

The Logarithmic Contributions to the $O(\alpha_s^3)$ Asymptotic Massive Wilson Coefficients and Operator Matrix Elements in Deeply Inelastic Scattering

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Abstract

We calculate the logarithmic contributions to the massive Wilson coefficients for deep-inelastic scattering in the asymptotic region $Q^2 \gg m^2$ to 3-loop order in the fixed-flavor number scheme and present the corresponding expressions for the massive operator matrix elements needed in the variable flavor number scheme. Explicit expressions are given both in Mellin- N space and z -space.

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1 Introduction

The heavy flavor corrections to deep-inelastic structure functions amount to sizeable contributions, in particular in the region of small values of the Bjorken variable x . Starting from lower values of the virtuality, over a rather wide kinematic range, their scaling violations are very different from those of the massless contributions. Currently the heavy flavor corrections are known in semi-analytic form to 2-loop (NLO) order [1]. The present accuracy of the deep-inelastic data reaches the order of 1% [2]. It therefore requires the next-to-next-to-leading order (NNLO) corrections for precision determinations of both the strong coupling constant $\alpha_s(M_Z^2)$ [3] and the parton distribution functions (PDFs) [4, 5], as well as the detailed understanding of the heavy flavor production cross sections in lepton–nucleon scattering [6]. The precise knowledge of these quantities is also of central importance for the interpretation of the physics results at the Large Hadron Collider, LHC, [7].

In the kinematic region at HERA, where the twist-2 contributions to the deep-inelastic scattering (DIS) dominate cf. [8]², i.e. $Q^2/m^2 \gtrsim 10$, with $m = m_c$ the charm quark mass, it has been proven in Ref. [10] that the heavy flavor Wilson coefficients factorize into massive operator matrix elements (OMEs) and the massless Wilson coefficients. The massless Wilson coefficients for the structure function $F_2(x, Q^2)$ are known to 3-loop order [11]. In the region $Q^2 \gg m^2$, where $Q^2 = -q^2$, with q the space-like 4-momentum transfer and m the heavy quark mass, the power corrections $O((m^2/Q^2)^k)$, $k \geq 1$ to the heavy quark structure functions become very small.

In Ref. [12] a series of fixed Mellin moments N up to $N = 10, \dots, 14$, depending on the respective transition, has been calculated for all the OMEs at 3-loop order³. Also the moments of the transition coefficients needed in the variable flavor scheme (VFNS) have been calculated. Here, the massive OMEs for given total spin N were mapped onto massive tadpoles which have been computed using MATAD [14].

In the present paper, we calculate the logarithmic contributions to the unpolarized massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$ to 3-loop order and the massive OMEs needed in the VFNS. These include the logarithmic terms $\log(Q^2/m^2)$. In the following, we set the factorization and renormalization scales equal $\mu_F = \mu_R \equiv \mu$ and exhibit the $\log(m^2/\mu^2)$ dependence on the Wilson coefficients, besides their dependence on the virtuality Q^2 . The logarithmic contributions are determined by the lower order massive OMEs [15–19], the mass- and coupling constant renormalization constants, and the anomalous dimensions [20, 21], as has been worked out in Ref. [12]. For the structure function $F_L(x, Q^2)$ the asymptotic heavy flavor Wilson coefficients at $O(\alpha_s^3)$ were calculated in [22]. They are also presented here, for inclusive hadronic final states. In this case the corrections, however, become effective only at much higher scales of Q^2 [10] compared to the case of $F_2(x, Q^2)$. We first choose the fixed flavor number scheme to express the heavy flavor contributions to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$. This scheme has to be considered as the genuine scheme in quantum field theoretic calculations since the initial states, the twist-2 *massless* partons can, at least to a good approximation, be considered as LSZ-states. The representations in the VFNS can be obtained using the respective transition coefficients within the appropriate regions, where one single heavy quark flavor becomes effectively massless. Here, appropriate matching scales have to be applied, which vary in dependence on the observable considered, cf. [23].

Two of the OMEs, $A_{qg,Q}^{\text{PS}}(N)$ and $A_{qg,Q}(N)$, have already been calculated completely including the constant contribution in Ref. [24]. They and the corresponding massive Wilson coefficients

²For higher order corrections to the gluonic contributions in the threshold region, cf. [9].

³For the corresponding contributions in case of transversity see [13].

contribute first at 2- and 3-loop order, respectively. For these quantities we also derive numerical results. The quantities being presented in the present paper derive from OMEs which were computed in terms of generalized hypergeometric functions [25] and sums thereof, prior to the expansion in the dimensional variable $\varepsilon = D - 4$, cf. [26–28]. Finally, they are represented in terms of nested sums over products of hypergeometric terms and harmonic sums, which can be calculated using modern summation techniques [29–33]. They are based on a refined difference field of [34] and generalize the summation paradigms presented in [35] to multi-summation. The results of this computation can be expressed in terms of nested harmonic sums [36, 37]. The corresponding representations in z -space are obtained in terms of harmonic polylogarithms [38]. Here, the variable z denotes the partonic momentum fraction. The results in Mellin N -space can be continued to complex values of N as has been described in Refs. [26, 39].

It is the aim of the present paper to provide a detailed documentation of formulae both in N - and z -space for all logarithmic contributions to the heavy flavor Wilson coefficients of the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ and the massive OMEs needed in the variable flavor number scheme up to $O(\alpha_s^3)$. Here, we refer to a minimal representation, i.e. we use all the algebraic relations between the harmonic sums and the harmonic polylogarithms, respectively, leading to a minimal number of basic functions. Based on the known Mellin moments [12] we also perform numerical comparisons between the different contributions to the Wilson coefficients and massive OMEs at $O(\alpha_s^3)$ referring to the parton distributions [5].

The paper is organized as follows. In Section 2, we summarize the basic formalism. The Wilson coefficients $L_{q,2}^{\text{PS}}$ and $L_{g,2}^{\text{S}}$ are discussed in Section 3. As they are known in complete form we also present numerical results. In Section 4, the logarithmic contributions to the Wilson coefficients $H_{q,2}^{\text{PS}}$ and $H_{g,2}^{\text{S}}$ ⁴ are derived. The corresponding Wilson coefficients for the longitudinal structure function $F_L(x, Q^2)$ in the asymptotic region are presented in Section 5. In Section 6, we compare the different loop contributions to the massive Wilson coefficients and OMEs for a series of Mellin moments in dependence on the virtuality Q^2 . Section 7 contains the conclusions. In Appendix A the massive OMEs needed in the VFNS are given in Mellin N -space. The asymptotic heavy flavor Wilson coefficients contributing to the structure function $F_2(x, Q^2)$ are presented in z -space in Appendix B, retaining all contributions except for the 3-loop constant part of the unrenormalized OMEs $a_{ij}^{(3)}$ being not yet known. Likewise, in Appendix C and D, the asymptotic heavy flavor Wilson coefficients for the structure function $F_L(x, Q^2)$ and the massive OMEs are given in z -space.

2 The heavy flavor Wilson coefficients in the asymptotic region

We consider the heavy flavor contributions to the inclusive unpolarized structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in deep-inelastic scattering, cf. [41, 42], in case of single electro-weak gauge-boson exchange at large virtualities Q^2 . At higher orders in the strong coupling constant these corrections receive both contributions from massive and massless partons in the hadronic final state, which is summed over completely. In the latter case, the heavy flavor corrections are also due to virtual contributions. We consider the situation in which the contributions to the twist-2 operators dominate in the Bjorken limit. Here, no transverse momentum effects of the initial state contribute. In the present paper, we consider only heavy flavor contributions

⁴The expressions for the non-singlet Wilson-coefficient, are presented elsewhere together with the OME for transversity [40].

due to N_F massless and one massive flavor of mass m .⁵ The Wilson coefficients are calculable perturbatively and are denoted by

$$C_{i,(2,L)}^{\mathbf{S,PS,NS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right). \quad (1)$$

Here, x denotes the Bjorken variable, the index i refers to the respective initial state on-shell parton $i = q, g$ being a quark or gluon, and $\mathbf{S, PS, NS}$ label the flavor singlet, pure-singlet and non-singlet contributions, respectively. In the twist-2 approximation the Bjorken variable x and the parton momentum fraction z are identical. Representations in momentum fraction space are therefore also called z -space representation in what follows.

The massless flavor contributions in (1) may be identified and separated in the Wilson coefficients into a purely light part $C_{i,(2,L)}$, and a heavy part by :

$$\begin{aligned} C_{i,(2,L)}^{\mathbf{S,PS,NS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) &= C_{i,(2,L)}^{\mathbf{S,PS,NS}}\left(x, N_F, \frac{Q^2}{\mu^2}\right) \\ &+ H_{i,(2,L)}^{\mathbf{S,PS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) + L_{i,(2,L)}^{\mathbf{S,PS,NS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right). \end{aligned} \quad (2)$$

The heavy flavor Wilson coefficients are defined by $L_{i,j}$ and $H_{i,j}$, depending on whether the exchanged electro-weak gauge boson couples to a light (L) or heavy (H) quark line. From this it follows that the light flavor Wilson coefficients $C_{i,j}$ depend on N_F light flavors only, whereas $H_{i,j}$ and $L_{i,j}$ may contain light flavors in addition to the heavy quark, indicated by the argument $N_F + 1$. The perturbative series of the heavy flavor Wilson coefficients read

$$H_{g,(2,L)}^{\mathbf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=1}^{\infty} a_s^i H_{g,(2,L)}^{(i),\mathbf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right), \quad (3)$$

$$H_{q,(2,L)}^{\mathbf{PS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=2}^{\infty} a_s^i H_{q,(2,L)}^{(i),\mathbf{PS}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right), \quad (4)$$

$$L_{g,(2,L)}^{\mathbf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=2}^{\infty} a_s^i L_{g,(2,L)}^{(i),\mathbf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right), \quad (5)$$

$$L_{q,(2,L)}^{\mathbf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_{i=2}^{\infty} a_s^i L_{q,(2,L)}^{(i),\mathbf{S}}\left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right). \quad (6)$$

Here, we defined $a_s = \alpha_s/(4\pi)$. At leading order, only the term $H_{g,(2,L)}$ contributes via the photon-gluon fusion process, [44–49],

$$\gamma^* + g \rightarrow Q + \bar{Q}. \quad (7)$$

At $O(a_s^2)$, the terms $H_{q,(2,L)}^{\mathbf{PS}}$, $L_{q,(2,L)}^{\mathbf{S}}$ and $L_{g,(2,L)}^{\mathbf{S}}$ contribute as well. They result from the processes

$$\gamma^* + q(\bar{q}) \rightarrow q(\bar{q}) + X, \quad (8)$$

$$\gamma^* + g \rightarrow q(\bar{q}) + X, \quad (9)$$

⁵At 3-loop order there are also contributions by graphs carrying heavy quark lines of different mass. These are dealt with elsewhere [43].

where X may contain heavy flavor contributions. $L_{q,(2,L)}^S$ can be split into the flavor non-singlet and pure-singlet contributions

$$L_{q,(2,L)}^S = L_{q,(2,L)}^{\text{NS}} + L_{q,(2,L)}^{\text{PS}}, \quad (10)$$

and at $O(\alpha_s^2)$ only the non-singlet term contributes. The pure-singlet term emerges at 3-loop order.

The heavy quark contribution to the structure functions $F_{(2,L)}(x, Q^2)$ for one heavy quark of mass m and N_F light flavors is then given by, cf. [15], in case of pure photon exchange⁶

$$\begin{aligned} \frac{1}{x} F_{(2,L)}^{Q\bar{Q}}(x, N_F + 1, Q^2, m^2) = & \\ & \sum_{k=1}^{N_F} e_k^2 \left\{ L_{q,(2,L)}^{\text{NS}} \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes [f_k(x, \mu^2, N_F) + f_{\bar{k}}(x, \mu^2, N_F)] \right. \\ & + \frac{1}{N_F} L_{q,(2,L)}^{\text{PS}} \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) \\ & \left. + \frac{1}{N_F} L_{g,(2,L)}^S \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \right\} \\ + e_Q^2 & \left[H_{q,(2,L)}^{\text{PS}} \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes \Sigma(x, \mu^2, N_F) \right. \\ & \left. + H_{g,(2,L)}^S \left(x, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) \otimes G(x, \mu^2, N_F) \right], \quad (11) \end{aligned}$$

The meaning of the argument $(N_F + 1)$ in Eqs. (11) in the massive Wilson coefficients shall be interpreted as N_F massless and one massive flavor. N_F denotes the number of massless flavors. The symbol \otimes denotes the Mellin convolution,⁷

$$[A \otimes B](x) = \int_0^1 \int_0^1 dx_1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2). \quad (12)$$

The charges of the light quarks are denoted by e_k and that of the heavy quark by e_Q . The scale μ^2 is the factorization scale, and $f_k, f_{\bar{k}}, \Sigma$ and G are the quark, anti-quark, flavor singlet and gluon distribution functions, with

$$\Sigma(x, \mu^2, N_F) = \sum_{k=1}^{N_F} [f_k(x, \mu^2, N_F) + f_{\bar{k}}(x, \mu^2, N_F)]. \quad (13)$$

An important part of the kinematic region in case of heavy flavor production in DIS is located at larger values of Q^2 , cf. e.g. [54, 55]. As has been shown in Ref. [10], the heavy flavor Wilson coefficients $H_{i,j}$, $L_{i,j}$ factorize in the limit $Q^2 \gg m^2$ into massive operator matrix elements A_{ki} and the massless Wilson coefficients $C_{i,j}$, if one heavy quark flavor and N_F light flavors are considered. The massive OMEs are process independent quantities and contain all the mass dependence except for the power corrections $\propto (m^2/Q^2)^k$, $k \geq 1$. The process dependence is implied by the massless Wilson coefficients. This allows the analytic calculation of the NLO

⁶For the heavy flavor corrections in case of W^\pm -boson exchange up to $O(\alpha_s^2)$ see [50–53].

⁷Note that the heavy flavor threshold in the limit $Q^2 \gg m^2$ is again x and not $x(1 + 4m^2/Q^2)$, which is the case retaining also power corrections.

heavy flavor Wilson coefficients, [10, 17]. Comparing these asymptotic expressions with the exact LO and NLO results obtained in Refs. [44–47, 49] and [1], respectively, one finds that this approximation becomes valid in case of $F_2^{Q\bar{Q}}$ for $Q^2/m^2 \gtrsim 10$. These scales are sufficiently low and match with the region analyzed in deeply inelastic scattering for precision measurements. In case of $F_L^{Q\bar{Q}}$, this approximation is only valid for $Q^2/m^2 \gtrsim 800$, [10]. For the latter case, the 3-loop corrections were calculated in Ref. [22]. This difference is due to the emergence of terms $\propto (m^2/Q^2) \ln(m^2/Q^2)$, which only vanish slowly in the limit $Q^2/m^2 \rightarrow \infty$.

In order to derive the factorization formula, one considers the inclusive Wilson coefficients $\mathcal{C}_{i,j}^{\text{S,PS,NS}}$, which have been defined in Eq. (1). After applying the light cone expansion (LCE) [56] to the partonic tensor, or the forward Compton amplitude, corresponding to the respective Wilson coefficients, one arrives at the factorization relation,

$$\mathcal{C}_{j,(2,L)}^{\text{S,PS,NS,asympt}} \left(N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2} \right) = \sum_i A_{ij}^{\text{S,PS,NS}} \left(N, N_F + 1, \frac{m^2}{\mu^2} \right) C_{i,(2,L)}^{\text{S,PS,NS}} \left(N, N_F + 1, \frac{Q^2}{\mu^2} \right) + O\left(\frac{m^2}{Q^2}\right). \quad (14)$$

Here, μ refers to the factorization scale between the heavy and light contributions in $\mathcal{C}_{j,i}$ and 'asympt' denotes the limit $Q^2 \gg m^2$. The $C_{i,j}$ are the light Wilson coefficients, cf. [11], taken at $N_F + 1$ flavors. This can be inferred from the fact that in the LCE the Wilson coefficients describe the singularities for very large values of Q^2 , which can not depend on the presence of a quark mass. The mass dependence is given by the OMEs A_{ij} , between partonic states. Eq. (14) accounts for all mass effects but corrections which are power suppressed, $(m^2/Q^2)^k$, $k \geq 1$. This factorization is only valid if the heavy quark coefficient functions are defined in such a way that all radiative corrections containing heavy quark loops are included. Otherwise, (14) would not show the correct asymptotic Q^2 -behavior, [15, 19]. An equivalent way of describing Eq. (14) is obtained by considering the calculation of the massless Wilson coefficients. Here, the initial state collinear singularities are given by evaluating the massless OMEs between off-shell partons, leading to transition functions Γ_{ij} . The Γ_{ij} are given in terms of the anomalous dimensions of the twist-2 operators and transfer the initial state singularities to the bare parton-densities due to mass factorization, cf. e.g. [10, 15]. In the case at hand, something similar happens: The initial state collinear singularities are transferred to the parton densities except for those which are regulated by the quark mass and described by the OMEs. Instead of absorbing these terms into the parton densities as well, they are used to reconstruct the asymptotic behavior of the heavy flavor Wilson coefficients. Here,

$$A_{ij}^{\text{S,NS}} \left(N, N_F + 1, \frac{m^2}{\mu^2} \right) = \langle j | O_i^{\text{S,NS}} | j \rangle = \delta_{ij} + \sum_{i=k}^{\infty} a_s^k A_{ij}^{(k),\text{S,NS}} \quad (15)$$

are the operator matrix elements of the local twist-2 operators between on-shell partonic states $|j\rangle$, $j = q, g$.

Let us now derive the explicit expressions for the massive Wilson coefficients in the asymptotic region. One may split Eq. (14) into parts by considering the different N_F contributions. We define

$$\tilde{f}(N_F) \equiv \frac{f(N_F)}{N_F}. \quad (16)$$

This is necessary in order to separate the different types of contributions in Eq. (11), weighted by the electric charges of the light and heavy flavors, respectively. Since we would like to derive

the heavy flavor part, we define as well for later use

$$\hat{f}(N_F) \equiv f(N_F + 1) - f(N_F) , \quad (17)$$

where $\hat{f}(N_F) \equiv [\widehat{f}(N_F)]$. The following Eqs. (18)–(22) are the same as Eqs. (2.31)–(2.35) in Ref. [15]. We present these terms here again, however, since Ref. [15] contains a few inconsistencies regarding the \tilde{f} -description. Contrary to the latter reference, the argument corresponding to the number of flavors stands for all flavors, light or heavy. The separation for the NS-term is obtained by

$$C_{q,(2,L)}^{\text{NS}}\left(N, N_F, \frac{Q^2}{\mu^2}\right) + L_{q,(2,L)}^{\text{NS}}\left(N, N_F + 1, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = A_{qq,Q}^{\text{NS}}\left(N, N_F + 1, \frac{m^2}{\mu^2}\right) C_{q,(2,L)}^{\text{NS}}\left(N, N_F + 1, \frac{Q^2}{\mu^2}\right) . \quad (18)$$

Here and in the following, we omit the index "asyp" to denote the asymptotic heavy flavor Wilson coefficients. For the remaining terms, we suppress the arguments N , Q^2/μ^2 and m^2/μ^2 for brevity, all of which can be inferred from Eqs. (2, 14). Additionally, we will suppress from now on the index S and label only the NS and PS terms explicitly. The contributions to $L_{i,j}$ read

$$C_{q,(2,L)}^{\text{PS}}(N_F) + L_{q,(2,L)}^{\text{PS}}(N_F + 1) = \left[A_{qq,Q}^{\text{NS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1) + A_{Qq}^{\text{PS}}(N_F + 1) \right] \times N_F \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1) C_{q,(2,L)}^{\text{NS}}(N_F + 1) + A_{gq,Q}(N_F + 1) N_F \tilde{C}_{g,(2,L)}(N_F + 1) , \quad (19)$$

$$C_{g,(2,L)}(N_F) + L_{g,(2,L)}(N_F + 1) = A_{gg,Q}(N_F + 1) N_F \tilde{C}_{g,(2,L)}(N_F + 1) + A_{qq,Q}(N_F + 1) C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \left[A_{qq,Q}(N_F + 1) + A_{Qg}(N_F + 1) \right] N_F \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) . \quad (20)$$

The terms $H_{i,j}$ are given by

$$H_{q,(2,L)}^{\text{PS}}(N_F + 1) = A_{Qq}^{\text{PS}}(N_F + 1) \left[C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) \right] + \left[A_{qq,Q}^{\text{NS}}(N_F + 1) + A_{qq,Q}^{\text{PS}}(N_F + 1) \right] \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) + A_{gq,Q}(N_F + 1) \tilde{C}_{g,(2,L)}(N_F + 1) , \quad (21)$$

$$H_{g,(2,L)}(N_F + 1) = A_{gg,Q}(N_F + 1) \tilde{C}_{g,(2,L)}(N_F + 1) + A_{qq,Q}(N_F + 1) \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) + A_{Qg}(N_F + 1) \left[C_{q,(2,L)}^{\text{NS}}(N_F + 1) + \tilde{C}_{q,(2,L)}^{\text{PS}}(N_F + 1) \right] . \quad (22)$$

Expanding the above relations up to $O(a_s^3)$, we obtain, using Eqs. (16, 17), the heavy flavor Wilson coefficients in the asymptotic limit, cf. [12] :

$$L_{q,(2,L)}^{\text{NS}}(N_F + 1) = a_s^2 \left[A_{qq,Q}^{(2),\text{NS}}(N_F + 1) \delta_2 + \hat{C}_{q,(2,L)}^{(2),\text{NS}}(N_F) \right] + a_s^3 \left[A_{qq,Q}^{(3),\text{NS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2),\text{NS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right]$$

$$+ \hat{C}_{q,(2,L)}^{(3),\text{NS}}(N_F) \Big] , \quad (23)$$

$$L_{q,(2,L)}^{\text{PS}}(N_F + 1) = a_s^3 \left[A_{qq,Q}^{(3),\text{PS}}(N_F + 1) \delta_2 + A_{qq,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + N_F \hat{C}_{q,(2,L)}^{(3),\text{PS}}(N_F) \right] , \quad (24)$$

$$L_{g,(2,L)}^{\text{S}}(N_F + 1) = a_s^2 A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + a_s^3 \left[A_{gg,Q}^{(3)}(N_F + 1) \delta_2 + A_{gg,Q}^{(1)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) N_F \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) N_F \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) + N_F \hat{C}_{g,(2,L)}^{(3)}(N_F) \right] , \quad (25)$$

$$H_{q,(2,L)}^{\text{PS}}(N_F + 1) = a_s^2 \left[A_{Qq}^{(2),\text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right] + a_s^3 \left[A_{Qq}^{(3),\text{PS}}(N_F + 1) \delta_2 + \tilde{C}_{q,(2,L)}^{(3),\text{PS}}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qq}^{(2),\text{PS}}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) \right] , \quad (26)$$

$$H_{g,(2,L)}^{\text{S}}(N_F + 1) = a_s \left[A_{Qg}^{(1)}(N_F + 1) \delta_2 + \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) \right] + a_s^2 \left[A_{Qg}^{(2)}(N_F + 1) \delta_2 + A_{Qg}^{(1)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) \right] + a_s^3 \left[A_{Qg}^{(3)}(N_F + 1) \delta_2 + A_{Qg}^{(2)}(N_F + 1) C_{q,(2,L)}^{(1),\text{NS}}(N_F + 1) + A_{gg,Q}^{(2)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(1)}(N_F + 1) + A_{Qg}^{(1)}(N_F + 1) \left\{ C_{q,(2,L)}^{(2),\text{NS}}(N_F + 1) + \tilde{C}_{q,(2,L)}^{(2),\text{PS}}(N_F + 1) \right\} + A_{gg,Q}^{(1)}(N_F + 1) \tilde{C}_{g,(2,L)}^{(2)}(N_F + 1) + \tilde{C}_{g,(2,L)}^{(3)}(N_F + 1) \right] , \quad (27)$$

with $\delta_2 = 1$ for F_2 and $\delta_2 = 0$ for F_L . Again, the argument $(N_F + 1)$ in the massive OMEs signals that these functions depend on N_F massless and one massive flavor, while the setting of N_F in the massless Wilson coefficients is a functional one. The above equations include radiative corrections due to heavy quark loops to the Wilson coefficients. Therefore, in order to compare e.g. with the calculation in Refs. [1], these terms still have to be subtracted. Since the light flavor Wilson coefficients were calculated in the $\overline{\text{MS}}$ -scheme, the **same** scheme has to be used for the massive OMEs. It should also be thoroughly used for renormalization to derive consistent results in QCD analyses of deep-inelastic scattering data and to be able to compare to other analyses of hard scattering data directly. This requests special attendance w.r.t. the choice of the scheme in which a_s is defined, cf. [12].

The renormalized massive OMEs depend on the ratio m^2/μ^2 , while the scale ratio in the massless Wilson coefficients is μ^2/Q^2 . The latter are pure functions of the momentum fraction z , or the Mellin variable N , if one sets $\mu^2 = Q^2$. The mass dependence on the heavy flavor Wilson coefficients in the asymptotic region derives from the unrenormalized massive OMEs

$$\hat{A}_{ij}^{(3)}(\varepsilon) = \frac{1}{\varepsilon^3} \hat{a}_{ij}^{(3),3} + \frac{1}{\varepsilon^2} \hat{a}_{ij}^{(3),2} + \frac{1}{\varepsilon} \hat{a}_{ij}^{(3),1} + \hat{a}_{ij}^{(3),0} , \quad (28)$$

applying mass, coupling constant, and operator-renormalization, as well as mass factorization,

cf. Ref. [12]. The renormalized massive OMEs obey then the general structure

$$A_{ij}^{(3)}\left(\frac{m^2}{Q^2}\right) = a_{ij}^{(3),3} \ln^3\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),2} \ln^2\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),1} \ln\left(\frac{m^2}{Q^2}\right) + a_{ij}^{(3),0}. \quad (29)$$

The subsequent calculations will be performed in the $\overline{\text{MS}}$ scheme for the coupling constant and the on-shell scheme for the heavy quark mass m . The transition to the scheme in which m is renormalized in the $\overline{\text{MS}}$ -scheme is described in Ref. [12]. The strong coupling constant is obtained as the *perturbative* solution of the equation

$$\frac{da_s(\mu^2)}{d\ln(\mu^2)} = - \sum_{l=0}^{\infty} \beta_l a_s^{l+2}(\mu^2) \quad (30)$$

to 3-loop order, where β_k are the expansion coefficients of the QCD β -function and μ^2 denotes the renormalization scale. For simplicity we identify the factorization (μ_F) and renormalization (μ_R) scales from now on. In the subsequent sections we present explicit expressions of the asymptotic heavy flavor Wilson coefficients in Mellin- N space. They depend on the logarithms

$$L_Q = \ln\left(\frac{Q^2}{\mu^2}\right) \quad \text{and} \quad L_M = \ln\left(\frac{m^2}{\mu^2}\right), \quad (31)$$

where $\mu \equiv \mu_F = \mu_R$.

Besides the Wilson coefficients (23–27) the massive OMEs are important themselves to establish the matching conditions in the variable flavor number scheme in describing the process of a single massive quark becoming massless⁸ at large enough scales μ^2 , [12, 15]. Here, the PDFs for $N_F + 1$ massless quarks are related to the former N_F massless quarks process independently. The corresponding relations to 3-loop order read, cf. also [15]⁹ :

$$\begin{aligned} f_k(N_F + 1, \mu^2) + f_{\bar{k}}(N_F + 1, \mu^2) &= A_{qq,Q}^{\text{NS}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes [f_k(N_F, \mu^2) + f_{\bar{k}}(N_F, \mu^2)] \\ &+ \tilde{A}_{qq,Q}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Sigma(N_F, \mu^2) \\ &+ \tilde{A}_{qq,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes G(N_F, \mu^2) \end{aligned} \quad (32)$$

$$f_{Q+\bar{Q}}(N_F + 1, \mu^2) = A_{Qq}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Sigma(N_F, \mu^2) + A_{Qg}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes G(N_F, \mu^2) \quad (33)$$

$$G(N_F + 1, \mu^2) = A_{gq,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes \Sigma(N_F, \mu^2) + A_{gg,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \otimes G(N_F, \mu^2). \quad (34)$$

$$\begin{aligned} \Sigma(N_F + 1, \mu^2) &= \left[A_{qq,Q}^{\text{NS}}\left(N_F, \frac{\mu^2}{m^2}\right) + N_F \tilde{A}_{qq,Q}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) + A_{Qq}^{\text{PS}}\left(N_F, \frac{\mu^2}{m^2}\right) \right] \\ &\otimes \Sigma(N_F, \mu^2) \\ &+ \left[N_F \tilde{A}_{gq,Q}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) + A_{Qg}^{\text{S}}\left(N_F, \frac{\mu^2}{m^2}\right) \right] \otimes G(N_F, \mu^2) \end{aligned} \quad (35)$$

⁸For the VFNS in case of both the bottom and charm quarks transmuting into massless states, see [43].

⁹Here, we have corrected some typographical errors in (33–35) in [15], in accordance with the appendix of Ref. [15].

Here, the N_F -dependence of the OMEs is understood as functional and μ^2 denotes the matching scale, which for the heavy-to-light transitions is normally much larger than mass scale m^2 , [23]. We will present the corresponding OMEs in Appendix A. The results of the calculations being presented in the subsequent sections have been obtained making mutual use of the packages `HarmonicSums.m` [58] and `Sigma.m` [29].

3 The Wilson Coefficients $L_{q,2}^{\text{PS}}$ and $L_{g,2}^{\text{S}}$

The OMEs for these Wilson coefficients have been calculated in [24]. They contribute for the first time at 3- and 2-loop order, respectively, and stem from processes in which the virtual electro-weak gauge boson couples to a massless quark. As a shorthand notation we also define the function

$$\tilde{\gamma}_{qg}^0 = -4 \frac{N^2 + N + 2}{N(N+1)(N+2)} \quad (36)$$

denoting the kinetic part of the leading order anomalous dimensions separating off the corresponding color factor.

In Mellin- N space the Wilson coefficient $L_{q,2}^{\text{PS}}$ reads :

$$\begin{aligned} L_{q,2}^{\text{PS}} = & \frac{1}{2} [1 + (-1)^N] \\ & \times a_s^3 \left\{ C_F N_F T_F^2 \left[-\frac{32P_4 L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_Q \left[\frac{64P_6}{27(N-1)N^4(N+1)^4(N+2)^3} \right. \right. \right. \\ & - \frac{256P_1(-1)^N}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{2(\tilde{\gamma}_{qg}^0)^2(N+2)L_M^2}{3(N-1)} \\ & + \left. \left. \left. \left[\frac{64(N^2+N+2)(8N^3+13N^2+27N+16)}{9(N-1)N^2(N+1)^3(N+2)} - \frac{64(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] L_M \right. \right. \\ & + \left. \left. \left. \left. \frac{512S_{-2}}{3(N-1)N(N+1)(N+2)} \right] - \frac{32P_4 L_M^2}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \right. \\ & + \left. \left. \left. \left. \left[-\frac{32P_7}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64P_2 S_1}{3(N-1)N^3(N+1)^3(N+2)^2} \right. \right. \right. \\ & + \left. \left. \left. \left. \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \frac{32}{3} (S_1^2 - S_2) \right] L_M \right. \right. \\ & - \frac{32P_9}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{32P_8 S_1}{81(N-1)N^4(N+1)^4(N+2)^3} \\ & - \frac{16P_3 S_1^2}{27(N-1)N^3(N+1)^3(N+2)^2} - \frac{16P_5 S_2}{27(N-1)N^3(N+1)^3(N+2)^2} \\ & + \frac{32L_Q^3(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} \\ & + \left. \left. \left. \left. \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{64}{27} S_1^3 + \frac{32}{9} S_2 S_1 + \frac{160S_3}{27} + \frac{256\zeta_3}{9} \right] \right. \right. \right. \\ & \left. \left. \left. \left. + N_F \hat{C}_{2,q}^{\text{PS},(3)}(N_F) \right] \right\}, \quad (37) \end{aligned}$$

with the polynomials

$$P_1 = 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \quad (38)$$

$$P_2 = N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \quad (39)$$

$$P_3 = N^7 - 37N^6 - 248N^5 - 799N^4 - 1183N^3 - 970N^2 - 580N - 168 \quad (40)$$

$$P_4 = 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \quad (41)$$

$$P_5 = 49N^7 + 185N^6 + 340N^5 + 287N^4 + 65N^3 + 62N^2 - 196N - 168 \quad (42)$$

$$P_6 = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 + 1864N^2 + 1248N + 432 \quad (43)$$

$$P_7 = 143N^{10} + 838N^9 + 1995N^8 + 1833N^7 - 1609N^6 - 5961N^5 - 7503N^4 - 6928N^3 - 4024N^2 - 816N + 144 \quad (44)$$

$$P_8 = 176N^{10} + 973N^9 + 1824N^8 - 948N^7 - 10192N^6 - 19173N^5 - 20424N^4 - 16036N^3 - 7816N^2 - 1248N + 288 \quad (45)$$

$$P_9 = 1717N^{13} + 16037N^{12} + 66983N^{11} + 161797N^{10} + 241447N^9 + 216696N^8 + 86480N^7 - 67484N^6 - 170003N^5 - 165454N^4 - 81976N^3 - 15792N^2 - 1008N - 864 . \quad (46)$$

For the massless 3-loop Wilson coefficients $C_{i,j}^k$ we refer to Ref. [11]. Here and in the following, their expression will be kept symbolically. The corresponding z -space expressions are given in Appendix B.

Likewise the Wilson coefficient $L_{g,2}^S$ is given by :

$$\begin{aligned} L_{g,2}^S = & \frac{1}{2} [1 + (-1)^N] \left\{ a_s^2 T_F^2 N_F \left\{ L_M \left[\frac{4}{3} \tilde{\gamma}_{gg}^0 S_1 - \frac{16(N^3 - 4N^2 - N - 2)}{3N^2(N+1)(N+2)} \right] \right. \right. \\ & \left. \left. - \frac{4}{3} \tilde{\gamma}_{gg}^0 L_Q L_M \right\} + a_s^3 \left\{ N_F T_F^3 \left[L_M^2 \left[\frac{16}{9} \tilde{\gamma}_{gg}^0 S_1 - \frac{64(N^3 - 4N^2 - N - 2)}{9N^2(N+1)(N+2)} \right] - \frac{16}{9} \tilde{\gamma}_{gg}^0 L_Q L_M^2 \right] \right. \right. \\ & \left. \left. + C_{AN_F T_F^2} \left[\left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{9} \tilde{\gamma}_{gg}^0 S_1 \right] L_Q^3 + \left[-\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right. \right. \right. \\ & \left. \left. + \frac{8P_{25}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)^2(N+2)} \right. \right. \\ & \left. \left. + L_M \left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{3} \tilde{\gamma}_{gg}^0 S_1 \right] + \tilde{\gamma}_{gg}^0 \left[-\frac{4}{3} S_1^2 + \frac{4S_2}{3} + \frac{8}{3} S_{-2} \right] \right] L_Q^2 \right. \\ & \left. + \left[-\frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1^2}{9(N-1)N(N+1)^2(N+2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} \right. \right. \\ & \left. \left. - \frac{32P_{24}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^N P_{18}}{9(N-1)N^2(N+1)^4(N+2)^4} \right. \right. \\ & \left. \left. - \frac{16P_{32}}{27(N-1)N^3(N+1)^4(N+2)^4} + L_M^2 \left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)^2} \right. \right. \right. \\ & \left. \left. + \frac{8}{3} \tilde{\gamma}_{gg}^0 S_1 \right] + \frac{32(8N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N-1)N(N+1)(N+2)^2} + \frac{128(N^2 + N - 1)S_3}{9N(N+1)(N+2)} \right. \\ & \left. + \frac{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} + L_M \left[\frac{32P_{26}}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \right. \\ & \left. \left. - \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} - \frac{64(2N-1)(N^3 + 9N^2 + 7N + 7)S_1}{9(N-1)N(N+1)^2(N+2)} \right. \right. \end{aligned}$$

$$\begin{aligned}
& +\tilde{\gamma}_{qg}^0 \left[-\frac{8}{3}S_1^2 + \frac{8S_2}{3} + \frac{16}{3}S_{-2} \right] - \frac{128(N^2 + N + 3)S_{-3}}{3N(N+1)(N+2)} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{9}S_1^3 - 8S_2S_1 + \frac{32}{3}S_{2,1} \right] \\
& + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1) \left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3 \right]}{N(N+1)} \Big] L_Q + \frac{16P_{12}S_1^2}{81N(N+1)^3(N+2)^3} \\
& + \frac{8P_{39}}{243(N-1)N^5(N+1)^5(N+2)^5} + \frac{512}{9} \frac{(N^2 + N + 1)(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)^2} \zeta_3 \\
& + \frac{8P_{36}S_1}{243(N-1)N^4(N+1)^4(N+2)^4} + L_M^3 \left[-\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. - \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1 \right] - \frac{16P_{13}S_2}{81N(N+1)^3(N+2)^3} + \frac{64(5N^4 + 38N^3 + 59N^2 + 31N + 20)S_3}{81N(N+1)^2(N+2)^2} \\
& - \frac{32(121N^3 + 293N^2 + 414N + 224)S_{-2}}{81N(N+1)^2(N+2)} + L_M^2 \left[-\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right. \\
& + \frac{8P_{25}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0 \left[-\frac{4}{3}S_1^2 \right. \\
& \left. + \frac{4S_2}{3} + \frac{8}{3}S_{-2} \right] + \frac{128(5N^2 + 8N + 10)S_{-3}}{27N(N+1)(N+2)} \\
& + \frac{(5N^4 + 20N^3 + 41N^2 + 49N + 20) \left[\frac{32}{81}S_1^3 - \frac{32}{27}S_2S_1 + \frac{128}{27}S_{2,1} \right]}{N(N+1)^2(N+2)^2} \\
& + L_M \left[\frac{32(2N^5 + 21N^4 + 27N^3 + 11N^2 + 25N - 14)S_1^2}{9(N-1)N(N+1)^2(N+2)^2} + \frac{16P_{27}S_1}{27(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} - \frac{16}{3}\tilde{\gamma}_{qg}^0 S_2S_1 - \frac{64(-1)^N P_{14}}{9(N-1)N^2(N+1)^3(N+2)^4} \\
& + \frac{16P_{34}}{27(N-1)N^4(N+1)^4(N+2)^4} - \frac{32(2N^5 + 21N^4 + 51N^3 + 23N^2 - 11N - 14)S_2}{9(N-1)N(N+1)^2(N+2)^2} \\
& + \frac{64S_3}{3(N+2)} - \frac{64(2N^5 + 21N^4 + 36N^3 - 7N^2 - 68N - 56)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} + \frac{256}{3}S_{-2,1} - \frac{128}{3}S_{-3} \\
& + \frac{(N-1) \left(\frac{64}{3}S_{-2}S_1 - 32\zeta_3 \right)}{N(N+1)} \Big] + \tilde{\gamma}_{qg}^0 \left[\frac{1}{27}S_1^4 - \frac{2}{9}S_2S_1^2 + \left[\frac{16}{9}S_{2,1} - \frac{40S_3}{27} \right] S_1 \right. \\
& \left. + \frac{64}{9}\zeta_3 S_1 + \frac{1}{9}S_2^2 + \frac{14S_4}{9} + \frac{32}{9}S_{-4} + \frac{32}{9}S_{3,1} - \frac{16}{9}S_{2,1,1} \right] \\
& + C_{FN} C_{FT}^2 \left[\left[\frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^4 + 6N^3 - N^2 - 4N + 12)}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 \right] L_Q^3 \right. \\
& + \left[-\frac{4P_{31}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{16P_{21}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[\frac{20S_2}{3} - 4S_1^2 \right] \right. \\
& + L_M \left[\frac{8(N^2 + N + 2)P_{10}}{3(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1 \right] \Big] L_Q^2 + \left[\frac{16L_M^2(N^2 + N + 2)^3}{(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& - \frac{16P_{22}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{64(-1)^N P_{37}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3} \\
& \left. + \frac{4P_{42}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3} - \frac{8P_{30}S_1}{9(N-1)N^4(N+1)^4(N+2)^3} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{16P_{23}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + L_M \left[-\frac{16P_{28}}{3(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& + \left. \frac{16P_{17}S_1}{3(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{gg}^0 \left(\frac{16S_2}{3} - \frac{16}{3}S_1^2 \right) \right] - \frac{256(N^2+N+1)S_3}{3N(N+1)(N+2)} \\
& + \frac{64P_{16}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \tilde{\gamma}_{gg}^0 \left[\frac{8}{3}S_1^3 - 8S_2S_1 - \frac{32}{3}S_{2,1} \right] \\
& + \left. \frac{\frac{512}{3}S_1S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)} \right] L_Q - \frac{64}{9} \frac{(N^2+N+2)P_{10}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)S_1^2}{81N^2(N+1)^2(N+2)} + \frac{P_{40}}{243(N-1)N^6(N+1)^6(N+2)^5} \\
& - \frac{4P_{35}S_1}{243(N-1)N^5(N+1)^5(N+2)^2} + L_M^3 \left[\frac{8(N^2+N+2)P_{10}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{gg}^0S_1 \right] \\
& + L_M^2 \left[\frac{4P_{29}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{16P_{20}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{gg}^0 \left[-\frac{4}{3}S_1^2 - \frac{4S_2}{3} \right] \right] \\
& + \frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)S_2}{27N^2(N+1)^2(N+2)} + \frac{(10N^3 + 13N^2 + 29N + 6) \left[-\frac{16}{81}S_1^3 - \frac{16}{27}S_2S_1 \right]}{N^2(N+1)(N+2)} \\
& + \frac{32(5N^3 - 16N^2 + N - 6)S_3}{81N^2(N+1)(N+2)} + \tilde{\gamma}_{gg}^0 \left[-\frac{1}{27}S_1^4 - \frac{2}{9}S_2S_1^2 - \frac{8}{27}S_3S_1 - \frac{64}{9}\zeta_3S_1 - \frac{1}{9}S_2^2 + \frac{14S_4}{9} \right] \\
& + L_M \left[-\frac{8P_{19}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{16P_{33}S_1}{27(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& + \frac{64(-1)^N P_{38}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3} + \frac{8(N^2+N+2)P_{11}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{4P_{41}}{135(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3} + \frac{64P_{15}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} \\
& \left. + \tilde{\gamma}_{gg}^0 \left(\frac{8}{3}S_1^3 - \frac{8}{3}S_2S_1 - \frac{16}{3}S_{2,1} \right) + \frac{\frac{512}{3}S_1S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)} + \frac{(N-1)(64\zeta_3 - \frac{64S_3}{3})}{N(N+1)} \right] \Bigg\} \\
& + N_F \hat{C}_{2,g}^{\hat{S},(3)}(N_F) \Bigg\} , \tag{47}
\end{aligned}$$

where

$$P_{10} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \tag{48}$$

$$P_{11} = 47N^6 + 141N^5 + 59N^4 - 117N^3 + 2N^2 + 84N + 72 \tag{49}$$

$$P_{12} = 65N^6 + 455N^5 + 1218N^4 + 1820N^3 + 1968N^2 + 1460N + 448 \tag{50}$$

$$P_{13} = 139N^6 + 1093N^5 + 3438N^4 + 5776N^3 + 5724N^2 + 3220N + 752 \tag{51}$$

$$P_{14} = 9N^7 + 71N^6 + 214N^5 + 320N^4 + 275N^3 + 215N^2 + 160N + 32 \tag{52}$$

$$P_{15} = N^8 + 8N^7 - 2N^6 - 60N^5 - 23N^4 + 108N^3 + 96N^2 + 16N + 48 \tag{53}$$

$$P_{16} = N^8 + 8N^7 - 2N^6 - 60N^5 + N^4 + 156N^3 + 24N^2 - 80N - 240 \tag{54}$$

$$P_{17} = 3N^8 + 8N^7 - 2N^6 - 24N^5 + 15N^4 + 88N^3 + 152N^2 + 96N + 48 \tag{55}$$

$$P_{18} = 5N^8 - 8N^7 - 137N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 \tag{56}$$

$$P_{19} = 7N^8 + 4N^7 - 90N^6 - 224N^5 - 21N^4 + 388N^3 + 608N^2 + 336N + 144 \tag{57}$$

$$P_{20} = 10N^8 + 46N^7 + 105N^6 + 139N^5 + 87N^4 - 17N^3 + 50N^2 + 84N + 72 \tag{58}$$

$$P_{21} = 19N^8 + 70N^7 + 63N^6 - 41N^5 - 192N^4 - 221N^3 - 142N^2 - 60N - 72 \tag{59}$$

$$\begin{aligned}
P_{22} &= 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 & (60) \\
P_{23} &= 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 & (61) \\
P_{24} &= 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 & (62) \\
P_{25} &= 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 & (63) \\
P_{26} &= 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 & (64) \\
P_{27} &= 55N^9 + 336N^8 + 218N^7 - 2180N^6 - 6529N^5 - 9764N^4 - 9368N^3 - 6032N^2 \\
&\quad - 2448N - 576 & (65) \\
P_{28} &= N^{11} - 56N^9 - 236N^8 - 373N^7 + 82N^6 + 1244N^5 + 2330N^4 + 2560N^3 + 1712N^2 \\
&\quad + 896N + 288 & (66) \\
P_{29} &= 33N^{11} + 231N^{10} + 662N^9 + 1254N^8 + 1801N^7 + 2759N^6 + 5440N^5 + 9884N^4 \\
&\quad + 12512N^3 + 9200N^2 + 5184N + 1728 & (67) \\
P_{30} &= 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 \\
&\quad + 12976N^3 + 8016N^2 + 4608N + 1728 & (68) \\
P_{31} &= 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 - 4645N^6 - 8936N^5 - 11980N^4 \\
&\quad - 12352N^3 - 8272N^2 - 4800N - 1728 & (69) \\
P_{32} &= 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4 \\
&\quad - 6840N^3 + 120N^2 + 2432N + 960 & (70) \\
P_{33} &= 121N^{11} + 988N^{10} + 3554N^9 + 6972N^8 + 7131N^7 - 846N^6 - 14806N^5 - 25354N^4 \\
&\quad - 26096N^3 - 16752N^2 - 8352N - 2592 & (71) \\
P_{34} &= 27N^{12} + 441N^{11} + 2206N^{10} + 5360N^9 + 7445N^8 + 8555N^7 + 18766N^6 + 44852N^5 \\
&\quad + 67572N^4 + 63960N^3 + 39632N^2 + 15648N + 2880 & (72) \\
P_{35} &= 2447N^{12} + 16902N^{11} + 59649N^{10} + 125860N^9 + 128761N^8 - 36530N^7 - 248341N^6 \\
&\quad - 304460N^5 - 162188N^4 - 11724N^3 + 29160N^2 + 19440N + 7776 & (73) \\
P_{36} &= 3361N^{12} + 23769N^{11} + 62338N^{10} + 59992N^9 - 63303N^8 - 317823N^7 - 585520N^6 \\
&\quad - 640602N^5 - 430132N^4 - 167536N^3 - 27648N^2 + 9504N + 5184 & (74) \\
P_{37} &= 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} - 19377N^{10} - 58253N^9 - 26543N^8 + 170601N^7 \\
&\quad + 362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 117072N - 25920 & (75) \\
P_{38} &= 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} - 47693N^9 - 125723N^8 - 86079N^7 \\
&\quad + 36437N^6 + 22559N^5 - 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 & (76) \\
P_{39} &= 3180N^{15} + 38835N^{14} + 188728N^{13} + 456665N^{12} + 460954N^{11} - 406761N^{10} - 1972948N^9 \\
&\quad - 2827653N^8 - 1857970N^7 + 109786N^6 + 1302824N^5 + 1092456N^4 \\
&\quad + 265888N^3 - 227616N^2 - 194688N - 44928 & (77) \\
P_{40} &= 28503N^{17} + 297639N^{16} + 1232041N^{15} + 2461407N^{14} + 2169615N^{13} + 662941N^{12} \\
&\quad + 2110979N^{11} + 5346653N^{10} + 2021366N^9 - 7290864N^8 - 11721384N^7 - 3689680N^6 \\
&\quad + 15676192N^5 + 32276800N^4 + 31869312N^3 + 18809856N^2 + 6856704N + 1244160 & (78) \\
P_{41} &= 75N^{18} + 3330N^{17} + 35497N^{16} + 175010N^{15} + 486862N^{14} + 966996N^{13} + 2037362N^{12} \\
&\quad + 3604404N^{11} - 1625689N^{10} - 29506022N^9 - 78753403N^8 - 107977014N^7 - 71548880N^6 \\
&\quad + 18344016N^5 + 89016048N^4 + 92657952N^3 + 58942080N^2 + 25505280N + 5598720 & (79) \\
P_{42} &= 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} - 3216546N^{12} \\
&\quad - 1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 + 25361392N^7 + 9982840N^6 \\
&\quad - 10154128N^5 - 26098704N^4 - 26761536N^3 - 17642880N^2 - 8087040N - 1866240 . & (80)
\end{aligned}$$

In all the representations of the massive Wilson coefficients and OMEs in N -space we apply algebraic reduction [59]. The 2-loop term in (47) is purely multiplicative and induced by renormalization only, while the 3-loop contributions require the calculation of massive OMEs. The above Wilson coefficients depend on the harmonic sums

$$S_1, S_{-2}, S_2, S_{-3}, S_3, S_{-4}, S_4, S_{-2,1}, S_{2,1}, S_{3,1}, S_{2,1,1}, \quad (81)$$

apart of those defining the massless 3-loop Wilson coefficients [11]¹⁰. The harmonic sums are defined recursively by, cf. [36,37],

$$S_{b,\bar{a}}(N) = \sum_{k=1}^N \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\bar{a}}(k), \quad b, a_i \in \mathbb{Z} \setminus \{0\}, N \in \mathbb{N}, N \geq 1, S_{\emptyset} = 1. \quad (82)$$

In the above $\zeta_l = \sum_{k=1}^{\infty} 1/k^l$, $l \in \mathbb{N}$, $l \geq 2$ denote the Riemann ζ -values, which are convergent harmonic sums in the limit $N \rightarrow \infty$. In the constant part of the other Wilson coefficients it is expected that more complicated multiple zeta values emerge, which have been dealt with in [61].

In Eq. (47) denominator terms $\propto 1/(N-2)$ occur. They cancel in the complete expression and the rightmost singularity is located at $N = 1$ as expected for this Wilson coefficient. Let us now consider both the small- and large- x dominant terms for both Wilson coefficients. Those of the massless parts were given in [11] before. Both Wilson coefficients contain terms $\propto 1/(N-1)$. For simplicity we consider the choice of scale $Q^2 = \mu^2$ here. The expansion of the heavy flavor contribution, subtracting the massless 3-loop Wilson coefficients, denoted by \hat{L}_i , around $N = 1(x \rightarrow 0)$ and in the limit $N \rightarrow \infty(x \rightarrow 1)$, setting $Q^2 = \mu^2$, are given by

$$\hat{L}_{q,2}^{\text{PS}}(N \rightarrow 1) \propto \frac{1}{N-1} C_F T_F^2 N_F \left\{ \frac{1024}{27} \zeta_3 - \frac{64}{729} \left[54L_M^3 - 81L_M^2 + 342L_M + 500 \right] \right\} \quad (83)$$

$$\begin{aligned} \hat{L}_{g,2}^{\text{S}}(N \rightarrow 1) \propto & \frac{1}{N-1} \left\{ C_A T_F^2 N_F \left[\frac{512}{27} \zeta_3 - \frac{16}{729} \left[108L_M^3 + 540L_M^2 + 54L_M + 3091 \right] \right] \right. \\ & \left. + C_F T_F^2 N_F \left[\frac{1024}{27} \zeta_3 - \frac{32}{729} \left[108L_M^3 - 864L_M^2 + 1314L_M - 1091 \right] \right] \right\} \quad (84) \end{aligned}$$

$$\hat{L}_{q,2}^{\text{PS}}(N \rightarrow \infty) \propto -\frac{64}{27} C_F T_F^2 N_F \frac{\ln^3(\bar{N})}{N^2} \quad (85)$$

$$\hat{L}_{g,2}^{\text{S}}(N \rightarrow \infty) \propto -\frac{4}{27} \frac{(N-2) \ln^4(\bar{N})}{N^2} (C_A - C_F) T_F^2 N_F. \quad (86)$$

The corresponding limits for the contributions of the massless Wilson coefficients behave like

$$N_F \hat{C}_{2,q}^{\hat{\text{PS}},(3)}(N_F)(N \rightarrow 1) \propto \frac{4}{N-1} C_F T_F^2 N_F \left[\frac{22112}{729} - \frac{32}{9} \zeta_2 + \frac{128}{27} \zeta_3 \right] \quad [11] \quad (87)$$

$$\begin{aligned} N_F \hat{C}_{2,g}^{\hat{\text{S}},(3)}(N_F)(N \rightarrow 1) \propto & \frac{4}{N-1} \left\{ C_A T_F^2 N_F \left[-\frac{572}{729} + \frac{160}{27} \zeta_2 + \frac{64}{27} \zeta_3 \right] \right. \\ & \left. + C_F T_F^2 N_F \left[\frac{45468}{729} - \frac{512}{27} \zeta_2 + \frac{128}{27} \zeta_3 \right] \right\} \quad [11] \quad (88) \end{aligned}$$

$$N_F \hat{C}_{2,q}^{\hat{\text{PS}},(3)}(N_F)(N \rightarrow \infty) \propto \frac{\ln^3(\bar{N})}{N^3} C_F T_F^2 N_F \frac{64}{27} \quad (89)$$

¹⁰For the algebraically reduced representations see [60].

$$N_F \hat{C}_{2,g}^{\hat{S},(3)}(N_F)(N \rightarrow \infty) \propto \frac{\ln^4(\bar{N})}{N} \left[\frac{68}{27} C_F T_F^2 N_F + \frac{28}{27} C_A T_F^2 N_F \right], \quad (90)$$

where $\bar{N} = N \exp(\gamma_E)$ and γ_E denotes the Euler-Mascheroni constant.

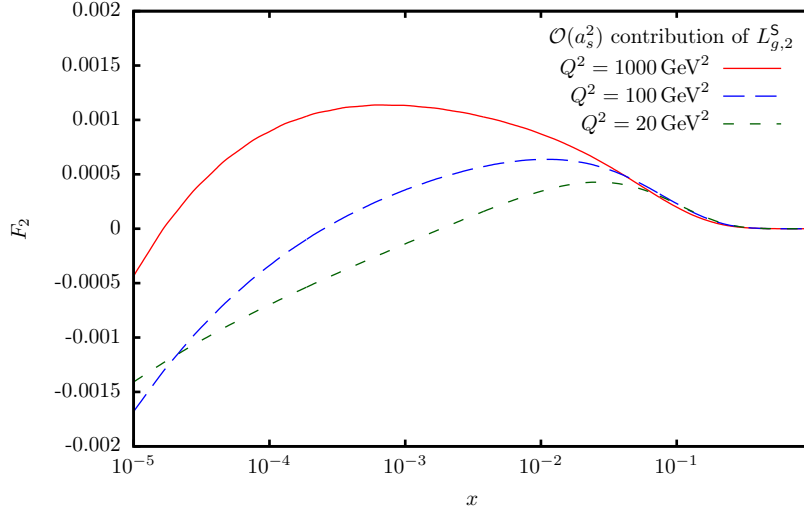


Figure 1: The $O(a_s^2)$ contribution by $L_{g,2}^S$ to the structure function $F_2(x, Q^2)$.

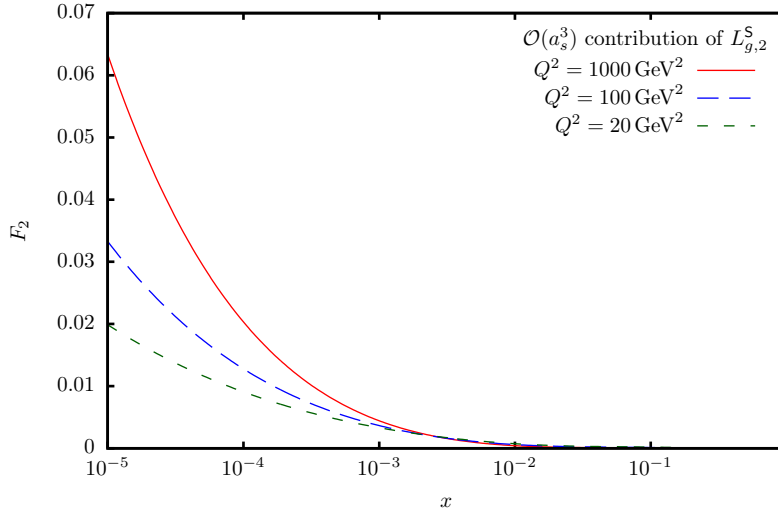


Figure 2: The $O(a_s^3)$ contribution by $L_{g,2}^S$ to the structure function $F_2(x, Q^2)$.

While the expression for $L_{q,2}^{\text{PS}}$ is the same in the $\overline{\text{MS}}$ - and on-mass-shell scheme to $O(a_s^3)$, $L_{g,2}^S$, in its 3-loop contribution, changes by the term

$$L_{g,2}^{\text{S},(3),\overline{\text{MS}}}(N) = L_{g,2}^{\text{S},(3),\text{OMS}}(N) + a_s^3 \frac{32}{3} C_F T_F^2 N_F [3L_M - 4]$$

$$\times \left[\frac{(N^2 + N + 2)}{N(N+1)(N+2)} S_1 + \frac{(N^3 - 4N^2 - N - 2)}{N^2(N+1)(N+2)} \right] \quad (91)$$

setting $Q^2 = \mu^2$. Here, we have identified the logarithms L_M in both schemes symbolically. In applications, either the on-shell or the $\overline{\text{MS}}$ mass has to be used here. The corresponding expression in z -space reads

$$L_{g,2}^{\text{S},(3),\overline{\text{MS}}}(z) = L_{g,2}^{\text{S},(3),\text{OMS}}(z) + a_s^3 \frac{32}{3} C_F T_F^2 N_F [3L_M - 4] \times \left[(2z^2 - 2z + 1) [H_0(z) + H_1(z)] + 8z^2 - 8z + 1 \right], \quad (92)$$

with $H_{\bar{a}}(z)$ harmonic polylogarithms, see Eq. (592).

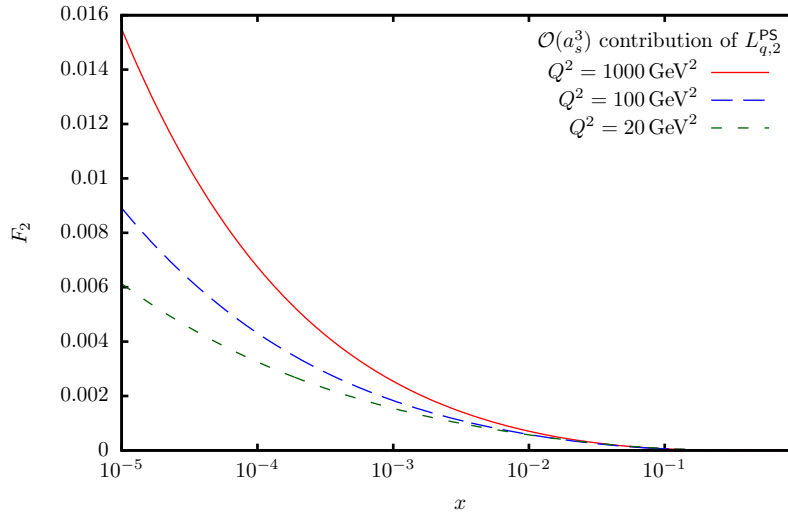


Figure 3: The $O(a_s^3)$ contribution by $L_{q,2}^{\text{PS}}$ to the structure function $F_2(x, Q^2)$.

x	$Q^2 = 20\text{GeV}^2$	$Q^2 = 100\text{GeV}^2$	$Q^2 = 1000\text{GeV}^2$
10^{-4}	1.946	3.200	5.340
10^{-3}	1.141	1.702	2.526
10^{-2}	0.641	0.825	1.040
10^{-1}	0.400	0.409	0.412

Table 1: Values of the structure function $F_2(x, Q^2)$ in the low x region using the PDF-parameterization [5].

In Figure 1 we illustrate the contribution of $L_{q,2}^{\text{PS}}$ to the structure function $F_2(x, Q^2)$ using the PDFs of Ref. [5], cf. Eq. (11). Likewise Figures 2 and 3 show the corresponding contributions by $L_{g,2}^{\text{S}}$ at $O(a_s^2)$ and $O(a_s^3)$, respectively. Note that the $O(a_s^2)$ -terms, cf. also Ref. [19] are smaller than those at $O(a_s^3)$, which is caused by terms $\propto 1/z$ in the 3-loop contribution to $L_{g,2}^{\text{S}}$, which are absent at 2-loop order.

These contributions emerging on the 2- and 3-loop level are minor compared to the values of the structure function $F_2(x, Q^2)$, for which typical values are given in Table 1.¹¹ A global comparison of all heavy flavor contributions up to 3-loop order can presently only be performed using the known number of Mellin moments, cf. [12], given in Section 6.

4 The Logarithmic Contributions to $H_{q,2}^{\text{PS}}$ and $H_{g,2}^{\text{S}}$ to $O(a_s^3)$

The pure-singlet Wilson coefficient $H_{q,2}^{\text{PS}}$, except for the constant part $a_{Qq}^{\text{PS},(3)}$ of the unrenormalized operator matrix element in the on-shell scheme can be expressed by harmonic sums and rational functions in N only. As before we reduce to a basis eliminating the algebraic relations [59]. It is given by :

$$\begin{aligned}
H_{q,2}^{\text{PS}} = & \frac{1}{2}[1 + (-1)^N] \\
& \times \left\{ a_s^2 \left[C_{FTF} \left[-\frac{4L_M^2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{(4S_1^2 - 12S_2)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \right. \\
& + \frac{4L_Q^2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(-1)^N P_{45}}{3(N-1)N^2(N+1)^3(N+2)^3} \\
& + \frac{8P_{75}}{3(N-1)N^4(N+1)^4(N+2)^3} + \frac{8P_{57}S_1}{(N-1)N^3(N+1)^3(N+2)^2} \\
& \left. \left. \left. + L_Q \left[-\frac{8S_1(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{8P_{57}}{(N-1)N^3(N+1)^3(N+2)^2} \right] \right. \right. \\
& \left. \left. \left. + \frac{64S_{-2}}{(N-1)N(N+1)(N+2)} - \frac{8(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)L_M}{(N-1)N^3(N+1)^3(N+2)^2} \right] \right\} \\
& + a_s^3 \left\{ C_{FTF}^2 \left[L_Q^3 \left[\frac{8(N^2 + N + 2)^2(3N^2 + 3N + 2)}{3(N-1)N^3(N+1)^3(N+2)} - \frac{32(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] \right. \right. \\
& + L_Q^2 \left[\frac{(24S_1^2 - 24S_2)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{4P_{73}}{(N-1)N^4(N+1)^4(N+2)^2} \right. \\
& \left. \left. \left. + \frac{8P_{62}S_1}{(N-1)N^3(N+1)^3(N+2)^2} \right] + L_Q \left[\frac{(104S_1S_2 - \frac{56}{3}S_1^3)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& - \frac{16(N^2 + N - 22)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{(128S_{-3} - 256S_{-2,1} - 384\zeta_3)(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{4P_{67}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2} - \frac{64(-1)^N P_{98}}{15(N-2)(N-1)^3N^3(N+1)^5(N+2)^4(N+3)^3} \\
& \left. \left. \left. + \frac{4P_{100}}{15(N-1)^3N^5(N+1)^5(N+2)^4(N+3)^3} + \frac{128(-1)^N P_{45}S_1}{3(N-1)N^2(N+1)^3(N+2)^3} \right] \right\}
\end{aligned}$$

¹¹Note that the kinematic region at small x probed at HERA is limited to values $x \geq Q^2/(yS)$, with $S \simeq 10^5 \text{GeV}^2$ and $y \in [0, 1]$.

$$\begin{aligned}
& -\frac{8P_{79}S_1}{3(N-1)N^4(N+1)^4(N+2)^3} + \frac{512S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{65}S_2}{(N-1)N^3(N+1)^3(N+2)^2} \\
& + L_M^2 \left[\frac{16(N^2+N+2)^2S_1}{(N-1)N^2(N+1)^2(N+2)} - \frac{4(N^2+N+2)^2(3N^2+3N+2)}{(N-1)N^3(N+1)^3(N+2)} \right] \\
& + L_M \left[\frac{32(N^2+5N+2)(5N^3+7N^2+4N+4)S_1}{(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& \left. - \frac{8(N^2+5N+2)(3N^2+3N+2)(5N^3+7N^2+4N+4)}{(N-1)N^4(N+1)^4(N+2)^2} \right] \\
& + \frac{32P_{69}S_{-2}}{(N-2)(N-1)N^3(N+1)^3(N+2)(N+3)} \left] - \frac{8(3N+2)(N^2+N+2)S_1^3}{3(N-1)N^3(N+1)(N+2)} \right. \\
& + \frac{4P_{74}S_1^2}{(N-1)N^4(N+1)^4(N+2)^3} - 2\frac{(N^2+N+2)\zeta_2}{(N-1)N^4(N+1)^4(N+2)}P_{55} - \frac{4P_{94}}{(N-1)N^5(N+1)^6(N+2)^3} \\
& - \frac{4(N^2+N+2)^2(3N^2+3N+2)\zeta_3}{3(N-1)N^3(N+1)^3(N+2)} + 4\frac{(N^2+N+2)(5N^4+4N^3+N^2-10N-8)\zeta_2}{(N-1)N^3(N+1)^3(N+2)}S_1 \\
& + \frac{4P_{91}S_1}{(N-1)N^5(N+1)^5(N+2)^3} + L_M^3 \left[\frac{4(N^2+N+2)^2(3N^2+3N+2)}{3(N-1)N^3(N+1)^3(N+2)} \right. \\
& \left. - \frac{16(N^2+N+2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] - \frac{8(N^2+N+2)(3N^4+9N^3+15N^2+11N-2)S_1S_2}{(N-1)N^3(N+1)^3(N+2)} \\
& - \frac{4P_{76}S_2}{(N-1)N^4(N+1)^4(N+2)^3} + L_M^2 \left[-\frac{4(13N^2+5N-6)S_1(N^2+N+2)^2}{(N-1)N^3(N+1)^3(N+2)} \right. \\
& \left. + \frac{(24S_2-8S_1^2)(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{4P_{51}(N^2+N+2)}{(N-1)N^4(N+1)^4(N+2)} \right] \\
& - \frac{8(N^2+N+2)(3N^4+48N^3+43N^2-22N-8)S_3}{3(N-1)N^3(N+1)^3(N+2)} + \frac{32(N^2-3N-2)(N^2+N+2)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)} \\
& + \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{2}{3}S_1^4 - 12S_2S_1^2 + \left[\frac{16S_3}{3} + 32S_{2,1} \right] S_1 + \frac{16}{3}\zeta_3S_1 + 18S_2^2 - 12S_4 \right. \\
& \left. + 32S_{3,1} - 64S_{2,1,1} + (4S_1^2 - 12S_2)\zeta_2 \right] + L_M \left[\frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{8}{3}S_1^3 - 24S_2S_1 \right. \right. \\
& \left. \left. - \frac{80S_3}{3} + 32S_{2,1} + 96\zeta_3 \right] - \frac{4P_{60}S_1^2}{(N-1)N^3(N+1)^3(N+2)^2} - \frac{4P_{92}}{(N-1)N^5(N+1)^5(N+2)^3} \right. \\
& \left. + \frac{8P_{77}S_1}{(N-1)N^4(N+1)^4(N+2)^3} - \frac{4P_{59}S_2}{(N-1)N^3(N+1)^3(N+2)^2} \right] \left. \right] \\
& + C_F T_F^2 \left[\left[\frac{32(N^2+N+2)^2L_Q^3}{9(N-1)N^2(N+1)^2(N+2)} - \frac{32P_{66}L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \right. \\
& \left. \left. + L_Q \left[\frac{32(N^2+N+2)^2L_M^2}{3(N-1)N^2(N+1)^2(N+2)} + \left[\frac{64(N^2+N+2)(8N^3+13N^2+27N+16)}{9(N-1)N^2(N+1)^3(N+2)} \right. \right. \right. \right. \\
& \left. \left. - \frac{64(N^2+N+2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] L_M - \frac{256(-1)^N P_{45}}{9(N-1)N^2(N+1)^3(N+2)^3} \right. \\
\end{aligned}$$

$$\begin{aligned}
& \left. + \frac{64P_{82}}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{512S_{-2}}{3(N-1)N(N+1)(N+2)} \right] \\
& - \frac{128(N^2+N+2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} + \frac{16(N^2+N+2)(7N^4+16N^3+32N^2+19N+2)S_1^2}{3(N-1)N^3(N+1)^3(N+2)} \\
& - \frac{32(11N^5+26N^4+57N^3+142N^2+84N+88)L_M^2}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{32\zeta_2 P_{48}}{9(N-1)N^3(N+1)^2(N+2)^2} \\
& + \frac{32P_{95}}{81(N-1)N^5(N+1)^5(N+2)^4} - \frac{32P_{68}S_1}{27(N-1)N^3(N+1)^4(N+2)} \\
& + L_M \left[\frac{\left[\frac{32}{3}S_1^2 - 32S_2 \right] (N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{64P_{80}}{27(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& \left. + \frac{64P_{57}S_1}{3(N-1)N^3(N+1)^3(N+2)^2} \right] + \frac{16P_{61}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{64}{9}S_1^3 - \frac{32}{3}S_2S_1 - \frac{32}{3}\zeta_2S_1 + \frac{160S_3}{9} + \frac{128\zeta_3}{9} \right] \Bigg] \\
& + N_F T_F^2 C_F \left[\frac{32(N^2+N+2)^2 L_Q^3}{9(N-1)N^2(N+1)^2(N+2)} - \frac{32P_{66}L_Q^2}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& \left. + \left[\frac{-\frac{16}{3}S_1^2 - \frac{16S_2}{3}}{(N-1)N^2(N+1)^2(N+2)} (N^2+N+2)^2 + \frac{32(8N^3+13N^2+27N+16)S_1(N^2+N+2)}{9(N-1)N^2(N+1)^3(N+2)} \right] \right. \\
& - \frac{256(-1)^N P_{45}}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{32P_{84}}{27(N-1)N^4(N+1)^4(N+2)^3} \\
& \left. + \frac{512S_{-2}}{3(N-1)N(N+1)(N+2)} \right] L_Q - \frac{32(N^2+N+2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{16}{9} \frac{\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} P_{63} - \frac{32P_{87}}{3(N-1)N^5(N+1)^5(N+2)^4} \\
& + L_M^2 \left[\frac{32P_{64}}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{32(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] \\
& + L_M \left[\frac{\left[-\frac{16}{3}S_1^2 - \frac{80S_2}{3} \right] (N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{32P_{78}}{27(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& \left. + \frac{32P_{63}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right] + \frac{64(N^2+5N+2)(5N^3+7N^2+4N+4)S_2}{3(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{(N^2+N+2)^2 \left[\frac{64S_3}{3} + \frac{16}{3}S_1\zeta_2 + \frac{32\zeta_3}{9} \right]}{(N-1)N^2(N+1)^2(N+2)} \Bigg] \\
& + C_A C_F T_F \left[L_Q^3 \left[-\frac{16S_1(N^2+N+2)^2}{3(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. - \frac{8(11N^4+22N^3-23N^2-34N-12)(N^2+N+2)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2} \right] \right. \\
& \left. + L_Q^2 \left[-\frac{16(5N^2-1)S_1(N^2+N+2)^2}{(N-1)^2N^3(N+1)^3(N+2)} + \frac{(16S_1^2-16S_2-32S_{-2})(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right] \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{16(-1)^N(N^5 + 9N^4 + 24N^3 + 36N^2 + 32N + 8)(N^2 + N + 2)}{(N-1)N^3(N+1)^4(N+2)^3} \\
& + \frac{8P_{85}}{9(N-1)^2N^4(N+1)^3(N+2)^3} \Bigg] + L_Q \left[\frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{40}{3}S_1^3 + 40S_2S_1 \right. \right. \\
& - 144S_{-3} + 96S_{-2,1} \Bigg] + \frac{4P_{47}S_1^2(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^3(N+2)^2} + 48 \frac{(N^2 + N + 6)(N^2 + N + 2)\zeta_3}{(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32S_{-2}S_1(N^2 + N + 2)}{N^2(N+1)^2} + \frac{4P_{52}S_2(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^3(N+2)^2} + \frac{32(-1)^N P_{93}}{9(N-1)^2N^4(N+1)^5(N+2)^4} \\
& - \frac{8(13N^2 + 13N + 62)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} - \frac{8P_{97}}{27(N-1)^2N^5(N+1)^5(N+2)^4} \\
& - \frac{32(-1)^N P_{72}S_1}{3(N-1)N^3(N+1)^4(N+2)^3} - \frac{8P_{88}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^3} \\
& + \frac{16P_{71}S_{-2}}{3(N-1)^2N^3(N+1)^3(N+2)^2} \Bigg] + \frac{8(N^2 + N + 2)(N^3 + 8N^2 + 11N + 2)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2} \\
& + \frac{4(N^2 + N + 2)P_{43}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3} + 8(N^2 + N + 2) \frac{(-1)^N \zeta_2}{(N-1)N^3(N+1)^4(N+2)^3} P_{44} \\
& - \frac{8}{9} \frac{(N^2 + N + 2)\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2} P_{49} + \frac{4}{9} \frac{\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3} P_{89} \\
& + \frac{8P_{99}}{3(N-1)^2N^6(N+1)^6(N+2)^5} - \frac{4}{3} \frac{(N^2 + N + 2)\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2} P_{53}S_1 \\
& - \frac{8(N^2 + N + 2)P_{70}S_1}{(N-1)N^2(N+1)^5(N+2)^4} + L_M^3 \left[\frac{16S_1(N^2 + N + 2)^2}{3(N-1)N^2(N+1)^2(N+2)} \right. \\
& + \left. \frac{8(11N^4 + 22N^3 - 23N^2 - 34N - 12)(N^2 + N + 2)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2} \right] + \frac{4P_{86}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^3} \\
& - \frac{8(N^2 + N + 2)(3N^3 - 12N^2 - 27N - 2)S_1S_2}{(N-1)N^2(N+1)^3(N+2)^2} - \frac{16(N^2 + N + 2)P_{50}S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2} \\
& + \frac{32(-1)^N(N^2 + N + 2)(N^4 + 2N^3 + 7N^2 + 22N + 20)S_{-2}}{(N-1)N(N+1)^4(N+2)^3} \\
& + L_M^2 \left[\frac{(16S_2 + 32S_{-2})(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{54}S_1(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^3(N+2)^2} \right. \\
& - \left. \frac{16(-1)^N(N^5 + 9N^4 + 24N^3 + 36N^2 + 32N + 8)(N^2 + N + 2)}{(N-1)N^3(N+1)^4(N+2)^3} \right. \\
& - \left. \frac{8P_{83}}{9(N-1)^2N^4(N+1)^3(N+2)^3} \right] + \frac{(N^2 - N - 4)(N^2 + N + 2)}{(N-1)N(N+1)^3(N+2)^2} \left[-64(-1)^N S_1S_{-2} \right. \\
& - 32(-1)^N S_{-3} + 64S_{-2,1} - 32(-1)^N S_1\zeta_2 - 24(-1)^N \zeta_3 \Bigg] \\
& + \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{2}{3}S_1^4 - 20S_2S_1^2 - 32(-1)^N S_{-3}S_1 + \left(64S_{-2,1} - \frac{160S_3}{3} \right) S_1 \right. \\
& - \frac{8}{3}(-7 + 9(-1)^N)\zeta_3S_1 - 2S_2^2 + S_{-2}(-32(-1)^N S_1^2 - 32(-1)^N S_2) - 36S_4 - 16(-1)^N S_{-4} \\
& + 16S_{3,1} + 32S_{-2,2} + 32S_{-3,1} + 16S_{2,1,1} - 64S_{-2,1,1} + (-4(-3 + 4(-1)^N)S_1^2 \\
& \left. - 4(-1 + 4(-1)^N)S_2 - 8(1 + 2(-1)^N)S_{-2})\zeta_2 \right] + L_M \left[\frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{8}{3}S_1^3 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +40S_2S_1 + 32(1 + (-1)^N)S_{-2}S_1 + 16(-1)^NS_{-3} - 32S_{2,1} + 12(-9 + (-1)^N)\zeta_3 \Big] \\
& + \frac{4(17N^4 - 6N^3 + 41N^2 - 16N - 12)S_1^2(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^2(N+2)} + \frac{4P_{56}S_2(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^3(N+2)^2} \\
& + \frac{8(31N^2 + 31N + 74)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{16(7N^2 + 7N + 10)S_{-3}(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{128(N^2 + N + 1)S_{-2,1}(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} + \frac{(N^2 - N - 4)(N^2 + N + 2)32(-1)^NS_{-2}}{(N-1)N(N+1)^3(N+2)^2} \\
& - \frac{64(-1)^NP_{81}}{9(N-1)N^3(N+1)^5(N+2)^4} + \frac{8P_{96}}{27(N-1)^2N^5(N+1)^5(N+2)^4} \\
& + \frac{64(-1)^NP_{46}S_1}{3(N-1)N^2(N+1)^3(N+2)^3} - \frac{8P_{90}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^3} \\
& + \frac{16P_{58}S_{-2}}{(N-1)N^3(N+1)^3(N+2)^2} \Big] + a_{Qq}^{\text{PS},(3)} + \tilde{C}_{2,q}^{\text{PS},(3)}(N_F + 1) \Big\} \Big\} , \tag{93}
\end{aligned}$$

with the polynomials

$$P_{43} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \tag{94}$$

$$P_{44} = 2N^6 + 7N^5 + 31N^4 + 82N^3 + 86N^2 + 32N + 8 \tag{95}$$

$$P_{45} = 4N^6 + 22N^5 + 48N^4 + 53N^3 + 45N^2 + 36N + 8 \tag{96}$$

$$P_{46} = 5N^6 + 29N^5 + 78N^4 + 118N^3 + 114N^2 + 72N + 16 \tag{97}$$

$$P_{47} = 5N^6 + 135N^5 + 327N^4 + 329N^3 + 220N^2 - 176N - 120 \tag{98}$$

$$P_{48} = 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24 \tag{99}$$

$$P_{49} = 11N^6 + 6N^5 + 75N^4 + 68N^3 - 200N^2 - 80N - 24 \tag{100}$$

$$P_{50} = 11N^6 + 29N^5 - 7N^4 - 25N^3 - 56N^2 - 72N - 24 \tag{101}$$

$$P_{51} = 16N^6 + 35N^5 + 33N^4 - 11N^3 - 41N^2 - 36N - 12 \tag{102}$$

$$P_{52} = 17N^6 - 57N^5 - 213N^4 - 175N^3 - 140N^2 + 64N + 72 \tag{103}$$

$$P_{53} = 17N^6 + 27N^5 + 75N^4 + 149N^3 - 20N^2 - 80N - 24 \tag{104}$$

$$P_{54} = 17N^6 + 51N^5 + 51N^4 + 89N^3 + 40N^2 - 80N - 24 \tag{105}$$

$$P_{55} = 38N^6 + 108N^5 + 151N^4 + 106N^3 + 21N^2 - 28N - 12 \tag{106}$$

$$P_{56} = 73N^6 + 189N^5 + 45N^4 + 31N^3 - 238N^2 - 412N - 120 \tag{107}$$

$$P_{57} = N^7 - 15N^5 - 58N^4 - 92N^3 - 76N^2 - 48N - 16 \tag{108}$$

$$P_{58} = 2N^7 + 14N^6 + 37N^5 + 102N^4 + 155N^3 + 158N^2 + 132N + 40 \tag{109}$$

$$P_{59} = 3N^7 - 15N^6 - 153N^5 - 577N^4 - 854N^3 - 652N^2 - 408N - 128 \tag{110}$$

$$P_{60} = 5N^7 + 19N^6 + 61N^5 + 197N^4 + 266N^3 + 212N^2 + 136N + 32 \tag{111}$$

$$P_{61} = 5N^7 + 37N^6 + 188N^5 + 643N^4 + 925N^3 + 742N^2 + 460N + 120 \tag{112}$$

$$P_{62} = 7N^7 + 21N^6 + 5N^5 - 117N^4 - 244N^3 - 232N^2 - 192N - 80 \tag{113}$$

$$P_{63} = 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \tag{114}$$

$$P_{64} = 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \tag{115}$$

$$P_{65} = 9N^7 + 15N^6 - 103N^5 - 575N^4 - 998N^3 - 948N^2 - 696N - 256 \tag{116}$$

$$P_{66} = 11N^7 + 37N^6 + 53N^5 + 7N^4 - 68N^3 - 56N^2 - 80N - 48 \tag{117}$$

$$P_{67} = 25N^7 + 91N^6 + 101N^5 - 195N^4 - 546N^3 - 556N^2 - 520N - 224 \tag{118}$$

$$P_{68} = 62N^7 + 329N^6 + 986N^5 + 1790N^4 + 2242N^3 + 1653N^2 + 650N + 96 \tag{119}$$

$$P_{69} = N^8 + 8N^7 + 8N^6 - 14N^5 - 53N^4 - 82N^3 + 60N^2 + 104N + 96 \tag{120}$$

$$P_{70} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \tag{121}$$

$$P_{71} = 6N^8 - 42N^7 - 241N^6 - 579N^5 - 307N^4 + 477N^3 + 602N^2 + 492N + 168 \quad (122)$$

$$P_{72} = 10N^8 + 71N^7 + 244N^6 + 497N^5 + 698N^4 + 720N^3 + 512N^2 + 248N + 48 \quad (123)$$

$$P_{73} = 19N^9 + 86N^8 + 144N^7 - 38N^6 - 535N^5 - 1016N^4 - 1180N^3 - 872N^2 - 416N - 96 \quad (124)$$

$$P_{74} = N^{10} + 15N^9 + 105N^8 + 361N^7 + 660N^6 + 828N^5 + 814N^4 + 384N^3 - 112N^2 - 128N - 32 \quad (125)$$

$$P_{75} = 6N^{10} + 49N^9 + 197N^8 + 472N^7 + 833N^6 + 1469N^5 + 2142N^4 + 1904N^3 + 1040N^2 + 432N + 96 \quad (126)$$

$$P_{76} = 11N^{10} + 123N^9 + 541N^8 + 1273N^7 + 1806N^6 + 1672N^5 + 1006N^4 + 320N^3 - 16N^2 - 64N - 32 \quad (127)$$

$$P_{77} = 19N^{10} + 143N^9 + 412N^8 + 426N^7 - N^6 + 159N^5 + 1066N^4 + 1552N^3 + 1456N^2 + 848N + 224 \quad (128)$$

$$P_{78} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288 \quad (129)$$

$$P_{79} = 60N^{10} + 397N^9 + 1073N^8 + 1111N^7 + 623N^6 + 4328N^5 + 12432N^4 + 15944N^3 + 12704N^2 + 6816N + 1728 \quad (130)$$

$$P_{80} = 67N^{10} + 383N^9 + 867N^8 + 696N^7 - 755N^6 - 2391N^5 - 3027N^4 - 2744N^3 - 1256N^2 - 48N + 144 \quad (131)$$

$$P_{81} = 77N^{10} + 646N^9 + 2553N^8 + 6903N^7 + 14498N^6 + 22898N^5 + 24861N^4 + 17068N^3 + 7040N^2 + 1760N + 192 \quad (132)$$

$$P_{82} = 85N^{10} + 530N^9 + 1458N^8 + 2112N^7 + 1744N^6 + 2016N^5 + 3399N^4 + 2968N^3 + 1864N^2 + 1248N + 432 \quad (133)$$

$$P_{83} = 118N^{10} + 675N^9 + 1588N^8 + 1652N^7 + 326N^6 + 357N^5 + 876N^4 + 1672N^3 + 3440N^2 + 2544N + 576 \quad (134)$$

$$P_{84} = 127N^{10} + 740N^9 + 1737N^8 + 1308N^7 - 1592N^6 - 2226N^5 + 1386N^4 + 3064N^3 + 3040N^2 + 2496N + 864 \quad (135)$$

$$P_{85} = 151N^{10} + 708N^9 + 1156N^8 + 464N^7 - 967N^6 + 372N^5 + 3672N^4 + 5236N^3 + 6152N^2 + 3792N + 864 \quad (136)$$

$$P_{86} = 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3 + 6512N^2 + 2432N + 384 \quad (137)$$

$$P_{87} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32 \quad (138)$$

$$P_{88} = 118N^{11} + 529N^{10} + 1264N^9 + 3846N^8 + 11353N^7 + 23684N^6 + 32793N^5 + 31801N^4 + 22836N^3 + 10448N^2 + 2592N + 432 \quad (139)$$

$$P_{89} = 127N^{11} + 820N^{10} + 2197N^9 + 1890N^8 - 1847N^7 - 1960N^6 + 3843N^5 + 9730N^4 + 13632N^3 + 10688N^2 + 4944N + 864 \quad (140)$$

$$P_{90} = 136N^{11} + 1039N^{10} + 3100N^9 + 3534N^8 - 1295N^7 - 6352N^6 - 8421N^5 - 11729N^4 - 7644N^3 + 1376N^2 + 1920N + 144 \quad (141)$$

$$P_{91} = 7N^{12} + 47N^{11} + 123N^{10} + 76N^9 - 598N^8 - 2178N^7 - 3626N^6 - 3933N^5 - 3254N^4 - 1608N^3 - 144N^2 + 112N + 32 \quad (142)$$

$$P_{92} = 37N^{12} + 305N^{11} + 1017N^{10} + 1462N^9 + 592N^8 + 408N^7 + 4064N^6 + 9645N^5 + 12222N^4 + 10280N^3 + 6064N^2 + 2192N + 352 \quad (143)$$

$$P_{93} = 242N^{12} + 1853N^{11} + 6173N^{10} + 12711N^9 + 18608N^8 + 17040N^7 - 302N^6 - 24986N^5 - 32225N^4 - 20010N^3 - 7904N^2 - 2016N - 288 \quad (144)$$

$$P_{94} = 5N^{13} + 27N^{12} - 97N^{11} - 1410N^{10} - 5754N^9 - 12428N^8 - 16530N^7 - 14531N^6 - 7956N^5 - 1038N^4 + 2176N^3 + 1632N^2 + 448N + 32 \quad (145)$$

$$P_{95} = 119N^{13} + 1897N^{12} + 12595N^{11} + 48221N^{10} + 124877N^9 + 239946N^8 + 345670N^7 + 356234N^6 + 253043N^5 + 129982N^4 + 55768N^3 + 20112N^2 + 5616N + 864 \quad (146)$$

$$P_{96} = 686N^{14} + 8408N^{13} + 39228N^{12} + 89257N^{11} + 113445N^{10} + 109336N^9 + 76360N^8 - 109649N^7 - 393915N^6 - 482272N^5 - 376932N^4 - 263440N^3 - 155472N^2 - 56448N - 8640 \quad (147)$$

$$P_{97} = 1790N^{14} + 15938N^{13} + 56250N^{12} + 90805N^{11} + 43917N^{10} - 38450N^9 - 42314N^8 - 169217N^7 - 616623N^6 - 992860N^5 - 964980N^4 - 697072N^3 - 376464N^2 - 127872N - 19008 \quad (148)$$

$$P_{98} = 30N^{16} + 397N^{15} + 1996N^{14} + 3786N^{13} - 3905N^{12} - 30084N^{11} - 44372N^{10} + 5100N^9 + 71344N^8 + 27709N^7 - 104744N^6 - 146534N^5 - 30293N^4 + 77346N^3 + 33768N^2 - 23544N - 3888 \quad (149)$$

$$P_{99} = 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} - 17182N^{10} - 36527N^9 - 27469N^8 - 11770N^7 + 5554N^6 + 32640N^5 + 46456N^4 + 34528N^3 + 14816N^2 + 3584N + 384 \quad (150)$$

$$P_{100} = 1245N^{18} + 19980N^{17} + 133282N^{16} + 461805N^{15} + 787161N^{14} + 185392N^{13} - 1368400N^{12} - 225082N^{11} + 6978631N^{10} + 13143336N^9 + 5808466N^8 - 11433627N^7 - 19928573N^6 - 12013164N^5 + 1462668N^4 + 8209584N^3 + 6906384N^2 + 2980800N + 544320. \quad (151)$$

