^{a_1+a_2}\,(z_2^*)^{\widetilde{a_1}+\widetilde{a_2}} \int \frac{d^2y}{|y|^2}\,y^{-a_2}(y^*)^{-\widetilde{a_2}} \left(\frac{y-1}{y-x}\right)^{a_1} \left(\frac{y^*-1}{y^*-x^*}\right)^{\widetilde{a_1}}\,,\ x = \frac{z_2}{z_1}\,, \tag{120}$$

where the integration is performed over the two-dimensional plane $\vec{y}$. Note, that the integrand has no ambiguity in the points $y = 0, 1, x$ due to the additional constraints for the parameters $a_i, \widetilde{a}_i$ [45]

$$a_1 - \widetilde{a_1} = N_{a_1}\,,\ a_2 - \widetilde{a_2} = N_{a_2}\,, \tag{121}$$

where N_{a_1}, N_{a_2} are integers. Moreover, the function Ψ near the points $x = 0, 1, \infty$ can be presented in terms of the sum of products of hypergeometric functions in (117).

4.3 Hamiltonian and integrals of motion

The holomorphic hamiltonian for composite states of two reggeized gluons can be written as follows

$$\widetilde{h} = \ln(z_1^2\partial_1) + \ln(\partial_1) + 2\gamma = \psi(z_1\partial_1) + \psi(-z_1\partial_1) + 2\gamma\,. \tag{122}$$

Acting by $\widetilde{h}$ on the function z_1^δ we obtain

$$\widetilde{h}z_1^\delta = \epsilon(\delta)\, z_1^\delta\,,\ \epsilon(\delta) = \psi(\delta) + \psi(-\delta) + 2\gamma\,. \tag{123}$$

In the case of wave function (112) satisfying the single-valuedness and orthonormality conditions in the two-dimensional space one derives the following expression for the total energy [42]

$$E_{m,\widetilde{m}} = \epsilon_m + \epsilon_{\widetilde{m}}\,,\ \epsilon_m = \psi(-\frac{1}{2}+m) + \psi(\frac{1}{2}-m) + 2\gamma\,. \tag{124}$$

Note, that it does not coincide with the corresponding result (13) for the Pomeron state.

For the case of composite states of n reggeized gluons the holomorphic hamiltonian (86) in the region

$$z_1 \ll z_2 \ll z_3 \ll ... \ll z_{n-1}\,. \tag{125}$$

becomes the sum of the disconnected hamiltonians

$$h' = \sum_{r=1}^{n-1}\left(\psi(z_r\partial_1) + \psi(-z_r\partial_1) + 2\gamma\right)\,. \tag{126}$$

As a result, we obtain, that the wave function in this limit is factorized [45]

$$\Psi_{a_1,a_2,...,a_{n-1}} = \prod_{r=1}^{n-1} z_r^{a_r} \tag{127}$$

and the energy is the sum of the particle energies

$$\epsilon = \sum_{r=1}^{n-1}\epsilon(a_r)\,. \tag{128}$$

The parameters a_r for these solutions and $\widetilde{a_r}$ for anti-holomorphic solutions are obtained from the single-valuedness condition and the normalizability

$$a_r = i\nu_r + \frac{n_r}{2}\,,\ \widetilde{a_r} = i\nu_r - \frac{n_r}{2}\,,$$

where ν_r are real and n_r are integer numbers.

The eigenvalue of the integral of motion $D(u)$ is expressed also in terms of these parameters

$$\Lambda(u) = \prod_{r=1}^{n-1}(u - ia_r)\,. \tag{129}$$

4.4 The Baxter-Sklyanin approach

To find a solution of the Yang-Baxter equation for the open spin chain one can use the Bethe ansatz. For this purpose it is convenient to work in the transposed representation for the monodromy matrix

$$T^t(u) = \begin{pmatrix} j_0^t(u) + j_3^t(u) & j_+^t(u) \\ j_-^t(u) & j_0^t(u) - j_3^t(u) \end{pmatrix} = L_1^t(u)L_2^t(u)...L_{n-1}^t(u)\,, \tag{130}$$

where the L-operator can be chosen as follows

$$L_r^t(u) = \left(\begin{array}{cc} u + i\partial_r z_r & i\partial_r \\ -i\partial_r z_r^2 & u - i\partial_r z_r \end{array} \right) . \tag{131}$$

The pseudo-vacuum state is defined as a solution of the equation

$$j_-^t(u)\Psi_0 = 0 . \tag{132}$$

It can be written in the form [12]

$$\Psi_0 = \prod_{r=1}^{n-1} z_r^{-2} . \tag{133}$$

Note, that the function $|\Psi_0|^2$ does not belong to the principal series of the unitary representations. As a result, the states constructed in the framework of the Bethe ansatz by applying the product of the operators $j_+^r(u_r)$ to Ψ_0

$$\Psi_k^t = \prod_{r=1}^{k} j_+^t(u_r)\, \Psi_0 \tag{134}$$

are non-physical. Nevertheless, these states are eigenfunctions of the integral of motion

$$D^t(u)\Psi_k^t = (j_0^t(u) - j_3^t(u))\Psi_k^t = \Lambda(u)\Psi_k^t \tag{135}$$

providing that

$$\Lambda(u) = (u+i)^{n-1} \prod_{t=1}^{k} \frac{u - u_t + i}{u - u_t} \equiv (u+i)^{n-1} \frac{Q(u+i)}{Q(u)} \tag{136}$$

is a polynomial, which leads to a quantization condition for the Bethe roots u_t. If we parameterize this polynomial as follows

$$\Lambda(u) = \prod_{l=1}^{n-1} (u - ia_l) , \tag{137}$$

the above defined Baxter function $Q(u)$ can be calculated

$$Q(u) = \prod_{l=1}^{n-1} \frac{\Gamma(-iu - a_l)}{\Gamma(-iu+1)} . \tag{138}$$

As it was mentioned above, the polynomial solutions for $Q(u)$ are non-physical, because the corresponding wave functions Ψ do not belong to the principal series of unitary representations of the Möbius group. We should find a set of non-polynomial solutions $Q_s(u)$ satisfying this physical requirement.

According to E. Sklyanin [15] the correct variables in which the dynamics of the Heisenberg spin model is drastically simplified are the zeroes $\hat{b}_r$ of the operator $B(u) = j_+^t(u)$ entering in the monodromy matrix

$$B(u) = P_{n-1} \prod_{k=1}^{n-2} (u - \hat{b}_r) , \; P_{n-1} = i \sum_{r=1}^{n-1} \partial_r , \tag{139}$$

where the operators $\hat{b}_r$ and P_{n-1} commute each with others

$$[\hat{b}_r, \hat{b}_s] = [\hat{b}_r, P_{n-1}] = 0 . \tag{140}$$

It is convenient to pass from the coordinate representation $\vec{z}$ to the Baxter-Sklyanin representation [13], in which the currents $j_+^t(u)$ and $(j_+^t(u))^*$ (together with the operators $\hat{b}_r, \hat{b}_r^*$ and P_{n-1}, P_{n-1}^*)

are diagonal. We denote the eigenvalues of the Sklyanin operators by b_r, b_r^*. The kernel of the unitary transformation to the Baxter-Sklyanin representation is known explicitly for the cases $n = 2$, $n = 3$ and $n = 4$ [13]. For general n this integral operator can be presented as a multi-dimensional integral [14].

In the Baxter-Sklyanin representation the wave function in the holomorphic subspace can be expressed as a product of the pseudo-vacuum state in this representation $\Psi_0(P_{n-1}, b_1, b_2, ..., b_{n-2})$ and the Baxter functions $Q(u_t)$

$$\Psi^t(P_{n-1}; b_1, ..., b_{n-2}) = P_{n-1}^{-\frac{n-1}{2}-m} \prod_{k=1}^{n-2} Q(b_k)\, \Psi_0(P_{n-1}, b_1, ..., b_{n-2})\,, \tag{141}$$

where the power-like behavior in the variable P_{n-1} is in an agreement with the normalization condition.