The Wilson coefficient $H_{g,2}^S$, except for the constant contribution $a_{Qg}^{(3)}$, has a similar structure. It is given by :

$$\begin{aligned} H_{g,2}^S &= \frac{1}{2}[1 + (-1)^N] \\ &\times \left\{ a_s T_F \left\{ -\tilde{\gamma}_{qg}^0 L_Q - \frac{4(N^3 - 4N^2 - N - 2)}{N^2(N+1)(N+2)} + \tilde{\gamma}_{qg}^0 S_1 + \tilde{\gamma}_{qg}^0 L_M \right\} \right. \\ &+ a_s^2 \left\{ T_F^2 \left[\frac{4}{3}\tilde{\gamma}_{qg}^0 L_M^2 - \frac{4}{3}\tilde{\gamma}_{qg}^0 L_Q L_M + \left[\frac{4}{3}\tilde{\gamma}_{qg}^0 S_1 - \frac{16(N^3 - 4N^2 - N - 2)}{3N^2(N+1)(N+2)} \right] L_M \right] \right. \\ &+ C_F T_F \left[\left[\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} + 2\tilde{\gamma}_{qg}^0 S_1 \right] L_Q^2 + \left[-\frac{4P_{111}}{N^3(N+1)^3(N+2)} \right. \right. \\ &+ \left. \frac{4(3N^4 + 2N^3 - 9N^2 - 16N - 12)S_1}{N^2(N+1)^2(N+2)} + L_M \left[-\frac{4(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \right. \right. \\ &\left. \left. - 4\tilde{\gamma}_{qg}^0 S_1 \right] + \tilde{\gamma}_{qg}^0 (4S_2 - 4S_1^2) \right] L_Q - \frac{2(9N^4 + 6N^3 - 15N^2 - 28N - 20)S_1^2}{N^2(N+1)^2(N+2)} \\ &\left. + \frac{16(-1)^N P_{252}}{15(N-2)(N-1)^2 N^2 (N+1)^4 (N+2)^4 (N+3)^3} + L_M^2 \left[\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + 2\tilde{\gamma}_{qg}^0 S_1 \Big] + \frac{4P_{265}}{15(N-1)^2 N^4 (N+1)^4 (N+2)^4 (N+3)^3} + \frac{4P_{106} S_1}{N^3 (N+1)^3 (N+2)} \\
& + L_M \left[\frac{4P_{111}}{N^3 (N+1)^3 (N+2)} - \frac{4(3N^4 + 2N^3 - 9N^2 - 16N - 12)S_1}{N^2 (N+1)^2 (N+2)} + \tilde{\gamma}_{qg}^0 [4S_1^2 - 4S_2] \right] \\
& + \frac{2(11N^4 + 42N^3 + 43N^2 - 4N - 12)S_2}{N^2 (N+1)^2 (N+2)} + \frac{16P_{103} S_{-2}}{(N-2)N^2 (N+1)^2 (N+2)(N+3)} \\
& + \tilde{\gamma}_{qg}^0 \left[2S_1^3 - 2S_2 S_1 - 4S_{2,1} \right] + \frac{128S_1 S_{-2} + 64S_{-3} - 128S_{-2,1}}{N(N+1)(N+2)} + \frac{(N-1)[48\zeta_3 - 16S_3]}{N(N+1)} \Big] \\
& + C_{AT_F} \left[\left[\frac{16(N^2 + N + 1)(N^2 + N + 2)}{(N-1)N^2 (N+1)^2 (N+2)^2} + 2\tilde{\gamma}_{qg}^0 S_1 \right] L_Q^2 + \left[-\frac{32(-1)^N (N^3 + 4N^2 + 7N + 5)}{(N+1)^3 (N+2)^3} \right. \right. \\
& + \frac{8P_{160}}{(N-1)N^3 (N+1)^2 (N+2)^3} + \frac{16(N^2 + 1)(N^2 - 4N - 1)S_1}{(N-1)N(N+1)^2 (N+2)} + \tilde{\gamma}_{qg}^0 [-2S_1^2 + 2S_2 + 4S_{-2}] \Big] L_Q \\
& - \frac{16P_{165} S_1}{(N-1)N^3 (N+1)^2 (N+2)^3} - \frac{4(2N^5 - 3N^4 - 5N^3 - 3N^2 - 33N - 6)S_1^2}{(N-1)N(N+1)^2 (N+2)^2} \\
& - \frac{16(-1)^N P_{167}}{3(N-1)N^2 (N+1)^4 (N+2)^4} - \frac{8P_{239}}{3(N-1)N^4 (N+1)^4 (N+2)^4} \\
& + \frac{32(-1)^N (N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3 (N+2)^3} + L_M^2 \left[-\frac{16(N^2 + N + 1)(N^2 + N + 2)}{(N-1)N^2 (N+1)^2 (N+2)^2} - 2\tilde{\gamma}_{qg}^0 S_1 \right] \\
& + \frac{4P_{105} S_2}{(N-1)N^2 (N+1)^2 (N+2)^2} + \frac{8(3N^2 + 3N + 2)S_3}{N(N+1)(N+2)} + L_M \left[\frac{32(-1)^N (N^3 + 4N^2 + 7N + 5)}{(N+1)^3 (N+2)^3} \right. \\
& - \frac{8P_{161}}{(N-1)N^3 (N+1)^2 (N+2)^3} - \frac{32(2N + 3)S_1}{(N+1)^2 (N+2)^2} + \tilde{\gamma}_{qg}^0 [-2S_1^2 - 2S_2 - 4S_{-2}] \Big] \\
& + \frac{16(N^5 - N^4 - 5N^3 + 3N^2 + 14N + 12)S_{-2}}{(N-1)N(N+1)^2 (N+2)^2} - \frac{16(N^2 + N + 4)S_{-3}}{N(N+1)(N+2)} \\
& + \frac{(N-1)[16S_{-2} S_1 - 16S_{-2,1}]}{N(N+1)} + \frac{(N^2 - N - 4)16(-1)^N S_{-2}}{(N+1)^2 (N+2)^2} - \frac{6(5N^2 + 5N - 6)\zeta_3}{N(N+1)(N+2)} \\
& + \tilde{\gamma}_{qg}^0 \left[-8S_1 S_2 - 4(-1)^N S_{-2} S_1 - 2(-1)^N S_{-3} + 4S_{2,1} - \frac{3}{2}(-1)^N \zeta_3 \right] \Big] \Big\} \\
& + a_s^3 \left\{ T_F^3 \left[\frac{16}{9}\tilde{\gamma}_{qg}^0 L_M^3 - \frac{16}{9}\tilde{\gamma}_{qg}^0 L_Q L_M^2 + \left[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 - \frac{64(N^3 - 4N^2 - N - 2)}{9N^2 (N+1)(N+2)} \right] L_M^2 - \frac{16\tilde{\gamma}_{qg}^0 \zeta_3}{9} \right] \right. \\
& + C_{AT_F}^2 \left[\left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2 (N+1)^2 (N+2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1 \right] L_Q^3 + \left[-\frac{64(-1)^N (N^3 + 4N^2 + 7N + 5)}{3(N+1)^3 (N+2)^3} \right. \right. \\
& + \frac{8P_{194}}{9(N-1)N^3 (N+1)^3 (N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)^2 (N+2)} \\
& + L_M \left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{3(N-1)N^2 (N+1)^2 (N+2)^2} + \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1 \right] + \tilde{\gamma}_{qg}^0 \left[-\frac{4}{3}S_1^2 + \frac{4S_2}{3} + \frac{8}{3}S_{-2} \right] \Big] L_Q^2 \\
& + \left[-\frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1^2}{9(N-1)N(N+1)^2 (N+2)} + \frac{128(-1)^N (N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3 (N+2)^3} \right. \\
& - \frac{32P_{186} S_1}{27(N-1)N^2 (N+1)^3 (N+2)^3} + \frac{64(-1)^N P_{170}}{9(N-1)N^2 (N+1)^4 (N+2)^4}
\end{aligned}$$

$$\begin{aligned}
& -\frac{16P_{233}}{27(N-1)N^3(N+1)^4(N+2)^4} + L_M^2 \left[\frac{64(N^2+N+1)(N^2+N+2)}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{3}\tilde{\gamma}_{ag}^0 S_1 \right] \\
& + \frac{32(8N^4+13N^3-22N^2-9N-26)S_2}{9(N-1)N(N+1)(N+2)^2} + \frac{64(8N^5+15N^4+6N^3+11N^2+16N+16)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} \\
& + \frac{128(N^2+N-1)S_3}{9N(N+1)(N+2)} + L_M \left[-\frac{128(-1)^N(N^3+4N^2+7N+5)}{3(N+1)^3(N+2)^3} + \frac{32P_{195}}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{64(2N-1)(N^3+9N^2+7N+7)S_1}{9(N-1)N(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[-\frac{8}{3}S_1^2 + \frac{8S_2}{3} + \frac{16}{3}S_{-2} \right] \right] - \frac{128(N^2+N+3)S_{-3}}{3N(N+1)(N+2)} \\
& + \tilde{\gamma}_{ag}^0 \left[\frac{8}{9}S_1^3 - 8S_2S_1 + \frac{32}{3}S_{2,1} \right] + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1) \left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3 \right]}{N(N+1)} \Big] L_Q \\
& + \frac{32(N^3+8N^2+11N+2)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{152}S_1^2}{27N(N+1)^3(N+2)^3} + \frac{4}{9} \frac{P_{201}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^3} \\
& + \frac{8P_{264}}{81(N-1)N^5(N+1)^5(N+2)^5} + \frac{32}{3}(2N^4+3N^3+10N^2+37N+35) \frac{(-1)^N\zeta_2}{(N+1)^3(N+2)^3} \\
& + \frac{32(9N^5-4N^4+N^3+92N^2+42N+28)\zeta_3}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{8P_{249}S_1}{81(N-1)N^4(N+1)^4(N+2)^4} \\
& - \frac{32(5N^4+8N^3+17N^2+43N+20)\zeta_2}{9N(N+1)^2(N+2)^2} S_1 + L_M^3 \left[-\frac{448(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. - \frac{56}{9}\tilde{\gamma}_{ag}^0 S_1 \right] + \frac{8P_{192}S_2}{3(N-1)N^3(N+1)^3(N+2)^3} - \frac{32(3N^3-12N^2-27N-2)S_1S_2}{3N(N+1)^2(N+2)^2} \\
& + \frac{256(N^5+10N^4+9N^3+3N^2+7N+6)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + L_M^2 \left[\frac{64(-1)^N(N^3+4N^2+7N+5)}{(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{8P_{199}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^5+9N^4-57N^3-31N^2+25N-26)S_1}{9(N-1)N(N+1)^2(N+2)^2} \right. \\
& \left. + \tilde{\gamma}_{ag}^0 \left[-\frac{20}{3}S_1^2 - 4S_2 - 8S_{-2} \right] \right] + \frac{128(-1)^N(N^4+2N^3+7N^2+22N+20)S_{-2}}{3(N+1)^3(N+2)^3} \\
& + \frac{(N^2-N-4)}{(N+1)^2(N+2)^2} \left[-\frac{256}{3}(-1)^N S_1S_{-2} - \frac{128}{3}(-1)^N S_{-3} + \frac{256}{3}S_{-2,1} - \frac{128}{3}(-1)^N S_1\zeta_2 \right. \\
& \left. - 32(-1)^N\zeta_3 \right] + \tilde{\gamma}_{ag}^0 \left[\frac{2}{9}S_1^4 + \frac{20}{3}S_2S_1^2 + \frac{32}{3}(-1)^N S_{-3}S_1 + \left(\frac{160S_3}{9} - \frac{64}{3}S_{-2,1} \right) S_1 + \frac{2}{3}S_2^2 \right. \\
& \left. + \frac{8}{9}(-2+9(-1)^N)\zeta_3S_1 + S_{-2} \left(\frac{32}{3}(-1)^N S_1^2 + \frac{32}{3}(-1)^N S_2 \right) + 12S_4 \right. \\
& \left. + \frac{16}{3}(-1)^N S_{-4} - \frac{16}{3}S_{3,1} - \frac{32}{3}S_{-2,2} - \frac{32}{3}S_{-3,1} - \frac{16}{3}S_{2,1,1} + \frac{64}{3}S_{-2,1,1} + \left[\frac{2}{3}(-3+8(-1)^N)S_1^2 \right. \right. \\
& \left. \left. + \frac{2}{3}(-3+8(-1)^N)S_2 + \frac{4}{3}(1+4(-1)^N)S_{-2} \right] \zeta_2 \right] + L_M \left[\frac{64(N^5+9N^4+3N^3+N^2+26N-4)S_1^2}{9(N-1)N(N+1)^2(N+2)^2} \right. \\
& \left. + \frac{128(-1)^N(N^3+4N^2+7N+5)S_1}{3(N+1)^3(N+2)^3} + \frac{16P_{198}S_1}{27(N-1)N^3(N+1)^3(N+2)^3} + \frac{64(N-1)S_{-2}S_1}{3N(N+1)} \right. \\
& \left. - \frac{64(-1)^N P_{157}}{9(N-1)N^2(N+1)^3(N+2)^4} + \frac{16P_{242}}{27(N-1)N^4(N+1)^4(N+2)^4} + \frac{64(7N^2+7N+8)S_3}{9N(N+1)(N+2)} \right. \\
& \left. - \frac{64P_{104}S_2}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{64(2N^5+21N^4+36N^3-7N^2-68N-56)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{128S_{-3}}{3N(N+1)(N+2)} - \frac{128S_{-2,1}}{3(N+2)} + \frac{(N^2 - N - 4)\frac{128}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2} - \frac{16(3N^2 + 3N - 2)\zeta_3}{N(N+1)(N+2)} \\
& + \tilde{\gamma}_{ag}^0 \left[-\frac{8}{9}S_1^3 - \frac{40}{3}S_2S_1 - \frac{32}{3}(-1)^N S_{-2}S_1 - \frac{16}{3}(-1)^N S_{-3} - 4(-1)^N \zeta_3 \right] \Bigg] \\
& + C_{AT_F^2 N_F} \left[\left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{ag}^0 S_1 \right] L_Q^3 + \left[-\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right. \right. \\
& + \frac{8P_{194}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1}{9(N-1)N(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[-\frac{4}{3}S_1^2 + \frac{4S_2}{3} \right. \\
& \left. \left. + \frac{8}{3}S_{-2} \right] \right] L_Q^2 + \left[-\frac{32(8N^4 - 7N^3 + 5N^2 - 17N - 13)S_1^2}{9(N-1)N(N+1)^2(N+2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{3(N+1)^3(N+2)^3} \right. \\
& - \frac{16P_{187}S_1}{27(N-1)N^2(N+1)^3(N+2)^3} + \frac{64(-1)^N P_{170}}{9(N-1)N^2(N+1)^4(N+2)^4} + \frac{128(N^2 + N - 1)S_3}{9N(N+1)(N+2)} \\
& - \frac{8P_{248}}{27(N-1)N^4(N+1)^4(N+2)^4} + \frac{32(8N^4 + 13N^3 - 22N^2 - 9N - 26)S_2}{9(N-1)N(N+1)(N+2)^2} \\
& + \frac{64(8N^5 + 15N^4 + 6N^3 + 11N^2 + 16N + 16)S_{-2}}{9(N-1)N(N+1)^2(N+2)^2} - \frac{128(N^2 + N + 3)S_{-3}}{3N(N+1)(N+2)} \\
& + \tilde{\gamma}_{ag}^0 \left[\frac{8}{9}S_1^3 - 8S_2S_1 + \frac{32}{3}S_{2,1} \right] + \frac{256S_{-2,1}}{3N(N+1)(N+2)} + \frac{(N-1)\left[\frac{64}{3}S_{-2}S_1 - 32\zeta_3\right]}{N(N+1)} \Bigg] L_Q \\
& + \frac{16(N^3 + 8N^2 + 11N + 2)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{102}S_1^2}{3N(N+1)^3(N+2)^3} + \frac{4}{9} \frac{P_{175}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{16P_{261}}{3(N-1)N^5(N+1)^5(N+2)^5} + \frac{16}{9} \frac{(9N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16P_{166}S_1}{3N(N+1)^4(N+2)^4} + \frac{16}{9} \frac{(5N^4 + 32N^3 + 47N^2 + 28N + 20)\zeta_2}{N(N+1)^2(N+2)^2} S_1 \\
& + L_M^3 \left[-\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{8}{9}\tilde{\gamma}_{ag}^0 S_1 \right] + \frac{8P_{191}S_2}{3(N-1)N^3(N+1)^3(N+2)^3} \\
& - \frac{16(3N^3 - 12N^2 - 27N - 2)S_1S_2}{3N(N+1)^2(N+2)^2} + \frac{128(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& + L_M^2 \left[\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} - \frac{8P_{172}}{9(N-1)N^2(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9N(N+1)^2(N+2)^2} + \tilde{\gamma}_{ag}^0 \left[-\frac{4}{3}S_1^2 - \frac{4S_2}{3} - \frac{8}{3}S_{-2} \right] \right] \\
& + \frac{(N^4 + 2N^3 + 7N^2 + 22N + 20)\left[\frac{64}{3}(-1)^N S_{-2} + \frac{32}{3}(-1)^N \zeta_2\right]}{(N+1)^3(N+2)^3} \\
& + \frac{(N^2 - N - 4)\left[-\frac{128}{3}(-1)^N S_1S_{-2} - \frac{64}{3}(-1)^N S_{-3} + \frac{128}{3}S_{-2,1} - \frac{64}{3}(-1)^N S_1\zeta_2 - 16(-1)^N \zeta_3\right]}{(N+1)^2(N+2)^2} \\
& + \tilde{\gamma}_{ag}^0 \left[\frac{1}{9}S_1^4 + \frac{10}{3}S_2S_1^2 + \frac{16}{3}(-1)^N S_{-3}S_1 + \left[\frac{80S_3}{9} - \frac{32}{3}S_{-2,1} \right] S_1 + \frac{4}{9}(-7 + 9(-1)^N)\zeta_3S_1 + \frac{1}{3}S_2^2 \right. \\
& + S_{-2} \left[\frac{16}{3}(-1)^N S_1^2 + \frac{16}{3}(-1)^N S_2 \right] + 6S_4 + \frac{8}{3}(-1)^N S_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} - \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} \\
& \left. + \frac{32}{3}S_{-2,1,1} + \left[\frac{4}{3}(-1 + 2(-1)^N)S_1^2 + \frac{4}{3}(-1 + 2(-1)^N)S_2 + \frac{8}{3}(-1)^N S_{-2} \right] \zeta_2 \right]
\end{aligned}$$

$$\begin{aligned}
& +L_M \left[-\frac{16(10N^4 + 43N^3 + 106N^2 + 131N + 46)S_1^2}{9N(N+1)^2(N+2)^2} + \frac{16P_{140}S_1}{27N(N+1)^3(N+2)^3} \right. \\
& - \frac{64(-1)^N(7N^5 + 43N^4 + 117N^3 + 166N^2 + 107N + 16)}{9(N+1)^4(N+2)^4} + \frac{8P_{246}}{27(N-1)N^4(N+1)^4(N+2)^4} \\
& - \frac{16P_{122}S_2}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{64(5N^2 + 8N + 10)S_{-2}}{9N(N+1)(N+2)} + \frac{(N^2 - N - 4)\frac{64}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2} \\
& \left. + \tilde{\gamma}_{ag}^0 \left[-\frac{8}{9}S_1^3 - \frac{8}{3}S_2S_1 - \frac{16}{3}(-1)^N S_{-2}S_1 - \frac{40S_3}{9} - \frac{8}{3}(2 + (-1)^N)S_{-3} - \frac{16}{3}S_{2,1} + \frac{16}{3}S_{-2,1} \right] \right] \\
& + C_F^2 T_F \left[-\frac{(15N^4 + 6N^3 - 25N^2 - 32N - 28)S_1^4}{3N^2(N+1)^2(N+2)} + \frac{2P_{159}S_1^2}{N^3(N+1)^4(N+2)} \right. \\
& - \frac{4(3N^5 - 47N^4 - 147N^3 - 93N^2 + 8N + 12)S_1^3}{3N^3(N+1)^2(N+2)} - 4\frac{(3N^4 + 14N^3 + 43N^2 + 48N + 20)\zeta_2}{N^2(N+1)^2(N+2)} S_1^2 \\
& - \frac{2(5N^4 - 14N^3 + 53N^2 + 120N + 28)S_2S_1^2}{N^2(N+1)^2(N+2)} - \frac{2P_{211}S_1}{N^5(N+1)^5(N+2)} - \frac{4P_{118}S_2S_1}{N^3(N+1)^3(N+2)} \\
& - \frac{8(3N^4 + 90N^3 + 83N^2 - 44N - 4)S_3S_1}{3N^2(N+1)^2(N+2)} - \frac{16(3N^4 + 2N^3 + 19N^2 + 28N + 12)S_{2,1}S_1}{N^2(N+1)^2(N+2)} \\
& - \frac{8P_{132}\zeta_2S_1}{N^3(N+1)^3(N+2)} - \frac{(11N^4 + 142N^3 + 147N^2 - 32N - 12)S_2^2}{N^2(N+1)^2(N+2)} - \frac{P_{247}}{N^6(N+1)^6(N+2)} \\
& + 16\frac{(-1)^N(N^2 + N + 2)\zeta_2}{N(N+1)^4(N+2)} + \frac{2}{3}(N^2 + N + 2)\frac{(153N^4 + 306N^3 + 165N^2 + 12N + 4)\zeta_3}{N^3(N+1)^3(N+2)} \\
& + L_Q^3 \left[\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N+1)^3(N+2)} - \frac{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)}{3N^2(N+1)^2(N+2)} \right. \\
& - \left. \frac{8}{3}\tilde{\gamma}_{ag}^0 S_1^2 \right] + L_M^3 \left[-\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{3N^3(N+1)^3(N+2)} + \frac{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)}{3N^2(N+1)^2(N+2)} \right. \\
& + \left. \frac{8}{3}\tilde{\gamma}_{ag}^0 S_1^2 \right] - \frac{2P_{182}S_2}{N^4(N+1)^4(N+2)} + \frac{4(21N^5 + 217N^4 + 415N^3 + 351N^2 + 152N - 4)S_3}{3N^2(N+1)^3(N+2)} \\
& - \frac{16(N^2 - 3N - 2)(3N^2 + 3N + 2)S_{2,1}}{N^3(N+1)^2(N+2)} + L_M^2 \left[\frac{32(N^3 + 5N^2 + 6N + 4)S_1^2}{N^2(N+1)^2(N+2)} \right. \\
& + \frac{2P_{138}S_1}{N^3(N+1)^3(N+2)} - \frac{32(-1)^N(N^2 + N + 2)}{N(N+1)^4(N+2)} - \frac{P_{178}}{N^4(N+1)^4(N+2)} \\
& - \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)S_2}{N^2(N+1)^2(N+2)} - \frac{32(N^2 + N + 2)S_{-2}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[8S_1^3 - 16S_2S_1 \right. \\
& \left. - 16S_{-2}S_1 - 8S_3 - 8S_{-3} + 16S_{-2,1} \right] + L_Q^2 \left[\frac{32(N^3 + 5N^2 + 6N + 4)S_1^2}{N^2(N+1)^2(N+2)} + \frac{2P_{138}S_1}{N^3(N+1)^3(N+2)} \right. \\
& - \frac{32(-1)^N(N^2 + N + 2)}{N(N+1)^4(N+2)} - \frac{P_{178}}{N^4(N+1)^4(N+2)} + L_M \left[-\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)^2}{N^3(N+1)^3(N+2)} \right. \\
& + \left. \frac{16(N^2 + N + 2)S_1(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} + 8\tilde{\gamma}_{ag}^0 S_1^2 \right] - \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)S_2}{N^2(N+1)^2(N+2)} \\
& - \left. \frac{32(N^2 + N + 2)S_{-2}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[8S_1^3 - 16S_2S_1 - 16S_{-2}S_1 - 8S_3 - 8S_{-3} + 16S_{-2,1} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{P_{190}\zeta_2}{2N^4(N+1)^4(N+2)} + \frac{16(N^2+N+2)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} + 96\tilde{\gamma}_{ag}^0 \log(2)\zeta_2 \\
& + \frac{(N^2+N+2)(3N^2+3N+2)\left[6S_4 - 16S_{3,1} + 32S_{2,1,1} + 8S_2\zeta_2 - \frac{16}{3}S_1\zeta_3\right]}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[\frac{1}{3}S_1^5\right. \\
& - \frac{2}{3}S_2S_1^3 + \left(-\frac{16}{3}S_3 - 16S_{2,1}\right)S_1^2 - \frac{8}{3}\zeta_3S_1^2 + \left[-3S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1}\right]S_1 \\
& \left. + \frac{8}{3}S_2S_3 + \left[-4S_1^3 + 8S_2S_1 + 8S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1}\right]\zeta_2\right] \\
& + L_Q \left[\frac{16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N+1)^2(N+2)} - \frac{2P_{136}S_1^2}{N^3(N+1)^3(N+2)} + \frac{64(-1)^N(N^2+N+2)S_1}{N(N+1)^4(N+2)} \right. \\
& - \frac{2P_{139}S_1}{N^4(N+1)^3} - \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2S_1}{N^2(N+1)^2(N+2)} - \frac{64(2N^3 + N^2 + 3N - 10)S_{-2}S_1}{N^2(N+1)^2(N+2)} \\
& - \frac{32(-1)^N P_{266}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^5(N+3)^3} + \frac{2P_{277}}{5(N-1)^2N^5(N+1)^5(N+2)^5(N+3)^3} \\
& - 48 \frac{(9N^4 + 10N^3 + 9N^2 - 8N + 12)\zeta_3}{N^2(N+1)^2(N+2)} + L_M^2 \left[\frac{2(N^2+N+2)(3N^2+3N+2)^2}{N^3(N+1)^3(N+2)} \right. \\
& \left. - \frac{16(N^2+N+2)S_1(3N^2+3N+2)}{N^2(N+1)^2(N+2)} - 8\tilde{\gamma}_{ag}^0 S_1^2 \right] + \frac{2P_{147}S_2}{N^3(N+1)^3(N+2)} \\
& - \frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)S_3}{N^2(N+1)^2(N+2)} + \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_{-2}}{(N+1)^3(N+2)^3} \\
& - \frac{64P_{212}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)} - \frac{64(N^2+3N+4)S_{-3}}{N(N+1)^2(N+2)} \\
& + \frac{16(N^2+N+2)(3N^2+3N+2)S_{2,1}}{N^2(N+1)^2(N+2)} + L_M \left[-\frac{64(N^3+5N^2+6N+4)S_1^2}{N^2(N+1)^2(N+2)} \right. \\
& - \frac{4P_{138}S_1}{N^3(N+1)^3(N+2)} + \frac{64(-1)^N(N^2+N+2)}{N(N+1)^4(N+2)} + \frac{2P_{178}}{N^4(N+1)^4(N+2)} \\
& + \frac{32(N^2+N+2)(3N^2+3N+2)S_2}{N^2(N+1)^2(N+2)} + \frac{64(N^2+N+2)S_{-2}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[-16S_1^3 + 32S_2S_1 + 32S_{-2}S_1 \right. \\
& \left. + 16S_3 + 16S_{-3} - 32S_{-2,1} \right] + \frac{128(N-1)(N^2+2N+4)S_{-2,1}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[-8S_1^4 + 24S_2S_1^2 - 48S_{-3}S_1 \right. \\
& \left. + \left[16S_{2,1} - 32S_{-2,1} \right] S_1 - 48\zeta_3S_1 - 16S_2^2 - 32S_{-2}^2 + S_{-2} \left[32S_1^2 - 32S_2 \right] - 32S_4 - 80S_{-4} - 32S_{3,1} \right. \\
& \left. + 32S_{-2,2} + 64S_{-3,1} \right] + L_M \left[-\frac{16(3N^4 - 13N^2 - 18N - 12)S_1^3}{N^2(N+1)^2(N+2)} + \frac{2P_{136}S_1^2}{N^3(N+1)^3(N+2)} \right. \\
& - \frac{64(-1)^N(N^2+N+2)S_1}{N(N+1)^4(N+2)} + \frac{2P_{139}S_1}{N^4(N+1)^3} + \frac{16(7N^4 + 20N^3 + 7N^2 - 22N - 20)S_2S_1}{N^2(N+1)^2(N+2)} \\
& + \frac{64(2N^3 + N^2 + 3N - 10)S_{-2}S_1}{N^2(N+1)^2(N+2)} + \frac{32(-1)^N P_{266}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^5(N+3)^3} \\
& - \frac{2P_{277}}{5(N-1)^2N^5(N+1)^5(N+2)^5(N+3)^3} + 48 \frac{(9N^4 + 10N^3 + 9N^2 - 8N + 12)\zeta_3}{N^2(N+1)^2(N+2)} \\
& - \frac{2P_{147}S_2}{N^3(N+1)^3(N+2)} + \frac{16(3N^4 + 10N^3 + 15N^2 + 16N - 12)S_3}{N^2(N+1)^2(N+2)}
\end{aligned}$$

$$\begin{aligned}
& - \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_{-2}}{(N+1)^3(N+2)^3} + \frac{64P_{212}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)} \\
& + \frac{64(N^2 + 3N + 4)S_{-3}}{N(N+1)^2(N+2)} - \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)S_{2,1}}{N^2(N+1)^2(N+2)} - \frac{128(N-1)(N^2 + 2N + 4)S_{-2,1}}{N^2(N+1)^2(N+2)} \\
& + \tilde{\gamma}_{gg}^0 \left[8S_1^4 - 24S_2S_1^2 + 48S_{-3}S_1 + \left[32S_{-2,1} - 16S_{2,1} \right] S_1 + 48\zeta_3S_1 + 16S_2^2 + 32S_{-2}^2 \right. \\
& \left. + S_{-2} \left[32S_2 - 32S_1^2 \right] + 32S_4 + 80S_{-4} + 32S_{3,1} - 32S_{-2,2} - 64S_{-3,1} \right] \Bigg] \\
& + C_{AF}^2 \left[\frac{P_{123}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{176}S_1^3}{9(N-1)N^2(N+1)^3(N+2)^3} \right. \\
& - \frac{4}{3} \frac{P_{131}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{2P_{215}S_1^2}{3(N-1)N^2(N+1)^4(N+2)^4} \\
& + \frac{2(N-2)(55N^5 + 347N^4 + 379N^3 + 213N^2 + 326N + 120)S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{4}{9} \frac{P_{217}S_1\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^3} - \frac{4P_{262}S_1}{3(N-1)N^5(N+1)^5(N+2)^5} \\
& - \frac{4P_{193}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3} + \frac{16P_{144}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{8}{3} \frac{(-1)^N P_{213}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^4} - \frac{2}{9} \frac{P_{251}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^4} \\
& - \frac{4(11N^4 + 22N^3 - 35N^2 - 46N - 24)P_{261}}{3(N-1)^2N^6(N+1)^6(N+2)^6} - \frac{4}{9} \frac{\zeta_3 P_{148}S_1}{(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{4}{9} \frac{(11N^4 + 22N^3 - 35N^2 - 46N - 24)(9N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^3} \\
& + L_Q^3 \left[-\frac{8}{3} \tilde{\gamma}_{gg}^0 S_1^2 + \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& - \frac{16(N^2 + N + 1)(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{9(N-1)^2N^3(N+1)^3(N+2)^3} \Bigg] + L_M^3 \left[\frac{8}{3} \tilde{\gamma}_{gg}^0 S_1^2 \right. \\
& - \frac{8(N^2 + N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& \left. + \frac{16(N^2 + N + 1)(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right] \\
& - \frac{2(11N^4 + 22N^3 - 35N^2 - 46N - 24)P_{191}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^4} \\
& - \frac{32(11N^4 + 22N^3 - 35N^2 - 46N - 24)(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
& - \frac{16(-1)^N(N^4 + 2N^3 + 7N^2 + 22N + 20)(11N^4 + 22N^3 - 35N^2 - 46N - 24)S_{-2}}{3(N-1)N(N+1)^4(N+2)^4} \\
& - 32 \frac{(N^2 + N + 1)(N^2 + N + 2)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2} + L_M^2 \left[-\frac{4P_{127}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. - \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} + \frac{8P_{216}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4} + \frac{16P_{241}}{9(N-1)^2N^4(N+1)^3(N+2)^4} \\
& -\frac{4(N^2+N+2)(11N^4+22N^3-83N^2-94N-72)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& -\frac{8(N^2+N+2)(11N^4+22N^3-59N^2-70N-48)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[4S_1^3 + 12S_2S_1 + 16S_{-2}S_1 \right. \\
& \left. + 4S_3 + 4S_{-3} - 8S_{-2,1} \right] + L_Q^2 \left[-\frac{4P_{141}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. + \frac{64(-1)^N(N^3+4N^2+7N+5)S_1}{(N+1)^3(N+2)^3} - \frac{8P_{218}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right. \\
& \left. + \frac{16(-1)^N P_{196}}{3(N-1)N^3(N+1)^4(N+2)^4} - \frac{16P_{240}}{9(N-1)^2N^4(N+1)^3(N+2)^4} \right. \\
& \left. + \frac{4(N^2+N+2)(11N^4+22N^3-83N^2-94N-72)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. + \frac{8(N^2+N+2)(11N^4+22N^3-59N^2-70N-48)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. + \tilde{\gamma}_{qg}^0 \left[4S_1^3 - 12S_2S_1 - 16S_{-2}S_1 - 4S_3 - 4S_{-3} + 8S_{-2,1} \right] \right] \\
& + \frac{(5N^5 - 131N^3 - 58N^2 + 232N + 96)}{(N-1)N(N+1)^2(N+2)^3} \left[\frac{32}{3}(-1)^N S_1S_{-2} + \frac{16}{3}(-1)^N S_1\zeta_2 \right] \\
& + \frac{(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)}{(N-1)N^2(N+1)^2(N+2)^2} \left[\frac{1}{3}S_2^2 + \frac{16}{3}(-1)^N S_{-2}S_2 \right. \\
& \left. + 6S_4 + \frac{8}{3}(-1)^N S_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} - \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} + \frac{32}{3}S_{-2,1,1} \right. \\
& \left. + \left(\frac{8}{3}(-1)^N S_2 + \frac{8}{3}(-1)^N S_{-2} \right) \zeta_2 \right] + \frac{(N^2-N-4)(11N^4+22N^3-35N^2-46N-24)}{(N-1)N(N+1)^3(N+2)^3} \\
& \times \left[4(-1)^N \zeta_3 + \frac{16}{3}(-1)^N S_{-3} - \frac{32}{3}S_{-2,1} \right] + \frac{P_{130} \left[\frac{16}{3}(-1)^N S_{-2}S_1^2 + \frac{8}{3}(-1)^N \zeta_2 S_1^2 \right]}{(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{(11N^5 + 34N^4 - 49N^3 - 24N^2 - 68N - 48) \left[\frac{16}{3}(-1)^N S_1S_{-3} - \frac{32}{3}S_1S_{-2,1} + 4(-1)^N S_1\zeta_3 \right]}{(N-1)N^2(N+1)(N+2)^2} \\
& + L_Q \left[\frac{8(47N^4 - 133N^3 + 11N^2 - 155N - 58)S_1^3}{9(N-1)N(N+1)^2(N+2)} + \frac{8P_{223}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right. \\
& - \frac{32(N-2)(N+3)S_{-2}S_1^2}{N(N+1)(N+2)} - \frac{32(-1)^N P_{197}S_1}{3(N-1)N^3(N+1)^4(N+2)^4} + \frac{8P_{257}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4} \\
& - \frac{8P_{137}S_2S_1}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{32(N^2+N+6)S_3S_1}{N(N+1)(N+2)} - \frac{16P_{142}S_{-2}S_1}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{32(9N^2+9N+22)S_{-3}S_1}{N(N+1)(N+2)} - \frac{64(3N^2+3N+10)S_{-2,1}S_1}{N(N+1)(N+2)} - \frac{384\zeta_3S_1}{N(N+1)(N+2)} \\
& - 8\frac{P_{117}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{16(-1)^N P_{256}}{9(N-1)^2N^4(N+1)^5(N+2)^5} \\
& + \frac{4P_{269}}{27(N-1)^2N^5(N+1)^5(N+2)^5} - \frac{8P_{219}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
& - \frac{8P_{149}S_3}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{16P_{220}S_{-2}}{9(N-1)^2N^3(N+1)^3(N+2)^3}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(N^3 + 4N^2 + 7N + 5)(-64(-1)^N S_1^2 + 64(-1)^N S_2 + 192(-1)^N S_{-2})}{(N+1)^3(N+2)^3} \\
& + \frac{16P_{133}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{32P_{112}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \tilde{\gamma}_{gg}^0 \left[-2S_1^4 + 16S_2S_1^2 - 2S_2^2 - 12S_{-2}^2 - 16S_{-2}S_2 - 4S_4 - 44S_{-4} - \frac{88}{3}S_{2,1} - 16S_{3,1} + 56S_{-2,2} \right. \\
& \left. + 64S_{-3,1} - 96S_{-2,1,1} \right] + \tilde{\gamma}_{gg}^0 \left[-\frac{1}{3}S_1^5 - 10S_2S_1^3 - 16(-1)^N S_{-3}S_1^2 + \left[32S_{-2,1} - \frac{80S_3}{3} \right] S_1^2 \right. \\
& - \frac{4}{3}(-7 + 9(-1)^N)\zeta_3 S_1^2 - 8(-1)^N S_{-4}S_1 + \left[-S_2^2 - 18S_4 + 8S_{3,1} + 16S_{-2,2} + 16S_{-3,1} + 8S_{2,1,1} \right. \\
& \left. - 32S_{-2,1,1} \right] S_1 + S_{-2} \left[-16(-1)^N S_1^3 - 16(-1)^N S_2S_1 \right] + \left[-4(-1 + 2(-1)^N)S_1^3 - 8(-1)^N S_2S_1 \right. \\
& \left. - 4(1 + 2(-1)^N)S_{-2}S_1 + \frac{11S_2}{3} - 2S_3 - 2S_{-3} + 4S_{-2,1} \right] \zeta_2 \left. \right] + L_M \left[\frac{128(-1)^N P_{171}S_1}{3(N-1)N^2(N+1)^4(N+2)^4} \right. \\
& - \frac{8(11N^4 + 26N^3 - 139N^2 - 218N + 8)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{4P_{221}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
& - \frac{8P_{254}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4} - \frac{8P_{124}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{32(2N^5 - 23N^4 - 32N^3 + 13N^2 + 4N - 12)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)^2} + 2 \frac{P_{151}\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16(-1)^N P_{237}}{9(N-1)N^3(N+1)^5(N+2)^5} - \frac{4P_{268}}{27(N-1)^2N^5(N+1)^5(N+2)^5} \\
& + \frac{4P_{222}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(-64(-1)^N S_1^2 - 64(-1)^N S_2)}{(N+1)^3(N+2)^3} \\
& - \frac{8P_{145}S_3}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{16(-1)^N P_{129}S_{-2}}{3(N-1)N(N+1)^3(N+2)^3} + \frac{16P_{202}S_{-2}}{9(N-1)N^3(N+1)^3(N+2)^3} \\
& - \frac{16P_{126}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{16(N^2 + N + 2)(11N^4 + 22N^3 + 13N^2 + 2N + 24)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{(N-1)N^2(N+1)^2(N+2)^2} \left[-\frac{8}{3}(-1)^N S_{-3} - 2(-1)^N \zeta_3 \right] \\
& + \frac{(11N^5 + 34N^4 - 49N^3 - 24N^2 - 68N - 48) \left[-\frac{16}{3}(-1)^N S_1S_{-2} \right]}{(N-1)N^2(N+1)(N+2)^2} \\
& + \tilde{\gamma}_{gg}^0 \left[2S_1^4 + 32S_2S_1^2 + 8(8 + (-1)^N)S_{-3}S_1 + \left[40S_3 - 16S_{2,1} - 80S_{-2,1} \right] S_1 \right. \\
& + 6(-5 + (-1)^N)\zeta_3 S_1 + 2S_2^2 + 12S_{-2}^2 + S_{-2} \left[8(3 + 2(-1)^N)S_1^2 + 16S_2 \right] + 4S_4 + 44S_{-4} + 16S_{3,1} \\
& \left. - 56S_{-2,2} - 64S_{-3,1} + 96S_{-2,1,1} \right] + \frac{16P_{125}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \left. \right] \\
& + C_F T_F^2 \left[\left[\frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^4 + 6N^3 - N^2 - 4N + 12)}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{16}{9}\tilde{\gamma}_{gg}^0 S_1 \right] L_Q^3 \right. \\
& + \left[-\frac{16L_M(N^2 + N + 2)^3}{(N-1)N^3(N+1)^3(N+2)^2} - \frac{4P_{232}}{9(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& \left. + \frac{16P_{177}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{gg}^0 \left[\frac{20S_2}{3} - 4S_1^2 \right] \right] L_Q^2 + \left[-\frac{16P_{181}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{8P_{229}S_1}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{64(-1)^N P_{259}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3} \\
& + \frac{4P_{275}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3} + L_M^2 \left[-\frac{16(N^2+N+2)P_{109}}{3(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& \left. - \frac{16}{3}\tilde{\gamma}_{qg}^0 S_1 \right] + L_M \left[\frac{8P_{231}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{32P_{173}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& \left. + \tilde{\gamma}_{qg}^0 \left[-\frac{8}{3}S_1^2 - \frac{8S_2}{3} \right] \right] + \frac{16P_{183}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{256(N^2+N+1)S_3}{3N(N+1)(N+2)} \\
& + \frac{64P_{163}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{3}S_1^3 - 8S_2S_1 - \frac{32}{3}S_{2,1} \right] \\
& + \frac{\frac{512}{3}S_1S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)} + \frac{64(N-1)\zeta_3}{N(N+1)} \Big] L_Q - \frac{16(N^4-5N^3-32N^2-18N-4)S_1^2}{3N^2(N+1)^2(N+2)} \\
& - \frac{16}{9}(N^2+N+2)\frac{\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2}P_{113} + \frac{2}{9}\frac{P_{234}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3} \\
& - \frac{2P_{253}}{3(N-1)N^5(N+1)^6(N+2)^2} + \frac{4P_{244}S_1}{3(N-1)N^5(N+1)^5(N+2)^2} \\
& - \frac{80}{9}\frac{(N^3+4N^2+11N+6)\zeta_2}{N^2(N+1)(N+2)}S_1 + L_M^3 \left[\frac{16(N^2+N+2)P_{113}}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& \left. + \frac{32}{9}\tilde{\gamma}_{qg}^0 S_1 \right] + L_M^2 \left[-\frac{4P_{227}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{16P_{164}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& \left. + \tilde{\gamma}_{qg}^0 \left[\frac{20}{3}S_1^2 - 4S_2 \right] \right] - \frac{16P_{110}S_2}{3N^3(N+1)^3(N+2)} + \frac{(3N+2)\left[-\frac{32}{9}S_1^3 - \frac{32}{3}S_2S_1\right]}{N^2(N+2)} \\
& - \frac{32(3N^4+48N^3+43N^2-22N-8)S_3}{9N^2(N+1)^2(N+2)} + \frac{128(N^2-3N-2)S_{2,1}}{3N^2(N+1)(N+2)} \\
& + L_M \left[\frac{16P_{174}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8P_{230}S_1}{9(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& + \frac{64(-1)^N P_{260}}{45(N-2)(N-1)^2N^3(N+1)^4(N+2)^4(N+3)^3} + \frac{4P_{274}}{45(N-1)^2N^5(N+1)^5(N+2)^4(N+3)^3} \\
& - \frac{16P_{179}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{3}S_1^3 + \frac{8}{3}S_2S_1 \right] + \frac{128(N^2+N+4)S_3}{3N(N+1)(N+2)} \\
& + \frac{64P_{162}S_{-2}}{3(N-2)(N-1)N^2(N+1)^2(N+2)^2(N+3)} + \frac{\frac{512}{3}S_1S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N+1)(N+2)} \\
& + \frac{64(N-1)\zeta_3}{N(N+1)} \Big] + \tilde{\gamma}_{qg}^0 \left[-\frac{2}{9}S_1^4 - \frac{4}{3}S_2S_1^2 + \left[-\frac{16}{9}S_3 - \frac{32}{3}S_{2,1} \right] S_1 - \frac{32}{9}\zeta_3S_1 - \frac{2}{3}S_2^2 + 4S_4 \right. \\
& \left. - \frac{32}{3}S_{3,1} + \frac{64}{3}S_{2,1,1} + \left[2S_2 - \frac{10}{3}S_1^2 \right] \zeta_2 \right] \\
& + C_{FNFT_F}^2 \left[\left[\frac{16}{9}\tilde{\gamma}_{qg}^0 S_1 + \frac{16(N^2+N+1)(N^2+N+2)(3N^4+6N^3-N^2-4N+12)}{9(N-1)N^3(N+1)^3(N+2)^2} \right] L_Q^3 \right. \\
& \left. + \left[-\frac{4P_{232}}{9(N-1)N^4(N+1)^4(N+2)^3} + \frac{16P_{177}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& +L_M \left[-\frac{8(N^2 + N + 2)(3N^2 + 3N + 2)}{3N^2(N + 1)^2(N + 2)} - \frac{8}{3}\tilde{\gamma}_{qg}^0 S_1 \right] + \tilde{\gamma}_{qg}^0 \left[\frac{20S_2}{3} - 4S_1^2 \right] L_Q^2 \\
& + \left[-\frac{16P_{181}S_1^2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} - \frac{8P_{229}S_1}{9(N - 1)N^4(N + 1)^4(N + 2)^3} \right. \\
& + \frac{64(-1)^N P_{259}}{45(N - 2)(N - 1)^2 N^3(N + 1)^4(N + 2)^4(N + 3)^3} + \frac{8P_{276}}{45(N - 1)^2 N^5(N + 1)^5(N + 2)^4(N + 3)^3} \\
& + L_M \left[\frac{8(N^2 + N + 2)(57N^4 + 72N^3 + 29N^2 - 22N - 24)}{9N^3(N + 1)^3(N + 2)} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{3}S_1^2 - 8S_2 \right] \right. \\
& \left. - \frac{16(N^2 + N + 2)(29N^2 + 29N - 6)S_1}{9N^2(N + 1)^2(N + 2)} \right] - \frac{256(N^2 + N + 1)S_3}{3N(N + 1)(N + 2)} + \frac{16P_{183}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \\
& + \frac{64P_{163}S_{-2}}{3(N - 2)(N - 1)N^2(N + 1)^2(N + 2)^2(N + 3)} + \tilde{\gamma}_{qg}^0 \left[\frac{8}{3}S_1^3 - 8S_2S_1 - \frac{32}{3}S_{2,1} \right] \\
& + \frac{\frac{512}{3}S_1S_{-2} + \frac{256}{3}S_{-3} - \frac{512}{3}S_{-2,1}}{N(N + 1)(N + 2)} + \frac{64(N - 1)\zeta_3}{N(N + 1)} \left] L_Q - \frac{2}{9} \frac{P_{235}\zeta_2}{(N - 1)N^4(N + 1)^4(N + 2)^3} \right. \\
& - \frac{8(N^4 - 5N^3 - 32N^2 - 18N - 4)S_1^2}{3N^2(N + 1)^2(N + 2)} - \frac{8}{9} \frac{(N^2 + N + 2)\zeta_3}{(N - 1)N^3(N + 1)^3(N + 2)^2} P_{108} \\
& - \frac{4P_{271}}{3(N - 1)N^6(N + 1)^6(N + 2)^5} + \frac{16(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12)S_1}{3N^2(N + 1)^3(N + 2)} \\
& - \frac{16}{9} \frac{(5N^3 + 11N^2 + 28N + 12)\zeta_2}{N^2(N + 1)(N + 2)} S_1 + L_M^3 \left[\frac{8(N^2 + N + 2)P_{108}}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1 \right] \\
& - \frac{8P_{225}S_2}{3(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{(3N + 2) \left[-\frac{16}{9}S_1^3 - \frac{16}{3}S_2S_1 \right]}{N^2(N + 2)} \\
& + L_M^2 \left[-\frac{4P_{228}}{9(N - 1)N^4(N + 1)^4(N + 2)^3} + \frac{32(5N^3 + 8N^2 + 19N + 6)S_1}{9N^2(N + 1)(N + 2)} + \tilde{\gamma}_{qg}^0 \left[\frac{4}{3}S_1^2 + \frac{4S_2}{3} \right] \right] \\
& - \frac{16P_{169}S_3}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{64(N^2 - 3N - 2)S_{2,1}}{3N^2(N + 1)(N + 2)} \\
& + L_M \left[\frac{8(49N^4 + 122N^3 + 213N^2 + 164N + 12)S_1^2}{9N^2(N + 1)^2(N + 2)} - \frac{8P_{258}}{9(N - 1)N^5(N + 1)^5(N + 2)^4} \right. \\
& + \frac{8P_{146}S_1}{9N^3(N + 1)^3(N + 2)} - \frac{8P_{184}S_2}{9(N - 1)N^3(N + 1)^3(N + 2)^2} + \tilde{\gamma}_{qg}^0 \left[\frac{16}{3}S_1S_2 - 16S_3 + \frac{16}{3}S_{2,1} \right] \\
& \left. + \tilde{\gamma}_{qg}^0 \left[-\frac{1}{9}S_1^4 - \frac{2}{3}S_2S_1^2 - \frac{4}{3}\zeta_2S_1^2 + \left[-\frac{8}{9}S_3 - \frac{16}{3}S_{2,1} \right] S_1 - \frac{8}{9}\zeta_3S_1 - \frac{1}{3}S_2^2 + 2S_4 - \frac{16}{3}S_{3,1} + \frac{32}{3}S_{2,1,1} \right] \right] \\
& + C_{ACFT_F} \left[\frac{2P_{134}S_1^4}{9(N - 1)N^2(N + 1)^2(N + 2)^2} + \frac{4P_{180}S_1^3}{9(N - 1)N^3(N + 1)^3(N + 2)^2} \right. \\
& - \frac{4P_{245}S_1^2}{3(N - 1)N^4(N + 1)^4(N + 2)^4} - \frac{8}{3} \frac{(19N^5 - 11N^4 - 8N^3 - 49N^2 - 17N + 18)\zeta_2}{(N - 1)N^2(N + 1)^2(N + 2)} S_1^2 \\
& + \frac{4P_{143}S_2S_1^2}{3(N - 1)N^2(N + 1)^2(N + 2)^2} - \frac{2}{9} \frac{P_{155}S_1\zeta_3}{(N - 1)N^2(N + 1)^2(N + 2)^2} \\
& \left. + \frac{4}{9} \frac{P_{189}S_1\zeta_2}{(N - 1)N^3(N + 1)^3(N + 2)^2} + \frac{4P_{263}S_1}{3(N - 1)N^5(N + 1)^5(N + 2)^5} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{4P_{200}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3} + \frac{8P_{153}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16(11N^5 + 45N^4 - 3N^3 - 145N^2 - 176N - 20)S_{2,1}S_1}{3(N-1)N(N+1)^2(N+2)^2} - \frac{2P_{101}S_2^2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N+1)^3(N+2)^3} + 8 \frac{(-1)^N \zeta_2}{N^2(N+1)^4(N+2)^3} P_{115} - \frac{2}{9} \frac{P_{188} \zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} \\
& - \frac{1}{18} \frac{P_{208} \zeta_2}{(N-1)N^3(N+1)^4(N+2)^2} + \frac{P_{273}}{3(N-1)N^6(N+1)^6(N+2)^5} \\
& + L_M^3 \left[-\frac{16}{3} \tilde{\gamma}_{ag}^0 S_1^2 - \frac{8(N^2 + N + 2)(N^2 + N + 6)(7N^2 + 7N + 4)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)}{9(N-1)N^3(N+1)^3(N+2)^2} \right] + L_Q^3 \left[-\frac{8}{3} \tilde{\gamma}_{ag}^0 S_1^2 \right. \\
& \left. + \frac{8(N^2 + N + 2)(13N^4 + 26N^3 - 43N^2 - 56N - 12)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. - \frac{4(N^2 + N + 2)(3N^2 + 3N + 2)(11N^4 + 22N^3 - 23N^2 - 34N - 12)}{9(N-1)N^3(N+1)^3(N+2)^2} \right] \\
& - 4 \frac{\zeta_2 P_{107} S_2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{243} S_2}{3(N-1)N^4(N+1)^4(N+2)^4} + \frac{4P_{185} S_3}{9(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{4(N^2 + N + 2)(19N^4 + 38N^3 - 22N^2 - 41N - 30)S_4}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{16(-1)^N P_{116} S_{-2}}{N^2(N+1)^4(N+2)^3} \\
& - \frac{16(N^2 - 3N - 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)S_{2,1}}{3(N-1)N^3(N+1)^2(N+2)^2} \\
& - \frac{8(N^2 + N + 2)(31N^4 + 62N^3 - 73N^2 - 104N - 60)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + L_Q^2 \left[-\frac{8(10N^5 + 6N^4 + 5N^3 - 38N^2 - 17N + 2)S_1^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{4P_{204} S_1}{9(N-1)N^3(N+1)^3(N+2)^3} - \frac{16(-1)^N(3N^4 + 11N^3 + 19N^2 + 15N + 2)}{N(N+1)^3(N+2)^3} \right. \\
& \left. + \frac{P_{238}}{9(N-1)N^4(N+1)^4(N+2)^3} + L_M \left[\frac{22(N^2 + N + 2)(3N^2 + 3N + 2)}{3N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. + \frac{22}{3} \tilde{\gamma}_{ag}^0 S_1 \right] + \frac{8(N^2 + N + 2)(23N^4 + 46N^3 - 50N^2 - 73N - 18)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} + \tilde{\gamma}_{ag}^0 \left[4S_1^3 - 12S_2S_1 \right. \right. \\
& \left. \left. + 4S_3 + 6S_{-2} + 4S_{-3} - 8S_{-2,1} \right] \right] + L_M^2 \left[\frac{8P_{121} S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. + \frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} - \frac{8P_{203} S_1}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{16(-1)^N(3N^4 + 11N^3 + 19N^2 + 15N + 2)}{N(N+1)^3(N+2)^3} + \frac{P_{224}}{9(N-1)N^4(N+1)^3(N+2)^3} \right. \\
& \left. + \frac{8(N^2 + N + 2)(N^4 + 2N^3 + 20N^2 + 19N + 30)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} + \tilde{\gamma}_{ag}^0 \left[-12S_1^3 + 4S_2S_1 + 4S_3 + 6S_{-2} \right. \right. \\
& \left. \left. + 4S_{-3} - 8S_{-2,1} \right] \right] + \frac{8(N^2 + N + 2)(35N^4 + 70N^3 - 137N^2 - 172N - 84)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)^2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(N^2 + N + 2)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} - 48\tilde{\gamma}_{gg}^0 \log(2)\zeta_2 + \frac{P_{168} \left[16(-1)^N S_1 S_{-2} + 8(-1)^N S_1 \zeta_2 \right]}{N^3(N+1)^3(N+2)^3} \\
& + \frac{(N^4 + 3N^3 + 8N^2 + 12N + 4) \left[96(-1)^N S_{-2} S_1^2 + 48(-1)^N \zeta_2 S_1^2 \right]}{N(N+1)^2(N+2)^2} \\
& + \frac{(3N^5 + 10N^4 + 25N^3 + 38N^2 + 20N + 8) \left[16(-1)^N S_2 S_{-2} + 8(-1)^N S_2 \zeta_2 \right]}{N^2(N+1)^2(N+2)^2} \\
& + \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \left[8(-1)^N S_{-2} \zeta_2 + 8(-1)^N S_{-4} - 16S_{-2,2} - 16S_{-3,1} + 32S_{-2,1,1} \right] \\
& + \frac{P_{156} \left[8(-1)^N S_{-3} - 16S_{-2,1} + 6(-1)^N \zeta_3 \right]}{N^3(N+1)^3(N+2)^2} \\
& + \frac{(9N^5 + 28N^4 + 73N^3 + 110N^2 + 44N + 8) \left[8(-1)^N S_1 S_{-3} - 16S_1 S_{-2,1} + 6(-1)^N S_1 \zeta_3 \right]}{N^2(N+1)^2(N+2)^2} \\
& + \tilde{\gamma}_{gg}^0 \left[\frac{20}{3} S_2 S_1^3 + \left[24S_3 + 16S_{2,1} - 24S_{-2,1} \right] S_1^2 + 8(-1)^N S_{-4} S_1 + \left[8S_2^2 + 12S_4 + 8S_{3,1} - 16S_{-2,2} \right. \right. \\
& \left. \left. - 16S_{-3,1} - 40S_{2,1,1} + 32S_{-2,1,1} \right] S_1 + S_{-3} \left[12(-1)^N S_1^2 + 4(-1)^N S_2 \right] + S_2 \left[\frac{16S_3}{3} - 8S_{-2,1} \right] \right. \\
& \left. + S_{-2} \left[8(-1)^N S_1^3 + 24(-1)^N S_2 S_1 + 32 \right] + \left[4(1 + (-1)^N) S_1^3 + 4(-1 + 2(-1)^N) S_{-2} S_1 - 2S_3 \right. \right. \\
& \left. \left. - 2S_{-3} + 4S_{-2,1} \right] \zeta_2 + \left[\frac{1}{3} (-11 + 27(-1)^N) S_1^2 \right] \zeta_3 \right] \\
& + L_Q \left[\frac{8(29N^5 + 81N^4 + 117N^3 - 49N^2 - 362N - 104) S_1^3}{3(N-1)N(N+1)^2(N+2)^2} + \frac{4P_{206} S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{32(-1)^N P_{255} S_1}{15(N-2)(N-1)^2 N^2 (N+1)^4 (N+2)^4 (N+3)^3} + \frac{2P_{270} S_1}{45(N-1)^2 N^4 (N+1)^4 (N+2)^4 (N+3)^3} \right. \\
& \left. - \frac{8(17N^5 + 33N^4 - 43N^3 - 153N^2 - 254N - 80) S_2 S_1}{(N-1)N(N+1)^2(N+2)^2} + \frac{32(N-2)(N+3) S_3 S_1}{N(N+1)(N+2)} \right. \\
& \left. - \frac{16P_{158} S_{-2} S_1}{3(N-2)(N-1)N(N+1)^2(N+2)^2(N+3)} + \frac{512S_{-2,1} S_1}{N(N+1)(N+2)} - \frac{24(11N^2 + 11N - 10) \zeta_3 S_1}{N(N+1)(N+2)} \right. \\
& \left. + 2 \frac{\zeta_3 P_{154}}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{16(-1)^N P_{272}}{45(N-2)(N-1)^3 N^3 (N+1)^5 (N+2)^5 (N+3)^3} \right. \\
& \left. - \frac{2P_{279}}{45(N-1)^3 N^5 (N+1)^5 (N+2)^5 (N+3)^3} \right. \\
& \left. + L_M^2 \left[8\tilde{\gamma}_{gg}^0 S_1^2 + \frac{8(N^2 + N + 2)(3N^4 + 6N^3 + 7N^2 + 4N + 4) S_1}{(N-1)N^2(N+1)^2(N+2)^2} \right. \right. \\
& \left. \left. - \frac{16(N^2 + N + 1)(N^2 + N + 2)(3N^2 + 3N + 2)}{(N-1)N^3(N+1)^3(N+2)^2} \right] + \frac{(N^2 + N + 10) \left[-64S_{-2} S_1^2 - 32S_{-3} S_1 \right]}{N(N+1)(N+2)} \right. \\
& \left. - \frac{4P_{207} S_2}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(128(-1)^N S_2 - 128(-1)^N S_1^2)}{(N+1)^3(N+2)^3} \right. \\
& \left. + \frac{32P_{135} S_3}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{16(-1)^N (3N^5 - 6N^4 - 61N^3 - 124N^2 - 96N - 16) S_{-2}}{N(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{16P_{119} S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{32P_{226} S_{-2}}{3(N-2)(N-1)N^3(N+1)^3(N+2)^3(N+3)} \right]
\end{aligned}$$

$$\begin{aligned}
& - \frac{16(N^2 + N + 2)(31N^2 + 31N + 6)S_{2,1}}{3N^2(N+1)^2(N+2)} - \frac{16P_{120}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + L_M \left[\frac{16(10N^5 + 40N^4 + 121N^3 + 161N^2 + 52N + 12)S_1^2}{3N^2(N+1)^2(N+2)^2} + \frac{4P_{205}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& - \frac{128(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} + \frac{32(-1)^N(3N^4 + 11N^3 + 19N^2 + 15N + 2)}{N(N+1)^3(N+2)^3} \\
& - \frac{2P_{236}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{16(N^2 + N + 2)(4N^2 + 4N - 1)S_2}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0 \left[8S_1^3 + 8S_2S_1 - 8S_3 \right. \\
& \left. - 12S_{-2} - 8S_{-3} + 16S_{-2,1} \right] + \frac{(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8)16(-1)^N S_1 S_{-2}}{N^2(N+1)^2(N+2)^2} \\
& + \frac{(N^2 + N + 2)(3N^2 + 3N + 2) \left[8(-1)^N S_{-3} + 6(-1)^N \zeta_3 \right]}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0 \left[40S_2S_1^2 + 16(-1)^N S_{-2}S_1^2 \right. \\
& + 8(-1)^N S_{-3}S_1 - 16S_{2,1}S_1 + 6(-1)^N \zeta_3 S_1 - 8S_2^2 + 24S_{-2}^2 + 8S_4 + 40S_{-4} \\
& \left. + 32S_{3,1} - 16S_{-3,1} - 32S_{-2,1,1} \right] + L_M \left[\frac{8(3N^5 + 2N^4 - 61N^3 - 112N^2 - 56N - 24)S_1^3}{N^2(N+1)^2(N+2)^2} \right. \\
& - \frac{2P_{209}S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{32(-1)^N(15N^5 + 97N^4 + 260N^3 + 328N^2 + 158N - 4)S_1}{N(N+1)^3(N+2)^4} \\
& - \frac{2P_{250}S_1}{9(N-1)N^4(N+1)^4(N+2)^4} + \frac{8P_{114}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{16(3N^3 + 2N^2 - 47N - 62)S_{-2}S_1}{N(N+1)^2(N+2)} - 6 \frac{\zeta_3 P_{150}}{(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16(-1)^N P_{267}}{5(N-2)(N-1)^2 N^3 (N+1)^5 (N+2)^5 (N+3)^3} + \frac{2P_{278}}{45(N-1)^2 N^5 (N+1)^5 (N+2)^5 (N+3)^3} \\
& + \frac{2P_{210}S_2}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{(N^3 + 4N^2 + 7N + 5)(128(-1)^N S_1^2 - 128(-1)^N S_2)}{(N+1)^3(N+2)^3} \\
& - \frac{16P_{128}S_3}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{16(-1)^N(3N^5 - 6N^4 - 61N^3 - 124N^2 - 96N - 16)S_{-2}}{N(N+1)^3(N+2)^3} \\
& - \frac{16P_{214}S_{-2}}{(N-2)N^3(N+1)^3(N+2)^3(N+3)} + \frac{16(3N^4 + 4N^3 - 9N^2 - 14N + 8)S_{-3}}{N^2(N+1)^2(N+2)} \\
& + \frac{32(N^2 + N + 2)(10N^4 + 20N^3 + 5N^2 - 5N + 6)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{16(3N^4 + 10N^3 + 43N^2 + 44N - 20)S_{-2,1}}{N^2(N+1)^2(N+2)} \\
& + \frac{(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8) \left[-16(-1)^N S_1 S_{-2} \right]}{N^2(N+1)^2(N+2)^2} \\
& + \frac{(N^2 + N + 2)(3N^2 + 3N + 2) \left[-8(-1)^N S_{-3} - 6(-1)^N \zeta_3 \right]}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{qg}^0 \left[-8S_1^4 - 32S_2S_1^2 \right. \\
& \left. - 16(1 + (-1)^N)S_{-2}S_1^2 - 8(1 + (-1)^N)S_{-3}S_1 + \left[32S_{2,1} - 8S_3 \right] S_1 \right. \\
& \left. - 6(3 + (-1)^N)\zeta_3 S_1 + 8S_2^2 - 24S_{-2}^2 - 8S_4 - 40S_{-4} - 32S_{3,1} + 16S_{-3,1} + 32S_{-2,1,1} \right] \left. \right] \left. \right]
\end{aligned}$$

$$+a_{Qg}^{(3)} + \tilde{C}_{2,g}^{S,(3)}(N_F + 1) \left. \right\} \left. \right\}, \quad (152)$$

with the polynomials

$$P_{101} = N^6 - 81N^5 - 264N^4 - 185N^3 - 307N^2 - 256N - 204 \quad (153)$$

$$P_{102} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \quad (154)$$

$$P_{103} = N^6 + 7N^5 - 7N^4 - 39N^3 + 14N^2 + 40N + 48 \quad (155)$$

$$P_{104} = N^6 + 21N^5 + 57N^4 + 31N^3 + 26N^2 + 20N + 24 \quad (156)$$

$$P_{105} = 2N^6 - 7N^5 - 41N^4 - 31N^3 - 29N^2 - 22N - 16 \quad (157)$$

$$P_{106} = 2N^6 - 7N^5 - 24N^4 - 35N^3 - 44N^2 - 44N - 16 \quad (158)$$

$$P_{107} = 3N^6 + 5N^5 + 27N^4 + 35N^3 + 6N^2 + 12N + 8 \quad (159)$$

$$P_{108} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \quad (160)$$

$$P_{109} = 3N^6 + 9N^5 + 2N^4 - 11N^3 - 23N^2 - 16N - 12 \quad (161)$$

$$P_{110} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \quad (162)$$

$$P_{111} = 4N^6 + 5N^5 - 10N^4 - 39N^3 - 40N^2 - 24N - 8 \quad (163)$$

$$P_{112} = 6N^6 - 12N^5 + 17N^4 + 106N^3 + 127N^2 + 104N + 84 \quad (164)$$

$$P_{113} = 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12 \quad (165)$$

$$P_{114} = 7N^6 - 93N^5 - 327N^4 - 287N^3 - 316N^2 - 112N - 24 \quad (166)$$

$$P_{115} = 7N^6 - 20N^5 - 176N^4 - 335N^3 - 276N^2 - 116N - 16 \quad (167)$$

$$P_{116} = 7N^6 - 19N^5 - 171N^4 - 325N^3 - 264N^2 - 108N - 16 \quad (168)$$

$$P_{117} = 7N^6 + 21N^5 + 5N^4 - 25N^3 - 204N^2 - 188N - 192 \quad (169)$$

$$P_{118} = 8N^6 + 13N^5 - 111N^4 - 193N^3 - 89N^2 - 56N - 20 \quad (170)$$

$$P_{119} = 9N^6 + 21N^5 + 11N^4 - 5N^3 - 104N^2 - 76N - 144 \quad (171)$$

$$P_{120} = 9N^6 + 39N^5 + 53N^4 + 25N^3 + 94N^2 + 44N + 312 \quad (172)$$

$$P_{121} = 10N^6 + 18N^5 - 111N^4 - 164N^3 - 61N^2 - 16N + 36 \quad (173)$$

$$P_{122} = 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48 \quad (174)$$

$$P_{123} = 11N^6 - 15N^5 - 327N^4 - 181N^3 + 292N^2 - 20N - 48 \quad (175)$$

$$P_{124} = 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N - 240 \quad (176)$$

$$P_{125} = 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72 \quad (177)$$

$$P_{126} = 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N - 108 \quad (178)$$

$$P_{127} = 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N - 48 \quad (179)$$

$$P_{128} = 11N^6 + 35N^5 + 59N^4 + 57N^3 - 38N^2 - 68N + 40 \quad (180)$$

$$P_{129} = 11N^6 + 47N^5 + 7N^4 + 9N^3 + 90N^2 + 28N + 96 \quad (181)$$

$$P_{130} = 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 \quad (182)$$

$$P_{131} = 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 \quad (183)$$

$$P_{132} = 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 \quad (184)$$

$$P_{133} = 16N^6 + 78N^5 - 23N^4 - 228N^3 - 503N^2 - 408N - 228 \quad (185)$$

$$P_{134} = 17N^6 + 111N^5 + 234N^4 + 203N^3 - 89N^2 - 296N - 36 \quad (186)$$

$$P_{135} = 22N^6 + 69N^5 + 71N^4 + 23N^3 - 57N^2 - 68N + 84 \quad (187)$$

$$P_{136} = 23N^6 - 7N^5 - 237N^4 - 593N^3 - 678N^2 - 548N - 200 \quad (188)$$

$$P_{137} = 23N^6 + 9N^5 - 71N^4 - 53N^3 - 184N^2 - 92N - 16 \quad (189)$$

$$P_{138} = 25N^6 + 35N^5 - 55N^4 - 243N^3 - 286N^2 - 204N - 72 \quad (190)$$

$$P_{139} = 29N^6 + 91N^5 + 235N^4 + 405N^3 + 272N^2 + 288N + 120 \quad (191)$$

$$\begin{aligned}
P_{140} &= 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 & (192) \\
P_{141} &= 35N^6 - 15N^5 - 183N^4 - 133N^3 - 356N^2 - 164N - 48 & (193) \\
P_{142} &= 35N^6 - 15N^5 - 101N^4 + 31N^3 + 54N^2 + 164N + 120 & (194) \\
P_{143} &= 44N^6 + 96N^5 + 369N^4 + 290N^3 - 695N^2 - 428N - 108 & (195) \\
P_{144} &= 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 & (196) \\
P_{145} &= 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 & (197) \\
P_{146} &= 57N^6 + 161N^5 - 25N^4 - 193N^3 - 172N^2 - 36N + 48 & (198) \\
P_{147} &= 65N^6 + 199N^5 + 197N^4 - 143N^3 - 330N^2 - 316N - 120 & (199) \\
P_{148} &= 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 & (200) \\
P_{149} &= 80N^6 + 60N^5 + 9N^4 + 230N^3 + 901N^2 + 988N + 1188 & (201) \\
P_{150} &= 81N^6 + 211N^5 - 23N^4 - 355N^3 - 334N^2 - 4N - 344 & (202) \\
P_{151} &= 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 & (203) \\
P_{152} &= 130N^6 + 865N^5 + 2316N^4 + 3811N^3 + 4434N^2 + 2884N + 536 & (204) \\
P_{153} &= 133N^6 + 699N^5 + 1395N^4 + 217N^3 - 880N^2 + 164N + 288 & (205) \\
P_{154} &= 155N^6 + 369N^5 + 211N^4 - 65N^3 - 1002N^2 - 556N - 1416 & (206) \\
P_{155} &= 215N^6 + 429N^5 + 891N^4 + 491N^3 - 2486N^2 - 1436N - 408 & (207) \\
P_{156} &= 3N^7 + 28N^6 + 66N^5 + 90N^4 + 107N^3 + 78N^2 + 36N + 8 & (208) \\
P_{157} &= 9N^7 + 71N^6 + 214N^5 + 320N^4 + 275N^3 + 215N^2 + 160N + 32 & (209) \\
P_{158} &= 21N^7 + 120N^6 - 128N^5 - 1038N^4 - 89N^3 + 2382N^2 + 1636N - 600 & (210) \\
P_{159} &= 81N^7 + 247N^6 + 291N^5 + 277N^4 + 108N^3 - 56N^2 + 20N + 24 & (211) \\
P_{160} &= N^8 + 5N^7 + 10N^6 + 27N^5 + 65N^4 + 112N^3 + 124N^2 + 80N + 32 & (212) \\
P_{161} &= N^8 + 5N^7 + 14N^6 + 23N^5 + 25N^4 + 52N^3 + 56N^2 + 48N + 16 & (213) \\
P_{162} &= N^8 + 8N^7 - 2N^6 - 60N^5 - 23N^4 + 108N^3 + 96N^2 + 16N + 48 & (214) \\
P_{163} &= N^8 + 8N^7 - 2N^6 - 60N^5 + N^4 + 156N^3 + 24N^2 - 80N - 240 & (215) \\
P_{164} &= N^8 + 22N^7 + 111N^6 + 211N^5 + 42N^4 - 281N^3 - 406N^2 - 204N - 72 & (216) \\
P_{165} &= 2N^8 + N^7 - 6N^6 + 26N^5 + 64N^4 + 51N^3 + 54N^2 + 28N + 8 & (217) \\
P_{166} &= 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 & (218) \\
P_{167} &= 2N^8 + 44N^7 + 211N^6 + 485N^5 + 654N^4 + 581N^3 + 391N^2 + 192N + 32 & (219) \\
P_{168} &= 3N^8 + 41N^7 + 136N^6 + 233N^5 + 331N^4 + 360N^3 + 208N^2 + 80N + 16 & (220) \\
P_{169} &= 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 & (221) \\
P_{170} &= 5N^8 - 8N^7 - 137N^6 - 436N^5 - 713N^4 - 672N^3 - 407N^2 - 192N - 32 & (222) \\
P_{171} &= 7N^8 + 40N^7 + 110N^6 + 193N^5 + 261N^4 + 313N^3 + 260N^2 + 96N + 16 & (223) \\
P_{172} &= 9N^8 + 54N^7 + 80N^6 - 110N^5 - 645N^4 - 1168N^3 - 1132N^2 - 672N - 160 & (224) \\
P_{173} &= 10N^8 + 46N^7 + 87N^6 + 85N^5 - 75N^4 - 251N^3 - 274N^2 - 132N - 72 & (225) \\
P_{174} &= 11N^8 + 74N^7 + 213N^6 + 281N^5 - 30N^4 - 427N^3 - 446N^2 - 180N - 72 & (226) \\
P_{175} &= 15N^8 + 36N^7 + 50N^6 - 252N^5 - 357N^4 + 152N^3 - 68N^2 + 88N + 48 & (227) \\
P_{176} &= 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 & (228) \\
P_{177} &= 19N^8 + 70N^7 + 63N^6 - 41N^5 - 192N^4 - 221N^3 - 142N^2 - 60N - 72 & (229) \\
P_{178} &= 21N^8 + 42N^7 - 38N^6 - 360N^5 - 631N^4 - 730N^3 - 472N^2 - 216N - 48 & (230) \\
P_{179} &= 23N^8 + 2N^7 - 135N^6 + 29N^5 + 210N^4 - 151N^3 - 350N^2 - 132N - 72 & (231) \\
P_{180} &= 27N^8 - 36N^7 - 956N^6 - 1724N^5 + 187N^4 + 1288N^3 + 70N^2 - 224N - 72 & (232) \\
P_{181} &= 38N^8 + 146N^7 + 177N^6 + 35N^5 - 249N^4 - 373N^3 - 218N^2 - 60N - 72 & (233)
\end{aligned}$$

$$\begin{aligned}
P_{182} &= 41N^8 + 5N^7 - 195N^6 - 97N^5 + 326N^4 + 424N^3 + 208N^2 + 72N + 16 & (234) \\
P_{183} &= 56N^8 + 194N^7 + 213N^6 + 83N^5 - 231N^4 - 469N^3 - 290N^2 - 60N - 72 & (235) \\
P_{184} &= 79N^8 + 196N^7 + 132N^6 + 274N^5 + 465N^4 + 82N^3 + 332N^2 + 456N + 288 & (236) \\
P_{185} &= 105N^8 + 978N^7 + 1688N^6 - 1330N^5 - 5245N^4 - 4672N^3 - 2212N^2 - 544N - 288 & (237) \\
P_{186} &= 113N^8 + 348N^7 + 109N^6 - 289N^5 - 272N^4 - 859N^3 - 778N^2 - 172N + 72 & (238) \\
P_{187} &= 170N^8 + 369N^7 - 521N^6 - 1393N^5 - 761N^4 - 952N^3 - 544N^2 + 32N + 144 & (239) \\
P_{188} &= 264N^8 + 1407N^7 + 2246N^6 + 1746N^5 + 804N^4 - 1069N^3 - 674N^2 - 92N - 24 & (240) \\
P_{189} &= 283N^8 + 838N^7 + 1482N^6 + 628N^5 - 1497N^4 - 1130N^3 - 772N^2 + 456N + 288 & (241) \\
P_{190} &= 633N^8 + 2532N^7 + 5036N^6 + 6142N^5 + 4275N^4 + 1118N^3 - 176N^2 - 184N - 48 & (242) \\
P_{191} &= N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 & (243) \\
P_{192} &= 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 1172N^3 + 904N^2 + 512N + 128 & (244) \\
P_{193} &= 6N^9 + 93N^8 + 576N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2 \\
&\quad + 1488N + 384 & (245) \\
P_{194} &= 9N^9 + 54N^8 + 56N^7 - 110N^6 - 381N^5 - 568N^4 - 364N^3 - 72N^2 + 128N + 96 & (246) \\
P_{195} &= 9N^9 + 54N^8 + 167N^7 + 397N^6 + 780N^5 + 1241N^4 + 1448N^3 + 1200N^2 + 608N + 144 & (247) \\
P_{196} &= 11N^9 + 78N^8 + 214N^7 + 335N^6 + 383N^5 + 571N^4 + 916N^3 + 876N^2 + 480N + 96 & (248) \\
P_{197} &= 35N^9 + 150N^8 + 232N^7 + 137N^6 + 119N^5 + 661N^4 + 1174N^3 + 876N^2 + 480N + 96 & (249) \\
P_{198} &= 37N^9 + 210N^8 - 52N^7 - 2738N^6 - 7249N^5 - 9368N^4 - 8216N^3 - 5888N^2 \\
&\quad - 2448N - 576 & (250) \\
P_{199} &= 45N^9 + 270N^8 + 820N^7 + 1478N^6 + 1683N^5 + 1996N^4 + 2356N^3 + 2328N^2 \\
&\quad + 1408N + 288 & (251) \\
P_{200} &= 57N^9 + 624N^8 + 1756N^7 + 1092N^6 - 1803N^5 - 1512N^4 + 966N^3 + 1116N^2 \\
&\quad + 920N + 528 & (252) \\
P_{201} &= 69N^9 + 366N^8 + 1124N^7 + 1966N^6 + 2523N^5 + 5228N^4 + 7340N^3 + 5352N^2 \\
&\quad + 3008N + 672 & (253) \\
P_{202} &= 94N^9 + 597N^8 + 1616N^7 + 2410N^6 + 1841N^5 + 1165N^4 + 2191N^3 + 3802N^2 \\
&\quad + 2916N + 648 & (254) \\
P_{203} &= 121N^9 + 696N^8 + 1535N^7 + 1585N^6 + 416N^5 - 749N^4 - 836N^3 + 16N^2 \\
&\quad + 528N + 144 & (255) \\
P_{204} &= 197N^9 + 1242N^8 + 2938N^7 + 3524N^6 + 2713N^5 + 2234N^4 + 3680N^3 + 6176N^2 \\
&\quad + 4080N + 864 & (256) \\
P_{205} &= 439N^9 + 2634N^8 + 6008N^7 + 6694N^6 + 3545N^5 + 736N^4 + 2008N^3 + 6208N^2 \\
&\quad + 5136N + 1152 & (257) \\
P_{206} &= 538N^9 + 3333N^8 + 7802N^7 + 7630N^6 + 458N^5 - 1415N^4 + 7786N^3 + 12340N^2 \\
&\quad + 5592N + 864 & (258) \\
P_{207} &= 664N^9 + 3861N^8 + 9038N^7 + 11830N^6 + 9344N^5 + 3793N^4 + 3874N^3 + 11044N^2 \\
&\quad + 9624N + 2592 & (259) \\
P_{208} &= 891N^9 + 4455N^8 + 16078N^7 + 28774N^6 + 37047N^5 + 45835N^4 + 42192N^3 + 28888N^2 \\
&\quad + 10640N + 1776 & (260) \\
P_{209} &= 923N^9 + 5208N^8 + 11824N^7 + 12854N^6 + 2185N^5 - 7030N^4 + 1436N^3 + 15032N^2 \\
&\quad + 12864N + 3456 & (261) \\
P_{210} &= 965N^9 + 4884N^8 + 10816N^7 + 20810N^6 + 36895N^5 + 40442N^4 + 27692N^3 + 22712N^2
\end{aligned}$$

$$\begin{aligned}
& +14496N + 3456 & (262) \\
P_{211} & = 2N^{10} - 46N^9 - 98N^8 + 282N^7 + 1063N^6 + 1569N^5 + 1275N^4 + 403N^3 - 94N^2 \\
& -108N - 24 & (263) \\
P_{212} & = 2N^{10} + 12N^9 + 24N^8 + 11N^7 - 48N^6 - 151N^5 - 282N^4 - 480N^3 - 664N^2 \\
& -576N - 288 & (264) \\
P_{213} & = 11N^{10} + 44N^9 + 74N^8 + 196N^7 + 31N^6 - 1426N^5 - 3044N^4 - 2762N^3 - 1476N^2 \\
& -480N - 96 & (265) \\
P_{214} & = 11N^{10} + 76N^9 + 138N^8 - 204N^7 - 1041N^6 - 988N^5 + 752N^4 + 1896N^3 + 944N^2 \\
& -384N - 576 & (266) \\
P_{215} & = 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 9218N^4 - 6116N^3 - 72N^2 \\
& -752N - 192 & (267) \\
P_{216} & = 85N^{10} + 425N^9 + 902N^8 + 932N^7 - 521N^6 - 685N^5 + 2022N^4 + 2928N^3 + 968N^2 \\
& -1296N - 576 & (268) \\
P_{217} & = 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2 \\
& -2160N - 864 & (269) \\
P_{218} & = 118N^{10} + 425N^9 + 197N^8 + 86N^7 + 1240N^6 + 2489N^5 + 4401N^4 + 3480N^3 + 524N^2 \\
& -1728N - 864 & (270) \\
P_{219} & = 118N^{10} + 557N^9 + 461N^8 - 94N^7 + 1300N^6 + 3521N^5 + 4509N^4 + 1920N^3 \\
& -1132N^2 - 2376N - 1008 & (271) \\
P_{220} & = 127N^{10} + 536N^9 + 611N^8 + 602N^7 + 1474N^6 + 2099N^5 + 798N^4 - 2301N^3 \\
& -4486N^2 - 3708N - 936 & (272) \\
P_{221} & = 170N^{10} + 883N^9 + 2041N^8 + 2998N^7 - 448N^6 - 5465N^5 + 129N^4 + 6624N^3 \\
& +1132N^2 - 2016N - 864 & (273) \\
P_{222} & = 170N^{10} + 1213N^9 + 3235N^8 + 2794N^7 - 2692N^6 - 3767N^5 - 1293N^4 \\
& -1632N^3 - 5324N^2 - 6240N - 2016 & (274) \\
P_{223} & = 226N^{10} + 317N^9 - 811N^8 + 662N^7 + 4552N^6 + 3857N^5 + 3933N^4 + 2364N^3 \\
& +236N^2 - 1656N - 720 & (275) \\
P_{224} & = 489N^{10} + 2934N^9 + 9364N^8 + 18830N^7 + 18627N^6 + 124N^5 - 19856N^4 - 19296N^3 \\
& -10640N^2 - 2880N - 1152 & (276) \\
P_{225} & = 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3 \\
& -1232N^2 - 736N - 192 & (277) \\
P_{226} & = 11N^{11} + 37N^{10} - 27N^9 - 118N^8 + 21N^7 - 249N^6 - 1097N^5 - 1138N^4 + 552N^3 \\
& +3448N^2 + 3456N + 2016 & (278) \\
P_{227} & = 21N^{11} + 231N^{10} + 1334N^9 + 4086N^8 + 6277N^7 + 1775N^6 - 9488N^5 - 18076N^4 \\
& -18208N^3 - 11344N^2 - 5568N - 1728 & (279) \\
P_{228} & = 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 \\
& +7328N^3 + 5456N^2 + 3456N + 1152 & (280) \\
P_{229} & = 45N^{11} + 383N^{10} + 958N^9 + 526N^8 - 763N^7 + 1375N^6 + 7808N^5 + 13028N^4 \\
& +12976N^3 + 8016N^2 + 4608N + 1728 & (281) \\
P_{230} & = 51N^{11} + 269N^{10} + 46N^9 - 1934N^8 - 3973N^7 - 875N^6 + 7364N^5 + 14972N^4 \\
& +16768N^3 + 10896N^2 + 5376N + 1728 & (282) \\
P_{231} & = 51N^{11} + 357N^{10} + 1238N^9 + 2586N^8 + 2755N^7 - 1435N^6 - 9212N^5 - 15028N^4
\end{aligned}$$