The analogous representation is valid for the total wave function

$$\Psi^+(\vec{P}_{n-1}; \vec{b}_1, ..., \vec{b}_{n-2}) = P_{n-1}^{-\frac{n-1}{2}-m} \,(P_{n-1}^*)^{-\frac{n-1}{2}-\tilde{m}} \prod_{k=1}^{n-2} Q(\vec{b}_r)\, \Psi_0(\vec{P}_{n-1}; \vec{b}_1, ..., \vec{b}_{n-2}) \tag{142}$$

with the use of the generalized Baxter function $Q(\vec{u})$ being a bilinear combination of the usual Baxter functions in the holomorphic and anti-holomorphic subspaces

$$Q(\vec{u}) = \sum_{s,t} d_{s,t}\, Q_s(u)\, Q_t(u^*)\,. \tag{143}$$

Here $Q_s(u)$ are different solutions of the Baxter equation with the same eigenvalue $\Lambda(u)$. The coefficients $d_{s,t}$ are chosen from the requirement, that the function $Q(\vec{u})$ satisfies the normalization condition everywhere including the points where the functions $Q_s(u)$ and $Q_t(u^*)$ have the poles [13, 14]. For the periodic spin chain this condition leads to the quantization of the eigenvalue of the operator $A(u) + B(u)$ although a simpler method of quantization is based on the requirement, that all Baxter functions corresponding to the same eigenvalue should have the same holomorphic energy [13]. In the case of the open Heisenberg spin model the situation is simpler and will be discussed below.

4.5 Baxter-Sklyanin representation for three gluon states

For the states composed from three reggeized gluons the transposed integral of motion in the holomorphic subspace is

$$D_3^t(u) = u^2 - iu(\partial_1\, z_1 + \partial_2\, z_2) + \partial_1\partial_2\, z_1 z_{1,2} \tag{144}$$

and the operator j_+^t is given below

$$j_+^t = iu(\partial_1 + \partial_2) - \partial_1\partial_2 z_{12} = i(\partial_1 + \partial_2)\,(u - \hat{b}_1)\,, \tag{145}$$

where

$$\hat{b}_1 = -i\frac{\partial_1\partial_2}{\partial_1 + \partial_2}\, z_{12}\,. \tag{146}$$

The operator j_+^t is easily diagonalized after a transition to the momentum representation, where

$$i\partial_1\, f_{p_1,p_2} = p_1\, f_{p_1,p_2}\,,\ i\partial_2\, f_{p_1,p_2} = p_2\, f_{p_1,p_2}\,. \tag{147}$$

In this case the eigenvalue equation for j_-^t has the form

$$\left(u(p_1 + p_2) - i\, p_1 p_2(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2})\right) f = (p_1 + p_2)(u - b_1)\, f\,, \tag{148}$$

where b_1 is the eigenvalue of $\hat{b}_1$. Its solution is given below

$$f = \chi(p_1 + p_2, b_1) \left(\frac{p_1}{p_2}\right)^{-ib_1}, \tag{149}$$

where χ is an arbitrary function of $p_1 + p_2$ and b_1. The dependence of Ψ^t from $p_1 + p_2$ is fixed by the normalization condition

$$\Psi^t \sim (p_1 + p_2)^{-a_1-a_2}. \tag{150}$$

On the other hand, the eigenvalue equation for the integral of motion in the momentum space can be written in the form

$$p_1 p_2 \frac{\partial}{\partial p_1} \left(\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}\right) \Psi(p_1, p_2) = a_1 a_2 \Psi(p_1, p_2). \tag{151}$$

Using the ansatz

$$\Psi(p_1, p_2) = (p_1 + p_2)^{-a_1-a_2}\, \eta(y), \; y = \frac{p_2}{p_1}, \tag{152}$$

we obtain the following equation for the function $\eta(y)$

$$\left(y^2\,\partial^2 + (a_1 + a_2 + 1)\, y\, \partial - a_1 a_2\right) \eta(y) = \left(-y^3\,\partial^2 - 2\, y^2\,\partial\right) \eta(y). \tag{153}$$

There are two independent solutions of this equation [45]

$$\eta_1(y) = \frac{1}{y}\, F(1 - a_1, 1 - a_2, 2; -\frac{1}{y}) \tag{154}$$

and

$$\eta_2(y) = -\frac{\Gamma(-a_1)\,\Gamma(+a_2)}{\Gamma(1 + a_2 - a_1)}\, y^{-a_1}\, F(-a_1, 1 - a_1, 1 + a_2 - a_1; -y). \tag{155}$$

One can construct the bilinear combination of these solutions having the single-valuedness property in the $\vec{y}$-space. Finally with the use of the integral representation for the hypergeometric function the wave function Ψ^t in the momentum space can be written as follows [45]

$$\Psi^t(\vec{p}_1, \vec{p}_2) = (p_1 + p_2)^{-a_1-a_2} (p_1^* + p_2^*)^{-\widetilde{a}_1-\widetilde{a}_2}\, \phi(\vec{y}), \tag{156}$$

where $\phi(\vec{y})$ is given below

$$\phi(\vec{y}) = \int d^2t \left(\frac{1}{t\,y} + 1\right)^{a_1} \left(\frac{1}{t^*\,y^*} + 1\right)^{\widetilde{a}_1} (1 - t)^{a_2-1}\, (1 - t^*)^{\widetilde{a}_2-1} \tag{157}$$

and satisfies the single valuedness condition in the $\vec{y}$-space due to the quantization conditions for a_r and $\widetilde{a}_r$.

The transition to the Baxter-Sklyanin representation $(u, \widetilde{u})$ corresponds to the Mellin-type transformation of $\phi(\vec{y})$

$$\phi(u, \widetilde{u}) = \int \frac{d^2y}{|y|^2}\, y^{-iu}\, (y^*)^{-i\widetilde{u}} \phi(\vec{y}). \tag{158}$$

The inverse transformation corresponds to the Baxter-Sklyanin representation for the wave function

$$\Psi^t(\vec{p}_1, \vec{p}_2) = (p_1 + p_2)^{-a_1-a_2} (p_1^* + p_2^*)^{-\widetilde{a}_1-\widetilde{a}_2} \int d^2u\, \phi(u, \widetilde{u}) \left(\frac{p_1}{p_2}\right)^{-iu} \left(\frac{p_1^*}{p_2^*}\right)^{-i\widetilde{u}}, \tag{159}$$

where

$$-\, iu = i\nu_u + \frac{N_u}{2}, \; -i\widetilde{u} = i\nu_u - \frac{N_u}{2}, \; \int d^2u \equiv \int_{-\infty}^{\infty} d\nu_u \sum_{N_u=-\infty}^{\infty}. \tag{160}$$

One can interpret the wave function $\phi(u,\widetilde{u})$ in the Baxter-Sklyanin representation as a product of the pseudo-vacuum state $u\,\widetilde{u}$ and the total Baxter function [45]

$$\phi(u,\widetilde{u}) = u\,\widetilde{u}\,Q(u,\widetilde{u})\,, \tag{161}$$

where

$$Q(u,\widetilde{u}) \sim \frac{\Gamma(iu)\Gamma(i\widetilde{u})}{\Gamma(1-iu)\,\Gamma(1-i\widetilde{u})}\,\frac{\Gamma(-iu-a_1)\,\Gamma(-iu-a_2)}{\Gamma(1+i\widetilde{u}+\widetilde{a}_1)\Gamma(1+i\widetilde{u}+\widetilde{a}_2)}\,. \tag{162}$$

This expression can be written in the factorized form

$$Q(u,\widetilde{u}) \sim Q(u,a_1,a_2)\,Q(\widetilde{u},\widetilde{a}_1,\widetilde{a}_2)\,, \tag{163}$$

where

$$Q(u,a_1,a_2) = \frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma^2(1-iu)}\,\Phi(u)\,, \tag{164}$$

$$\Phi(u) = \sqrt{\frac{\sin(\pi(-iu-a_1))\,\sin(\pi(-iu-a_2))}{\sin^2(-i\pi u)}}\,. \tag{165}$$

The expression $Q(u,a_1,a_2)$ differs from the Baxter function in the holomorphic space

$$Q(u) = \frac{\Gamma(-iu-a_1)\Gamma(-iu-a_2)}{\Gamma^2(1-iu)} \tag{166}$$

only by the periodic function $\Phi(u)$ and therefore it can be considered also as a Baxter function. This additional factor can be included in the definition of a new pseudo-vacuum state. Really this pseudo-vacuum state can be considered as the additional factor for the wave function in the Baxter-Sklyanin representation providing correct hermicity properties of the hamiltonian and integrals of motion in this representation[2] (see also ref. [14]).