$$\begin{aligned}
& -15280N^3 - 9808N^2 - 5184N - 1728 & (283) \\
P_{232} = & 81N^{11} + 483N^{10} + 1142N^9 + 1086N^8 - 767N^7 - 4645N^6 - 8936N^5 - 11980N^4 \\
& -12352N^3 - 8272N^2 - 4800N - 1728 & (284) \\
P_{233} = & 120N^{11} + 1017N^{10} + 2737N^9 + 1292N^8 - 8086N^7 - 20743N^6 - 24563N^5 - 16702N^4 \\
& -6840N^3 + 120N^2 + 2432N + 960 & (285) \\
P_{234} = & 243N^{11} + 1701N^{10} + 5378N^9 + 10350N^8 + 11479N^7 + 1193N^6 - 14684N^5 - 20572N^4 \\
& -16288N^3 - 8944N^2 - 4992N - 1728 & (286) \\
P_{235} = & 333N^{11} + 2331N^{10} + 6556N^9 + 9270N^8 + 5081N^7 - 6701N^6 - 17554N^5 - 20036N^4 \\
& -15680N^3 - 9200N^2 - 5664N - 1728 & (287) \\
P_{236} = & 753N^{11} + 4809N^{10} + 13174N^9 + 20466N^8 + 17717N^7 + 6829N^6 + 3908N^5 \\
& +15304N^4 + 25408N^3 + 20272N^2 + 8448N + 1152 & (288) \\
P_{237} = & 837N^{11} + 7757N^{10} + 30120N^9 + 68575N^8 + 119176N^7 + 191350N^6 + 262979N^5 \\
& +258308N^4 + 163106N^3 + 63360N^2 + 14848N + 1536 & (289) \\
P_{238} = & 1017N^{11} + 6195N^{10} + 14050N^9 + 12738N^8 - 2023N^7 - 5093N^6 + 27548N^5 \\
& +69760N^4 + 80752N^3 + 54064N^2 + 20928N + 3456 & (290) \\
P_{239} = & 3N^{12} + 21N^{11} + 17N^{10} - 202N^9 - 842N^8 - 1924N^7 - 3378N^6 - 5059N^5 \\
& -6008N^4 - 4860N^3 - 2536N^2 - 960N - 192 & (291) \\
P_{240} = & 9N^{12} + 63N^{11} + 38N^{10} - 414N^9 - 1035N^8 - 1341N^7 - 1511N^6 - 2972N^5 \\
& -6011N^4 - 8038N^3 - 6892N^2 - 3432N - 864 & (292) \\
P_{241} = & 9N^{12} + 63N^{11} + 71N^{10} - 381N^9 - 1536N^8 - 2529N^7 - 1946N^6 - 1331N^5 \\
& -2096N^4 - 4036N^3 - 4144N^2 - 2304N - 576 & (293) \\
P_{242} = & 39N^{12} + 585N^{11} + 2938N^{10} + 7136N^9 + 9083N^8 + 7745N^7 + 14668N^6 + 38246N^5 \\
& +59856N^4 + 55560N^3 + 32144N^2 + 12480N + 2304 & (294) \\
P_{243} = & 48N^{12} + 459N^{11} + 2322N^{10} + 8290N^9 + 20159N^8 + 30862N^7 + 28247N^6 + 16109N^5 \\
& +9312N^4 + 7488N^3 + 4064N^2 + 1328N + 192 & (295) \\
P_{244} = & 61N^{12} + 302N^{11} + 531N^{10} + 348N^9 - 349N^8 - 786N^7 + 457N^6 + 2524N^5 \\
& +2012N^4 + 204N^3 - 360N^2 - 240N - 96 & (296) \\
P_{245} = & 92N^{12} + 796N^{11} + 3089N^{10} + 7550N^9 + 10547N^8 + 1029N^7 - 19496N^6 \\
& -24199N^5 - 8960N^4 + 736N^3 + 1744N^2 + 816N + 192 & (297) \\
P_{246} = & 201N^{12} + 1845N^{11} + 6910N^{10} + 12854N^9 + 8915N^8 - 7741N^7 - 17126N^6 \\
& -4294N^5 + 16260N^4 + 22080N^3 + 12416N^2 + 4128N + 576 & (298) \\
P_{247} = & 239N^{12} + 1338N^{11} + 3137N^{10} + 3164N^9 - 983N^8 - 6640N^7 - 8123N^6 \\
& -4526N^5 - 342N^4 + 1232N^3 + 848N^2 + 256N + 32 & (299) \\
P_{248} = & 255N^{12} + 2169N^{11} + 6496N^{10} + 7694N^9 - 127N^8 - 6973N^7 + 4132N^6 \\
& +25502N^5 + 31956N^4 + 22656N^3 + 9632N^2 + 864N - 576 & (300) \\
P_{249} = & 581N^{12} + 7035N^{11} + 37826N^{10} + 112904N^9 + 190293N^8 + 174327N^7 + 92032N^6 \\
& +69438N^5 + 78364N^4 + 44464N^3 + 11520N^2 - 3168N - 1728 & (301) \\
P_{250} = & 825N^{12} + 7363N^{11} + 25396N^{10} + 40686N^9 + 26213N^8 - 12749N^7 - 55498N^6 \\
& -89796N^5 - 110552N^4 - 134960N^3 - 127584N^2 - 64704N - 12672 & (302) \\
P_{251} = & 69N^{13} + 420N^{12} + 794N^{11} - 1357N^{10} - 10401N^9 - 15678N^8 + 532N^7 + 239N^6 \\
& -40018N^5 - 69432N^4 - 69152N^3 - 43792N^2 - 18336N - 3456 & (303) \\
P_{252} = & 76N^{13} + 922N^{12} + 4479N^{11} + 9107N^{10} - 3747N^9 - 52973N^8 - 76133N^7 + 42261N^6
\end{aligned}$$

$$\begin{aligned}
& +199307N^5 + 123839N^4 - 77470N^3 - 84132N^2 - 2160N - 432 & (304) \\
P_{253} = & 295N^{13} + 2387N^{12} + 8005N^{11} + 13687N^{10} + 10883N^9 + 389N^8 - 2641N^7 + 6029N^6 \\
& +11034N^5 + 6644N^4 + 1384N^3 + 80N^2 + 128N + 64 & (305) \\
P_{254} = & 296N^{13} + 2368N^{12} + 10916N^{11} + 27006N^{10} + 23644N^9 - 19764N^8 - 61931N^7 \\
& -63733N^6 - 52001N^5 - 56865N^4 - 38104N^3 + 2664N^2 + 7344N + 432 & (306) \\
P_{255} = & 377N^{13} + 4649N^{12} + 21813N^{11} + 38539N^{10} - 39339N^9 - 272611N^8 - 332971N^7 \\
& +220377N^6 + 801934N^5 + 384958N^4 - 362030N^3 - 297864N^2 - 1080N - 864 & (307) \\
P_{256} = & 859N^{13} + 7376N^{12} + 25294N^{11} + 47088N^{10} + 63868N^9 + 80876N^8 + 63648N^7 \\
& -35856N^6 - 146697N^5 - 157168N^4 - 91320N^3 - 34800N^2 - 8640N - 1152 & (308) \\
P_{257} = & 1211N^{13} + 5680N^{12} + 3338N^{11} - 17355N^{10} - 31517N^9 - 48486N^8 - 139667N^7 \\
& -278026N^6 - 340745N^5 - 269457N^4 - 138568N^3 - 34632N^2 + 9072N + 3888 & (309) \\
P_{258} = & 70N^{14} + 555N^{13} + 1599N^{12} + 1192N^{11} - 4430N^{10} - 13305N^9 - 11835N^8 + 8440N^7 \\
& +35816N^6 + 57126N^5 + 60340N^4 + 44464N^3 + 27808N^2 + 12768N + 2880 & (310) \\
P_{259} = & 76N^{14} + 802N^{13} + 2979N^{12} + 1847N^{11} - 19377N^{10} - 58253N^9 - 26543N^8 + 170601N^7 \\
& +362177N^6 + 225119N^5 - 103240N^4 - 193092N^3 - 137160N^2 - 117072N - 25920 & (311) \\
P_{260} = & 76N^{14} + 1042N^{13} + 5979N^{12} + 16367N^{11} + 11883N^{10} - 47693N^9 - 125723N^8 - 86079N^7 \\
& +36437N^6 + 22559N^5 - 51700N^4 + 24828N^3 + 132840N^2 + 116208N + 25920 & (312) \\
P_{261} = & 4N^{15} + 50N^{14} + 267N^{13} + 765N^{12} + 1183N^{11} + 682N^{10} - 826N^9 - 1858N^8 - 1116N^7 \\
& +457N^6 + 1500N^5 + 2268N^4 + 2400N^3 + 1392N^2 + 448N + 64 & (313) \\
P_{262} = & 26N^{15} + 314N^{14} + 1503N^{13} + 3222N^{12} + 2510N^{11} + 1996N^{10} + 15041N^9 + 40728N^8 \\
& +54008N^7 + 44956N^6 + 31936N^5 + 30416N^4 + 29568N^3 + 16704N^2 + 5376N + 768 & (314) \\
P_{263} = & 101N^{15} + 1234N^{14} + 6867N^{13} + 21904N^{12} + 40098N^{11} + 32226N^{10} - 22057N^9 \\
& -86972N^8 - 114557N^7 - 111416N^6 - 89204N^5 - 37312N^4 + 13392N^3 + 23040N^2 \\
& +9792N + 1536 & (315) \\
P_{264} = & 390N^{15} + 5121N^{14} + 30556N^{13} + 114173N^{12} + 321958N^{11} + 771597N^{10} \\
& +1583594N^9 + 2637549N^8 + 3381542N^7 + 3199120N^6 + 2183360N^5 + 1123200N^4 \\
& +489952N^3 + 178176N^2 + 48384N + 6912 & (316) \\
P_{265} = & 75N^{16} + 1245N^{15} + 8291N^{14} + 27609N^{13} + 43437N^{12} + 14221N^{11} - 5995N^{10} \\
& +182937N^9 + 488696N^8 + 296818N^7 - 452292N^6 - 730430N^5 - 186180N^4 \\
& +259728N^3 + 241056N^2 + 116640N + 25920 & (317) \\
P_{266} = & 115N^{16} + 1838N^{15} + 11829N^{14} + 36114N^{13} + 30900N^{12} - 133946N^{11} - 454068N^{10} \\
& -457420N^9 + 249211N^8 + 864716N^7 + 312979N^6 - 634466N^5 - 587862N^4 \\
& -19556N^3 + 104832N^2 + 9504N + 1728 & (318) \\
P_{267} = & 185N^{16} + 2988N^{15} + 19694N^{14} + 62954N^{13} + 64470N^{12} - 207876N^{11} - 792388N^{10} \\
& -861230N^9 + 437231N^8 + 1750616N^7 + 869954N^6 - 1016136N^5 - 1130122N^4 \\
& -96596N^3 + 199872N^2 + 31104N + 1728 & (319) \\
P_{268} = & 939N^{16} + 10527N^{15} + 37207N^{14} + 18679N^{13} - 202006N^{12} - 617170N^{11} - 930025N^{10} \\
& -882917N^9 - 157123N^8 + 1388549N^7 + 2739376N^6 + 2837500N^5 + 2088640N^4 \\
& +1259696N^3 + 622464N^2 + 211392N + 34560 & (320) \\
P_{269} = & 1155N^{16} + 12417N^{15} + 37693N^{14} - 12293N^{13} - 285754N^{12} - 613900N^{11} - 571735N^{10} \\
& -134309N^9 + 778901N^8 + 2698745N^7 + 4995724N^6 + 5915740N^5 + 4978144N^4 \\
& +3161840N^3 + 1498752N^2 + 479808N + 76032 & (321)
\end{aligned}$$

$$\begin{aligned}
P_{270} = & 1665N^{16} + 33005N^{15} + 287646N^{14} + 1402624N^{13} + 4031902N^{12} + 6199846N^{11} \\
& + 1054640N^{10} - 16668628N^9 - 37272559N^8 - 38892027N^7 - 17387942N^6 + 3962700N^5 \\
& + 15625800N^4 + 26960688N^3 + 27379296N^2 + 12985920N + 2332800 \quad (322)
\end{aligned}$$

$$\begin{aligned}
P_{271} = & 87N^{17} + 1099N^{16} + 6055N^{15} + 19019N^{14} + 37119N^{13} + 45159N^{12} + 29583N^{11} - 2639N^{10} \\
& - 30218N^9 - 40778N^8 - 39994N^7 - 35844N^6 - 30808N^5 - 30384N^4 - 28256N^3 \\
& - 16064N^2 - 5248N - 768 \quad (323)
\end{aligned}$$

$$\begin{aligned}
P_{272} = & 829N^{17} + 13413N^{16} + 83461N^{15} + 226391N^{14} + 55508N^{13} - 1239070N^{12} - 2862466N^{11} \\
& - 1217372N^{10} + 3372689N^9 + 2779147N^8 - 2705687N^7 + 171733N^6 \\
& + 8617302N^5 + 5817902N^4 - 3127236N^3 - 3652560N^2 - 336096N - 25920 \quad (324)
\end{aligned}$$

$$\begin{aligned}
P_{273} = & 1407N^{17} + 18107N^{16} + 103463N^{15} + 347083N^{14} + 760095N^{13} + 1142715N^{12} + 1220067N^{11} \\
& + 983393N^{10} + 702746N^9 + 533822N^8 + 337702N^7 - 3552N^6 \\
& - 300296N^5 - 332160N^4 - 188128N^3 - 63232N^2 - 13184N - 1536 \quad (325)
\end{aligned}$$

$$\begin{aligned}
P_{274} = & 95N^{18} + 3940N^{17} + 48989N^{16} + 308380N^{15} + 1166094N^{14} + 2843192N^{13} + 4428234N^{12} \\
& + 3171928N^{11} - 4692053N^{10} - 19875244N^9 - 34305831N^8 - 34774388N^7 - 16392680N^6 \\
& + 11584912N^5 + 30493776N^4 + 29700864N^3 + 18783360N^2 + 8294400N + 1866240 \quad (326)
\end{aligned}$$

$$\begin{aligned}
P_{275} = & 325N^{18} + 4280N^{17} + 17759N^{16} - 14880N^{15} - 412326N^{14} - 1696848N^{13} - 3216546N^{12} \\
& - 1169232N^{11} + 8956857N^{10} + 23914216N^9 + 31536899N^8 + 25361392N^7 + 9982840N^6 \\
& - 10154128N^5 - 26098704N^4 - 26761536N^3 - 17642880N^2 - 8087040N - 1866240 \quad (327)
\end{aligned}$$

$$\begin{aligned}
P_{276} = & 500N^{18} + 8215N^{17} + 56287N^{16} + 201810N^{15} + 361782N^{14} + 98826N^{13} - 759348N^{12} \\
& - 495786N^{11} + 3942186N^{10} + 11896133N^9 + 16709737N^8 + 13315736N^7 + 3779660N^6 \\
& - 7306454N^5 - 14232852N^4 - 13254768N^3 - 8367840N^2 - 3771360N - 855360 \quad (328)
\end{aligned}$$

$$\begin{aligned}
P_{277} = & 150N^{19} + 2815N^{18} + 24285N^{17} + 131358N^{16} + 511310N^{15} + 1515954N^{14} \\
& + 3372978N^{13} + 5213980N^{12} + 4715522N^{11} + 980739N^{10} - 2709391N^9 - 3741506N^8 \\
& - 4630558N^7 - 5623132N^6 - 2333736N^5 + 3419632N^4 + 5238496N^3 + 3231936N^2 \\
& + 1123200N + 172800 \quad (329)
\end{aligned}$$

$$\begin{aligned}
P_{278} = & 5410N^{19} + 98215N^{18} + 764965N^{17} + 3280996N^{16} + 8031920N^{15} + 8939378N^{14} \\
& - 7608074N^{13} - 44964380N^{12} - 74768226N^{11} - 57879177N^{10} - 5243187N^9 + 13745888N^8 \\
& - 28158216N^7 - 49672024N^6 + 14757808N^5 + 94650144N^4 + 100507392N^3 \\
& + 53764992N^2 + 15655680N + 1866240 \quad (330)
\end{aligned}$$

$$\begin{aligned}
P_{279} = & 7060N^{20} + 123495N^{19} + 898682N^{18} + 3394183N^{17} + 6222824N^{16} + 376386N^{15} \\
& - 22032204N^{14} - 39912378N^{13} - 13976964N^{12} + 31985011N^{11} + 4994394N^{10} \\
& - 91499501N^9 - 97243208N^8 + 54501988N^7 + 183103272N^6 + 127073120N^5 \\
& - 20272608N^4 - 88410816N^3 - 62225280N^2 - 21772800N - 3110400 . \quad (331)
\end{aligned}$$

The corresponding expressions in z -space are given in Appendix B.

Note that our result for $H_{g,2}^S$ differs from the one given in Eq. (B.7) in z -space in Ref. [9] by the term

$$C_F T_F^2 N_F \frac{4(N^2 + N + 2)}{N(N+1)(N+2)} [28\zeta_2 - 69] \quad (332)$$

in N -space. This result of [9] is based on the calculation carried out in Ref. [12], including the renormalization formulae derived there. We have checked, however, that our result Eq. (152) is in full agreement with Eq. (27) and the moments having been calculated by part of the present authors in Ref. [12]. The corresponding expression in z -space is presented in Appendix B.

5 The Asymptotic Wilson Coefficients for the Longitudinal Structure Function

The Wilson coefficients have been calculated in Ref. [22] for exclusive heavy flavor production, retaining three contributions only. In total also here five Wilson coefficients contribute and the expressions are slightly modified in the inclusive case of the complete structure function $F_L(x, Q^2)$, cf. [12]. In Mellin- N space they read :

$$\begin{aligned}
L_{q,L}^{\text{PS},(3)} &= \frac{1}{2}[1 + (-1)^N] \\
&\times \left\{ a_s^3 \left\{ C_F N_F T_F^2 \left[\frac{128 L_Q^2 (N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)} + \frac{128(N^2 + N + 2)L_M^2}{3(N-1)N(N+1)^2(N+2)} \right. \right. \right. \\
&\quad - \frac{256 L_Q (11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)}{9(N-1)N^2(N+1)^3(N+2)^2} + \left. \left. \left. \frac{256(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^3(N+2)} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{256(N^2 + N + 2)S_1}{3(N-1)N(N+1)^2(N+2)} \right] L_M + \frac{64(N^2 + N + 2)[S_1^2 + S_2]}{3(N-1)N(N+1)^2(N+2)} \right. \\
&\quad \left. \left. - \frac{128(8N^3 + 13N^2 + 27N + 16)S_1}{9(N-1)N(N+1)^3(N+2)} + \frac{128(43N^4 + 105N^3 + 224N^2 + 230N + 86)}{27(N-1)N(N+1)^4(N+2)} \right] \right. \\
&\quad \left. + N_F \hat{C}_{L,q}^{\text{PS},(3)}(N_F) \right\}, \tag{333}
\end{aligned}$$

$$\begin{aligned}
L_{g,L}^{\text{S}} &= \frac{1}{2}[1 + (-1)^N] \\
&\times \left\{ a_s^2 \frac{64 N_F T_F^2 L_M}{3(N+1)(N+2)} + a_s^3 \left\{ N_F T_F^3 \frac{256 L_M^2}{9(N+1)(N+2)} + C_A N_F T_F^2 \left[\right. \right. \right. \\
&\quad \left. \left. \left. \frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128 S_1}{3(N+1)(N+2)} \right] L_Q^2 + \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{64 Q_2}{9(N-1)N(N+1)^3(N+2)^3} + \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} \right. \right. \right. \\
&\quad \left. \left. \left. + L_M \left[\frac{512(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256 S_1}{3(N+1)(N+2)} \right] \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{1}{(N+1)(N+2)} \left[\frac{128}{3} S_1^2 - \frac{128 S_2}{3} - \frac{256}{3} S_{-2} \right] L_Q + \frac{32 Q_5}{27(N-1)N^3(N+1)^4(N+2)^2} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{64(56N + 47)S_1}{27(N+1)^2(N+2)} + L_M^2 \left[\frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128 S_1}{3(N+1)(N+2)} \right] \right. \right. \right. \\
&\quad \left. \left. \left. + L_M \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} + \frac{128 Q_4}{9(N-1)N^2(N+1)^3(N+2)^3} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{256(N^3 - 6N^2 + 2N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\frac{128}{3} S_1^2 - \frac{128 S_2}{3} - \frac{256}{3} S_{-2}}{(N+1)(N+2)} \right] \right. \right. \right. \\
&\quad \left. \left. \left. + C_F T_F^2 N_F \left[\frac{64(N^2 + N + 2)(N^4 + 2N^3 + 2N^2 + N + 6)L_Q^2}{3(N-1)N^2(N+1)^3(N+2)^2} \right. \right. \right. \right. \\
&\quad \left. \left. \left. + \left[\frac{256(-1)^N Q_6}{45(N-2)(N-1)^2 N^2(N+1)^3(N+2)^2(N+3)^3} \right. \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left. \begin{aligned}
& -\frac{32Q_9}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} - \frac{128Q_1S_1}{3(N-1)N^2(N+1)^3(N+2)^2} \\
& + \frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)} + \frac{64(N^2+N+2)(N^4+2N^3-7N^2-8N-12)L_M}{3(N-1)N^2(N+1)^3(N+2)^2} \Big] L_Q \\
& + \frac{64(N^2+N+2)^2L_M^2}{(N-1)N^2(N+1)^3(N+2)^2} - \frac{16Q_7}{(N-1)N^4(N+1)^5(N+2)^2} \\
& + L_M \left[\frac{256(-1)^N Q_6}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} \right. \\
& + \left. \frac{64Q_8}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} - \frac{64Q_3S_1}{3(N-1)N^2(N+1)^3(N+2)^2} \right. \\
& \left. \left. + \frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)} \right] \right] + N_F \hat{C}_{L,g}^{\hat{S},(3)}(N_F) \Big\} \Big\}, \tag{334}
\end{aligned}
\end{aligned}$$

with

$$Q_1 = 2N^6 + 6N^5 + 7N^4 + 4N^3 + 9N^2 + 8N + 12 \tag{335}$$

$$Q_2 = 3N^6 + 3N^5 - 121N^4 - 391N^3 - 474N^2 - 308N - 80 \tag{336}$$

$$Q_3 = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \tag{337}$$

$$Q_4 = 6N^7 + 24N^6 + 47N^5 + 104N^4 + 219N^3 + 316N^2 + 208N + 48 \tag{338}$$

$$Q_5 = 15N^8 + 60N^7 + 572N^6 + 1470N^5 + 2135N^4 + 1794N^3 + 722N^2 - 24N - 72 \tag{339}$$

$$Q_6 = N^{10} - 13N^9 - 39N^8 + 222N^7 + 1132N^6 + 1787N^5 + 913N^4 + 392N^3 + 645N^2 - 324N - 108 \tag{340}$$

$$Q_7 = 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2 - 32N - 16 \tag{341}$$

$$Q_8 = 45N^{13} + 656N^{12} + 4397N^{11} + 17513N^{10} + 43665N^9 + 63005N^8 + 27977N^7 - 71993N^6 - 140386N^5 - 78985N^4 + 25350N^3 + 80460N^2 + 100008N + 38880 \tag{342}$$

$$Q_9 = 95N^{13} + 1218N^{12} + 6096N^{11} + 14484N^{10} + 11570N^9 - 28440N^8 - 117844N^7 - 225884N^6 - 238953N^5 - 83290N^4 + 57660N^3 + 122040N^2 + 182304N + 77760, \tag{343}$$

$$L_{q,L}^{\text{NS}} = \frac{1}{2}[1 + (-1)^N]$$

$$\begin{aligned}
& \times \left\{ a_s^2 C_F T_F \left\{ \frac{16L_Q}{3(N+1)} - \frac{8(19N^2+7N-6)}{9N(N+1)^2} - \frac{16S_1}{3(N+1)} \right\} \right. \\
& + a_s^3 \left\{ C_F^2 T_F \left[\left[\frac{8(3N^2+3N+2)}{N(N+1)^2} - \frac{32S_1}{N+1} \right] L_Q^2 \right. \right. \\
& + \left. \left. \left[\frac{256(-1)^N Q_{11}}{15(N-2)(N-1)^2N^2(N+1)^4(N+2)^2(N+3)^3} \right. \right. \right. \\
& - \frac{32Q_{12}}{45(N-1)^2N^2(N+1)^4(N+2)^2(N+3)^3} - \frac{16(N+10)(5N+3)S_1}{9N(N+1)^2} \\
& + \frac{512(N^4+2N^3-N^2-2N-6)S_{-2}}{3(N-2)N(N+1)^2(N+3)} + \frac{1}{N+1} \left[\frac{128}{3}S_1^2 - \frac{512}{3}S_{-2}S_1 - \frac{128S_2}{3} - \frac{256S_3}{3} \right. \\
& \left. \left. - \frac{256}{3}S_{-3} + \frac{512}{3}S_{-2,1} + 256\zeta_3 \right] \right. L_Q + \frac{2Q_{10}}{27N^3(N+1)^4} + L_M^2 \left[\frac{8(3N^2+3N+2)}{3N(N+1)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{32S_1}{3(N+1)} \Big] + L_M \left[\frac{8(3N^4 + 6N^3 + 47N^2 + 20N - 12)}{9N^2(N+1)^3} + \frac{64S_2 - \frac{320S_1}{9}}{N+1} \right] \\
& + \frac{-\frac{896}{27}S_1 + \frac{160S_2}{9} - \frac{32S_3}{3}}{N+1} \Big] + C_F T_F^2 \left[\frac{64L_Q^2}{9(N+1)} + \left[-\frac{64(19N^2 + 7N - 6)}{27N(N+1)^2} - \frac{128S_1}{9(N+1)} \right] L_Q \right] \\
& + C_F T_F^2 N_F \left[\frac{128L_Q^2}{9(N+1)} + \left[-\frac{128(19N^2 + 7N - 6)}{27N(N+1)^2} - \frac{256S_1}{9(N+1)} \right] L_Q \right] + C_F C_A T_F \left[L_Q \left[\right. \right. \\
& \left. \left. -\frac{128(-1)^N Q_{11}}{15(N-2)(N-1)^2 N^2 (N+1)^4 (N+2)^2 (N+3)^3} - \frac{256(N^4 + 2N^3 - N^2 - 2N - 6)S_{-2}}{3(N-2)N(N+1)^2(N+3)} \right. \right. \\
& \left. \left. + \frac{16Q_{13}}{135(N-1)^2 N^2 (N+1)^4 (N+2)^2 (N+3)^3} + \frac{1}{N+1} \left[\frac{256}{3}S_{-2}S_1 + \frac{1088S_1}{9} + \frac{128S_3}{3} \right. \right. \right. \\
& \left. \left. \left. + \frac{128}{3}S_{-3} - \frac{256}{3}S_{-2,1} - 128\zeta_3 \right] \right] - \frac{352L_Q^2}{9(N+1)} \right] + \hat{C}_{L,q}^{\text{NS},(3)}(N_F) \Big\} \Big\} , \tag{344}
\end{aligned}$$

with

$$Q_{10} = 219N^6 + 657N^5 + 1193N^4 + 763N^3 - 40N^2 - 48N + 72 \tag{345}$$

$$\begin{aligned}
Q_{11} &= 2N^{11} + 41N^{10} + 226N^9 + 556N^8 + 963N^7 + 2733N^6 + 7160N^5 + 8610N^4 + 1969N^3 \\
&\quad - 2748N^2 - 864N - 216 \tag{346}
\end{aligned}$$

$$\begin{aligned}
Q_{12} &= 180N^{12} + 2385N^{11} + 11798N^{10} + 23030N^9 - 10466N^8 - 131068N^7 - 245294N^6 \\
&\quad - 196786N^5 - 22282N^4 + 86571N^3 + 50688N^2 - 7236N - 3888 \tag{347}
\end{aligned}$$

$$\begin{aligned}
Q_{13} &= 2345N^{12} + 31510N^{11} + 163614N^{10} + 380250N^9 + 208092N^8 - 794874N^7 - 1604762N^6 \\
&\quad - 833938N^5 + 451419N^4 + 584028N^3 + 113724N^2 - 36288N + 7776, \tag{348}
\end{aligned}$$

$$\begin{aligned}
H_{q,L}^{\text{PS}} &= \frac{1}{2}[1 + (-1)^N] \\
&\times \left\{ a_s^2 C_F T_F \left\{ -\frac{32S_1(N^2 + N + 2)}{(N-1)N(N+1)^2(N+2)} + \frac{32L_Q(N^2 + N + 2)}{(N-1)N(N+1)^2(N+2)} \right. \right. \\
&\quad \left. \left. - \frac{32(N^5 + 2N^4 + 2N^3 - 5N^2 - 12N - 4)}{(N-1)N^2(N+1)^3(N+2)^2} \right\} + a_s^3 \left\{ C_F^2 T_F \left[\left[\frac{64(N^2 + N + 1)(N^2 + N + 2)}{(N-1)N^2(N+1)^3(N+2)} \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{64(N^2 + N + 2)S_1}{(N-1)N(N+1)^2(N+2)} \right] L_Q^2 + \left[\frac{128(-1)^N(N^2 + N + 2)Q_{16}}{15(N-2)(N-1)^3N^3(N+1)^4(N+2)^2(N+3)^3} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{32Q_{18}}{15(N-1)^3N^3(N+1)^4(N+2)^2(N+3)^3} + \frac{128(2N^5 + 5N^4 + 7N^3 + 2N^2 - 12N - 8)S_1}{(N-1)N^2(N+1)^3(N+2)^2} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{(N^2 + N + 2)(64S_1^2 - 64S_2)}{(N-1)N(N+1)^2(N+2)} + \frac{128(N^2 + N + 2)S_{-2}}{(N-2)N(N+1)^2(N+3)} \right] L_Q \right. \right. \\
&\quad \left. \left. - \frac{16(N^2 + N + 2)^2 L_M^2}{(N-1)N^2(N+1)^3(N+2)} + \frac{16Q_{17}}{(N-1)N^4(N+1)^5(N+2)^3} - \frac{32(N^2 + N + 2)^2 S_2}{(N-1)N^2(N+1)^3(N+2)} \right. \right. \\
&\quad \left. \left. - \frac{32(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)L_M}{(N-1)N^3(N+1)^4(N+2)^2} \right] + C_F T_F^2 N_F \left[\frac{128L_Q^2(N^2 + N + 2)}{3(N-1)N(N+1)^2(N+2)} \right. \right. \\
&\quad \left. \left. - \frac{256L_Q(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)}{9(N-1)N^2(N+1)^3(N+2)^2} \right] + C_F T_F^2 \left[\frac{128(N^2 + N + 2)L_Q^2}{3(N-1)N(N+1)^2(N+2)} \right. \right. \\
&\quad \left. \left. - \frac{256(11N^5 + 35N^4 + 59N^3 + 55N^2 - 4N - 12)L_Q}{9(N-1)N^2(N+1)^3(N+2)^2} + \frac{128(N^2 + N + 2)L_M^2}{3(N-1)N(N+1)^2(N+2)} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{128(43N^4 + 105N^3 + 224N^2 + 230N + 86)}{27(N-1)N(N+1)^4(N+2)} - \frac{128(8N^3 + 13N^2 + 27N + 16)S_1}{9(N-1)N(N+1)^3(N+2)} \\
& + L_M \left[\frac{256(8N^3 + 13N^2 + 27N + 16)}{9(N-1)N(N+1)^3(N+2)} - \frac{256(N^2 + N + 2)S_1}{3(N-1)N(N+1)^2(N+2)} \right] \\
& + \frac{(N^2 + N + 2) \left[\frac{64}{3}S_1^2 + \frac{64S_2}{3} \right]}{(N-1)N(N+1)^2(N+2)} \\
& + C_F C_{AT_F} \left[\left[-\frac{32(N^2 + N + 2)(11N^4 + 22N^3 - 23N^2 - 34N - 12)}{3(N-1)^2N^2(N+1)^3(N+2)^2} \right. \right. \\
& \left. \left. - \frac{64(N^2 + N + 2)S_1}{(N-1)N(N+1)^2(N+2)} \right] L_Q^2 + \left[\frac{128(-1)^N Q_{14}}{(N-1)N^2(N+1)^4(N+2)^3} \right. \right. \\
& \left. \left. + \frac{64Q_{15}}{9(N-1)^2N^3(N+1)^3(N+2)^3} + \frac{128(N^4 - N^3 - 4N^2 - 11N - 1)S_1}{(N-1)^2N(N+1)^3(N+2)} \right. \right. \\
& \left. \left. + \frac{(N^2 + N + 2) \left[128S_1^2 - 128S_2 - 256S_{-2} \right]}{(N-1)N(N+1)^2(N+2)} \right] L_Q \right] + \tilde{C}_{L,q}^{\text{PS},(3)}(N_F + 1) \left. \right\}, \tag{349}
\end{aligned}$$

with

$$Q_{14} = N^6 + 8N^5 + 30N^4 + 58N^3 + 65N^2 + 42N + 8 \tag{350}$$

$$Q_{15} = 142N^8 + 593N^7 + 801N^6 + 199N^5 - 1067N^4 - 900N^3 + 976N^2 + 1128N + 288 \tag{351}$$

$$Q_{16} = N^{10} - 13N^9 - 39N^8 + 222N^7 + 1132N^6 + 1787N^5 + 913N^4 + 392N^3 + 645N^2 - 324N - 108 \tag{352}$$

$$Q_{17} = N^{10} + 8N^9 + 29N^8 + 49N^7 - 11N^6 - 131N^5 - 161N^4 - 160N^3 - 168N^2 - 80N - 16 \tag{353}$$

$$Q_{18} = 225N^{12} + 2494N^{11} + 9980N^{10} + 14480N^9 - 11602N^8 - 68380N^7 - 86828N^6 - 15080N^5 + 67401N^4 + 60334N^3 - 312N^2 - 33912N - 12528, \tag{354}$$

and

$$\begin{aligned}
H_{g,L}^S &= \frac{1}{2}[1 + (-1)^N] \\
& \times \left\{ a_s T_F \frac{16}{(N+1)(N+2)} + a_s^2 \left\{ \frac{64L_M T_F^2}{3(N+1)(N+2)} + C_{AT_F} \left[\frac{64(-1)^N(N^3 + 4N^2 + 7N + 5)}{(N+1)^3(N+2)^3} \right. \right. \right. \\
& \left. \left. - \frac{32(2N^5 + 9N^4 + 5N^3 - 12N^2 - 20N - 8)}{(N-1)N^2(N+1)^2(N+2)^3} + \frac{64(2N^3 - 2N^2 - N - 1)S_1}{(N-1)N(N+1)^2(N+2)} \right. \right. \\
& \left. \left. + L_Q \left[\frac{128(N^2 + N + 1)}{(N-1)N(N+1)^2(N+2)^2} - \frac{64S_1}{(N+1)(N+2)} \right] + \frac{32S_1^2 - 32S_2 - 64S_{-2}}{(N+1)(N+2)} \right] \right. \\
& \left. + C_F T_F \left[-\frac{16L_M(N^2 + N + 2)}{N(N+1)^2(N+2)} + \frac{16L_Q(N^2 + N + 2)}{N(N+1)^2(N+2)} \right. \right. \\
& \left. \left. + \frac{16Q_{29}}{15(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} + \frac{64(-1)^N Q_{30}}{15(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} \right. \right. \\
& \left. \left. - \frac{16(3N^2 + 3N + 2)S_1}{N(N+1)^2(N+2)} + \frac{64(N-1)S_{-2}}{(N-2)(N+1)(N+3)} \right] \right\} \\
& + a_s^3 \left\{ \frac{256L_M^2 T_F^3}{9(N+1)(N+2)} + C_{AT_F}^2 \left[\left[\frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \right] L_Q^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} + \frac{64Q_{22}}{9(N-1)N(N+1)^3(N+2)^3} \right. \\
& + \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} + L_M \left[\frac{512(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256S_1}{3(N+1)(N+2)} \right] \\
& + \left. \frac{\frac{128}{3}S_1^2 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \right] L_Q + \frac{32Q_{27}}{27(N-1)N^3(N+1)^4(N+2)^2} - \frac{64(56N+47)S_1}{27(N+1)^2(N+2)} \\
& + L_M^2 \left[\frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \right] + L_M \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} \right. \\
& + \left. \frac{128Q_{24}}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{256(N^3 - 6N^2 + 2N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\frac{128}{3}S_1^2 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \right] \\
& + C_{AT_F}^2 N_F \left[\left[\frac{256(N^2 + N + 1)}{3(N-1)N(N+1)^2(N+2)^2} - \frac{128S_1}{3(N+1)(N+2)} \right] L_Q^2 \right. \\
& + \left[\frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)}{3(N+1)^3(N+2)^3} + \frac{64Q_{22}}{9(N-1)N(N+1)^3(N+2)^3} \right. \\
& + \left. \frac{256(11N^3 - 6N^2 - 8N - 3)S_1}{9(N-1)N(N+1)^2(N+2)} + \frac{\frac{128}{3}S_1^2 - \frac{128S_2}{3} - \frac{256}{3}S_{-2}}{(N+1)(N+2)} \right] L_Q \\
& + C_{AT_F}^2 \left[\left[\frac{128S_1^2}{(N+1)(N+2)} + \frac{32(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_1}{3(N-1)N(N+1)^2(N+2)^2} \right. \right. \\
& - \left. \left. \frac{64(N^2 + N + 1)(11N^4 + 22N^3 - 35N^2 - 46N - 24)}{3(N-1)^2N^2(N+1)^3(N+2)^3} \right] L_Q^2 \right. \\
& + \left[-\frac{32(59N^4 + 70N^3 - 155N^2 - 118N - 72)S_1^2}{3(N-1)N(N+1)^2(N+2)^2} - \frac{256(-1)^N(N^3 + 4N^2 + 7N + 5)S_1}{(N+1)^3(N+2)^3} \right. \\
& - \frac{64Q_{28}S_1}{9(N-1)^2N^2(N+1)^3(N+2)^3} - \frac{64(-1)^N Q_{25}}{3(N-1)N^2(N+1)^4(N+2)^4} \\
& - \frac{32Q_{31}}{9(N-1)^2N^3(N+1)^3(N+2)^4} + \frac{32(11N^4 + 22N^3 - 83N^2 - 94N - 72)S_2}{3(N-1)N(N+1)^2(N+2)^2} \\
& + \frac{64(11N^4 + 22N^3 - 59N^2 - 70N - 48)S_{-2}}{3(N-1)N(N+1)^2(N+2)^2} + \frac{1}{(N+1)(N+2)} \left[-128S_1^3 \right. \\
& + 384S_2S_1 + 512S_{-2}S_1 + 128S_3 + 128S_{-3} - 256S_{-2,1} \left. \right] L_Q \\
& + C_{FT_F}^2 \left[\left[\frac{8(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^3(N+2)} - \frac{32(N^2 + N + 2)S_1}{N(N+1)^2(N+2)} \right] L_Q^2 \right. \\
& + \left[\frac{128(-1)^N(N^2 + N + 2)Q_{34}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} - \frac{8Q_{39}}{5(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} \right. \\
& - \frac{16(9N^4 + 26N^3 + 49N^2 + 48N + 12)S_1}{N^2(N+1)^3(N+2)} + L_M \left[\frac{64(N^2 + N + 2)S_1}{N(N+1)^2(N+2)} \right. \\
& - \left. \frac{16(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^3(N+2)} \right] + \frac{256(N^2 + N + 2)(N^4 + 2N^3 - N^2 - 2N - 6)S_{-2}}{(N-2)N^2(N+1)^3(N+2)(N+3)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(N^2 + N + 2)}{N(N+1)^2(N+2)} \left[64S_1^2 - 256S_{-2}S_1 - 64S_2 - 128S_3 - 128S_{-3} + 256S_{-2,1} + 384\zeta_3 \right] L_Q \\
& + \frac{16(3N+2)S_1^2}{N^2(N+1)(N+2)} + \frac{8Q_{26}}{N^4(N+1)^5(N+2)} + \frac{16(N^4 - N^3 - 20N^2 - 10N - 4)S_1}{N^2(N+1)^3(N+2)} \\
& + L_M^2 \left[\frac{8(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^3(N+2)} - \frac{32(N^2 + N + 2)S_1}{N(N+1)^2(N+2)} \right] \\
& + \frac{16(N^4 + 17N^3 + 17N^2 - 5N - 2)S_2}{N^2(N+1)^3(N+2)} + \frac{(N^2 + N + 2) \left[-\frac{16}{3}S_1^3 - 16S_2S_1 + \frac{64S_3}{3} \right]}{N(N+1)^2(N+2)} \\
& + L_M \left[-\frac{128(-1)^N(N^2 + N + 2)Q_{34}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} \right. \\
& + \frac{8Q_{39}}{5(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} + \frac{16(9N^4 + 26N^3 + 49N^2 + 48N + 12)S_1}{N^2(N+1)^3(N+2)} \\
& \left. - \frac{256(N^2 + N + 2)(N^4 + 2N^3 - N^2 - 2N - 6)S_{-2}}{(N-2)N^2(N+1)^3(N+2)(N+3)} + \frac{(N^2 + N + 2)}{N(N+1)^2(N+2)} \left[-64S_1^2 \right. \right. \\
& \left. \left. + 256S_{-2}S_1 + 64S_2 + 128S_3 + 128S_{-3} - 256S_{-2,1} - 384\zeta_3 \right] \right] \\
& + C_F T_F^2 \left[\frac{64(N^2 + N + 2)(N^4 + 2N^3 + 2N^2 + N + 6)L_Q^2}{3(N-1)N^2(N+1)^3(N+2)^2} + \left[-\frac{128L_M(N^2 + N + 2)^2}{(N-1)N^2(N+1)^3(N+2)^2} \right. \right. \\
& + \frac{256(-1)^N Q_{30}}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} - \frac{32Q_{37}}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} \\
& \left. - \frac{128Q_{21}S_1}{3(N-1)N^2(N+1)^3(N+2)^2} + \frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)} \right] L_Q \\
& - \frac{64(N-2)(N+3)(N^2 + N + 1)(N^2 + N + 2)L_M^2}{3(N-1)N^2(N+1)^3(N+2)^2} - \frac{16Q_{32}}{(N-1)N^4(N+1)^5(N+2)^2} \\
& + L_M \left[\frac{256(-1)^N Q_{30}}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} + \frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)} \right. \\
& \left. + \frac{32Q_{38}}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} - \frac{128Q_{19}S_1}{3(N-1)N^2(N+1)^3(N+2)^2} \right] \\
& + C_F T_F^2 N_F \left[\frac{64(N^2 + N + 2)(N^4 + 2N^3 + 2N^2 + N + 6)L_Q^2}{3(N-1)N^2(N+1)^3(N+2)^2} \right. \\
& + \left[\frac{256(-1)^N Q_{30}}{45(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} - \frac{128Q_{21}S_1}{3(N-1)N^2(N+1)^3(N+2)^2} \right. \\
& \left. - \frac{32Q_{37}}{45(N-1)^2N^3(N+1)^4(N+2)^3(N+3)^3} + \frac{256(N-1)S_{-2}}{3(N-2)(N+1)(N+3)} - \frac{64(N^2 + N + 2)L_M}{3N(N+1)^2(N+2)} \right] L_Q \\
& + L_M \left[\frac{32(N^2 + N + 2)(19N^2 + 7N - 6)}{9N^2(N+1)^3(N+2)} + \frac{64(N^2 + N + 2)S_1}{3N(N+1)^2(N+2)} \right] \\
& + C_F C_A T_F \left[\left[-\frac{16(N^2 + N + 2)(11N^4 + 22N^3 - 23N^2 - 34N - 12)}{3(N-1)N^2(N+1)^3(N+2)^2} - \frac{32(N^2 + N + 2)S_1}{N(N+1)^2(N+2)} \right] L_Q^2 \right. \\
& \left. + \left[\frac{32(5N^2 + 5N + 2)S_1^2}{N(N+1)^2(N+2)} - \frac{256(-1)^N Q_{30}S_1}{15(N-2)(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{128Q_{33}S_1}{15(N-1)^2N^2(N+1)^3(N+2)^2(N+3)^3} - \frac{128(N^4+2N^3+N^2+12)S_{-2}S_1}{(N-2)N(N+1)^2(N+2)(N+3)} \\
& - \frac{64(-1)^N Q_{41}}{45(N-2)(N-1)^3N^3(N+1)^5(N+2)^3(N+3)^3} + \frac{8Q_{42}}{45(N-1)^3N^3(N+1)^5(N+2)^3(N+3)^3} \\
& - \frac{128Q_{23}S_{-2}}{3(N-2)N^2(N+1)^3(N+2)(N+3)} + \frac{176(N^2+N+2)L_M}{3N(N+1)^2(N+2)} \\
& + \frac{(N^2+N+2)\left[-32S_2+64S_3+64S_{-3}-128S_{-2,1}-192\zeta_3\right]}{N(N+1)^2(N+2)} \Big] L_Q \\
& - \frac{16(N^3+8N^2+11N+2)S_1^2}{N(N+1)^3(N+2)^2} + \frac{16Q_{35}}{(N-1)N^4(N+1)^5(N+2)^4} - \frac{16Q_{20}S_1}{N(N+1)^4(N+2)^3} \\
& + L_M^2 \left[\frac{32(N^2+N+2)S_1}{N(N+1)^2(N+2)} - \frac{64(N^2+N+1)(N^2+N+2)}{(N-1)N^2(N+1)^3(N+2)^2} \right] \\
& - \frac{16(7N^5+21N^4+13N^3+21N^2+18N+16)S_2}{(N-1)N^2(N+1)^3(N+2)^2} + \frac{(N^2-N-4)64(-1)^N S_{-2}}{(N+1)^3(N+2)^2} \\
& + \frac{(N^2+N+2)}{N(N+1)^2(N+2)} \left[\frac{16}{3}S_1^3 + 48S_2S_1 + 64(-1)^N S_{-2}S_1 + \frac{128S_3}{3} + 32(-1)^N S_{-3} - 64S_{-2,1} \right] \\
& + L_M \left[\frac{64(-1)^N Q_{36}}{5(N-2)(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} \right. \\
& - \frac{8Q_{40}}{45(N-1)^2N^3(N+1)^5(N+2)^3(N+3)^3} - \frac{16(23N^4+92N^3+209N^2+256N+92)S_1}{3N(N+1)^3(N+2)^2} \\
& + \frac{64(N^2+N+2)(3N^4+6N^3-7N^2-10N-12)S_{-2}}{(N-2)N^2(N+1)^3(N+2)(N+3)} \\
& \left. + \frac{(N^2+N+2)\left[32S_1^2-128S_{-2}S_1+32S_2-64S_3-64S_{-3}+128S_{-2,1}+192\zeta_3\right]}{N(N+1)^2(N+2)} \right] \Big] \\
& + \tilde{C}_{L,g}^{S,(3)}(N_F+1) \Big\} \Big\}, \tag{355}
\end{aligned}$$

with

$$Q_{19} = N^6 + 3N^5 - 2N^4 - 9N^3 - 17N^2 - 12N - 12 \tag{356}$$

$$Q_{20} = N^6 + 8N^5 + 23N^4 + 54N^3 + 94N^2 + 72N + 8 \tag{357}$$

$$Q_{21} = 2N^6 + 6N^5 + 7N^4 + 4N^3 + 9N^2 + 8N + 12 \tag{358}$$

$$Q_{22} = 3N^6 + 3N^5 - 121N^4 - 391N^3 - 474N^2 - 308N - 80 \tag{359}$$

$$Q_{23} = 10N^6 + 30N^5 + N^4 - 48N^3 - 89N^2 - 60N - 36 \tag{360}$$

$$Q_{24} = 6N^7 + 24N^6 + 47N^5 + 104N^4 + 219N^3 + 316N^2 + 208N + 48 \tag{361}$$

$$Q_{25} = 11N^8 + 66N^7 + 106N^6 - 121N^5 - 775N^4 - 1325N^3 - 1130N^2 - 552N - 96 \tag{362}$$

$$Q_{26} = 12N^8 + 52N^7 + 132N^6 + 216N^5 + 191N^4 + 54N^3 - 25N^2 - 20N - 4 \tag{363}$$

$$Q_{27} = 15N^8 + 60N^7 + 572N^6 + 1470N^5 + 2135N^4 + 1794N^3 + 722N^2 - 24N - 72 \tag{364}$$

$$Q_{28} = 133N^8 + 430N^7 - 271N^6 - 1361N^5 + 110N^4 + 2023N^3 + 1684N^2 - 12N - 144 \tag{365}$$

$$\begin{aligned}
Q_{29} &= 26N^9 + 539N^8 + 3244N^7 + 8465N^6 + 9342N^5 + 841N^4 - 5720N^3 - 2193N^2 \\
&\quad + 2484N + 1404 \tag{366}
\end{aligned}$$

$$\begin{aligned}
Q_{30} &= N^{10} - 13N^9 - 39N^8 + 222N^7 + 1132N^6 + 1787N^5 + 913N^4 + 392N^3 \\
&\quad + 645N^2 - 324N - 108 \tag{367}
\end{aligned}$$

$$Q_{31} = 3N^{10} - 48N^9 - 856N^8 - 2702N^7 - 1961N^6 + 2142N^5 + 3122N^4 - 1924N^3 - 5552N^2 - 4032N - 1152 \quad (368)$$

$$Q_{32} = 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2 - 32N - 16 \quad (369)$$

$$Q_{33} = 35N^{10} + 372N^9 + 1263N^8 + 673N^7 - 5090N^6 - 11596N^5 - 8413N^4 + 2305N^3 + 8049N^2 + 3078N + 108 \quad (370)$$

$$Q_{34} = 2N^{11} + 41N^{10} + 226N^9 + 556N^8 + 963N^7 + 2733N^6 + 7160N^5 + 8610N^4 + 1969N^3 - 2748N^2 - 864N - 216 \quad (371)$$

$$Q_{35} = 2N^{12} + 20N^{11} + 86N^{10} + 192N^9 + 199N^8 - N^7 - 297N^6 - 495N^5 - 514N^4 - 488N^3 - 416N^2 - 176N - 32 \quad (372)$$

$$Q_{36} = 12N^{13} + 143N^{12} + 591N^{11} + 954N^{10} + 371N^9 + 1658N^8 + 11559N^7 + 26626N^6 + 29129N^5 + 14011N^4 - 2374N^3 - 6576N^2 - 1944N - 432 \quad (373)$$

$$Q_{37} = 95N^{13} + 1218N^{12} + 6096N^{11} + 14484N^{10} + 11570N^9 - 28440N^8 - 117844N^7 - 225884N^6 - 238953N^5 - 83290N^4 + 57660N^3 + 122040N^2 + 182304N + 77760 \quad (374)$$

$$Q_{38} = 185N^{13} + 2582N^{12} + 15584N^{11} + 53036N^{10} + 109190N^9 + 124040N^8 + 12604N^7 - 200836N^6 - 294247N^5 - 116270N^4 + 85260N^3 + 158760N^2 + 193536N + 77760 \quad (375)$$

$$Q_{39} = 35N^{14} + 465N^{13} + 1962N^{12} - 348N^{11} - 32130N^{10} - 131686N^9 - 280396N^8 - 363984N^7 - 290209N^6 - 122547N^5 + 6730N^4 + 47316N^3 + 11928N^2 - 21600N - 5184 \quad (376)$$

$$Q_{40} = 1255N^{14} + 18165N^{13} + 107824N^{12} + 331744N^{11} + 515430N^{10} + 132498N^9 - 1057432N^8 - 2202648N^7 - 1979173N^6 - 534079N^5 + 350880N^4 - 29088N^3 - 519264N^2 - 382320N - 62208 \quad (377)$$

$$Q_{41} = 11N^{15} - 2N^{14} + 308N^{13} + 5275N^{12} + 24535N^{11} + 52925N^{10} + 50941N^9 - 5977N^8 - 85550N^7 - 191059N^6 - 294877N^5 - 248414N^4 - 64728N^3 + 57636N^2 + 28944N + 6480 \quad (378)$$

$$Q_{42} = 1255N^{15} + 16338N^{14} + 76085N^{13} + 117654N^{12} - 198422N^{11} - 971844N^{10} - 1002678N^9 + 1019372N^8 + 3525323N^7 + 3236906N^6 + 272625N^5 - 1523746N^4 - 632844N^3 + 606888N^2 + 635904N + 129600 \quad (379)$$

The expressions in z -space are presented in Appendix C.

As has been outlined for the 2-loop results in Ref. [10] already, the scales at which the asymptotic expressions are dominating are estimated to be $Q^2/m^2 \gtrsim 800$. They are far outside the kinematic region in which the structure function $F_L(x, Q^2)$ can presently be measured in deep-inelastic scattering. The corresponding expressions are therefore of merely theoretical character and cannot be used in current phenomenological analyses.

6 Comparison of Mellin Moments for the Wilson Coefficients and OMEs

In order to compare the relative impact of the different Wilson coefficients on the structure function $F_2(x, Q^2)$ we will consider the Mellin moments for $N = 2$ to 10 in the following, folded

with the moments of the respective parton distribution functions in the flavor singlet case, i.e. the gluon $G(x, Q^2)$ and quark-singlet density $\Sigma(x, Q^2)$ for $N_F = 3$ and characteristic values of Q^2 . Since only a series of Mellin moments has been calculated at large momentum transfer Q^2 in Ref. [12], a detailed numerical comparison is only possible in this way at the moment. The numerical results for the moments of the contributing parton densities are given in Table 2. Note that for $N \geq 2$ the moments for the singlet-distribution are mostly larger than those of the gluon. We apply these parton densities to study the relative contributions of the different Wilson coefficients, normalizing to $H_{g,2}^S$ within the respective order in a_s using the following ratios:

$$\begin{aligned} R(L_{g,2}^S, H_{g,2}^S) &= \frac{c_{N_F} L_{g,2}^S G}{c_Q H_{g,2}^S G} \\ R(L_{q,2}^{\text{PS}}, H_{g,2}^S) &= \frac{c_{N_F} L_{q,2}^{\text{PS}} \Sigma}{c_Q H_{g,2}^S G} \\ R(H_{q,2}^{\text{PS}}, H_{g,2}^S) &= \frac{c_Q H_{q,2}^{\text{PS}} \Sigma}{c_Q H_{g,2}^S G}, \end{aligned}$$

where

$$c_{N_F} = \frac{1}{N_F} \sum_{k=0}^{N_F} e_k^2, \quad c_Q = e_Q^2. \quad (380)$$

In the numerical examples we set $e_Q = e_c = 2/3$.

Q^2	20 GeV ²				
N	2	4	6	8	10
G	0.4583	0.0044	0.0003	3.62×10^{-5}	7.78×10^{-6}
Σ	0.5417	0.0353	0.0070	2.10×10^{-3}	8.01×10^{-4}
Q^2	100 GeV ²				
N	2	4	6	8	10
G	0.4819	0.0038	0.0002	3.60×10^{-5}	8.55×10^{-6}
Σ	0.5181	0.0296	0.0056	1.61×10^{-3}	5.97×10^{-4}
Q^2	1000 GeV ²				
N	2	4	6	8	10
G	0.5042	0.0032	0.0002	3.14×10^{-5}	7.60×10^{-6}
Σ	0.4958	0.0244	0.0043	1.20×10^{-3}	4.32×10^{-3}

Table 2: The moments $N = 2, \dots, 10$ of the gluon and quark-singlet momentum density using the parton distribution functions [5].

Q^2	20 GeV ²				
N	2	4	6	8	10
Σ/G	1.1821	7.9967	25.847	57.965	103.06
$O(a_s^2) : R(L_{g,2}^S, H_{g,2}^S)$	0.0387	0.1349	1.7000	-0.2592	-0.1384
$R(H_{q,2}^{PS}, H_{g,2}^S)$	-0.2588	-0.3018	1.9946	-1.6153	-1.7222
$O(a_s^3) : R(L_{g,2}^S, H_{g,2}^S)$	0.0829	0.1983	0.6628	-2.5018	-0.5957
$R(L_{q,2}^{PS}, H_{g,2}^S)$	0.0438	0.1042	0.7483	-5.7476	-2.3762
$R(H_{q,2}^{PS}, H_{g,2}^S)$	-0.2259	-0.3472	0.3387	-9.3371	-4.5870
Q^2	100 GeV ²				
Σ/G	1.0753	7.7514	22.797	44.660	69.888
$O(a_s^2) R(L_{g,2}^S, H_{g,2}^S)$	0.0313	0.0687	0.1071	0.1587	0.2418
$R(H_{q,2}^{PS}, H_{g,2}^S)$	-0.2496	-0.3753	-0.5429	-0.6531	-0.6743
$O(a_s^3) R(L_{g,2}^S, H_{g,2}^S)$	0.0533	0.0853	0.1449	0.2186	0.3195
$R(L_{q,2}^{PS}, H_{g,2}^S)$	0.0340	0.0378	0.1006	0.2600	0.5828
$R(H_{q,2}^{PS}, H_{g,2}^S)$	-0.3062	-0.5165	-0.7070	-0.7471	-0.5637
Q^2	1000 GeV ²				
Σ/G	0.9833	7.5948	20.958	38.236	56.876
$O(a_s^2) R(L_{g,2}^S, H_{g,2}^S)$	0.0243	0.0420	0.0531	0.0615	0.0687
$R(H_{q,2}^{PS}, H_{g,2}^S)$	-0.2837	-0.4085	-0.5690	-0.6707	-0.7237
$O(a_s^3) R(L_{g,2}^S, H_{g,2}^S)$	0.0337	0.0392	0.0597	0.0764	0.0907
$R(L_{q,2}^{PS}, H_{g,2}^S)$	0.0313	0.0209	0.0297	0.0505	0.0828
$R(H_{q,2}^{PS}, H_{g,2}^S)$	-0.3825	-0.5903	-0.8058	-0.9253	-0.9679

Table 3: Relative impact of the moments $N = 2, \dots, 10$ of the individual massive Wilson coefficients, weighted by moments of the corresponding parton distributions [5], at $O(a_s^2)$ and $O(a_s^3)$ normalized to the contribution to $H_{g,2}^S$ for $Q^2 = 20, 100$ and 1000 GeV².

Before we discuss quantitative results, a remark on the contributions by the color factor $d_{abc}d_{abc}$ to the massless Wilson coefficients Refs. [62–64] and [11,57] used in the present analysis, is in order. For $SU(N)$ one obtains

$$d_{abc}d_{abc} = \frac{(N^2 - 1)(N^2 - 4)}{N}. \quad (381)$$

It emerges weighted by $1/N_c$ and $1/N_A$ for external quark and gluon lines, respectively, with $N_c = N$ and $N_A = N^2 - 1$. In Refs. [62–64] this group-theoretic expression has been used, while in [11, 57] a factor of 16 has been taken out and was absorbed into the Lorentz structure of the corresponding contribution to the Wilson coefficient. We agree with the N_F -dependence as given in Refs. [62–64]. Furthermore, we note a typographical error in Eq. (4.13) of [11]. Here,

the corresponding term reads correctly¹²

$$c_{2,g}^{(3)}(x) \simeq -932.089 N_F \frac{L_0}{x} \dots, \quad \text{with } L_0 = \ln(x) . \quad (382)$$

Also in the pure-singlet case the massless Wilson coefficients contain terms $\propto d_{abc}d_{abc}$, although with a generally different charge-weight factor, cf. [62–64].

Q^2	20 GeV ²				
N	2	4	6	8	10
Σ/G	1.1821	7.9967	25.847	57.965	103.06
$O(a_s) : R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.8182	-2.5455	-3.2432	-3.9286
$O(a_s^2) : R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.6395	-2.3808	-3.1781	-4.0262
$R(A_{Qq}^{\text{PS}}, A_{Qg})$	-0.1259	-0.3656	-0.7822	-1.3339	-1.9352
$R(A_{qq,Q}^{\text{NS}}, A_{Qg})$	-0.0584	-1.1306	-6.3206	-20.735	-49.508
$R(A_{gq,Q}, A_{Qg})$	0.1843	1.1422	3.9956	9.8073	18.995
$O(a_s^3) : R(A_{gg,Q}, A_{Qg})$	-1.0051	-1.3397	-1.8466	-2.4306	-3.0890
$R(A_{Qq}^{\text{PS}}, A_{Qg})$	-0.1604	-0.4838	-0.9635	-1.5449	-2.1295
$R(A_{qq,Q}^{\text{NS}}, A_{Qg})$	-0.0404	-0.5832	-2.9406	-9.1817	-21.375
$R(A_{gq,Q}, A_{Qg})$	0.1473	1.1265	3.7972	8.9925	16.961
$R(A_{qg,Q}, A_{Qg})$	0.0051	-0.0202	-0.0326	-0.0445	-0.0567
$R(A_{qq,Q}^{\text{PS}}, A_{Qg})$	0.0534	0.1093	0.2460	0.4678	0.7619
Q^2	100 GeV ²				
Σ/G	1.0753	7.7514	22.797	44.660	69.888
$O(a_s) : R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.8182	-2.5455	-3.2432	-3.9286
$O(a_s^2) : R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.7746	-2.6448	-3.5805	-4.5771
$R(A_{Qq}^{\text{PS}}, A_{Qg})$	-0.1884	-0.4246	-0.7627	-1.0887	-1.3514
$R(A_{qq,Q}^{\text{NS}}, A_{Qg})$	-0.1247	-2.4385	-12.377	-35.460	-74.510
$R(A_{gq,Q}, A_{Qg})$	0.3131	1.3950	3.9075	7.7692	12.560
$O(a_s^3) : R(A_{gg,Q}, A_{Qg})$	-1.0048	-1.6120	-2.3201	-3.0734	-3.8667
$R(A_{Qq}^{\text{PS}}, A_{Qg})$	-0.2799	-0.5808	-0.9473	-1.2540	-1.4615
$R(A_{qq,Q}^{\text{NS}}, A_{Qg})$	-0.1772	-2.8698	-13.694	-37.928	-77.874
$R(A_{gq,Q}, A_{Qg})$	0.3924	1.4140	3.4078	6.0520	8.9253
$R(A_{qg,Q}, A_{Qg})$	0.0048	-0.0375	-0.0491	-0.0580	-0.0657
$R(A_{qq,Q}^{\text{PS}}, A_{Qg})$	0.0647	0.0984	0.1726	0.2553	0.3319

Table 4: Relative impact of the moments $N = 2, \dots, 10$ of the individual massive OMEs, weighted by moments of the corresponding parton distributions [5], at the different orders in a_s normalized to the contribution to A_{Qg} for $Q^2 = 20$ and 100 GeV².

¹²The expression in the parameterization given at <http://www.liv.ac.uk/~avogt/> is correct, however.

Let us now consider the relative impact of the individual massive Wilson coefficients. The ratios at $O(a_s^2)$ and $O(a_s^3)$ for different values of Q^2 and the moments $N = 2$ to 10 are given in Table 3. One first notes that at low values of Q^2 the moments of $L_{g,2}^S$ change sign, which is also the case for $H_{g,2}^{\text{PS}}$ in the whole region up to $Q^2 = 1000 \text{ GeV}^2$. At $O(a_s^2)$ $L_{g,2}^S$ is small for low moments and grows 24% for $N = 10$ compared to $H_{g,2}^S$ at $Q^2 = 100 \text{ GeV}^2$, with lower values at higher Q^2 . A comparable tendency is observed at $O(a_s^2)$. The fraction $|R(H_{g,2}^{\text{PS}}, H_{g,2}^S)|$ moves between 25% and 170% comparing the moments $N = 2$ to 10 at $Q^2 = 20 \text{ GeV}^2$ and upper values of $\sim 70\%$ at $Q^2 = 1000 \text{ GeV}^2$.

In the case of the comparison of the massive OMEs we normalize to A_{Qg} with PDFs according to their appearance in the singlet and gluon transitions from $N_F \rightarrow N_F + 1$ massless flavors in the variable flavor number scheme, cf. Eqs. (33–35):

$$\begin{aligned}
R(A_{gg,Q}, A_{Qg}) &= \frac{A_{gg,Q} G}{A_{Qg} G} & R(A_{Qq}^{\text{PS}}, A_{Qg}) &= \frac{A_{Qq}^{\text{PS}} \Sigma}{A_{Qg} G} \\
R(A_{qq,Q}^{\text{NS}}, A_{Qg}) &= \frac{A_{qq,Q}^{\text{NS}} \Sigma}{A_{Qg} G} & R(A_{gq,Q}, A_{Qg}) &= \frac{A_{gq,Q} \Sigma}{A_{Qg} G} \\
R(A_{gq,Q}, A_{Qg}) &= \frac{A_{gq,Q} G}{A_{Qg} G} & R(A_{qq,Q}^{\text{PS}}, A_{Qg}) &= \frac{A_{qq,Q}^{\text{PS}} \Sigma}{A_{Qg} G}.
\end{aligned}$$

These ratios describe the relative impact, within the corresponding order in a_s , of the massive OMEs in the variable flavor number scheme for the flavor singlet contributions.

Q^2	1000 GeV ²					
N	2	4	6	8	10	
Σ/G	0.9833	7.5948	20.958	38.236	56.876	
$O(a_s) : R(A_{gg,Q}, A_{Qg})$	-1.0000	-1.8182	-2.5455	-3.2432	-3.9286	
$O(a_s^2) : R(A_{gg,Q}, A_{Qg})$	-1.0000	-2.0101	-3.0170	-4.0596	-5.1403	
	$R(A_{Qq}^{\text{PS}}, A_{Qg})$	-0.2555	-0.4521	-0.6997	-0.8850	-1.0072
	$R(A_{qq,Q}^{\text{NS}}, A_{Qg})$	-0.2048	-3.7111	-16.978	-44.037	-85.945
	$R(A_{gq,Q}, A_{Qg})$	0.4603	1.5427	3.5816	6.1302	8.8857
$O(a_s^3) : R(A_{gg,Q}, A_{Qg})$	-1.0054	-1.7515	-2.4980	-3.2560	-4.0291	
	$R(A_{Qq}^{\text{PS}}, A_{Qg})$	-0.3731	-0.6275	-0.9067	-1.0906	-1.1928
	$R(A_{qq,Q}^{\text{NS}}, A_{Qg})$	-0.2930	-4.1599	-17.532	-43.424	-82.110
	$R(A_{gq,Q}, A_{Qg})$	0.5934	1.5060	2.9692	4.5100	5.9397
	$R(A_{gg,Q}, A_{Qg})$	0.0054	-0.0469	-0.0561	-0.0624	-0.0675
	$R(A_{qq,Q}^{\text{PS}}, A_{Qg})$	0.0727	0.0902	0.1341	0.1721	0.2011

Table 5: The same as Table 4 for $Q^2 = 1000 \text{ GeV}^2$.

The numerical values for different scales of Q^2 are given in Tables 4 and 5. $|A_{gg,Q}/A_{Qg}|$ rises from about 1 to higher values from $N = 2$ to 10, irrespectively of the values of Q^2 and the order in a_s . The smallest contributions are $|A_{gq,Q}|$ and $A_{qq,Q}^{\text{PS}}$ contributing the ratios R by $\sim 0.5\%$ to 5% and

form ~ 5 to $\sim 10\%$, respectively, for $N = 2$ and 4 , i.e. in the region dominated by lower values of the Bjorken variable x . The OMEs $|A_{Q,q}^{\text{PS}}|$ and $A_{gq,Q}$ have contributions of 16–62% and 14–150%, respectively, for $N = 2$ and 4 . Also the flavor non-singlet Wilson coefficient $|A_{qq,Q}^{\text{NS}}|$ contributes in the flavor singlet transitions and is weighted by the distribution Σ here. Its relative impact rises with Q^2 and amounts from $\sim 4\%$ to 370% for the R -ratio considering the lower moments $N = 2$ and $N = 4$ only.

Right after having obtained a series of moments for the massive OMEs at 3-loops in [12], it became clear that the logarithmic contributions are of comparable order to the constant term. Moreover, there is a strong functional dependence w.r.t. N , as displayed in Tables 2–5. To obtain a definite answer, the calculation of the constant parts of the unrenormalized OMEs $a_{ij}^{(3)}$ as a function of $N \in \mathbb{C}$ is necessary. In particular predictions for the range of small values $x \simeq 10^{-4}$ appear to be rather difficult otherwise.

7 Conclusions

We have derived the contributions of the massive Wilson coefficients to the structure functions $F_2(x, Q^2)$ and $F_L(x, Q^2)$ in deep-inelastic scattering and the corresponding massive OMEs to 3-loop order in the asymptotic region $Q^2 \gg m^2$ both in Mellin- N and z -space except for the constant parts a_{ij} of the unrenormalized OMEs, which are not known for all quantities yet. Here, we retained both the scale-dependence due to the virtuality Q^2 and the factorization and renormalization scales μ^2 , which were set equal. This allows for scale variation studies in applications. Two of the Wilson coefficients, $L_{q,2}^{\text{PS}}$ and $L_{g,2}^{\text{S}}$, are known in complete form, and the corresponding results for $L_{q,2}^{\text{NS}}$ will be given in [40]. In the variable flavor number scheme being applied to describe the process through which an initially massive quark transmutes into a massless one at high momentum scales, moreover, the matching coefficients A_{ij} are needed. Here, $A_{qq,Q}^{\text{PS}}$ and $A_{gq,Q}$ are known in complete form to 3-loop order and the results for A_{gq} and $A_{qq,Q}^{\text{NS}}$ are given in [65] and [40], respectively.

We have given numerical results for the Wilson coefficients $L_{q,2}^{\text{PS}}$ and $L_{g,2}^{\text{S}}$. Using the available Mellin moments we have performed a numerical comparison of the different Wilson coefficients and operator matrix elements inside the respective order in the coupling constant for the moments $N = 2$ to 10 and in the Q^2 range between 20 and 1000 GeV². While some of the quantities studied are of minor importance, several others of the Wilson coefficients and OMEs are of similar size, which is varying in the kinematic range of experimental interest for present and future precision measurements. Even in case of the charm-quark contributions the logarithmic terms are not dominant over the constant contributions in wide kinematic ranges, as. e.g. at HERA.

The expression which were derived in the present paper are available in form of computer-readable files on request via e-mail to Johannes.Bluemlein@desy.de.

A The massive operator matrix elements in N -space

In this appendix we present the massive OMEs in Mellin-space to be used in the matching coefficients in the variable flavor number scheme Eqs. (32–35). The corresponding representations in z -space are given in Appendix D. Thus far the OMEs $A_{qq,Q}^{\text{PS}}$ and $A_{gq,Q}$ are known completely. The other OMEs are presented except for the 3-loop constant part a_{ij} in the unrenormalized OMEs. The OMEs $A_{qq,Q}^{\text{NS}}$ and $A_{gq,Q}^{\text{S}}$ are presented elsewhere [40, 65].

The transition matrix elements are given by $A_{qq,Q}^{\text{PS}}$ and $A_{qg,Q}$:

$$\begin{aligned}
A_{qq,Q}^{\text{PS}} &= \frac{1}{2}[1 + (-1)^N] \\
&\times \left\{ a_s^3 C_F N_F T_F^2 \left[L_M^2 \left[\frac{32(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} - \frac{32P_{280}}{9(N-1)N^3(N+1)^3(N+2)^2} \right] \right. \right. \\
&+ \left[-\frac{32P_{282}}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64P_{280}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \\
&+ \left. \left. \frac{(N^2 + N + 2)^2 \left[-\frac{32}{3}S_1^2 - \frac{32S_2}{3} \right]}{(N-1)N^2(N+1)^2(N+2)} \right] L_M - \frac{32P_{284}}{243(N-1)N^5(N+1)^5(N+2)^4} \right. \\
&+ \frac{32P_{283}S_1}{81(N-1)N^4(N+1)^4(N+2)^3} - \frac{32(N^2 + N + 2)^2}{9(N-1)N^2(N+1)^2(N+2)} L_M^3 \\
&+ \left. \left. \frac{P_{281} \left[-\frac{16}{27}S_1^2 - \frac{16S_2}{27} \right]}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{(N^2 + N + 2)^2 \left[\frac{80}{27}S_1^3 + \frac{80}{9}S_2S_1 + \frac{160S_3}{27} + \frac{256\zeta_3}{9} \right]}{(N-1)N^2(N+1)^2(N+2)} \right] \right\}, \quad (383)
\end{aligned}$$

with

$$P_{280} = 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \quad (384)$$

$$P_{281} = 40N^7 + 185N^6 + 430N^5 + 521N^4 + 452N^3 + 404N^2 - 16N - 96 \quad (385)$$

$$\begin{aligned}
P_{282} &= 95N^{10} + 712N^9 + 2379N^8 + 4269N^7 + 4763N^6 + 4569N^5 + 3309N^4 + 200N^3 \\
&\quad - 808N^2 - 48N + 144 \quad (386)
\end{aligned}$$

$$\begin{aligned}
P_{283} &= 233N^{10} + 1744N^9 + 5937N^8 + 11454N^7 + 14606N^6 + 15396N^5 + 12030N^4 + 3272N^3 \\
&\quad - 928N^2 - 96N + 288 \quad (387)
\end{aligned}$$

$$\begin{aligned}
P_{284} &= 1330N^{13} + 13931N^{12} + 66389N^{11} + 187681N^{10} + 354532N^9 + 492456N^8 + 532664N^7 \\
&\quad + 423970N^6 + 204541N^5 + 34274N^4 - 11704N^3 - 3408N^2 - 1008N - 864 \quad (388)
\end{aligned}$$

and

$$\begin{aligned}
A_{qg,Q} &= \frac{1}{2}[1 + (-1)^N] \\
&\times \left\{ a_s^3 \left\{ C_F N_F T_F^2 \left[\left[\frac{8(N^2 + N + 2)P_{285}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{qg}^0 S_1 \right] L_M^3 \right. \right. \right. \\
&+ \left[\frac{4P_{291}}{9(N-1)N^4(N+1)^4(N+2)^3} - \frac{32(5N^3 + 8N^2 + 19N + 6)S_1}{9N^2(N+1)(N+2)} + \tilde{\gamma}_{qg}^0 \left[-\frac{4}{3}S_1^2 - \frac{4S_2}{3} \right] \right] L_M^2 \\
&+ \left[\frac{16(10N^3 + 13N^2 + 29N + 6)S_1^2}{9N^2(N+1)(N+2)} - \frac{16(103N^4 + 257N^3 + 594N^2 + 524N + 120)S_1}{27N^2(N+1)^2(N+2)} \right. \\
&+ \frac{4P_{293}}{27(N-1)N^5(N+1)^5(N+2)^4} + \frac{16(10N^3 + 25N^2 + 29N + 6)S_2}{9N^2(N+1)(N+2)} \\
&+ \left. \left. \tilde{\gamma}_{qg}^0 \left[\frac{4}{9}S_1^3 + \frac{4}{3}S_2S_1 - \frac{16S_3}{9} \right] \right] L_M + \frac{8(215N^4 + 481N^3 + 930N^2 + 748N + 120)S_1^2}{81N^2(N+1)^2(N+2)} \right. \\
&- \frac{64}{9} \frac{(N^2 + N + 2)P_{285}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{P_{295}}{243(N-1)N^6(N+1)^6(N+2)^5} \\
&- \frac{16(1523N^5 + 5124N^4 + 11200N^3 + 14077N^2 + 7930N + 1344)S_1}{243N^2(N+1)^3(N+2)} \\
&+ \frac{8(109N^4 + 291N^3 + 478N^2 + 324N + 40)S_2}{27N^2(N+1)^2(N+2)} + \left. \frac{(10N^3 + 13N^2 + 29N + 6) \left[-\frac{16}{81}S_1^3 - \frac{16}{27}S_2S_1 \right]}{N^2(N+1)(N+2)} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{32(5N^3 - 16N^2 + N - 6)S_3}{81N^2(N+1)(N+2)} + \tilde{\gamma}_{ag}^0 \left[-\frac{1}{27}S_1^4 - \frac{2}{9}S_2S_1^2 - \frac{8}{27}S_3S_1 - \frac{64}{9}\zeta_3S_1 - \frac{1}{9}S_2^2 + \frac{14S_4}{9} \right] \\
& + C_{ANFT_F} \left[L_M^3 \left[-\frac{64(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{8}{9}\tilde{\gamma}_{ag}^0S_1 \right] \right. \\
& + \left[\frac{8P_{289}}{9(N-1)N^2(N+1)^3(N+2)^3} + \frac{32(5N^4 + 20N^3 + 47N^2 + 58N + 20)S_1}{9N(N+1)^2(N+2)^2} \right. \\
& + \left. \tilde{\gamma}_{ag}^0 \left[\frac{4}{3}S_1^2 + \frac{4S_2}{3} + \frac{8}{3}S_{-2} \right] \right] L_M^2 + \left[-\frac{32(5N^4 + 20N^3 + 41N^2 + 49N + 20)S_1^2}{9N(N+1)^2(N+2)^2} \right. \\
& + \frac{16P_{292}}{27(N-1)N^4(N+1)^4(N+2)^4} - \frac{32(5N^4 + 26N^3 + 47N^2 + 43N + 20)S_2}{9N(N+1)^2(N+2)^2} \\
& + \frac{16P_{286}S_1}{27N(N+1)^3(N+2)^3} - \frac{64(5N^2 + 8N + 10)S_{-2}}{9N(N+1)(N+2)} + \tilde{\gamma}_{ag}^0 \left[-\frac{4}{9}S_1^3 + \frac{4}{3}S_2S_1 - \frac{8S_3}{9} - \frac{16}{3}S_{-3} \right. \\
& \left. \left. - \frac{16}{3}S_{2,1} \right] \right] L_M - \frac{16P_{287}S_1^2}{81N(N+1)^3(N+2)^3} + \frac{8P_{294}}{243(N-1)N^5(N+1)^5(N+2)^5} \\
& + \frac{512}{9}(N^2 + N + 1)(N^2 + N + 2) \frac{\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{16P_{290}S_1}{243(N-1)N^2(N+1)^4(N+2)^4} \\
& - \frac{16P_{288}S_2}{81N(N+1)^3(N+2)^3} + \frac{64(5N^4 + 38N^3 + 59N^2 + 31N + 20)S_3}{81N(N+1)^2(N+2)^2} \\
& - \frac{32(121N^3 + 293N^2 + 414N + 224)S_{-2}}{81N(N+1)^2(N+2)} + \frac{128(5N^2 + 8N + 10)S_{-3}}{27N(N+1)(N+2)} \\
& + \frac{(5N^4 + 20N^3 + 41N^2 + 49N + 20) \left[\frac{32}{81}S_1^3 - \frac{32}{27}S_2S_1 + \frac{128}{27}S_{2,1} \right]}{N(N+1)^2(N+2)^2} + \tilde{\gamma}_{ag}^0 \left[\frac{1}{27}S_1^4 - \frac{2}{9}S_2S_1^2 \right. \\
& \left. + \left[\frac{16}{9}S_{2,1} - \frac{40S_3}{27} \right] S_1 + \frac{64}{9}\zeta_3S_1 + \frac{1}{9}S_2^2 + \frac{14S_4}{9} + \frac{32}{9}S_{-4} + \frac{32}{9}S_{3,1} - \frac{16}{9}S_{2,1,1} \right] \left. \right\} \left. \right\}, \quad (389)
\end{aligned}$$

with the polynomials

$$P_{285} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \quad (390)$$

$$P_{286} = 94N^6 + 631N^5 + 2106N^4 + 4243N^3 + 4878N^2 + 2812N + 680 \quad (391)$$

$$P_{287} = 103N^6 + 694N^5 + 2148N^4 + 3991N^3 + 4494N^2 + 2704N + 680 \quad (392)$$

$$P_{288} = 139N^6 + 1093N^5 + 3438N^4 + 5776N^3 + 5724N^2 + 3220N + 752 \quad (393)$$

$$P_{289} = 9N^8 + 54N^7 + 56N^6 - 182N^5 - 717N^4 - 1120N^3 - 1012N^2 - 672N - 160 \quad (394)$$

$$\begin{aligned}
P_{290} &= 1244N^{10} + 10557N^9 + 40547N^8 + 90323N^7 + 114495N^6 + 49344N^5 - 69902N^4 \\
&\quad - 115200N^3 - 64352N^2 - 11264N + 864 \quad (395)
\end{aligned}$$

$$\begin{aligned}
P_{291} &= 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 + 7328N^3 \\
&\quad + 5456N^2 + 3456N + 1152 \quad (396)
\end{aligned}$$

$$\begin{aligned}
P_{292} &= 99N^{12} + 891N^{11} + 2902N^{10} + 3392N^9 - 4300N^8 - 20914N^7 - 33059N^6 - 28357N^5 \\
&\quad - 11406N^4 + 3840N^3 + 7568N^2 + 4176N + 864 \quad (397)
\end{aligned}$$

$$\begin{aligned}
P_{293} &= 159N^{14} + 1590N^{13} + 7223N^{12} + 20982N^{11} + 43703N^{10} + 65162N^9 + 62553N^8 + 30282N^7 \\
&\quad - 28286N^6 - 145968N^5 - 257720N^4 - 241760N^3 - 158112N^2 - 73728N - 17280 \quad (398)
\end{aligned}$$

$$\begin{aligned}
P_{294} &= 3315N^{15} + 39780N^{14} + 194011N^{13} + 471164N^{12} + 416251N^{11} - 860568N^{10} - 3525799N^9 \\
&\quad - 6015120N^8 - 6333994N^7 - 4373672N^6 - 1907512N^5 - 499824N^4 - 217952N^3 \\
&\quad - 264192N^2 - 160128N - 34560 \quad (399)
\end{aligned}$$

$$\begin{aligned}
P_{295} = & 13923N^{17} + 180999N^{16} + 1064857N^{15} + 3812487N^{14} + 9348807N^{13} + 16391845N^{12} \\
& + 20248499N^{11} + 17070917N^{10} + 11536274N^9 + 11303496N^8 + 13846104N^7 + 16104128N^6 \\
& + 22643488N^5 + 29337472N^4 + 26395008N^3 + 15388416N^2 + 5612544N + 995328. \quad (400)
\end{aligned}$$

Next we present the OMEs, which are known except for the constant term in the unrenormalized massive OME at 3-loop order, $a_{ij}^{(3)}$. The matrix element A_{Qq}^{PS} is given by :

$$\begin{aligned}
A_{Qq}^{\text{PS}} = & \frac{1}{2}[1 + (-1)^N] \\
& \times \left\{ a_s^2 C_F T_F \left[-\frac{4L_M^2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{8S_2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. + \frac{4P_{312}}{(N-1)N^4(N+1)^4(N+2)^3} - \frac{8(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)L_M}{(N-1)N^3(N+1)^3(N+2)^2} \right] \right. \\
& \left. + a_s^3 \left\{ T_F C_F^2 \left[\left[\frac{4(N^2 + N + 2)^2(3N^2 + 3N + 2)}{3(N-1)N^3(N+1)^3(N+2)} - \frac{16(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] L_M^3 \right. \right. \right. \\
& \left. \left. + \left[-\frac{8(5N^2 + N - 2)S_1(N^2 + N + 2)^2}{(N-1)N^3(N+1)^3(N+2)} + \frac{16S_2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \right. \\
& \left. \left. + \frac{4P_{297}(N^2 + N + 2)}{(N-1)N^4(N+1)^4(N+2)} \right] L_M^2 + \left[\frac{\left[\frac{8}{3}S_1^3 - 24S_2S_1 - \frac{80S_3}{3} + 32S_{2,1} + 96\zeta_3 \right](N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. - \frac{4(5N^3 + 4N^2 + 9N + 6)S_1^2(N^2 + N + 2)}{(N-1)N^2(N+1)^3(N+2)} - \frac{4P_{321}}{(N-1)N^5(N+1)^5(N+2)^3} \right. \right. \\
& \left. \left. + \frac{8P_{313}S_1}{(N-1)N^4(N+1)^4(N+2)^3} - \frac{4P_{307}S_2}{(N-1)N^3(N+1)^3(N+2)^2} \right] L_M - \frac{2(N^2 + N + 2)\zeta_2 P_{304}}{(N-1)N^4(N+1)^4(N+2)} \right. \\
& \left. - \frac{4(N^2 + N + 2)(N^4 - 5N^3 - 32N^2 - 18N - 4)S_1^2}{(N-1)N^3(N+1)^3(N+2)} + \frac{4P_{323}}{(N-1)N^6(N+1)^6(N+2)^3} \right. \\
& \left. - \frac{4(N^2 + N + 2)^2(3N^2 + 3N + 2)\zeta_3}{3(N-1)N^3(N+1)^3(N+2)} + \frac{4(N^2 + N + 2)(5N^4 + 4N^3 + N^2 - 10N - 8)\zeta_2}{(N-1)N^3(N+1)^3(N+2)} S_1 \right. \\
& \left. + \frac{8(N^2 + N + 2)(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12)S_1}{(N-1)N^3(N+1)^4(N+2)} - \frac{4(N^2 + N + 2)P_{303}S_2}{(N-1)N^4(N+1)^4(N+2)} \right. \\
& \left. + \frac{(3N + 2)(N^2 + N + 2) \left[-\frac{8}{3}S_1^3 - 8S_2S_1 \right]}{(N-1)N^3(N+1)(N+2)} - \frac{8(N^2 + N + 2)(3N^4 + 48N^3 + 43N^2 - 22N - 8)S_3}{3(N-1)N^3(N+1)^3(N+2)} \right. \\
& \left. + \frac{32(N^2 - 3N - 2)(N^2 + N + 2)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)} + \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{2}{3}S_1^4 + 4S_2S_1^2 \right. \right. \\
& \left. \left. + \left(\frac{16S_3}{3} + 32S_{2,1} \right) S_1 + \frac{16}{3}\zeta_3 S_1 + 2S_2^2 - 12S_4 + 32S_{3,1} - 64S_{2,1,1} + (4S_1^2 - 12S_2)\zeta_2 \right] \right. \\
& \left. + C_F T_F^2 \left[-\frac{128(N^2 + N + 2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} + \left[\frac{32(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right. \right. \right. \\
& \left. \left. - \frac{32P_{298}}{9(N-1)N^3(N+1)^2(N+2)^2} \right] L_M^2 + \left[\frac{\left(-\frac{32}{3}S_1^2 - 32S_2 \right)(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. - \frac{64P_{315}}{27(N-1)N^4(N+1)^4(N+2)^3} + \frac{64P_{309}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} \right] L_M \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{16(N^2 + N + 2)(8N^3 + 13N^2 + 27N + 16)S_1^2}{9(N-1)N^2(N+1)^3(N+2)} + \frac{32}{9} \frac{P_{298}\zeta_2}{(N-1)N^3(N+1)^2(N+2)^2} \\
& + \frac{32P_{322}}{81(N-1)N^5(N+1)^5(N+2)^4} - \frac{32(N^2 + N + 2)(43N^4 + 105N^3 + 224N^2 + 230N + 86)S_1}{27(N-1)N^2(N+1)^4(N+2)} \\
& + \frac{16P_{310}S_2}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{(N^2 + N + 2)^2 \left[-\frac{16}{9}S_1^3 - \frac{16}{3}S_2S_1 - \frac{32}{3}\zeta_2S_1 + \frac{160S_3}{9} + \frac{128\zeta_3}{9} \right]}{(N-1)N^2(N+1)^2(N+2)} \Big] \\
& + C_F T_F^2 N_F \left[-\frac{16}{9} \frac{P_{326}\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} + L_M^2 \left[\frac{32P_{327}}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \right. \\
& \left. \left. - \frac{32(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right] + L_M \left[-\frac{32P_{328}}{27(N-1)N^4(N+1)^4(N+2)^3} \right. \right. \\
& \left. \left. + \frac{32P_{326}S_1}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{(N^2 + N + 2)^2 \left[-\frac{16}{3}S_1^2 - \frac{80S_2}{3} \right]}{(N-1)N^2(N+1)^2(N+2)} \right] \right. \\
& \left. - \frac{32P_{329}}{3(N-1)N^5(N+1)^5(N+2)^4} + \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{16}{3}\zeta_2S_1 + \frac{64S_3}{3} + \frac{32\zeta_3}{9} \right] \right. \\
& \left. - \frac{32(N^2 + N + 2)^2 L_M^3}{9(N-1)N^2(N+1)^2(N+2)} + \frac{64(N^2 + 5N + 2)(5N^3 + 7N^2 + 4N + 4)S_2}{3(N-1)N^3(N+1)^3(N+2)^2} \right] \\
& + C_F C_A T_F \left[\frac{8(N^2 + N + 2)(N^3 + 8N^2 + 11N + 2)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2} + \frac{4(N^2 + N + 2)P_{296}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3} \right. \\
& \left. - \frac{4}{3} \frac{(N^2 + N + 2)P_{301}\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2} S_1 - \frac{8(N^2 + N + 2)P_{311}S_1}{(N-1)N^2(N+1)^5(N+2)^4} \right. \\
& \left. - \frac{8(N^2 + N + 2)(3N^3 - 12N^2 - 27N - 2)S_2S_1}{(N-1)N^2(N+1)^3(N+2)^2} - \frac{8}{9} \frac{(N^2 + N + 2)P_{299}\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2} \right. \\
& \left. + \frac{4}{9} \frac{P_{320}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3} + \frac{8P_{325}}{3(N-1)^2N^6(N+1)^6(N+2)^5} \right. \\
& \left. + L_M^3 \left[\frac{8(11N^4 + 22N^3 - 23N^2 - 34N - 12)(N^2 + N + 2)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2} + \frac{16S_1(N^2 + N + 2)^2}{3(N-1)N^2(N+1)^2(N+2)} \right] \right. \\
& \left. + \frac{4P_{317}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^3} - \frac{16(N^2 + N + 2)P_{300}S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2} \right. \\
& \left. + L_M^2 \left[\frac{\left[16S_2 + 32S_{-2} \right] (N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} + \frac{8P_{302}S_1(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^3(N+2)^2} \right. \right. \\
& \left. \left. - \frac{8P_{319}}{9(N-1)^2N^4(N+1)^4(N+2)^3} \right] \right. \\
& \left. + \frac{(N^2 + N + 2)(N^4 + 2N^3 + 7N^2 + 22N + 20) \left[32(-1)^N S_{-2} + 16(-1)^N \zeta_2 \right]}{(N-1)N(N+1)^4(N+2)^3} \right. \\
& \left. + \frac{(N^2 - N - 4)(N^2 + N + 2)}{(N-1)N(N+1)^3(N+2)^2} \left[-64(-1)^N S_1 S_{-2} - 32(-1)^N S_{-3} + 64S_{-2,1} - 32(-1)^N S_1 \zeta_2 \right. \right. \\
& \left. \left. - 24(-1)^N \zeta_3 \right] + \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{2}{3}S_1^4 - 20S_2S_1^2 - 32(-1)^N S_{-3}S_1 + (64S_{-2,1} \right. \right. \\
& \left. \left. - \frac{160S_3}{3})S_1 - \frac{8}{3}(-7 + 9(-1)^N)\zeta_3S_1 - 2S_2^2 + S_{-2}(-32(-1)^N S_1^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -32(-1)^N S_2 - 36S_4 - 16(-1)^N S_{-4} + 16S_{3,1} + 32S_{-2,2} + 32S_{-3,1} + 16S_{2,1,1} \\
& -64S_{-2,1,1} + \left(-4(-3 + 4(-1)^N)S_1^2 - 4(-1 + 4(-1)^N)S_2 - 8(1 + 2(-1)^N)S_{-2} \right) \zeta_2 \Big] \\
& + L_M \left[\frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{8}{3}S_1^3 + 40S_2S_1 + 32(1 + (-1)^N)S_{-2}S_1 \right. \right. \\
& + 16(-1)^N S_{-3} - 32S_{2,1} + 12(-9 + (-1)^N)\zeta_3 \Big] + \frac{8P_{324}}{27(N-1)^2N^5(N+1)^5(N+2)^4} \\
& + \frac{4(17N^4 - 6N^3 + 41N^2 - 16N - 12)S_1^2(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^2(N+2)} + \frac{4P_{305}S_2(N^2 + N + 2)}{3(N-1)^2N^3(N+1)^3(N+2)^2} \\
& + \frac{8(31N^2 + 31N + 74)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} + \frac{16(7N^2 + 7N + 10)S_{-3}(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{128(N^2 + N + 1)S_{-2,1}(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} + \frac{(N^2 - N - 4)32(-1)^N S_{-2}(N^2 + N + 2)}{(N-1)N(N+1)^3(N+2)^2} \\
& \left. - \frac{8P_{316}S_1}{9(N-1)^2N^4(N+1)^4(N+2)^2} + \frac{16P_{306}S_{-2}}{(N-1)N^3(N+1)^3(N+2)^2} \right] + a_{Q^q}^{\text{PS},(3)} \Big\} \Big\}, \tag{401}
\end{aligned}$$

with the polynomials

$$P_{296} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \tag{402}$$

$$P_{297} = 7N^6 + 15N^5 + 7N^4 - 23N^3 - 26N^2 - 20N - 8 \tag{403}$$

$$P_{298} = 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24 \tag{404}$$

$$P_{299} = 11N^6 + 6N^5 + 75N^4 + 68N^3 - 200N^2 - 80N - 24 \tag{405}$$

$$P_{300} = 11N^6 + 29N^5 - 7N^4 - 25N^3 - 56N^2 - 72N - 24 \tag{406}$$

$$P_{301} = 17N^6 + 27N^5 + 75N^4 + 149N^3 - 20N^2 - 80N - 24 \tag{407}$$

$$P_{302} = 17N^6 + 51N^5 + 51N^4 + 89N^3 + 40N^2 - 80N - 24 \tag{408}$$

$$P_{303} = 27N^6 + 102N^5 + 135N^4 + 56N^3 - 8N^2 - 20N - 8 \tag{409}$$

$$P_{304} = 38N^6 + 108N^5 + 151N^4 + 106N^3 + 21N^2 - 28N - 12 \tag{410}$$

$$P_{305} = 73N^6 + 189N^5 + 45N^4 + 31N^3 - 238N^2 - 412N - 120 \tag{411}$$

$$P_{306} = 2N^7 + 14N^6 + 37N^5 + 102N^4 + 155N^3 + 158N^2 + 132N + 40 \tag{412}$$

$$P_{307} = 3N^7 - 15N^6 - 133N^5 - 449N^4 - 658N^3 - 500N^2 - 296N - 96 \tag{413}$$

$$P_{308} = 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \tag{414}$$

$$P_{309} = 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \tag{415}$$

$$P_{310} = 8N^7 + 37N^6 + 158N^5 + 565N^4 + 796N^3 + 628N^2 + 400N + 96 \tag{416}$$

$$P_{311} = 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 \tag{417}$$

$$P_{312} = N^{10} + 8N^9 + 29N^8 + 49N^7 - 11N^6 - 131N^5 - 161N^4 - 160N^3 - 168N^2 - 80N - 16 \tag{418}$$

$$\begin{aligned}
P_{313} &= 19N^{10} + 143N^9 + 427N^8 + 567N^7 + 454N^6 + 822N^5 + 1560N^4 + 1784N^3 + 1488N^2 \\
&+ 768N + 192 \tag{419}
\end{aligned}$$

$$\begin{aligned}
P_{314} &= 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 \\
&+ 672N + 288 \tag{420}
\end{aligned}$$

$$\begin{aligned}
P_{315} &= 43N^{10} + 320N^9 + 1059N^8 + 1914N^7 + 2431N^6 + 2874N^5 + 2379N^4 + 820N^3 + 352N^2 \\
&+ 336N + 144 \tag{421}
\end{aligned}$$

$$\begin{aligned}
P_{316} &= 136N^{10} + 647N^9 + 1110N^8 - 438N^7 - 2555N^6 - 2106N^5 - 3105N^4 - 3167N^3 + 418N^2 \\
&+ 924N + 72 \tag{422}
\end{aligned}$$

$$P_{317} = 3N^{11} + 66N^{10} + 104N^9 - 1152N^8 - 3801N^7 - 2510N^6 + 3318N^5 + 8076N^4 + 9608N^3$$

$$+6512N^2 + 2432N + 384 \quad (423)$$

$$P_{318} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32 \quad (424)$$

$$P_{319} = 118N^{11} + 793N^{10} + 2281N^9 + 3402N^8 + 2428N^7 + 1457N^6 + 1917N^5 + 2476N^4 + 4392N^3 + 4976N^2 + 2832N + 576 \quad (425)$$

$$P_{320} = 127N^{11} + 820N^{10} + 2251N^9 + 2196N^8 - 1109N^7 - 934N^6 + 4491N^5 + 9334N^4 + 12552N^3 + 9680N^2 + 4656N + 864 \quad (426)$$

$$P_{321} = 37N^{12} + 305N^{11} + 1107N^{10} + 2328N^9 + 3520N^8 + 5020N^7 + 7642N^6 + 10519N^5 + 10938N^4 + 8248N^3 + 4656N^2 + 1712N + 288 \quad (427)$$

$$P_{322} = 248N^{13} + 2599N^{12} + 12793N^{11} + 39593N^{10} + 87182N^9 + 148026N^8 + 196942N^7 + 192416N^6 + 128195N^5 + 63406N^4 + 32344N^3 + 15984N^2 + 5616N + 864 \quad (428)$$

$$P_{323} = 4N^{14} + 56N^{13} + 443N^{12} + 2139N^{11} + 6049N^{10} + 10762N^9 + 13272N^8 + 11692N^7 + 6106N^6 + 339N^5 - 1254N^4 - 72N^3 + 496N^2 + 240N + 32 \quad (429)$$

$$P_{324} = 686N^{14} + 6560N^{13} + 25572N^{12} + 43489N^{11} + 9045N^{10} - 72944N^9 - 125240N^8 - 156761N^7 - 206883N^6 - 241600N^5 - 250212N^4 - 225808N^3 - 150864N^2 - 56448N - 8640 \quad (430)$$

$$P_{325} = 12N^{17} + 162N^{16} + 1030N^{15} + 4188N^{14} + 11527N^{13} + 19051N^{12} + 11176N^{11} - 17182N^{10} - 36527N^9 - 27469N^8 - 11770N^7 + 5554N^6 + 32640N^5 + 46456N^4 + 34528N^3 + 14816N^2 + 3584N + 384 \quad (431)$$

$$P_{326} = 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \quad (432)$$

$$P_{327} = 8N^7 + 37N^6 + 83N^5 + 85N^4 + 61N^3 + 58N^2 - 20N - 24 \quad (433)$$

$$P_{328} = 43N^{10} + 320N^9 + 939N^8 + 912N^7 - 218N^6 - 510N^5 - 654N^4 - 1232N^3 + 16N^2 + 672N + 288 \quad (434)$$

$$P_{329} = 5N^{11} + 62N^{10} + 252N^9 + 374N^8 + 38N^7 - 400N^6 - 473N^5 - 682N^4 - 904N^3 - 592N^2 - 208N - 32 . \quad (435)$$