5 Conclusion

In this review article we have outlined the role of Mandelstam-cut contributions in the remainder functions for the BDS amplitudes. Particular emphasis has been given to the integrability of the Hamiltonian which describes the energy spectrum of the states of n reggeized gluons. These cut contributions appear in multi-Regge kinematics in special physical regions, where some energies are negative. For the cut corresponding to the composite states of n reggeized gluons the number of external particles should be $k \geq 2+2n$. The wave functions of these states in the adjoint representation satisfy the BFKL-like equations, which have the property of holomorphic factorization and are integrable in LLA. The corresponding holomorphic hamiltonian coincides with the local hamiltonian for an integrable open Heisenberg spin model. The Baxter equation for this model is reduced to a simple recurrence relation and can be solved in terms of the product of the Γ-functions. We constructed the wave functions of composite states of 2 and 3 gluons explicitly.

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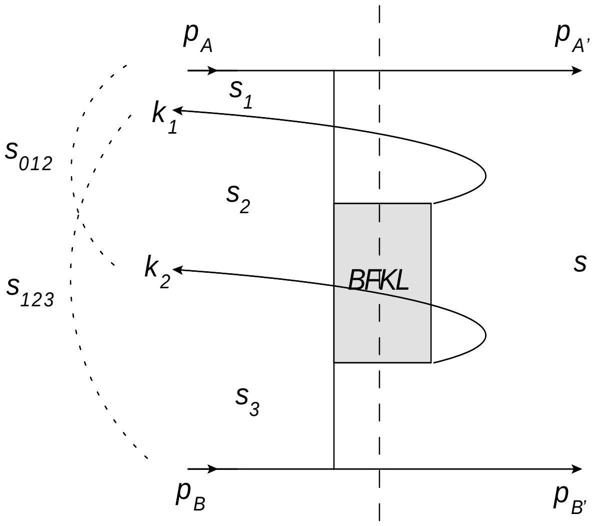

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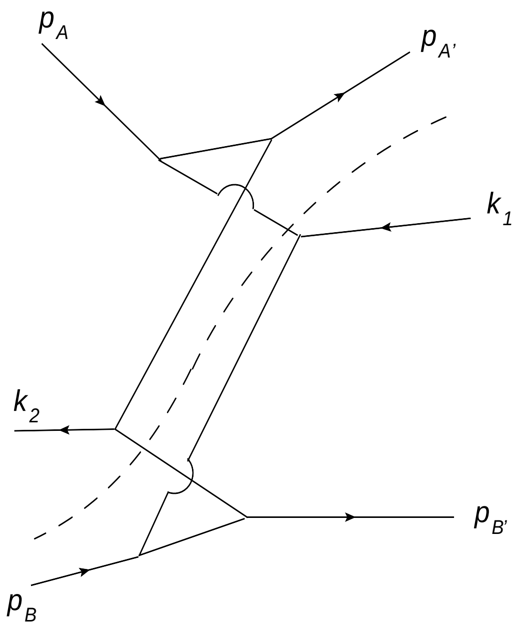