The OME A_{Qg} except for the term $a_{Qg}^{(3)}$ reads :

$$\begin{aligned} A_{Qg} &= \frac{1}{2}[1 + (-1)^N] \\ &\times \left\{ a_s \tilde{\gamma}_{gg}^0 T_F L_M + a_s^2 \left\{ \frac{4}{3} \tilde{\gamma}_{gg}^0 T_F^2 L_M^2 + C_F T_F \left[\frac{4(3N+2)S_1^2}{N^2(N+2)} + \frac{4(N^4 - N^3 - 20N^2 - 10N - 4)S_1}{N^2(N+1)^2(N+2)} \right. \right. \right. \\ &+ \left. \frac{2P_{375}}{N^4(N+1)^4(N+2)} + L_M^2 \left[\frac{2(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} + 2\tilde{\gamma}_{gg}^0 S_1 \right] \right. \\ &+ \left. L_M \left[-\frac{4P_{337}}{N^3(N+1)^3(N+2)} + \frac{16S_1}{N^2} + \tilde{\gamma}_{gg}^0 [2S_1^2 - 2S_2] \right] + \frac{4(N^4 + 17N^3 + 17N^2 - 5N - 2)S_2}{N^2(N+1)^2(N+2)} \right. \\ &+ \left. \tilde{\gamma}_{gg}^0 \left[\frac{1}{3}S_1^3 + S_2S_1 - \frac{4S_3}{3} \right] \right\} + C_A T_F \left[-\frac{4(N^3 + 8N^2 + 11N + 2)S_1^2}{N(N+1)^2(N+2)^2} - \frac{4P_{333}S_1}{N(N+1)^3(N+2)^3} \right. \\ &+ \left. \frac{4P_{413}}{(N-1)N^4(N+1)^4(N+2)^4} + L_M^2 \left[-\frac{16(N^2 + N + 1)(N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)^2} - 2\tilde{\gamma}_{gg}^0 S_1 \right] \right. \\ &- \left. \frac{4(7N^5 + 21N^4 + 13N^3 + 21N^2 + 18N + 16)S_2}{(N-1)N^2(N+1)^2(N+2)^2} + L_M \left[-\frac{8P_{383}}{(N-1)N^3(N+1)^3(N+2)^3} \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{32(2N+3)S_1}{(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[-2S_1^2 - 2S_2 - 4S_{-2} \right] + \frac{(N^2 - N - 4)}{(N+1)^2(N+2)^2} 16(-1)^N S_{-2} \\
& + \tilde{\gamma}_{qg}^0 \left[-\frac{1}{3}S_1^3 - 3S_2S_1 - 4(-1)^N S_{-2}S_1 - \frac{8S_3}{3} - 2(-1)^N S_{-3} + 4S_{-2,1} \right] \Bigg\} \\
& + a_s^3 \left\{ T_F^3 \left[\frac{16}{9}\tilde{\gamma}_{qg}^0 L_M^3 - \frac{16\tilde{\gamma}_{qg}^0 \zeta_3}{9} \right] + C_{AT_F}^2 \left[\frac{32(N^3 + 8N^2 + 11N + 2)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{334}S_1^2}{3N(N+1)^3(N+2)^3} \right. \right. \\
& - \frac{16P_{406}S_1}{81(N-1)N^2(N+1)^4(N+2)^4} - \frac{32(5N^4 + 8N^3 + 17N^2 + 43N + 20)\zeta_2}{9N(N+1)^2(N+2)^2} S_1 \\
& - \frac{32(3N^3 - 12N^2 - 27N - 2)S_2S_1}{3N(N+1)^2(N+2)^2} + \frac{4P_{392}\zeta_2}{9(N-1)N^3(N+1)^3(N+2)^3} \\
& + \frac{8P_{427}}{81(N-1)N^5(N+1)^5(N+2)^5} + \frac{32(9N^5 - 4N^4 + N^3 + 92N^2 + 42N + 28)\zeta_3}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& + L_M^3 \left[-\frac{448(N^2 + N + 1)(N^2 + N + 2)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{56}{9}\tilde{\gamma}_{qg}^0 S_1 \right] + \frac{8P_{385}S_2}{3(N-1)N^3(N+1)^3(N+2)^3} \\
& + \frac{256(N^5 + 10N^4 + 9N^3 + 3N^2 + 7N + 6)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + L_M^2 \left[-\frac{8P_{390}}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& + \left. \frac{32(5N^4 + 20N^3 - N^2 - 14N + 20)S_1}{9N(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[-4S_1^2 - 4S_2 - 8S_{-2} \right] \right] \\
& + \frac{(N^4 + 2N^3 + 7N^2 + 22N + 20) \left[\frac{128}{3}(-1)^N S_{-2} + \frac{64}{3}(-1)^N \zeta_2 \right]}{(N+1)^3(N+2)^3} \\
& + \frac{(N^2 - N - 4) \left[-\frac{256}{3}(-1)^N S_1 S_{-2} - \frac{128}{3}(-1)^N S_{-3} + \frac{256}{3}S_{-2,1} - \frac{128}{3}(-1)^N S_1 \zeta_2 - 32(-1)^N \zeta_3 \right]}{(N+1)^2(N+2)^2} \\
& + \tilde{\gamma}_{qg}^0 \left[\frac{2}{9}S_1^4 + \frac{20}{3}S_2S_1^2 + \frac{32}{3}(-1)^N S_{-3}S_1 + \left[\frac{160S_3}{9} - \frac{64}{3}S_{-2,1} \right] S_1 + \frac{8}{9}(-2 + 9(-1)^N)\zeta_3 S_1 + \frac{2}{3}S_2^2 \right. \\
& + S_{-2} \left(\frac{32}{3}(-1)^N S_1^2 + \frac{32}{3}(-1)^N S_2 \right) + 12S_4 + \frac{16}{3}(-1)^N S_{-4} - \frac{16}{3}S_{3,1} - \frac{32}{3}S_{-2,2} - \frac{32}{3}S_{-3,1} \\
& - \frac{16}{3}S_{2,1,1} + \frac{64}{3}S_{-2,1,1} + \left. \left[\frac{2}{3}(-3 + 8(-1)^N)S_1^2 + \frac{2}{3}[-3 + 8(-1)^N]S_2 + \frac{4}{3}(1 + 4(-1)^N)S_{-2} \right] \zeta_2 \right] \\
& + L_M \left[\frac{16P_{415}}{27(N-1)N^4(N+1)^4(N+2)^4} - \frac{32(5N^4 + 23N^3 + 65N^2 + 82N + 26)S_1^2}{9N(N+1)^2(N+2)^2} \right. \\
& + \frac{16P_{364}S_1}{27N(N+1)^3(N+2)^3} - \frac{32P_{340}S_2}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{64(5N^2 + 8N + 10)S_{-2}}{9N(N+1)(N+2)} \\
& + \frac{(N^2 - N - 4)\frac{128}{3}(-1)^N S_{-2}}{(N+1)^2(N+2)^2} + \tilde{\gamma}_{qg}^0 \left[-\frac{4}{3}S_1^3 - \frac{20}{3}S_2S_1 - \frac{32}{3}(-1)^N S_{-2}S_1 - 8S_3 \right. \\
& \left. - \frac{16}{3}(1 + (-1)^N)S_{-3} - \frac{16}{3}S_{2,1} + \frac{32}{3}S_{-2,1} \right] \Bigg] \\
& + C_{AT_F}^2 N_F \left[\frac{16(N^3 + 8N^2 + 11N + 2)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{8P_{332}S_1^2}{3N(N+1)^3(N+2)^3} + \frac{4P_{376}\zeta_2}{9(N-1)N^3(N+1)^3(N+2)^2} \right. \\
& - \frac{16P_{371}S_1}{3N(N+1)^4(N+2)^4} + \frac{16P_{424}}{3(N-1)N^5(N+1)^5(N+2)^5} - \frac{16(3N^3 - 12N^2 - 27N - 2)S_2S_1}{3N(N+1)^2(N+2)^2} \\
& \left. + \frac{16(5N^4 + 32N^3 + 47N^2 + 28N + 20)\zeta_2}{9N(N+1)^2(N+2)^2} S_1 + \frac{16(9N^5 - 14N^4 - 19N^3 + 52N^2 + 12N + 8)\zeta_3}{9(N-1)N^2(N+1)^2(N+2)^2} \right]
\end{aligned}$$

$$\begin{aligned}
& +L_M^3 \left[-\frac{64(N^2+N+1)(N^2+N+2)}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{8}{9}\tilde{\gamma}_{gg}^0 S_1 \right] + \frac{8P_{384}S_2}{3(N-1)N^3(N+1)^3(N+2)^3} \\
& + \frac{128(N^5+10N^4+9N^3+3N^2+7N+6)S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + L_M^2 \left[-\frac{8P_{374}}{9(N-1)N^2(N+1)^3(N+2)^3} \right. \\
& \left. - \frac{32(5N^4+20N^3+47N^2+58N+20)S_1}{9N(N+1)^2(N+2)^2} + \tilde{\gamma}_{gg}^0 \left[-\frac{4}{3}S_1^2 - \frac{4S_2}{3} - \frac{8}{3}S_{-2} \right] \right] \\
& + \frac{(N^4+2N^3+7N^2+22N+20) \left[\frac{64}{3}(-1)^N S_{-2} + \frac{32}{3}(-1)^N \zeta_2 \right]}{(N+1)^3(N+2)^3} \\
& + \frac{(N^2-N-4)}{(N+1)^2(N+2)^2} \left[-\frac{128}{3}(-1)^N S_1 S_{-2} - \frac{64}{3}(-1)^N S_{-3} + \frac{128}{3}S_{-2,1} - \frac{64}{3}(-1)^N S_1 \zeta_2 - 16(-1)^N \zeta_3 \right] \\
& + \tilde{\gamma}_{gg}^0 \left[\frac{1}{9}S_1^4 + \frac{10}{3}S_2 S_1^2 + \frac{16}{3}(-1)^N S_{-3} S_1 + \left[\frac{80S_3}{9} - \frac{32}{3}S_{-2,1} \right] S_1 + \frac{4}{9}(-7+9(-1)^N)\zeta_3 S_1 + \frac{1}{3}S_2^2 \right. \\
& + S_{-2} \left[\frac{16}{3}(-1)^N S_1^2 + \frac{16}{3}(-1)^N S_2 \right] + 6S_4 + \frac{8}{3}(-1)^N S_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} - \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} \\
& \left. + \frac{32}{3}S_{-2,1,1} + \left[\frac{4}{3}(-1+2(-1)^N)S_1^2 + \frac{4}{3}(-1+2(-1)^N)S_2 + \frac{8}{3}(-1)^N S_{-2} \right] \zeta_2 \right] \\
& + L_M \left[-\frac{16(10N^4+43N^3+106N^2+131N+46)S_1^2}{9N(N+1)^2(N+2)^2} + \frac{8P_{416}}{27(N-1)N^4(N+1)^4(N+2)^4} \right. \\
& + \frac{16P_{357}S_1}{27N(N+1)^3(N+2)^3} - \frac{16P_{345}S_2}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{64(5N^2+8N+10)S_{-2}}{9N(N+1)(N+2)} \\
& + \frac{(N^2-N-4)}{(N+1)^2(N+2)^2} \frac{64}{3}(-1)^N S_{-2} + \tilde{\gamma}_{gg}^0 \left[-\frac{8}{9}S_1^3 - \frac{8}{3}S_2 S_1 - \frac{16}{3}(-1)^N S_{-2} S_1 - \frac{40S_3}{9} \right. \\
& \left. - \frac{8}{3}(2+(-1)^N)S_{-3} - \frac{16}{3}S_{2,1} + \frac{16}{3}S_{-2,1} \right] \left. \right] + C_F^2 T_F \left[-\frac{(3N^4+54N^3+139N^2+120N+36)S_1^4}{3N^2(N+1)^2(N+2)} \right. \\
& - \frac{4(18N^5-15N^4-180N^3-111N^2-40N-4)S_1^3}{3N^3(N+1)^2(N+2)} + \frac{2P_{359}S_1^2}{N^3(N+1)^3(N+2)} \\
& - \frac{4(3N^4+14N^3+43N^2+48N+20)\zeta_2}{N^2(N+1)^2(N+2)} S_1^2 - \frac{2(3N^3+24N^2+27N+10)S_2 S_1^2}{N^2(N+1)^2} \\
& - \frac{4P_{398}S_1}{N^5(N+1)^5(N+2)} - \frac{4P_{355}S_2 S_1}{N^3(N+1)^3(N+2)} - \frac{8(9N^4+102N^3+97N^2-36N-12)S_3 S_1}{3N^2(N+1)^2(N+2)} \\
& - \frac{16(3N^4+2N^3+19N^2+28N+12)S_{2,1} S_1}{N^2(N+1)^2(N+2)} - \frac{8P_{353}\zeta_2 S_1}{N^3(N+1)^3(N+2)} - \frac{P_{414}}{N^6(N+1)^6(N+2)} \\
& + \frac{2}{3}(N^2+N+2)(153N^4+306N^3+165N^2+12N+4) \frac{\zeta_3}{N^3(N+1)^3(N+2)} \\
& + L_M^3 \left[-\frac{2(N^2+N+2)(3N^2+3N+2)^2}{3N^3(N+1)^3(N+2)} + \frac{16(N^2+N+2)S_1(3N^2+3N+2)}{3N^2(N+1)^2(N+2)} + \frac{8}{3}\tilde{\gamma}_{gg}^0 S_1^2 \right] \\
& + \frac{2P_{370}S_2}{N^4(N+1)^4(N+2)} + \frac{4P_{362}S_3}{3N^3(N+1)^3(N+2)} - \frac{16(N^2-3N-2)(3N^2+3N+2)S_{2,1}}{N^3(N+1)^2(N+2)} \\
& + L_M^2 \left[\frac{4(3N^4+14N^3+43N^2+48N+20)S_1^2}{N^2(N+1)^2(N+2)} - \frac{8P_{338}S_1}{N^3(N+1)^3(N+2)} + \frac{P_{378}}{N^4(N+1)^4(N+2)} \right. \\
& \left. - \frac{12(N^2+N+2)(3N^2+3N+2)S_2}{N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)S_{-2}}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{gg}^0 \left[4S_1^3 - 12S_2 S_1 - 16S_{-2} S_1 \right] \right]
\end{aligned}$$

$$\begin{aligned}
& \left. -8S_3 - 8S_{-3} + 16S_{-2,1} \right] + \frac{P_{382}\zeta_2}{2N^4(N+1)^4(N+2)} + \frac{16(N^2+N+2)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} + 96\tilde{\gamma}_{qg}^0 \log(2)\zeta_2 \\
& + L_M \left[\frac{64(N+1)S_1^3}{N^2(N+2)} - \frac{8P_{339}S_1^2}{N^3(N+1)^3(N+2)} + \frac{16(N^4+10N^3+3N^2-18N-4)S_2S_1}{N^2(N+1)^2(N+2)} \right. \\
& + \frac{8P_{372}S_1}{N^4(N+1)^4(N+2)} - \frac{P_{403}}{2N^5(N+1)^5(N+2)} - \frac{8(N^2+5N+2)(3N^2-N+2)S_3}{N^2(N+1)^2(N+2)} \\
& + \frac{4P_{356}S_2}{N^3(N+1)^3(N+2)} + \frac{32(2N^5+4N^4-N^3+N^2-2N+8)S_{-2}}{N^3(N+1)^2(N+2)} - \frac{64(N-1)S_{-3}}{(N+1)^2(N+2)} \\
& - \frac{64(N^2+N+2)S_{2,1}}{N^2(N+1)^2(N+2)} + \frac{(N^2-N+2)[128S_{-2,1}-128S_{-2}S_1]}{N^2(N+1)(N+2)} \\
& + \tilde{\gamma}_{qg}^0 \left[2S_1^4 + 4S_2S_1^2 + 32S_{-3}S_1 + (24S_3 - 32S_{2,1})S_1 + 6S_2^2 + 8S_{-2}^2 + 16S_{-2}S_2 + 20S_4 + 40S_{-4} \right. \\
& \left. - 8S_{3,1} - 16S_{-2,2} - 32S_{-3,1} + 24S_{2,1,1} \right] + \frac{48(N-1)(3N^2+3N-2)\zeta_3}{N^2(N+1)^2} \\
& + \frac{(N^2+N+2)(3N^2+3N+2)[-S_2^2+8\zeta_2S_2+6S_4-16S_{3,1}+32S_{2,1,1}-\frac{16}{3}S_1\zeta_3]}{N^2(N+1)^2(N+2)} \\
& + \tilde{\gamma}_{qg}^0 \left[-\frac{1}{3}S_1^5 - 2S_2S_1^3 + \left[-\frac{8}{3}S_3 - 16S_{2,1} \right] S_1^2 - \frac{8}{3}\zeta_3S_1^2 + \left[-S_2^2 + 6S_4 - 16S_{3,1} + 32S_{2,1,1} \right] S_1 \right. \\
& \left. + \left[-4S_1^3 + 8S_2S_1 + 8S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1} \right] \zeta_2 \right] + C_A^2 T_F \left[\frac{P_{346}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& - \frac{4P_{377}S_1^3}{9(N-1)N^2(N+1)^3(N+2)^3} - \frac{4}{3} \frac{P_{352}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{2P_{400}S_1^2}{3(N-1)N^2(N+1)^4(N+2)^4} \\
& + \frac{2(N-2)(55N^5+347N^4+379N^3+213N^2+326N+120)S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{4}{9} \frac{P_{366}S_1\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4}{9} \frac{P_{402}S_1\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^3} \\
& - \frac{4P_{425}S_1}{3(N-1)N^5(N+1)^5(N+2)^5} - \frac{4P_{386}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3} + \frac{16P_{360}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{2}{9} \frac{P_{419}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^4} - \frac{4(11N^4+22N^3-35N^2-46N-24)P_{424}}{3(N-1)^2N^6(N+1)^6(N+2)^6} \\
& - \frac{4(11N^4+22N^3-35N^2-46N-24)(9N^5-14N^4-19N^3+52N^2+12N+8)\zeta_3}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
& + L_M^3 \left[\frac{8}{3}\tilde{\gamma}_{qg}^0 S_1^2 - \frac{8(N^2+N+2)(11N^4+22N^3-59N^2-70N-48)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. + \frac{16(N^2+N+1)(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right] \\
& - \frac{2(11N^4+22N^3-35N^2-46N-24)P_{384}S_2}{3(N-1)^2N^4(N+1)^4(N+2)^4} - 32 \frac{(N^2+N+1)(N^2+N+2)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} S_{-2} \\
& - \frac{32(11N^4+22N^3-35N^2-46N-24)(N^5+10N^4+9N^3+3N^2+7N+6)S_3}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
& + L_M^2 \left[-\frac{4P_{350}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{401}S_1}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{16P_{418}}{9(N-1)^2N^4(N+1)^4(N+2)^4} - \frac{4(N^2+N+2)(11N^4+22N^3-83N^2-94N-72)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{8(N^2+N+2)(11N^4+22N^3-59N^2-70N-48)S_{-2}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \tilde{\gamma}_{gg}^0 \left[4S_1^3 + 12S_2S_1 + 16S_{-2}S_1 + 4S_3 + 4S_{-3} - 8S_{-2,1} \right] \\
& + \frac{(N^4+2N^3+7N^2+22N+20)(11N^4+22N^3-35N^2-46N-24)}{(N-1)N(N+1)^4(N+2)^4} \left[-\frac{16}{3}(-1)^N S_{-2} \right. \\
& - \left. \frac{8}{3}(-1)^N \zeta_2 \right] + \frac{(5N^5-131N^3-58N^2+232N+96) \left[\frac{32}{3}(-1)^N S_1S_{-2} + \frac{16}{3}(-1)^N S_1\zeta_2 \right]}{(N-1)N(N+1)^2(N+2)^3} \\
& + \frac{P_{351} \left[\frac{16}{3}(-1)^N S_{-2}S_1^2 + \frac{8}{3}(-1)^N \zeta_2 S_1^2 \right]}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{(N^2+N+2)(11N^4+22N^3-35N^2-46N-24)}{(N-1)N^2(N+1)^2(N+2)^2} \\
& \times \left[\frac{1}{3}S_2^2 + \frac{16}{3}(-1)^N S_{-2}S_2 + 6S_4 + \frac{8}{3}(-1)^N S_{-4} - \frac{8}{3}S_{3,1} - \frac{16}{3}S_{-2,2} \right. \\
& - \left. \frac{16}{3}S_{-3,1} - \frac{8}{3}S_{2,1,1} + \frac{32}{3}S_{-2,1,1} + \left(\frac{8}{3}(-1)^N S_2 + \frac{8}{3}(-1)^N S_{-2} \right) \zeta_2 \right] \\
& + \frac{(N^2-N-4)(11N^4+22N^3-35N^2-46N-24) \left[\frac{16}{3}(-1)^N S_{-3} - \frac{32}{3}S_{-2,1} + 4(-1)^N \zeta_3 \right]}{(N-1)N(N+1)^3(N+2)^3} \\
& + \frac{(11N^5+34N^4-49N^3-24N^2-68N-48) \left[\frac{16}{3}(-1)^N S_1S_{-3} - \frac{32}{3}S_1S_{-2,1} + 4(-1)^N S_1\zeta_3 \right]}{(N-1)N^2(N+1)(N+2)^2} \\
& + \tilde{\gamma}_{gg}^0 \left[-\frac{1}{3}S_1^5 - 10S_2S_1^3 - 16(-1)^N S_{-3}S_1^2 + \left[32S_{-2,1} - \frac{80S_3}{3} \right] S_1^2 - \frac{4}{3}(-7+9(-1)^N) \zeta_3 S_1^2 \right. \\
& - 8(-1)^N S_{-4}S_1 + \left[-S_2^2 - 18S_4 + 8S_{3,1} + 16S_{-2,2} + 16S_{-3,1} + 8S_{2,1,1} - 32S_{-2,1,1} \right] S_1 \\
& + S_{-2} \left[-16(-1)^N S_1^3 - 16(-1)^N S_2S_1 \right] + \left[-4(-1+2(-1)^N) S_1^3 - 8(-1)^N S_2S_1 \right. \\
& - 4(1+2(-1)^N) S_{-2}S_1 + \left. \frac{11S_2}{3} - 2S_3 - 2S_{-3} + 4S_{-2,1} \right] \zeta_2 \Big] \\
& + L_M \left[-\frac{8(11N^4+26N^3-139N^2-218N+8)S_1^3}{9N(N+1)^2(N+2)^2} + \frac{4P_{404}S_1^2}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right. \\
& - \frac{8P_{420}S_1}{27(N-1)^2N^4(N+1)^4(N+2)^4} - \frac{8P_{347}S_2S_1}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{32(2N^5-23N^4-32N^3+13N^2+4N-12)S_{-2}S_1}{(N-1)N^2(N+1)^2(N+2)^2} + 2 \frac{\zeta_3 P_{367}}{(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{4P_{428}}{27(N-1)^2N^5(N+1)^5(N+2)^5} + \frac{4P_{405}S_2}{9(N-1)^2N^3(N+1)^3(N+2)^3} \\
& - \frac{8P_{361}S_3}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{16P_{394}S_{-2}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{16P_{348}S_{-2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16P_{349}S_{-3}}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{16(N^2+N+2)(11N^4+22N^3+13N^2+2N+24)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{(N^2-N-4)(11N^4+22N^3-35N^2-46N-24) \frac{16}{3}(-1)^N S_{-2}}{(N-1)N(N+1)^3(N+2)^3} \\
& - \frac{(11N^5+34N^4-49N^3-24N^2-68N-48) \frac{16}{3}(-1)^N S_1S_{-2}}{(N-1)N^2(N+1)(N+2)^2}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(N^2 + N + 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24) \left[-\frac{8}{3}(-1)^N S_{-3} - 2(-1)^N \zeta_3 \right]}{(N-1)N^2(N+1)^2(N+2)^2} + \tilde{\gamma}_{gg}^0 \left[2S_1^4 + 2S_2^2 \right. \\
& + 32S_2S_1^2 + 8(8 + (-1)^N)S_{-3}S_1 + \left[40S_3 - 16S_{2,1} - 80S_{-2,1} \right] S_1 + 6(-5 + (-1)^N)\zeta_3S_1 + 12S_{-2}^2 \\
& \left. + S_{-2} \left[8(3 + 2(-1)^N)S_1^2 + 16S_2 \right] + 4S_4 + 44S_{-4} + 16S_{3,1} - 56S_{-2,2} - 64S_{-3,1} + 96S_{-2,1,1} \right] \Bigg] \\
& + C_F T_F^2 \left[\left[\frac{16(N^2 + N + 2)P_{341}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{32}{9}\tilde{\gamma}_{gg}^0 S_1 \right] L_M^3 + \left[-\frac{4P_{410}}{9(N-1)N^4(N+1)^4(N+2)^3} \right. \right. \\
& + \frac{32(5N^3 + 14N^2 + 37N + 18)S_1}{9N^2(N+1)(N+2)} + \tilde{\gamma}_{gg}^0 \left[4S_1^2 - \frac{4S_2}{3} \right] \left. \right] L_M^2 + \left[\frac{16(10N^3 + 31N^2 + 59N + 18)S_1^2}{9N^2(N+1)(N+2)} \right. \\
& - \frac{16(29N^4 + 163N^3 + 786N^2 + 592N + 192)S_1}{27N^2(N+1)^2(N+2)} + \frac{2P_{423}}{27(N-1)N^5(N+1)^5(N+2)^4} \\
& + \frac{16(2N^4 + 39N^3 + 42N^2 - 5N - 2)S_2}{3N^2(N+1)^2(N+2)} + \tilde{\gamma}_{gg}^0 \left[\frac{4}{3}S_1^3 + 4S_2S_1 - 8S_3 \right] \left. \right] L_M \\
& - \frac{16(N^4 - 5N^3 - 32N^2 - 18N - 4)S_1^2}{3N^2(N+1)^2(N+2)} - \frac{16}{9} \frac{(N^2 + N + 2)P_{341}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} \\
& + \frac{2}{9} \frac{P_{411}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3} + \frac{32(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12)S_1}{3N^2(N+1)^3(N+2)} \\
& - \frac{2P_{421}}{3(N-1)N^6(N+1)^6(N+2)^2} - \frac{80}{9} \frac{(N^3 + 4N^2 + 11N + 6)\zeta_2}{N^2(N+1)(N+2)} S_1 - \frac{16P_{336}S_2}{3N^3(N+1)^3(N+2)} \\
& + \frac{(3N+2) \left(-\frac{32}{9}S_1^3 - \frac{32}{3}S_2S_1 \right)}{N^2(N+2)} - \frac{32(3N^4 + 48N^3 + 43N^2 - 22N - 8)S_3}{9N^2(N+1)^2(N+2)} \\
& + \frac{128(N^2 - 3N - 2)S_{2,1}}{3N^2(N+1)(N+2)} + \tilde{\gamma}_{gg}^0 \left[-\frac{2}{9}S_1^4 - \frac{4}{3}S_2S_1^2 + \left(-\frac{16}{9}S_3 - \frac{32}{3}S_{2,1} \right) S_1 - \frac{32}{9}\zeta_3S_1 - \frac{2}{3}S_2^2 + 4S_4 \right. \\
& \left. - \frac{32}{3}S_{3,1} + \frac{64}{3}S_{2,1,1} + \left(2S_2 - \frac{10}{3}S_1^2 \right) \zeta_2 \right] \Bigg] \\
& + C_F T_F^2 N_F \left[\left[\frac{8(N^2 + N + 2)P_{335}}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{8}{9}\tilde{\gamma}_{gg}^0 S_1 \right] L_M^3 + \left[-\frac{4P_{408}}{9(N-1)N^4(N+1)^4(N+2)^3} \right. \right. \\
& + \frac{32(5N^3 + 8N^2 + 19N + 6)S_1}{9N^2(N+1)(N+2)} + \tilde{\gamma}_{gg}^0 \left[\frac{4}{3}S_1^2 + \frac{4S_2}{3} \right] \left. \right] L_M^2 + L_M \left[\frac{32(5N^3 + 11N^2 + 22N + 6)S_1^2}{9N^2(N+1)(N+2)} \right. \\
& - \frac{32(19N^4 + 77N^3 + 303N^2 + 251N + 78)S_1}{27N^2(N+1)^2(N+2)} + \frac{2P_{422}}{27(N-1)N^5(N+1)^5(N+2)^4} \\
& + \frac{16P_{369}S_2}{3(N-1)N^3(N+1)^3(N+2)^2} + \tilde{\gamma}_{gg}^0 \left[\frac{8}{9}S_1^3 + \frac{8}{3}S_2S_1 - \frac{56S_3}{9} \right] \left. \right] - \frac{8}{9} \frac{(N^2 + N + 2)P_{335}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} \\
& - \frac{8(N^4 - 5N^3 - 32N^2 - 18N - 4)S_1^2}{3N^2(N+1)^2(N+2)} - \frac{2}{9} \frac{P_{412}\zeta_2}{(N-1)N^4(N+1)^4(N+2)^3} \\
& + \frac{16(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12)S_1}{3N^2(N+1)^3(N+2)} - \frac{16}{9} \frac{(5N^3 + 11N^2 + 28N + 12)\zeta_2}{N^2(N+1)(N+2)} S_1 \\
& - \frac{4P_{430}}{3(N-1)N^6(N+1)^6(N+2)^5} - \frac{8P_{407}S_2}{3(N-1)N^4(N+1)^4(N+2)^3} + \frac{(3N+2) \left[-\frac{16}{9}S_1^3 - \frac{16}{3}S_2S_1 \right]}{N^2(N+2)} \\
& - \frac{16P_{373}S_3}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{64(N^2 - 3N - 2)S_{2,1}}{3N^2(N+1)(N+2)} + \tilde{\gamma}_{gg}^0 \left[-\frac{1}{9}S_1^4 - \frac{2}{3}S_2S_1^2 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{4}{3}\zeta_2 S_1^2 + \left[-\frac{8}{9}S_3 - \frac{16}{3}S_{2,1} \right] S_1 - \frac{8}{9}\zeta_3 S_1 - \frac{1}{3}S_2^2 + 2S_4 - \frac{16}{3}S_{3,1} + \frac{32}{3}S_{2,1,1} \Big] \\
& + C_{FCATF} \left[-\frac{2P_{330}S_1^4}{9(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{388}S_1^3}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& - \frac{8}{3} \frac{P_{344}S_1^2\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{4P_{409}S_1^2}{3(N-1)N^3(N+1)^4(N+2)^4} \\
& + \frac{4P_{354}S_2S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{4}{9} \frac{P_{363}S_1\zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} + \frac{8P_{426}S_1}{3(N-1)N^5(N+1)^5(N+2)^5} \\
& + \frac{4}{9} \frac{P_{379}S_1\zeta_2}{(N-1)N^3(N+1)^3(N+2)^2} + \frac{8P_{391}S_2S_1}{3(N-1)N^3(N+1)^3(N+2)^3} + \frac{8P_{368}S_3S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16(11N^5 + 45N^4 - 3N^3 - 145N^2 - 176N - 20)S_{2,1}S_1}{3(N-1)N(N+1)^2(N+2)^2} - \frac{2}{9} \frac{P_{381}\zeta_3}{(N-1)N^3(N+1)^3(N+2)^2} \\
& - \frac{2(N^2 + N + 2)(N^4 + 2N^3 - 16N^2 - 17N - 6)S_2^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{1}{18} \frac{P_{397}\zeta_2}{(N-1)N^3(N+1)^4(N+2)^2} \\
& + \frac{P_{429}}{3(N-1)N^5(N+1)^6(N+2)^5} + L_M^3 \left[-\frac{8(N^2 + N + 2)(N^2 + N + 6)(7N^2 + 7N + 4)S_1}{9(N-1)N^2(N+1)^2(N+2)^2} \right. \\
& \left. - \frac{16}{3} \tilde{\gamma}_{gg}^0 S_1^2 - \frac{2(N^2 + N + 2)(3N^2 + 3N + 2)(11N^4 + 22N^3 - 59N^2 - 70N - 48)}{9(N-1)N^3(N+1)^3(N+2)^2} \right] \\
& - \frac{4P_{399}S_2}{3(N-1)N^3(N+1)^4(N+2)^3} - 4 \frac{(N^2 + N + 2)(3N^4 + 6N^3 + 7N^2 + 4N + 4)\zeta_2}{(N-1)N^2(N+1)^2(N+2)^2} S_2 \\
& + \frac{4P_{380}S_3}{9(N-1)N^3(N+1)^3(N+2)^2} + \frac{4(N^2 + N + 2)(19N^4 + 38N^3 - 22N^2 - 41N - 30)S_4}{(N-1)N^2(N+1)^2(N+2)^2} \\
& - \frac{16(N^2 - 3N - 2)(11N^4 + 22N^3 - 35N^2 - 46N - 24)S_{2,1}}{3(N-1)N^3(N+1)^2(N+2)^2} \\
& - \frac{8(N^2 + N + 2)(31N^4 + 62N^3 - 73N^2 - 104N - 60)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + L_M^2 \left[\frac{8P_{331}S_1^2}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8P_{389}S_1}{9(N-1)N^3(N+1)^3(N+2)^3} \right. \\
& + \frac{P_{396}}{9(N-1)N^3(N+1)^3(N+2)^3} + \frac{8(N^2 + N + 2)(N^4 + 2N^3 + 8N^2 + 7N + 18)S_2}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& \left. + \tilde{\gamma}_{gg}^0 \left[-8S_1^3 + 4S_3 + 6S_{-2} + 4S_{-3} - 8S_{-2,1} \right] \right] \\
& + \frac{8(N^2 + N + 2)(35N^4 + 70N^3 - 137N^2 - 172N - 84)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)^2} - \frac{8(N^2 + N + 2)S_{-2}\zeta_2}{N^2(N+1)^2(N+2)} \\
& - 48\tilde{\gamma}_{gg}^0 \log(2)\zeta_2 + \frac{P_{343} \left[16(-1)^N S_{-2} + 8(-1)^N \zeta_2 \right]}{N(N+1)^4(N+2)^3} + \frac{P_{342} \left[32(-1)^N S_1 S_{-2} + 16(-1)^N S_1 \zeta_2 \right]}{N(N+1)^3(N+2)^3} \\
& + \frac{(3N^5 + 4N^4 + 31N^3 + 62N^2 + 20N + 8) \left[16(-1)^N S_{-2} S_1^2 + 8(-1)^N \zeta_2 S_1^2 \right]}{N^2(N+1)^2(N+2)^2} \\
& + \frac{(N^2 + N + 2)(3N^2 + 3N + 2)}{N^2(N+1)^2(N+2)} \left[\left[8(-1)^N S_2 + 8(-1)^N S_{-2} \right] \zeta_2 + 16(-1)^N S_{-2} S_2 + 8(-1)^N S_{-4} \right. \\
& \left. - 16S_{-2,2} - 16S_{-3,1} + 32S_{-2,1,1} \right] + \frac{(6N^5 + 33N^4 + 66N^3 + 77N^2 + 58N + 16)}{N(N+1)^3(N+2)^2} \left[16(-1)^N S_{-3} \right.
\end{aligned}$$

$$\begin{aligned}
& -32S_{-2,1} + 12(-1)^N \zeta_3 \Big] + \frac{(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8)}{N^2(N+1)^2(N+2)^2} \Big[16(-1)^N S_1 S_{-3} - 32S_1 S_{-2,1} \\
& + 12(-1)^N S_1 \zeta_3 \Big] + \tilde{\gamma}_{ag}^0 \left[\frac{2}{3} S_1^5 + 12S_2 S_1^3 + 16(-1)^N S_{-3} S_1^2 + \left[\frac{88S_3}{3} + 16S_{2,1} - 32S_{-2,1} \right] S_1^2 \right. \\
& + \frac{4}{3} (-5 + 9(-1)^N) \zeta_3 S_1^2 + 8(-1)^N S_{-4} S_1 + \left[2S_2^2 + 12S_4 + 8S_{3,1} - 16S_{-2,2} - 16S_{-3,1} - 40S_{2,1,1} \right. \\
& + 32S_{-2,1,1} \Big] S_1 + S_{-2} \left[16(-1)^N S_1^3 + 16(-1)^N S_2 S_1 + 32 \right] + \left[8(-1)^N S_1^3 + 4(-1 + 2(-1)^N) S_{-2} S_1 \right. \\
& - 2S_3 - 2S_{-3} + 4S_{-2,1} \Big] \zeta_2 \Big] + L_M \left[\frac{8(11N^5 - 46N^4 - 499N^3 - 866N^2 - 496N - 144) S_1^3}{9N^2(N+1)^2(N+2)^2} \right. \\
& - \frac{4P_{393} S_1^2}{9(N-1)N^3(N+1)^3(N+2)^3} - \frac{8(N+3)(13N^5 + 114N^4 + 93N^3 + 52N^2 + 8N + 8) S_2 S_1}{3(N-1)N^2(N+1)^2(N+2)^2} \\
& + \frac{8P_{395} S_1}{27(N-1)N^4(N+1)^3(N+2)^2} + \frac{64(N^3 - 13N^2 - 14N - 2) S_{-2} S_1}{N^2(N+1)^2(N+2)} \\
& - \frac{6P_{358} \zeta_3}{(N-1)N^2(N+1)^2(N+2)^2} - \frac{P_{417}}{54(N-1)N^5(N+1)^4(N+2)^3} + \frac{4P_{387} S_2}{3(N-1)N^3(N+1)^3(N+2)^3} \\
& - \frac{8P_{365} S_3}{9(N-1)N^2(N+1)^2(N+2)^2} - \frac{16(3N^5 - 40N^4 - 87N^3 - 54N^2 - 10N + 12) S_{-2}}{N^3(N+1)^3(N+2)} \\
& + \frac{16(3N^4 + 8N^3 - 5N^2 - 6N + 8) S_{-3}}{N^2(N+1)^2(N+2)} + \frac{16(3N^4 + 2N^3 + 39N^2 + 32N - 12) S_{-2,1}}{N^2(N+1)^2(N+2)} \\
& + \frac{48(N^2 + N + 2)^2 S_{2,1}}{(N-1)N(N+1)(N+2)^2} - \frac{(N^2 - N - 4)(3N^2 + 3N + 2) 16(-1)^N S_{-2}}{N(N+1)^3(N+2)^2} \\
& - \frac{(3N^5 + 8N^4 + 27N^3 + 46N^2 + 20N + 8) 16(-1)^N S_1 S_{-2}}{N^2(N+1)^2(N+2)^2} \\
& + \frac{(N^2 + N + 2)(3N^2 + 3N + 2)(-8(-1)^N S_{-3} - 6(-1)^N \zeta_3)}{N^2(N+1)^2(N+2)} + \tilde{\gamma}_{ag}^0 \left[-4S_1^4 - 36S_2 S_1^2 \right. \\
& - 8(3 + 2(-1)^N) S_{-2} S_1^2 - 8(-1)^N S_{-3} S_1 + (-32S_3 + 48S_{2,1} + 16S_{-2,1}) S_1 - 6(-5 + (-1)^N) \zeta_3 S_1 \\
& \left. + 8S_2^2 - 12S_{-2}^2 - 20S_{-4} - 8S_{3,1} - 8S_{-2,2} - 24S_{2,1,1} + 32S_{-2,1,1} \right] \Big] \Big] + a_{Qg}^{(3)} \Big\} \Big\}, \tag{436}
\end{aligned}$$

where

$$P_{330} = N^6 - 93N^5 - 444N^4 - 317N^3 + 329N^2 + 296N + 84 \tag{437}$$

$$P_{331} = N^6 - 9N^5 - 120N^4 - 137N^3 + 29N^2 + 56N + 36 \tag{438}$$

$$P_{332} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \tag{439}$$

$$P_{333} = N^6 + 8N^5 + 23N^4 + 54N^3 + 94N^2 + 72N + 8 \tag{440}$$

$$P_{334} = 2N^6 + 11N^5 + 8N^4 - 7N^3 + 14N^2 + 12N - 24 \tag{441}$$

$$P_{335} = 3N^6 + 9N^5 - N^4 - 17N^3 - 38N^2 - 28N - 24 \tag{442}$$

$$P_{336} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \tag{443}$$

$$P_{337} = 5N^6 + 15N^5 + 36N^4 + 51N^3 + 25N^2 + 8N + 4 \tag{444}$$

$$P_{338} = 5N^6 + 18N^5 + 51N^4 + 84N^3 + 60N^2 + 34N + 12 \tag{445}$$

$$P_{339} = 5N^6 + 26N^5 + 97N^4 + 160N^3 + 135N^2 + 79N + 22 \tag{446}$$

$$P_{340} = 5N^6 + 42N^5 + 84N^4 + 35N^3 + 40N^2 + 34N + 48 \tag{447}$$

$$P_{341} = 6N^6 + 18N^5 + 7N^4 - 16N^3 - 31N^2 - 20N - 12 \tag{448}$$

$$P_{342} = 6N^6 + 47N^5 + 136N^4 + 223N^3 + 256N^2 + 172N + 32 \tag{449}$$

$$P_{343} = 9N^6 + 27N^5 - 65N^4 - 319N^3 - 404N^2 - 200N - 40 \tag{450}$$

$$\begin{aligned}
P_{344} &= 10N^6 - 6N^5 - 39N^4 - 44N^3 - 97N^2 + 20N + 12 & (451) \\
P_{345} &= 10N^6 + 63N^5 + 105N^4 + 31N^3 + 17N^2 + 14N + 48 & (452) \\
P_{346} &= 11N^6 - 15N^5 - 327N^4 - 181N^3 + 292N^2 - 20N - 48 & (453) \\
P_{347} &= 11N^6 + 15N^5 - 285N^4 - 319N^3 - 254N^2 - 368N - 240 & (454) \\
P_{348} &= 11N^6 + 33N^5 - 189N^4 - 361N^3 - 194N^2 - 92N - 72 & (455) \\
P_{349} &= 11N^6 + 33N^5 - 114N^4 - 247N^3 - 263N^2 - 176N - 108 & (456) \\
P_{350} &= 11N^6 + 33N^5 - 87N^4 - 85N^3 + 4N^2 - 116N - 48 & (457) \\
P_{351} &= 11N^6 + 57N^5 - 39N^4 - 109N^3 - 44N^2 - 116N - 48 & (458) \\
P_{352} &= 11N^6 + 81N^5 + 9N^4 - 133N^3 - 92N^2 - 116N - 48 & (459) \\
P_{353} &= 13N^6 + 36N^5 + 39N^4 + 8N^3 - 21N^2 - 29N - 10 & (460) \\
P_{354} &= 17N^6 + 51N^5 + 390N^4 + 359N^3 - 389N^2 - 200N - 84 & (461) \\
P_{355} &= 18N^6 + 87N^5 + 57N^4 - 119N^3 - 131N^2 - 60N - 20 & (462) \\
P_{356} &= 23N^6 + 39N^5 + 75N^4 + 157N^3 + 96N^2 + 70N + 28 & (463) \\
P_{357} &= 29N^6 + 176N^5 + 777N^4 + 1820N^3 + 1878N^2 + 776N + 232 & (464) \\
P_{358} &= 45N^6 + 135N^5 - 91N^4 - 407N^3 - 214N^2 + 12N - 248 & (465) \\
P_{359} &= 51N^6 + 140N^5 + 227N^4 + 208N^3 - 202N^2 - 96N - 8 & (466) \\
P_{360} &= 55N^6 + 141N^5 - 195N^4 - 401N^3 - 772N^2 - 748N - 384 & (467) \\
P_{361} &= 55N^6 + 165N^5 - 420N^4 - 899N^3 - 1561N^2 - 1336N - 1188 & (468) \\
P_{362} &= 57N^6 + 297N^5 + 519N^4 + 399N^3 + 92N^2 - 68N - 16 & (469) \\
P_{363} &= 67N^6 + 93N^5 + 351N^4 + 259N^3 - 1054N^2 - 556N - 312 & (470) \\
P_{364} &= 76N^6 + 487N^5 + 1692N^4 + 3271N^3 + 3186N^2 + 1516N + 536 & (471) \\
P_{365} &= 77N^6 + 195N^5 + 627N^4 + 977N^3 - 128N^2 - 452N + 432 & (472) \\
P_{366} &= 77N^6 + 339N^5 - 105N^4 - 487N^3 - 356N^2 - 668N - 240 & (473) \\
P_{367} &= 83N^6 + 249N^5 - 111N^4 - 637N^3 - 956N^2 - 596N - 624 & (474) \\
P_{368} &= 97N^6 + 591N^5 + 1311N^4 + 229N^3 - 712N^2 + 308N + 192 & (475) \\
P_{369} &= N^8 + 24N^7 + 62N^6 + 8N^5 - 123N^4 - 128N^3 - 108N^2 - 72N - 48 & (476) \\
P_{370} &= N^8 + 427N^7 + 1133N^6 + 697N^5 - 434N^4 - 636N^3 - 244N^2 - 64N - 16 & (477) \\
P_{371} &= 2N^8 + 22N^7 + 117N^6 + 386N^5 + 759N^4 + 810N^3 + 396N^2 + 72N + 32 & (478) \\
P_{372} &= 2N^8 + 29N^7 + 179N^6 + 441N^5 + 529N^4 + 332N^3 + 172N^2 + 92N + 24 & (479) \\
P_{373} &= 3N^8 + 54N^7 + 118N^6 - 44N^5 - 353N^4 - 314N^3 - 272N^2 - 200N - 144 & (480) \\
P_{374} &= 9N^8 + 54N^7 + 56N^6 - 182N^5 - 717N^4 - 1120N^3 - 1012N^2 - 672N - 160 & (481) \\
P_{375} &= 12N^8 + 52N^7 + 132N^6 + 216N^5 + 191N^4 + 54N^3 - 25N^2 - 20N - 4 & (482) \\
P_{376} &= 15N^8 + 36N^7 + 50N^6 - 252N^5 - 357N^4 + 152N^3 - 68N^2 + 88N + 48 & (483) \\
P_{377} &= 18N^8 + 101N^7 + 128N^6 + 208N^5 + 190N^4 - 769N^3 - 1200N^2 - 212N - 48 & (484) \\
P_{378} &= 33N^8 + 132N^7 + 350N^6 + 636N^5 + 685N^4 + 528N^3 + 292N^2 + 128N + 32 & (485) \\
P_{379} &= 121N^8 + 370N^7 + 924N^6 + 358N^5 - 381N^4 + 184N^3 - 1096N^2 - 48N + 144 & (486) \\
P_{380} &= 321N^8 + 1674N^7 + 2360N^6 - 1378N^5 - 6565N^4 - 5992N^3 - 1972N^2 + 128N - 96 & (487) \\
P_{381} &= 507N^8 + 2190N^7 + 3002N^6 + 1692N^5 - 681N^4 - 2554N^3 - 404N^2 + 664N + 192 & (488) \\
P_{382} &= 633N^8 + 2532N^7 + 5036N^6 + 6174N^5 + 4307N^4 + 1182N^3 - 176N^2 - 184N - 48 & (489) \\
P_{383} &= N^9 + 6N^8 + 15N^7 + 25N^6 + 36N^5 + 85N^4 + 128N^3 + 104N^2 + 64N + 16 & (490) \\
P_{384} &= N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 & (491) \\
P_{385} &= 4N^9 + 53N^8 + 193N^7 + 233N^6 + 87N^5 + 554N^4 + 1172N^3 + 904N^2 + 512N + 128 & (492)
\end{aligned}$$

$$P_{386} = 6N^9 + 93N^8 + 576N^7 + 1296N^6 + 586N^5 + 359N^4 + 2000N^3 + 1996N^2 + 1488N + 384 \quad (493)$$

$$P_{387} = 25N^9 - 43N^8 - 424N^7 + 462N^6 + 4345N^5 + 7513N^4 + 6446N^3 + 4020N^2 + 1944N + 480 \quad (494)$$

$$P_{388} = 36N^9 + 156N^8 - 115N^7 - 1116N^6 - 1251N^5 - 78N^4 + 300N^3 + 84N^2 - 128N - 48 \quad (495)$$

$$P_{389} = 40N^9 + 273N^8 + 635N^7 + 613N^6 + 119N^5 - 2N^4 - 314N^3 - 668N^2 + 24N + 144 \quad (496)$$

$$P_{390} = 45N^9 + 270N^8 + 724N^7 + 1262N^6 + 1731N^5 + 2740N^4 + 3484N^3 + 2928N^2 + 1696N + 384 \quad (497)$$

$$P_{391} = 66N^9 + 534N^8 + 1409N^7 + 1080N^6 - 933N^5 - 1116N^4 + 588N^3 + 996N^2 + 736N + 240 \quad (498)$$

$$P_{392} = 69N^9 + 366N^8 + 1100N^7 + 1894N^6 + 2451N^5 + 5276N^4 + 7460N^3 + 5352N^2 + 3008N + 672 \quad (499)$$

$$P_{393} = 80N^9 + 441N^8 + 568N^7 - 592N^6 - 1202N^5 + 2003N^4 + 4106N^3 + 3116N^2 + 2712N + 864 \quad (500)$$

$$P_{394} = 94N^9 + 597N^8 + 1508N^7 + 2086N^6 + 1517N^5 + 1381N^4 + 2731N^3 + 3802N^2 + 2916N + 648 \quad (501)$$

$$P_{395} = 251N^9 + 1586N^8 + 4206N^7 + 6764N^6 + 4008N^5 - 2242N^4 + 13N^3 + 7122N^2 + 6156N + 1944 \quad (502)$$

$$P_{396} = 489N^9 + 2934N^8 + 7636N^7 + 12206N^6 + 6675N^5 - 12692N^4 - 24608N^3 - 16272N^2 - 2864N + 2304 \quad (503)$$

$$P_{397} = 891N^9 + 5751N^8 + 15070N^7 + 21430N^6 + 37623N^5 + 55339N^4 + 44064N^3 + 25144N^2 + 9488N + 1776 \quad (504)$$

$$P_{398} = 10N^{10} + 62N^9 + 407N^8 + 1119N^7 + 1405N^6 + 889N^5 + 240N^4 - 90N^3 - 114N^2 - 48N - 8 \quad (505)$$

$$P_{399} = 36N^{10} + 456N^9 + 2448N^8 + 7171N^7 + 12399N^6 + 13213N^5 + 8997N^4 + 5000N^3 + 2888N^2 + 992N + 112 \quad (506)$$

$$P_{400} = 37N^{10} + 392N^9 + 2106N^8 + 6514N^7 + 9211N^6 + 1258N^5 - 9218N^4 - 6116N^3 - 72N^2 - 752N - 192 \quad (507)$$

$$P_{401} = 85N^{10} + 425N^9 + 830N^8 + 788N^7 - 521N^6 - 325N^5 + 2238N^4 + 2568N^3 + 968N^2 - 1296N - 576 \quad (508)$$

$$P_{402} = 103N^{10} + 575N^9 + 1124N^8 - 334N^7 - 1505N^6 + 3755N^5 + 4926N^4 + 36N^3 - 472N^2 - 2160N - 864 \quad (509)$$

$$P_{403} = 149N^{10} + 793N^9 + 2368N^8 + 5026N^7 + 6853N^6 + 6277N^5 + 5062N^4 + 3168N^3 + 1296N^2 + 400N + 96 \quad (510)$$

$$P_{404} = 170N^{10} + 883N^9 + 1897N^8 + 2710N^7 - 448N^6 - 4745N^5 + 561N^4 + 5904N^3 + 1132N^2 - 2016N - 864 \quad (511)$$

$$P_{405} = 170N^{10} + 1213N^9 + 3091N^8 + 2506N^7 - 2692N^6 - 3047N^5 - 861N^4 - 2352N^3 - 5324N^2 - 6240N - 2016 \quad (512)$$

$$P_{406} = 436N^{10} + 3960N^9 + 15787N^8 + 36343N^7 + 46431N^6 + 17745N^5 - 28270N^4 - 33648N^3 - 11056N^2 - 1936N + 864 \quad (513)$$

$$P_{407} = 3N^{11} + 42N^{10} + 144N^9 + 74N^8 - 459N^7 - 1060N^6 - 1152N^5 - 1424N^4 - 1688N^3 - 1232N^2 - 736N - 192 \quad (514)$$

$$P_{408} = 33N^{11} + 231N^{10} + 698N^9 + 1290N^8 + 1513N^7 + 1463N^6 + 2236N^5 + 5096N^4 + 7328N^3 + 5456N^2 + 3456N + 1152 \quad (515)$$

$$P_{409} = 95N^{11} + 853N^{10} + 3599N^9 + 9245N^8 + 12320N^7 - 282N^6 - 23342N^5 - 26920N^4 - 10832N^3 - 1712N^2 - 416N - 192 \quad (516)$$

$$P_{410} = 129N^{11} + 903N^{10} + 2894N^9 + 5730N^8 + 6505N^7 + 383N^6 - 9464N^5 - 13912N^4 - 11680N^3 - 6640N^2 - 3648N - 1152 \quad (517)$$

$$P_{411} = 243N^{11} + 1701N^{10} + 5378N^9 + 10350N^8 + 11479N^7 + 1193N^6 - 14684N^5 - 20572N^4 - 16288N^3 - 8944N^2 - 4992N - 1728 \quad (518)$$

$$P_{412} = 333N^{11} + 2331N^{10} + 6556N^9 + 9270N^8 + 5081N^7 - 6701N^6 - 17554N^5 - 20036N^4 - 15680N^3 - 9200N^2 - 5664N - 1728 \quad (519)$$

$$P_{413} = 2N^{12} + 20N^{11} + 86N^{10} + 192N^9 + 199N^8 - N^7 - 297N^6 - 495N^5 - 514N^4 - 488N^3 - 416N^2 - 176N - 32 \quad (520)$$

$$P_{414} = 23N^{12} + 138N^{11} - 311N^{10} - 3148N^9 - 7605N^8 - 8462N^7 - 4163N^6 + 246N^5 + 1540N^4 + 1066N^3 + 444N^2 + 120N + 16 \quad (521)$$

$$P_{415} = 111N^{12} + 1035N^{11} + 3634N^{10} + 5168N^9 - 2662N^8 - 21724N^7 - 37157N^6 - 34963N^5 - 19122N^4 - 4560N^3 + 80N^2 + 1008N + 288 \quad (522)$$

$$P_{416} = 201N^{12} + 1845N^{11} + 6742N^{10} + 11990N^9 + 7139N^8 - 8917N^7 - 15710N^6 - 2110N^5 + 16644N^4 + 22080N^3 + 12416N^2 + 4128N + 576 \quad (523)$$

$$P_{417} = 7299N^{12} + 53973N^{11} + 206656N^{10} + 532170N^9 + 820775N^8 + 650149N^7 + 204230N^6 + 189820N^5 + 606016N^4 + 664624N^3 + 372192N^2 + 143424N + 27648 \quad (524)$$

$$P_{418} = 9N^{13} + 72N^{12} + 101N^{11} - 511N^{10} - 2325N^9 - 4428N^8 - 4619N^7 - 3841N^6 - 4462N^5 - 6012N^4 - 6992N^3 - 5296N^2 - 2592N - 576 \quad (525)$$

$$P_{419} = 69N^{13} + 420N^{12} + 794N^{11} - 1501N^{10} - 11265N^9 - 18414N^8 - 4436N^7 - 5017N^6 - 41818N^5 - 65616N^4 - 62960N^3 - 39184N^2 - 17184N - 3456 \quad (526)$$

$$P_{420} = 296N^{13} + 2368N^{12} + 9908N^{11} + 22254N^{10} + 13564N^9 - 31716N^8 - 71723N^7 - 71221N^6 - 44369N^5 - 33249N^4 - 26584N^3 + 4968N^2 + 7344N + 432 \quad (527)$$

$$P_{421} = 385N^{14} + 2567N^{13} + 6877N^{12} + 9235N^{11} + 5375N^{10} - 1207N^9 - 3313N^8 + 905N^7 + 3876N^6 + 1676N^5 + 256N^4 + 1544N^3 + 1616N^2 + 736N + 192 \quad (528)$$

$$P_{422} = 531N^{14} + 5454N^{13} + 25877N^{12} + 77604N^{11} + 159437N^{10} + 205070N^9 + 82971N^8 - 207408N^7 - 490544N^6 - 694320N^5 - 735104N^4 - 562304N^3 - 355584N^2 - 158976N - 34560 \quad (529)$$

$$P_{423} = 1773N^{14} + 18018N^{13} + 80795N^{12} + 214620N^{11} + 371423N^{10} + 398930N^9 + 154773N^8 - 228072N^7 - 435356N^6 - 492936N^5 - 534656N^4 - 453440N^3 - 299712N^2 - 144000N - 34560 \quad (530)$$

$$P_{424} = 4N^{15} + 50N^{14} + 267N^{13} + 765N^{12} + 1183N^{11} + 682N^{10} - 826N^9 - 1858N^8 - 1116N^7 + 457N^6 + 1500N^5 + 2268N^4 + 2400N^3 + 1392N^2 + 448N + 64 \quad (531)$$

$$P_{425} = 26N^{15} + 314N^{14} + 1503N^{13} + 3222N^{12} + 2510N^{11} + 1996N^{10} + 15041N^9 + 40728N^8 + 54008N^7 + 44956N^6 + 31936N^5 + 30416N^4 + 29568N^3 + 16704N^2 + 5376N + 768 \quad (532)$$

$$P_{426} = 28N^{15} + 335N^{14} + 1953N^{13} + 6497N^{12} + 11508N^{11} + 6624N^{10} - 11753N^9 - 27541N^8 - 33352N^7 - 40915N^6 - 40468N^5 - 16628N^4 + 7584N^3 + 10416N^2 + 4032N + 576 \quad (533)$$

$$P_{427} = 435N^{15} + 5436N^{14} + 32317N^{13} + 119006N^{12} + 307057N^{11} + 620328N^{10} + 1065977N^9 + 1575060N^8 + 1889534N^7 + 1704634N^6 + 1113248N^5 + 592440N^4 + 328672N^3$$

$$+165984N^2 + 59904N + 10368 \quad (534)$$

$$\begin{aligned} P_{428} = & 939N^{16} + 10527N^{15} + 47251N^{14} + 101719N^{13} + 66350N^{12} - 155710N^{11} \\ & -322813N^{10} - 16829N^9 + 702425N^8 + 1332497N^7 + 1596952N^6 + 1640548N^5 \\ & +1506496N^4 + 1099952N^3 + 604032N^2 + 211392N + 34560 \end{aligned} \quad (535)$$

$$\begin{aligned} P_{429} = & 1623N^{16} + 20963N^{15} + 119399N^{14} + 394315N^{13} + 831483N^{12} + 1160715N^{11} + 1086519N^{10} \\ & +712841N^9 + 425270N^8 + 337718N^7 + 207634N^6 - 73752N^5 - 261272N^4 \\ & -200160N^3 - 64672N^2 - 4480N + 1408 \end{aligned} \quad (536)$$

$$\begin{aligned} P_{430} = & 87N^{17} + 1099N^{16} + 6055N^{15} + 19019N^{14} + 37119N^{13} + 45159N^{12} + 29583N^{11} - 2639N^{10} \\ & -30218N^9 - 40778N^8 - 39994N^7 - 35844N^6 - 30808N^5 - 30384N^4 - 28256N^3 \\ & -16064N^2 - 5248N - 768 . \end{aligned} \quad (537)$$

The operator matrix element $A_{gg,Q}$ except for the term $a_{gg,Q}^{(3)}$ is given by :

$$\begin{aligned} A_{gg,Q} = & \frac{1}{2}[1 + (-1)^N] \times \\ & \left\{ a_s \frac{4}{3} T_F L_M + a_s^2 \left\{ \frac{16}{9} T_F^2 L_M^2 + C_A T_F \left[\left[\frac{16(N^2 + N + 1)}{3(N-1)N(N+1)(N+2)} - \frac{8S_1}{3} \right] L_M^2 \right. \right. \right. \\ & + \left[\frac{16P_{433}}{9(N-1)N^2(N+1)^2(N+2)} - \frac{80S_1}{9} \right] L_M + \frac{2P_{451}}{27(N-1)N^3(N+1)^3(N+2)} - \frac{4(56N+47)S_1}{27(N+1)} \left. \right. \\ & + C_F T_F \left[\frac{4(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} L_M^2 + \frac{4P_{445}}{(N-1)N^3(N+1)^3(N+2)} L_M \right. \\ & \left. \left. \left. - \frac{P_{461}}{(N-1)N^4(N+1)^4(N+2)} \right] \right\} \right\} \\ & + a_s^3 \left\{ \left[T_F^3 \left[\frac{64}{27} L_M^3 - \frac{64\zeta_3}{27} \right] + C_A T_F^2 \left[\left[\frac{448(N^2 + N + 1)}{27(N-1)N(N+1)(N+2)} - \frac{224S_1}{27} \right] L_M^3 \right. \right. \right. \\ & + \left[\frac{8P_{441}}{27(N-1)N^2(N+1)^2(N+2)} - \frac{640S_1}{27} \right] L_M^2 + \left[-\frac{2P_{454}}{27(N-1)N^3(N+1)^3(N+2)} \right. \\ & \left. \left. - \frac{8P_{440}S_1}{9(N-1)N^2(N+1)^2(N+2)} \right] L_M + \frac{8S_1^2}{3(N+1)} - \frac{4}{27} \frac{P_{442}\zeta_2}{(N-1)N^2(N+1)^2(N+2)} \right. \\ & \left. - \frac{8P_{460}}{81(N-1)N^4(N+1)^4(N+2)} + \frac{16(328N^4 + 256N^3 - 247N^2 - 175N + 54)S_1}{81(N-1)N(N+1)^2} \right. \\ & \left. - \frac{448}{27} \frac{(N^2 + N + 1)\zeta_3}{(N-1)N(N+1)(N+2)} - \frac{8(2N+1)S_2}{3(N+1)} + \frac{560}{27} S_1\zeta_2 + \frac{224}{27} S_1\zeta_3 \right. \\ & + N_F \left[\left[\frac{128(N^2 + N + 1)}{27(N-1)N(N+1)(N+2)} - \frac{64S_1}{27} \right] L_M^3 + \left[-\frac{4P_{457}}{81(N-1)N^3(N+1)^3(N+2)} \right. \right. \\ & \left. \left. - \frac{16P_{443}S_1}{81(N-1)N^2(N+1)^2(N+2)} \right] L_M + \frac{16S_1^2}{9(N+1)} - \frac{4}{27} \frac{\zeta_2 P_{437}}{(N-1)N^2(N+1)^2(N+2)} \right. \\ & \left. + \frac{32P_{465}}{243(N-1)N^4(N+1)^4(N+2)} + \frac{32(328N^4 + 256N^3 - 247N^2 - 175N + 54)S_1}{243(N-1)N(N+1)^2} \right. \\ & \left. - \frac{128}{27} \frac{(N^2 + N + 1)\zeta_3}{(N-1)N(N+1)(N+2)} - \frac{16(2N+1)S_2}{9(N+1)} + \frac{160}{27} S_1\zeta_2 + \frac{64}{27} S_1\zeta_3 \right] \end{aligned}$$

$$\begin{aligned}
& + C_A^2 T_F \left[\left[\frac{176S_1}{27} - \frac{352(N^2 + N + 1)}{27(N-1)N(N+1)(N+2)} \right] L_M^3 + \left[-\frac{2P_{469}}{9(N-1)^2N^3(N+1)^3(N+2)^3} \right. \right. \\
& - \frac{8P_{452}S_1}{9(N-1)^2N^2(N+1)^2(N+2)^2} + \frac{64}{3}S_{-2}S_1 + \frac{64}{3}S_1S_2 + \frac{(N^2 + N + 1) \left[-\frac{128}{3}S_2 - \frac{128}{3}S_{-2} \right]}{(N-1)N(N+1)(N+2)} \\
& + \left. \frac{32S_3}{3} + \frac{32}{3}S_{-3} - \frac{64}{3}S_{-2,1} \right] L_M^2 + \left[\frac{32}{3}S_{-2}^2 - \frac{16P_{456}S_{-2}}{9(N-1)^2N^3(N+1)^3(N+2)^2} \right. \\
& + \frac{32P_{439}S_1S_{-2}}{9(N-1)N^2(N+1)^2(N+2)} + 64 \frac{(2N^4 + 4N^3 + 7N^2 + 5N + 6)\zeta_3}{(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{P_{479}}{81(N-1)^2N^4(N+1)^4(N+2)^3} - \frac{4P_{474}S_1}{81(N-1)^2N^4(N+1)^4(N+2)^2} + \left[\frac{640S_2}{9} - \frac{32S_3}{3} \right] S_1 \\
& + \frac{16P_{432}S_2}{9(N-1)N^2(N+1)^2(N+2)} + \frac{8(40N^4 + 80N^3 + 73N^2 + 33N + 54)S_3}{9N^2(N+1)^2} - 64S_1\zeta_3 \\
& + \frac{P_{438} \left(\frac{16}{9}S_{-3} - \frac{32}{9}S_{-2,1} \right)}{(N-1)N^2(N+1)^2(N+2)} \left] L_M - \frac{44S_1^2}{9(N+1)} - \frac{8P_{468}}{243(N-1)N^4(N+1)^4(N+2)} \right. \\
& + \frac{4}{27} \frac{P_{470}\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^3} - \frac{8(2834N^4 + 2042N^3 - 1943N^2 - 1151N + 594)S_1}{243(N-1)N(N+1)^2} \\
& + \frac{16}{27} \frac{P_{444}S_1\zeta_2}{(N-1)^2N^2(N+1)^2(N+2)^2} + \frac{44(2N+1)S_2}{9(N+1)} + \left[-\frac{32}{3}S_1S_2 - \frac{32}{3}S_{-2}S_1 - \frac{16S_3}{3} \right. \\
& - \left. \frac{16}{3}S_{-3} + \frac{32}{3}S_{-2,1} \right] \zeta_2 - \frac{176}{27}S_1\zeta_3 + \frac{(N^2 + N + 1) \left(\left(\frac{64S_2}{3} + \frac{64}{3}S_{-2} \right) \zeta_2 + \frac{352\zeta_3}{27} \right)}{(N-1)N(N+1)(N+2)} \left. \right] \\
& + C_F^2 T_F \left[\left[\frac{16(N^2 + N + 2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} - \frac{4(N^2 + N + 2)^2(3N^2 + 3N + 2)}{3(N-1)N^3(N+1)^3(N+2)} \right] L_M^3 \right. \\
& + \left[\frac{8(5N^2 + N - 2)S_1(N^2 + N + 2)^2}{(N-1)N^3(N+1)^3(N+2)} - \frac{16S_2(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \\
& - \left. \frac{4P_{436}(N^2 + N + 2)}{(N-1)N^4(N+1)^4(N+2)} \right] L_M^2 + \left[\frac{\left(-\frac{8}{3}S_1^3 + 24S_2S_1 - 32S_{2,1} \right) (N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \\
& + \frac{4(5N^3 + 4N^2 + 9N + 6)S_1^2(N^2 + N + 2)}{(N-1)N^2(N+1)^3(N+2)} + \frac{16(5N^2 + 5N - 14)S_3(N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{4(17N^4 + 48N^3 + 69N^2 + 10N - 8)S_2(N^2 + N + 2)}{(N-1)N^3(N+1)^3(N+2)} - \frac{96\zeta_3(N^2 + N + 2)}{N^2(N+1)^2} \\
& + \frac{\left(-256S_1S_{-2} - 128S_{-3} + 256S_{-2,1} \right) (N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} - \frac{2P_{471}}{(N-1)N^5(N+1)^5(N+2)} \\
& - \frac{8P_{448}S_1}{(N-1)N^4(N+1)^4(N+2)} - \frac{32(N^6 + 3N^5 + N^4 - 3N^3 - 26N^2 - 24N - 16)S_{-2}}{(N-1)N^3(N+1)^3(N+2)} \left. \right] L_M \\
& + \frac{4(N^2 + N + 2)(N^4 - 5N^3 - 32N^2 - 18N - 4)S_1^2}{(N-1)N^3(N+1)^3(N+2)} - \frac{4}{3} \frac{P_{453}\zeta_3}{(N-1)N^3(N+1)^3(N+2)} \\
& - 2 \frac{P_{464}\zeta_2}{(N-1)N^4(N+1)^4(N+2)} - \frac{8(N^2 + N + 2)(2N^5 - 2N^4 - 11N^3 - 19N^2 - 44N - 12)S_1}{(N-1)N^3(N+1)^4(N+2)} \\
& - \frac{P_{480}}{(N-1)N^6(N+1)^6(N+2)} - 4 \frac{(N^2 + N + 2)(5N^4 + 4N^3 + N^2 - 10N - 8)\zeta_2}{(N-1)N^3(N+1)^3(N+2)} S_1
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(N^2 + N + 2)P_{434}S_2}{(N-1)N^4(N+1)^4(N+2)} + \frac{(3N+2)(N^2+N+2)(\frac{8}{3}S_1^3 + 8S_2S_1)}{(N-1)N^3(N+1)(N+2)} + 128 \log(2)\zeta_2 \\
& + \frac{8(N^2+N+2)(3N^4+48N^3+43N^2-22N-8)S_3}{3(N-1)N^3(N+1)^3(N+2)} - \frac{32(N^2-3N-2)(N^2+N+2)S_{2,1}}{(N-1)N^3(N+1)^2(N+2)} \\
& + \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[-\frac{2}{3}S_1^4 - 4S_2S_1^2 + \left[-\frac{16}{3}S_3 - 32S_{2,1} \right] S_1 - \frac{16}{3}\zeta_3S_1 - 2S_2^2 + 12S_4 \right. \\
& \left. - 32S_{3,1} + 64S_{2,1,1} + \left[12S_2 - 4S_1^2 \right] \zeta_2 \right] \\
& + C_F T_F^2 \left[\frac{80(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} L_M^3 + \left[\frac{32(N^2+N+2)^2 S_1}{3(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. + \frac{8P_{449}}{9(N-1)N^3(N+1)^3(N+2)} \right] L_M^2 + \left[\frac{(\frac{16}{3}S_1^2 - 16S_2)(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \\
& \left. \left. - \frac{8P_{467}}{27(N-1)N^4(N+1)^4(N+2)} + \frac{32P_{435}S_1}{9(N-1)N^3(N+1)^3(N+2)} \right] L_M \right. \\
& \left. - \frac{8}{9} \frac{P_{450}\zeta_2}{(N-1)N^3(N+1)^3(N+2)} + \frac{2P_{475}}{9(N-1)N^5(N+1)^5(N+2)} \right. \\
& \left. + N_F \left[\frac{64(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} L_M^3 + \left[\frac{[16S_1^2 - \frac{80S_2}{3}](N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \right. \right. \right. \\
& \left. \left. - \frac{4P_{466}}{9(N-1)N^4(N+1)^4(N+2)} - \frac{32(5N^5+52N^4+109N^3+90N^2+48N+16)S_1}{3(N-1)N^3(N+1)^3(N+2)} \right] L_M \right. \\
& \left. + \frac{4}{9} \frac{P_{455}\zeta_2}{(N-1)N^3(N+1)^3(N+2)} - \frac{16(N^2+N+2)(8N^3+13N^2+27N+16)(S_1^2+S_2)}{9(N-1)N^2(N+1)^3(N+2)} \right. \\
& \left. + \frac{2P_{476}}{81(N-1)N^5(N+1)^5(N+2)} + \frac{32(N^2+N+2)(43N^4+105N^3+224N^2+230N+86)S_1}{27(N-1)N^2(N+1)^4(N+2)} \right. \\
& \left. + \frac{(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{16}{9}S_1^3 + \frac{16}{3}S_2S_1 + \frac{16}{3}\zeta_2S_1 + \frac{32S_3}{9} - \frac{64\zeta_3}{9} \right] \right. \\
& \left. + \frac{(N^2+N+2)^2(-\frac{16}{3}S_1\zeta_2 - \frac{80\zeta_3}{9})}{(N-1)N^2(N+1)^2(N+2)} \right] \\
& + C_A C_F T_F \left[-\frac{8(N^2+N+2)(N^3+8N^2+11N+2)S_1^3}{3(N-1)N^2(N+1)^3(N+2)^2} - \frac{4(N^2+N+2)P_{431}S_1^2}{(N-1)N^2(N+1)^4(N+2)^3} \right. \\
& \left. - \frac{4}{3} \frac{P_{463}\zeta_2}{(N-1)^2N^3(N+1)^3(N+2)^2} S_1 - \frac{2P_{473}S_1}{9(N-1)N^2(N+1)^5(N+2)^4} \right. \\
& \left. + \frac{8(N^2+N+2)(3N^3-12N^2-27N-2)S_2S_1}{(N-1)N^2(N+1)^3(N+2)^2} + \frac{8}{9} \frac{\zeta_3}{(N-1)^2N^3(N+1)^3(N+2)^2} P_{462} \right. \\
& \left. + \frac{4}{9} \frac{P_{477}\zeta_2}{(N-1)^2N^4(N+1)^4(N+2)^3} - \frac{P_{482}}{18(N-1)^2N^6(N+1)^6(N+2)^5} \right. \\
& \left. + L_M^3 \left[-\frac{8(11N^4+22N^3-23N^2-34N-12)(N^2+N+2)^2}{9(N-1)^2N^3(N+1)^3(N+2)^2} - \frac{16S_1(N^2+N+2)^2}{3(N-1)N^2(N+1)^2(N+2)} \right] \right. \\
& \left. - \frac{4(N^2+N+2)P_{458}S_2}{(N-1)^2N^4(N+1)^4(N+2)^3} - \frac{64(N^2+N+2)(N^5+10N^4+9N^3+3N^2+7N+6)S_3}{3(N-1)^2N^3(N+1)^3(N+2)^2} \right]
\end{aligned}$$

$$\begin{aligned}
& + L_M^2 \left[\frac{[-16S_2 - 32S_{-2}](N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} - \frac{2P_{478}}{9(N-1)^2N^4(N+1)^4(N+2)^3} \right. \\
& \left. - \frac{8P_{459}S_1}{3(N-1)^2N^3(N+1)^3(N+2)^2} \right] - 64 \log(2)\zeta_2 \\
& + \frac{(N^2 + N + 2)(N^4 + 2N^3 + 7N^2 + 22N + 20) [-32(-1)^N S_{-2} - 16(-1)^N \zeta_2]}{(N-1)N(N+1)^4(N+2)^3} \\
& + \frac{(N^2 - N - 4)(N^2 + N + 2)}{(N-1)N(N+1)^3(N+2)^2} \left[64(-1)^N S_1 S_{-2} + 32(-1)^N S_{-3} - 64S_{-2,1} + 32(-1)^N S_1 \zeta_2 \right. \\
& + 24(-1)^N \zeta_3 \left. \right] + \frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{2}{3} S_1^4 + 20S_2 S_1^2 + 32(-1)^N S_{-3} S_1 \right. \\
& + \left[\frac{160S_3}{3} - 64S_{-2,1} \right] S_1 + \frac{8}{3} (-7 + 9(-1)^N) \zeta_3 S_1 + 2S_2^2 + S_{-2} \left[32(-1)^N S_1^2 + 32(-1)^N S_2 \right] \\
& + 36S_4 + 16(-1)^N S_{-4} - 16S_{3,1} - 32S_{-2,2} - 32S_{-3,1} - 16S_{2,1,1} + 64S_{-2,1,1} + \left[4(-3 + 4(-1)^N) S_1^2 \right. \\
& + 4(-1 + 4(-1)^N) S_2 + 8(1 + 2(-1)^N) S_{-2} \left. \right] \zeta_2 \left. \right] + L_M \left[\frac{(N^2 + N + 2)^2}{(N-1)N^2(N+1)^2(N+2)} \left[\frac{8}{3} S_1^3 - 40S_2 S_1 \right. \right. \\
& - 32(-1)^N S_{-2} S_1 - 16(-1)^N S_{-3} + 32S_{2,1} - 12(-1)^N \zeta_3 \left. \right] - \frac{16(5N^2 + 5N - 26) S_3 (N^2 + N + 2)}{3(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{4(17N^4 - 6N^3 + 41N^2 - 16N - 12) S_1^2 (N^2 + N + 2)}{3(N-1)^2 N^3 (N+1)^2 (N+2)} + \frac{96(N^2 + N + 4) S_{-3} (N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{32(N^2 + N + 14) S_{-2,1} (N^2 + N + 2)}{(N-1)N^2(N+1)^2(N+2)} - \frac{(N^2 - N - 4) 32(-1)^N S_{-2} (N^2 + N + 2)}{(N-1)N(N+1)^3(N+2)^2} \\
& + \frac{8P_{481}}{27(N-1)^2 N^5 (N+1)^5 (N+2)^4} - \frac{4(N^2 + N + 10)(5N^2 + 5N + 18) \zeta_3}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{8P_{472} S_1}{9(N-1)^2 N^4 (N+1)^4 (N+2)^2} + \frac{64(N^4 + 2N^3 + 7N^2 + 6N + 16) S_{-2} S_1}{(N-1)N^2(N+1)^2(N+2)} \\
& \left. - \frac{4P_{446} S_2}{(N-1)^2 N^3 (N+1)^3 (N+2)^2} - \frac{32P_{447} S_{-2}}{(N-1)^2 N^3 (N+1)^3 (N+2)^2} + 64S_1 \zeta_3 \right] \left. \right] + a_{gg,Q}^{(3)} \left. \right\} , \quad (538)
\end{aligned}$$

with

$$P_{431} = N^6 + 6N^5 + 7N^4 + 4N^3 + 18N^2 + 16N - 8 \quad (539)$$

$$P_{432} = 3N^6 + 9N^5 - 113N^4 - 241N^3 - 274N^2 - 152N - 24 \quad (540)$$

$$P_{433} = 3N^6 + 9N^5 + 22N^4 + 29N^3 + 41N^2 + 28N + 6 \quad (541)$$

$$P_{434} = 3N^6 + 30N^5 + 15N^4 - 64N^3 - 56N^2 - 20N - 8 \quad (542)$$

$$P_{435} = 4N^6 + 3N^5 - 50N^4 - 129N^3 - 100N^2 - 56N - 24 \quad (543)$$

$$P_{436} = 7N^6 + 15N^5 + 7N^4 - 23N^3 - 26N^2 - 20N - 8 \quad (544)$$

$$P_{437} = 9N^6 + 27N^5 + 161N^4 + 277N^3 + 358N^2 + 224N + 48 \quad (545)$$

$$P_{438} = 20N^6 + 60N^5 + 11N^4 - 78N^3 - 121N^2 - 72N - 108 \quad (546)$$

$$P_{439} = 20N^6 + 60N^5 + 11N^4 - 78N^3 - 85N^2 - 36N - 108 \quad (547)$$

$$P_{440} = 40N^6 + 114N^5 + 19N^4 - 132N^3 - 147N^2 - 70N - 32 \quad (548)$$

$$P_{441} = 63N^6 + 189N^5 + 367N^4 + 419N^3 + 626N^2 + 448N + 96 \quad (549)$$

$$P_{442} = 99N^6 + 297N^5 + 631N^4 + 767N^3 + 1118N^2 + 784N + 168 \quad (550)$$

$$P_{443} = 136N^6 + 390N^5 + 19N^4 - 552N^3 - 947N^2 - 630N - 288 \quad (551)$$

$$\begin{aligned}
P_{444} &= N^8 + 4N^7 + 2N^6 + 64N^5 + 173N^4 + 292N^3 + 256N^2 - 72N - 72 & (552) \\
P_{445} &= N^8 + 4N^7 + 8N^6 + 6N^5 - 3N^4 - 22N^3 - 10N^2 - 8N - 8 & (553) \\
P_{446} &= 3N^8 - 14N^7 - 164N^6 - 454N^5 - 527N^4 - 204N^3 - 112N^2 + 80N + 48 & (554) \\
P_{447} &= 3N^8 + 10N^7 + 13N^6 + N^5 + 28N^4 + 81N^3 + 4N^2 - 12N - 32 & (555) \\
P_{448} &= 3N^8 + 23N^7 + 51N^6 + 95N^5 + 142N^4 + 158N^3 + 56N^2 - 32N - 16 & (556) \\
P_{449} &= 15N^8 + 60N^7 + 76N^6 - 18N^5 - 275N^4 - 546N^3 - 400N^2 - 224N - 96 & (557) \\
P_{450} &= 15N^8 + 60N^7 + 86N^6 + 12N^5 - 166N^4 - 378N^3 - 245N^2 - 148N - 84 & (558) \\
P_{451} &= 15N^8 + 60N^7 + 572N^6 + 1470N^5 + 2135N^4 + 1794N^3 + 722N^2 - 24N - 72 & (559) \\
P_{452} &= 23N^8 + 92N^7 + 46N^6 - 88N^5 + 79N^4 + 476N^3 + 428N^2 - 96N - 96 & (560) \\
P_{453} &= 24N^8 + 96N^7 + 93N^6 - 57N^5 - 143N^4 - 79N^3 - 34N^2 - 20N - 8 & (561) \\
P_{454} &= 27N^8 + 108N^7 - 1440N^6 - 4554N^5 - 5931N^4 - 3762N^3 - 256N^2 + 1184N + 480 & (562) \\
P_{455} &= 63N^8 + 252N^7 + 196N^6 - 258N^5 - 551N^4 - 282N^3 - 220N^2 - 80N + 48 & (563) \\
P_{456} &= 131N^8 + 524N^7 + 691N^6 + 239N^5 - 848N^4 - 1483N^3 - 586N^2 + 108N + 360 & (564) \\
P_{457} &= 297N^8 + 1188N^7 + 640N^6 - 2094N^5 - 1193N^4 + 2874N^3 + 5008N^2 + 3360N + 864 & (565) \\
P_{458} &= N^9 + 21N^8 + 85N^7 + 105N^6 + 42N^5 + 290N^4 + 600N^3 + 456N^2 + 256N + 64 & (566) \\
P_{459} &= 3N^{10} + 15N^9 + 35N^8 + 50N^7 + 91N^6 + 233N^5 + 255N^4 + 150N^3 - 24N^2 \\
&\quad - 184N - 48 & (567) \\
P_{460} &= 3N^{10} + 15N^9 + 3316N^8 + 12778N^7 + 22951N^6 + 23815N^5 + 14212N^4 + 3556N^3 \\
&\quad - 30N^2 + 288N + 216 & (568) \\
P_{461} &= 15N^{10} + 75N^9 + 112N^8 + 14N^7 - 61N^6 + 107N^5 + 170N^4 + 36N^3 - 36N^2 \\
&\quad - 32N - 16 & (569) \\
P_{462} &= 18N^{10} + 90N^9 + 119N^8 - 91N^7 - 167N^6 + 101N^5 + 162N^4 - 72N^3 - 504N^2 \\
&\quad - 184N - 48 & (570) \\
P_{463} &= 30N^{10} + 150N^9 + 163N^8 - 224N^7 - 586N^6 - 368N^5 - 39N^4 - 78N^3 + 144N^2 \\
&\quad + 184N + 48 & (571) \\
P_{464} &= 40N^{10} + 200N^9 + 282N^8 - 66N^7 - 615N^6 - 753N^5 - 509N^4 - 205N^3 - 2N^2 \\
&\quad + 68N + 24 & (572) \\
P_{465} &= 63N^{10} + 315N^9 - 1142N^8 - 6260N^7 - 11927N^6 - 12359N^5 - 7235N^4 - 1778N^3 \\
&\quad + 15N^2 - 144N - 108 & (573) \\
P_{466} &= 67N^{10} + 335N^9 + 368N^8 - 762N^7 - 3349N^6 - 6669N^5 - 8310N^4 - 7656N^3 \\
&\quad - 4648N^2 - 1600N - 288 & (574) \\
P_{467} &= 219N^{10} + 1095N^9 + 1640N^8 - 82N^7 - 2467N^6 - 2947N^5 - 3242N^4 - 4326N^3 \\
&\quad - 3466N^2 - 1488N - 360 & (575) \\
P_{468} &= 693N^{10} + 3465N^9 - 11014N^8 - 62668N^7 - 120361N^6 - 125113N^5 - 73393N^4 \\
&\quad - 18010N^3 + 165N^2 - 1584N - 1188 & (576) \\
P_{469} &= 3N^{11} + 21N^{10} - 124N^9 - 1014N^8 - 2185N^7 - 2099N^6 - 934N^5 - 2060N^4 - 4632N^3 \\
&\quad - 4256N^2 - 2688N - 768 & (577) \\
P_{470} &= 27N^{11} + 189N^{10} + 631N^9 + 1356N^8 + 2155N^7 + 2207N^6 + 211N^5 - 4984N^4 - 8400N^3 \\
&\quad - 5824N^2 - 2544N - 576 & (578) \\
P_{471} &= N^{12} + 6N^{11} - 5N^{10} - 80N^9 - 379N^8 - 846N^7 - 1057N^6 - 786N^5 + 84N^4 + 490N^3 \\
&\quad + 324N^2 + 152N + 48 & (579) \\
P_{472} &= 15N^{12} + 90N^{11} + 80N^{10} - 452N^9 - 1401N^8 - 2298N^7 - 5002N^6 - 6516N^5 - 1116N^4
\end{aligned}$$

$$+2960N^3 + 3464N^2 + 2400N + 864 \quad (580)$$

$$P_{473} = 233N^{12} + 2724N^{11} + 13349N^{10} + 34680N^9 + 46703N^8 + 12096N^7 - 69461N^6 - 137724N^5 - 141176N^4 - 91776N^3 - 34832N^2 - 6336N - 2304 \quad (581)$$

$$P_{474} = 310N^{12} + 2058N^{11} + 5939N^{10} + 17235N^9 + 44700N^8 + 93240N^7 + 140861N^6 + 113169N^5 + 14578N^4 - 40374N^3 - 33372N^2 - 12312N - 3888 \quad (582)$$

$$P_{475} = 391N^{12} + 2346N^{11} + 4795N^{10} + 2758N^9 - 2243N^8 + 1150N^7 + 7713N^6 + 4546N^5 - 792N^4 + 1224N^2 + 864N + 288 \quad (583)$$

$$P_{476} = 1593N^{12} + 9558N^{11} + 15013N^{10} - 8758N^9 - 62269N^8 - 82318N^7 - 79041N^6 - 90898N^5 - 70928N^4 - 15872N^3 + 7344N^2 + 5184N + 1728 \quad (584)$$

$$P_{477} = 15N^{13} + 120N^{12} + 530N^{11} + 1562N^{10} + 2042N^9 - 1680N^8 - 9220N^7 - 12524N^6 - 7911N^5 - 5230N^4 - 5880N^3 - 3344N^2 - 2544N - 864 \quad (585)$$

$$P_{478} = 33N^{13} + 264N^{12} + 479N^{11} - 1366N^{10} - 8809N^9 - 23124N^8 - 34351N^7 - 26198N^6 - 3624N^5 + 5240N^4 - 2496N^3 - 7232N^2 - 7104N - 2304 \quad (586)$$

$$P_{479} = 2493N^{13} + 19944N^{12} + 79295N^{11} + 208394N^{10} + 375431N^9 + 531516N^8 + 623697N^7 + 733338N^6 + 963340N^5 + 1047352N^4 + 895648N^3 + 559488N^2 + 222336N + 41472 \quad (587)$$

$$P_{480} = 39N^{14} + 273N^{13} + 741N^{12} + 1025N^{11} + 1343N^{10} + 3479N^9 + 6707N^8 + 6555N^7 + 2258N^6 - 1520N^5 - 1944N^4 - 532N^3 + 280N^2 + 208N + 32 \quad (588)$$

$$P_{481} = 276N^{16} + 3036N^{15} + 13660N^{14} + 30172N^{13} + 22446N^{12} - 50653N^{11} - 171627N^{10} - 246412N^9 - 204934N^8 - 83791N^7 + 28263N^6 + 43144N^5 - 33372N^4 - 82640N^3 - 79152N^2 - 47232N - 12096 \quad (589)$$

$$P_{482} = 3135N^{19} + 43890N^{18} + 257636N^{17} + 794084N^{16} + 1224418N^{15} + 244448N^{14} - 2371724N^{13} - 3594388N^{12} - 792201N^{11} + 2719198N^{10} + 2284064N^9 - 85568N^8 - 227344N^7 + 952768N^6 + 1160704N^5 + 807552N^4 + 574464N^3 + 305664N^2 + 104448N + 18432 \quad (590)$$

B The asymptotic Heavy Flavor Wilson Coefficients contributing to $F_2(x, Q^2)$ in z -space

The representation of the Wilson coefficients in momentum fraction or z -space of the contributions being known at present can be obtained in terms of harmonic polylogarithms [38] over the alphabet

$$\mathfrak{A} = \left\{ \frac{dz}{z}, \frac{dz}{1-z}, \frac{dz}{1+z} \right\} \equiv \{f_0(z), f_1(z), f_{-1}(z)\} \quad (591)$$

as iterated integrals

$$H_{b,\bar{a}}(z) = \int_0^z f_b(y) H_{\bar{a}}(y), \quad f_b \in \mathfrak{A}, \quad H_\emptyset = 1, \quad H_{\underbrace{0,\dots,0}_k}(z) := \frac{1}{k!} \ln^k(z). \quad (592)$$

For brevity we will drop the argument z of the harmonic polylogarithms in the following. As a shorthand notation we introduce the Mellin-inversion of (36)

$$\gamma_{qg} = -4 [z^2 + (1-z)^2], \quad (593)$$

where the Mellin transform is defined by

$$\mathbf{M}[f(x)](N) = \int_0^1 dx x^{N-1} f(x) . \quad (594)$$

The Wilson coefficients $L_{g,2}^{\text{PS}}$ and $L_{g,2}^{\text{S}}$ are given in z -space by :

$$\begin{aligned} L_{g,2}^{\text{PS}} = & \alpha_s^3 \left\{ C_F N_F T_F^2 \left[\left[\frac{64}{9}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{27z} \right] L_Q^3 + \left[-\frac{64}{3}(z+1)H_0^2 \right. \right. \right. \\ & + \frac{64}{9}(4z^2-11z-8)H_0 + \left. \left. \frac{32(z-1)(10z^2+33z-2)}{9z} \right] L_Q^2 + L_Q \left[\frac{64}{3}H_0^3(z+1) \right. \right. \\ & + \left. \left. \frac{\left[\frac{256}{9}H_{0,-1} - \frac{256}{9}H_{-1}H_0 \right](z+1)^3}{z} - \frac{32}{9}(12z^2-59z-29)H_0^2 \right. \right. \\ & + \left. \left. \left[\frac{64}{3}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{9z} \right] L_M^2 - \frac{64(z-1)(304z^2+811z+124)}{81z} \right. \right. \\ & - \frac{64}{27}(60z^2-155z-233)H_0 - \frac{256(3z^2+1)\zeta_2}{9z} + L_M \left[-\frac{64(z-1)(38z^2+47z+20)}{27z} \right. \\ & + \left. \left. \frac{128}{9}(2z^2+11z+8)H_0 + \frac{64(z-1)(4z^2+7z+4)H_1}{9z} + (z+1) \left[\frac{128\zeta_2}{3} - \frac{128}{3}H_{0,1} \right] \right] \right] \\ & + L_M^3 \left[\frac{32(z-1)(4z^2+7z+4)}{27z} - \frac{64}{9}(z+1)H_0 \right] - \frac{32}{81}(57z^2+367z+295)H_0^2 \\ & - \frac{16(z-1)(118z^2-107z+118)H_1^2}{81z} + L_M^2 \left[-\frac{64}{3}(z+1)H_0^2 + \frac{64}{9}(4z^2-11z-8)H_0 \right. \\ & + \left. \frac{32(z-1)(10z^2+33z-2)}{9z} \right] + \frac{64(z-1)(3821z^2+6698z+500)}{729z} \\ & - \frac{32}{27}(68z^3+53z^2+5z-32)\frac{\zeta_3}{z} - \frac{32}{243}(48z^2+5087z+2783)H_0 \\ & - \frac{32(z-1)(256z^2+521z-14)H_1}{81z} - \frac{32(z-1)(38z^2+47z+20)H_0H_1}{27z} \\ & - \frac{64(42z^3-227z^2-74z+30)H_{0,1}}{81z} + \frac{64}{9}(2z^2+11z+8)H_0H_{0,1} \\ & - \frac{64}{27}(6z^2+4z-5)H_{0,0,1} + \frac{128(9z^3+7z^2+7z+3)H_{0,1,1}}{27z} \\ & + \frac{32}{81}(198z^2-427z-229)\zeta_2 - \frac{64}{27}(6z^2+62z+53)H_0\zeta_2 + \frac{(z-1)(4z^2+7z+4)}{z} \left[\frac{64}{81}H_1^3 \right. \\ & + \frac{16}{9}H_0H_1^2 - \frac{32}{9}\zeta_2H_1 \left. \right] + L_M \left[\frac{256\zeta_2z^2}{3} - \frac{16}{9}(8z^2+73z+61)H_0^2 \right. \\ & + \frac{64}{27}(28z^2-221z-107)H_0 + (z+1) \left[-\frac{32}{9}H_0^3 + \frac{128}{3}H_{0,1}H_0 - \frac{256}{3}\zeta_2H_0 + \frac{128}{3}H_{0,1,1} \right. \\ & - \left. \frac{128\zeta_3}{3} \right] + \frac{32(z-1)(832z^2+1213z+76)}{81z} + \frac{128(z-1)(4z^2-26z+13)H_1}{27z} \\ & + \left. \frac{(z-1)(4z^2+7z+4) \left[-\frac{32}{9}H_1^2 - \frac{64}{9}H_0H_1 \right]}{z} - \frac{64(8z^3-3z^2+3z+4)H_{0,1}}{9z} \right] \end{aligned}$$

$$\begin{aligned}
& +(z+1) \left[-\frac{8}{27}H_0^4 - \frac{464}{81}H_0^3 - \frac{64}{3}H_{0,1,1}H_0 + \frac{832}{9}\zeta_3H_0 - \frac{224\zeta_2^2}{15} + \frac{128}{9}H_{0,0,0,1} \right. \\
& \left. - \frac{64}{9}H_{0,0,1,1} - \frac{256}{9}H_{0,1,1,1} + \left(\frac{64}{3}H_{0,1} - \frac{64}{9}H_0^2 \right) \zeta_2 \right] + N_F \hat{C}_{2,q}^{\text{PS},(3)}(N_F) \Big\} \quad (595)
\end{aligned}$$

and

$$L_{g,2}^S =$$

$$\begin{aligned}
& \alpha_s^2 N_F T_F^2 \left\{ \left[\gamma_{qg}^0 \left[\frac{4H_0}{3} + \frac{4H_1}{3} \right] - \frac{16}{3}(8z^2 - 8z + 1) \right] L_M - \frac{4}{3} \gamma_{qg}^0 L_Q L_M \right\} \\
& + \alpha_s^3 \left\{ N_F T_F^3 \left[\left(\gamma_{qg}^0 \left[\frac{16H_0}{9} + \frac{16H_1}{9} \right] - \frac{64}{9}(8z^2 - 8z + 1) \right) L_M^2 - \frac{16}{9} \gamma_{qg}^0 L_Q L_M^2 \right] \right\} \\
& + C_A N_F T_F^2 \left[-\frac{4}{27}(30z - 13)H_0^4 - \frac{8}{81}(32z^2 + 628z - 169)H_0^3 - \frac{16}{81}(532z^2 + 2586z - 193)H_0^2 \right. \\
& - \frac{64}{27}(7z^2 + 7z + 5)H_{-1}H_0^2 + \frac{16}{3}(2z - 1)\zeta_2 H_0^2 + \frac{8}{243}(4641z^2 - 67330z + 3473)H_0 \\
& + \frac{32}{27}(59z^2 - 50z + 25)H_1H_0 + \frac{64}{27}(14z^2 + 11z + 10)H_{0,-1}H_0 - \frac{64}{27}(7z^2 - 4z + 5)H_{0,1}H_0 \\
& + \frac{16}{9}(62z - 19)\zeta_2 H_0 + \frac{32}{9}(38z + 5)\zeta_3 H_0 + L_M^3 \left[\frac{16(z-1)(31z^2 + 7z + 4)}{27z} - \frac{32}{9}(4z + 1)H_0 \right. \\
& \left. - \frac{8}{9} \gamma_{qg}^0 H_1 \right] + \frac{16}{81}(172z^2 - 163z + 56)H_1^2 - \frac{64}{45}(8z + 3)\zeta_2^2 \\
& + \frac{8(276317z^3 - 271875z^2 + 11280z - 6182)}{729z} - \frac{16}{27}(248z^3 - 438z^2 + 33z - 32) \frac{\zeta_3}{z} \\
& + \frac{8(28805z^3 - 28460z^2 + 4612z - 1596)H_1}{243z} + L_Q^3 \left[-\frac{16(z-1)(31z^2 + 7z + 4)}{27z} \right. \\
& \left. + \frac{32}{9}(4z + 1)H_0 + \frac{8}{9} \gamma_{qg}^0 H_1 \right] + (200z^2 + 191z + 112) \left[\frac{32}{81}H_{0,-1} - \frac{32}{81}H_{-1}H_0 \right] \\
& - \frac{32}{27}(98z^2 + 347z + 16)H_{0,1} - \frac{128}{27}(7z^2 + 4z + 5)H_{0,0,-1} + \frac{16}{27}(28z^2 - 202z + 77)H_{0,0,1} \\
& + \frac{32}{9}(14z^2 - 15z + 10)H_{0,1,1} + (2z^2 + 2z + 1) \left[-\frac{64}{27}H_{-1}H_0^3 + \frac{64}{9}H_{0,-1}H_0^2 - \frac{128}{9}H_{0,0,-1}H_0 \right. \\
& \left. + \frac{128}{9}H_{0,0,0,-1} \right] - \frac{64}{3}z^2 H_{0,0,0,1} + \frac{32}{81}(117z^2 + 1000z - 27)\zeta_2 + L_M^2 \left[-\frac{16}{3}(8z + 1)H_0^2 \right. \\
& + \frac{16}{9}(49z^2 - 136z - 13)H_0 + \frac{8(1048z^3 - 894z^2 - 87z - 40)}{27z} + \frac{16(87z^3 - 80z^2 + 13z - 4)H_1}{9z} \\
& + \gamma_{qg}^0 \left[-\frac{4}{3}H_1^2 - \frac{8}{3}H_0H_1 \right] + (2z^2 + 2z + 1) \left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0 \right] - \frac{32}{3}(4z + 1)H_{0,1} \\
& - \frac{64}{3}(z - 2)z\zeta_2 + L_Q^2 \left[-\frac{16}{3}(8z + 1)H_0^2 + \frac{16}{9}(49z^2 - 136z - 13)H_0 \right. \\
& + \frac{8(1048z^3 - 894z^2 - 87z - 40)}{27z} + \frac{16(87z^3 - 80z^2 + 13z - 4)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{4}{3}H_1^2 \right. \\
& \left. - \frac{8}{3}H_0H_1 \right] + (2z^2 + 2z + 1) \left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0 \right] - \frac{32}{3}(4z + 1)H_{0,1}
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{16(z-1)(31z^2+7z+4)}{9z} + \frac{32}{3}(4z+1)H_0 + \frac{8}{3}\gamma_{qg}^0 H_1 \right] L_M - \frac{64}{3}(z-2)z\zeta_2 \Big] \\
& + (7z^2-7z+5) \left[\frac{32}{81}H_1^3 + \frac{32}{27}H_0H_1^2 + \frac{32}{27}H_0^2H_1 - \frac{128}{27}H_{0,1}H_1 + \frac{64}{27}\zeta_2H_1 \right] \\
& + L_Q \left[\frac{448}{9}zH_0^3 - \frac{16}{9}(63z^2-296z-10)H_0^2 + \frac{32}{3}(5z^2+4z+2)H_{-1}H_0^2 \right. \\
& - \frac{32}{3}(3z^2-4z+2)H_1H_0^2 - \frac{64}{27}(325z^2-832z-31)H_0 - \frac{64}{3}(3z^2+4z+2)H_{0,-1}H_0 \\
& \left. + \frac{64}{3}(z^2+2z+2)H_{0,1}H_0 + \frac{128}{3}(z-5)z\zeta_2H_0 + \left[-\frac{16(z-1)(31z^2+7z+4)}{9z} \right. \right. \\
& \left. \left. + \frac{32}{3}(4z+1)H_0 + \frac{8}{3}\gamma_{qg}^0 H_1 \right] L_M^2 - \frac{8(25657z^3-23556z^2-969z-412)}{81z} \right. \\
& \left. + \frac{32}{9}(105z^3-184z^2+3z-4)\frac{\zeta_2}{z} - \frac{32(1032z^3-964z^2+65z-20)H_1}{27z} \right. \\
& \left. + \frac{(87z^3-80z^2+13z-4)\left[-\frac{16}{9}H_1^2 - \frac{32}{9}H_0H_1\right]}{z} + \frac{(17z^3+7z^2-4z-2)\left[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1}\right]}{z} \right. \\
& - \frac{64(9z^3-59z^2-5z+2)H_{0,1}}{9z} + \gamma_{qg}^0 \left[\frac{8}{9}H_1^3 + \frac{16}{3}H_0H_1^2 - \frac{32}{3}H_{0,1}H_1 \right] + (2z+1) \left[\frac{32}{3}H_0H_{-1}^2 \right. \\
& \left. - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1} \right] + \frac{64}{3}(z^2+4z+2)H_{0,0,-1} - \frac{64}{3}(z^2-2z+2)H_{0,0,1} \\
& + (2z^2+2z+1) \left[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1} \right] - \frac{128}{3}(3z^2-5z+1)H_{0,1,1} \\
& - \frac{32}{3}(4z^2+2z+1)H_{-1}\zeta_2 - \frac{32}{3}(2z-1)H_1\zeta_2 + L_M \left[-\frac{32}{3}(4z+3)H_0^2 \right. \\
& \left. + \frac{32}{9}(101z^2-8z+13)H_0 - \frac{32(8z^3-14z^2+11z-8)}{3z} + \frac{32(47z^3-40z^2-7z-4)H_1}{9z} \right. \\
& \left. + \gamma_{qg}^0 \left[-\frac{8}{3}H_1^2 - \frac{16}{3}H_0H_1 \right] + (2z^2+2z+1) \left[\frac{64}{3}H_{0,-1} - \frac{64}{3}H_{-1}H_0 \right] - \frac{64}{3}(4z+1)H_{0,1} \right. \\
& \left. - \frac{128}{3}(z-2)z\zeta_2 \right] + \frac{32}{3}(6z^2-22z+3)\zeta_3 \Big] + L_M \left[\frac{160}{9}H_0^3 - \frac{16}{9}(131z^2+32z+13)H_0^2 \right. \\
& - \frac{32}{3}(z-1)^2H_1H_0^2 + \frac{8}{3}\gamma_{qg}^0 H_1^2H_0 - \frac{16}{27}(880z^2+1486z-353)H_0 - \frac{32(47z^3-40z^2-7z-4)H_1H_0}{9z} \\
& - \frac{64}{3}(z^2-4z-1)H_{0,1}H_0 + \frac{128}{3}(z^2-3z-1)\zeta_2H_0 - \frac{16(35z^3-28z^2-7z-4)H_1^2}{9z} \\
& \left. + \frac{8(4933z^3-5084z^2+113z+92)}{27z} + \frac{32}{9}(117z^3-32z^2+9z-4)\frac{\zeta_2}{z} \right. \\
& \left. + \frac{16(404z^3-476z^2+271z-144)H_1}{27z} + \frac{(3z^3-7z^2-14z-2)\left[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1}\right]}{z} \right. \\
& - \frac{64(35z^3+11z^2+8z+2)H_{0,1}}{9z} + (2z+1) \left[\frac{32}{3}H_0H_{-1}^2 - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1} \right] \\
& + z^2 \left[\frac{32}{3}H_{-1}H_0^2 + \frac{64}{3}H_{0,-1}H_0 - 64H_{0,0,-1} \right] + \frac{64}{3}(z^2-4z+1)H_{0,0,1} + (2z^2+2z+1) \left[\frac{64}{3}H_{-1}H_{0,1} \right. \\
& \left. - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1} \right] + \frac{64}{3}(4z+1)H_{0,1,1} - \frac{32}{3}(4z^2+2z+1)H_{-1}\zeta_2 + \frac{32}{3}(4z^2-6z+3)H_1\zeta_2
\end{aligned}$$

$$\begin{aligned}
& + \frac{32}{3}(6z^2 - 2z - 1)\zeta_3 \Big] + \gamma_{qg}^0 \left[\frac{1}{27}H_1^4 - \frac{8}{9}H_{0,1}H_1^2 - \frac{4}{9}H_0^3H_1 + \left[\frac{16}{9}H_{0,0,1} + \frac{16}{9}H_{0,1,1} \right]H_1 \right. \\
& + \frac{16}{3}\zeta_3H_1 - \frac{8}{9}H_{0,1}^2 + H_0^2 \left[\frac{4}{3}H_{0,1} - \frac{2}{9}H_1^2 \right] + H_0 \left[\frac{4}{27}H_1^3 - \frac{8}{3}H_{0,0,1} + \frac{8}{9}H_{0,1,1} \right] - \frac{8}{9}H_{0,0,1,1} \\
& - \frac{8}{9}H_{0,1,1,1} + \left. \left[\frac{4}{9}H_1^2 - \frac{8}{9}H_0H_1 + \frac{8}{9}H_{0,1} \right] \zeta_2 \right] + C_F N_F T_F^2 \left[-\frac{2}{27}(56z^2 + 448z - 179)H_0^4 \right. \\
& + \frac{4}{81}(288z^2 - 6524z + 3259)H_0^3 - \frac{4}{81}(4096z^2 + 23771z - 21328)H_0^2 + 8(12z - 5)\zeta_2H_0^2 \\
& + \frac{56}{243}(1491z^2 - 4715z + 17578)H_0 - \frac{448}{81}(z^2 - z + 2)H_1H_0 + 112(5z - 2)\zeta_2H_0 \\
& + \frac{16}{9}(96z^2 + 92z - 109)\zeta_3H_0 + \left[-\frac{16}{3}(2z - 1)H_0^2 - \frac{32}{9}(6z^2 - z - 4)H_0 \right. \\
& + \left. \frac{8(124z^3 - 258z^2 + 159z - 16)}{27z} + \frac{8}{9}\gamma_{qg}^0H_1 \right] L_M^3 + \frac{8}{81}(364z^2 - 373z + 224)H_1^2 \\
& - \frac{32}{45}(4z^2 - 112z + 47)\zeta_2^2 + \frac{2540132z^3 - 7301946z^2 + 4812411z + 34912}{729z} \\
& - \frac{16}{27}(468z^3 - 1949z^2 + 994z - 64)\frac{\zeta_3}{z} + \frac{4(43898z^3 - 106070z^2 + 58429z + 1296)H_1}{243z} \\
& + L_Q^3 \left[\frac{16}{3}(2z - 1)H_0^2 + \frac{16}{9}(4z - 11)H_0 - \frac{16(62z^3 - 147z^2 + 84z - 8)}{27z} + \frac{16}{9}\gamma_{qg}^0H_1 \right] \\
& - \frac{16}{81}(580z^2 + 797z - 3196)H_{0,1} + (16z^2 - 16z + 5) \left[\frac{16}{27}H_0^2H_1 - \frac{32}{27}H_0H_{0,1} \right] \\
& + \frac{16}{27}(32z^2 - 977z + 388)H_{0,0,1} + (7z^2 - 7z + 5) \left[-\frac{32}{81}H_1^3 - \frac{64}{27}H_{0,1,1} \right] \\
& - \frac{16}{9}(8z^2 + 100z - 41)H_{0,0,0,1} + \frac{16}{81}(608z^2 + 769z - 3140)\zeta_2 + \gamma_{qg}^0 \left[-\frac{1}{27}H_1^4 - \frac{8}{27}H_0^3H_1 \right. \\
& - \frac{64}{9}\zeta_3H_1 + \frac{8}{9}H_0^2H_{0,1} - \frac{16}{9}H_0H_{0,0,1} - \frac{8}{9}H_{0,1,1,1} \Big] + L_Q^2 \left[-\frac{8}{3}(8z^2 + 56z - 31)H_0^2 \right. \\
& + \frac{8}{9}(244z^2 - 236z + 571)H_0 + \frac{4(2156z^3 - 7632z^2 + 4977z + 256)}{27z} \\
& + \frac{16(130z^3 - 215z^2 + 112z - 8)H_1}{9z} + \gamma_{qg}^0 \left[-4H_1^2 - \frac{32}{3}H_0H_1 \right] - \frac{16}{3}(12z^2 - 8z - 5)H_{0,1} \\
& + \left. \left[-16(2z - 1)H_0^2 - \frac{32}{3}(6z^2 - z - 4)H_0 + \frac{8(124z^3 - 258z^2 + 159z - 16)}{9z} + \frac{8}{3}\gamma_{qg}^0H_1 \right] L_M \right. \\
& - \left. \frac{16}{3}(4z^2 - 8z + 13)\zeta_2 + (2z - 1) \left[-16H_0^3 + 32\zeta_2H_0 - 32H_{0,0,1} + 32\zeta_3 \right] \right] \\
& + L_M^2 \left[-8(8z^2 + 16z - 9)H_0^2 + \frac{8}{9}(92z^2 - 160z + 521)H_0 - \frac{4}{3}\gamma_{qg}^0H_1^2 \right. \\
& + \frac{4(2372z^3 - 7140z^2 + 4611z + 256)}{27z} + \frac{16(54z^3 - 127z^2 + 71z - 8)H_1}{9z} + (4z^2 + 4z - 11) \left[\frac{16\zeta_2}{3} \right. \\
& - \left. \frac{16}{3}H_{0,1} \right] + (2z - 1) \left[-16H_0^3 + 32\zeta_2H_0 - 32H_{0,0,1} + 32\zeta_3 \right] \Big] + L_Q \left[\frac{64}{3}(2z^2 + 15z - 5)H_0^3 \right. \\
& - \left. \frac{16}{45}(72z^3 + 560z^2 - 2660z + 2975)H_0^2 - \frac{64}{3}(5z^2 - 4z + 2)H_1H_0^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{16(1444z^3 - 5632z^2 + 9213z + 4)H_0}{45z} - \frac{32(198z^3 - 283z^2 + 140z - 8)H_1H_0}{9z} \\
& + \frac{128}{3}(z-2)(3z-2)H_{0,-1}H_0 + \frac{32}{3}(16z^2 - 8z - 5)H_{0,1}H_0 - \frac{32}{3}(4z^2 + 84z - 33)\zeta_2H_0 \\
& - \frac{16(168z^3 - 253z^2 + 131z - 8)H_1^2}{9z} + \left[16(2z-1)H_0^2 + \frac{16}{3}(8z^2 - 9)H_0 \right. \\
& \left. - \frac{16(z-1)(62z^2 - 73z + 8)}{9z} \right] L_M^2 - \frac{4(269954z^3 - 828996z^2 + 567861z - 11744)}{405z} \\
& + \frac{16}{45}(144z^4 + 1600z^3 - 1400z^2 + 3045z - 80)\frac{\zeta_2}{z} - \frac{8(3080z^3 - 8448z^2 + 5247z + 256)H_1}{27z} \\
& + \frac{(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} \\
& + \frac{16(76z^3 - 254z^2 - 329z - 16)H_{0,1}}{9z} + \gamma_{qg}^0 \left[\frac{8}{3}H_1^3 + 8H_0H_1^2 + \frac{32}{3}H_{0,1}H_1 \right] \\
& - \frac{128}{3}(7z^2 - 14z + 9)H_{0,0,-1} - \frac{32}{3}(8z^2 - 60z + 15)H_{0,0,1} + \frac{32}{3}(24z^2 - 20z + 1)H_{0,1,1} \\
& + \frac{64}{3}(8z^2 - 6z + 3)H_1\zeta_2 + (z+1)^2 \left[-\frac{128}{3}H_0H_{-1}^2 + \left(\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1}\right)H_{-1} - \frac{128}{3}\zeta_2H_{-1} \right. \\
& \left. - \frac{256}{3}H_{0,-1,-1} \right] + L_M \left[\frac{16}{3}(32z^2 + 44z - 25)H_0^2 - \frac{32}{3}(8z^2 - 21z + 89)H_0 \right. \\
& \left. - \frac{16(1006z^3 - 3333z^2 + 2208z + 128)}{27z} - \frac{16(64z^3 - 198z^2 + 141z - 16)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{16}{3}H_1^2 \right. \right. \\
& \left. \left. - \frac{32}{3}H_0H_1 \right] + \frac{64}{3}(z-1)(4z+5)H_{0,1} - \frac{64}{3}(8z^2 - 3z - 3)\zeta_2 + (2z-1) \left[32H_0^3 - 64\zeta_2H_0 + 64H_{0,0,1} \right. \right. \\
& \left. \left. - 64\zeta_3 \right] \right] + \frac{128}{3}(4z^2 - 21z + 11)\zeta_3 + (2z-1) \left[16H_0^4 - 96\zeta_2H_0^2 + 64H_{0,0,1}H_0 - 128\zeta_3H_0 - \frac{32\zeta_2^2}{5} \right. \\
& \left. + 64H_{0,0,1,1} \right] \left. \right] + L_M \left[-\frac{64}{9}(14z^2 + 41z - 13)H_0^3 - \frac{16}{45}(72z^3 + 360z^2 + 2390z - 2835)H_0^2 \right. \\
& \left. - \frac{64}{3}z^2H_1H_0^2 - \frac{8(524z^3 + 29468z^2 - 50797z + 24)H_0}{135z} + \frac{16(28z^3 - 162z^2 + 123z - 16)H_1H_0}{9z} \right. \\
& \left. - \frac{128}{3}(z^2 - 4z + 2)H_{0,-1}H_0 - 64(z-1)(2z+1)H_{0,1}H_0 + \frac{32}{3}(36z^2 + 64z - 23)\zeta_2H_0 \right. \\
& \left. + \frac{8(44z^3 - 178z^2 + 143z - 16)H_1^2}{9z} + \frac{4(259856z^3 - 763164z^2 + 514989z - 11456)}{405z} \right. \\
& \left. + \frac{32}{45}(72z^4 - 70z^3 + 345z^2 - 1340z + 40)\frac{\zeta_2}{z} + \frac{16(794z^3 - 3157z^2 + 2114z + 128)H_1}{27z} \right. \\
& \left. + \frac{(36z^5 + 155z^4 + 40z^3 - 45z^2 + 20z + 1)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} + \frac{16(56z^2 + 413z + 16)H_{0,1}}{9z} \right. \\
& \left. + \gamma_{qg}^0 \left[\frac{8}{3}H_1^3 + \frac{16}{3}H_0H_1^2 + \frac{16}{3}H_{0,1}H_1 \right] + \frac{128}{3}(z^2 - 10z + 3)H_{0,0,-1} - \frac{32}{3}(8z^2 + 52z - 11)H_{0,0,1} \right. \\
& \left. - \frac{64}{3}(2z^2 + 3z - 6)H_{0,1,1} + \frac{64}{3}(4z^2 - 2z + 1)H_1\zeta_2 + (z+1)^2 \left[-\frac{128}{3}H_0H_{-1}^2 + \left(\frac{64}{3}H_0^2 \right. \right. \right. \\
& \left. \left. + \frac{256}{3}H_{0,-1}\right)H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,-1} \right] + \frac{32}{3}(24z^2 + 82z - 27)\zeta_3 \\
& \left. + (2z-1) \left[-16H_0^4 + 96\zeta_2H_0^2 - 64H_{0,0,1}H_0 + 128\zeta_3H_0 + \frac{32\zeta_2^2}{5} - 64H_{0,0,1,1} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + (2z-1) \left[-\frac{4}{3}H_0^5 + \frac{16}{3}\zeta_2 H_0^3 + \frac{176}{3}\zeta_3 H_0^2 + \frac{64}{5}\zeta_2^2 H_0 - 32H_{0,0,0,0,1} \right. \\
& \left. + 32\zeta_5 \right] + N_F \hat{C}_{2,g}^{S,(3)}(N_F) \Big\} \tag{596}
\end{aligned}$$

The pure-singlet Wilson coefficient $H_{q,2}^{\text{PS}}$ reads :

$$\begin{aligned}
H_{q,2}^{\text{PS}} = & a_s^2 C_F T_F \left\{ \frac{\left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0 \right] (z+1)^3}{z} + \left[\frac{16}{3}H_0^3 + 32H_{0,1}H_0 - 32\zeta_2 H_0 - 32H_{0,0,1} \right. \right. \\
& + 16H_{0,1,1} + 16\zeta_3 \Big] (z+1) - 8z(2z-5)H_0^2 + \left[\frac{4(z-1)(4z^2+7z+4)}{3z} - 8(z+1)H_0 \right] L_M^2 \\
& + \frac{16(z-1)(52z^2-24z-5)}{9z} + \frac{32}{3}(3z^3-3z^2-1) \frac{\zeta_2}{z} - \frac{8}{9}(88z^2+99z-105)H_0 \\
& + L_Q^2 \left[8(z+1)H_0 - \frac{4(z-1)(4z^2+7z+4)}{3z} \right] + \frac{16(z-1)(4z^2-26z+13)H_1}{9z} \\
& + \frac{(z-1)(4z^2+7z+4)}{z} \left[-\frac{4}{3}H_1^2 - \frac{16}{3}H_0H_1 \right] - \frac{16(2z^3-3z^2+3z+4)H_{0,1}}{3z} \\
& + \left[8(z+1)H_0^2 - \frac{8}{3}(8z^2+15z+3)H_0 + \frac{16(z-1)(28z^2+z+10)}{9z} \right] L_M \\
& + L_Q \left[32H_0z^2 + (z+1) \left[-16H_0^2 - 16H_{0,1} + 16\zeta_2 \right] - \frac{16(z-1)(4z^2-26z+13)}{9z} \right. \\
& \left. + \frac{8(z-1)(4z^2+7z+4)H_1}{3z} \right] \Big\} \\
& + a_s^3 \left\{ C_F^2 T_F \left[\left[\frac{32(4z^2+7z+4)H_1(z-1)}{9z} + \frac{184(z-1)}{9} + \frac{32}{9}z(4z+3)H_0 \right. \right. \right. \\
& + (z+1) \left[-\frac{16}{3}H_0^2 - \frac{64}{3}H_{0,1} + \frac{64\zeta_2}{3} \right] L_Q^3 + \left[-\frac{8}{3}(28z^2+3z+6)H_0^2 \right. \\
& - \frac{8}{9}(4z^2+315z-198)H_0 + \frac{4(z-1)(120z^2-289z-36)}{9z} \\
& - \frac{8(z-1)(4z^2+337z-32)H_1}{9z} + \frac{(z-1)(4z^2+7z+4) \left[-8H_1^2 - 16H_0H_1 \right]}{z} \\
& - \frac{16(16z^3-15z^2-9z+12)H_{0,1}}{3z} + \frac{32}{3}(14z^2-3z-9)\zeta_2 \\
& \left. \left. \left. + (z+1) \left[16H_0^3 + 96H_{0,1}H_0 - 160\zeta_2 H_0 - 32H_{0,0,1} + 96H_{0,1,1} - 64\zeta_3 \right] L_Q^2 \right. \right. \right. \\
& + \left[\frac{(z+1)^3}{z} \left[\frac{128}{3}H_{-1}H_{0,1} - \frac{128}{3}H_{0,-1,1} - \frac{128}{3}H_{0,1,-1} \right] - \frac{64}{3}(z^2+14z+1) \frac{\zeta_2}{z} H_{-1}(z+1) \right. \\
& + \frac{(36z^4-36z^3-1069z^2+4z-4)(z+1)}{z^2} \left[\frac{32}{45}H_{0,-1} - \frac{32}{45}H_{-1}H_0 \right] \\
& + \frac{(z^2-10z+1)(z+1)}{z} \left[\frac{64}{3}H_0H_{-1}^2 - \frac{128}{3}H_{0,-1}H_{-1} + \frac{128}{3}H_{0,-1,-1} \right] + \left[-14H_0^4 \right. \\
& \left. \left. \left. + 128H_{-1}H_0^2 - 64H_{0,1}H_0^2 + \left[-224H_{0,0,1} - 320H_{0,1,1} \right] H_0 - 384H_{0,0,0,-1} \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +480H_{0,0,0,1} + 192H_{0,0,1,1} - 224H_{0,1,1,1} + \left[288H_0^2 + 192H_{0,1}\right]\zeta_2](z+1) \\
& + \frac{16}{9}(52z^2 - 22z + 15)H_0^3 + \frac{2}{45}(288z^3 + 3680z^2 + 1525z - 7695)H_0^2 \\
& - \frac{4(z-1)(8z^2 - 305z + 20)H_1^2}{3z} - \frac{32}{5}(33z + 23)\zeta_2^2 - \frac{4(z-1)(19536z^2 - 1103z + 4056)}{135z} \\
& - \frac{8(6298z^3 - 32859z^2 + 606z + 48)H_0}{135z} + \frac{64(z-1)(z^2 + z + 1)H_0^2H_1}{z} \\
& - \frac{8(z-1)(986z^2 - 4747z - 292)H_1}{27z} - 32(z-1)(6z^2 + 5z + 6)\frac{\zeta_2}{z}H_1 \\
& - \frac{8(z-1)(32z^2 - 299z + 32)H_0H_1}{3z} + \frac{(z-1)(4z^2 + 7z + 4)\left[\frac{56}{9}H_1^3 + \frac{80}{3}H_0H_1^2\right]}{z} \\
& - \frac{64}{3}(2z^2 + 7z - 3)H_0H_{0,-1} + \frac{8(536z^3 - 327z^2 - 453z - 96)H_{0,1}}{9z} \\
& + \frac{32(32z^3 + 6z^2 - 21z + 12)H_0H_{0,1}}{3z} + \frac{128}{3}(2z^2 + z - 9)H_{0,0,-1} + 256zH_0H_{0,0,-1} \\
& - \frac{32(20z^3 + 63z^2 - 12z + 12)H_{0,0,1}}{3z} + \frac{16(36z^3 - 45z^2 - 27z + 40)H_{0,1,1}}{3z} \\
& + L_M^2 \left[-\frac{16(4z^2 + 7z + 4)H_1(z-1)}{3z} - \frac{92(z-1)}{3} - \frac{16}{3}z(4z+3)H_0 \right. \\
& \left. + (z+1)(8H_0^2 + 32H_{0,1} - 32\zeta_2) \right] - \frac{16}{45}(72z^3 + 1100z^2 - 545z - 3615)\zeta_2 \\
& - \frac{32}{3}(56z^2 - 61z - 30)H_0\zeta_2 + (z-1)\left[-64H_{0,-1}H_0^2 + 256H_{0,-1,-1}H_0 - 128H_{0,-1}^2 \right. \\
& \left. + 128H_{0,-1}\zeta_2\right] + \frac{16(219z^2 + 51z - 16)\zeta_3}{3z} - 64(3z-1)H_0\zeta_3 \\
& + L_M \left[\frac{16}{3}(8z^2 + 9z + 3)H_0^2 - \frac{8}{9}(224z^2 - 99z + 81)H_0 - \frac{16}{9}(z-1)(30z+23) \right. \\
& \left. - \frac{64(z-1)(28z^2 + z + 10)H_1}{9z} + (8z^2 + 15z + 3)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right] + (z+1)\left[-\frac{16}{3}H_0^3 \right. \right. \\
& \left. \left. + 64\zeta_2H_0 - 64H_{0,0,1} + 64\zeta_3\right] \right] L_Q - \frac{4}{27}(212z^2 + 225z + 207)H_0^3 \\
& - \frac{8(z-1)(6z^2 - z - 6)H_1^3}{9z} + \frac{2}{27}(88z^2 - 1899z - 1548)H_0^2 \\
& + \frac{4(z-1)(584z^2 + 359z + 71)H_1^2}{27z} - \frac{4(z-1)(24z^2 + 65z + 24)H_0H_1^2}{3z} \\
& + \frac{16}{15}(74z^2 + 69z - 60)\zeta_2^2 + \frac{4(z-1)(1312z^2 + 3535z + 699)}{27z} \\
& + \frac{4}{27}(1532z^3 + 3033z^2 + 1305z - 540)\frac{\zeta_2}{z} + \frac{4}{27}(2040z^2 + 2015z - 2297)H_0 \\
& - \frac{8(z-1)(22z^2 + 53z + 10)H_0^2H_1}{3z} + \frac{4(z-1)(2844z^2 + 3572z + 753)H_1}{27z} \\
& + \frac{4}{3}(z-1)(8z^2 + 97z + 44)\frac{\zeta_2}{z}H_1 + \frac{8(z-1)(944z^2 + 809z - 370)H_0H_1}{27z} \\
& + \frac{8(4z^3 + 45z^2 + 15z - 12)H_0^2H_{0,1}}{3z} - \frac{8(1548z^3 + 1395z^2 - 648z + 370)H_{0,1}}{27z}
\end{aligned}$$

$$\begin{aligned}
& -\frac{8}{3}(12z^3 + 51z^2 + 33z - 8)\frac{\zeta_2}{z}H_{0,1} - \frac{16(82z^3 + 135z^2 + 180z + 30)H_0H_{0,1}}{9z} \\
& -\frac{32}{3}(z-1)H_1H_{0,1} + (4z^2 + 9z + 3)\left[\frac{2}{9}H_0^4 + \frac{16}{3}H_{0,1}^2\right] + \frac{32(9z^2 + 6z - 4)H_0H_{0,1,1}}{3z} \\
& + \frac{4(616z^3 + 1917z^2 + 1017z + 120)H_{0,0,1}}{9z} + \frac{64(z+1)(z^2 - 5z + 1)H_0H_{0,0,1}}{z} \\
& - \frac{8(188z^3 - 60z^2 + 141z + 72)H_{0,1,1}}{9z} - \frac{32(14z^3 - 9z^2 - 42z + 6)H_{0,0,0,1}}{3z} \\
& - \frac{16(33z^2 + 45z - 8)H_{0,0,1,1}}{3z} + \frac{16(28z^3 + 21z^2 - 33z - 32)H_{0,1,1,1}}{3z} \\
& - 8(z-1)(4z+3)H_0^2\zeta_2 + \frac{2}{9}(368z^2 - 1401z - 417)H_0\zeta_2 + L_M^3 \left[\frac{16(4z^2 + 7z + 4)H_1(z-1)}{9z} \right. \\
& \left. + \frac{92(z-1)}{9} + \frac{16}{9}z(4z+3)H_0 + (z+1)\left[-\frac{8}{3}H_0^2 - \frac{32}{3}H_{0,1} + \frac{32\zeta_2}{3}\right] \right] \\
& + L_M^2 \left[\frac{8}{3}z(8z-9)H_0^2 + \frac{4}{3}(88z^2 + 91z - 37)H_0 + \frac{4(z-1)(61z+12)}{3z} \right. \\
& \left. + \frac{4(z-1)(88z^2 + 135z + 40)H_1}{3z} + \frac{(z-1)(4z^2 + 7z + 4)\left[\frac{8}{3}H_1^2 + \frac{32}{3}H_0H_1\right]}{z} \right. \\
& \left. - \frac{16(4z^3 + 30z^2 + 15z - 8)H_{0,1}}{3z} - \frac{16}{3}(4z^2 - 24z - 21)\zeta_2 + (z+1)\left[-8H_0^3 - 64H_{0,1}H_0 \right. \right. \\
& \left. \left. + 48\zeta_2H_0 + 80H_{0,0,1} - 32H_{0,1,1} - 48\zeta_3\right] \right] - \frac{8}{9}(180z^2 + 1069z + 245)\zeta_3 \\
& - \frac{16}{9}(4z^2 - 96z + 27)H_0\zeta_3 + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[-\frac{2}{9}H_1^4 - \frac{16}{9}H_0H_1^3 - \frac{8}{3}H_0^2H_1^2 \right. \\
& \left. + \frac{16}{3}H_{0,1}H_1^2 - \frac{8}{3}H_0^3H_1 - \frac{64}{3}H_{0,1,1}H_1 - \frac{16}{9}\zeta_3H_1 + \left[\frac{16}{3}H_0H_1 - \frac{4}{3}H_1^2\right]\zeta_2 \right] \\
& + L_M \left[-\frac{64}{9}(8z^2 + 3z + 3)H_0^3 + \frac{2}{9}(568z^2 - 1071z + 585)H_0^2 \right. \\
& \left. + \frac{16}{27}(1003z^2 - 1527z - 291)H_0 + \frac{8(z-1)(364z^2 - 119z + 184)H_1H_0}{9z} \right. \\
& \left. - \frac{64}{3}(10z^2 + 12z + 3)H_{0,1}H_0 + 32(2z-1)(4z+5)\zeta_2H_0 + \frac{4(z-1)(308z^2 + 71z + 92)H_1^2}{9z} \right. \\
& \left. + \frac{4(z-1)(1128z^2 + 1655z + 180)}{27z} - 16(12z^3 + 9z^2 + 7z - 8)\frac{\zeta_3}{z} \right. \\
& \left. + \frac{16(z-1)(823z^2 - 1088z - 59)H_1}{27z} - \frac{16(68z^3 - 75z^2 - 159z - 92)H_{0,1}}{9z} \right. \\
& \left. + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[-\frac{8}{9}H_1^3 - \frac{16}{3}H_0H_1^2 + \frac{32}{3}H_{0,1}H_1\right] \right. \\
& \left. + \frac{32}{3}(16z^2 + 30z + 27)H_{0,0,1} - \frac{16(28z^3 + 57z^2 + 9z - 8)H_{0,1,1}}{3z} \right. \\
& \left. - \frac{8}{3}(76z^2 - 111z + 207)\zeta_2 + (z+1)\left[\frac{22}{3}H_0^4 - 112\zeta_2H_0^2 + (160H_{0,0,1} + 64H_{0,1,1})H_0 \right. \right. \\
& \left. \left. + 224\zeta_3H_0 - 32H_{0,1}^2 + \frac{576\zeta_2^2}{5} - 256H_{0,0,0,1} + 64H_{0,0,1,1} + 32H_{0,1,1,1}\right] \right] \\
& + (z+1)\left[16H_{0,1}H_0^3 + \left[32H_{0,1,1} - 96H_{0,0,1}\right]H_0^2 - 80\zeta_2^2H_0 + \left[256H_{0,0,0,1} - 96H_{0,0,1,1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +64H_{0,1,1,1}]H_0 - 64H_{0,1}H_{0,1,1} - 272H_{0,0,0,0,1} + 64H_{0,0,0,1,1} - 64H_{0,0,1,0,1} + 288H_{0,0,1,1,1} \\
& +192H_{0,1,0,1,1} + 32H_{0,1,1,1,1} + \zeta_2 \left[\frac{20}{3}H_0^3 - 32H_{0,1}H_0 + 128H_{0,0,1} + 16H_{0,1,1} - \frac{368\zeta_3}{3} \right] \\
& + \left[\frac{32}{3}H_{0,1} - \frac{16}{3}H_0^2 \right] \zeta_3 + 240\zeta_5 \Big] + C_F T_F^2 \left[\left[\left[\frac{64}{9}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{27z} \right] L_Q^3 \right. \right. \\
& + \left[-\frac{64}{3}(z+1)H_0^2 + \frac{64}{9}(4z^2-11z-8)H_0 + \frac{32(z-1)(10z^2+33z-2)}{9z} \right] L_Q^2 \\
& + \left[\frac{(256H_{0,-1} - \frac{256}{9}H_{-1}H_0)(z+1)^3}{z} + \frac{64}{3}H_0^3(z+1) - \frac{32}{9}(12z^2-59z-29)H_0^2 \right. \\
& + \left. \left. \left(\frac{64}{3}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{9z} \right) L_M^2 - \frac{64(z-1)(304z^2+811z+124)}{81z} \right. \right. \\
& - \frac{64}{27}(60z^2-155z-233)H_0 - \frac{256(3z^2+1)\zeta_2}{9z} + L_M \left[-\frac{64(z-1)(38z^2+47z+20)}{27z} \right. \\
& + \frac{128}{9}(2z^2+11z+8)H_0 + \frac{64(z-1)(4z^2+7z+4)H_1}{9z} + (z+1) \left[\frac{128\zeta_2}{3} - \frac{128}{3}H_{0,1} \right] \Big] \Big] L_Q \\
& + \left[\frac{128(z-1)(4z^2+7z+4)}{27z} - \frac{256}{9}(z+1)H_0 \right] L_M^3 - \frac{32}{27}(75z^2+100z+64)H_0^2 \\
& - \frac{16(z-1)(22z^2+29z+22)H_1^2}{9z} + \left[\frac{32(z-1)(142z^2+103z+34)}{27z} \right. \\
& - \left. \frac{64}{9}(4z^2+26z+11)H_0 \right] L_M^2 - \frac{64(z-1)(461z^2-460z+740)}{243z} \\
& - \frac{64}{27}(14z^3-62z^2-77z-20) \frac{\zeta_2}{z} - \frac{32}{27}(148z^3+279z^2+111z-16) \frac{\zeta_3}{z} \\
& + \frac{32}{81}(784z^2-473z+463)H_0 - \frac{32(z-1)(16z^2-161z-254)H_1}{81z} \\
& + \frac{32(z-1)(74z^2-43z+20)H_0H_1}{27z} + \frac{(z-1)(4z^2+7z+4) \left[\frac{64}{27}H_1^3 + \frac{16}{9}H_0H_1^2 - \frac{32}{9}H_0^2H_1 \right]}{z} \\
& - \frac{64(70z^3-19z^2+104z-10)H_{0,1}}{27z} - \frac{64(2z^3+z^2-2z+4)H_0H_{0,1}}{9z} \\
& + \frac{64(10z^3+16z^2+z+4)H_{0,0,1}}{9z} + \frac{128(3z^3+12z^2+12z+1)H_{0,1,1}}{9z} \\
& + (8z^2+15z+3) \left[\frac{16}{27}H_0^3 + \frac{32}{9}\zeta_2H_0 \right] + (z+1) \left[-\frac{8}{9}H_0^4 + \frac{64}{3}H_{0,1}H_0^2 - \frac{32}{3}\zeta_2H_0^2 \right. \\
& + \left. \left[-\frac{128}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1} \right] H_0 + \frac{1216}{9}\zeta_3H_0 + 32\zeta_2^2 + \frac{64}{3}H_{0,0,1,1} - \frac{256}{3}H_{0,1,1,1} \right] \\
& + L_M \left[\frac{256\zeta_2z^2}{3} - \frac{128}{27}(14z^2+127z+64)H_0 + (z+1) \left[-\frac{64}{9}H_0^3 - \frac{928}{9}H_0^2 \right. \right. \\
& + \frac{256}{3}H_{0,1}H_0 - \frac{256}{3}\zeta_2H_0 - \frac{256}{3}H_{0,0,1} + \frac{128}{3}H_{0,1,1} + \frac{128\zeta_3}{3} \Big] + \frac{64(z-1)(616z^2+667z+94)}{81z} \\
& + \frac{128(z-1)(4z^2-26z+13)H_1}{27z} + \frac{(z-1)(4z^2+7z+4) \left[-\frac{32}{9}H_1^2 - \frac{128}{9}H_0H_1 \right]}{z} \\
& \left. \left. \left. - \frac{128(2z^3-3z^2+3z+4)H_{0,1}}{9z} \right] \right] \right] \Big] \Big]
\end{aligned}$$

$$\begin{aligned}
& + C_F N_F T_F^2 \left[\left[\frac{64}{9}(z+1)H_0 - \frac{32(z-1)(4z^2+7z+4)}{27z} \right] L_Q^3 \right. \\
& + \left[-\frac{64}{3}(z+1)H_0^2 + \frac{64}{9}(4z^2-11z-8)H_0 + \frac{32(z-1)(10z^2+33z-2)}{9z} \right] L_Q^2 \\
& + \left[\frac{(256H_{0,-1} - \frac{256}{9}H_{-1}H_0)(z+1)^3}{z} + \left[\frac{64}{3}H_0^3 - \frac{64}{3}H_{0,1,1} + \frac{64\zeta_3}{3} \right] (z+1) \right. \\
& - \frac{32}{9}(12z^2-59z-29)H_0^2 + \frac{16(z-1)(4z^2+7z+4)H_1^2}{9z} - \frac{64(z-1)(194z^2+683z+68)}{81z} \\
& - \frac{64}{9}(2z^3+23z^2+8z+4)\frac{\zeta_2}{z} - \frac{64}{27}(79z^2-88z-190)H_0 - \frac{32(z-1)(38z^2+47z+20)H_1}{27z} \\
& + \frac{64}{9}(2z^2+11z+8)H_{0,1} \left. \right] L_Q + \frac{16}{27}(8z^2+15z+3)H_0^3 + \left[\frac{32(z-1)(4z^2+7z+4)}{27z} \right. \\
& - \frac{64}{9}(z+1)H_0 \left. \right] L_M^3 - \frac{32}{27}(56z^2+33z+21)H_0^2 - \frac{64(z-1)(1156z^2-203z+328)}{243z} \\
& - \frac{16}{27}(z-1)(74z^2-43z+20)\frac{\zeta_2}{z} - \frac{32}{27}(100z^3+183z^2+33z-4)\frac{\zeta_3}{z} \\
& + \frac{32}{81}(800z^2-57z+111)H_0 + \frac{(z-1)(28z^2+z+10) \left[\frac{128}{27}H_0H_1 - \frac{128}{27}H_{0,1} \right]}{z} \\
& - \frac{128(2z^3+6z^2+3z+2)H_0H_{0,1}}{9z} + \frac{64(12z^3+27z^2+9z+4)H_{0,0,1}}{9z} \\
& + \frac{32}{9}(6z^2+4z-5)H_0\zeta_2 + \frac{(z-1)(4z^2+7z+4) \left[-\frac{32}{9}H_1H_0^2 - \frac{16}{9}H_1\zeta_2 \right]}{z} \\
& + L_M^2 \left[\frac{32}{3}(z-1)(2z-5) - \frac{32}{9}(4z^2-7z-13)H_0 + \frac{32(z-1)(4z^2+7z+4)H_1}{9z} \right. \\
& + (z+1) \left[\frac{32}{3}H_0^2 - \frac{64}{3}H_{0,1} + \frac{64\zeta_2}{3} \right] \left. \right] + (z+1) \left[-\frac{8}{9}H_0^4 + \frac{64}{3}H_{0,1}H_0^2 - \frac{128}{3}H_{0,0,1}H_0 \right. \\
& + \frac{832}{9}\zeta_3H_0 - \frac{32\zeta_2^2}{3} + \left[\frac{32}{3}H_{0,1} - \frac{32}{3}H_0^2 \right] \zeta_2 \left. \right] + L_M \left[\frac{32}{9}(4z^2-7z-13)H_0^2 \right. \\
& + \frac{64}{27}(z^2+2z-58)H_0 + \frac{128(z-1)(25z^2+94z+34)}{81z} + \frac{32(z-1)(74z^2-43z+20)H_1}{27z} \\
& + \frac{(z-1)(4z^2+7z+4) \left[\frac{16}{9}H_1^2 - \frac{64}{9}H_0H_1 \right]}{z} - \frac{64(2z^3+z^2-2z+4)H_{0,1}}{9z} \\
& + \frac{64}{9}(6z^2+4z-5)\zeta_2 + (z+1) \left[-\frac{64}{9}H_0^3 + \frac{128}{3}H_{0,1}H_0 - \frac{128}{3}\zeta_2H_0 - \frac{128}{3}H_{0,0,1} \right. \\
& - \frac{64}{3}H_{0,1,1} + 64\zeta_3 \left. \right] \left. \right] + C_F C_A T_F \left[-\frac{2}{9}(4z-17)H_0^4 - \frac{4}{9}(36z^2+47z+36)H_0^3 \right. \\
& + \frac{4}{27}(2132z^2-681z+855)H_0^2 + \frac{8(z-1)(122z^2-19z+113)H_1H_0^2}{9z} \\
& - \frac{8(19z^2+19z+8)H_{0,1}H_0^2}{3z} - 32(2z-1)H_{0,0,1}H_0^2 - \frac{8}{3}(10z-17)\zeta_2H_0^2 - \frac{16}{3}(14z-13)\zeta_3H_0^2 \\
& - \frac{16(z-1)(19z^2+16z+10)H_1^2H_0}{9z} + \frac{16}{5}(38z-5)\zeta_2^2H_0 + \frac{16}{9}(13z^2-215z-4)\frac{\zeta_3}{z}H_0
\end{aligned}$$

$$\begin{aligned}
& -\frac{4(54204z^3 + 9339z^2 + 17082z + 2624)H_0}{81z} - \frac{4}{9}(474z^3 + 121z^2 + 295z + 80)\frac{\zeta_2}{z}H_0 \\
& -\frac{32(z-1)(769z^2 - 62z + 337)H_1H_0}{27z} - \frac{32(19z^3 - 24z^2 - 6z + 10)H_{0,-1}H_0}{9z} \\
& +\frac{16(18z^3 + 127z^2 + 6z + 51)H_{0,1}H_0}{3z} + \frac{64(10z^2 + z + 4)H_{0,0,1}H_0}{3z} + 64(5z - 2)H_{0,0,0,1}H_0 \\
& +\frac{8(z-1)(2z+1)(14z+1)H_1^3}{27z} + \frac{4(z-1)(328z^2 + 313z + 67)H_1^2}{27z} \\
& +\frac{4(z-1)(75516z^2 - 7654z + 21765)}{81z} + \frac{8}{27}(1841z^3 - 1719z^2 + 1230z - 515)\frac{\zeta_2}{z} \\
& +\frac{8}{15}(20z^3 + 340z^2 - 137z + 152)\frac{\zeta_2^2}{z} + \frac{8}{9}(768z^3 + 1158z^2 + 687z + 76)\frac{\zeta_3}{z} \\
& +\frac{4(z-1)(2500z^2 + 2755z + 1771)H_1}{81z} + \frac{4}{9}(z-1)(154z^2 + 163z + 46)\frac{\zeta_2}{z}H_1 \\
& +\frac{(z+1)(182z^2 - 122z + 47)\left[\frac{32}{27}H_{-1}H_0 - \frac{32}{27}H_{0,-1}\right]}{z} \\
& +\frac{8(2964z^3 - 3393z^2 + 1368z - 1348)H_{0,1}}{27z} + \frac{32(19z^3 - 51z^2 - 6z + 10)H_{0,0,-1}}{9z} \\
& -\frac{16(230z^3 + 621z^2 + 168z + 193)H_{0,0,1}}{9z} + \frac{8(56z^3 - 105z^2 - 66z - 40)H_{0,1,1}}{9z} \\
& -\frac{16(21z^2 - 15z + 8)H_{0,0,0,1}}{z} - \frac{16(20z^3 + 18z^2 - 15z - 20)H_{0,0,1,1}}{3z} \\
& -\frac{32(z-1)(z+2)(2z+1)H_{0,1,1,1}}{3z} - 128(4z-1)H_{0,0,0,0,1} + L_M^3\left[\frac{16}{3}(2z-1)H_0^2\right. \\
& +\frac{16(8z^2 + 11z + 4)H_0}{9z} - \frac{8(z-1)(44z^2 - z + 44)}{9z} - \frac{16(z-1)(4z^2 + 7z + 4)H_1}{9z} \\
& \left.+(z+1)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right]\right] - \frac{8}{3}(23z + 14)H_{0,1}\zeta_2 + \frac{(z+1)(19z^2 - 16z + 10)}{z}\left[-\frac{32}{9}H_0H_{-1}^2\right. \\
& \left.+\left(\frac{16}{9}H_0^2 + \frac{64}{9}H_{0,-1}\right)H_{-1} - \frac{32}{9}\zeta_2H_{-1} - \frac{64}{9}H_{0,-1,-1}\right] + L_Q^3\left[-\frac{16}{3}(2z-1)H_0^2\right. \\
& -\frac{16(8z^2 + 11z + 4)H_0}{9z} + \frac{8(z-1)(44z^2 - z + 44)}{9z} + \frac{16(z-1)(4z^2 + 7z + 4)H_1}{9z} \\
& \left.+(z+1)\left[\frac{32\zeta_2}{3} - \frac{32}{3}H_{0,1}\right]\right] + \frac{8}{3}(43z + 37)\zeta_2\zeta_3 + L_Q^2\left[\frac{16}{3}(4z-3)H_0^3\right. \\
& -\frac{32(101z^3 - 53z^2 + 82z + 13)H_0}{9z} - 32(2z+1)\zeta_2H_0 + \frac{8(z-1)(181z^2 - 413z + 901)}{27z} \\
& -\frac{16(z-1)(62z^2 - 7z + 44)H_1}{9z} + \frac{(z-1)(4z^2 + 7z + 4)\left[-\frac{16}{3}H_1^2 - \frac{32}{3}H_0H_1\right]}{z} \\
& \left.+\frac{(z+1)(4z^2 - 7z + 4)\left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\right]}{z}\right] + 16(2z-1)H_{0,1} + (z-1)\left[64H_0H_{0,-1}\right. \\
& \left.-128H_{0,0,-1}\right] - 96H_{0,0,1} + (z+1)\left[\frac{224}{3}H_0^2 + 64H_{0,1}H_0 + 64H_{0,1,1}\right] + \frac{16}{3}(8z^2 - 6z - 3)\zeta_2 \\
& \left.+32(z-2)\zeta_3\right] + L_M^2\left[\frac{8}{3}(16z-17)H_0^2 + \frac{8(272z^3 + 103z^2 + 139z + 40)H_0}{9z}\right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{16(z-1)(4z^2+7z+4)H_1H_0}{3z} - \frac{8(z-1)(1864z^2-485z+694)}{27z} \\
& - \frac{16}{3}(4z^3+17z^2+11z+4)\frac{\zeta_2}{z} - \frac{8(z-1)(104z^2+119z+32)H_1}{9z} \\
& + \frac{(z+1)(4z^2-7z+4)\left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0\right]}{z} + \frac{32}{3}(10z+7)H_{0,1} + (z+1)\left[32H_{0,0,1} \right. \\
& \left. - 32H_0H_{0,1}\right] + (z-1)\left[-\frac{32}{3}H_0^3 - 64H_{0,-1}H_0 - 32\zeta_2H_0 + 128H_{0,0,-1}\right] - 64(2z-1)\zeta_3 \\
& + \frac{(z+1)(4z^2-7z+4)}{z}\left[\frac{32}{9}H_0H_{-1}^3 + \left[-\frac{8}{3}H_0^2 - \frac{32}{3}H_{0,-1}\right]H_{-1}^2 + \left[-\frac{8}{9}H_0^3 + \left(\frac{32}{3}H_{0,-1} \right. \right. \right. \\
& \left. \left. - \frac{32}{3}H_{0,1}\right)H_0 + \frac{64}{3}H_{0,-1,-1} - \frac{32}{3}H_{0,0,-1} + \frac{64}{3}H_{0,0,1}\right]H_{-1} - 16\zeta_3H_{-1} + \frac{8}{3}H_0^2H_{0,-1} \\
& + H_0\left[-\frac{32}{3}H_{0,-1,-1} + \frac{32}{3}H_{0,-1,1} - \frac{16}{3}H_{0,0,-1} + \frac{32}{3}H_{0,1,-1}\right] - \frac{64}{3}H_{0,-1,-1,-1} - \frac{32}{3}H_{0,-1,0,1} \\
& + \frac{32}{3}H_{0,0,-1,-1} - \frac{64}{3}H_{0,0,-1,1} + \frac{16}{3}H_{0,0,0,-1} - \frac{64}{3}H_{0,0,1,-1} + \left[\frac{16}{3}H_{-1}^2 + 8H_0H_{-1} - 8H_{0,-1}\right]\zeta_2 \\
& + L_Q\left[-\frac{4}{3}(12z-11)H_0^4 - \frac{8}{3}(41z+34)H_0^3 + \frac{4}{9}(1508z^2-1040z+1297)H_0^2 \right. \\
& \left. - \frac{24(z+1)(4z^2-5z+4)H_{-1}H_0^2}{z} + \frac{4(z-1)(56z^2+125z+56)H_1H_0^2}{3z} + 8(19z-3)\zeta_2H_0^2 \right. \\
& \left. - \frac{16(466z^3+1961z^2+4007z+284)H_0}{27z} - \frac{8}{3}(40z^3-93z^2-21z-16)\frac{\zeta_2}{z}H_0 \right. \\
& + \frac{32(z-1)(76z^2+19z+31)H_1H_0}{9z} - \frac{32(2z^3+3z^2+24z-22)H_{0,-1}H_0}{3z} \\
& \left. - \frac{8(31z^2-19z-8)H_{0,1}H_0}{z} + 32(11z-7)H_{0,0,-1}H_0 + 32(7z+13)H_{0,0,1}H_0 \right. \\
& + 32(14z+15)\zeta_3H_0 + \frac{4(z-1)(286z^2+25z+106)H_1^2}{9z} + \frac{8}{5}(41z+35)\zeta_2^2 \\
& + \frac{8(z-1)(3335z^2+2808z+4604)}{27z} - \frac{8}{9}(846z^3-371z^2+901z-264)\frac{\zeta_2}{z} \\
& - \frac{16}{3}(76z^3+35z^2+14z-48)\frac{\zeta_3}{z} + 16(z+1)(8z^2-5z+8)\frac{\zeta_2}{z}H_{-1} \\
& - \frac{8(z-1)(968z^2+1301z+2210)H_1}{27z} - \frac{64}{3}(z-1)(5z^2+11z+5)\frac{\zeta_2}{z}H_1 \\
& + \frac{(z-1)(4z^2+7z+4)\left[\frac{40}{9}H_1^3 + \frac{40}{3}H_0H_1^2\right]}{z} + \frac{(z+1)(73z^2+38z+118)}{z}\left[\frac{32}{9}H_{-1}H_0 \right. \\
& \left. - \frac{32}{9}H_{0,-1}\right] + \frac{8(542z^3+301z^2+853z+332)H_{0,1}}{9z} + \frac{16(44z^3+3z^2+87z-52)H_{0,0,-1}}{3z} \\
& - \frac{16(8z^3-15z^2+33z-4)H_{0,0,1}}{3z} + \frac{(z+1)(z^2-z+1)}{z}\left[-128H_{-1}H_{0,1} + 128H_{0,-1,1} \right. \\
& \left. + 128H_{0,1,-1}\right] + \frac{16}{3}(8z^2-19z+14)H_{0,1,1} - 384zH_{0,0,0,-1} - 128(2z+3)H_{0,0,0,1} \\
& - 32(z-5)H_{0,0,1,1} + (z-1)\left[-112H_{0,-1}H_0^2 - 64H_{0,-1,-1}H_0 + 32H_{0,-1}^2 - 128H_{0,-1,0,1} \right. \\
& \left. + 96H_{0,-1}\zeta_2\right] + (z+1)\left[48H_0H_{-1}^2 - 96H_{0,-1}H_{-1} - 136H_0^2H_{0,1} + 96H_{0,-1,-1} - 160H_0H_{0,1,1} \right. \\
& \left. - 160H_{0,1,1,1} + 192H_{0,1}\zeta_2\right] + L_M\left[\frac{4}{3}(4z-5)H_0^4 - \frac{8}{9}(35z-46)H_0^3 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{4}{9}(606z^2 - 346z + 377)H_0^2 + \frac{32(z+1)(z^2 - z + 1)H_{-1}H_0^2}{z} \\
& -\frac{4(z-1)(8z^2 + 17z + 8)H_1H_0^2}{z} + \frac{16(3729z^3 + 2093z^2 + 2330z + 224)H_0}{z} \\
& + \frac{16(z-1)(203z^2 + 47z + 140)H_1H_0}{9z} + \frac{16(4z^3 - 13z^2 + 3z - 4)H_{0,-1}H_0}{z} \\
& + \frac{8(16z^3 - 41z^2 - 77z - 40)H_{0,1}H_0}{3z} - 32(3z+1)H_{0,0,-1}H_0 - 32(11z+5)H_{0,0,1}H_0 \\
& + \frac{8}{3}(8z^2 - 13z + 59)\zeta_2H_0 - 32(16z+9)\zeta_3H_0 - \frac{4(z-1)(20z^2 + 21z + 2)H_1^2}{3z} \\
& - \frac{8}{5}(209z + 87)\zeta_2^2 - \frac{8(z-1)(11542z^2 + 399z + 3892)}{27z} - \frac{8}{9}(132z^3 + 313z^2 - 473z + 168)\frac{\zeta_2}{z} \\
& + \frac{16}{3}(52z^3 - 78z^2 - 33z - 48)\frac{\zeta_3}{z} - \frac{8(z-1)(210z^2 - 559z - 186)H_1}{9z} \\
& + \frac{(z+1)(125z^2 - 20z + 62)\left[\frac{32}{9}H_{0,-1} - \frac{32}{9}H_{-1}H_0\right]}{z} - \frac{8(274z^3 - 205z^2 + 659z - 200)H_{0,1}}{9z} \\
& + \frac{(z+1)(4z^2 - 7z + 4)\left[\frac{32}{3}H_0H_{-1}^2 - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1}\right]}{z} \\
& - \frac{32(z-2)(6z^2 - z + 1)H_{0,0,-1}}{z} - \frac{32(4z^3 - 11z^2 - 17z - 14)H_{0,0,1}}{3z} \\
& + \frac{(z+1)(2z^2 + z + 2)\left[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1}\right]}{z} \\
& + \frac{16(8z^3 + 23z^2 + 5z - 8)H_{0,1,1}}{3z} + 96(z+3)H_{0,0,0,-1} + 32(23z+5)H_{0,0,0,1} \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[\frac{8}{9}H_1^3 + 8H_0H_1^2 - \frac{32}{3}H_{0,1}H_1 + \frac{16}{3}\zeta_2H_1\right] \\
& + (z-1)\left[32H_{0,-1}H_0^2 + 128H_{0,-1,-1}H_0 - 64H_{0,-1}^2 + (8H_0^2 + 64H_{0,-1})\zeta_2\right] \\
& + (z+1)\left[56H_{0,1}H_0^2 - 96H_{0,1,1}H_0 + 32H_{0,1}^2 + 32H_{0,0,1,1} - 32H_{0,1,1,1}\right. \\
& \left. + \left[-96H_{-1} - 32H_{0,1}\right]\zeta_2\right] + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[\frac{2}{9}H_1^4 + \frac{4}{3}H_0^2H_1^2\right. \\
& \left. + \left[\frac{80}{3}H_{0,0,1} + \frac{16}{3}H_{0,1,1}\right]H_1 - \frac{80}{9}\zeta_3H_1 - \frac{16}{3}H_{0,1}^2 + H_0\left[-\frac{16}{9}H_1^3 - 16H_{0,1}H_1\right.\right. \\
& \left. + \frac{80}{3}H_{0,1,1}\right] + \left[\frac{4}{3}H_1^2 - \frac{8}{3}H_0H_1\right]\zeta_2 + (z-1)\left[\frac{4}{15}H_0^5 - \frac{16}{3}H_{0,-1}H_0^3 + \left[32H_{0,0,-1}\right.\right. \\
& \left. - 32H_{0,-1,-1}\right]H_0^2 + \left[32H_{0,-1}^2 + 128H_{0,-1,-1,-1} - 64H_{0,-1,0,1} - 96H_{0,0,0,-1}\right]H_0 \\
& + H_{0,-1}\left[-128H_{0,-1,-1} - 64H_{0,0,-1} + 128H_{0,0,1}\right] + 256H_{0,-1,0,-1,-1} + 512H_{0,0,-1,-1,-1} \\
& + 64H_{0,0,-1,0,-1} - 128H_{0,0,-1,0,1} + 192H_{0,0,0,-1,-1} - 384H_{0,0,0,-1,1} + 128H_{0,0,0,0,-1} \\
& - 384H_{0,0,0,1,-1} - 128H_{0,0,1,0,-1} + \left[8H_0^3 + 48H_{0,-1}H_0 + 64H_{0,-1,-1} - 96H_{0,0,-1}\right]\zeta_2 \\
& - 96H_{0,-1}\zeta_3 + (z+1)\left[-16H_{0,1,1}H_0^2 + \left[48H_{0,1}^2 - 128H_{0,0,1,1} + 64H_{0,1,1,1}\right]H_0\right. \\
& \left. - 160H_{0,1}H_{0,0,1} + 864H_{0,0,0,1,1} + 288H_{0,0,1,0,1} - 128H_{0,0,1,1,1} - 32H_{0,1,0,1,1} - 32H_{0,1,1,1,1}\right. \\
& \left. + \left[16H_0H_{0,1} - 16H_{0,0,1} - 16H_{0,1,1}\right]\zeta_2 + \frac{160}{3}H_{0,1}\zeta_3\right] + 80(7z-3)\zeta_5 \Big]
\end{aligned}$$

$$+a_{Qq}^{\text{PS},(3)} + \tilde{C}_{2,q}^{\text{PS},(3)}(N_F + 1) \Big\}. \quad (597)$$