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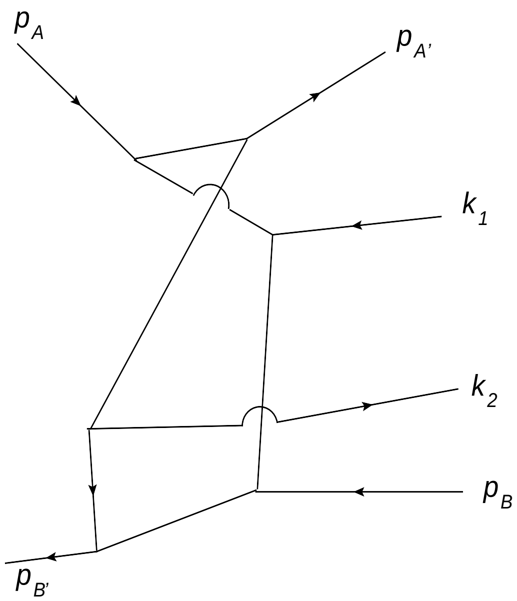
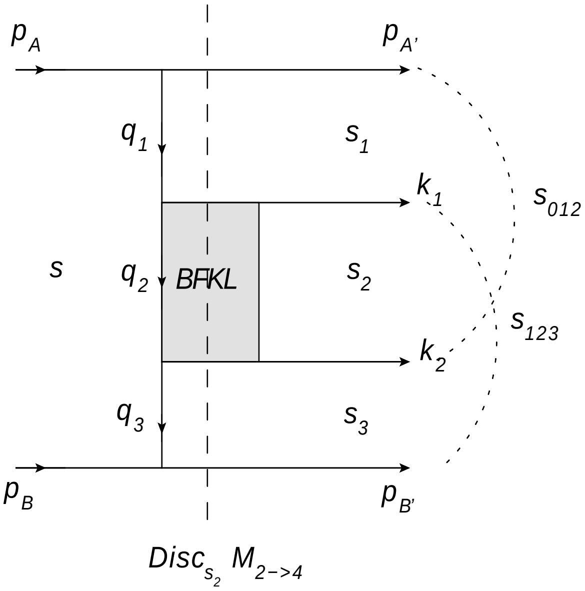
k_2
$p_{B'}$
p_B



p_A
$p_{A'}$
k_1
k_2
p_B
$p_{B'}$



p_A
$p_{A'}$
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k_1
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s
q_2
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s_3
p_B
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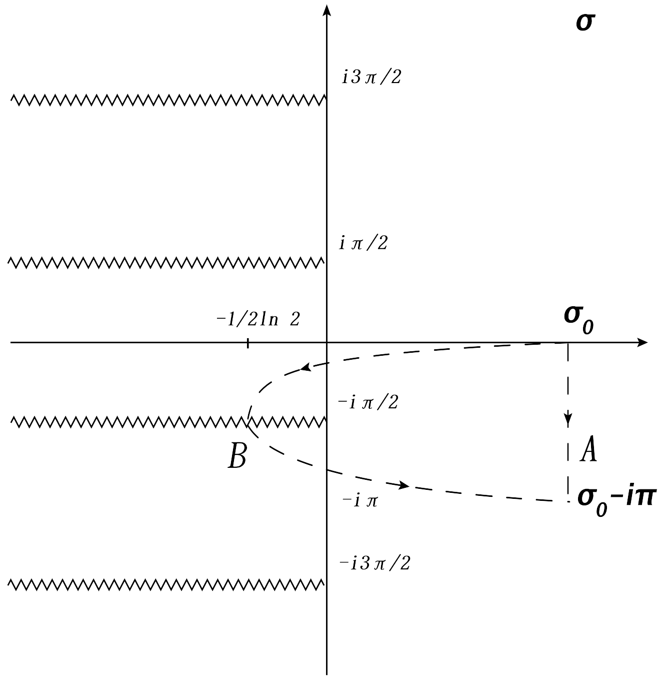

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B
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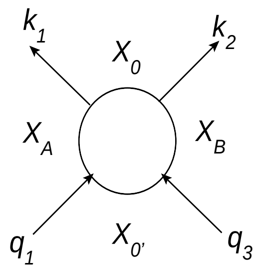

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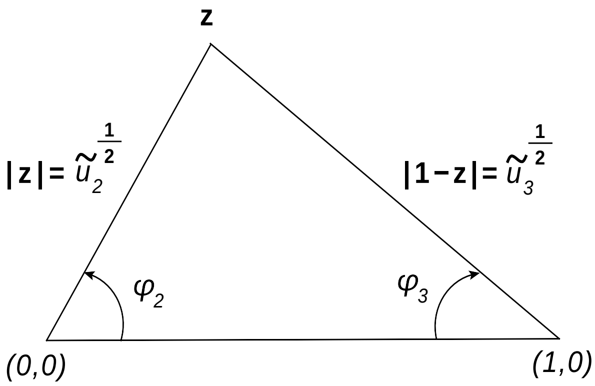
q1
q3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φ_3
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(1,0)