The Wilson coefficient $H_{g,2}^S$ is given by :

$$\begin{aligned}
H_{g,2}^S = & \\
& a_s T_F \left\{ -\gamma_{qg}^0 L_Q - 4(8z^2 - 8z + 1) + \gamma_{qg}^0 [H_0 + H_1] + \gamma_{qg}^0 L_M \right\} \\
& + a_s^2 \left\{ T_F^2 \left[\frac{4}{3} \gamma_{qg}^0 L_M^2 - \frac{4}{3} \gamma_{qg}^0 L_Q L_M + \left[\gamma_{qg}^0 \left[\frac{4H_0}{3} + \frac{4H_1}{3} \right] - \frac{16}{3} (8z^2 - 8z + 1) \right] L_M \right] \right. \\
& + C_{AT_F} \left[\frac{16}{3} (3z + 1) H_0^3 - 2z(57z - 92) H_0^2 + 4(8z^2 + 6z + 3) H_{-1} H_0^2 - 4(4z^2 - 6z + 3) H_1 H_0^2 \right. \\
& - \frac{4}{9} (1445z^2 - 747z - 219) H_0 - \frac{4(199z^3 - 168z^2 - 3z - 16) H_1 H_0}{3z} - 8(4z^2 + 6z + 3) H_{0,-1} H_0 \\
& + 8(14z + 5) H_{0,1} H_0 + 16(2z^2 - 8z - 1) \zeta_2 H_0 - \frac{2(107z^3 - 96z^2 + 9z - 8) H_1^2}{3z} \\
& + \left[\frac{4(z-1)(31z^2 + 7z + 4)}{3z} - 8(4z + 1) H_0 - 2\gamma_{qg}^0 H_1 \right] L_M^2 - \frac{2(439z^3 - 130z^2 - 233z - 40)}{9z} \\
& + \frac{4}{3} (219z^3 - 204z^2 + 12z - 8) \frac{\zeta_2}{z} - \frac{4(749z^3 - 645z^2 - 84z + 52) H_1}{9z} \\
& + L_Q^2 \left[-\frac{4(z-1)(31z^2 + 7z + 4)}{3z} + 8(4z + 1) H_0 + 2\gamma_{qg}^0 H_1 \right] + \frac{(13z^3 + 3z^2 - 9z - 2)}{z} \left[\frac{16}{3} H_{-1} H_0 \right. \\
& - \left. \frac{16}{3} H_{0,-1} \right] - \frac{4(20z^3 - 48z^2 + 15z + 16) H_{0,1}}{3z} + \gamma_{qg}^0 [4H_0 H_1^2 - 4H_1 H_{0,1}] + z^2 [-16H_0 H_{-1}^2 \\
& + 32H_{0,-1} H_{-1} - 32H_{0,-1,-1}] + 24(2z + 1) H_{0,0,-1} - 8(18z + 5) H_{0,0,1} + (2z^2 + 2z + 1) [16H_{-1} H_{0,1} \\
& - 16H_{0,-1,1} - 16H_{0,1,-1}] - 32(z - 3) z H_{0,1,1} - 16(3z^2 + 2z + 1) H_{-1} \zeta_2 + 16(z - 1)^2 H_1 \zeta_2 \\
& + L_M \left[8(2z + 1) H_0^2 - \frac{8}{3} (44z^2 + 24z + 3) H_0 - 2\gamma_{qg}^0 H_1^2 + \frac{8(218z^3 - 225z^2 + 18z - 20)}{9z} \right. \\
& + 32(z - 1) z H_1 + (2z^2 + 2z + 1) [16H_{-1} H_0 - 16H_{0,-1}] + 32z \zeta_2 \left. \right] + L_Q \left[-16(3z + 1) H_0^2 \right. \\
& + 8z(25z - 24) H_0 + \frac{4(407z^3 - 276z^2 - 165z + 52)}{9z} + \frac{8(67z^3 - 60z^2 + 3z - 4) H_1}{3z} + \gamma_{qg}^0 [-2H_1^2 \\
& - 4H_0 H_1] + (2z^2 + 2z + 1) [16H_{0,-1} - 16H_{-1} H_0] - 16(4z + 1) H_{0,1} - 32(z - 2) z \zeta_2 \left. \right] \\
& + 8(6z^2 + 2z + 5) \zeta_3 \left. \right] + C_{FT_F} \left[\left[2(4z - 1) - 4(4z^2 - 2z + 1) H_0 + 2\gamma_{qg}^0 H_1 \right] L_Q^2 \right. \\
& + \left[8(4z^2 - 2z + 1) H_0^2 + 8(10z^2 - 6z + 1) H_0 + 4(4z^2 - 17z + 9) + 4(2z - 1)(10z - 7) H_1 \right. \\
& + \gamma_{qg}^0 [-4H_1^2 - 8H_0 H_1] + 8(2z - 1) H_{0,1} + \left. \left[-4(4z - 1) + 8(4z^2 - 2z + 1) H_0 - 4\gamma_{qg}^0 H_1 \right] L_M \right. \\
& \left. - 8(8z^2 - 6z + 3) \zeta_2 \right] L_Q - \frac{4}{15} (72z^3 + 195z^2 - 10z + 15) H_0^2 - 2(42z^2 - 44z + 11) H_1^2
\end{aligned}$$

$$\begin{aligned}
& + \left[2(4z-1) - 4(4z^2-2z+1)H_0 + 2\gamma_{qg}^0 H_1 \right] L_M^2 + \frac{4(246z^3 + 51z^2 - 226z + 4)}{15z} \\
& - \frac{8(252z^3 - 51z^2 + 89z + 2)H_0}{15z} - 16z^2 H_0^2 H_1 - 4[24z^2 - 33z + 7]H_1 \\
& - 8(13z^2 - 14z + 6)H_0 H_1 + \frac{(36z^5 + 40z^3 + 90z^2 + 1) \left[\frac{16}{15}H_{-1}H_0 - \frac{16}{15}H_{0,-1} \right]}{z^2} \\
& + (z-1)^2 H_0 [32H_{0,-1} - 32H_{0,1}] - 8(8z^2 - 6z - 3)H_{0,1} + \gamma_{qg}^0 [2H_1^3 + 4H_0 H_1^2 + 4H_{0,1} H_1] \\
& - 32(3z^2 - 2z + 3)H_{0,0,-1} - 32(2z-1)H_{0,0,1} + 8(4z^2 - 6z + 3)H_{0,1,1} \\
& + \frac{8}{15}(72z^3 + 315z^2 - 220z + 45)\zeta_2 + 32(3z^2 - 2z + 1)H_0 \zeta_2 + L_M \left[-8(4z^2 - 2z + 1)H_0^2 \right. \\
& - 8(10z^2 - 6z + 1)H_0 - 4(4z^2 - 17z + 9) - 4(2z-1)(10z-7)H_1 + \gamma_{qg}^0 [4H_1^2 + 8H_0 H_1] \\
& \left. - 8(2z-1)H_{0,1} + 8(8z^2 - 6z + 3)\zeta_2 \right] + (z+1)^2 \left[-32H_0 H_{-1}^2 + [16H_0^2 + 64H_{0,-1}]H_{-1} \right. \\
& \left. - 32\zeta_2 H_{-1} - 64H_{0,-1,-1} \right] + (4z^2 - 2z + 1) \left[16H_1 \zeta_2 - \frac{8}{3}H_0^3 \right] + 8(20z^2 + 2z + 7)\zeta_3 \left. \right\} \\
& + a_s^3 \left\{ T_F^3 \left[\frac{16}{9}\gamma_{qg}^0 L_M^3 - \frac{16}{9}\gamma_{qg}^0 L_Q L_M^2 + \left[\gamma_{qg}^0 \left[\frac{16H_0}{9} + \frac{16H_1}{9} \right] - \frac{64}{9}(8z^2 - 8z + 1) \right] L_M^2 - \frac{16\gamma_{qg}^0 \zeta_3}{9} \right] \right. \\
& + C_{AT_F}^2 \left[-\frac{8}{9}(2z+1)H_0^4 + \frac{16}{27}(23z^2 - 6z + 3)H_0^3 - \frac{8}{27}(822z^2 + 592z + 229)H_0^2 \right. \\
& \left. - \frac{16(z-1)(65z^2 + 17z + 8)H_1 H_0^2}{9z} + \frac{64}{3}(4z+1)H_{0,1} H_0^2 - \frac{8}{3}(2z+9)\zeta_2 H_0^2 + \frac{64}{3}(z-1)z H_1^2 H_0 \right. \\
& \left. + \frac{8}{81}(16303z^2 - 4390z + 2000)H_0 + \frac{16(1128z^3 - 1147z^2 + 146z - 80)H_1 H_0}{27z} \right. \\
& \left. - \frac{32(23z^3 + 99z^2 + 15z + 8)H_{0,1} H_0}{9z} - \frac{128}{3}(6z+1)H_{0,0,1} H_0 + \frac{8}{9}(272z^2 + 360z + 3)\zeta_2 H_0 \right. \\
& \left. + \frac{64}{9}(41z+14)\zeta_3 H_0 - \frac{16}{9}(z-1)(5z+1)H_1^3 + L_M^3 \left[\frac{112(z-1)(31z^2 + 7z + 4)}{27z} - \frac{56}{9}\gamma_{qg}^0 H_1 \right. \right. \\
& \left. \left. - \frac{224}{9}(4z+1)H_0 \right] + \frac{8}{27}(206z^2 - 143z + 67)H_1^2 - \frac{4(30335z^3 - 36798z^2 + 17367z - 13244)}{243z} \right. \\
& \left. - \frac{32}{15}(77z+2)\zeta_2^2 - \frac{4}{9}(776z^3 - 1628z^2 + 59z - 128)\frac{\zeta_2}{z} - \frac{16}{27}(1255z^3 + 150z^2 + 108z - 28)\frac{\zeta_3}{z} \right. \\
& \left. + \frac{8(6819z^3 - 6878z^2 + 106z - 628)H_1}{81z} + L_Q^3 \left[-\frac{16(z-1)(31z^2 + 7z + 4)}{27z} \right. \right. \\
& \left. \left. + \frac{32}{9}(4z+1)H_0 + \frac{8}{9}\gamma_{qg}^0 H_1 \right] - \frac{32(672z^3 - 257z^2 + 76z - 40)H_{0,1}}{27z} \right. \\
& \left. + \frac{16(222z^3 + 244z^2 + 61z + 16)H_{0,0,1}}{9z} - \frac{32}{3}z(5z-1)H_{0,1,1} + \frac{32}{3}(22z+1)H_{0,0,0,1} \right. \\
& \left. - \frac{160}{9}(5z^2 - 5z + 1)H_1 \zeta_2 + L_M^2 \left[16H_0^2 - \frac{16}{9}(127z^2 + 232z + 25)H_0 \right. \right. \\
& \left. \left. + \frac{8(2612z^3 - 2514z^2 - 33z - 200)}{27z} + \frac{16(135z^3 - 128z^2 + 13z - 4)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{20}{3}H_1^2 \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{8}{3}H_0H_1 \Big] + (2z^2 + 2z + 1) \left[32H_{-1}H_0 - 32H_{0,-1} \right] - \frac{32}{3}(4z + 1)H_{0,1} - \frac{64}{3}(z - 6)z\zeta_2 \Big] \\
& + L_Q^2 \left[-\frac{16}{3}(8z + 1)H_0^2 + \frac{16}{9}(49z^2 - 136z - 13)H_0 + \frac{8(1048z^3 - 894z^2 - 87z - 40)}{27z} \right. \\
& + \frac{16(87z^3 - 80z^2 + 13z - 4)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{4}{3}H_1^2 - \frac{8}{3}H_0H_1 \right] \\
& + (2z^2 + 2z + 1) \left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0 \right] - \frac{32}{3}(4z + 1)H_{0,1} + L_M \left[-\frac{16(z - 1)(31z^2 + 7z + 4)}{9z} \right. \\
& + \frac{32}{3}(4z + 1)H_0 + \frac{8}{3}\gamma_{qg}^0H_1 \Big] - \frac{64}{3}(z - 2)z\zeta_2 \Big] + z(z + 1) \left[\frac{128}{3}H_0H_{-1}^2 \right. \\
& + \left[-\frac{64}{3}H_0^2 - 128H_0 - \frac{256}{3}H_{0,-1} \right] H_{-1} + \frac{128}{3}\zeta_2H_{-1} + \frac{128}{3}H_0H_{0,-1} + 128H_{0,-1} \\
& + \frac{256}{3}H_{0,-1,-1} - \frac{128}{3}H_{0,0,-1} \Big] + L_Q \left[\frac{448}{9}zH_0^3 - \frac{16}{9}(63z^2 - 296z - 10)H_0^2 \right. \\
& + \frac{32}{3}(5z^2 + 4z + 2)H_{-1}H_0^2 - \frac{32}{3}(3z^2 - 4z + 2)H_1H_0^2 - \frac{64}{27}(325z^2 - 832z - 31)H_0 \\
& - \frac{64}{3}(3z^2 + 4z + 2)H_{0,-1}H_0 + \frac{64}{3}(z^2 + 2z + 2)H_{0,1}H_0 + \frac{128}{3}(z - 5)z\zeta_2H_0 \\
& + \left[-\frac{16(z - 1)(31z^2 + 7z + 4)}{9z} + \frac{32}{3}(4z + 1)H_0 + \frac{8}{3}\gamma_{qg}^0H_1 \right] L_M^2 \\
& - \frac{8(25657z^3 - 23556z^2 - 969z - 412)}{81z} + \frac{32}{9}(105z^3 - 184z^2 + 3z - 4)\frac{\zeta_2}{z} \\
& - \frac{32(1032z^3 - 964z^2 + 65z - 20)H_1}{27z} + \frac{(87z^3 - 80z^2 + 13z - 4) \left[-\frac{16}{9}H_1^2 - \frac{32}{9}H_0H_1 \right]}{z} \\
& + \frac{(17z^3 + 7z^2 - 4z - 2) \left[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1} \right]}{z} - \frac{64(9z^3 - 59z^2 - 5z + 2)H_{0,1}}{9z} \\
& + \gamma_{qg}^0 \left[\frac{8}{9}H_1^3 + \frac{16}{3}H_0H_1^2 - \frac{32}{3}H_{0,1}H_1 \right] + (2z + 1) \left[\frac{32}{3}H_0H_{-1}^2 - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1} \right] \\
& + \frac{64}{3}(z^2 + 4z + 2)H_{0,0,-1} - \frac{64}{3}(z^2 - 2z + 2)H_{0,0,1} + (2z^2 + 2z + 1) \left[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} \right. \\
& - \left. \frac{64}{3}H_{0,1,-1} \right] - \frac{128}{3}(3z^2 - 5z + 1)H_{0,1,1} - \frac{32}{3}(4z^2 + 2z + 1)H_{-1}\zeta_2 - \frac{32}{3}(2z - 1)H_1\zeta_2 \\
& + L_M \left[-\frac{32}{3}(4z + 3)H_0^2 + \frac{32}{9}(101z^2 - 8z + 13)H_0 - \frac{32(8z^3 - 14z^2 + 11z - 8)}{3z} \right. \\
& + \frac{32(47z^3 - 40z^2 - 7z - 4)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{8}{3}H_1^2 - \frac{16}{3}H_0H_1 \right] + (2z^2 + 2z + 1) \left[\frac{64}{3}H_{0,-1} \right. \\
& - \left. \frac{64}{3}H_{-1}H_0 \right] - \frac{64}{3}(4z + 1)H_{0,1} - \frac{128}{3}(z - 2)z\zeta_2 \Big] + \frac{32}{3}(6z^2 - 22z + 3)\zeta_3 \Big] \\
& + L_M \left[-\frac{64}{9}(z - 2)H_0^3 - \frac{32}{9}(54z^2 + 10z + 5)H_0^2 + \frac{32}{3}(3z^2 + 2z + 1)H_{-1}H_0^2 \right. \\
& - \frac{32}{3}(z - 1)^2H_1H_0^2 - \frac{16}{27}(1680z^2 + 1744z - 269)H_0 - \frac{128(28z^3 - 22z^2 - 4z - 3)H_1H_0}{9z} \\
& - \left. \frac{64}{3}(z + 1)^2H_{0,-1}H_0 - \frac{64}{3}(z^2 - 12z - 3)H_{0,1}H_0 + \frac{128}{3}(z^2 - 3z - 1)\zeta_2H_0 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{64(5z^3 - 4z^2 - z - 1)H_1^2}{9z} + \frac{8(20863z^3 - 20616z^2 + 159z - 172)}{81z} \\
& + \frac{32}{9}(114z^3 - 20z^2 + 9z - 4)\frac{\zeta_2}{z} + \frac{16(476z^3 - 548z^2 + 253z - 144)H_1}{27z} \\
& + \gamma_{qg}^0 \left[\frac{16}{3}H_0H_1^2 - \frac{8}{9}H_1^3 \right] + \frac{(9z^3 - z^2 - 14z - 2) \left[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1} \right]}{z} \\
& - \frac{32(2z^3 + 70z^2 + 25z + 12)H_{0,1}}{9z} + (4z^2 + 2z + 1) \left[-\frac{32}{3}H_0H_{-1}^2 + \frac{64}{3}H_{0,-1}H_{-1} \right. \\
& \left. - \frac{64}{3}H_{0,-1,-1} \right] - \frac{64}{3}(z^2 - 2z - 1)H_{0,0,-1} + \frac{64}{3}(z^2 - 20z - 3)H_{0,0,1} + (2z^2 + 2z + 1) \left[\frac{64}{3}H_{-1}H_{0,1} \right. \\
& \left. - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1} \right] + \frac{128}{3}(z^2 + z + 1)H_{0,1,1} - \frac{32}{3}(8z^2 + 6z + 3)H_{-1}\zeta_2 \\
& + \frac{32}{3}(4z^2 - 6z + 3)H_1\zeta_2 + \frac{32}{3}(6z^2 + 26z + 7)\zeta_3 \left. \right] + (2z^2 + 2z + 1) \left[-\frac{128}{9}H_0H_{-1}^3 \right. \\
& + \left[\frac{32}{3}H_0^2 + \frac{128}{3}H_{0,-1} \right]H_{-1}^2 + \left[\frac{32}{9}H_0^3 + \left[\frac{128}{3}H_{0,1} - \frac{128}{3}H_{0,-1} \right]H_0 - \frac{256}{3}H_{0,-1,-1} + \frac{128}{3}H_{0,0,-1} \right. \\
& \left. - \frac{256}{3}H_{0,0,1} \right]H_{-1} + 64\zeta_3H_{-1} - \frac{32}{3}H_0^2H_{0,-1} + H_0 \left[\frac{128}{3}H_{0,-1,-1} - \frac{128}{3}H_{0,-1,1} + \frac{64}{3}H_{0,0,-1} \right. \\
& \left. - \frac{128}{3}H_{0,1,-1} \right] + \frac{256}{3}H_{0,-1,-1,-1} + \frac{128}{3}H_{0,-1,0,1} - \frac{128}{3}H_{0,0,-1,-1} + \frac{256}{3}H_{0,0,-1,1} - \frac{64}{3}H_{0,0,0,-1} \\
& + \frac{256}{3}H_{0,0,1,-1} + \left[-\frac{64}{3}H_{-1}^2 - \frac{80}{3}H_0H_{-1} + \frac{80}{3}H_{0,-1} \right]\zeta_2 + \gamma_{qg}^0 \left[\frac{2}{9}H_1^4 + \frac{4}{3}H_0^2H_1^2 + \frac{10}{3}\zeta_2H_1^2 \right. \\
& + \left[\frac{80}{3}H_{0,0,1} + \frac{16}{3}H_{0,1,1} \right]H_1 - \frac{40}{9}\zeta_3H_1 - \frac{16}{3}H_{0,1}^2 + H_0 \left[-\frac{16}{9}H_1^3 - 16H_{0,1}H_1 + \frac{80}{3}H_{0,1,1} \right] \\
& \left. - \frac{80}{3}H_{0,0,1,1} - \frac{16}{3}H_{0,1,1,1} \right] \left. \right] + C_{ANFT_F}^2 \left[-\frac{4}{9}(2z + 1)H_0^4 + \frac{8}{27}(23z^2 + 12z + 3)H_0^3 \right. \\
& \left. - \frac{8}{27}(292z^2 + 39z + 42)H_0^2 - \frac{8(z - 1)(65z^2 + 17z + 8)H_1H_0^2}{9z} + \frac{32}{3}(4z + 1)H_{0,1}H_0^2 \right. \\
& + \frac{32}{3}(z - 1)zH_1^2H_0 + \frac{16}{81}(3392z^2 + 645z + 111)H_0 + \frac{32(z - 1)(254z^2 - 7z + 20)H_1H_0}{27z} \\
& \left. - \frac{16(23z^3 + 96z^2 + 15z + 8)H_{0,1}H_0}{9z} + \frac{8}{9}(62z^2 - 16z - 7)\zeta_2H_0 + \frac{32}{9}(28z + 13)\zeta_3H_0 \right. \\
& \left. - \frac{8}{9}(z - 1)(5z + 1)H_1^3 + \left[\frac{16(z - 1)(31z^2 + 7z + 4)}{27z} - \frac{32}{9}(4z + 1)H_0 - \frac{8}{9}\gamma_{qg}^0H_1 \right] L_M^3 \right. \\
& \left. - \frac{8}{3}(4z^2 - 6z + 1)H_1^2 - \frac{32(5854z^3 - 6219z^2 + 531z - 328)}{243z} + \frac{4}{9}(19z^3 - 50z^2 + 2z - 4)\frac{\zeta_2}{z} \right. \\
& \left. - \frac{16}{27}(550z^3 + 228z^2 + 33z - 4)\frac{\zeta_3}{z} + \frac{32}{3}(2z^2 - 2z - 1)H_1 + L_Q^3 \left[-\frac{16(z - 1)(31z^2 + 7z + 4)}{27z} \right. \right. \\
& \left. \left. + \frac{32}{9}(4z + 1)H_0 + \frac{8}{9}\gamma_{qg}^0H_1 \right] - \frac{32(290z^3 - 261z^2 + 27z - 20)H_{0,1}}{27z} \right. \\
& + \frac{16(111z^3 + 144z^2 + 21z + 8)H_{0,0,1}}{9z} - \frac{16}{3}z(5z - 4)H_{0,1,1} - \frac{16}{9}(2z^2 - 2z - 5)H_1\zeta_2 \\
& + L_Q^2 \left[-\frac{16}{3}(8z + 1)H_0^2 + \frac{16}{9}(49z^2 - 136z - 13)H_0 + \frac{8(1048z^3 - 894z^2 - 87z - 40)}{27z} \right. \\
& \left. + \frac{16(87z^3 - 80z^2 + 13z - 4)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{4}{3}H_1^2 - \frac{8}{3}H_0H_1 \right] + (2z^2 + 2z + 1) \left[\frac{32}{3}H_{0,-1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{32}{3}H_{-1}H_0] - \frac{32}{3}(4z+1)H_{0,1} - \frac{64}{3}(z-2)z\zeta_2] + z(z+1)\left[\frac{64}{3}H_0H_{-1}^2 + \left[-\frac{32}{3}H_0^2 - 64H_0\right. \right. \\
& \left. \left. - \frac{128}{3}H_{0,-1}\right]H_{-1} + \frac{64}{3}\zeta_2H_{-1} + \frac{64}{3}H_0H_{0,-1} + 64H_{0,-1} + \frac{128}{3}H_{0,-1,-1} - \frac{64}{3}H_{0,0,-1}\right] \\
& + (6z+1)\left[-\frac{8}{3}\zeta_2H_0^2 - \frac{64}{3}H_{0,0,1}H_0\right] + z\left[128H_{0,0,0,1} - \frac{1216\zeta_2^2}{15}\right] + L_M^2\left[-\frac{4}{3}\gamma_{qg}^0H_1^2\right. \\
& \left. - \frac{32}{9}(4z^2 - 4z + 5)H_1 - \frac{8(205z^3 - 168z^2 + 42z - 52)}{27z} - \frac{32}{9}(9z^2 - 20z - 5)H_0\right. \\
& \left. + (2z^2 + 2z + 1)\left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\right] + z\left[\frac{64}{3}H_0^2 + \frac{64\zeta_2}{3}\right]\right] + L_M\left[-\frac{16}{9}(10z - 1)H_0^3\right. \\
& \left. + \frac{8}{9}(7z^2 - 66z - 13)H_0^2 + \frac{16}{27}(58z^2 - 269z - 2)H_0 - \frac{16(z-1)(65z^2 + 17z + 8)H_1H_0}{9z}\right. \\
& \left. - \frac{32}{3}(2z^2 - 10z - 1)H_{0,1}H_0 - \frac{8}{9}(13z^2 - 16z + 23)H_1^2 + \frac{8(592z^3 - 268z^2 - 119z - 4)}{27z}\right. \\
& \left. + \frac{16}{27}(64z^2 - 64z + 29)H_1 + (4z^2 + 4z + 5)\left[\frac{64}{9}H_{0,-1} - \frac{64}{9}H_{-1}H_0\right]\right. \\
& \left. + \frac{16(68z^3 - 54z^2 - 9z - 8)H_{0,1}}{9z} + \frac{32}{3}(2z^2 - 18z - 3)H_{0,0,1} - \frac{16}{9}z(3z + 10)\zeta_2\right. \\
& \left. + (2z^2 + 2z + 1)\left[-\frac{32}{3}H_0H_{-1}^2 + \left(\frac{64}{3}H_{0,-1} - \frac{16}{3}H_0^2\right)H_{-1} - \frac{32}{3}\zeta_2H_{-1} + \frac{32}{3}H_0H_{0,-1} - \frac{64}{3}H_{0,-1,-1}\right. \right. \\
& \left. \left. - \frac{32}{3}H_{0,0,-1}\right] + \gamma_{qg}^0\left[-\frac{8}{9}H_1^3 - \frac{4}{3}H_0^2H_1 + \frac{16}{3}H_{0,1}H_1 - \frac{8}{3}\zeta_2H_1 - \frac{32}{3}H_{0,1,1}\right] + \frac{128}{3}(5z + 1)\zeta_3\right] \\
& + L_Q\left[\frac{16}{9}(26z + 1)H_0^3 - \frac{8}{9}(126z^2 - 542z - 39)H_0^2 + \frac{32}{3}(5z^2 + 4z + 2)H_{-1}H_0^2\right. \\
& \left. - \frac{32}{3}(3z^2 - 4z + 2)H_1H_0^2 - \frac{16}{27}(1520z^2 - 2822z - 89)H_0 - \frac{64}{3}(3z^2 + 4z + 2)H_{0,-1}H_0\right. \\
& \left. + \frac{64}{3}(z^2 + 2z + 2)H_{0,1}H_0 + \frac{128}{3}(z - 5)z\zeta_2H_0 - \frac{8(2306z^3 - 2056z^2 - 181z + 16)}{9z}\right. \\
& \left. + \frac{32}{9}(105z^3 - 181z^2 + 3z - 4)\frac{\zeta_2}{z} - \frac{16(1934z^3 - 1807z^2 + 83z - 40)H_1}{27z}\right. \\
& \left. + \frac{(87z^3 - 80z^2 + 13z - 4)\left[-\frac{16}{9}H_1^2 - \frac{32}{9}H_0H_1\right]}{z} + \frac{(17z^3 + 7z^2 - 4z - 2)\left[\frac{64}{9}H_{-1}H_0 - \frac{64}{9}H_{0,-1}\right]}{z}\right. \\
& \left. - \frac{32(18z^3 - 115z^2 - 10z + 4)H_{0,1}}{9z} + \gamma_{qg}^0\left[\frac{8}{9}H_1^3 + \frac{16}{3}H_0H_1^2 - \frac{32}{3}H_{0,1}H_1\right] + (2z + 1)\left[\frac{32}{3}H_0H_{-1}^2\right. \right. \\
& \left. \left. - \frac{64}{3}H_{0,-1}H_{-1} + \frac{64}{3}H_{0,-1,-1}\right] + \frac{64}{3}(z^2 + 4z + 2)H_{0,0,-1} - \frac{64}{3}(z^2 - 2z + 2)H_{0,0,1}\right. \\
& \left. + (2z^2 + 2z + 1)\left[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1}\right] - \frac{128}{3}(3z^2 - 5z + 1)H_{0,1,1}\right. \\
& \left. - \frac{32}{3}(4z^2 + 2z + 1)H_{-1}\zeta_2 - \frac{32}{3}(2z - 1)H_1\zeta_2 + \frac{32}{3}(6z^2 - 22z + 3)\zeta_3\right] \\
& + (2z^2 + 2z + 1)\left[-\frac{64}{9}H_0H_{-1}^3 + \left[\frac{16}{3}H_0^2 + \frac{64}{3}H_{0,-1}\right]H_{-1}^2 + \left[\frac{16}{9}H_0^3 + \left[\frac{64}{3}H_{0,1} - \frac{64}{3}H_{0,-1}\right]H_0\right. \right. \\
& \left. \left. - \frac{128}{3}H_{0,-1,-1} + \frac{64}{3}H_{0,0,-1} - \frac{128}{3}H_{0,0,1}\right]H_{-1} + 32\zeta_3H_{-1} - \frac{16}{3}H_0^2H_{0,-1} + H_0\left[\frac{64}{3}H_{0,-1,-1}\right. \right. \\
& \left. \left. - \frac{64}{3}H_{0,-1,1} + \frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,1,-1}\right] + \frac{128}{3}H_{0,-1,-1,-1} + \frac{64}{3}H_{0,-1,0,1} - \frac{64}{3}H_{0,0,-1,-1}\right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{128}{3}H_{0,0,-1,1} - \frac{32}{3}H_{0,0,0,-1} + \frac{128}{3}H_{0,0,1,-1} + \left[-\frac{32}{3}H_{-1}^2 - \frac{32}{3}H_0H_{-1} + \frac{32}{3}H_{0,-1} \right] \zeta_2 + \gamma_{ag}^0 \left[\frac{1}{9}H_1^4 \right. \\
& + \frac{2}{3}H_0^2H_1^2 + \frac{4}{3}\zeta_2H_1^2 + \left. \left[\frac{40}{3}H_{0,0,1} + \frac{8}{3}H_{0,1,1} \right] H_1 - \frac{40}{9}\zeta_3H_1 - \frac{8}{3}H_{0,1}^2 + H_0 \left[-\frac{8}{9}H_1^3 - 8H_{0,1}H_1 \right. \right. \\
& + \left. \left. \frac{40}{3}H_{0,1,1} \right] - \frac{40}{3}H_{0,0,1,1} - \frac{8}{3}H_{0,1,1,1} \right] + C_{ATF}^2 \left[\frac{1}{9}(18z^2 - 26z + 23)H_0^4 + \frac{2}{9}(225z^2 - 12z - 35)H_0^3 \right. \\
& - \frac{4(60z^3 + 46z^2 + 5z + 8)H_{-1}H_0^3}{9z} + \frac{8(z-1)(65z^2 + 17z + 8)H_1H_0^3}{9z} - \frac{32}{3}(4z+1)H_{0,1}H_0^3 \\
& + \frac{8}{3}(8z-5)\zeta_2H_0^3 - \frac{4(120z^3 + 106z^2 + 5z + 8)H_{-1}^2H_0^2}{3z} + \frac{2(190z^3 - 142z^2 - 13z - 24)H_1^2H_0^2}{3z} \\
& + \frac{2}{27}(5248z^2 - 6573z + 738)H_0^2 + \frac{8(z+1)(359z^2 - 32z + 20)H_{-1}H_0^2}{9z} \\
& + \frac{2(z-1)(1619z^2 - 169z + 308)H_1H_0^2}{9z} + \frac{4(96z^3 + 70z^2 + 5z + 8)H_{0,-1}H_0^2}{3z} \\
& + \frac{8}{3}(32z^2 - 62z + 7)H_{0,1}H_0^2 - 16(2z^2 - 6z + 3)H_{0,0,-1}H_0^2 - 64(z-1)H_{0,0,1}H_0^2 \\
& + 96(z-3)zH_{0,1,1}H_0^2 + \frac{2}{3}(38z^2 - 6z + 47)\zeta_2H_0^2 - \frac{32}{3}(31z-5)\zeta_3H_0^2 \\
& - \frac{8(93z^3 - 82z^2 + 8z - 8)H_1^3H_0}{9z} - \frac{8(241z^3 - 229z^2 + 11z - 20)H_1^2H_0}{3z} + 32(z^2 + 9z + 3)H_{0,1}^2H_0 \\
& + \frac{16}{5}(124z - 19)\zeta_2^2H_0 - \frac{4(225962z^3 + 100242z^2 + 16377z + 2624)H_0}{81z} \\
& - \frac{2}{9}(6474z^3 + 1864z^2 + 1165z + 160)\frac{\zeta_2}{z}H_0 + \frac{8}{9}(210z^3 + 340z^2 - 269z - 8)\frac{\zeta_3}{z}H_0 \\
& + \frac{8}{3}(67z^3 + 46z^2 + 2z + 12)\frac{\zeta_2}{z}H_{-1}H_0 - \frac{8(16040z^3 - 16335z^2 + 1575z - 1388)H_1H_0}{27z} \\
& - \frac{8}{3}(z-1)(137z^2 + 41z + 20)\frac{\zeta_2}{z}H_1H_0 - \frac{16(323z^3 + 111z^2 - 12z + 20)H_{0,-1}H_0}{9z} \\
& + \frac{4(1501z^3 + 5892z^2 + 93z + 468)H_{0,1}H_0}{9z} - \frac{8(406z^3 - 330z^2 - 3z - 40)H_1H_{0,1}H_0}{3z} \\
& - \frac{16(108z^3 + 82z^2 + 5z + 8)H_{0,-1,-1}H_0}{3z} - \frac{8(132z^3 + 94z^2 + 5z + 8)H_{0,0,-1}H_0}{3z} \\
& - \frac{16(129z^3 - 210z^2 + 28z - 8)H_{0,0,1}H_0}{3z} + \frac{8(768z^3 - 398z^2 + 19z - 56)H_{0,1,1}H_0}{3z} \\
& - 128(4z^2 + 3)H_{0,-1,-1,-1}H_0 + 128(7z - 1)H_{0,0,0,1}H_0 - 32(6z^2 + 14z + 9)H_{0,0,1,1}H_0 \\
& - 32(6z^2 - 14z + 1)H_{0,1,1,1}H_0 + 8(2z^2 + 38z + 11)H_{0,1}\zeta_2H_0 + \frac{(204z^3 - 166z^2 - 19z - 8)H_1^4}{9z} \\
& + \frac{2(1321z^3 - 1440z^2 + 15z - 4)H_1^3}{27z} + \frac{8}{3}(132z^3 + 118z^2 + 5z + 8)\frac{\zeta_2}{z}H_{-1}^2 \\
& + \frac{2(3536z^3 - 3108z^2 + 39z - 134)H_1^2}{27z} + \frac{4}{3}(102z^3 - 94z^2 + 11z - 8)\frac{\zeta_2}{z}H_1^2 + 32z^2H_{0,-1}^2 \\
& - \frac{8(161z^3 - 130z^2 - 4z - 16)H_{0,1}^2}{3z} + \frac{4(1158802z^3 - 1178838z^2 + 83079z - 66607)}{243z} \\
& + \frac{2}{27}(57548z^3 - 10209z^2 + 4590z - 2176)\frac{\zeta_2}{z} + \frac{16}{15}(86z^3 + 1673z^2 - 90z + 76)\frac{\zeta_2^2}{z} \\
& + \frac{4}{27}(16364z^3 + 25128z^2 + 3537z + 448)\frac{\zeta_3}{z} - \frac{16}{9}(z+1)(413z^2 - 32z + 20)\frac{\zeta_2}{z}H_{-1} \\
& - 8(112z^3 + 98z^2 + 5z + 8)\frac{\zeta_3}{z}H_{-1} + \frac{4(18046z^3 - 18101z^2 + 1286z - 1465)H_1}{27z}
\end{aligned}$$

$$\begin{aligned}
& + \frac{4}{9}(3111z^3 - 3052z^2 + 116z - 242) \frac{\zeta_2}{z} H_1 - \frac{8}{9}(874z^3 - 698z^2 - 41z - 80) \frac{\zeta_3}{z} H_1 \\
& + \frac{(z+1)(2443z^2 - 244z + 94) \left[\frac{16}{27} H_{-1} H_0 - \frac{16}{27} H_{0,-1} \right]}{z} - \frac{8}{3}(103z^3 + 70z^2 + 2z + 12) \frac{\zeta_2}{z} H_{0,-1} \\
& + \frac{8(9594z^3 - 17343z^2 + 1125z - 1388) H_{0,1}}{27z} - \frac{8}{3}(124z^3 + 258z^2 + 33z + 16) \frac{\zeta_2}{z} H_{0,1} \\
& + \frac{(z+1)(305z^2 - 32z + 20) \left[-\frac{16}{9} H_0 H_{-1}^2 + \frac{32}{9} H_{0,-1} H_{-1} - \frac{32}{9} H_{0,-1,-1} \right]}{z} \\
& + \frac{16(287z^3 - 105z^2 - 12z + 20) H_{0,0,-1}}{9z} - \frac{4(3525z^3 + 9840z^2 + 495z + 628) H_{0,0,1}}{9z} \\
& + \frac{8(728z^3 - 590z^2 - 11z - 72) H_1 H_{0,0,1}}{3z} - 192(z^2 + 5z + 2) H_{0,1} H_{0,0,1} \\
& + \frac{(84z^3 + 70z^2 + 5z + 8) \left[H_0 \left[\frac{16}{3} H_{0,-1,1} + \frac{16}{3} H_{0,1,-1} \right] - \frac{16}{3} H_{-1} H_0 H_{0,1} \right]}{z} \\
& + \frac{4(3425z^3 - 3312z^2 + 48z - 240) H_{0,1,1}}{9z} + \frac{8(84z^3 - 70z^2 + 5z - 8) H_1 H_{0,1,1}}{3z} \\
& + \frac{(108z^3 + 94z^2 + 5z + 8) \left[\frac{16}{9} H_0 H_{-1}^3 - \frac{16}{3} H_{0,-1} H_{-1}^2 + \frac{32}{3} H_{0,-1,-1} H_{-1} - \frac{32}{3} H_{0,-1,-1,-1} \right]}{z} \\
& + \frac{(96z^3 + 82z^2 + 5z + 8) \left[\frac{16}{3} H_{-1} H_0 H_{0,-1} - \frac{16}{3} H_{0,-1,0,1} \right]}{z} + (4z - 1) \left[\frac{4}{15} H_0^5 - 64 H_0 H_{0,-1,0,1} \right] \\
& + \frac{(72z^3 + 58z^2 + 5z + 8) \left[\frac{16}{3} H_{0,0,-1,-1} - \frac{16}{3} H_{-1} H_{0,0,-1} \right]}{z} + \frac{8(168z^3 + 118z^2 + 5z + 8) H_{0,0,0,-1}}{3z} \\
& + \frac{16(203z^3 - 408z^2 + 69z - 16) H_{0,0,0,1}}{3z} + \frac{(90z^3 + 76z^2 + 5z + 8)}{z} \left[\frac{32}{3} H_{-1} H_{0,0,1} - \frac{32}{3} H_{0,0,-1,1} \right. \\
& \left. - \frac{32}{3} H_{0,0,1,-1} \right] - \frac{8(932z^3 - 266z^2 + 31z - 56) H_{0,0,1,1}}{3z} + z(z+1) \left[-64 H_{0,1} H_{-1}^2 \right. \\
& \left. + \left[192 H_{0,1} + 128 H_{0,-1,1} + 128 H_{0,1,-1} \right] H_{-1} - 192 H_{0,-1,1} - 192 H_{0,1,-1} - 128 H_{0,-1,-1,1} \right. \\
& \left. - 128 H_{0,-1,1,-1} - 128 H_{0,1,-1,-1} \right] - \frac{8(30z^3 - 10z^2 - 7z - 8) H_{0,1,1,1}}{3z} + (z^2 - z + 1) \left[\frac{32}{3} H_{0,-1} H_0^3 \right. \\
& \left. - 64 H_{0,-1}^2 H_0 + H_{0,-1} \left[256 H_{0,-1,-1} + 128 H_{0,0,-1} \right] - 512 H_{0,-1,0,-1,-1} - 1024 H_{0,0,-1,-1,-1} \right. \\
& \left. - 128 H_{0,0,-1,0,-1} \right] - 32(2z^2 - 22z + 7) H_{0,0,0,-1,-1} - 32(54z - 1) H_{0,0,0,0,1} \\
& + (z^2 - 5z + 2) \left[64 H_0 H_{0,0,0,-1} + 256 H_{0,0,0,-1,1} + 256 H_{0,0,0,1,-1} \right] + 64(13z^2 + 83z + 32) H_{0,0,0,1,1} \\
& + (2z^2 - 14z + 5) \left[-32 H_{0,-1} H_{0,0,1} + 32 H_{0,0,-1,0,1} - 32 H_{0,0,0,0,-1} + 32 H_{0,0,1,0,-1} \right] \\
& + 64(5z^2 + 29z + 11) H_{0,0,1,0,1} - 128(12z^2 - 8z + 7) H_{0,0,1,1,1} - 64(9z^2 - 7z + 5) H_{0,1,0,1,1} \\
& - 64(z^2 + z + 1) H_{0,1,1,1,1} + (6z^2 - 10z + 7)(24 H_{0,0,-1} - 16 H_{0,-1,-1}) \zeta_2 \\
& - 8(2z^2 + 46z + 7) H_{0,0,1} \zeta_2 - 16(30z^2 - 22z + 17) H_{0,1,1} \zeta_2 + L_M^3 \left[\frac{16}{3} (8z - 1) H_0^2 \right. \\
& \left. - \frac{8(18z^3 - 152z^2 - 11z - 8) H_0}{9z} - \frac{4(z-1)(1883z^2 - 97z + 272)}{27z} - \frac{8(146z^3 - 118z^2 - z - 16) H_1}{9z} \right. \\
& \left. + \gamma_{qg}^0 \left[\frac{8}{3} H_1^2 + \frac{8}{3} H_0 H_1 \right] + \frac{32}{3} (2z^2 + 6z + 3) H_{0,1} - \frac{64}{3} (4z + 1) \zeta_2 \right] + L_Q^3 \left[-\frac{16}{3} (8z - 1) H_0^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{8(18z^3 - 152z^2 - 11z - 8)H_0}{9z} + \frac{4(z-1)(1883z^2 - 97z + 272)}{27z} + \frac{8(146z^3 - 118z^2 - z - 16)H_1}{9z} \\
& + \gamma_{qg}^0 \left[-\frac{8}{3}H_1^2 - \frac{8}{3}H_0H_1 \right] - \frac{32}{3}(2z^2 + 6z + 3)H_{0,1} + \frac{64}{3}(4z + 1)\zeta_2 \Big] + (6z^2 - 2z + 5) \left[16H_0^2H_{0,-1,-1} \right. \\
& - 24H_0H_{0,-1}\zeta_2 \Big] + 32(4z^2 - 8z + 5)H_{0,-1}\zeta_3 + \frac{160}{3}(2z^2 + 6z + 3)H_{0,1}\zeta_3 \\
& + \frac{8}{3}(278z + 83)\zeta_2\zeta_3 + L_Q^2 \left[\frac{16}{3}(16z - 3)H_0^3 - \frac{4}{3}(54z^2 - 436z - 23)H_0^2 \right. \\
& - \frac{4(5183z^3 - 3512z^2 + 1009z + 104)H_0}{9z} - \frac{8(280z^3 - 238z^2 + 5z - 24)H_1H_0}{3z} \\
& - 16(6z^2 - 10z + 7)H_{0,-1}H_0 + 16(2z^2 + 22z + 7)H_{0,1}H_0 - 32(14z + 3)\zeta_2H_0 \\
& - \frac{4(352z^3 - 310z^2 + 17z - 24)H_1^2}{3z} - \frac{2(6639z^3 - 7318z^2 - 509z + 1260)}{9z} \\
& - \frac{4(4549z^3 - 4204z^2 + 59z - 168)H_1}{9z} + \frac{(146z^3 + 118z^2 - z + 16) \left[\frac{8}{3}H_{-1}H_0 - \frac{8}{3}H_{0,-1} \right]}{z} \\
& + \frac{8(28z^3 + 272z^2 + 23z - 8)H_{0,1}}{3z} + 16(6z^2 - 26z + 11)H_{0,0,-1} + 16(2z^2 - 6z - 5)H_{0,0,1} \\
& + 64(z^2 + 5z + 2)H_{0,1,1} + \frac{16}{3}(126z^2 - 196z - 9)\zeta_2 + (2z^2 + 2z + 1) \left[-32H_0H_{-1}^2 \right. \\
& + \left. \left[24H_0^2 + 64H_{0,-1} + 32H_{0,1} \right] H_{-1} - 64\zeta_2H_{-1} - 64H_{0,-1,-1} - 32H_{0,-1,1} \right. \\
& - \left. 32H_{0,1,-1} \right] + \gamma_{qg}^0 \left[4H_1^3 + 12H_0H_1^2 + 6H_0^2H_1 - 16\zeta_2H_1 \right] + 32(z-3)\zeta_3 \Big] \\
& + L_M^2 \left[-\frac{32}{3}(4z-1)H_0^3 + \frac{16}{3}(9z^2 + z - 3)H_0^2 + \frac{8(1335z^3 + 710z^2 + 266z + 40)H_0}{9z} \right. \\
& + \frac{16(z-1)(19z^2 + 7z + 4)H_1H_0}{3z} + 16(6z^2 - 10z + 7)H_{0,-1}H_0 - 16(2z^2 + 6z + 3)H_{0,1}H_0 \\
& - 64(z-1)\zeta_2H_0 - \frac{4(132z^3 - 118z^2 + 5z - 8)H_1^2}{3z} - \frac{8(2695z^3 - 2661z^2 + 192z - 244)}{9z} \\
& - \frac{16}{3}(44z^3 + 83z^2 - 3z + 4)\frac{\zeta_2}{z} - \frac{8(617z^3 - 604z^2 - 70z - 28)H_1}{9z} \\
& + \frac{(146z^3 + 118z^2 - z + 16) \left[\frac{8}{3}H_{0,-1} - \frac{8}{3}H_{-1}H_0 \right]}{z} + \frac{16}{3}z(25z + 36)H_{0,1} \\
& - 16(6z^2 - 26z + 11)H_{0,0,-1} + 32(4z + 1)H_{0,1,1} + (2z^2 + 2z + 1) \left[32H_0H_{-1}^2 + \left[-24H_0^2 \right. \right. \\
& - \left. \left. 64H_{0,-1} - 32H_{0,1} \right] H_{-1} + 64\zeta_2H_{-1} + 64H_{0,-1,-1} + 32H_{0,-1,1} + 16H_{0,0,1} + 32H_{0,1,-1} \right] \\
& + \gamma_{qg}^0 \left[4H_1^3 - 2H_0^2H_1 - 8\zeta_2H_1 \right] - 32(17z - 2)\zeta_3 \Big] + (2z^2 + 2z + 1) \left[-16H_0H_{-1}^4 + \left[\frac{64}{3}H_0^2 \right. \right. \\
& + 64H_{0,-1} + \frac{64}{3}H_{0,1} \Big] H_{-1}^3 + \left[-\frac{8}{3}H_0^3 + \left[32H_{0,1} - 64H_{0,-1} \right] H_0 - 192H_{0,-1,-1} - 64H_{0,-1,1} \right. \\
& - \left. 96H_{0,0,1} - 64H_{0,1,-1} \right] H_{-1}^2 - \frac{376}{5}\zeta_2^2H_{-1} + \left[-\frac{4}{3}H_0^4 + \left[-16H_{0,-1} - 32H_{0,1} \right] H_0^2 \right. \\
& + \left. \left[128H_{0,-1,-1} - 64H_{0,-1,1} + 96H_{0,0,-1} + 64H_{0,0,1} - 64H_{0,1,-1} - 64H_{0,1,1} \right] H_0 \right. \\
& + 384H_{0,-1,-1,-1} + 128H_{0,-1,-1,1} + 128H_{0,-1,0,1} + 128H_{0,-1,1,-1} + 192H_{0,0,-1,1} - 160H_{0,0,0,-1} \\
& - \left. 32H_{0,0,0,1} + 192H_{0,0,1,-1} + 128H_{0,0,1,1} + 128H_{0,1,-1,-1} \right] H_{-1} + H_0^2 \left[32H_{0,-1,1} + 32H_{0,1,-1} \right]
\end{aligned}$$

$$\begin{aligned}
& +H_0 \left[64H_{0,-1,-1,1} + 64H_{0,-1,1,-1} + 64H_{0,-1,1,1} - 96H_{0,0,-1,-1} - 64H_{0,0,-1,1} - 64H_{0,0,1,-1} \right. \\
& + 64H_{0,1,-1,-1} + 64H_{0,1,-1,1} + 64H_{0,1,1,-1} \left. \right] - 384H_{0,-1,-1,-1,-1} \\
& - 128H_{0,-1,-1,-1,1} - 128H_{0,-1,-1,0,1} - 128H_{0,-1,-1,1,-1} - 128H_{0,-1,0,-1,1} - 128H_{0,-1,0,1,-1} \\
& - 64H_{0,-1,0,1,1} - 128H_{0,-1,1,-1,-1} - 192H_{0,0,-1,-1,1} - 192H_{0,0,-1,1,-1} \\
& - 128H_{0,0,-1,1,1} - 192H_{0,0,1,-1,-1} - 128H_{0,0,1,-1,1} - 128H_{0,0,1,1,-1} - 128H_{0,1,-1,-1,-1} \\
& + \left[-\frac{160}{3}H_{-1}^3 + 40H_0H_{-1}^2 + \left[36H_0^2 - 16H_{0,-1} + 96H_{0,1} \right] H_{-1} - 96H_{0,-1,1} - 96H_{0,1,-1} \right] \zeta_2 \\
& + \left[160H_{-1}^2 - 64H_{-1}H_0 \right] \zeta_3 + \gamma_{gg}^0 \left[-\frac{1}{3}H_1^5 - \frac{2}{3}H_0^3H_1^2 + \left[-44H_{0,0,1} - 4H_{0,1,1} \right] H_1^2 \right. \\
& + 46\zeta_2^2H_1 + 8H_0^2H_{0,1}H_1 + \left[20H_{0,1}^2 - 16H_{0,0,0,1} + 24H_{0,0,1,1} \right] H_1 + H_0 \left[\frac{5}{3}H_1^4 + 24H_{0,1}H_1^2 \right. \\
& + \left. \left. \left[-8H_{0,0,1} - 56H_{0,1,1} \right] H_1 \right] - 40H_{0,1}H_{0,1,1} + \left[-\frac{8}{3}H_1^3 + 2H_0H_1^2 + H_0^2H_1 - 32H_{0,1}H_1 \right] \zeta_2 \right. \\
& + \left. \left[\frac{52}{3}H_1^2 + \frac{40}{3}H_0H_1 \right] \zeta_3 \right] + L_M \left[\frac{4}{3}(19z - 5)H_0^4 + \frac{4(136z^3 + 160z^2 + 59z + 24)H_{-1}H_0^2}{3z} \right. \\
& - \frac{4}{9}(54z^2 + 40z - 37)H_0^3 - \frac{2}{9}(4303z^2 + 216z + 1303)H_0^2 - \frac{4(18z^3 + 59z^2 - 58z - 8)H_1H_0^2}{3z} \\
& + 16(z^2 + 8z - 1)H_{0,-1}H_0^2 + 4(36z^2 - 2z + 23)H_{0,1}H_0^2 + \frac{40(z - 1)(31z^2 + 7z + 4)H_1^2H_0}{3z} \\
& + 4(24z^2 - 22z - 1)\zeta_2H_0^2 + \frac{8(30972z^3 + 13409z^2 + 3023z + 448)H_0}{27z} \\
& + \frac{4(z - 1)(1583z^2 - 5z + 192)H_1H_0}{3z} + \frac{8(128z^3 - 346z^2 - 5z - 24)H_{0,-1}H_0}{3z} \\
& + \frac{8(40z^3 - 329z^2 - 86z - 24)H_{0,1}H_0}{3z} - 64(5z^2 - 4z + 5)H_{0,-1,-1}H_0 \\
& - 32(3z^2 + 16z + 3)H_{0,0,-1}H_0 - 32(z + 1)(13z + 7)H_{0,0,1}H_0 \\
& - 32(12z^2 + 8z + 11)H_{0,1,1}H_0 + 8(30z^2 - 31z + 13)\zeta_2H_0 - 96(22z + 3)\zeta_3H_0 \\
& - \frac{8}{9}(103z^2 - 94z + 2)H_1^3 - \frac{2(2023z^3 - 2032z^2 - 247z - 84)H_1^2}{9z} \\
& + 32(3z^2 - 6z + 4)H_{0,-1}^2 + 64(3z^2 - z + 2)H_{0,1}^2 - \frac{4}{5}(112z^2 + 1758z + 245)\zeta_2^2 \\
& - \frac{4(290560z^3 - 295527z^2 + 31944z - 24160)}{81z} - \frac{4}{9}(1333z^3 + 2860z^2 - 762z + 336)\frac{\zeta_2}{z} \\
& + \frac{32}{3}(180z^3 - 343z^2 - 14z - 24)\frac{\zeta_3}{z} - \frac{4(9884z^3 - 13409z^2 + 3625z + 492)H_1}{27z} \\
& - \frac{8}{3}(54z^3 - 76z^2 + 41z - 8)\frac{\zeta_2}{z}H_1 + \frac{(541z^3 + 476z^2 - 50z + 62)}{z} \left[\frac{32}{9}H_{0,-1} - \frac{32}{9}H_{-1}H_0 \right] \\
& - \frac{4(3416z^3 - 3816z^2 + 1353z - 416)H_{0,1}}{9z} - \frac{16(40z^3 - 26z^2 - 17z - 8)H_1H_{0,1}}{3z} \\
& + \frac{(176z^3 + 112z^2 - 37z + 16)}{z} \left[\frac{8}{3}H_0H_{-1}^2 - \frac{16}{3}H_{0,-1}H_{-1} + \frac{16}{3}H_{0,-1,-1} \right] \\
& - \frac{8(392z^3 - 532z^2 + 49z - 24)H_{0,0,-1}}{3z} - \frac{8(152z^3 - 624z^2 - 75z - 40)H_{0,0,1}}{3z} \\
& + \frac{(z + 1)(5z^2 + 4z + 2)}{z} \left[\frac{64}{3}H_{-1}H_{0,1} - \frac{64}{3}H_{0,-1,1} - \frac{64}{3}H_{0,1,-1} \right] + \frac{32}{3}(8z^2 + 49z - 11)H_{0,1,1} \\
& - 64(2z^2 - 4z + 1)H_{0,-1,0,1} + 16(10z^2 + 46z + 23)H_{0,0,0,-1} + 16(22z^2 + 126z + 13)H_{0,0,0,1}
\end{aligned}$$

$$\begin{aligned}
& +32(12z^2 + 4z + 9)H_{0,0,1,1} + \frac{8}{3}(136z^2 + 40z - 85)H_{-1}\zeta_2 - 16(16z^2 - 2z + 13)H_{0,-1}\zeta_2 \\
& -16(20z^2 - 26z + 9)H_{0,1}\zeta_2 + (2z^2 + 2z + 1)\left[-\frac{160}{3}H_0H_{-1}^3 + [8H_0^2 + 160H_{0,-1}]H_{-1}^2\right. \\
& + \left[\frac{8}{3}H_0^3 + 64H_{0,-1}H_0 - 320H_{0,-1,-1} - 160H_{0,0,-1} + 32H_{0,0,1} - 64H_{0,1,1}\right]H_{-1} \\
& + 192\zeta_3H_{-1} + 320H_{0,-1,-1,-1} + 64H_{0,-1,1,1} + 160H_{0,0,-1,-1} - 32H_{0,0,-1,1} - 32H_{0,0,1,-1} \\
& + 64H_{0,1,-1,1} + 64H_{0,1,1,-1} + [64H_{-1}H_0 - 80H_{-1}^2]\zeta_2 + \gamma_{qg}^0[2H_1^4 - 4H_0^2H_1^2 + 8H_{0,1}H_1^2 \\
& + \frac{10}{3}H_0^3H_1 + [48H_{0,0,-1} + 64H_{0,0,1}]H_1 - 136\zeta_3H_1 - 24H_{0,-1}H_{0,1} + H_0\left[-\frac{20}{3}H_1^3\right. \\
& + \left.[-24H_{0,-1} - 16H_{0,1}]H_1 + 24H_{0,-1,1} + 24H_{0,1,-1}\right] - 8H_{0,1,1,1} + [-8H_1^2 - 40H_0H_1]\zeta_2 \left. \right] \\
& + L_Q \left[-\frac{4}{3}(51z - 11)H_0^4 + \frac{4}{9}(198z^2 - 1648z - 83)H_0^3 - \frac{8}{3}(22z^2 + 18z + 9)H_{-1}H_0^3 \right. \\
& + \frac{8}{3}(22z^2 - 26z + 13)H_1H_0^3 + 8(16z^2 + 10z + 5)H_{-1}^2H_0^2 + 16(7z^2 - 8z + 4)H_1^2H_0^2 \\
& + \frac{2}{9}(18688z^2 - 14584z + 3849)H_0^2 - \frac{8(389z^3 + 317z^2 + 7z + 28)H_{-1}H_0^2}{3z} \\
& + \frac{4(822z^3 - 631z^2 - 61z - 72)H_1H_0^2}{3z} + 16(3z^2 - 22z + 9)H_{0,-1}H_0^2 \\
& - 4(36z^2 + 134z + 63)H_{0,1}H_0^2 - 4(24z^2 - 230z - 9)\zeta_2H_0^2 \\
& + \frac{8(477z^3 - 428z^2 + 37z - 28)H_1^2H_0}{3z} + \frac{4(37209z^3 - 68257z^2 - 7837z - 1136)H_0}{27z} \\
& - \frac{8}{3}(708z^3 - 1801z^2 - 21z - 16)\frac{\zeta_2}{z}H_0 + \frac{16(3243z^3 - 3023z^2 + 10z - 58)H_1H_0}{9z} \\
& + \frac{16(129z^3 + 194z^2 - 50z + 36)H_{0,-1}H_0}{3z} - 64(7z^2 + 6z + 3)H_{-1}H_{0,-1}H_0 \\
& - \frac{8(42z^3 + 833z^2 - 49z - 40)H_{0,1}H_0}{3z} - 64(z - 1)^2H_1H_{0,1}H_0 + 128(5z^2 + 2z + 3)H_{0,-1,-1}H_0 \\
& + 32(5z^2 + 44z - 7)H_{0,0,-1}H_0 + 32(11z^2 + 24z + 19)H_{0,0,1}H_0 + (10z^2 - 2z + 5)\left[32H_{0,-1,1}\right. \\
& + \left.32H_{0,1,-1}\right]H_0 + 32(2z^2 - 32z - 5)H_{0,1,1}H_0 + 32(14z^2 + 10z + 5)H_{-1}\zeta_2H_0 \\
& - 128(5z^2 - 6z + 3)H_1\zeta_2H_0 + 32(64z + 17)\zeta_3H_0 + \frac{8(424z^3 - 382z^2 + 29z - 24)H_1^3}{9z} \\
& + \frac{8(3661z^3 - 3443z^2 + 79z - 71)H_1^2}{9z} - 32(3z^2 - 2z + 3)H_{0,-1}^2 - 32(3z^2 - 4z + 2)H_{0,1}^2 \\
& + \frac{4}{5}(352z^2 - 18z + 157)\zeta_2^2 + \frac{2(503837z^3 - 529368z^2 + 89781z - 57320)}{81z} \\
& - \frac{16}{9}(4600z^3 - 3718z^2 + 405z - 136)\frac{\zeta_2}{z} - \frac{8}{3}(1252z^3 - 1106z^2 + 15z - 96)\frac{\zeta_3}{z} \\
& + \frac{8}{3}(536z^3 + 424z^2 - 7z + 24)\frac{\zeta_2}{z}H_{-1} + \frac{4(52691z^3 - 54734z^2 + 469z + 3996)H_1}{27z} \\
& - \frac{8}{3}(766z^3 - 676z^2 + 11z - 64)\frac{\zeta_2}{z}H_1 + \frac{(272z^3 + 257z^2 - 128z - 240)\left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0\right]}{z} \\
& + \frac{16(1357z^3 - 952z^2 + 395z + 162)H_{0,1}}{9z} + \frac{(100z^3 + 92z^2 + 19z - 8)}{z}\left[\frac{8}{3}H_0H_{-1}^2 - \frac{16}{3}H_{0,-1}H_{-1}\right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{16}{3} H_{0,-1,-1} \Big] + \gamma_{qg}^0 \left[-2H_1^4 + \left[\frac{88}{3} H_{0,1} - 48H_{0,0,-1} \right] H_1 + H_0 \left[24H_1 H_{0,-1} - \frac{28}{3} H_1^3 \right] + 24H_{0,-1} H_{0,1} \right] \\
& + \frac{16(131z^3 - 71z^2 + 107z - 44) H_{0,0,-1}}{3z} - \frac{16(15z^3 - 125z^2 + 29z + 4) H_{0,0,1}}{3z} \\
& + 64(4z^2 - 6z + 3) H_1 H_{0,0,1} + \frac{(218z^3 + 166z^2 - 13z + 16)}{z} \left[-\frac{16}{3} H_{-1} H_{0,1} + \frac{16}{3} H_{0,-1,1} \right. \\
& + \left. \frac{16}{3} H_{0,1,-1} \right] + \frac{16(60z^3 - 488z^2 + z + 8) H_{0,1,1}}{3z} + (2z^2 + 6z + 3) \left[\frac{32}{3} H_0 H_{-1}^3 - 32H_{0,-1} H_{-1}^2 \right. \\
& + 64H_{0,-1,-1} H_{-1} - 64H_{0,-1,-1,-1} \Big] + 64(5z^2 - 8z + 5) H_{0,-1,0,1} + (12z^2 + 14z + 7) \left[32H_{-1} H_{0,0,-1} \right. \\
& - 32H_{0,0,-1,-1} \Big] - 16(26z^2 + 114z + 3) H_{0,0,0,-1} - 16(22z^2 + 50z + 37) H_{0,0,0,1} \\
& + (8z^2 + 6z + 3) \left[-32H_{-1} H_{0,0,1} + 32H_{0,0,-1,1} + 32H_{0,0,1,-1} \right] - 32(6z^2 - 8z - 1) H_{0,0,1,1} \\
& + (3z^2 + 2z + 1) \left[64H_{0,1} H_{-1}^2 + \left[-128H_{0,-1,1} - 128H_{0,1,-1} \right] H_{-1} + 128H_{0,-1,-1,1} + 128H_{0,-1,1,-1} \right. \\
& + 128H_{0,1,-1,-1} \Big] + (2z^2 + 2z + 1) \left[H_{-1} \left[-64H_0 H_{0,1} - 64H_{0,1,1} \right] + 64H_{0,-1,1,1} + 64H_{0,1,-1,1} \right. \\
& + 64H_{0,1,1,-1} \Big] - 32(2z^2 + 22z + 7) H_{0,1,1,1} - 16(10z^2 + 2z + 1) H_{-1}^2 \zeta_2 - 16(4z^2 - 14z + 9) H_{0,-1} \zeta_2 \\
& + 16(24z^2 + 38z + 31) H_{0,1} \zeta_2 + 16(20z^2 + 6z + 3) H_{-1} \zeta_3 + (8z^2 - 10z + 5) \left[-32\zeta_2 H_1^2 - 112\zeta_3 H_1 \right] \Big] \\
& + 272(8z - 1) \zeta_5 \Big] + C_F^2 T_F \left[\frac{2}{15} (8z^2 - 2z + 1) H_0^5 + \frac{1}{3} (52z^2 - 16z - 3) H_0^4 - \frac{2}{3} (36z^2 + 2z + 5) H_0^3 \right. \\
& + \frac{16}{3} (13z^2 - 17z + 3) H_1 H_0^3 + 16(2z - 1) H_{0,1} H_0^3 - \frac{2}{3} (2z - 3)(4z - 1) \zeta_2 H_0^3 \\
& + 2(40z^2 - 56z + 15) H_1^2 H_0^2 + (384z^2 - 196z + 63) H_0^2 - 4(18z^2 - 18z - 7) H_1 H_0^2 \\
& - 4(12z^2 - 60z + 17) H_{0,1} H_0^2 - 8(12z^2 + 10z - 5) H_{0,0,1} H_0^2 + 8(16z^2 - 6z + 3) H_{0,1,1} H_0^2 \\
& - 2(52z^2 - 37z + 4) \zeta_2 H_0^2 - \frac{16}{3} (22z^2 - 10z + 5) \zeta_3 H_0^2 - \frac{4}{3} (4z - 3)(4z + 1) H_1^3 H_0 \\
& - 2(88z^2 - 86z - 7) H_1^2 H_0 - 8(20z^2 - 14z + 7) H_{0,1}^2 H_0 - \frac{8}{5} (284z^2 - 94z + 57) \zeta_2^2 H_0 \\
& + 4(136z^2 - 91z - 13) H_0 + 4(192z^2 - 156z + 25) H_1 H_0 - 16(13z^2 + 7z + 3) H_{0,1} H_0 \\
& - 8(64z - 25) H_{0,0,1} H_0 - 16(z - 1)(16z - 3) H_{0,1,1} H_0 + 32(12z^2 + 2z - 1) H_{0,0,0,1} H_0 \\
& + 192(z - 1)^2 H_{0,0,1,1} H_0 - 16(36z^2 - 38z + 19) H_{0,1,1,1} H_0 + 2(122z^2 + 38z + 29) \zeta_2 H_0 \\
& - 4(4z - 3)(16z - 9) H_1 \zeta_2 H_0 - 32z^2 H_{0,-1} \zeta_2 H_0 + 8(44z^2 - 58z + 29) H_{0,1} \zeta_2 H_0 \\
& + 8(4z - 3)(8z + 1) \zeta_3 H_0 - \frac{1}{3} (72z^2 - 76z + 19) H_1^4 - \frac{4}{3} (62z^2 - 28z - 31) H_1^3 \\
& - 78z^2 + 2(136z^2 - 129z + 74) H_1^2 - 4(12z^2 - 24z + 7) H_{0,1}^2 - \frac{8}{5} (176z^2 - 79z + 1) \zeta_2^2 \\
& - 84z + 2(272z^2 - 313z + 39) H_1 - 2(112z^2 - 7z + 22) H_{0,1} + 2(104z^2 + 30z - 17) H_{0,0,1} \\
& + 64(5z^2 - 2z + 1) H_{0,1} H_{0,0,1} + (2z - 3)(2z - 1) \left[48H_1 H_{0,0,1} - 24H_0 H_1 H_{0,1} \right] \\
& - 4(36z^2 + 102z - 37) H_{0,1,1} + 64(3z^2 - 4z + 2) H_{0,1} H_{0,1,1} + (4z - 1) \left[8H_1^2 H_{0,1} - 32H_1 H_{0,1,1} \right] \\
& + 8(24z^2 + 67z - 33) H_{0,0,0,1} + 8(32z^2 - 60z + 17) H_{0,0,1,1} - 32(2z - 1)(7z - 5) H_{0,1,1,1} \\
& - 8(48z^2 - 2z + 1) H_{0,0,0,0,1} - 16(92z^2 - 62z + 31) H_{0,0,0,1,1} - 32(20z^2 - 6z + 3) H_{0,0,1,0,1} \\
& - 16(36z^2 - 50z + 25) H_{0,0,1,1,1} - 96(4z^2 - 6z + 3) H_{0,1,0,1,1} - 16(40z^2 - 34z + 17) H_{0,1,1,1,1} \\
& + 16z(8z - 9) H_1^2 \zeta_2 - \frac{1}{2} (232z^2 - 510z - 375) \zeta_2 + 4(61z^2 - 82z - 10) H_1 \zeta_2
\end{aligned}$$

$$\begin{aligned}
& +(z+1)(3z+2) \left[16H_{0,-1} - 16H_{-1}H_0 \right] \zeta_2 + 4(128z^2 - 166z + 47)H_{0,1}\zeta_2 \\
& + 16(6z^2 + 2z + 1)H_{0,0,-1}\zeta_2 - 8(20z^2 - 74z + 37)H_{0,0,1}\zeta_2 - 16(2z^2 - 6z + 3)H_{0,1,1}\zeta_2 \\
& + 96\gamma_{qg}^0 \log(2)\zeta_2 + L_Q^3 \left[\frac{64}{3}H_{0,1}z^2 - 8H_0z + \frac{4}{3}(8z^2 - 2z + 1)H_0^2 - \frac{2}{3}(2z - 11) - \frac{16}{3}(4z - 1)H_1 \right. \\
& + \gamma_{qg}^0 \left[-\frac{8}{3}H_1^2 - \frac{8}{3}H_0H_1 \right] - \frac{32}{3}(4z^2 - 2z + 1)\zeta_2 \left. \right] + L_M^3 \left[-\frac{64}{3}H_{0,1}z^2 + 8H_0z \right. \\
& - \frac{4}{3}(8z^2 - 2z + 1)H_0^2 + \frac{2}{3}(2z - 11) + \frac{16}{3}(4z - 1)H_1 + \gamma_{qg}^0 \left[\frac{8}{3}H_1^2 + \frac{8}{3}H_0H_1 \right] \\
& + \frac{32}{3}(4z^2 - 2z + 1)\zeta_2 \left. \right] + (2z^2 + 2z + 1) \left[16H_{-1}\zeta_2^2 + \left[16H_0H_{-1}^2 + \left[-8H_0^2 - 32H_{0,-1} \right] H_{-1} \right. \right. \\
& + 32H_{0,-1,-1} \left. \right] \zeta_2 \left. \right] + \frac{4}{3}(252z^2 - 34z + 13)\zeta_3 + \frac{64}{3}(12z^2 - 7z - 2)H_1\zeta_3 \\
& + \frac{128}{3}(5z^2 - 6z + 3)H_{0,1}\zeta_3 - \frac{16}{3}(20z^2 + 14z - 1)\zeta_2\zeta_3 + L_M^2 \left[-\frac{4}{3}(24z^2 + 2z + 3)H_0^3 \right. \\
& - 4z(32z + 9)H_0^2 - 4(18z^2 - 6z + 11)H_0 - 16(10z^2 - 18z + 5)H_1H_0 + 64z^2H_{0,-1}H_0 \\
& - 32(6z^2 - 2z + 1)H_{0,1}H_0 + 8(40z^2 - 10z + 9)\zeta_2H_0 + 104z^2 - 32(z - 1)(5z - 2)H_1^2 \\
& - 92z - 2(36z^2 - 126z + 65)H_1 + (z + 1)(3z + 2) \left[32H_{-1}H_0 - 32H_{0,-1} \right] - 8(20z^2 + 5z - 4)H_{0,1} \\
& - 32(6z^2 + 2z + 1)H_{0,0,-1} + 8(24z^2 - 14z + 7)H_{0,0,1} - 32(2z^2 + 2z - 1)H_{0,1,1} \\
& + 8(40z^2 - 11z + 6)\zeta_2 + (2z^2 + 2z + 1) \left[-32H_0H_{-1}^2 + \left[16H_0^2 + 64H_{0,-1} \right] H_{-1} \right. \\
& \left. - 32\zeta_2H_{-1} - 64H_{0,-1,-1} \right] + \gamma_{qg}^0 \left[8H_1^3 + 20H_0H_1^2 + 8H_0^2H_1 - 32\zeta_2H_1 \right] + 8(32z^2 + 2z + 7)\zeta_3 - 33 \left. \right] \\
& + L_Q^2 \left[-\frac{4}{3}(24z^2 + 2z + 3)H_0^3 - 4z(32z + 9)H_0^2 - 4(18z^2 - 6z + 11)H_0 - 16(10z^2 - 18z + 5)H_1H_0 \right. \\
& + 64z^2H_{0,-1}H_0 - 32(6z^2 - 2z + 1)H_{0,1}H_0 + 8(40z^2 - 10z + 9)\zeta_2H_0 + 104z^2 \\
& - 32(z - 1)(5z - 2)H_1^2 - 92z - 2(36z^2 - 126z + 65)H_1 + (z + 1)(3z + 2) \left[32H_{-1}H_0 - 32H_{0,-1} \right] \\
& - 8(20z^2 + 5z - 4)H_{0,1} - 32(6z^2 + 2z + 1)H_{0,0,-1} + 8(24z^2 - 14z + 7)H_{0,0,1} \\
& - 32(2z^2 + 2z - 1)H_{0,1,1} + 8(40z^2 - 11z + 6)\zeta_2 + L_M \left[-64H_{0,1}z^2 + 24H_0z - 4(8z^2 - 2z + 1)H_0^2 \right. \\
& + 2(2z - 11) + 16(4z - 1)H_1 + \gamma_{qg}^0 \left[8H_1^2 + 8H_0H_1 \right] + 32(4z^2 - 2z + 1)\zeta_2 \left. \right] \\
& + (2z^2 + 2z + 1) \left[-32H_0H_{-1}^2 + \left[16H_0^2 + 64H_{0,-1} \right] H_{-1} - 32\zeta_2H_{-1} - 64H_{0,-1,-1} \right] \\
& + \gamma_{qg}^0 \left[8H_1^3 + 20H_0H_1^2 + 8H_0^2H_1 - 32\zeta_2H_1 \right] + 8(32z^2 + 2z + 7)\zeta_3 - 33 \left. \right] + \gamma_{qg}^0 \left[\frac{1}{3}H_1^5 + \frac{16}{3}H_{0,1}H_1^3 \right. \\
& - \frac{4}{3}H_0^3H_1^2 + \left[-32H_{0,0,1} - 16H_{0,1,1} \right] H_1^2 - \frac{2}{3}H_0^4H_1 + 20H_{0,1}^2H_1 + \frac{264}{5}\zeta_2^2H_1 + 12H_0^2H_{0,1}H_1 \\
& + H_0 \left[H_1^4 + 20H_{0,1}H_1^2 + \left[-24H_{0,0,1} - 56H_{0,1,1} \right] H_1 \right] + \left[-\frac{40}{3}H_1^3 - 18H_0H_1^2 \right. \\
& \left. + 4H_0^2H_1 - 24H_{0,1}H_1 \right] \zeta_2 + \left[\frac{4}{3}H_1^2 + \frac{88}{3}H_0H_1 \right] \zeta_3 \left. \right] + L_M \left[-\frac{2}{3}(40z^2 + 6z + 5)H_0^4 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{8}{3}z(82z+29)H_0^3 + \frac{2}{15}(576z^3 - 4620z^2 - 5510z - 585)H_0^2 + 128(z+1)(2z+3)H_{-1}H_0^2 \\
& -8(50z^2 - 40z - 3)H_1H_0^2 + 32(10z^2 - 2z + 3)H_{0,-1}H_0^2 - 16(8z^2 - 14z - 1)H_{0,1}H_0^2 \\
& +8(72z^2 - 22z + 15)\zeta_2H_0^2 - 64(11z^2 - 13z + 4)H_1^2H_0 - \frac{4(1464z^3 + 1593z^2 + 308z - 16)H_0}{15z} \\
& -16(55z^2 - 95z + 29)H_1H_0 - 64(7z^2 + 18z - 2)H_{0,-1}H_0 - 128(4z^2 + 1)H_{0,0,-1}H_0 \\
& -64(4z^2 + 10z + 3)H_{0,0,1}H_0 + 128(4z^2 - 6z + 3)H_{0,1,1}H_0 + 16(100z^2 - 31z + 9)\zeta_2H_0 \\
& +32(44z^2 - 30z + 9)\zeta_3H_0 - 48(7z^2 - 8z + 2)H_1^3 - 2(368z^2 - 546z + 155)H_1^2 \\
& -32(14z - 1)H_{0,-1}^2 - 16(8z^2 - 6z + 3)H_{0,1}^2 - \frac{8}{5}(192z^2 - 346z + 57)\zeta_2^2 \\
& + \frac{2(1092z^3 - 1893z^2 + 383z - 32)}{15z} - 2(272z^2 - 216z - 85)H_1 - 16(37z^2 - 6z - 13)H_{0,1} \\
& + \frac{(72z^5 - 165z^4 - 1090z^3 - 915z^2 + 2) \left[\frac{32}{15}H_{0,-1} - \frac{32}{15}H_{-1}H_0 \right]}{z^2} + 16(4z - 1)H_1H_{0,1} \\
& + (z+1)(z+14) \left[-32H_0H_{-1}^2 + 64H_{0,-1}H_{-1} - 64H_{0,-1,-1} \right] + 128(3z^2 + 8z - 8)H_{0,0,-1} \\
& + 16(26z^2 - 77z + 26)H_{0,0,1} + (z+1)(3z+2) \left[64H_{-1}H_{0,1} - 64H_{0,-1,1} - 64H_{0,1,-1} \right] \\
& -16(38z^2 - 3z - 15)H_{0,1,1} + (3z^2 - 4z + 2) \left[-128H_0H_{0,-1,-1} - 128H_{0,-1,0,1} \right] \\
& -64(10z^2 - 2z + 5)H_{0,0,0,-1} + 16(48z^2 + 34z + 27)H_{0,0,0,1} - 16(32z^2 - 34z + 17)H_{0,0,1,1} \\
& -192(2z - 1)H_{0,1,1,1} - \frac{16}{15}(144z^3 - 1380z^2 - 665z - 240)\zeta_2 - 32(z+1)(7z+18)H_{-1}\zeta_2 \\
& +16(86z^2 - 78z + 5)H_1\zeta_2 - 32(12z^2 - 6z + 5)H_{0,-1}\zeta_2 - 64(2z+3)H_{0,1}\zeta_2 \\
& +16(128z^2 - 54z + 23)\zeta_3 + (2z^2 + 2z + 1) \left[\frac{128}{3}H_0H_{-1}^3 + \left[-112H_0^2 - 128H_{0,-1} - 64H_{0,1} \right] H_{-1}^2 \right. \\
& \left. + \left[\frac{64}{3}H_0^3 + 192H_{0,-1}H_0 + 256H_{0,-1,-1} + 128H_{0,-1,1} + 64H_{0,0,-1} + 64H_{0,0,1} + 128H_{0,1,-1} \right] H_{-1} \right. \\
& \left. -224\zeta_3H_{-1} - 256H_{0,-1,-1,-1} - 128H_{0,-1,-1,1} - 128H_{0,-1,1,-1} - 64H_{0,0,-1,-1} - 64H_{0,0,-1,1} \right. \\
& \left. -64H_{0,0,1,-1} - 128H_{0,1,-1,-1} + \left[128H_{-1}^2 - 64H_{-1}H_0 \right] \zeta_2 \right] + \gamma_{qg}^0 \left[8H_1^4 + 20H_0^2H_1^2 \right. \\
& \left. +8H_{0,1}H_1^2 + 8H_0^3H_1 + \left[128H_{0,0,-1} - 80H_{0,0,1} \right] H_1 - 120\zeta_3H_1 - 64H_{0,-1}H_{0,1} \right. \\
& \left. +H_0 \left[24H_1^3 + \left[64H_{0,1} - 64H_{0,-1} \right] H_1 + 76H_{0,1} + 64H_{0,-1,1} + 64H_{0,1,-1} \right] \right. \\
& \left. + \left[-64H_1^2 - 128H_0H_1 \right] \zeta_2 \right] + L_Q \left[\frac{2}{3}(40z^2 + 6z + 5)H_0^4 + \frac{8}{3}z(82z+29)H_0^3 \right. \\
& -\frac{2}{15}(576z^3 - 4620z^2 - 5510z - 585)H_0^2 - 128(z+1)(2z+3)H_{-1}H_0^2 + 8(50z^2 - 40z - 3)H_1H_0^2 \\
& -32(10z^2 - 2z + 3)H_{0,-1}H_0^2 + 16(8z^2 - 14z - 1)H_{0,1}H_0^2 - 8(72z^2 - 22z + 15)\zeta_2H_0^2 \\
& +64(11z^2 - 13z + 4)H_1^2H_0 + \frac{4(1464z^3 + 1593z^2 + 308z - 16)H_0}{15z} + 16(55z^2 - 95z + 29)H_1H_0 \\
& +64(7z^2 + 18z - 2)H_{0,-1}H_0 + 128(4z^2 + 1)H_{0,0,-1}H_0 + 64(4z^2 + 10z + 3)H_{0,0,1}H_0 \\
& -128(4z^2 - 6z + 3)H_{0,1,1}H_0 - 16(100z^2 - 31z + 9)\zeta_2H_0 - 32(44z^2 - 30z + 9)\zeta_3H_0 \\
& +48(7z^2 - 8z + 2)H_1^3 + 2(368z^2 - 546z + 155)H_1^2 + 32(14z - 1)H_{0,-1}^2 + 16(8z^2 - 6z + 3)H_{0,1}^2 \\
& + \frac{8}{5}(192z^2 - 346z + 57)\zeta_2^2 - \frac{2(1092z^3 - 1893z^2 + 383z - 32)}{15z} + 2(272z^2 - 216z - 85)H_1 \\
& + \frac{(72z^5 - 165z^4 - 1090z^3 - 915z^2 + 2) \left[\frac{32}{15}H_{-1}H_0 - \frac{32}{15}H_{0,-1} \right]}{z^2} + 16(37z^2 - 6z - 13)H_{0,1}
\end{aligned}$$

$$\begin{aligned}
& -16(4z-1)H_1H_{0,1} + (z+1)(z+14)\left[32H_0H_{-1}^2 - 64H_{0,-1}H_{-1} + 64H_{0,-1,-1}\right] \\
& -128(3z^2+8z-8)H_{0,0,-1} - 16(26z^2-77z+26)H_{0,0,1} + (z+1)(3z+2)\left[-64H_{-1}H_{0,1} \right. \\
& \left. + 64H_{0,-1,1} + 64H_{0,1,-1}\right] + 16(38z^2-3z-15)H_{0,1,1} + (3z^2-4z+2)\left[128H_0H_{0,-1,-1} \right. \\
& \left. + 128H_{0,-1,0,1}\right] \\
& + 64(10z^2-2z+5)H_{0,0,0,-1} - 16(48z^2+34z+27)H_{0,0,0,1} + 16(32z^2-34z+17)H_{0,0,1,1} \\
& + 192(2z-1)H_{0,1,1,1} + \frac{16}{15}(144z^3-1380z^2-665z-240)\zeta_2 + 32(z+1)(7z+18)H_{-1}\zeta_2 \\
& - 16(86z^2-78z+5)H_1\zeta_2 + 32(12z^2-6z+5)H_{0,-1}\zeta_2 + 64(2z+3)H_{0,1}\zeta_2 \\
& + L_M^2\left[64H_{0,1}z^2 - 24H_0z + 4(8z^2-2z+1)H_0^2 - 2(2z-11) - 16(4z-1)H_1\right. \\
& \left. + \gamma_{qg}^0\left[-8H_1^2 - 8H_0H_1\right] - 32(4z^2-2z+1)\zeta_2\right] - 16(128z^2-54z+23)\zeta_3 \\
& + L_M\left[\frac{8}{3}(24z^2+2z+3)H_0^3 + 8z(32z+9)H_0^2 + 8(18z^2-6z+11)H_0\right. \\
& + 32(10z^2-18z+5)H_1H_0 - 128z^2H_{0,-1}H_0 + 64(6z^2-2z+1)H_{0,1}H_0 - 16(40z^2-10z+9)\zeta_2H_0 \\
& + 64(z-1)(5z-2)H_1^2 - 2(104z^2-92z-33) + 4(36z^2-126z+65)H_1 \\
& + (z+1)(3z+2)\left[64H_{0,-1} - 64H_{-1}H_0\right] + 16(20z^2+5z-4)H_{0,1} + 64(6z^2+2z+1)H_{0,0,-1} \\
& - 16(24z^2-14z+7)H_{0,0,1} + 64(2z^2+2z-1)H_{0,1,1} - 16(40z^2-11z+6)\zeta_2 \\
& + (2z^2+2z+1)\left[64H_0H_{-1}^2 + \left[-32H_0^2 - 128H_{0,-1}\right]H_{-1} + 64\zeta_2H_{-1} + 128H_{0,-1,-1}\right] \\
& \left. + \gamma_{qg}^0\left[-16H_1^3 - 40H_0H_1^2 - 16H_0^2H_1 + 64\zeta_2H_1\right] - 16(32z^2+2z+7)\zeta_3\right] \\
& + (2z^2+2z+1)\left[-\frac{128}{3}H_0H_{-1}^3 + \left[112H_0^2 + 128H_{0,-1} + 64H_{0,1}\right]H_{-1}^2 + \left[-\frac{64}{3}H_0^3 \right. \right. \\
& \left. - 192H_{0,-1}H_0 - 256H_{0,-1,-1} - 128H_{0,-1,1} - 64H_{0,0,-1} - 64H_{0,0,1} - 128H_{0,1,-1}\right]H_{-1} \\
& + 224\zeta_3H_{-1} + 256H_{0,-1,-1,-1} + 128H_{0,-1,-1,1} + 128H_{0,-1,1,-1} + 64H_{0,0,-1,-1} + 64H_{0,0,-1,1} \\
& + 64H_{0,0,1,-1} + 128H_{0,1,-1,-1} + \left[64H_{-1}H_0 - 128H_{-1}^2\right]\zeta_2\left. + \gamma_{qg}^0\left[-8H_1^4 - 20H_0^2H_1^2 - 8H_{0,1}H_1^2 \right. \right. \\
& \left. - 8H_0^3H_1 + \left[80H_{0,0,1} - 128H_{0,0,-1}\right]H_1 + 120\zeta_3H_1 + 64H_{0,-1}H_{0,1} + H_0\left[-24H_1^3 \right. \right. \\
& \left. + \left[64H_{0,-1} - 64H_{0,1}\right]H_1 - 76H_{0,1} - 64H_{0,-1,1} - 64H_{0,1,-1}\right] + \left[64H_1^2 + 128H_0H_1\right]\zeta_2\left. \right] \\
& - 8(56z^2-2z+1)\zeta_5 - 77\left. + C_F T_F^2\left[\frac{4}{9}(4z^2-20z+1)H_0^4 \right. \right. \\
& + \frac{16}{9}(10z^2-63z+4)H_0^3 - \frac{8}{3}(68z^2+75z-22)H_0^2 + \frac{16}{3}(10z^2-12z+1)H_1H_0^2 \\
& - \frac{64}{3}(z-1)^2H_{0,1}H_0^2 + \frac{4}{3}(32z^2+112z-59)\zeta_2H_0^2 + \frac{4}{3}(538z^2-1578z+21)H_0 \\
& - \frac{32}{3}(16z^2-27z+10)H_1H_0 - \frac{64}{3}(5z^2-4z+2)H_{0,1}H_0 - \frac{128}{3}(z^2+2z-1)H_{0,0,1}H_0 \\
& \left. + \frac{4}{9}(24z^2+1124z-1063)\zeta_2H_0 - \frac{32}{9}(36z^2-74z+37)\zeta_3H_0 + \frac{32}{9}(z-1)(3z+1)H_1^3 \right. \\
& \left. \right]
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{16}{3}(2z-1)H_0^2 - \frac{16}{9}(24z^2-8z-5)H_0 + \frac{16(62z^3-111z^2+75z-8)}{27z} \right. \\
& + \frac{32}{9}\gamma_{qg}^0 H_1 \left. \right] L_M^3 + \frac{16}{3}(12z^2-17z+4)H_1^2 - \frac{16}{15}(218z^2-194z+91)\zeta_2^2 \\
& + \frac{2(6994z^3-8610z^2+939z-208)}{9z} + \frac{2}{27}(1984z^3+7716z^2-12243z-256)\frac{\zeta_2}{z} \\
& - \frac{16}{27}(170z^3-1128z^2+525z-8)\frac{\zeta_3}{z} + \frac{4(810z^3-1558z^2+793z+16)H_1}{3z} \\
& + L_Q^3 \left[\frac{16}{3}(2z-1)H_0^2 + \frac{16}{9}(4z-11)H_0 - \frac{16(62z^3-147z^2+84z-8)}{27z} + \frac{16}{9}\gamma_{qg}^0 H_1 \right] \\
& + \frac{16}{3}(20z^2-113z+140)H_{0,1} + \frac{16}{3}(20z^2-113z+56)H_{0,0,1} + \frac{64}{3}(3z^2-6z-1)H_{0,1,1} \\
& + \frac{16}{3}(40z^2-28z+11)H_{0,0,0,1} + \frac{80}{9}(4z^2-4z-1)H_1\zeta_2 + \gamma_{qg}^0 \left[-\frac{2}{9}H_1^4 + \frac{16}{3}H_{0,1}H_1^2 \right. \\
& - \frac{8}{9}H_0^3H_1 + \left. \left[-\frac{64}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1} \right] H_1 + \frac{160}{9}\zeta_3H_1 + \frac{32}{3}H_{0,1}^2 + H_0 \left[\frac{32}{3}H_1H_{0,1} \right. \right. \\
& - \left. \left. \frac{64}{3}H_{0,1,1} \right] + \frac{80}{3}H_{0,1,1,1} + \left[-\frac{26}{3}H_1^2 - \frac{16}{3}H_0H_1 - 12H_{0,1} \right] \zeta_2 \right] + L_Q^2 \left[-\frac{8}{3}(8z^2+56z-31)H_0^2 \right. \\
& + \frac{8}{9}(244z^2-236z+571)H_0 + \frac{4(2156z^3-7632z^2+4977z+256)}{27z} \\
& + \frac{16(130z^3-215z^2+112z-8)H_1}{9z} + \gamma_{qg}^0 \left[-4H_1^2 - \frac{32}{3}H_0H_1 \right] - \frac{16}{3}(12z^2-8z-5)H_{0,1} \\
& + \left. \left[-16(2z-1)H_0^2 - \frac{16}{3}(8z^2-9)H_0 + \frac{16(z-1)(62z^2-73z+8)}{9z} \right] L_M \right. \\
& - \left. \frac{16}{3}(4z^2-8z+13)\zeta_2 + (2z-1) \left[-16H_0^3 + 32\zeta_2H_0 - 32H_{0,0,1} + 32\zeta_3 \right] \right] \\
& + L_M^2 \left[-\frac{8}{3}(40z^2+40z-23)H_0^2 + \frac{8}{9}(4z^2-92z+547)H_0 + \frac{4(1652z^3-6192z^2+4221z+256)}{27z} \right. \\
& + \frac{16(10z^3-71z^2+70z-8)H_1}{9z} + \gamma_{qg}^0 \left[\frac{20}{3}H_1^2 + \frac{32}{3}H_0H_1 \right] - 16(4z^2-3)H_{0,1} \\
& + \left. \frac{16}{3}(28z^2-16z-1)\zeta_2 + (2z-1) \left[-16H_0^3 + 32\zeta_2H_0 - 32H_{0,0,1} + 32\zeta_3 \right] \right] \\
& + L_Q \left[\frac{64}{3}(2z^2+15z-5)H_0^3 - \frac{16}{45}(72z^3+560z^2-2660z+2975)H_0^2 \right. \\
& - \frac{64}{3}(5z^2-4z+2)H_1H_0^2 - \frac{16(1444z^3-5632z^2+9213z+4)H_0}{45z} \\
& - \frac{32(198z^3-283z^2+140z-8)H_1H_0}{9z} + \frac{128}{3}(z-2)(3z-2)H_{0,-1}H_0 \\
& + \frac{32}{3}(16z^2-8z-5)H_{0,1}H_0 - \frac{32}{3}(4z^2+84z-33)\zeta_2H_0 - \frac{16(168z^3-253z^2+131z-8)H_1^2}{9z} \\
& + \left. \left[-\frac{16(62z^3-123z^2+78z-8)}{9z} + 16(2z-1)H_0^2 + \frac{16}{3}(16z^2-4z-7)H_0 - \frac{16}{3}\gamma_{qg}^0 H_1 \right] L_M^2 \right. \\
& - \left. \frac{4(269954z^3-828996z^2+567861z-11744)}{405z} + \frac{16}{45}(144z^4+1600z^3-1400z^2+3045z-80)\frac{\zeta_2}{z} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(3080z^3 - 8448z^2 + 5247z + 256)H_1}{27z} - \frac{128}{3}(7z^2 - 14z + 9)H_{0,0,-1} \\
& + \frac{(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} \\
& + \frac{16(76z^3 - 254z^2 - 329z - 16)H_{0,1}}{9z} + \gamma_{qg}^0 \left[\frac{8}{3}H_1^3 + 8H_0H_1^2 + \frac{32}{3}H_{0,1}H_1 \right] \\
& - \frac{32}{3}(8z^2 - 60z + 15)H_{0,0,1} + \frac{32}{3}(24z^2 - 20z + 1)H_{0,1,1} + \frac{64}{3}(8z^2 - 6z + 3)H_1\zeta_2 \\
& + (z+1)^2 \left[-\frac{128}{3}H_0H_{-1}^2 + \left[\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1} \right] H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,-1} \right] \\
& + L_M \left[16(8z^2 + 16z - 9)H_0^2 - \frac{16}{9}(124z^2 - 164z + 559)H_0 - \frac{8}{3}\gamma_{qg}^0H_1^2 \right. \\
& - \frac{8(1904z^3 - 6912z^2 + 4599z + 256)}{27z} - \frac{32(70z^3 - 143z^2 + 91z - 8)H_1}{9z} + (12z^2 - 4z - 7) \left[\frac{32}{3}H_{0,1} \right. \\
& \left. - \frac{32\zeta_2}{3} \right] + (2z-1) \left[32H_0^3 - 64\zeta_2H_0 + 64H_{0,0,1} - 64\zeta_3 \right] \left. + \frac{128}{3}(4z^2 - 21z + 11)\zeta_3 \right. \\
& \left. + (2z-1) \left[16H_0^4 - 96\zeta_2H_0^2 + 64H_{0,0,1}H_0 - 128\zeta_3H_0 - \frac{32\zeta_2^2}{5} + 64H_{0,0,1,1} \right] \right] \\
& + L_M \left[-\frac{64}{9}(10z^2 + 43z - 14)H_0^3 - \frac{16}{45}(72z^3 - 170z^2 + 2640z - 2945)H_0^2 \right. \\
& + \frac{64}{3}(3z^2 - 4z + 2)H_1H_0^2 + \frac{32(173z^3 - 2669z^2 + 4406z - 2)H_0}{45z} \\
& + \frac{32(120z^3 - 199z^2 + 104z - 8)H_1H_0}{9z} - \frac{128}{3}(z^2 - 4z + 2)H_{0,-1}H_0 - 32(8z^2 - 8z + 1)H_{0,1}H_0 \\
& + \frac{32}{3}(28z^2 + 68z - 25)\zeta_2H_0 + \frac{16(42z^3 - 121z^2 + 98z - 8)H_1^2}{9z} \\
& + \frac{4(274706z^3 - 812364z^2 + 549969z - 11456)}{405z} + \frac{16}{45}(144z^4 - 340z^3 + 520z^2 - 2865z + 80)\frac{\zeta_2}{z} \\
& + \frac{8(2108z^3 - 7152z^2 + 4941z + 256)H_1}{27z} + \gamma_{qg}^0 \left[\frac{8}{3}H_1^3 + \frac{8}{3}H_0H_1^2 \right] \\
& + \frac{(36z^5 + 155z^4 + 40z^3 - 45z^2 + 20z + 1)\left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1}\right]}{z^2} \\
& - \frac{16(172z^3 - 326z^2 - 365z - 16)H_{0,1}}{9z} + \frac{128}{3}(z^2 - 10z + 3)H_{0,0,-1} \\
& + \frac{32}{3}(8z^2 - 76z + 23)H_{0,0,1} - \frac{32}{3}(16z^2 - 8z - 5)H_{0,1,1} + (z+1)^2 \left[-\frac{128}{3}H_0H_{-1}^2 \right. \\
& + \left[\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1} \right] H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,-1} \left. + \frac{64}{3}(12z^2 + 44z - 15)\zeta_3 \right. \\
& \left. + (2z-1) \left[-16H_0^4 - 64H_{0,0,1}H_0 + 128\zeta_3H_0 + \frac{32\zeta_2^2}{5} - 64H_{0,0,1,1} + \left[96H_0^2 + \frac{64H_1}{3} \right] \zeta_2 \right] \right] \\
& + (2z-1) \left[\frac{40}{3}\zeta_2H_0^3 + \frac{64}{3}\zeta_3H_0^2 + \frac{64}{5}\zeta_2^2H_0 - 32H_{0,0,0,0,1} + 32\zeta_5 \right] \\
& + C_{FNFT}^2 \left[-\frac{4}{9}(z-2)(6z+1)H_0^4 + \frac{4}{27}(404z^2 - 54z + 69)H_0^3 - \frac{4}{27}(3112z^2 - 1329z - 210)H_0^2 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(94z^3 - 234z^2 + 159z - 16)H_1H_0^2}{9z} + \frac{16}{3}(6z^2 + 4z - 11)H_{0,1}H_0^2 \\
& -\frac{8}{3}(14z^2 - 5z - 11)\zeta_2H_0^2 + \frac{8}{81}(27824z^2 + 4929z + 2631)H_0 - \frac{32}{3}(2z^2 + 28z - 17)H_{0,0,1}H_0 \\
& + \frac{16(1556z^3 - 1539z^2 + 72z - 80)H_1H_0}{27z} - \frac{32(101z^3 + 222z^2 - 39z + 8)H_{0,1}H_0}{9z} \\
& + \frac{8}{9}(240z^2 + 166z + 193)\zeta_2H_0 + \frac{32}{9}(78z^2 - 133z - 28)\zeta_3H_0 + \frac{16}{9}(z-1)(3z+1)H_1^3 \\
& + \left[-\frac{16}{3}(2z-1)H_0^2 - \frac{32}{9}(6z^2 - z - 4)H_0 + \frac{8(124z^3 - 258z^2 + 159z - 16)}{27z} + \frac{8}{9}\gamma_{ag}^0H_1 \right] L_M^3 \\
& + \frac{8}{3}(12z^2 - 17z + 4)H_1^2 - \frac{64}{15}(2z^2 + 58z - 11)\zeta_2^2 - \frac{4(221158z^3 - 226026z^2 + 17163z - 5248)}{243z} \\
& - \frac{2}{27}(7280z^3 - 5646z^2 - 555z - 368)\frac{\zeta_2}{z} - \frac{8}{27}(3784z^3 + 2046z^2 + 1095z - 16)\frac{\zeta_3}{z} \\
& + \frac{16}{3}(20z^2 - 17z - 1)H_1 + L_Q^3 \left[\frac{16}{3}(2z-1)H_0^2 + \frac{16}{9}(4z-11)H_0 - \frac{16(62z^3 - 147z^2 + 84z - 8)}{27z} \right. \\
& \left. + \frac{16}{9}\gamma_{ag}^0H_1 \right] - \frac{16(1448z^3 - 1341z^2 + 18z - 80)H_{0,1}}{27z} + \frac{16(498z^3 + 654z^2 + 3z + 16)H_{0,0,1}}{9z} \\
& + \frac{32}{3}(3z^2 - 6z - 1)H_{0,1,1} - \frac{32}{3}(14z^2 - 74z + 19)H_{0,0,0,1} + L_M^2 \left[\frac{32}{3}(2z-1)H_0^3 \right. \\
& \left. + \frac{16}{3}(2z+3)(4z-3)H_0^2 - \frac{8}{9}(160z^2 + 146z + 305)H_0 + \frac{4(1000z^3 + 1356z^2 - 2247z - 208)}{27z} \right. \\
& \left. + \frac{32}{9}(4z^2 - 4z + 5)H_1 + \gamma_{ag}^0 \left[\frac{4}{3}H_1^2 + \frac{8}{3}H_{0,1} - \frac{8\zeta_2}{3} \right] \right] + \frac{16}{9}(2z^2 - 2z - 5)H_1\zeta_2 + \gamma_{ag}^0 \left[-\frac{1}{9}H_1^4 \right. \\
& \left. + \frac{8}{3}H_{0,1}H_1^2 - \frac{4}{9}H_0^3H_1 + \left[-\frac{32}{3}H_{0,0,1} - \frac{32}{3}H_{0,1,1} \right]H_1 + \frac{88}{9}\zeta_3H_1 + \frac{16}{3}H_{0,1}^2 + H_0 \left[\frac{16}{3}H_1H_{0,1} \right. \right. \\
& \left. \left. - \frac{32}{3}H_{0,1,1} \right] + \frac{40}{3}H_{0,1,1,1} + \left[-4H_1^2 - \frac{4}{3}H_0H_1 - \frac{20}{3}H_{0,1} \right]\zeta_2 \right] + L_Q^2 \left[-\frac{8}{3}(8z^2 + 56z - 31)H_0^2 \right. \\
& \left. + \frac{8}{9}(244z^2 - 236z + 571)H_0 + \frac{4(2156z^3 - 7632z^2 + 4977z + 256)}{27z} + \gamma_{ag}^0 \left[-4H_1^2 - \frac{32}{3}H_0H_1 \right] \right. \\
& \left. + \frac{16(130z^3 - 215z^2 + 112z - 8)H_1}{9z} - \frac{16}{3}(12z^2 - 8z - 5)H_{0,1} + \left[-\frac{8}{3}(4z-1) \right. \right. \\
& \left. \left. + \frac{16}{3}(4z^2 - 2z + 1)H_0 - \frac{8}{3}\gamma_{ag}^0H_1 \right] L_M - \frac{16}{3}(4z^2 - 8z + 13)\zeta_2 + (2z-1) \left[-16H_0^3 \right. \right. \\
& \left. \left. + 32\zeta_2H_0 - 32H_{0,0,1} + 32\zeta_3 \right] \right] + L_Q \left[\frac{8}{3}(16z^2 + 108z - 35)H_0^3 \right. \\
& \left. - \frac{8}{45}(144z^3 + 1120z^2 - 3745z + 5320)H_0^2 - \frac{16(1984z^3 - 5407z^2 + 7593z + 4)H_0}{45z} \right. \\
& \left. - \frac{64}{3}(5z^2 - 4z + 2)H_1H_0^2 - \frac{32(198z^3 - 283z^2 + 140z - 8)H_1H_0}{9z} + \frac{128}{3}(z-2)(3z-2)H_{0,-1}H_0 \right. \\
& \left. + \frac{32}{3}(16z^2 - 8z - 5)H_{0,1}H_0 - \frac{32}{3}(4z^2 + 84z - 33)\zeta_2H_0 - \frac{16(168z^3 - 253z^2 + 131z - 8)H_1^2}{9z} \right. \\
& \left. - \frac{8(91102z^3 - 318513z^2 + 229863z - 6952)}{405z} + \frac{16}{45}(144z^4 + 1600z^3 - 1400z^2 + 3045z - 80)\frac{\zeta_2}{z} \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(3080z^3 - 8448z^2 + 5247z + 256)H_1}{27z} + \frac{16(76z^3 - 254z^2 - 329z - 16)H_{0,1}}{9z} \\
& + \frac{(36z^5 - 155z^4 + 40z^3 + 225z^2 - 20z + 1) \left[\frac{64}{45}H_{-1}H_0 - \frac{64}{45}H_{0,-1} \right]}{z^2} + \gamma_{qg}^0 \left[\frac{8}{3}H_1^3 + 8H_0H_1^2 \right. \\
& + \frac{32}{3}H_{0,1}H_1 \left. \right] - \frac{128}{3}(7z^2 - 14z + 9)H_{0,0,-1} - \frac{32}{3}(8z^2 - 60z + 15)H_{0,0,1} \\
& + \frac{32}{3}(24z^2 - 20z + 1)H_{0,1,1} + \frac{64}{3}(8z^2 - 6z + 3)H_1\zeta_2 + (z+1)^2 \left[-\frac{128}{3}H_0H_{-1}^2 + \left[\frac{64}{3}H_0^2 \right. \right. \\
& + \frac{256}{3}H_{0,-1} \left. \right] H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,-1} \left. \right] + L_M \left[\frac{8}{9}(36z^2 + 82z - 61) \right. \\
& - \frac{16}{9}(76z^2 - 38z + 25)H_0 - \frac{16}{9}(76z^2 - 88z + 41)H_1 + \gamma_{qg}^0 \left[\frac{8}{3}H_1^2 + \frac{32}{3}H_0H_1 \right] \\
& + \frac{32}{3}(4z^2 - 6z + 3)H_{0,1} + (4z^2 - 2z + 1) \left[\frac{32\zeta_2}{3} - \frac{32}{3}H_0^2 \right] \left. \right] + \frac{128}{3}(4z^2 - 21z + 11)\zeta_3 \\
& + (2z-1) \left[\frac{44}{3}H_0^4 - 96\zeta_2H_0^2 + 64H_{0,0,1}H_0 - 128\zeta_3H_0 - \frac{32\zeta_2^2}{5} + 64H_{0,0,1,1} \right] \left. \right] \\
& + L_M \left[-\frac{8}{9}(16z^2 + 28z - 47)H_0^3 + \frac{8}{9}(422z^2 + 137z + 41)H_0^2 - \frac{80}{27}(176z^2 - 213z - 297)H_0 \right. \\
& + \frac{32(35z^3 + 29z^2 - 40z + 8)H_1H_0}{9z} - \frac{32}{3}(4z^2 - 16z + 17)H_{0,1}H_0 - \frac{64}{3}(4z^2 - 2z + 1)\zeta_2H_0 \\
& + \frac{8}{9}(86z^2 - 104z + 67)H_1^2 + \frac{8(3224z^3 - 19683z^2 + 16893z - 1064)}{81z} + \frac{8}{9}(152z^2 - 258z + 163)H_1 \\
& + \frac{16(16z^3 - 88z^2 + 131z - 16)H_{0,1}}{9z} - \frac{128}{3}(5z - 7)H_{0,0,1} - \frac{32}{3}(12z^2 - 14z + 7)H_{0,1,1} \\
& - \frac{16}{9}(86z^2 - 30z + 51)\zeta_2 + \gamma_{qg}^0 \left[-\frac{32}{3}H_1H_0^2 - \frac{8}{3}H_1^2H_0 - \frac{16}{3}H_1H_{0,1} + \frac{32}{3}H_1\zeta_2 \right] \\
& + \frac{32}{3}(16z^2 + 2z - 19)\zeta_3 + (2z-1) \left[-\frac{20}{3}H_0^4 + 64H_{0,0,1}H_0 + 128\zeta_3H_0 + \frac{384\zeta_2^2}{5} - 192H_{0,0,0,1} \right] \left. \right] \\
& + (2z-1) \left[-\frac{4}{15}H_0^5 - 8\zeta_2H_0^3 + 32H_{0,0,1}H_0^2 + \frac{208}{3}\zeta_3H_0^2 - 128H_{0,0,0,1}H_0 + 128H_{0,0,0,0,1} - 128\zeta_5 \right] \left. \right] \\
& + C_{FCATF} \left[-\frac{2}{15}H_0^5 + \frac{1}{18}(16z^2 + 178z + 13)H_0^4 - \frac{1}{27}(7624z^2 + 1752z + 1131)H_0^3 \right. \\
& + \frac{16}{3}(z+1)(3z+2)H_{-1}H_0^3 - \frac{4(404z^3 - 334z^2 - 25z - 40)H_1H_0^3}{9z} + \frac{16}{3}(2z^2 + 16z + 7)H_{0,1}H_0^3 \\
& + \frac{8}{3}(4z^2 + 5z + 4)\zeta_2H_0^3 - 2(20z^2 + 24z + 19)H_{-1}^2H_0^2 - \frac{4(151z^3 - 138z^2 - 16)H_1^2H_0^2}{3z} \\
& + \frac{1}{9}(10622z^2 - 2987z - 346)H_0^2 - 4(4z^2 - 7)H_{-1}H_0^2 + \frac{2(124z^3 - 1392z^2 + 1269z - 40)H_1H_0^2}{9z} \\
& - 12(4z+3)H_{0,-1}H_0^2 - \frac{8(13z^3 + 46z^2 - 23z + 20)H_{0,1}H_0^2}{3z} + 16(2z^2 - 28z - 9)H_{0,0,1}H_0^2 \\
& - 16(14z^2 - 26z + 1)H_{0,1,1}H_0^2 - \frac{1}{3}(128z^2 - 106z - 49)\zeta_2H_0^2 + \frac{16}{3}(6z^2 + 13z + 10)\zeta_3H_0^2 \\
& - \frac{4(94z^3 - 48z^2 - 3z - 16)H_1^3H_0}{9z} + \frac{4(728z^3 - 1068z^2 + 303z - 8)H_1^2H_0}{9z} \\
& + 16(6z^2 - 10z - 1)H_{0,1}^2H_0 - \frac{16}{5}(103z + 11)\zeta_2^2H_0 - \frac{2}{81}(9514z^2 - 34311z + 17499)H_0
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(12769z^3 - 12504z^2 - 1401z + 740)H_1H_0}{27z} + \frac{4}{3}(314z^3 - 242z^2 - 47z - 48)\frac{\zeta_2}{z}H_1H_0 \\
& + 8(2z^2 + 14z - 3)H_{0,-1}H_0 + 8(4z^2 + 5)H_{-1}H_{0,-1}H_0 - \frac{8(782z^3 + 387z^2 + 705z - 20)H_{0,1}H_0}{9z} \\
& + \frac{32(54z^3 - 56z^2 + z - 4)H_1H_{0,1}H_0}{3z} - 8(4z^2 - 16z - 11)H_{0,-1,-1}H_0 - 16(2z^2 - 6z - 5)H_{0,0,-1}H_0 \\
& - \frac{8}{3}(448z^2 - 328z - 49)H_{0,1,1}H_0 + \frac{8(370z^3 - 88z^2 - 133z + 40)H_{0,0,1}H_0}{3z} \\
& - 32(10z^2 - 34z - 7)H_{0,0,0,1}H_0 + 16(8z^2 - 22z + 1)H_{0,0,1,1}H_0 + 16(40z^2 - 22z + 23)H_{0,1,1,1}H_0 \\
& + \frac{2}{9}(1180z^2 - 1409z - 347)\zeta_2H_0 - 8(17z^2 + 27z + 13)H_{-1}\zeta_2H_0 + 80(z+1)^2H_{0,-1}\zeta_2H_0 \\
& - 8(26z^2 + 14z + 29)H_{0,1}\zeta_2H_0 - \frac{4}{9}(1828z^2 + 134z + 131)\zeta_3H_0 - \frac{2(12z^3 + 31z^2 - 56z - 4)H_1^4}{9z} \\
& - \frac{4(57z^3 - 94z^2 + 24z + 4)H_1^3}{3z} - \frac{2(2369z^3 - 3096z^2 + 2241z + 142)H_1^2}{27z} \\
& - \frac{4}{3}(290z^3 - 306z^2 + 57z - 8)\frac{\zeta_2}{z}H_1^2 - 64(z+1)H_{0,-1}^2 + \frac{4(384z^3 - 256z^2 + 35z - 16)H_{0,1}^2}{3z} \\
& + \frac{4}{15}(2354z^2 - 3310z + 101)\zeta_2^2 - \frac{54588z^3 - 98666z^2 - 2299z + 8388}{81z} - \frac{1}{18}(11000z^3 + 11030z^2 \\
& - 1283z + 1440)\frac{\zeta_2}{z} - \frac{4(30389z^3 - 44025z^2 + 11274z - 365)H_1}{81z} - \frac{4}{9}(2752z^3 - 3646z^2 + 593z \\
& - 72)\frac{\zeta_2}{z}H_1 + \frac{4}{9}(1088z^3 - 1036z^2 + 65z - 64)\frac{\zeta_3}{z}H_1 + (35z^2 + 40z + 21)\left[8H_{0,-1} - 8H_{-1}H_0\right] \\
& + \frac{8(84z^3 - 70z^2 + 5z - 8)H_1^2H_{0,1}}{3z} - \frac{4(9292z^3 - 15039z^2 - 1215z + 740)H_{0,1}}{27z} \\
& - \frac{4}{3}(150z^3 - 338z^2 + 169z - 48)\frac{\zeta_2}{z}H_{0,1} + \frac{16}{3}(z-1)(15z+1)H_1H_{0,1} + (3z^2 + z - 5)\left[8H_0H_{-1}^2 \right. \\
& \left. - 16H_{0,-1}H_{-1} + 16H_{0,-1,-1}\right] - 8(28z+1)H_{0,0,-1} + \frac{8(568z^3 + 2052z^2 + 801z - 20)H_{0,0,1}}{9z} \\
& - \frac{8(432z^3 - 436z^2 + 5z - 32)H_1H_{0,0,1}}{3z} + 48(6z+5)H_{0,1}H_{0,0,1} + (4z^2 + 8z + 1)\left[16H_{-1}H_0H_{0,1} \right. \\
& \left. + H_0\left[-16H_{0,-1,1} - 16H_{0,1,-1}\right]\right] + (z-2)(z+1)\left[-16H_{-1}H_{0,1} + 16H_{0,-1,1} + 16H_{0,1,-1}\right] \\
& - \frac{8(2599z^3 - 2553z^2 + 459z - 8)H_{0,1,1}}{9z} - \frac{8(336z^3 - 292z^2 + 23z - 32)H_1H_{0,1,1}}{3z} \\
& - 32(14z^2 - 6z + 9)H_{0,1}H_{0,1,1} + (z^2 + z + 1)\left[32H_0H_{-1}^3 - 96H_{0,-1}H_{-1}^2 \right. \\
& \left. + 192H_{0,-1,-1}H_{-1} - 192H_{0,-1,-1,-1}\right] + 8(2z-1)(2z+5)H_{0,-1,0,1} \\
& + (2z+1)(2z+3)\left[24H_{-1}H_{0,0,-1} - 24H_{0,0,-1,-1}\right] + (4z^2 + 2z + 1)\left[32H_0H_{0,0,-1,-1} \right. \\
& \left. - 8H_0^2H_{0,-1,-1}\right] - 8(20z+11)H_{0,0,0,-1} - \frac{8(590z^3 - 248z^2 - 233z + 40)H_{0,0,0,1}}{3z} \\
& + (4z^2 + 16z - 3)\left[-8H_{-1}H_{0,0,1} + 8H_{0,0,-1,1} + 8H_{0,0,1,-1}\right] + \frac{8}{3}(512z^2 - 240z - 183)H_{0,0,1,1} \\
& + (8z^2 + 12z + 7)\left[-8H_{0,1}H_{-1}^2 + \left[16H_{0,-1,1} + 16H_{0,1,-1}\right]H_{-1} - 16H_{0,-1,-1,1} - 16H_{0,-1,1,-1} \right. \\
& \left. - 16H_{0,1,-1,-1}\right] + z(z+1)\left[64H_{-1}H_{0,1,1} - 64H_{0,-1,1,1} - 64H_{0,1,-1,1} - 64H_{0,1,1,-1}\right] \\
& + 64(5z^2 + 6z + 3)H_{0,-1,-1,0,1} + \frac{16(251z^3 - 139z^2 + 17z - 32)H_{0,1,1,1}}{3z} + z^2\left[64H_0H_{0,-1,0,1} \right.
\end{aligned}$$

$$\begin{aligned}
& +64H_{0,0,-1,0,1}] + 32(20z^2 - 36z - 3)H_{0,0,0,0,1} + (16z^2 - 2z - 1)[16H_{0,0,0,-1,1} + 16H_{0,0,0,1,-1}] \\
& -16(8z^2 + 86z + 75)H_{0,0,0,1,1} + (4z^2 - 2z - 1)[-8H_{0,0,-1}H_0^2 + 16H_{0,0,0,-1}H_0 - 16H_{0,-1}H_{0,0,1} \\
& -80H_{0,0,0,-1,-1} + 16H_{0,0,1,0,-1}] - 16(8z^2 + 46z + 37)H_{0,0,1,0,1} + 16(120z^2 - 50z + 79)H_{0,0,1,1,1} \\
& +16(56z^2 - 10z + 41)H_{0,1,0,1,1} + 16(32z^2 - 18z + 15)H_{0,1,1,1,1} + 8(14z^2 + 18z + 13)H_{-1}^2\zeta_2 \\
& +8(5z^2 - z - 9)H_{-1}\zeta_2 + 8(z+1)(z+14)H_{0,-1}\zeta_2 - 16(2z^2 + 6z + 3)H_{0,-1,-1}\zeta_2 \\
& +8(2z^2 - 14z - 7)H_{0,0,-1}\zeta_2 + 8(26z^2 + 50z + 47)H_{0,0,1}\zeta_2 + 64(8z^2 - 7z + 5)H_{0,1,1}\zeta_2 \\
& -48\gamma_{qg}^0 \log(2)\zeta_2 + L_M^3 \left[-\frac{16}{3}(4z+1)H_0^2 + \frac{4}{9}(292z^2 - 94z + 11)H_0 + \frac{2}{3}(40z^2 - 20z - 31) \right. \\
& + \frac{8(146z^3 - 142z^2 + 5z - 16)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{16}{3}H_1^2 - \frac{16}{3}H_0H_1 \right] - 32(2z+1)H_{0,1} \\
& \left. - \frac{32}{3}(4z^2 - 10z - 1)\zeta_2 \right] + L_Q^3 \left[-\frac{8}{3}(4z+1)H_0^2 + \frac{8}{9}(106z^2 - 40z + 11)H_0 + \frac{4}{3}(10z^2 - 16z - 5) \right. \\
& + \frac{8(106z^3 - 104z^2 + 19z - 8)H_1}{9z} + \gamma_{qg}^0 \left[-\frac{8}{3}H_1^2 - \frac{8}{3}H_0H_1 \right] - 16(2z+1)H_{0,1} \\
& \left. - \frac{16}{3}(4z^2 - 10z - 1)\zeta_2 \right] + \frac{2}{3}(1360z^2 - 4980z + 33)\zeta_3 - 4(52z^2 + 56z + 43)H_{-1}\zeta_3 \\
& -8(48z^2 - 34z + 33)H_{0,1}\zeta_3 + \frac{4}{3}(236z^2 - 638z - 125)\zeta_2\zeta_3 \\
& + L_Q^2 \left[8(4z+1)H_0^3 - \frac{8}{3}(141z^2 - 49z + 14)H_0^2 - \frac{2}{9}(2320z^2 - 1126z - 109)H_0 \right. \\
& - \frac{8(248z^3 - 232z^2 + 53z - 16)H_1H_0}{3z} + 32(z^2 + 6z + 3)H_{0,1}H_0 + 16(8z^2 - 20z - 3)\zeta_2H_0 \\
& - \frac{8(131z^3 - 123z^2 + 30z - 8)H_1^2}{3z} - \frac{52z^3 - 536z^2 + 193z - 48}{3z} \\
& \left. - \frac{4(1052z^3 - 1078z^2 + 191z + 32)H_1}{9z} - \frac{8(66z^3 + 80z^2 - 31z + 16)H_{0,1}}{3z} \right. \\
& \left. - 32(z^2 + 7z + 2)H_{0,0,1} + 16(14z + 5)H_{0,1,1} + \left[\frac{22}{3}(4z - 1) - \frac{44}{3}(4z^2 - 2z + 1)H_0 \right. \right. \\
& \left. + \frac{22}{3}\gamma_{qg}^0 H_1 \right] L_M + \frac{16}{3}(157z^2 - 85z + 11)\zeta_2 + (2z^2 + 2z + 1) \left[-16H_0H_{0,-1} + 24H_{0,-1} \right. \\
& \left. + H_{-1} \left[8H_0^2 - 24H_0 + 32H_{0,1} \right] - 32H_{0,-1,1} + 16H_{0,0,-1} - 32H_{0,1,-1} - 32H_{-1}\zeta_2 \right] + \gamma_{qg}^0 \left[4H_1^3 \right. \\
& \left. + 12H_0H_1^2 + 10H_0^2H_1 - 24\zeta_2H_1 \right] + 64z^2\zeta_3 \left. \right] + L_M^2 \left[-\frac{8}{3}(4z+1)H_0^3 + 8(9z^2 - 11z - 1)H_0^2 \right. \\
& + \frac{2}{9}(3584z^2 - 1850z - 455)H_0 + \frac{8(208z^3 - 192z^2 + 9z - 16)H_1H_0}{3z} + 32(z^2 - 8z - 2)H_{0,1}H_0 \\
& - 16(8z^2 - 12z + 1)\zeta_2H_0 + \frac{8(157z^3 - 161z^2 + 22z - 8)H_1^2}{3z} + \frac{1556z^3 - 868z^2 - 477z - 48}{3z} \\
& \left. + \frac{8(1022z^3 - 1051z^2 - 28z - 64)H_1}{9z} - \frac{16(33z^3 + 35z^2 + 5z - 8)H_{0,1}}{3z} - 32(z^2 - 13z - 3)H_{0,0,1} \right. \\
& \left. - 16(10z + 7)H_{0,1,1} - \frac{8}{3}(142z^2 - 244z - 1)\zeta_2 + (2z^2 + 2z + 1) \left[-16H_0H_{0,-1} + 24H_{0,-1} \right. \right. \\
& \left. + H_{-1} \left[8H_0^2 - 24H_0 + 32H_{0,1} \right] - 32H_{0,-1,1} + 16H_{0,0,-1} - 32H_{0,1,-1} - 32H_{-1}\zeta_2 \right] + \gamma_{qg}^0 \left[-12H_1^3 \right.
\end{aligned}$$

$$\begin{aligned}
& -20H_0H_1^2 - 6H_0^2H_1 + 40\zeta_2H_1 \Big] - 96(2z^2 + 1)\zeta_3 \Big] + (2z^2 + 2z + 1) \Big[8H_0H_{-1}^4 + (-8H_0^2 \\
& -32H_{0,-1})H_{-1}^3 + \left[-\frac{16}{3}H_0^3 + [48H_{0,-1} - 64H_{0,1}]H_0 + 96H_{0,-1,-1} - 48H_{0,0,-1} + 112H_{0,0,1} \right. \\
& \left. -32H_{0,1,1} \right] H_{-1}^2 + 72\zeta_2^2H_{-1} + \left[\frac{8}{3}H_0^4 + [16H_{0,-1} + 32H_{0,1}]H_0^2 + [-96H_{0,-1,-1} \right. \\
& \left. +128H_{0,-1,1} - 64H_{0,0,-1} - 32H_{0,0,1} + 128H_{0,1,-1} + 64H_{0,1,1}]H_0 - 192H_{0,-1,-1,-1} \right. \\
& \left. -160H_{0,-1,0,1} + 64H_{0,-1,1,1} + 96H_{0,0,-1,-1} - 224H_{0,0,-1,1} + 160H_{0,0,0,-1} - 32H_{0,0,0,1} - 224H_{0,0,1,-1} \right. \\
& \left. -96H_{0,0,1,1} + 64H_{0,1,-1,1} + 64H_{0,1,1,-1} \right] H_{-1} + H_0^2 \left[-32H_{0,-1,1} - 32H_{0,1,-1} \right] + H_0 \left[96H_{0,-1,-1,-1} \right. \\
& \left. -128H_{0,-1,-1,1} - 128H_{0,-1,1,-1} - 64H_{0,-1,1,1} + 32H_{0,0,-1,1} + 32H_{0,0,1,-1} - 128H_{0,1,-1,-1} \right. \\
& \left. -64H_{0,1,-1,1} - 64H_{0,1,1,-1} \right] + 192H_{0,-1,-1,-1,-1} - 64H_{0,-1,-1,1,1} + 160H_{0,-1,0,-1,1} \\
& +160H_{0,-1,0,1,-1} \\
& +32H_{0,-1,0,1,1} - 64H_{0,-1,1,-1,1} - 64H_{0,-1,1,1,-1} - 96H_{0,0,-1,-1,-1} + 224H_{0,0,-1,-1,1} \\
& +224H_{0,0,-1,1,-1} \\
& +96H_{0,0,-1,1,1} + 224H_{0,0,1,-1,-1} + 96H_{0,0,1,-1,1} + 96H_{0,0,1,1,-1} - 64H_{0,1,-1,-1,1} - 64H_{0,1,-1,1,-1} \\
& -64H_{0,1,1,-1,-1} + \left[16H_{-1}^3 + 24H_0H_{-1}^2 + [-60H_0^2 + 16H_{0,-1} - 96H_{0,1}]H_{-1} + 96H_{0,-1,1} \right. \\
& \left. +96H_{0,1,-1} \right] \zeta_2 + [-56H_{-1}^2 - 16H_0H_{-1}] \zeta_3 + \gamma_{gg}^0 \left[\frac{1}{3}H_1H_0^4 + 2H_1^2H_0^3 \right. \\
& \left. + \left[\frac{2}{3}H_1^3 - 16H_1H_{0,1} \right] H_0^2 + \left[-\frac{5}{3}H_1^4 - 32H_{0,1}H_1^2 + [24H_{0,0,1} + 88H_{0,1,1}]H_1 \right] H_0 \right. \\
& \left. - \frac{434}{5}H_1\zeta_2^2 - \frac{16}{3}H_1^3H_{0,1} + H_1^2 [52H_{0,0,1} + 20H_{0,1,1}] + H_1 [-28H_{0,1}^2 + 16H_{0,0,0,1} - 24H_{0,0,1,1}] \right] \\
& + \left[16H_1^3 + 20H_0H_1^2 - H_0^2H_1 + 32H_{0,1}H_1 \right] \zeta_2 + \left[-\frac{14}{3}H_1^2 - \frac{116}{3}H_0H_1 \right] \zeta_3 \Big] \\
& + (2z + 1) \left[-\frac{16}{3}H_{0,-1}H_0^3 + 8H_{0,-1}^2H_0 - 48H_{0,-1}H_{0,0,-1} + 80H_{0,0,-1,0,-1} + 32H_{0,0,0,0,-1} + 8H_{0,-1}\zeta_3 \right] \\
& + L_Q \left[-\frac{4}{3}(18z + 5)H_0^4 + \frac{8}{45}(144z^3 + 2305z^2 - 610z + 225)H_0^3 - \frac{16}{3}(18z^2 + 22z + 11)H_{-1}H_0^3 \right. \\
& \left. + \frac{16}{3}(6z^2 - 2z + 1)H_1H_0^3 + 8(18z^2 - 14z + 7)H_1^2H_0^2 + 8(4z^2 - 6z + 9)H_{0,-1}H_0^2 \right. \\
& \left. + \frac{4(2016z^4 + 16274z^3 - 10977z^2 - 602z + 24)H_0^2}{45z} + \frac{8(303z^3 - 364z^2 + 161z - 16)H_1H_0^2}{3z} \right. \\
& \left. - \frac{4(288z^5 + 1140z^4 + 1000z^3 + 295z^2 + 80z + 8)H_{-1}H_0^2}{15z^2} - 24(4z^2 - 18z - 5)\zeta_2H_0^2 \right. \\
& \left. + \frac{8(398z^3 - 366z^2 + 69z - 32)H_1^2H_0}{3z} + \frac{4(242z^3 + 11758z^2 + 3445z + 64)H_0}{45z} \right. \\
& \left. + \frac{8(1834z^3 - 1520z^2 + 223z + 64)H_1H_0}{9z} - \frac{8(1260z^4 - 300z^3 - 1655z^2 - 80z - 8)H_{0,-1}H_0}{15z^2} \right. \\
& \left. + \frac{8(518z^3 + 172z^2 - 257z + 32)H_{0,1}H_0}{3z} + 32(2z^2 - 6z + 3)H_1H_{0,1}H_0 \right. \\
& \left. + 32(12z^2 + 38z - 5)H_{0,-1,-1}H_0 + 64(11z^2 + 12z + 8)H_{0,0,-1}H_0 - 32(18z^2 + 14z + 19)H_{0,0,1}H_0 \right. \\
& \left. + (2z^2 - 10z + 1) \left[-64H_{0,-1,1} - 64H_{0,1,-1} \right] H_0 - 32(2z^2 + 22z + 13)H_{0,1,1}H_0 \right. \\
& \left. - \frac{8}{5}(192z^3 + 2020z^2 - 320z + 85)\zeta_2H_0 + 32(34z^2 + 42z + 21)H_{-1}\zeta_2H_0 + 256(z - 1)^2H_1\zeta_2H_0 \right. \\
& \left. - 32(4z^2 + 58z + 3)\zeta_3H_0 + \frac{8(93z^3 - 82z^2 + 26z - 8)H_1^3}{3z} + \frac{8(518z^3 - 346z^2 + 65z + 32)H_1^2}{9z} \right]
\end{aligned}$$

$$\begin{aligned}
& -16(4z^2 + 26z - 11)H_{0,-1}^2 + 32(5z^2 - 8z + 4)H_{0,1}^2 + \frac{8}{5}(188z^2 - 234z + 39)\zeta_2^2 \\
& - \frac{2(67432z^3 - 39737z^2 - 18123z - 2512)}{45z} - \frac{4}{45}(4032z^4 + 29968z^3 - 22628z^2 + 1731z + 96)\frac{\zeta_2}{z} \\
& - \frac{8}{3}(144z^4 - 6z^3 + 646z^2 + 49z + 32)\frac{\zeta_3}{z} - \frac{2(32084z^3 - 22856z^2 - 10349z - 544)H_1}{45z} \\
& + \frac{16}{15}(216z^5 + 900z^4 + 1075z^3 + 560z^2 + 40z + 6)\frac{\zeta_2}{z^2}H_{-1} \\
& - \frac{16}{15}(72z^5 + 1610z^4 - 1585z^3 + 495z^2 - 120z - 2)\frac{\zeta_2}{z^2}H_1 \\
& + \frac{(1008z^5 - 558z^4 - 6071z^3 - 4851z^2 + 12z + 28)}{z^2}\left[\frac{16}{45}H_{0,-1} - \frac{16}{45}H_{-1}H_0\right] \\
& + \frac{4(11628z^3 + 16856z^2 - 499z - 592)H_{0,1}}{45z} + \frac{16}{3}(44z^2 - 32z + 19)H_1H_{0,1} \\
& + \frac{(72z^5 + 600z^4 + 405z^3 - 55z^2 + 40z + 2)\left[\frac{16}{15}H_0H_{-1}^2 - \frac{32}{15}H_{0,-1}H_{-1} + \frac{32}{15}H_{0,-1,-1}\right]}{z^2} \\
& + \frac{8(288z^5 + 3660z^4 + 400z^3 - 3015z^2 - 80z - 8)H_{0,0,-1}}{15z^2} + \gamma_{qg}^0\left[-24H_{0,1}H_0^2 + \left[-\frac{40}{3}H_1^3\right.\right. \\
& \left.\left.- 48H_{0,-1}H_1\right]H_0 + 8H_1^2H_{0,1} - 48H_{0,-1}H_{0,1} + 96H_1H_{0,0,-1}\right] - 128(4z^2 - 6z + 3)H_1H_{0,0,1} \\
& + \frac{8(288z^4 - 2150z^3 + 260z^2 + 1215z - 160)H_{0,0,1}}{15z} - 128(5z^2 - 6z + 3)H_{0,-1,0,1} \\
& + \frac{(144z^5 + 300z^4 + 670z^3 + 615z^2 + 4)\left[-\frac{16}{15}H_{-1}H_{0,1} + \frac{16}{15}H_{0,-1,1} + \frac{16}{15}H_{0,1,-1}\right]}{z^2} \\
& + \frac{8(34z^3 + 304z^2 - 107z + 64)H_{0,1,1}}{3z} + (4z^2 + 6z + 3)\left[-\frac{128}{3}H_0H_{-1}^3 + \left[48H_0^2 + 128H_{0,-1}\right]H_{-1}^2\right. \\
& \left.+ \left[-64H_0H_{0,-1} - 256H_{0,-1,-1} - 64H_{0,0,-1}\right]H_{-1} + 256H_{0,-1,-1,-1} + 64H_{0,0,-1,-1}\right] \\
& - 16(108z^2 + 82z + 101)H_{0,0,0,-1} + 16(36z^2 + 98z + 61)H_{0,0,0,1} + (z + 1)^2\left[-128H_{-1}H_{0,0,1}\right. \\
& \left.+ 128H_{0,0,-1,1} + 128H_{0,0,1,-1}\right] + 64(3z^2 + 13z + 6)H_{0,0,1,1} + (3z^2 + 4z + 2)\left[128H_{0,1}H_{-1}^2\right. \\
& \left.+ \left[-256H_{0,-1,1} - 256H_{0,1,-1}\right]H_{-1} + 256H_{0,-1,-1,1} + 256H_{0,-1,1,-1} + 256H_{0,1,-1,-1}\right] \\
& + (2z^2 + 2z + 1)\left[H_{-1}\left[-128H_0H_{0,1} - 128H_{0,1,1}\right] + 128H_{0,-1,1,1} + 128H_{0,1,-1,1} + 128H_{0,1,1,-1}\right] \\
& + 128(z^2 - 7z - 1)H_{0,1,1,1} - 64(10z^2 + 14z + 7)H_{-1}^2\zeta_2 - 64(2z - 1)H_1^2\zeta_2 \\
& - 16(8z^2 + 26z + 13)H_{0,-1}\zeta_2 - 16(28z^2 - 74z + 13)H_{0,1}\zeta_2 + L_M^2\left[8(4z + 1)H_0^2\right. \\
& \left.- \frac{16}{3}z(31z - 9)H_0 - \frac{8}{3}(z - 1)(15z + 13) - \frac{8(62z^3 - 60z^2 - 3z - 8)H_1}{3z} + \gamma_{qg}^0\left[8H_1^2 + 8H_0H_1\right]\right. \\
& \left.+ 48(2z + 1)H_{0,1} + 16(4z^2 - 10z - 1)\zeta_2\right] + 16(74z^2 + 102z + 51)H_{-1}\zeta_3 \\
& + 112(10z^2 - 14z + 7)H_1\zeta_3 + L_M\left[-\frac{16}{3}(4z + 1)H_0^3 + \frac{8}{3}(114z^2 - 16z + 17)H_0^2\right. \\
& \left.- \frac{4}{9}(632z^2 - 362z - 173)H_0 + \frac{32}{3}(10z^2 - 10z + 11)H_1H_0 + 64(2z + 1)\zeta_2H_0\right. \\
& \left.- \frac{16}{3}(13z^2 - 19z - 4)H_1^2 - \frac{2}{3}(752z^2 - 166z - 335) - \frac{4(992z^3 - 1024z^2 - 247z - 160)H_1}{9z}\right]
\end{aligned}$$

$$\begin{aligned}
& +8(44z^2 + 50z - 7)H_{0,1} + 32(2z^2 - 6z - 1)H_{0,0,1} - 32(2z - 1)H_{0,1,1} - \frac{8}{3}(172z^2 + 74z + 23)\zeta_2 \\
& + (2z^2 + 2z + 1) \left[32H_0H_{0,-1} - 48H_{0,-1} + H_{-1} \left[-16H_0^2 + 48H_0 - 64H_{0,1} \right] + 64H_{0,-1,1} \right. \\
& - 32H_{0,0,-1} + 64H_{0,1,-1} + 64H_{-1}\zeta_2 \left. \right] + \gamma_{gg}^0 \left[8H_1^3 - 4H_0^2H_1 - 16\zeta_2H_1 + H_0 \left[8H_1^2 + 8H_{0,1} \right] \right] \\
& + 32(4z^2 + 3)\zeta_3 \left. \right] + L_M \left[\frac{4}{3}(10z + 3)H_0^4 - \frac{8}{9}(293z^2 - 56z + 34)H_0^3 \right. \\
& - \frac{4}{45}(432z^3 + 8165z^2 - 5075z + 385)H_0^2 + 4(8z^2 - 16z - 27)H_{-1}H_0^2 \\
& - \frac{8(159z^3 - 232z^2 + 113z - 8)H_1H_0^2}{3z} - 8(12z^2 + 10z + 9)H_{0,-1}H_0^2 - 32(4z^2 - 2z + 1)H_{0,1}H_0^2 \\
& - 8(4z^2 + 14z + 11)\zeta_2H_0^2 - \frac{16(13z^3 - 5z^2 - 5z - 8)H_1^2H_0}{3z} \\
& + \frac{4(27226z^3 - 15808z^2 - 3393z - 24)H_0}{45z} + \frac{8(461z^3 - 676z^2 - 73z - 184)H_1H_0}{9z} \\
& + 8(16z^2 + 52z - 61)H_{0,-1}H_0 - \frac{8(218z^3 + 572z^2 - 193z + 16)H_{0,1}H_0}{3z} \\
& - 32(8z^2 + 14z - 1)H_{0,-1,-1}H_0 - 64(5z^2 + 2)H_{0,0,-1}H_0 + 32(18z^2 + 6z + 11)H_{0,0,1}H_0 \\
& + (2z^2 - 10z + 1) \left[64H_{0,-1,1} + 64H_{0,1,-1} \right] H_0 - 64(3z^2 - 10z - 1)H_{0,1,1}H_0 \\
& + \frac{8}{3}(552z^2 + 328z + 7)\zeta_2H_0 - 64(2z^2 - 29z - 4)\zeta_3H_0 + 8(39z^2 - 40z + 4)H_1^3 \\
& + \frac{2(5614z^3 - 6188z^2 - 77z - 272)H_1^2}{9z} + 16(4z^2 + 10z - 3)H_{0,-1}^2 - 96(z^2 + 1)H_{0,1}^2 \\
& - \frac{8}{5}(12z^2 - 162z - 79)\zeta_2^2 + \frac{2(53302z^3 - 22043z^2 - 25297z - 552)}{45z} \\
& - \frac{16}{3}(501z^3 - 490z^2 + 26z - 24)\frac{\zeta_3}{z} + \frac{2(14956z^3 - 14194z^2 - 1243z - 344)H_1}{9z} \\
& - \frac{16}{3}(32z^3 - z^2 - 49z + 8)\frac{\zeta_2}{z}H_1 + \frac{(72z^5 + 15z^4 - 1105z^3 - 1215z^2 + 2) \left[\frac{16}{15}H_{-1}H_0 - \frac{16}{15}H_{0,-1} \right]}{z^2} \\
& + \frac{4(252z^3 - 3316z^2 + 65z + 368)H_{0,1}}{9z} + \frac{32(20z^3 - 19z^2 - 7z - 4)H_1H_{0,1}}{3z} \\
& + (z + 1)(2z - 5) \left[-48H_0H_{-1}^2 + 96H_{0,-1}H_{-1} - 96H_{0,-1,-1} \right] - 8(40z^2 + 88z - 149)H_{0,0,-1} \\
& + \frac{8(202z^3 + 460z^2 - 167z + 16)H_{0,0,1}}{3z} + (20z^2 + 34z + 17) \left[16H_{-1}H_{0,1} - 16H_{0,-1,1} - 16H_{0,1,-1} \right] \\
& - \frac{16}{3}(53z^2 + 209z - 10)H_{0,1,1} + 16(68z^2 + 2z + 37)H_{0,0,0,-1} - 16(52z^2 + 42z + 41)H_{0,0,0,1} \\
& + 32(6z^2 - 10z + 1)H_{0,0,1,1} - 32(8z^2 - 14z + 7)H_{0,1,1,1} + \frac{4}{45}(864z^3 - 5870z^2 + 10080z + 405)\zeta_2 \\
& - 16(26z^2 + 25z + 2)H_{-1}\zeta_2 + 16(16z^2 + 26z + 13)H_{0,-1}\zeta_2 + 16(20z^2 - 30z + 11)H_{0,1}\zeta_2 \\
& + (2z^2 + 2z + 1) \left[\frac{128}{3}H_0H_{-1}^3 + \left[-48H_0^2 - 128H_{0,-1} - 128H_{0,1} \right] H_{-1}^2 + \left[\frac{112}{3}H_0^3 \right. \right. \\
& + \left. \left[64H_{0,-1} + 128H_{0,1} \right] H_0 + 256H_{0,-1,-1} + 256H_{0,-1,1} + 64H_{0,0,-1} + 256H_{0,1,-1} + 128H_{0,1,1} \right] H_{-1} \\
& - 368\zeta_3H_{-1} - 256H_{0,-1,-1,-1} - 256H_{0,-1,-1,1} - 256H_{0,-1,1,-1} - 128H_{0,-1,1,1} - 64H_{0,0,-1,-1} \\
& - 256H_{0,1,-1,-1} - 128H_{0,1,-1,1} - 128H_{0,1,1,-1} + \left[192H_{-1}^2 - 416H_{-1}H_0 \right] \zeta_2 \left. \right] + \gamma_{gg}^0 \left[-8H_1^4 + 6H_0^2H_1^2 \right. \\
& - 16H_{0,1}H_1^2 + \frac{4}{3}H_0^3H_1 + \left. \left[-96H_{0,0,-1} - 16H_{0,0,1} \right] H_1 + 196\zeta_3H_1 + H_0 \left[H_1 \left[48H_{0,-1} - 24H_{0,1} \right] \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16}{3}H_1^3 \Big] + 48H_{0,-1}H_{0,1} - 64H_{0,-1,0,1} + \left[48H_1^2 + 80H_0H_1 \right] \zeta_2 \Big] - 4(212z^2 - 174z + 1)\zeta_5 \Big] \\
& + a_{Qg}^{(3)} + \tilde{C}_{2,g}^{\mathcal{S},(3)}(N_F + 1) \Big\} .
\end{aligned} \tag{598}$$

Numerical implementations of the harmonic polylogarithms were given in Refs. [66].

C The Longitudinal Wilson Coefficients in z -space

The Wilson coefficients for the longitudinal structure function $F_L(x, Q^2)$ in the asymptotic region in z -space are presented in the following. $L_{q,L}^{\text{PS}}$ and $L_{g,L}^{\text{S}}$ read :

$$\begin{aligned}
L_{q,L}^{\text{PS}} = & a_s^3 \left\{ C_F N_F T_F^2 \left[L_M \left[z \left[\frac{256}{3} H_{0,1} - \frac{256 \zeta_2}{3} \right] - \frac{256(z-1)(2z^2+2z-1)H_1}{9z} \right. \right. \right. \\
& - \frac{256}{9} z(2z+11)H_0 + \left. \left. \frac{256(z-1)(19z^2+16z-5)}{27z} \right] + z(2z+11) \left[\frac{128}{9} H_{0,1} - \frac{128 \zeta_2}{9} \right] \right. \\
& + z \left[\frac{128 \zeta_3}{3} - \frac{128}{3} H_{0,1,1} \right] + L_Q^2 \left[\frac{128(z-1)(2z^2+2z-1)}{9z} - \frac{128}{3} z H_0 \right] \\
& + L_Q \left[-\frac{256}{9} (4z^2 - 8z - 3)H_0 + \frac{256}{3} z H_0^2 - \frac{256(z-1)(3z^2+6z-2)}{9z} \right] \\
& + \left[\frac{128(z-1)(2z^2+2z-1)}{9z} - \frac{128}{3} z H_0 \right] L_M^2 + \frac{64(z-1)(2z^2+2z-1)H_1^2}{9z} \\
& - \left. \left. \frac{128(z-1)(19z^2+16z-5)H_1}{27z} - \frac{128}{27} z(19z+67)H_0 + \frac{256(z-1)(55z^2+43z-14)}{81z} \right] \right. \\
& \left. + N_F \hat{C}_{L,q}^{\text{PS},(3)}(N_F) \right\} ,
\end{aligned} \tag{599}$$

$$\begin{aligned}
L_{g,L}^{\text{S}} = & -\frac{64}{3} a_s^2 N_F T_F^2 (z-1)z L_M + a_s^3 \left\{ -N_F T_F^3 \frac{256}{9} (z-1)z L_M^2 \right. \\
& + C_A T_F^2 N_F \left[\left[\frac{64(z-1)(17z^2+2z-1)}{9z} - \frac{256}{3} z H_0 + \frac{128}{3} (z-1)z H_1 \right] L_Q^2 \right. \\
& + \left[-\frac{64(z-1)(461z^2+11z-25)}{27z} - \frac{128}{9} z(26z-59)H_0 \right. \\
& - \frac{128(z-1)(39z^2+2z-1)H_1}{9z} + (z-1)z \left[-\frac{128}{3} H_1^2 - \frac{256}{3} H_0 H_1 \right] \\
& + z(z+1) \left[\frac{256}{3} H_{-1} H_0 - \frac{256}{3} H_{0,-1} \right] + z \left[\frac{512}{3} H_0^2 + \frac{512}{3} H_{0,1} \right] \\
& \left. \left. + \left[\frac{128(z-1)(17z^2+2z-1)}{9z} - \frac{512}{3} z H_0 + \frac{256}{3} (z-1)z H_1 \right] L_M \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{256}{3}(z-2)z\zeta_2 \Big] L_Q - \frac{32}{9}(28z-3)H_0^2 + \left[\frac{64(z-1)(17z^2+2z-1)}{9z} - \frac{256}{3}zH_0 \right. \\
& + \left. \frac{128}{3}(z-1)zH_1 \right] L_M^2 + \frac{32(z-1)(2714z^2-106z-139)}{81z} - \frac{64}{27}(110z^2+277z-33)H_0 \\
& + \frac{4160}{27}(z-1)zH_1 + z \left[-\frac{64}{9}H_0^3 - \frac{64}{3}H_{0,1} + \frac{64\zeta_2}{3} \right] + L_M \left[\frac{64(z-1)(68z^2+z-7)}{9z} \right. \\
& - \frac{128}{9}(4z-1)(13z+6)H_0 - \frac{128(z-1)(19z^2+2z-1)H_1}{9z} + (z-1)z \left[-\frac{128}{3}H_1^2 \right. \\
& - \left. \frac{256}{3}H_0H_1 \right] + z(z+1) \left[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1} \right] + z \left[\frac{256}{3}H_0^2 \right. \\
& + \left. \frac{512}{3}H_{0,1} \right] + \frac{256}{3}(z-2)z\zeta_2 \Big] + C_{FT_F^2} N_F \left[-\frac{16}{3}zH_0^4 - \frac{32}{3}(7z-1)H_0^3 - 32(19z-3)H_0^2 \right. \\
& - 64(6z^2+7z-8)H_0 + \left. \left[-64zH_0^2 - \frac{64}{3}(4z^2+3z-3)H_0 \right. \right. \\
& + \left. \left. \frac{128(z-1)(17z^2-10z-1)}{9z} \right] L_M^2 + \frac{16(z-1)(343z^2-242z+4)}{3z} \right. \\
& + L_Q^2 \left[-64zH_0^2 - \frac{64}{3}(z+1)(4z-3)H_0 + \frac{64(z-1)(28z^2-23z-2)}{9z} \right] \\
& + L_Q \left[-\frac{128(25z+2)H_1(z-1)^2}{9z} - \frac{32(2474z^2-4897z+44)(z-1)}{135z} \right. \\
& - \frac{64}{45}(12z^3-180z^2-265z+90)H_0^2 - \frac{64(354z^3-397z^2+388z+4)H_0}{45z} \\
& + \frac{(z+1)(6z^4-6z^3+z^2-z+1) \left(\frac{256}{45}H_{-1}H_0 - \frac{256}{45}H_{0,-1} \right)}{z^2} + \frac{128}{3}(z+1)(4z-3)H_{0,1} \\
& + \left. \left[128zH_0^2 + \frac{128}{3}(2z-1)(2z+3)H_0 - \frac{64(z-1)(74z^2-37z-4)}{9z} \right] L_M \right. \\
& + \left. \frac{128}{45}(12z^3-60z^2-25z+45)\zeta_2 + z \left[128H_0^3 - 256\zeta_2H_0 + 256H_{0,0,1} - 256\zeta_3 \right] \right] \\
& + L_M \left[-\frac{64}{45}(12z^3+180z^2+335z-90)H_0^2 + \frac{64(456z^3-708z^2+347z-4)H_0}{45z} \right. \\
& + \frac{64(z-1)(2368z^2-2189z-2)}{135z} + \frac{64(z-1)(80z^2-37z-4)H_1}{9z} \\
& + \frac{(z+1)(6z^4-6z^3+z^2-z+1) \left[\frac{256}{45}H_{-1}H_0 - \frac{256}{45}H_{0,-1} \right]}{z^2} - \frac{128}{3}(2z-1)(2z+3)H_{0,1} \\
& + \left. \frac{128}{45}(12z^3+60z^2+50z-45)\zeta_2 + z \left[-128H_0^3 + 256\zeta_2H_0 - 256H_{0,0,1} + 256\zeta_3 \right] \right] \Big] \\
& + N_F \hat{C}_{L,g}^{\mathcal{S},(3)}(N_F) \Big\} . \tag{600}
\end{aligned}$$

The flavor non-singlet Wilson coefficient is given by :

$$L_{q,L}^{\text{NS}} =$$

$$\begin{aligned}
& a_s^2 C_F T_F \left\{ \frac{16L_Q z}{3} + \left[-\frac{32}{3}H_0 - \frac{16H_1}{3} \right] z - \frac{8}{9}(25z - 6) \right\} \\
& + a_s^3 \left\{ C_F^2 T_F \left[\left[8(z+2) + z \left[-16H_0 - 32H_1 \right] \right] L_Q^2 + \left[\frac{16}{15}z(24z^2 - 5)H_0^2 + \frac{80}{9}(5z - 6)H_1 \right. \right. \right. \\
& + \frac{16(144z^3 - 227z^2 - 72z - 96)H_0}{45z} + \frac{32(72z^3 - 223z^2 - 77z + 48)}{45z} \\
& + \frac{(3z^5 - 5z^3 - 10z^2 - 2) \left[\frac{256}{15}H_{0,-1} - \frac{256}{15}H_{-1}H_0 \right]}{z^2} - \frac{32}{5}z(8z^2 + 5)\zeta_2 + z \left[-\frac{256}{3}H_0H_{-1}^2 \right. \\
& + \left. \left[\frac{128}{3}H_0^2 + \frac{512}{3}H_{0,-1} \right] H_{-1} + \frac{128}{3}H_1^2 - \frac{128}{3}H_0^2H_1 + 32H_{0,1} + H_0 \left[\frac{256H_1}{3} - \frac{256}{3}H_{0,-1} \right. \right. \\
& + \left. \left. \frac{256}{3}H_{0,1} \right] - \frac{512}{3}H_{0,-1,-1} + \frac{256}{3}H_{0,0,-1} - \frac{256}{3}H_{0,0,1} + \left[\frac{256H_1}{3} - \frac{256}{3}H_{-1} \right] \zeta_2 + \frac{512\zeta_3}{3} \right] \right] L_Q \\
& + \frac{8}{9}(z+3)H_0^2 + \left[\frac{8(z+2)}{3} + z \left[-\frac{16}{3}H_0 - \frac{32H_1}{3} \right] \right] L_M^2 - \frac{2}{27}(653z - 872) + \frac{16}{27}(11z + 42)H_0 \\
& + z \left[-\frac{8}{9}H_0^3 - \frac{16}{3}H_1H_0^2 + \left[\frac{32}{3}H_{0,1} - \frac{160H_1}{9} \right] H_0 - \frac{896H_1}{27} + \frac{160}{9}H_{0,1} - \frac{32}{3}H_{0,0,1} \right] \\
& + \left[-\frac{8}{9}(53z - 56) + \frac{32}{9}(z+3)H_0 + z \left[-\frac{16}{3}H_0^2 - \frac{64}{3}H_1H_0 - \frac{320H_1}{9} + \frac{64}{3}H_{0,1} \right] \right] L_M \\
& + \left[\left[\frac{64zL_Q^2}{9} + \left[z \left[-\frac{256}{9}H_0 - \frac{128H_1}{9} \right] - \frac{64}{27}(25z - 6) \right] L_Q + N_F \left[\frac{128zL_Q^2}{9} + \left[z \left[-\frac{512}{9}H_0 \right. \right. \right. \right. \right. \\
& - \left. \left. \frac{256H_1}{9} \right] - \frac{128}{27}(25z - 6) \right] L_Q \right] \right] T_F^2 + C_A \left[L_Q \left[-\frac{16(216z^3 - 3329z^2 + 624z + 144)}{135z} \right. \right. \\
& - \frac{64(18z^3 - 149z^2 + 6z - 12)H_0}{45z} + \frac{(3z^5 - 5z^3 - 10z^2 - 2) \left[\frac{128}{15}H_{-1}H_0 - \frac{128}{15}H_{0,-1} \right]}{z^2} \\
& + z(3z^2 - 5) \left[\frac{128\zeta_2}{15} - \frac{64}{15}H_0^2 \right] + z \left[\frac{128}{3}H_0H_{-1}^2 + \left[-\frac{64}{3}H_0^2 - \frac{256}{3}H_{0,-1} \right] H_{-1} + \frac{64}{3}H_0^2H_1 \right. \\
& + \frac{1088H_1}{9} + H_0 \left[\frac{128}{3}H_{0,-1} - \frac{128}{3}H_{0,1} \right] + \frac{256}{3}H_{0,-1,-1} - \frac{128}{3}H_{0,0,-1} + \frac{128}{3}H_{0,0,1} + \left[\frac{128}{3}H_{-1} \right. \\
& \left. \left. - \frac{128H_1}{3} \right] \zeta_2 - \frac{256\zeta_3}{3} \right] \left. \right] - \frac{352L_Q^2 z}{9} \left. \right] T_F \left. \right] C_F + \hat{C}_{L,q}^{\text{NS},(3)}(N_F) \left. \right\}. \tag{601}
\end{aligned}$$

The pure-singlet Wilson coefficient $H_{q,L}^{\text{PS}}$ reads :

$$\begin{aligned}
H_{q,L}^{\text{PS}} = & a_s^2 C_F T_F \left\{ \frac{32(z-1)(10z^2 - 2z + 1)}{9z} - 32(z+1)(2z-1)H_0 - \frac{32(z-1)(2z^2 + 2z - 1)H_1}{3z} \right. \\
& + L_Q \left[\frac{32(z-1)(2z^2 + 2z - 1)}{3z} - 32zH_0 \right] + z \left[32H_0^2 + 32H_{0,1} - 32\zeta_2 \right] \left. \right\} \\
& + a_s^3 \left\{ C_F^2 T_F \left[-\frac{8}{3}(5z+2)H_0^3 - \frac{8}{3}(8z^2+3)H_0^2 + \frac{16}{9}(160z^2+93z-39)H_0 \right. \right. \\
& + \left. \left[16zH_0^2 - 16(z+2)H_0 - \frac{16(z-1)(4z^2-11z-2)}{3z} \right] L_M^2 - \frac{32(z-1)(440z^2-91z-28)}{27z} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{(z-1)(4z^2-11z-2) \left[\frac{32}{3}H_0H_1 - \frac{32}{3}H_{0,1} \right]}{z} + \left[-\frac{32}{3}zH_0^3 + 16(5z+2)H_0^2 \right. \\
& + \frac{32}{3}(8z^2+18z+3)H_0 - \frac{32(z-1)(80z^2+17z-10)}{9z} \left. \right] L_M + L_Q^2 \left[-\frac{64}{3}(2z^2-3)H_0 \right. \\
& + \frac{32(z-1)(2z^2-9z-1)}{3z} - \frac{64(z-1)(2z^2+2z-1)H_1}{3z} + z \left. \left[64H_{0,1} - 64\zeta_2 \right] \right] \\
& + (z+2) \left[32H_0H_{0,1} - 64H_{0,0,1} + 64\zeta_3 \right] + z \left[\frac{4}{3}H_0^4 - 64H_{0,0,1}H_0 - 128\zeta_3H_0 - \frac{384\zeta_2^2}{5} \right. \\
& + 192H_{0,0,0,1} \left. \right] + L_Q \left[-\frac{32(z-1)(86z^2-33z+6)}{45z} - \frac{32(56z^3-813z^2+142z+16)H_0}{45z} \right. \\
& + \frac{64(z-1)(4z^2+55z-5)H_1}{9z} + \frac{(z-1)(2z^2+2z-1) \left[\frac{64}{3}H_1^2 + \frac{128}{3}H_0H_1 \right]}{z} \\
& + \frac{(z+1)(6z^4-6z^3+11z^2+4z-4) \left[\frac{128}{45}H_{0,-1} - \frac{128}{45}H_{-1}H_0 \right]}{z^2} \\
& + \frac{128(4z^3-6z^2-3z-1)H_{0,1}}{3z} + (6z^3+90z^2-85z-90) \left[\frac{64}{45}H_0^2 - \frac{128\zeta_2}{45} \right] \\
& + z \left[-\frac{64}{9}H_0^3 + \left[\frac{128}{3}H_{0,-1} - 128H_{0,1} \right] H_0 + \frac{896}{3}\zeta_2H_0 - \frac{256}{3}H_{0,0,-1} - 128H_{0,1,1} + 192\zeta_3 \right] \left. \right] \\
& + C_{FT_F^2} N_F \left[\left[\frac{128(z-1)(2z^2+2z-1)}{9z} - \frac{128}{3}zH_0 \right] L_Q^2 + \left[\frac{256}{3}zH_0^2 \right. \right. \\
& \left. \left. - \frac{256}{9}(4z^2-8z-3)H_0 - \frac{256(z-1)(3z^2+6z-2)}{9z} \right] L_Q \right] \\
& + C_{FT_F^2} \left[\left[\frac{128(z-1)(2z^2+2z-1)}{9z} - \frac{128}{3}zH_0 \right] L_Q^2 + \left[\frac{256}{3}zH_0^2 - \frac{256}{9}(4z^2-8z-3)H_0 \right. \right. \\
& \left. \left. - \frac{256(z-1)(3z^2+6z-2)}{9z} \right] L_Q + \frac{64(z-1)(2z^2+2z-1)H_1^2}{9z} + \left[\frac{128(z-1)(2z^2+2z-1)}{9z} \right. \right. \\
& \left. \left. - \frac{128}{3}zH_0 \right] L_M^2 + \frac{256(z-1)(55z^2+43z-14)}{81z} - \frac{128}{27}z(19z+67)H_0 \right. \\
& \left. - \frac{128(z-1)(19z^2+16z-5)H_1}{27z} + L_M \left[\frac{256(z-1)(19z^2+16z-5)}{27z} - \frac{256}{9}z(2z+11)H_0 \right. \right. \\
& \left. \left. - \frac{256(z-1)(2z^2+2z-1)H_1}{9z} + z \left[\frac{256}{3}H_{0,1} - \frac{256\zeta_2}{3} \right] \right] + z(2z+11) \left[\frac{128}{9}H_{0,1} - \frac{128\zeta_2}{9} \right] \right. \\
& + z \left[\frac{128\zeta_3}{3} - \frac{128}{3}H_{0,1,1} \right] + C_{FCAT_F} \left[\left[-\frac{16(z-1)(46z^2-z-21)}{3z} + \frac{64(7z^2-3z-1)H_0}{3z} \right. \right. \\
& \left. \left. - \frac{64(z-1)(2z^2+2z-1)H_1}{3z} + z \left[64H_0^2 + 64H_{0,1} - 64\zeta_2 \right] \right] L_Q^2 + \left[-\frac{32}{3}(19z-12)H_0^2 \right. \right. \\
& \left. \left. + \frac{32(422z^3-137z^2-114z+4)H_0}{9z} - \frac{32(z-1)(670z^2-245z+46)}{27z} \right. \right. \\
& \left. \left. + \frac{32(z-1)(106z^2-23z-65)H_1}{9z} + \frac{(z-1)(2z^2+2z-1) \left[\frac{128}{3}H_1^2 + \frac{256}{3}H_0H_1 \right]}{z} \right. \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{(z+1)(2z^2-2z-1) \left[\frac{256}{3} H_{0,-1} - \frac{256}{3} H_{-1} H_0 \right]}{z} - 64(z-4)H_{0,1} - \frac{64}{3} z(8z-3)\zeta_2 \\
& + z \left[-\frac{256}{3} H_0^3 + \left[-256H_{0,-1} - 256H_{0,1} \right] H_0 + 256\zeta_2 H_0 + 512H_{0,0,-1} - 256H_{0,1,1} \right. \\
& \left. - 128\zeta_3 \right] L_Q \left. + \tilde{C}_{L,q}^{\text{PS},(3)}(N_F+1) \right\}. \tag{602}
\end{aligned}$$

Finally, the gluonic Wilson coefficient is given by :

$$\begin{aligned}
H_{g,L}^S = & -16T_F(z-1)z a_s + a_s^2 \left\{ -\frac{64}{3}(z-1)z L_M T_F^2 + C_{AT_F} \left[-\frac{32(z-1)(53z^2+2z-1)}{9z} \right. \right. \\
& - 32(13z^2-8z-1)H_0 - \frac{32(z-1)(29z^2+2z-1)H_1}{3z} + L_Q \left[\frac{32(z-1)(17z^2+2z-1)}{3z} \right. \\
& \left. \left. - 128zH_0 + 64(z-1)zH_1 \right] + (z-1)z \left[-32H_1^2 - 64H_0H_1 \right] + z(z+1) \left[64H_{-1}H_0 - 64H_{0,-1} \right] \right. \\
& \left. + z \left[96H_0^2 + 128H_{0,1} \right] + 64(z-2)z\zeta_2 \right\} + C_{FT_F} \left[-\frac{64}{15}z(3z^2+5)H_0^2 \right. \\
& \left. + \frac{16(36z^3-78z^2-13z-4)H_0}{15z} + \frac{32(z-1)(63z^2+6z-2)}{15z} \right. \\
& \left. + L_Q \left[32zH_0 - 16(z-1)(2z+1) \right] + 16(z-1)(4z+1)H_1 \right. \\
& \left. + \frac{(z+1)(6z^4-6z^3+z^2-z+1)}{z^2} \left[\frac{64}{15}H_{-1}H_0 - \frac{64}{15}H_{0,-1} \right] - 32zH_{0,1} \right. \\
& \left. + \left[16(z-1)(2z+1) - 32zH_0 \right] L_M + \frac{32}{15}z(12z^2+5)\zeta_2 \right\} \\
& + a_s^3 \left\{ -T_F^3 \frac{256}{9}(z-1)z L_M^2 + C_{AT_F}^2 \left[\left[\frac{64(z-1)(17z^2+2z-1)}{9z} - \frac{256}{3}zH_0 \right. \right. \right. \\
& \left. \left. + \frac{128}{3}(z-1)zH_1 \right] L_Q^2 + \left[-\frac{64(z-1)(461z^2+11z-25)}{27z} - \frac{128}{9}z(26z-59)H_0 \right. \right. \\
& \left. \left. - \frac{128(z-1)(39z^2+2z-1)H_1}{9z} + (z-1)z \left[-\frac{128}{3}H_1^2 - \frac{256}{3}H_0H_1 \right] \right. \right. \\
& \left. \left. + z(z+1) \left[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1} \right] + z \left[\frac{512}{3}H_0^2 + \frac{512}{3}H_{0,1} \right] + \left[\frac{128(z-1)(17z^2+2z-1)}{9z} \right. \right. \right. \\
& \left. \left. - \frac{512}{3}zH_0 + \frac{256}{3}(z-1)zH_1 \right] L_M + \frac{256}{3}(z-2)z\zeta_2 \right] L_Q - \frac{32}{9}(28z-3)H_0^2 \\
& \left. + \left[\frac{64(z-1)(17z^2+2z-1)}{9z} - \frac{256}{3}zH_0 + \frac{128}{3}(z-1)zH_1 \right] L_M^2 \right. \\
& \left. + \frac{32(z-1)(2714z^2-106z-139)}{81z} - \frac{64}{27}(110z^2+277z-33)H_0 + \frac{4160}{27}(z-1)zH_1 \right. \\
& \left. + z \left[-\frac{64}{9}H_0^3 - \frac{64}{3}H_{0,1} + \frac{64\zeta_2}{3} \right] + L_M \left[\frac{64(z-1)(68z^2+z-7)}{9z} - \frac{128}{9}(4z-1)(13z+6)H_0 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{128(z-1)(19z^2+2z-1)H_1}{9z} + (z-1)z\left[-\frac{128}{3}H_1^2 - \frac{256}{3}H_0H_1\right] \\
& + z(z+1)\left[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1}\right] + z\left[\frac{256}{3}H_0^2 + \frac{512}{3}H_{0,1}\right] + \frac{256}{3}(z-2)z\zeta_2 \Big] \\
& + C_{AT_F^2}N_F \left[\left[\frac{64(z-1)(17z^2+2z-1)}{9z} - \frac{256}{3}zH_0 + \frac{128}{3}(z-1)zH_1 \right] L_Q^2 \right. \\
& + \left[-\frac{64(z-1)(461z^2+11z-25)}{27z} - \frac{128}{9}z(26z-59)H_0 - \frac{128(z-1)(39z^2+2z-1)H_1}{9z} \right. \\
& + (z-1)z\left[-\frac{128}{3}H_1^2 - \frac{256}{3}H_0H_1\right] + z(z+1)\left[\frac{256}{3}H_{-1}H_0 - \frac{256}{3}H_{0,-1}\right] + z\left[\frac{512}{3}H_0^2 + \frac{512}{3}H_{0,1}\right] \\
& \left. \left. + \frac{256}{3}(z-2)z\zeta_2 \right] L_Q \right] \\
& + C_{AT_F^2} \left[\left[-\frac{16(z-1)(1033z^2-26z-65)}{9z} - \frac{64(6z^3-47z^2+3z+1)H_0}{3z} \right. \right. \\
& - \frac{32(z-1)(79z^2+8z-4)H_1}{3z} + (z-1)z\left[-128H_1^2 - 128H_0H_1\right] + 128z(z+3)H_{0,1} \\
& \left. + z\left[256H_0^2 - 512\zeta_2\right] \right] L_Q^2 + \left[\frac{64}{3}(18z^2-91z+6)H_0^2 + \frac{32(2713z^3-1405z^2-60z+4)H_0}{9z} \right. \\
& + \frac{64(z-1)(137z^2+12z-6)H_1H_0}{3z} + 128z(3z-5)H_{0,-1}H_0 - 128z(z+11)H_{0,1}H_0 \\
& + \frac{32(z-1)(161z^2+12z-6)H_1^2}{3z} + \frac{32(z-1)(680z^2-60z-13)}{9z} \\
& + \frac{32(z-1)(1919z^2+30z-93)H_1}{9z} + \frac{(z+1)(79z^2-8z-4)\left[\frac{64}{3}H_{0,-1} - \frac{64}{3}H_{-1}H_0\right]}{z} \\
& - \frac{128(z^3+53z^2-6z+1)H_{0,1}}{3z} - 128z(3z-13)H_{0,0,-1} - 128(z-3)zH_{0,0,1} \\
& - 256z(z+5)H_{0,1,1} - \frac{64}{3}(135z^2-160z-6)\zeta_2 + z(z+1)\left[256H_0H_{-1}^2 + \left[-192H_0^2 \right. \right. \\
& \left. \left. - 512H_{0,-1} - 256H_{0,1}\right]H_{-1} + 512\zeta_2H_{-1} + 512H_{0,-1,-1} + 256H_{0,-1,1} + 256H_{0,1,-1}\right] \\
& \left. + (z-1)z\left[128H_1^3 + 384H_0H_1^2 + 192H_0^2H_1 - 512\zeta_2H_1\right] + z\left[-\frac{1024}{3}H_0^3 + 1792\zeta_2H_0 - 128\zeta_3\right] \right] L_Q \Big] \\
& + C_{FT_F^2} \left[-\frac{8}{3}(4z^2-4z-1)H_0^3 - 4(20z^2-11z-1)H_0^2 + 8(24z^2+37z-7)H_0 \right. \\
& - 16(z-1)(10z-1)H_1H_0 + 32(2z^2+5z-2)H_{0,1}H_0 + 48(z-1)zH_1^2 - 16(z-1)(20z+3) \\
& + 32(z-1)(6z-1)H_1 + 16(16z^2-24z+3)H_{0,1} - 32(2z^2+17z-3)H_{0,0,1} \\
& + (z-1)(2z+1)\left[\frac{16}{3}H_1^3 - 16H_0^2H_1 + 32H_{0,1,1}\right] - 16(z-2)(6z-1)\zeta_2 + L_Q^2 \left[32(2z+1)H_1(z-1) \right. \\
& \left. + 24(z-1) + 16(2z-1)(2z+1)H_0 + z\left[-16H_0^2 - 64H_{0,1} + 64\zeta_2\right] \right] \\
& \left. + L_M^2 \left[32(2z+1)H_1(z-1) + 24(z-1) + 16(2z-1)(2z+1)H_0 + z\left[-16H_0^2 \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \left. -64H_{0,1} + 64\zeta_2 \right] - 32(2z^2 - 19z + 1)\zeta_3 + z \left[\frac{4}{3}H_0^4 + 32H_{0,1}H_0^2 - 192H_{0,0,1}H_0 + 192\zeta_2H_0 \right. \\
& - 64\zeta_3H_0 - \frac{608\zeta_2^2}{5} + 384H_{0,0,0,1} - 64H_{0,0,1,1} - 64H_{0,1,1,1} \left. \right] + L_M \left[\frac{32}{15}(24z^3 + 90z^2 - 95z - 15)H_0^2 \right. \\
& + \frac{32(78z^3 + 141z^2 - 34z + 8)H_0}{15z} + 128(2z^2 - 3z - 1)H_{0,1}H_0 - \frac{8(z-1)(6z+1)(153z-32)}{15z} \\
& + 16(z-1)(4z-3)H_1 + \frac{(z+1)(12z^4 + 3z^3 - 73z^2 - 2z + 2) \left[\frac{128}{15}H_{0,-1} - \frac{128}{15}H_{-1}H_0 \right]}{z^2} \\
& + 32(6z+1)H_{0,1} - 64(4z^2 - 5z - 2)H_{0,0,1} - \frac{32}{15}(48z^3 + 120z^2 - 250z - 45)\zeta_2 \\
& + (z+1)(2z-1) \left[128H_0H_{-1}^2 + \left[-64H_0^2 - 256H_{0,-1} \right]H_{-1} + 128\zeta_2H_{-1} - 128H_0H_{0,-1} \right. \\
& + 256H_{0,-1,-1} + 384H_{0,0,-1} \left. \right] + (z-1)(2z+1) \left[-64H_1H_0^2 + 128H_1H_0 + 64H_1^2 + 128H_1\zeta_2 \right] \\
& + z \left[\frac{32}{3}H_0^3 + \left[128H_{0,1} - 128H_{0,-1} \right]H_0^2 + \left[512H_{0,-1,-1} + 512H_{0,0,-1} - 512H_{0,0,1} \right]H_0 \right. \\
& - 256H_{0,-1}^2 + \frac{768\zeta_2^2}{5} - 256H_{0,1,1} - 768H_{0,0,0,-1} + 768H_{0,0,0,1} + \left[64H_0 + 256H_{0,-1} - 256H_{0,1} \right] \zeta_2 \\
& + \left. \left[-512H_0 - 576 \right] \zeta_3 \right] + L_Q \left[-\frac{32}{15}(24z^3 + 90z^2 - 95z - 15)H_0^2 \right. \\
& - \frac{32(78z^3 + 141z^2 - 34z + 8)H_0}{15z} - 128(2z^2 - 3z - 1)H_{0,1}H_0 + \frac{8(z-1)(6z+1)(153z-32)}{15z} \\
& - 16(z-1)(4z-3)H_1 + \frac{(z+1)(12z^4 + 3z^3 - 73z^2 - 2z + 2) \left[\frac{128}{15}H_{-1}H_0 - \frac{128}{15}H_{0,-1} \right]}{z^2} \\
& - 32(6z+1)H_{0,1} + 64(4z^2 - 5z - 2)H_{0,0,1} + L_M \left[-64(2z+1)H_1(z-1) - 48(z-1) \right. \\
& - 32(2z-1)(2z+1)H_0 + z \left[32H_0^2 + 128H_{0,1} - 128\zeta_2 \right] \left. \right] + \frac{32}{15}(48z^3 + 120z^2 - 250z - 45)\zeta_2 \\
& + (z+1)(2z-1) \left[-128H_0H_{-1}^2 + \left[64H_0^2 + 256H_{0,-1} \right]H_{-1} - 128\zeta_2H_{-1} + 128H_0H_{0,-1} \right. \\
& - 256H_{0,-1,-1} - 384H_{0,0,-1} \left. \right] + (z-1)(2z+1) \left[64H_1H_0^2 - 128H_1H_0 - 64H_1^2 - 128H_1\zeta_2 \right] \\
& + z \left[-\frac{32}{3}H_0^3 + \left[128H_{0,-1} - 128H_{0,1} \right]H_0^2 + \left[-512H_{0,-1,-1} - 512H_{0,0,-1} + 512H_{0,0,1} \right]H_0 + 256H_{0,-1}^2 \right. \\
& - \frac{768\zeta_2^2}{5} + 256H_{0,1,1} + 768H_{0,0,0,-1} - 768H_{0,0,0,1} + \left[-64H_0 - 256H_{0,-1} + 256H_{0,1} \right] \zeta_2 \\
& + \left. \left[512H_0 + 576 \right] \zeta_3 \right] \left. \right] + C_F T_F^2 \left[-\frac{16}{3}zH_0^4 - \frac{32}{3}(7z-1)H_0^3 \right. \\
& - 32(19z-3)H_0^2 - 64(6z^2 + 7z - 8)H_0 + \left[-64zH_0^2 - \frac{64}{3}(4z^2 + 5z - 3)H_0 \right. \\
& + \frac{64(z-1)(40z^2 - 17z - 2)}{9z} \left. \right] L_M^2 + \frac{16(z-1)(343z^2 - 242z + 4)}{3z} + L_Q^2 \left[-64zH_0^2 \right. \\
& - \frac{64}{3}(z+1)(4z-3)H_0 + \frac{64(z-1)(28z^2 - 23z - 2)}{9z} \left. \right] + L_Q \left[-\frac{128(25z+2)H_1(z-1)^2}{9z} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{32(2474z^2 - 4897z + 44)(z-1)}{135z} - \frac{64}{45}(12z^3 - 180z^2 - 265z + 90)H_0^2 \\
& -\frac{64(354z^3 - 397z^2 + 388z + 4)H_0}{45z} + \frac{(z+1)(6z^4 - 6z^3 + z^2 - z + 1)}{z^2} \left[\frac{256}{45}H_{-1}H_0 \right. \\
& \left. - \frac{256}{45}H_{0,-1} \right] + \frac{128}{3}(z+1)(4z-3)H_{0,1} + \left[128zH_0^2 + \frac{128}{3}(4z^2 + 3z - 3)H_0 \right. \\
& \left. - \frac{256(z-1)(17z^2 - 10z - 1)}{9z} \right] L_M + \frac{128}{45}(12z^3 - 60z^2 - 25z + 45)\zeta_2 + z \left[128H_0^3 \right. \\
& \left. - 256\zeta_2H_0 + 256H_{0,0,1} - 256\zeta_3 \right] + L_M \left[-\frac{64}{45}(12z^3 + 180z^2 + 305z - 90)H_0^2 \right. \\
& \left. + \frac{64(426z^3 - 553z^2 + 362z - 4)H_0}{45z} + \frac{32(z-1)(3716z^2 - 4753z - 4)}{135z} \right. \\
& \left. + \frac{128(z-1)(37z^2 - 20z - 2)H_1}{9z} + \frac{(z+1)(6z^4 - 6z^3 + z^2 - z + 1)}{z^2} \left[\frac{256}{45}H_{-1}H_0 - \frac{256}{45}H_{0,-1} \right] \right. \\
& \left. - \frac{128}{3}(4z^2 + 3z - 3)H_{0,1} + \frac{128}{45}(12z^3 + 60z^2 + 35z - 45)\zeta_2 \right. \\
& \left. + z \left[-128H_0^3 + 256\zeta_2H_0 - 256H_{0,0,1} + 256\zeta_3 \right] \right] + C_F N_F T_F^2 \left[\left[-64zH_0^2 \right. \right. \\
& \left. - \frac{64}{3}(z+1)(4z-3)H_0 + \frac{64(z-1)(28z^2 - 23z - 2)}{9z} \right] L_Q^2 + \left[-\frac{128(25z+2)H_1(z-1)^2}{9z} \right. \\
& \left. - \frac{32(2474z^2 - 4897z + 44)(z-1)}{135z} - \frac{64}{45}(12z^3 - 180z^2 - 265z + 90)H_0^2 \right. \\
& \left. - \frac{64(354z^3 - 397z^2 + 388z + 4)H_0}{45z} + \frac{(z+1)(6z^4 - 6z^3 + z^2 - z + 1)}{z^2} \left[\frac{256}{45}H_{-1}H_0 \right. \right. \\
& \left. \left. - \frac{256}{45}H_{0,-1} \right] + \frac{128}{3}(z+1)(4z-3)H_{0,1} + \left[\frac{64}{3}(z-1)(2z+1) - \frac{128}{3}zH_0 \right] L_M \right. \\
& \left. + \frac{128}{45}(12z^3 - 60z^2 - 25z + 45)\zeta_2 + z \left[128H_0^3 - 256\zeta_2H_0 + 256H_{0,0,1} - 256\zeta_3 \right] \right] L_Q \\
& + L_M \left[-\frac{32}{9}(z-1)(68z+25) - \frac{64}{9}(6z^2 - 31z - 3)H_0 - \frac{64}{3}(z-1)(2z+1)H_1 \right. \\
& \left. + z \left[\frac{128}{3}H_0^2 + \frac{128}{3}H_{0,1} - \frac{128\zeta_2}{3} \right] \right] + C_A C_F T_F \left[-\frac{16}{3}(3z+1)H_0^3 \right. \\
& \left. - \frac{8}{3}(11z^2 - 18z + 3)H_0^2 + \frac{16}{9}(772z^2 + 480z - 39)H_0 + \frac{16(z-1)(77z^2 - 25z - 4)H_1H_0}{3z} \right. \\
& \left. - 32(7z-2)H_{0,1}H_0 - 8(z-1)(9z-1)H_1^2 - \frac{32(z-1)(2168z^2 - 91z - 28)}{27z} \right. \\
& \left. - 16(z-1)(16z-1)H_1 + (z-1)(2z+1) \left[16H_0H_1^2 - \frac{16}{3}H_1^3 \right] \right. \\
& \left. + z(z+1) \left[192H_{0,-1} - 192H_{-1}H_0 \right] \right. \\
& \left. - \frac{16(68z^3 - 117z^2 + 21z + 4)H_{0,1}}{3z} - 32(2z^2 - 2z - 1)H_{0,1,1} + L_M^2 \right. \\
& \left. - \frac{16(z-1)(43z^2 - 11z - 2)}{3z} + 32(3z-1)H_0 - 32(z-1)(2z+1)H_1 \right.
\end{aligned}$$

$$\begin{aligned}
& +z \left[64H_0^2 + 64H_{0,1} - 64\zeta_2 \right] - 16z(3z+17)\zeta_2 + (z+1)(2z-1) \left[32H_0H_{-1}^2 \right. \\
& + \left. \left[-16H_0^2 - 64H_{0,-1} \right] H_{-1} + 32\zeta_2H_{-1} + 32H_0H_{0,-1} + 64H_{0,-1,-1} - 32H_{0,0,-1} \right] \\
& + L_Q^2 \left[\frac{16(z-1)(65z^2-2)}{3z} - \frac{32}{3}(20z-3)H_0 + 32(z-1)(2z+1)H_1 + z \left[-64H_0^2 \right. \right. \\
& - 64H_{0,1} + 64\zeta_2 \left. \left. \right] \right] + (15z-4) \left[32H_{0,0,1} - 32\zeta_3 \right] + z \left[\frac{8}{3}H_0^4 - 32H_{0,-1}H_0^2 + \left[128H_{0,-1,-1} \right. \right. \\
& + 128H_{0,0,-1} - 256H_{0,0,1} - 64H_{0,1,1} \left. \left. \right] H_0 - 448\zeta_3H_0 - 64H_{0,-1}^2 - \frac{1472\zeta_2^2}{5} - 192H_{0,0,0,-1} \right. \\
& + 768H_{0,0,0,1} + 128H_{0,0,1,1} + 64H_{0,1,1,1} + \left. \left[64H_{0,-1} - 32H_0 \right] \zeta_2 \right] + L_Q \left[\frac{128}{45}z(6z^2+35)H_0^3 \right. \\
& + \frac{64(84z^4-9z^3+272z^2-48z+6)H_0^2}{45z} - \frac{32(z+1)(24z^4+6z^3-11z^2-4z+4)H_{-1}H_0^2}{15z^2} \\
& - 32(z-1)(2z+1)H_1H_0^2 - \frac{16(4668z^3-5233z^2-130z-64)H_0}{45z} - 64(z-1)(4z+1)H_1H_0 \\
& - \frac{64(30z^4-35z^3-15z^2-4)H_{0,-1}H_0}{15z^2} + 64(z+1)(2z-1)H_{0,1}H_0 - \frac{256}{5}z(4z^2+5)\zeta_2H_0 \\
& - 32(z-1)(6z+1)H_1^2 - \frac{8(z-1)(11062z^2+1335z-168)}{45z} \\
& - \frac{64}{45}(168z^4-108z^3+343z^2-6z+24)\frac{\zeta_2}{z} + \frac{64}{5}(z+1)(12z^4-2z^3-3z^2-2z+2)\frac{\zeta_2}{z^2}H_{-1} \\
& - \frac{64(z-1)(371z^2+37z-14)H_1}{15z} - \frac{64}{15}(z-1)(12z^4-18z^3-13z^2+2z+2)\frac{\zeta_2}{z^2}H_1 \\
& + \frac{(z+1)(42z^4-69z^3-35z^2-4z+7) \left[\frac{256}{45}H_{0,-1} - \frac{256}{45}H_{-1}H_0 \right]}{z^2} \\
& + \frac{64(24z^3+208z^2-17z+4)H_{0,1}}{15z} + \frac{(z+1)(12z^4+18z^3-13z^2-2z+2)}{z^2} \left[\frac{64}{15}H_0H_{-1}^2 \right. \\
& - \frac{128}{15}H_{0,-1}H_{-1} + \frac{128}{15}H_{0,-1,-1} \left. \right] + \frac{64(24z^5+90z^4-75z^3-45z^2-4)H_{0,0,-1}}{15z^2} \\
& + \frac{64}{15}(24z^3-30z^2+55z+15)H_{0,0,1} + \frac{(z+1)(6z^4-6z^3+z^2-z+1)}{z^2} \left[-\frac{256}{15}H_{-1}H_{0,1} \right. \\
& + \frac{256}{15}H_{0,-1,1} + \frac{256}{15}H_{0,1,-1} \left. \right] + \left[\frac{352}{3}zH_0 - \frac{176}{3}(z-1)(2z+1) \right] L_M - \frac{256}{3}z(3z^2+2)\zeta_3 \\
& + z \left[\left[64H_{0,1} - 64H_{0,-1} \right] H_0^2 + \left[256H_{0,-1,-1} + 256H_{0,0,-1} - 256H_{0,0,1} \right] H_0 - 256\zeta_3H_0 \right. \\
& - 128H_{0,-1}^2 + \frac{384\zeta_2^2}{5} + 128H_{0,1,1} - 384H_{0,0,0,-1} + 384H_{0,0,0,1} + \left. \left[128H_{0,-1} - 128H_{0,1} \right] \zeta_2 \right] \\
& + L_M \left[-\frac{32}{5}(4z^3+5z^2-10z-5)H_0^2 + \frac{16(2226z^3-43z^2-63z-24)H_0}{45z} \right. \\
& - \frac{8(z-1)(3758z^2-1299z-152)}{45z} - \frac{16}{3}(z-1)(2z-23)H_1 \\
& + \frac{(z+1)(12z^4-27z^3-58z^2-2z+2)}{z^2} \left[\frac{64}{15}H_{-1}H_0 - \frac{64}{15}H_{0,-1} \right] - 64(6z^2-z-3)H_{0,0,-1} \\
& + \frac{32}{15}z(24z^2-85)\zeta_2 + (z+1)(2z-1) \left[-64H_0H_{-1}^2 + \left[32H_0^2 + 128H_{0,-1} \right] H_{-1} - 64\zeta_2H_{-1} \right.
\end{aligned}$$

$$\begin{aligned}
& -128H_{0,-1,-1}] + (z-1)(2z+1) \left[32H_1H_0^2 + [64H_{0,-1} - 64H_{0,1}]H_0 - 32H_1^2 + 64H_{0,0,1} \right. \\
& - 64H_1\zeta_2] + z \left[-\frac{128}{3}H_0^3 + [64H_{0,-1} - 64H_{0,1}]H_0^2 + [-256H_{0,-1,-1} - 256H_{0,0,-1} \right. \\
& + 256H_{0,0,1}]H_0 + 256\zeta_3H_0 + 128H_{0,-1}^2 - \frac{384\zeta_2^2}{5} - \frac{544}{3}H_{0,1} + 128H_{0,1,1} + 384H_{0,0,0,-1} \\
& \left. - 384H_{0,0,0,1} + [128H_{0,1} - 128H_{0,-1}]\zeta_2 \right] \left. \right] + \tilde{C}_{L,g}^{S,(3)}(N_F + 1) \left. \right\}. \tag{603}
\end{aligned}$$

D The OMEs in z -Space

In the following, we present the massive operator matrix elements in z -space. They are given by :

$$\begin{aligned}
A_{qq,Q}^{\text{PS}} = & a_s^3 \left\{ C_F N_F T_F^2 \left[L_M^2 \left[(z+1) \left[\frac{64}{3}H_{0,1} - \frac{32}{3}H_0^2 - \frac{64\zeta_2}{3} \right] + \frac{32}{9}(4z^2 - 7z - 13)H_0 \right. \right. \right. \\
& - \frac{32(z-1)(4z^2 + 7z + 4)H_1}{9z} - \frac{32}{3}(z-1)(2z-5) \left. \right] + L_M \left[(4z^2 - 7z - 13) \left[\frac{64\zeta_2}{9} - \frac{64}{9}H_{0,1} \right] \right. \\
& + (z+1) \left[\frac{128}{3}H_{0,0,1} - \frac{128}{3}H_{0,1,1} - \frac{128}{3}H_0\zeta_2 - \frac{32}{9}H_0^3 - \frac{464}{9}H_0^2 \right] + \frac{32(z-1)(4z^2 + 7z + 4)H_1^2}{9z} \\
& + \frac{128}{27}(19z^2 - 16z - 40)H_0 + \frac{64}{3}(z-1)(2z-5)H_1 - \frac{32(z-1)(80z^2 - 511z - 136)}{81z} \left. \right] \\
& + (57z^2 - 131z - 203) \left[\frac{64\zeta_2}{81} - \frac{64}{81}H_{0,1} \right] + (z+1) \left[\frac{1856}{27}H_{0,0,1} + \frac{128}{9}H_{0,0,0,1} - \frac{256}{9}H_{0,0,1,1} \right. \\
& + \frac{320}{9}H_{0,1,1,1} + H_0^2 \left[-\frac{64}{9}\zeta_2 - \frac{5312}{81} \right] + H_0 \left[\frac{640\zeta_3}{9} - \frac{1856\zeta_2}{27} \right] - \frac{8}{27}H_0^4 - \frac{464}{81}H_0^3 \\
& - \frac{256}{15}\zeta_2^2 + \frac{64}{27}(6z^2 - 25z - 34)H_{0,1,1} + L_M^3 \left[\frac{32(z-1)(4z^2 + 7z + 4)}{27z} - \frac{64}{9}(z+1)H_0 \right] \\
& - \frac{80(z-1)(4z^2 + 7z + 4)H_1^3}{81z} - \frac{16(z-1)(34z^2 - 227z - 20)H_1^2}{81z} \\
& - \frac{64(z-1)(10z^2 + 213z + 64)H_1}{81z} + \frac{32}{243}(660z^2 - 1577z - 2351)H_0 \\
& \left. - \frac{64(22z^3 + 16z^2 - 17z - 16)\zeta_3}{27z} - \frac{64(z-1)(139z^2 - 3701z - 752)}{729z} \right] \left. \right\}, \tag{604}
\end{aligned}$$

$$\begin{aligned}
A_{qg,Q} = & a_s^3 \left\{ C_A T_F^2 N_F \left[-\frac{8}{27}(4z^2 + 16z - 5)H_0^4 + \frac{32}{81}z(14z + 29)H_0^3 \right. \right. \\
& - \frac{16}{81}(44z^2 + 243z - 56)H_0^3 + \frac{16}{81}z(200z + 347)H_0^2 - \frac{16}{81}(402z^2 + 1472z - 205)H_0^2 \\
& - \frac{64}{27}(7z^2 + 7z + 5)H_{-1}H_0^2 + \frac{8}{243}(5772z^2 - 27934z - 451)H_0 + z \left[\frac{96}{9}\zeta_2 \right. \\
& \left. \left. + \frac{11392}{81} \right] H_0 + \frac{256}{9}(4z+1)\zeta_3H_0 - \frac{16}{9}(4z^2 - 7z - 1)H_1H_0 \right. \\
& \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{27}(14z^2 + 11z + 10)H_{0,-1}H_0 - \frac{32}{27}(14z^2 - 17z + 10)H_{0,1}H_0 \\
& + L_M^3 \left[\frac{16(z-1)(31z^2 + 7z + 4)}{27z} - \frac{32}{9}(4z+1)H_0 - \frac{8}{9}\gamma_{qg}^0 H_1 \right] - \frac{16}{81}(218z^2 - 200z + 85)H_1^2 \\
& - \frac{11392}{81}(z-1)z + \frac{4(173275z^3 - 157668z^2 + 21651z - 17368)}{729z} + \frac{32}{81}(109z^2 + 47z + 47)\zeta_2 \\
& - \frac{16}{81}(254z^2 + 137z + 112)\zeta_2 - \frac{32(145z^3 - 123z^2 + 3z - 16)\zeta_3}{27z} \\
& + z(z+1) \left[\frac{32}{27}H_0^4 + \frac{0}{3}\zeta_2 H_0^2 \right] + \frac{32(1184z^3 - 1067z^2 + 487z + 18)H_1}{243z} \\
& + (200z^2 + 191z + 112) \left[\frac{32}{81}H_{0,-1} - \frac{32}{81}H_{-1}H_0 \right] \\
& + L_M^2 \left[-\frac{64}{3}zH_0^2 + \frac{32}{9}(9z^2 - 20z - 5)H_0 + \frac{8(205z^3 - 168z^2 + 42z - 52)}{27z} \right. \\
& + \frac{32}{9}(4z^2 - 4z + 5)H_1 + \gamma_{qg}^0 \left(\frac{4}{3}H_1^2 - \frac{4\zeta_2}{3} \right) + (2z^2 + 2z + 1) \left[-\frac{16}{3}\zeta_2 - \frac{32}{3}H_{-1}H_0 \right. \\
& \left. \left. + \frac{32}{3}H_{0,-1} \right] \right] + (7z^2 - 7z + 5) \left[\frac{32}{81}H_1^3 + \frac{32}{27}H_0H_1^2 + \frac{32}{27}H_0^2H_1 + \left[\frac{64\zeta_2}{27} - \frac{128}{27}H_{0,1} \right] H_1 \right] \\
& + \frac{16}{27}(24z^2 - 134z + 3)H_{0,1} + (7z^2 + 4z + 5) \left[\frac{32}{9}\zeta_3 - \frac{128}{27}H_{0,0,-1} \right] \\
& + \frac{32}{27}(14z^2 - 35z + 10)H_{0,0,1} + L_M \left[-\frac{16}{9}(10z - 3)H_0^3 - \frac{8}{9}(16z^2 + 128z - 3)H_0^2 \right. \\
& + \frac{16}{27}(238z^2 - 646z + 5)H_0 - \frac{32}{9}(7z^2 - 7z + 5)H_1^2 + \frac{8(3791z^3 - 3318z^2 + 465z - 344)}{81z} \\
& + \frac{32}{9}(7z^2 - z + 5)\zeta_2 + \frac{16}{27}(158z^2 - 149z + 85)H_1 + (7z^2 + 7z + 5) \left[-\frac{32}{9}\zeta_2 \right. \\
& \left. - \frac{64}{9}H_{-1}H_0 + \frac{64}{9}H_{0,-1} \right] - \frac{64}{3}zH_{0,1} + (2z^2 + 2z + 1) \left[-\frac{32}{3}H_{-1}H_0^2 \right. \\
& \left. + \frac{64}{3}H_{0,-1}H_0 + 16\zeta_3 - \frac{64}{3}H_{0,0,-1} \right] + \gamma_{qg}^0 \left[-\frac{4}{9}H_1^3 - \frac{4}{3}H_0^2H_1 + \left[\frac{16}{3}H_{0,1} \right. \right. \\
& \left. \left. - \frac{8\zeta_2}{3} \right] H_1 + 4\zeta_3 + H_0 \left[\frac{8}{3}H_{0,1} - \frac{4}{3}H_1^2 \right] - \frac{8}{3}H_{0,0,1} - 8H_{0,1,1} \right] + \frac{32}{9}(14z^2 - 9z + 10)H_{0,1,1} \\
& + (2z^2 + 2z + 1) \left[-\frac{64}{27}H_{-1}H_0^3 + \frac{64}{9}H_{0,-1}H_0^2 - \frac{128}{9}H_{0,0,-1}H_0 - \frac{224}{45}\zeta_2 \right. \\
& + \frac{128}{9}H_{0,0,0,-1} \left. \right] + \gamma_{qg}^0 \left[\frac{1}{27}H_1^4 + \left[\frac{4\zeta_2}{9} - \frac{8}{9}H_{0,1} \right] H_1^2 - \frac{4}{9}H_0^3H_1 + \left[\frac{16\zeta_3}{3} + \frac{16}{9}H_{0,0,1} \right. \right. \\
& \left. \left. + \frac{16}{9}H_{0,1,1} \right] H_1 - \frac{56}{45}\zeta_2^2 - \frac{8}{9}H_0^2 + \frac{8}{9}H_{0,1}\zeta_2 + H_0^2 \left[\frac{4}{3}H_{0,1} - \frac{2}{9}H_1^2 \right] + H_0 \left[\frac{4}{27}H_1^3 - \frac{8}{9}\zeta_2H_1 \right. \right. \\
& \left. \left. - \frac{8}{3}H_{0,0,1} + \frac{8}{9}H_{0,1,1} \right] + \frac{8}{3}H_{0,0,0,1} - \frac{8}{9}H_{0,0,1,1} - \frac{8}{9}H_{0,1,1,1} \right] \left. \right] \\
& + C_{FT_F^2 N_F} \left[-\frac{4}{27}(28z^2 + 116z - 67)H_0^4 + \frac{4}{81}(288z^2 - 2582z + 2341)H_0^3 \right. \\
& - \frac{4}{81}(2152z^2 + 5141z - 13876)H_0^2 + \frac{4}{243}(14880z^2 + 73472z \\
& \left. + 154967)H_0 + \frac{256}{9}(6z^2 - z - 4)\zeta_3H_0 - \frac{448}{81}(z^2 - z + 2)H_1H_0 + L_M^3 \left[-\frac{16}{3}(2z - 1)H_0^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{32}{9}(6z^2 - z - 4)H_0 + \frac{8(124z^3 - 258z^2 + 159z - 16)}{27z} + \frac{8}{9}\gamma_{qg}^0 H_1 \Big] \\
& + \frac{8}{81}(364z^2 - 373z + 224)H_1^2 - \frac{179524z^3 + 2535258z^2 - 2713863z - 42688}{729z} \\
& - \frac{448}{81}(13z^2 - 13z + 8)\zeta_2 - \frac{64(117z^3 - 251z^2 + 154z - 16)\zeta_3}{27z} + (2z - 1) \left[\frac{128}{3}H_0^2\zeta_3 \right. \\
& - \frac{16}{15}H_0^5 \Big] - \frac{16}{243}(2188z^2 - 2278z + 1613)H_1 + L_M^2 \left[-\frac{32}{3}(2z - 1)H_0^3 \right. \\
& - \frac{16}{3}(2z + 3)(4z - 3)H_0^2 + \frac{8}{9}(160z^2 + 146z + 305)H_0 \\
& - \frac{4(1000z^3 + 1356z^2 - 2247z - 208)}{27z} - \frac{32}{9}(4z^2 - 4z + 5)H_1 \\
& \left. + \gamma_{qg}^0 \left[-\frac{4}{3}H_1^2 + \frac{8\zeta_2}{3} - \frac{8}{3}H_{0,1} \right] \right] + (16z^2 - 16z + 5) \left[\frac{16}{27}H_1H_0^2 - \frac{32}{27}H_{0,1}H_0 + \frac{32}{27}H_{0,0,1} \right] \\
& + (7z^2 - 7z + 5) \left[-\frac{32}{81}H_1^3 + \frac{896}{81}H_{0,1} - \frac{64}{27}H_{0,1,1} \right] + L_M \left[-\frac{20}{3}(2z - 1)H_0^4 \right. \\
& - \frac{16}{9}(14z^2 + 37z - 26)H_0^3 + \frac{4}{9}(136z^2 - 174z + 909)H_0^2 - \frac{8}{27}(32z^2 - 2393z - 4145)H_0 \\
& + \frac{64}{3}(z - 1)zH_1H_0 - \frac{4(3556z^3 + 33342z^2 - 38175z + 800)}{81z} - \frac{16}{27}(140z^2 - 149z + 112)H_1 \\
& + (7z^2 - 7z + 5) \left[\frac{32}{9}H_1^2 - \frac{64\zeta_2}{9} \right] + \frac{64}{9}(4z^2 - 4z + 5)H_{0,1} + \gamma_{qg}^0 \left[\frac{4}{9}H_1^3 - \frac{4}{3}H_0^2H_1 - \frac{8\zeta_3}{3} \right. \\
& + \frac{8}{3}H_0H_{0,1} - \frac{8}{3}H_{0,0,1} + \frac{8}{3}H_{0,1,1} \Big] \left. + \gamma_{qg}^0 \left[-\frac{1}{27}H_1^4 - \frac{8}{27}H_0^3H_1 - \frac{64}{9}\zeta_3H_1 + \frac{16}{45}\zeta_2^2 \right. \right. \\
& \left. \left. + \frac{8}{9}H_0^2H_{0,1} - \frac{16}{9}H_0H_{0,0,1} + \frac{16}{9}H_{0,0,0,1} - \frac{8}{9}H_{0,1,1,1} \right] \right\} , \tag{605}
\end{aligned}$$

$$\begin{aligned}
A_{Qq}^{\text{PS}} = & a_s^2 \left\{ C_F T_F \left[\frac{2}{3}(8z^2 + 15z + 3)H_0^2 - \frac{8}{9}(56z^2 + 33z + 21)H_0 \right. \right. \\
& + L_M^2 \left[\frac{4(z - 1)(4z^2 + 7z + 4)}{3z} - 8(z + 1)H_0 \right] + \frac{4(z - 1)(400z^2 + 121z + 112)}{27z} \\
& + \frac{(z - 1)(4z^2 + 7z + 4) \left[\frac{8}{3}H_{0,1} - \frac{8}{3}H_0H_1 \right]}{z} + (z + 1) \left[-\frac{4}{3}H_0^3 + 16H_{0,1}H_0 + 32\zeta_3 - 32H_{0,0,1} \right] \\
& \left. \left. + L_M \left[8(z + 1)H_0^2 - \frac{8}{3}(8z^2 + 15z + 3)H_0 + \frac{16(z - 1)(28z^2 + z + 10)}{9z} \right] \right] \right\} \\
& + a_s^3 \left\{ a_{Qq}^{\text{PS},(3)} + C_F^2 T_F \left[-\frac{2}{9}(4z^2 - 3z + 3)H_0^4 - \frac{2}{9}(40z^2 + 149z + 115)H_0^3 \right. \right. \\
& + \frac{2}{3}(160z^2 + 191z - 117)H_0^2 - \frac{8}{3}(4z^2 - 3z - 3)\zeta_2H_0^2 - \frac{4(z - 1)(20z^2 + 41z - 4)H_1H_0^2}{3z} \\
& \left. \left. + \frac{8(4z^3 + 27z^2 + 3z - 4)H_{0,1}H_0^2}{3z} - \frac{4}{3}(400z^2 - 135z + 222)H_0 - \frac{2}{3}(80z^2 + 469z + 221)\zeta_2H_0 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{16}{9}(44z^2 + 51z - 18)\zeta_3 H_0 - \frac{8(z-1)(16z^2 - 43z + 66)H_1 H_0}{3z} \\
& + \frac{16(z-1)(10z^2 + 11z - 2)H_{0,1}H_0}{3z} + \frac{16(4z^3 - 33z^2 - 15z + 4)H_{0,0,1}H_0}{3z} \\
& - \frac{8(z-1)(6z^2 - z - 6)H_1^3}{9z} + \frac{8}{15}(188z^2 - 27z - 105)\zeta_2^2 - \frac{4(z-1)(24z^2 - 13z + 17)H_1^2}{3z} \\
& + \frac{64(z+1)^2(2z-1)H_{0,1}^2}{3z} + \frac{4(z-1)(2z+1)(352z+233)}{9z} + \frac{4(84z^3 + 79z^2 + 75z - 60)\zeta_2}{3z} \\
& - \frac{4(z-1)(40z^2 + 33z + 4)H_1\zeta_2}{3z} - \frac{8(12z^3 + 15z^2 + 9z + 8)H_{0,1}\zeta_2}{3z} \\
& + \frac{4}{9}(72z^2 - 809z - 145)\zeta_3 - \frac{4(z-1)(80z^2 - 181z - 9)H_1}{3z} + L_M^3 \left[\frac{16(4z^2 + 7z + 4)H_1(z-1)}{9z} \right. \\
& \left. + \frac{92(z-1)}{9} + \frac{16}{9}z(4z+3)H_0 + (z+1) \left[-\frac{8}{3}H_0^2 + \frac{32\zeta_2}{3} - \frac{32}{3}H_{0,1} \right] \right] \\
& - \frac{8(8z^3 + 100z^2 - 85z + 66)H_{0,1}}{3z} - \frac{32}{3}(z-1)H_1H_{0,1} - \frac{8(20z^3 - 95z^2 - 43z + 4)H_{0,0,1}}{3z} \\
& + L_M^2 \left[\frac{8}{3}(4z^2 - 9z - 3)H_0^2 + \frac{8}{3}(z+1)(32z - 31)H_0 + \frac{16(z-1)(4z^2 + 7z + 4)H_1H_0}{3z} \right. \\
& \left. + \frac{4(z-1)(32z^2 + 81z + 12)}{3z} + 32(3z+2)\zeta_2 + \frac{8(z-1)(32z^2 + 35z + 8)H_1}{3z} \right. \\
& \left. - \frac{16(4z^3 + 21z^2 + 9z - 4)H_{0,1}}{3z} + (z+1) \left[-\frac{16}{3}H_0^3 + [32\zeta_2 - 32H_{0,1}]H_0 - 32\zeta_3 + 32H_{0,0,1} \right] \right] \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z} \left[-\frac{2}{9}H_1^4 + \left[\frac{16}{3}H_{0,1} - \frac{20\zeta_2}{3} \right]H_1^2 - \frac{8}{9}H_0^3H_1 \right. \\
& \left. + \left[\frac{176\zeta_3}{9} - \frac{64}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1} \right]H_1 + H_0 \left[H_1 \left[\frac{32}{3}H_{0,1} - \frac{16\zeta_2}{3} \right] - \frac{64}{3}H_{0,1,1} \right] \right] \\
& - \frac{8}{3}(12z^2 - 23z - 22)H_{0,1,1} - \frac{16(20z^3 - 21z^2 - 33z + 4)H_{0,0,0,1}}{3z} - \frac{32(3z^2 + 15z + 8)H_{0,0,1,1}}{3z} \\
& + \frac{16(20z^3 + 15z^2 - 27z - 24)H_{0,1,1,1}}{3z} + L_M \left[-\frac{32}{3}(4z^2 + z + 1)H_0^3 \right. \\
& \left. + \frac{2}{3}(136z^2 - 111z + 213)H_0^2 + \frac{8}{27}(242z^2 - 3984z - 633)H_0 + \frac{32}{3}(2z+3)(8z-3)\zeta_2H_0 \right. \\
& \left. + \frac{8(z-1)(140z^2 - 127z + 104)H_1H_0}{9z} - 32(4z^2 + 3z + 1)H_{0,1}H_0 \right. \\
& \left. + \frac{4(z-1)(28z^2 + 21z + 4)H_1^2}{3z} + \frac{4(z-1)(3204z^2 + 1625z + 180)}{27z} - \frac{16}{9}(74z^2 + 18z + 297)\zeta_2 \right. \\
& \left. - \frac{16(12z^3 - z^2 + z - 8)\zeta_3}{z} + \frac{16(z-1)(229z^2 - 1175z - 239)H_1}{27z} \right. \\
& \left. + \frac{8(8z^3 + 303z^2 + 363z + 104)H_{0,1}}{9z} + \frac{(z-1)(4z^2 + 7z + 4) \left[-\frac{8}{9}H_1^3 - \frac{16}{3}H_0H_1^2 + \frac{32}{3}H_{0,1}H_1 \right]}{z} \right. \\
& \left. + \frac{32}{3}(8z^2 + 15)H_{0,0,1} - \frac{16(12z^3 + 27z^2 + 3z - 8)H_{0,1,1}}{3z} + (z+1) \left[6H_0^4 - 96\zeta_2H_0^2 \right. \right. \\
& \left. \left. + [192\zeta_3 + 96H_{0,0,1} + 64H_{0,1,1}]H_0 + \frac{288}{5}\zeta_2^2 - 32H_{0,1}^2 - 96H_{0,0,0,1} + 32H_{0,1,1,1} \right] \right] \\
& + (z+1) \left[\frac{2}{15}H_0^5 + \left[4\zeta_2 + \frac{16}{3}H_{0,1} \right]H_0^3 + \left[-\frac{88}{3}\zeta_3 - 48H_{0,0,1} \right]H_0^2 + \left[-\frac{448}{5}\zeta_2^2 + 32H_{0,1}\zeta_2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -32H_{0,1}^2 + 160H_{0,0,0,1} + 128H_{0,0,1,1}]H_0 + 32H_{0,0,1}\zeta_2 + 80H_{0,1,1}\zeta_2 - \frac{80}{3}\zeta_2\zeta_3 + 160\zeta_5 \\
& + H_{0,1}\left[-\frac{352}{3}\zeta_3 + 128H_{0,0,1} - 64H_{0,1,1}\right] - 192H_{0,0,0,0,1} - 768H_{0,0,0,1,1} - 320H_{0,0,1,0,1} \\
& + 416H_{0,0,1,1,1} + 192H_{0,1,0,1,1} + 32H_{0,1,1,1,1}] + C_F T_F^2 \left[\frac{16}{27}(8z^2 + 15z + 3)H_0^3 \right. \\
& - \frac{32}{27}(56z^2 + 33z + 21)H_0^2 + \frac{32}{81}(1020z^2 + 697z + 607)H_0 + \frac{32}{9}(12z^2 + 37z + 19)\zeta_2 H_0 \\
& + \frac{128(z-1)(28z^2 + z + 10)H_1 H_0}{27z} - \frac{128(2z^3 + 6z^2 + 3z + 2)H_{0,1} H_0}{9z} \\
& + L_M^3 \left[\frac{128(z-1)(4z^2 + 7z + 4)}{27z} - \frac{256}{9}(z+1)H_0 \right] \\
& - \frac{16(z-1)(38z^2 + 47z + 20)H_1^2}{27z} - \frac{64(z-1)(1781z^2 + 539z + 656)}{243z} \\
& - \frac{32(56z^3 - 179z^2 - 95z - 40)\zeta_2}{27z} - \frac{64(62z^3 + 129z^2 + 36z - 8)\zeta_3}{27z} \\
& + \frac{128(z-1)(55z^2 + 64z + 28)H_1}{81z} + \frac{(z-1)(4z^2 + 7z + 4)\left[\frac{16}{27}H_1^3 - \frac{32}{9}H_0^2 H_1 + \frac{32}{9}\zeta_2 H_1\right]}{z} \\
& - \frac{64(75z^3 + 13z^2 + 61z - 20)H_{0,1}}{27z} + L_M^2 \left[\frac{32(z-1)(94z^2 + 49z + 40)}{27z} \right. \\
& \left. - \frac{32}{9}(12z^2 + 37z + 19)H_0 - \frac{32(z-1)(4z^2 + 7z + 4)H_1}{9z} + (z+1)\left[\frac{32}{3}H_0^2 - \frac{64\zeta_2}{3} + \frac{64}{3}H_{0,1}\right] \right] \\
& + \frac{64(12z^3 + 27z^2 + 9z + 4)H_{0,0,1}}{9z} + L_M \left[\frac{256(z-1)(40z^2 + 79z + 31)}{81z} \right. \\
& - \frac{64}{27}(18z^2 + 65z + 101)H_0 + (4z^2 - 7z - 13)\left[\frac{32}{9}H_0^2 + \frac{64\zeta_2}{9}\right] + \frac{64}{3}(z-1)(2z-5)H_1 \\
& + \frac{(z-1)(4z^2 + 7z + 4)\left[\frac{32}{9}H_1^2 - \frac{64}{9}H_0 H_1\right]}{z} + \frac{128(5z^2 + 5z - 2)H_{0,1}}{9z} \\
& \left. + (z+1)\left[-\frac{64}{9}H_0^3 + \left[\frac{128}{3}H_{0,1} - \frac{128\zeta_2}{3}\right]H_0 + \frac{256\zeta_3}{3} - \frac{128}{3}H_{0,0,1} - \frac{128}{3}H_{0,1,1}\right] \right] \\
& + \frac{64}{9}(2z^2 + 11z + 8)H_{0,1,1} + (z+1)\left[-\frac{8}{9}H_0^4 + \left[\frac{64}{3}H_{0,1} - \frac{32\zeta_2}{3}\right]H_0^2 \right. \\
& \left. + \left[\frac{1024\zeta_3}{9} - \frac{128}{3}H_{0,0,1}\right]H_0 + \frac{448}{15}\zeta_2^2 - \frac{64}{3}H_{0,1}\zeta_2 - \frac{64}{3}H_{0,1,1,1}\right] \\
& + C_F T_F^2 N_F \left[L_M^2 \left[(z+1)\left[-\frac{64}{3}H_{0,1} + \frac{32}{3}H_0^2 + \frac{64\zeta_2}{3}\right] - \frac{32}{9}(4z^2 - 7z - 13)H_0 \right. \right. \\
& \left. + \frac{32(z-1)(4z^2 + 7z + 4)H_1}{9z} + \frac{32}{3}(z-1)(2z-5) \right] + L_M \left[(z+1)\left[\frac{128}{3}H_0 H_{0,1} \right. \right. \\
& \left. - \frac{128}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1} - \frac{128}{3}\zeta_2 H_0 - \frac{64}{9}H_0^3 + 64\zeta_3 \right] - \frac{64(2z^3 + z^2 - 2z + 4)H_{0,1}}{9z} \\
& + \frac{32}{9}(4z^2 - 7z - 13)H_0^2 + \frac{64}{27}(z^2 + 2z - 58)H_0 \\
& \left. + \frac{32(z-1)(74z^2 - 43z + 20)H_1}{27z} + \frac{(z-1)(4z^2 + 7z + 4)\left[\frac{16}{9}H_1^2 - \frac{64}{9}H_0 H_1\right]}{z} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{64}{9} \zeta_2 (6z^2 + 4z - 5) + \frac{128(z-1)(25z^2 + 94z + 34)}{81z} \Big] + (z+1) \left[\zeta_2 \left[\frac{32}{3} H_{0,1} - \frac{32}{3} H_0^2 \right] \right. \\
& + \frac{64}{3} H_0^2 H_{0,1} - \frac{128}{3} H_0 H_{0,0,1} + \frac{832}{9} \zeta_3 H_0 - \frac{8}{9} H_0^4 - \frac{32\zeta_2^2}{3} \Big] + \frac{(z-1)(28z^2 + z + 10)}{z} \left[\right. \\
& \frac{128}{27} H_0 H_1 - \frac{128}{27} H_{0,1} \Big] - \frac{128(2z^3 + 6z^2 + 3z + 2) H_0 H_{0,1}}{9z} + \frac{64(12z^3 + 27z^2 + 9z + 4) H_{0,0,1}}{9z} \\
& - \frac{16}{27} (z-1)(74z^2 - 43z + 20) \frac{\zeta_2}{z} - \frac{32}{27} (100z^3 + 183z^2 + 33z - 4) \frac{\zeta_3}{z} \\
& + L_M^3 \left[\frac{32(z-1)(4z^2 + 7z + 4)}{27z} - \frac{64}{9} (z+1) H_0 \right] + \frac{32}{9} \zeta_2 (6z^2 + 4z - 5) H_0 \\
& + \frac{(z-1)(4z^2 + 7z + 4) \left[-\frac{16}{9} \zeta_2 H_1 - \frac{32}{9} H_1 H_0^2 \right]}{z} + \frac{16}{27} (8z^2 + 15z + 3) H_0^3 \\
& - \frac{32}{27} (56z^2 + 33z + 21) H_0^2 + \frac{32}{81} (800z^2 - 57z + 111) H_0 \\
& - \frac{64(z-1)(1156z^2 - 203z + 328)}{243z} \Big] + C_A C_F T_F \left[-\frac{2}{9} (4z-17) H_0^4 - \frac{4}{9} (36z^2 + 47z + 36) H_0^3 \right. \\
& - \frac{8}{3} (z+3) \zeta_2 H_0^3 + \frac{64}{3} z^2 H_0^2 + \frac{4}{27} (1988z^2 - 681z + 855) H_0^2 + 8(z-1)(2z+1) \zeta_2 H_0^2 \\
& - \frac{8}{3} (2z+5)(3z-4) \zeta_2 H_0^2 - \frac{16}{3} (20z-13) \zeta_3 H_0^2 + \frac{8(z-1)(122z^2 - 19z + 113) H_1 H_0^2}{9z} \\
& - \frac{8(19z^2 + 19z + 8) H_{0,1} H_0^2}{3z} + \frac{16}{5} (9z-4) \zeta_2^2 H_0 + \frac{16}{5} (29z-1) \zeta_2^2 H_0 \\
& - \frac{16(z-1)(19z^2 + 16z + 10) H_1^2 H_0}{9z} - \frac{64}{9} (37z^2 + 6) H_0 \\
& - \frac{4(48876z^3 + 9339z^2 + 16218z + 2624) H_0}{81z} - \frac{8}{9} (152z^2 - 39z + 60) \zeta_2 H_0 \\
& - \frac{4(170z^3 + 199z^2 + 175z + 80) \zeta_2 H_0}{9z} - \frac{16(24z^3 - 31z^2 + 215z + 4) \zeta_3 H_0}{9z} \\
& - \frac{32(z-1)(733z^2 - 62z + 301) H_1 H_0}{27z} - \frac{32(19z^3 - 24z^2 - 6z + 10) H_{0,-1} H_0}{9z} \\
& + \frac{16(18z^3 + 119z^2 - 2z + 51) H_{0,1} H_0}{3z} - \frac{32(4z^3 - 23z^2 - 2z - 8) H_{0,0,1} H_0}{3z} \\
& + \frac{8(z-1)(2z+1)(14z+1) H_1^3}{27z} - \frac{8(96z^3 - 427z^2 + 134z - 148) \zeta_2^2}{15z} \\
& + \frac{8(116z^3 - 87z^2 - 3z + 4) \zeta_2^2}{15z} + \frac{4(z-1)(616z^2 + 313z + 355) H_1^2}{27z} \\
& + \frac{4(z-1)(75516z^2 - 7654z + 23205)}{81z} - \frac{8}{9} (9z^2 + 185z + 38) \zeta_2 \\
& + \frac{8(1868z^3 - 1164z^2 + 1344z - 515) \zeta_2}{27z} + \frac{4(z-1)(154z^2 + 163z + 46) H_1 \zeta_2}{9z} \\
& - \frac{8}{3} (23z + 14) H_{0,1} \zeta_2 - \frac{8(247z^3 - 9z^2 + 18z + 50) \zeta_3}{9z} \\
& + \frac{8(1015z^3 + 1149z^2 + 705z + 126) \zeta_3}{9z} - \frac{256}{3} (z-2) \zeta_2 \zeta_3 + 8(25z-9) \zeta_2 \zeta_3 \\
& + 8(3z+5) \zeta_5 + 8(67z-35) \zeta_5 - \frac{64(z-1)(37z^2 + 16) H_1}{9z}
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(z-1)(7828z^2 + 2755z + 4075)H_1}{81z} + \frac{(z-1)(z^2+1)\left[-\frac{128}{3}H_1^2 - \frac{128}{3}H_0H_1\right]}{z} \\
& + \frac{(z+1)(182z^2 - 122z + 47)\left[\frac{32}{27}H_{-1}H_0 - \frac{32}{27}H_{0,-1}\right]}{z} + \frac{64(6z^3 + 19z^2 + 10z - 6)H_{0,1}}{9z} \\
& + \frac{8(2820z^3 - 3849z^2 + 1128z - 1204)H_{0,1}}{27z} + L_M^3 \left[\frac{16}{3}(2z-1)H_0^2 + \frac{16(8z^2 + 11z + 4)H_0}{9z} \right. \\
& \left. - \frac{8(z-1)(44z^2 - z + 44)}{9z} - \frac{16(z-1)(4z^2 + 7z + 4)H_1}{9z} + (z+1)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right] \right] \\
& + \frac{32(19z^3 - 51z^2 - 6z + 10)H_{0,0,-1}}{9z} + \frac{64}{9}(19z^2 - 15z - 6)H_{0,0,1} \\
& - \frac{16(306z^3 + 561z^2 + 144z + 193)H_{0,0,1}}{9z} + L_M^2 \left[-\frac{8}{3}(4z^2 - 25z + 23)H_0^2 \right. \\
& \left. + \frac{8}{3}(4z^2 - 9z + 6)H_0^2 + \frac{16}{9}(13z^2 - 30z + 24)H_0 + \frac{8(246z^3 + 163z^2 + 91z + 40)H_0}{9z} \right. \\
& \left. + \frac{16(z-1)(4z^2 + 7z + 4)H_1H_0}{3z} - \frac{8(z-1)(35z^2 - 82z + 89)}{27z} \right. \\
& \left. - \frac{8(z-1)(1829z^2 - 403z + 605)}{27z} - \frac{8(z-1)(104z^2 + 119z + 32)H_1}{9z} \right. \\
& \left. + \frac{(z+1)(4z^2 - 7z + 4)\left[-\frac{16}{3}\zeta_2 - \frac{32}{3}H_{-1}H_0 + \frac{32}{3}H_{0,-1}\right]}{z} + (10z+7)\left[\frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3}\right] \right. \\
& \left. + (z-1)\left[-\frac{32}{3}H_0^3 + \left[-32\zeta_2 - 64H_{0,-1}\right]H_0 - 96\zeta_3 + 128H_{0,0,-1}\right] \right. \\
& \left. + (z+1)\left[-32\zeta_3 - 32H_0H_{0,1} + 32H_{0,0,1}\right] \right] + \frac{(z+1)(19z^2 - 16z + 10)}{z}\left[-\frac{32}{9}H_0H_{-1}^2\right. \\
& \left. + \left[\frac{16}{9}H_0^2 - \frac{32}{9}\zeta_2 + \frac{64}{9}H_{0,-1}\right]H_{-1} - \frac{64}{9}H_{0,-1,-1}\right] + \frac{8(56z^3 - 201z^2 - 162z - 40)H_{0,1,1}}{9z} \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[\frac{2}{9}H_1^4 + \frac{4}{3}H_0^2H_1^2 + \frac{4}{3}\zeta_2H_1^2 + \left[-\frac{80}{9}\zeta_3 + \frac{80}{3}H_{0,0,1} + \frac{16}{3}H_{0,1,1}\right]H_1 \right. \\
& \left. - \frac{16}{3}H_{0,1}^2 + H_0\left[-\frac{16}{9}H_1^3 + \left[-\frac{8}{3}\zeta_2 - 16H_{0,1}\right]H_1 + \frac{80}{3}H_{0,1,1}\right] \right] \\
& + \frac{16(32z^3 - 87z^2 + 45z - 24)H_{0,0,0,1}}{3z} + (2z-1)\left[128H_0H_{0,0,0,1} - 32H_0^2H_{0,0,1}\right] \\
& - \frac{16(36z^3 + 6z^2 - 15z - 20)H_{0,0,1,1}}{3z} + z(4z-3)\left[H_0\left[\frac{32}{3}\zeta_3 + \frac{32}{3}H_{0,0,1}\right] \right. \\
& \left. - \frac{128}{3}H_{0,0,0,1} + \frac{64}{3}H_{0,0,1,1}\right] + \frac{(z+1)(4z^2 - 7z + 4)}{z}\left[\frac{32}{9}H_0H_{-1}^3 \right. \\
& \left. + \left[-\frac{8}{3}H_0^2 - \frac{16}{3}\zeta_2 - \frac{32}{3}H_{0,-1}\right]H_{-1}^2 + \left[-\frac{8}{9}H_0^3 + \left[\frac{24}{3}\zeta_2 + \frac{32}{3}H_{0,-1} - \frac{32}{3}H_{0,1}\right]H_0 \right. \right. \\
& \left. \left. - 16\zeta_3 + \frac{64}{3}H_{0,-1,-1} - \frac{32}{3}H_{0,0,-1} + \frac{64}{3}H_{0,0,1}\right]H_{-1} - \frac{24}{3}H_{0,-1}\zeta_2 + \frac{8}{3}H_0^2H_{0,-1} \right. \\
& \left. + H_0\left[-\frac{32}{3}H_{0,-1,-1} + \frac{32}{3}H_{0,-1,1} - \frac{16}{3}H_{0,0,-1} + \frac{32}{3}H_{0,1,-1}\right] - \frac{64}{3}H_{0,-1,-1,-1} \right. \\
& \left. - \frac{32}{3}H_{0,-1,0,1} + \frac{32}{3}H_{0,0,-1,-1} - \frac{64}{3}H_{0,0,-1,1} + \frac{16}{3}H_{0,0,0,-1} - \frac{64}{3}H_{0,0,1,-1}\right] \\
& + L_M \left[\frac{4}{3}(4z-5)H_0^4 - \frac{8}{9}(35z-46)H_0^3 - \frac{4}{9}(606z^2 - 346z + 377)H_0^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{8(z+1)(16z^2 - 19z + 16)H_{-1}H_0^2}{3z} - \frac{4(z-1)(8z^2 + 17z + 8)H_1H_0^2}{z} \\
& + \frac{128}{3}z^2H_0 + \frac{16(3657z^3 + 2093z^2 + 2330z + 224)H_0}{27z} - \frac{16}{3}(16z^2 - 19z - 25)\zeta_2H_0 \\
& + \frac{8}{3}(40z^2 - 51z + 9)\zeta_2H_0 + 8(9z - 25)\zeta_3H_0 - 8(73z + 11)\zeta_3H_0 \\
& + \frac{16(z-1)(203z^2 + 47z + 140)H_1H_0}{9z} + \frac{64(2z^3 - 9z^2 + 3z - 4)H_{0,-1}H_0}{3z} \\
& + \frac{8(16z^3 - 41z^2 - 77z - 40)H_{0,1}H_0}{3z} - 32(5z - 1)H_{0,0,-1}H_0 - 32(11z + 5)H_{0,0,1}H_0 \\
& - \frac{16}{5}(7z + 27)\zeta_2^2 - \frac{24}{5}(65z + 11)\zeta_2^2 - \frac{4(z-1)(20z^2 + 21z + 2)H_1^2}{3z} \\
& - \frac{8(z-1)(11542z^2 + 399z + 4036)}{27z} - \frac{32(z+1)(53z^2 - 14z + 26)\zeta_2}{9z} \\
& + \frac{8(80z^3 - 157z^2 + 521z - 64)\zeta_2}{9z} - \frac{16(z+1)(4z^2 - z + 4)H_{-1}\zeta_2}{z} \\
& + \frac{4(44z^3 - 81z^2 - 213z - 180)\zeta_3}{3z} + \frac{4(164z^3 - 231z^2 + 81z - 12)\zeta_3}{3z} \\
& + \frac{128(z-1)(z^2 + 1)H_1}{3z} - \frac{8(z-1)(258z^2 - 559z - 138)H_1}{9z} \\
& + \frac{(z+1)(19z^2 - 16z + 10)}{z} \left[\frac{32}{9}H_{0,-1} - \frac{32}{9}H_{-1}H_0 \right] \\
& + \frac{(z+1)(53z^2 - 2z + 26)}{z} \left[\frac{64}{9}H_{0,-1} - \frac{64}{9}H_{-1}H_0 \right] \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z} \left[\frac{8}{9}H_1^3 + 8H_0H_1^2 + \left[\frac{16\zeta_2}{3} - \frac{32}{3}H_{0,1} \right] H_1 \right] \\
& - \frac{8(274z^3 - 253z^2 + 611z - 200)H_{0,1}}{9z} - \frac{16(32z^3 - 75z^2 + 21z - 16)H_{0,0,-1}}{3z} \\
& - \frac{32}{3}z(4z - 3)H_{0,0,1} + \frac{32(8z^2 + 17z + 14)H_{0,0,1}}{3z} \\
& + \frac{(z+1)(4z^2 - 7z + 4)}{z} \left[\frac{32}{3}H_0H_{-1}^2 + \left[-\frac{8}{3}H_0^2 + 16\zeta_2 - \frac{64}{3}H_{0,-1} \right. \right. \\
& \left. \left. - \frac{32}{3}H_{0,1} \right] H_{-1} + \frac{16}{3}H_0H_{0,-1} + \frac{64}{3}H_{0,-1,-1} + \frac{32}{3}H_{0,-1,1} - \frac{16}{3}H_{0,0,-1} + \frac{32}{3}H_{0,1,-1} \right] \\
& + \frac{(z+1)(8z^2 - 5z + 8)}{z} \left[\frac{32}{3}H_{-1}H_{0,1} - \frac{32}{3}H_{0,-1,1} - \frac{32}{3}H_{0,1,-1} \right] \\
& + \frac{16(8z^3 + 23z^2 + 5z - 8)H_{0,1,1}}{3z} + (z-1) \left[\left[8\zeta_2 + 32H_{0,-1} \right] H_0^2 \right. \\
& \left. + \left[128H_{0,-1,-1} + 64H_{0,0,-1} \right] H_0 - 64H_{0,-1}^2 + 64H_{0,-1}\zeta_2 \right. \\
& \left. - 96H_{0,0,0,-1} \right] - 64zH_{0,0,0,1} + 160(5z + 1)H_{0,0,0,1} \\
& + (z+1) \left[56H_{0,1}H_0^2 - 96H_{0,1,1}H_0 + 32H_{0,1}^2 + H_{0,1} \left[-32\zeta_2 - \frac{128}{3} \right] \right. \\
& \left. + 192H_{0,0,0,-1} + 32H_{0,0,1,1} - 32H_{0,1,1,1} \right] + \frac{128(z-1)}{3z} \\
& - \frac{32(z-1)(z+2)(2z+1)H_{0,1,1,1}}{3z} - 64(3z-2)H_{0,0,0,0,1} + 32(23z+27)H_{0,0,0,1,1}
\end{aligned}$$

$$\begin{aligned}
& +z \left[\frac{32}{3} \zeta_2 H_0^3 + 32 \zeta_3 H_0^2 + 64 H_{0,0,0,1} H_0 - 320 H_{0,0,0,0,1} + 128 H_{0,0,0,1,1} \right] \\
& + (z-1) \left[\frac{4}{15} H_0^5 - \frac{16}{3} H_{0,-1} H_0^3 + \left[32 H_{0,0,-1} - 32 H_{0,-1,-1} \right] H_0^2 \right. \\
& + \left[32 H_{0,-1}^2 + 16(-2+5) \zeta_2 H_{0,-1} + 128 H_{0,-1,-1,-1} - 64 H_{0,-1,0,1} - 96 H_{0,0,0,-1} \right] H_0 \\
& + 64 H_{0,-1,-1} \zeta_2 - 96 H_{0,0,-1} \zeta_2 + H_{0,-1} \left[48(-5+3) \zeta_3 - 128 H_{0,-1,-1} - 64 H_{0,0,-1} + 128 H_{0,0,1} \right] \\
& + 256 H_{0,-1,0,-1,-1} + 512 H_{0,0,-1,-1,-1} + 64 H_{0,0,-1,0,-1} - 128 H_{0,0,-1,0,1} + 192 H_{0,0,0,-1,-1} \\
& \left. - 384 H_{0,0,0,-1,1} + 128 H_{0,0,0,0,-1} - 384 H_{0,0,0,1,-1} - 128 H_{0,0,1,0,-1} \right] \\
& + (z+1) \left[-16 H_{0,1,1} H_0^2 + \left[48 H_{0,1}^2 + \left[16 \zeta_2 + \frac{128}{3} \right] H_{0,1} - 128 H_{0,0,1,1} + 64 H_{0,1,1,1} \right] H_0 \right. \\
& + H_{0,1,1} \left[\frac{256}{3} - 16 \zeta_2 \right] - 16 H_{0,0,1} \zeta_2 + H_{0,1} \left[\frac{160 \zeta_3}{3} - 160 H_{0,0,1} \right] + 288 H_{0,0,1,0,1} \\
& \left. - 128 H_{0,0,1,1,1} - 32 H_{0,1,0,1,1} - 32 H_{0,1,1,1,1} \right] - \frac{640(z-1)}{9z} \left. \right\}, \tag{606}
\end{aligned}$$

$A_{Qg} =$

$$\begin{aligned}
& a_s \gamma_{qg}^0 T_F L_M + a_s^2 \left\{ \frac{4}{3} \gamma_{qg}^0 T_F^2 L_M^2 \right. \\
& + C_{AT_F} \left[-\frac{4}{3} (2z+1) H_0^3 + \frac{2}{3} (23z^2 + 12z + 3) H_0^2 - 32z^2 H_0 - \frac{4}{9} (328z^2 + 129z + 42) H_0 \right. \\
& - \frac{4(z-1)(65z^2 + 17z + 8) H_1 H_0}{3z} + 16(4z+1) H_{0,1} H_0 + 2(z-1)(5z+1) H_1^2 \\
& + L_M^2 \left[\frac{4(z-1)(31z^2 + 7z + 4)}{3z} - 8(4z+1) H_0 - 2\gamma_{qg}^0 H_1 \right] + \frac{4(1588z^3 - 1413z^2 - 9z - 112)}{27z} \\
& - 4(z-12)z\zeta_2 - 2(26z^2 + 26z + 5)\zeta_3 + 2(26z^2 + 82z + 21)\zeta_3 - 32(z-1)zH_1 \\
& + 4(12z^2 - 12z - 1)H_1 + L_M \left[8(2z+1)H_0^2 - \frac{8}{3}(44z^2 + 24z + 3)H_0 + 32(z-1)zH_1 \right. \\
& + \frac{8(218z^3 - 225z^2 + 18z - 20)}{9z} + \gamma_{qg}^0 [2\zeta_2 - 2H_1^2] + (2z^2 + 2z + 1) [8\zeta_2 + 16H_{-1}H_0 - 16H_{0,-1}] \left. \right] \\
& + \frac{4(68z^3 - 72z^2 - 9z - 8)H_{0,1}}{3z} + z [32H_{0,1} - 32\zeta_2] - 32(z^2 + 5z + 1)H_{0,0,1} \\
& + z(z+1) [16H_{-1}H_0 - 16H_{0,-1} + 32H_{0,0,1}] + (2z^2 + 2z + 1) [-8H_0H_{-1}^2 + [4H_0^2 - 8\zeta_2 \\
& + 16H_{0,-1}]H_{-1} - 8H_0H_{0,-1} - 16H_{0,-1,-1} + 8H_{0,0,-1}] + \gamma_{qg}^0 \left[-\frac{1}{3}H_1^3 + H_0H_1^2 - 2H_{0,1,1} \right] \left. \right] \\
& + C_{FT_F} \left[\frac{2}{3} (4z^2 - 2z + 1) H_0^3 + (20z^2 - 12z - 1) H_0^2 - 2(24z^2 + 9z + 8) H_0 \right. \\
& + 4(10z^2 - 12z + 1) H_1 H_0 - 16(z-1)^2 H_{0,1} H_0 - 4(z-1)(3z+1) H_1^2 \\
& + L_M^2 \left[2(4z-1) - 4(4z^2 - 2z + 1) H_0 + 2\gamma_{qg}^0 H_1 \right] + 2(40z^2 - 41z + 13) + 8(3z^2 - 6z - 1)\zeta_2 \\
& + 8(2z^2 + 2z - 1)\zeta_3 - 4z(12z - 13)H_1 + L_M \left[-4(4z^2 - 2z + 1)H_0^2 - 4(8z^2 - 4z + 3)H_0 \right.
\end{aligned}$$

$$\begin{aligned}
& -4(20z^2 - 29z + 14) - 32(z-1)zH_1 + \gamma_{qg}^0 \left[2H_1^2 + 4H_0H_1 - 4\zeta_2 \right] - 4(16z^2 - 24z - 1)H_{0,1} \\
& + 8(2z^2 - 6z + 3)H_{0,0,1} + \gamma_{qg}^0 \left[\frac{1}{3}H_1^3 - H_0^2H_1 + 2H_{0,1,1} \right] \Bigg\} \\
& + a_s^3 \left\{ a_{Qg}^{(3)} + T_F^3 \left[\frac{16}{9}\gamma_{qg}^0 L_M^3 - \frac{16}{9}\gamma_{qg}^0 \zeta_3 \right] + C_{AT_F}^2 \left[-\frac{4}{9}(2z+3)H_0^4 \right. \right. \\
& + \frac{8}{27}(46z^2 + 74z - 13)H_0^3 - \frac{8}{27}(458z^2 - 382z + 221)H_0^2 + \frac{8}{3}(16z^2 + 10z - 7)\zeta_2 H_0^2 \\
& - \frac{16(z-1)(65z^2 + 17z + 8)H_1H_0^2}{9z} + \frac{64}{3}(4z+1)H_{0,1}H_0^2 + \frac{16}{81}(6612z^2 + 5083z + 346)H_0 \\
& + \frac{8}{9}(176z^2 + 248z + 41)\zeta_2 H_0 + \frac{32}{9}(24z^2 + 100z + 31)\zeta_3 H_0 \\
& + \frac{16(854z^3 - 882z^2 + 99z - 80)H_1H_0}{27z} - \frac{32(23z^3 + 72z^2 + 15z + 8)H_{0,1}H_0}{9z} \\
& + \frac{128}{3}(2z^2 - 4z - 1)H_{0,0,1}H_0 - \frac{16}{9}(z-1)(5z+1)H_1^3 + L_M^3 \left[\frac{112(z-1)(31z^2 + 7z + 4)}{27z} \right. \\
& \left. - \frac{224}{9}(4z+1)H_0 - \frac{56}{9}\gamma_{qg}^0 H_1 \right] + \frac{8}{15}(222z^2 - 102z - 1)\zeta_2^2 - \frac{8}{15}(222z^2 + 222z - 1)\zeta_2^2 \\
& - \frac{8}{3}(38z^2 - 43z + 3)H_1^2 - \frac{8(82666z^3 - 87018z^2 + 8835z - 5788)}{243z} - \frac{160}{9}(5z^2 - 5z + 1)H_1\zeta_2 \\
& - \frac{4(896z^3 - 336z^2 + 43z - 128)\zeta_2}{9z} + \frac{32}{3}z(13z+5)\zeta_3 - \frac{16(1489z^3 + 408z^2 + 51z - 28)\zeta_3}{27z} \\
& + z^2 \left[H_0 \left[\frac{256}{3}\zeta_2 + \frac{1024}{3} \right] - \frac{128}{3}H_0^2 \right] - \frac{16(2330z^3 - 2321z^2 + 391z + 36)H_1}{81z} \\
& + (z-1)z \left[\frac{256}{3}H_1^2 + \frac{1024}{3}H_1 + H_0 \left[\frac{64}{3}H_1^2 + \frac{256}{3}H_1 \right] \right] \\
& + L_M^2 \left[\frac{64}{3}(z+1)H_0^2 - \frac{32}{9}(79z^2 + 68z + 11)H_0 + \frac{8(1769z^3 - 1788z^2 + 96z - 212)}{27z} \right. \\
& + \frac{32}{9}(28z^2 - 28z + 5)H_1 + \gamma_{qg}^0 \left[4\zeta_2 - 4H_1^2 \right] + (2z^2 + 2z + 1) \left[16\zeta_2 + 32H_{-1}H_0 \right. \\
& \left. \left. - 32H_{0,-1} \right] \right] - \frac{16(980z^3 - 1218z^2 + 117z - 80)H_{0,1}}{27z} - \frac{256}{3}(z-2)zH_{0,0,1} \\
& + \frac{32(135z^3 + 96z^2 + 21z + 8)H_{0,0,1}}{9z} + L_M \left[-\frac{16}{9}(14z-1)H_0^3 + \frac{8}{9}(30z^2 - 104z + 9)H_0^2 \right. \\
& - \frac{256}{3}z^2H_0 - \frac{16}{27}(418z^2 + 904z + 79)H_0 - \frac{32(z-1)(65z^2 + 17z + 8)H_1H_0}{9z} \\
& - \frac{32}{3}(2z^2 - 18z - 3)H_{0,1}H_0 + \frac{16}{9}(z^2 + 2z - 13)H_1^2 + \frac{8(3285z^3 - 2894z^2 + 95z - 264)}{27z} \\
& + \frac{32}{9}(4z^2 + 35z + 5)\zeta_2 - \frac{32}{9}(7z^2 + 31z + 5)\zeta_2 - \frac{32}{3}(10z^2 + 10z + 1)\zeta_3 \\
& + \frac{32}{3}(10z^2 + 44z + 9)\zeta_3 - \frac{256}{3}(z-1)zH_1 + \frac{16}{27}(374z^2 - 365z + 67)H_1 \\
& + (7z^2 + 7z + 5) \left[\frac{64}{9}H_{0,-1} - \frac{64}{9}H_{-1}H_0 \right] + \frac{256}{3}zH_{0,1} + \frac{32(68z^3 - 78z^2 - 9z - 8)H_{0,1}}{9z} \\
& \left. - \frac{32}{3}(6z^2 + 42z + 7)H_{0,0,1} + z(z+1) \left[\frac{128}{3}H_{-1}H_0 - \frac{128}{3}H_{0,-1} + \frac{256}{3}H_{0,0,1} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + (2z^2 + 2z + 1) \left[-\frac{64}{3} H_0 H_{-1}^2 + \left[-\frac{64}{3} \zeta_2 + \frac{128}{3} H_{0,-1} \right] H_{-1} - \frac{128}{3} H_{0,-1,-1} \right] \\
& + \gamma_{qg}^0 \left[-\frac{4}{3} H_1^3 + \frac{4}{3} H_0 H_1^2 - \frac{4}{3} H_0^2 H_1 + \left[\frac{16}{3} H_{0,1} - \frac{8\zeta_2}{3} \right] H_1 - \frac{40}{3} H_{0,1,1} \right] - \frac{32}{3} z(5z - 19) H_{0,1,1} \\
& + z \left[\frac{512}{3} \zeta_2 - \frac{256}{3} H_0 H_{0,1} - \frac{512}{3} H_{0,1,1} \right] - \frac{256}{3} z(4z + 1) H_{0,0,0,1} \\
& + \frac{64}{3} (18z^2 - 2z + 5) H_{0,0,1,1} + z(z + 1) \left[\frac{128}{3} H_0 H_{-1}^2 + \left[-\frac{64}{3} H_0^2 - 128 H_0 - \frac{128}{3} \zeta_2 \right. \right. \\
& \left. \left. - \frac{256}{3} H_{0,-1} \right] H_{-1} - \frac{128}{3} H_0^2 \zeta_2 + 128 H_{0,-1} - \frac{256}{3} H_{0,1} + \frac{256}{3} H_{0,-1,-1} - \frac{128}{3} H_{0,0,-1} \right. \\
& \left. + H_0 \left[-\frac{256}{3} \zeta_3 + \frac{128}{3} H_{0,-1} - \frac{256}{3} H_{0,0,1} \right] + \frac{1024}{3} H_{0,0,0,1} - \frac{512}{3} H_{0,0,1,1} \right] \\
& + (2z^2 + 2z + 1) \left[-\frac{128}{9} H_0 H_{-1}^3 + \left[\frac{32}{3} H_0^2 - \frac{64}{3} \zeta_2 + \frac{128}{3} H_{0,-1} \right] H_{-1}^2 \right. \\
& + \left[\frac{32}{9} H_0^3 + \left[-\frac{80}{3} \zeta_2 - \frac{128}{3} H_{0,-1} + \frac{128}{3} H_{0,1} \right] H_0 + 64 \zeta_3 - \frac{256}{3} H_{0,-1,-1} + \frac{128}{3} H_{0,0,-1} \right. \\
& \left. - \frac{256}{3} H_{0,0,1} \right] H_{-1} + \frac{80}{3} H_{0,-1} \zeta_2 - \frac{32}{3} H_0^2 H_{0,-1} + H_0 \left[\frac{128}{3} H_{0,-1,-1} - \frac{128}{3} H_{0,-1,1} + \frac{64}{3} H_{0,0,-1} \right. \\
& \left. - \frac{128}{3} H_{0,1,-1} \right] + \frac{256}{3} H_{0,-1,-1,-1} + \frac{128}{3} H_{0,-1,0,1} - \frac{128}{3} H_{0,0,-1,-1} + \frac{256}{3} H_{0,0,-1,1} - \frac{64}{3} H_{0,0,0,-1} \\
& \left. + \frac{256}{3} H_{0,0,1,-1} \right] + \gamma_{qg}^0 \left[\frac{2}{9} H_1^4 + \frac{4}{3} H_0^2 H_1^2 + \frac{10}{3} \zeta_2 H_1^2 + \left[-\frac{40}{9} \zeta_3 + \frac{80}{3} H_{0,0,1} + \frac{16}{3} H_{0,1,1} \right] H_1 \right. \\
& \left. - \frac{16}{3} H_{0,1}^2 + H_0 \left[-\frac{16}{9} H_1^3 - 16 H_{0,1} H_1 + \frac{80}{3} H_{0,1,1} \right] - \frac{16}{3} H_{0,1,1,1} \right] \\
& + C_{AT_F^2 N_F} \left[-\frac{4}{9} (2z + 1) H_0^4 + \frac{8}{27} (23z^2 + 12z + 3) H_0^3 - \frac{8}{27} (292z^2 + 39z + 42) H_0^2 \right. \\
& \left. - \frac{8(z - 1)(65z^2 + 17z + 8) H_1 H_0^2}{9z} + \frac{32}{3} (4z + 1) H_{0,1} H_0^2 + \frac{32}{3} (z - 1) z H_1^2 H_0 \right. \\
& \left. + \frac{16}{81} (3392z^2 + 645z + 111) H_0 + \frac{32(z - 1)(254z^2 - 7z + 20) H_1 H_0}{27z} \right. \\
& \left. - \frac{16(23z^3 + 96z^2 + 15z + 8) H_{0,1} H_0}{9z} + \frac{8}{9} (62z^2 - 16z - 7) \zeta_2 H_0 + \frac{32}{9} (28z + 13) \zeta_3 H_0 \right. \\
& \left. - \frac{8}{9} (z - 1)(5z + 1) H_1^3 + L_M^3 \left[\frac{16(z - 1)(31z^2 + 7z + 4)}{27z} - \frac{32}{9} (4z + 1) H_0 \right. \right. \\
& \left. \left. - \frac{8}{9} \gamma_{qg}^0 H_1 \right] - \frac{8}{3} (4z^2 - 6z + 1) H_1^2 - \frac{32(5854z^3 - 6219z^2 + 531z - 328)}{243z} \right. \\
& \left. + \frac{4}{9} (19z^3 - 50z^2 + 2z - 4) \frac{\zeta_2}{z} - \frac{16}{27} (550z^3 + 228z^2 + 33z - 4) \frac{\zeta_3}{z} \right. \\
& \left. + \frac{32}{3} (2z^2 - 2z - 1) H_1 - \frac{32(290z^3 - 261z^2 + 27z - 20) H_{0,1}}{27z} - \frac{16}{3} z(5z - 4) H_{0,1,1} \right. \\
& \left. + \frac{16(111z^3 + 144z^2 + 21z + 8) H_{0,0,1}}{9z} - \frac{16}{9} (2z^2 - 2z - 5) H_1 \zeta_2 + z(z + 1) \left[\frac{64}{3} H_0 H_{-1}^2 \right. \right. \\
& \left. \left. + \left[-\frac{32}{3} H_0^2 - 64 H_0 - \frac{128}{3} H_{0,-1} \right] H_{-1} + \frac{64}{3} \zeta_2 H_{-1} + \frac{64}{3} H_0 H_{0,-1} + 64 H_{0,-1} + \frac{128}{3} H_{0,-1,-1} \right. \right. \\
& \left. \left. - \frac{64}{3} H_{0,0,-1} \right] + (6z + 1) \left[-\frac{8}{3} \zeta_2 H_0^2 - \frac{64}{3} H_{0,0,1} H_0 \right] + z \left[128 H_{0,0,0,1} - \frac{1216 \zeta_2^2}{15} \right] \right. \\
& \left. + L_M^2 \left[-\frac{4}{3} \gamma_{qg}^0 H_1^2 - \frac{32}{9} (4z^2 - 4z + 5) H_1 - \frac{8(205z^3 - 168z^2 + 42z - 52)}{27z} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{32}{9}(9z^2 - 20z - 5)H_0 + (2z^2 + 2z + 1) \left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1} \right] + z \left[\frac{64}{3}H_0^2 + \frac{64\zeta_2}{3} \right] \\
& + L_M \left[-\frac{16}{9}(10z - 1)H_0^3 + \frac{8}{9}(7z^2 - 66z - 13)H_0^2 + \frac{16}{27}(58z^2 - 269z - 2)H_0 \right. \\
& - \frac{16(z - 1)(65z^2 + 17z + 8)H_1H_0}{9z} - \frac{32}{3}(2z^2 - 10z - 1)H_{0,1}H_0 - \frac{8}{9}(13z^2 - 16z + 23)H_1^2 \\
& + \frac{8(592z^3 - 268z^2 - 119z - 4)}{27z} + \frac{16}{27}(64z^2 - 64z + 29)H_1 + (4z^2 + 4z + 5) \left[\frac{64}{9}H_{0,-1} \right. \\
& - \left. \frac{64}{9}H_{-1}H_0 \right] + \frac{16(68z^3 - 54z^2 - 9z - 8)H_{0,1}}{9z} + \frac{32}{3}(2z^2 - 18z - 3)H_{0,0,1} - \frac{16}{9}z(3z + 10)\zeta_2 \\
& + (2z^2 + 2z + 1) \left[-\frac{32}{3}H_0H_{-1}^2 + \left[\frac{64}{3}H_{0,-1} - \frac{16}{3}H_0^2 \right]H_{-1} - \frac{32}{3}\zeta_2H_{-1} + \frac{32}{3}H_0H_{0,-1} \right. \\
& - \left. \frac{64}{3}H_{0,-1,-1} - \frac{32}{3}H_{0,0,-1} \right] + \gamma_{qg}^0 \left[-\frac{8}{9}H_1^3 - \frac{4}{3}H_0^2H_1 + \frac{16}{3}H_{0,1}H_1 - \frac{8}{3}\zeta_2H_1 - \frac{32}{3}H_{0,1,1} \right] \\
& + \frac{128}{3}(5z + 1)\zeta_3 \left. \right] + (2z^2 + 2z + 1) \left[-\frac{64}{9}H_0H_{-1}^3 + \left[\frac{16}{3}H_0^2 + \frac{64}{3}H_{0,-1} \right]H_{-1}^2 + \left[\frac{16}{9}H_0^3 \right. \right. \\
& + \left. \left[\frac{64}{3}H_{0,1} - \frac{64}{3}H_{0,-1} \right]H_0 - \frac{128}{3}H_{0,-1,-1} + \frac{64}{3}H_{0,0,-1} - \frac{128}{3}H_{0,0,1} \right]H_{-1} + 32\zeta_3H_{-1} \\
& - \frac{16}{3}H_0^2H_{0,-1} + H_0 \left[\frac{64}{3}H_{0,-1,-1} - \frac{64}{3}H_{0,-1,1} + \frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,1,-1} \right] + \frac{128}{3}H_{0,-1,-1,-1} \\
& + \frac{64}{3}H_{0,-1,0,1} - \frac{64}{3}H_{0,0,-1,-1} + \frac{128}{3}H_{0,0,-1,1} - \frac{32}{3}H_{0,0,0,-1} + \frac{128}{3}H_{0,0,1,-1} + \left[-\frac{32}{3}H_{-1}^2 \right. \\
& - \left. \frac{32}{3}H_0H_{-1} + \frac{32}{3}H_{0,-1} \right] \zeta_2 + \gamma_{qg}^0 \left[\frac{1}{9}H_1^4 + \frac{2}{3}H_0^2H_1^2 + \frac{4}{3}\zeta_2H_1^2 + \left[\frac{40}{3}H_{0,0,1} + \frac{8}{3}H_{0,1,1} \right]H_1 \right. \\
& - \left. \frac{40}{9}\zeta_3H_1 - \frac{8}{3}H_{0,1}^2 + H_0 \left[-\frac{8}{9}H_1^3 - 8H_{0,1}H_1 + \frac{40}{3}H_{0,1,1} \right] - \frac{40}{3}H_{0,0,1,1} - \frac{8}{3}H_{0,1,1,1} \right] \\
& + C_{AT}^2 F \left[\frac{1}{9}(18z^2 - 26z + 23)H_0^4 + \frac{2}{9}(225z^2 - 12z - 35)H_0^3 - \frac{8}{3}(z - 15)z\zeta_2H_0^3 \right. \\
& + \frac{8}{3}(z^2 - 7z - 5)\zeta_2H_0^3 - \frac{4(60z^3 + 46z^2 + 5z + 8)H_{-1}H_0^3}{9z} + \frac{8(z - 1)(65z^2 + 17z + 8)H_1H_0^3}{9z} \\
& - \frac{32}{3}(4z + 1)H_{0,1}H_0^3 - \frac{4(120z^3 + 106z^2 + 5z + 8)H_{-1}^2H_0^2}{3z} + \frac{2(190z^3 - 142z^2 - 13z - 24)H_1^2H_0^2}{3z} \\
& + \frac{2}{27}(5248z^2 - 6573z + 738)H_0^2 + \frac{8}{3}(37z^2 + 46z - 3)\zeta_2H_0^2 \\
& - \frac{2}{3}(110z^2 + 190z - 59)\zeta_2H_0^2 - \frac{32}{3}(43z - 5)\zeta_3H_0^2 + \frac{8(z + 1)(359z^2 - 32z + 20)H_{-1}H_0^2}{9z} \\
& + \frac{2(z - 1)(1907z^2 - 169z + 308)H_1H_0^2}{9z} + \frac{4(96z^3 + 70z^2 + 5z + 8)H_{0,-1}H_0^2}{3z} \\
& + \frac{8}{3}(32z^2 - 86z + 7)H_{0,1}H_0^2 + 16(6z^2 - 2z + 5)H_{0,-1,-1}H_0^2 - 64(z^2 + 2z - 1)H_{0,0,1}H_0^2 \\
& + 96(z - 3)zH_{0,1,1}H_0^2 - \frac{8(93z^3 - 82z^2 + 8z - 8)H_1^3H_0}{9z} + \frac{32}{5}(2z + 3)(10z - 3)\zeta_2^2H_0 \\
& - \frac{16}{5}(40z^2 - 76z + 1)\zeta_2^2H_0 - \frac{8(169z^3 - 157z^2 + 11z - 20)H_1^2H_0}{3z} + 32(z^2 + 9z + 3)H_{0,1}^2H_0 \\
& - \frac{32}{9}(701z^2 + 12)H_0 - \frac{4(175490z^3 + 100242z^2 + 15513z + 2624)H_0}{81z} \\
& - \frac{4}{9}(2359z^2 + 462z + 120)\zeta_2H_0 - \frac{2(1756z^3 + 940z^2 + 925z + 160)\zeta_2H_0}{9z}
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(180z^3 + 166z^2 + 5z + 8)H_{-1}\zeta_2H_0}{3z} + \frac{8(247z^3 + 212z^2 + 7z + 20)H_{-1}\zeta_2H_0}{3z} \\
& -\frac{8(z-1)(137z^2 + 41z + 20)H_1\zeta_2H_0}{3z} - 8(18z^2 - 22z + 19)H_{0,-1}\zeta_2H_0 \\
& + 8(2z^2 + 38z + 11)H_{0,1}\zeta_2H_0 + \frac{8}{3}z(51z + 62)\zeta_3H_0 \\
& + \frac{8(57z^3 + 154z^2 - 269z - 8)\zeta_3H_0}{9z} - \frac{8(11936z^3 - 12231z^2 + 1431z - 1244)H_1H_0}{27z} \\
& -\frac{16(323z^3 + 111z^2 - 12z + 20)H_{0,-1}H_0}{9z} + \frac{64}{3}(6z^2 + 33z + 2)H_{0,1}H_0 \\
& + \frac{4(1213z^3 + 4308z^2 - 3z + 468)H_{0,1}H_0}{9z} - \frac{8(406z^3 - 330z^2 - 3z - 40)H_1H_{0,1}H_0}{3z} \\
& -\frac{16(108z^3 + 82z^2 + 5z + 8)H_{0,-1,-1}H_0}{3z} - \frac{8(132z^3 + 94z^2 + 5z + 8)H_{0,0,-1}H_0}{3z} \\
& + \frac{32}{3}z(54z + 11)H_{0,0,1}H_0 - \frac{16(237z^3 - 188z^2 + 28z - 8)H_{0,0,1}H_0}{3z} \\
& + \frac{8(768z^3 - 542z^2 + 19z - 56)H_{0,1,1}H_0}{3z} + 64(z^2 - 5z + 2)H_{0,0,0,-1}H_0 \\
& - 256z(2z + 1)H_{0,0,0,1}H_0 + 128(4z^2 + 9z - 1)H_{0,0,0,1}H_0 - 32(18z^2 + 26z + 9)H_{0,0,1,1}H_0 \\
& - 32(6z^2 - 14z + 1)H_{0,1,1,1}H_0 + \frac{(204z^3 - 166z^2 - 19z - 8)H_1^4}{9z} + \frac{2(3049z^3 - 3168z^2 + 15z - 4)H_1^3}{27z} \\
& -\frac{2(3846z^3 - 10226z^2 + 675z - 592)\zeta_2^2}{15z} + \frac{2(4534z^3 + 3158z^2 - 45z + 16)\zeta_2^2}{15z} \\
& + \frac{2(19952z^3 - 19524z^2 + 615z - 710)H_1^2}{27z} - \frac{8(161z^3 - 130z^2 - 4z - 16)H_{0,1}^2}{3z} \\
& + \frac{4(1158802z^3 - 1178838z^2 + 87399z - 70927)}{243z} + \frac{4(102z^3 - 94z^2 + 11z - 8)H_1^2\zeta_2}{3z} \\
& + \frac{2}{9}(5643z^2 - 5900z - 143)\zeta_2 + \frac{2(40619z^3 + 7491z^2 + 5019z - 2176)\zeta_2}{27z} \\
& + \frac{4(2247z^3 - 2188z^2 + 116z - 242)H_1\zeta_2}{9z} + \frac{8(72z^3 + 94z^2 + 5z + 8)H_{0,-1}\zeta_2}{3z} \\
& -\frac{8(175z^3 + 164z^2 + 7z + 20)H_{0,-1}\zeta_2}{3z} - \frac{8(124z^3 + 114z^2 + 33z + 16)H_{0,1}\zeta_2}{3z} \\
& + 16(6z^2 - 26z + 11)H_{0,-1,-1}\zeta_2 + 8(18z^2 - 62z + 29)H_{0,0,-1}\zeta_2 \\
& + 8(46z^2 + 2z - 7)H_{0,0,1}\zeta_2 - 16(30z^2 - 22z + 17)H_{0,1,1}\zeta_2 \\
& -\frac{4(3551z^3 + 1551z^2 + 36z + 100)\zeta_3}{9z} + \frac{4(27017z^3 + 29781z^2 + 3645z + 748)\zeta_3}{27z} \\
& - 4(66z^2 - 134z + 19)\zeta_2\zeta_3 + \frac{4}{3}(198z^2 + 154z + 223)\zeta_2\zeta_3 \\
& + \frac{4(304z^3 + 262z^2 + 15z + 24)H_{-1}\zeta_3}{z} - \frac{4(528z^3 + 458z^2 + 25z + 40)H_{-1}\zeta_3}{z} \\
& -\frac{8(874z^3 - 698z^2 - 41z - 80)H_1\zeta_3}{9z} + 32(13z^2 - 17z + 14)H_{0,-1}\zeta_3 \\
& + \frac{160}{3}(2z^2 + 6z + 3)H_{0,1}\zeta_3 + 8(12z + 5)\zeta_5 + 104(20z - 3)\zeta_5 \\
& + (4z - 1)\left[\frac{4}{15}H_0^5 - 32H_{0,-1}\zeta_2H_0 + 64H_{0,0,-1}\zeta_2\right] - \frac{32(z-1)(701z^2 + 32)H_1}{9z}
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(34870z^3 - 34925z^2 + 2054z - 2233)H_1}{27z} + \frac{(z-1)(57z^2 + 2) \left[-\frac{64}{3}H_1^2 - \frac{64}{3}H_0H_1 \right]}{z} \\
& + (z-1)z \left[-128H_1^3 - 192H_0H_1^2 - 64H_0^2H_1 + 384\zeta_2H_1 \right] \\
& + \frac{(z+1)(2443z^2 - 244z + 94) \left[\frac{16}{27}H_{-1}H_0 - \frac{16}{27}H_{0,-1} \right]}{z} + \frac{32(17z^2 + 20z - 12)H_{0,1}}{9z} \\
& + \frac{8(9594z^3 - 17547z^2 + 885z - 1244)H_{0,1}}{27z} + \frac{8(84z^3 - 70z^2 + 5z - 8)H_1H_{0,1,1}}{3z} \\
& + L_M^3 \left[\frac{16}{3}(8z-1)H_0^2 - \frac{8(18z^3 - 152z^2 - 11z - 8)H_0}{9z} - \frac{4(z-1)(1883z^2 - 97z + 272)}{27z} \right. \\
& \left. - \frac{64}{3}(4z+1)\zeta_2 - \frac{8(146z^3 - 118z^2 - z - 16)H_1}{9z} + \gamma_{qg}^0 \left[\frac{8}{3}H_1^2 + \frac{8}{3}H_0H_1 \right] + \frac{32}{3}(2z^2 + 6z + 3)H_{0,1} \right] \\
& + \frac{16(287z^3 - 105z^2 - 12z + 20)H_{0,0,-1}}{9z} + \frac{32}{9}(251z^2 - 375z - 12)H_{0,0,1} \\
& - \frac{4(5533z^3 + 6840z^2 + 399z + 628)H_{0,0,1}}{9z} + \frac{8(728z^3 - 590z^2 - 11z - 72)H_1H_{0,0,1}}{3z} \\
& + \frac{(z+1)(305z^2 - 32z + 20)}{z} \left[-\frac{16}{9}H_0H_{-1}^2 + \left[-\frac{16}{3}\zeta_2 + \frac{32}{9}H_{0,-1} + \frac{32}{9}H_{0,1} \right]H_{-1} \right. \\
& \left. - \frac{32}{9}H_{0,-1,-1} - \frac{32}{9}H_{0,-1,1} - \frac{32}{9}H_{0,1,-1} \right] - 192(z^2 + 5z + 2)H_{0,1}H_{0,0,1} \\
& + \frac{(z+1)(251z^2 - 32z + 20)}{z} \left[H_{-1} \left[\frac{32}{9}\zeta_2 - \frac{32}{9}H_{0,1} \right] + \frac{32}{9}H_{0,-1,1} + \frac{32}{9}H_{0,1,-1} \right] \\
& + \frac{128}{3}(9z^2 + 30z + 2)H_{0,1,1} + \frac{4(2561z^3 - 6192z^2 - 144z - 240)H_{0,1,1}}{9z} \\
& + L_M^2 \left[\frac{16}{3}z(z+1)H_0^3 - \frac{16}{3}(z^2 + 9z - 2)H_0^3 - \frac{8}{3}(13z^2 - 8z + 12)H_0^2 \right. \\
& \left. + \frac{8}{3}(31z^2 - 6z + 6)H_0^2 + \frac{8}{9}(197z^2 - 42z + 48)H_0 + \frac{16(569z^3 + 376z^2 + 109z + 20)H_0}{9z} \right. \\
& \left. + 32(2z+1)\zeta_2H_0 - 32(4z-1)\zeta_2H_0 + \frac{16(z-1)(19z^2 + 7z + 4)H_1H_0}{3z} \right. \\
& \left. + 16(6z^2 - 10z + 7)H_{0,-1}H_0 - 16(2z^2 + 6z + 3)H_{0,1}H_0 - \frac{4(132z^3 - 118z^2 + 5z - 8)H_1^2}{3z} \right. \\
& \left. - \frac{4(z-1)(1231z^2 - 191z + 178)}{27z} - \frac{4(14939z^3 - 14544z^2 + 783z - 1286)}{27z} \right. \\
& \left. - \frac{4}{3}(30z^2 + 214z - 11)\zeta_2 + 8(2z^2 - 46z + 13)\zeta_3 \right. \\
& \left. - 8(2z^2 + 22z + 5)\zeta_3 - \frac{8(617z^3 - 604z^2 - 70z - 28)H_1}{9z} + \gamma_{qg}^0 \left[4H_1^3 - 2H_0^2H_1 - 8\zeta_2H_1 \right] \right. \\
& \left. + \frac{(146z^3 + 118z^2 - z + 16)}{z} \left[-\frac{4}{3}\zeta_2 - \frac{8}{3}H_{-1}H_0 + \frac{8}{3}H_{0,-1} \right] + \frac{16}{3}z(25z + 36)H_{0,1} \right. \\
& \left. - 16(6z^2 - 26z + 11)H_{0,0,-1} + (2z^2 + 2z + 1) \left[32H_0H_{-1}^2 + \left[-24H_0^2 \right. \right. \right. \\
& \left. \left. + 64\zeta_2 - 64H_{0,-1} - 32H_{0,1} \right] H_{-1} + 64H_{0,-1,-1} + 32H_{0,-1,1} \right. \\
& \left. \left. + 16H_{0,0,1} + 32H_{0,1,-1} \right] + 32(4z+1)H_{0,1,1} \right] - 64z(3z+2)H_{0,-1,0,1}
\end{aligned}$$

$$\begin{aligned}
& -\frac{16(60z^3 + 58z^2 + 5z + 8)H_{0,-1,0,1}}{3z} + \frac{8(168z^3 + 118z^2 + 5z + 8)H_{0,0,0,-1}}{3z} \\
& -\frac{64}{3}z(93z + 46)H_{0,0,0,1} + \frac{16(575z^3 - 224z^2 + 69z - 16)H_{0,0,0,1}}{3z} \\
& + \frac{(72z^3 + 58z^2 + 5z + 8)}{z} \left[H_{-1} \left[\frac{16}{3}H_{0,0,1} - \frac{16}{3}H_{0,0,-1} \right] + \frac{16}{3}H_{0,0,-1,-1} \right. \\
& - \left. \frac{16}{3}H_{0,0,-1,1} - \frac{16}{3}H_{0,0,1,-1} \right] + \frac{64}{3}z(54z - 1)H_{0,0,1,1} - \frac{8(1364z^3 - 274z^2 + 31z - 56)H_{0,0,1,1}}{3z} \\
& + \gamma_{gg}^0 \left[-\frac{1}{3}H_1^5 - \frac{8}{3}\zeta_2 H_1^3 - \frac{2}{3}H_0^3 H_1^2 + \left[\frac{52\zeta_3}{3} - 44H_{0,0,1} - 4H_{0,1,1} \right] H_1^2 + H_0^2 \left[\zeta_2 + 8H_{0,1} \right] H_1 \right. \\
& + \left. \left[46\zeta_2^2 - 32H_{0,1}\zeta_2 + 20H_{0,1}^2 - 16H_{0,0,0,1} + 24H_{0,0,1,1} \right] H_1 + H_0 \left[\frac{5}{3}H_1^4 + \left[2\zeta_2 + 24H_{0,1} \right] H_1^2 \right. \right. \\
& + \left. \left. \left(\frac{40\zeta_3}{3} - 8H_{0,0,1} - 56H_{0,1,1} \right) H_1 \right] - 40H_{0,1}H_{0,1,1} \right] \\
& + \frac{(96z^3 + 82z^2 + 5z + 8)}{z} \left[\left[\frac{16}{3}H_{0,1} - \frac{16}{3}\zeta_2 \right] H_{-1}^2 + \left[\frac{16}{3}H_0H_{0,-1} - \frac{32}{3}H_{0,-1,1} \right. \right. \\
& - \left. \left. \frac{32}{3}H_{0,1,-1} \right] H_{-1} + \frac{32}{3}H_{0,-1,-1,1} + \frac{32}{3}H_{0,-1,1,-1} + \frac{32}{3}H_{0,1,-1,-1} \right] \\
& + \frac{(108z^3 + 94z^2 + 5z + 8)}{z} \left[\frac{16}{9}H_0H_{-1}^3 + \left[8\zeta_2 - \frac{16}{3}H_{0,-1} - \frac{16}{3}H_{0,1} \right] H_{-1}^2 + \left[-\frac{16}{3}H_0H_{0,1} \right. \right. \\
& + \left. \left. \frac{32}{3}H_{0,-1,-1} + \frac{32}{3}H_{0,-1,1} + \frac{16}{3}H_{0,0,1} + \frac{32}{3}H_{0,1,-1} \right] H_{-1} \right. \\
& + \left. H_0 \left[\frac{16}{3}H_{0,-1,1} + \frac{16}{3}H_{0,1,-1} \right] - \frac{32}{3}H_{0,-1,-1,-1} - \frac{32}{3}H_{0,-1,-1,1} - \frac{32}{3}H_{0,-1,1,-1} \right. \\
& - \left. \frac{16}{3}H_{0,0,-1,1} - \frac{16}{3}H_{0,0,1,-1} - \frac{32}{3}H_{0,1,-1,-1} \right] + L_M \left[\frac{4}{3}(19z - 5)H_0^4 - \frac{4}{9}(54z^2 \right. \\
& + 40z - 37)H_0^3 - \frac{2}{9}(4303z^2 + 216z + 1303)H_0^2 + 4(34z - 9)\zeta_2 H_0^2 + 32(3z^2 - 7z + 1) \\
& \times \zeta_2 H_0^2 - \frac{4(108z^3 + 94z^2 + 5z + 8)H_{-1}H_0^2}{3z} + \frac{8(122z^3 + 127z^2 + 32z + 16)H_{-1}H_0^2}{3z} \\
& - \frac{4(18z^3 + 59z^2 - 58z - 8)H_1H_0^2}{3z} - 16(z^2 - 10z + 3)H_{0,-1}H_0^2 \\
& + 4(36z^2 - 2z + 23)H_{0,1}H_0^2 + \frac{40(z - 1)(31z^2 + 7z + 4)H_1^2H_0}{3z} \\
& + \frac{8(28596z^3 + 13409z^2 + 3023z + 448)H_0}{27z} + \frac{4(z - 1)(1487z^2 - 5z + 192)H_1H_0}{3z} \\
& - \frac{32}{3}(57z^2 + 38z - 3)\zeta_2 H_0 + \frac{8}{3}(318z^2 + 59z + 27)\zeta_2 H_0 + 8(4z^2 + 24z - 21)\zeta_3 H_0 \\
& - 8(4z^2 + 288z + 15)\zeta_3 H_0 + \frac{16(28z^3 - 208z^2 - 5z - 16)H_{0,-1}H_0}{3z} \\
& + \frac{8(72z^3 + 70z^2 + 5z + 8)H_{0,-1}H_0}{3z} + \frac{8(40z^3 - 281z^2 - 86z - 24)H_{0,1}H_0}{3z} \\
& - 32(8z^2 - 2z + 7)H_{0,-1,-1}H_0 + (2z^2 - 6z + 3) \left[-32H_{0,-1,-1} - 32H_{0,0,-1} \right] H_0 \\
& - 32(z + 1)(9z + 7)H_{0,0,1}H_0 + (2z^2 - 10z + 1) \left[-32H_{0,-1,1} - 32H_{0,1,-1} \right] H_0 \\
& - 32(12z^2 + 8z + 11)H_{0,1,1}H_0 - \frac{8}{9}(103z^2 - 94z + 2)H_1^3 + \frac{8}{5}(24z^2 - 696z - 71)\zeta_2^2 \\
& - \frac{4}{5}(160z^2 + 366z + 103)\zeta_2^2 - \frac{2(2599z^3 - 2608z^2 - 247z - 84)H_1^2}{9z} - 32z(z + 22)H_{0,0,-1}H_0
\end{aligned}$$

$$\begin{aligned}
& +32(z^2 - 4z + 2)H_{0,-1}^2 + 64(3z^2 - z + 2)H_{0,1}^2 - \frac{4(290560z^3 - 295527z^2 + 32808z - 25024)}{81z} \\
& + \frac{4(617z^3 - 2690z^2 + 490z - 128)\zeta_2}{9z} - \frac{8(975z^3 + 85z^2 - 136z + 104)\zeta_2}{9z} \\
& - \frac{16(76z^3 + 103z^2 + 50z + 12)H_{-1}\zeta_2}{3z} - \frac{8(54z^3 - 76z^2 + 41z - 8)H_1\zeta_2}{3z} \\
& - 16(6z - 1)H_{0,-1}\zeta_2 - 32(8z^2 - 4z + 7)H_{0,-1}\zeta_2 - 16(20z^2 - 26z + 9)H_{0,1}\zeta_2 \\
& + \frac{2(640z^3 - 4874z^2 - 543z - 360)\zeta_3}{3z} + \frac{2(2240z^3 - 614z^2 + 319z - 24)\zeta_3}{3z} \\
& + \frac{64(z - 1)(33z^2 + 2)H_1}{3z} - \frac{4(14636z^3 - 18161z^2 + 3913z + 204)H_1}{27z} \\
& + (z - 1)z \left[128H_1^2 + 128H_0H_1 \right] + \frac{(z + 1)(251z^2 - 32z + 20)}{z} \left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0 \right] \\
& + \frac{(831z^3 + 733z^2 - 88z + 104)}{z} \left[\frac{16}{9}H_{0,-1} - \frac{16}{9}H_{-1}H_0 \right] + (z^2 - z + 1) \left[32H_{0,-1}H_0^2 \right. \\
& \left. + 64H_{0,-1}^2 \right] - \frac{64}{3}(21z + 2)H_{0,1} - \frac{4(3416z^3 - 4824z^2 + 1257z - 416)H_{0,1}}{9z} \\
& - \frac{16(40z^3 - 26z^2 - 17z - 8)H_1H_{0,1}}{3z} + \frac{(z + 1)(40z^2 - 25z + 4)}{z} \left[\frac{16}{3}H_0H_{-1}^2 - \frac{32}{3}H_{0,-1}H_{-1} \right. \\
& \left. + \frac{32}{3}H_{0,-1,-1} \right] - \frac{8(36z^3 + 46z^2 + 5z + 8)H_{0,0,-1}}{3z} - \frac{16(178z^3 - 289z^2 + 22z - 16)H_{0,0,-1}}{3z} \\
& - \frac{32}{3}z(42z + 11)H_{0,0,1} + \frac{8(16z^3 + 668z^2 + 75z + 40)H_{0,0,1}}{3z} \\
& + \frac{(96z^3 + 82z^2 + 5z + 8)}{z} \left[\frac{8}{3}H_0H_{-1}^2 + \left[8\zeta_2 - \frac{16}{3}H_{0,-1} - \frac{16}{3}H_{0,1} \right] H_{-1} + \frac{16}{3}H_{0,-1,-1} \right. \\
& \left. + \frac{16}{3}H_{0,-1,1} + \frac{16}{3}H_{0,1,-1} \right] + \frac{(116z^3 + 118z^2 + 29z + 16)}{z} \left[\frac{16}{3}H_{-1}H_{0,1} - \frac{16}{3}H_{0,-1,1} - \frac{16}{3}H_{0,1,-1} \right] \\
& + \frac{32}{3}(8z^2 + 73z - 11)H_{0,1,1} + z \left[-128H_0H_{0,1} - 256H_{0,1,1} \right] + z^2 \left[704H_0 \right. \\
& \left. - 128H_{0,-1,0,1} \right] + 64(4z - 1)H_{0,-1,0,1} + 64(z^2 - 5z + 2)H_{0,0,0,-1} \\
& + 48(2z^2 + 22z + 5)H_{0,0,0,-1} + 128z(3z + 1)H_{0,0,0,1} - 16(2z^2 - 118z - 13)H_{0,0,0,1} \\
& + 32(20z^2 + 12z + 9)H_{0,0,1,1} + z(z + 1) \left[-128H_0H_{0,0,1} - 256H_{0,0,1,1} \right] \\
& + (2z^2 + 2z + 1) \left[-\frac{160}{3}H_0H_{-1}^3 + \left[8H_0^2 - 80\zeta_2 + 160H_{0,-1} \right] H_{-1}^2 + \left[\frac{8}{3}H_0^3 + \left[64\zeta_2 + 64H_{0,-1} \right] H_0 \right. \right. \\
& \left. \left. + 192\zeta_3 - 320H_{0,-1,-1} - 160H_{0,0,-1} + 32H_{0,0,1} - 64H_{0,1,1} \right] H_{-1} + H_0 \left[-64H_{0,-1,1} - 64H_{0,1,-1} \right] \right. \\
& \left. + 320H_{0,-1,-1,-1} + 64H_{0,-1,1,1} + 160H_{0,0,-1,-1} - 32H_{0,0,-1,1} - 32H_{0,0,1,-1} \right. \\
& \left. + 64H_{0,1,-1,1} + 64H_{0,1,1,-1} \right] + \gamma_{gg}^0 \left[2H_1^4 - 4H_0^2H_1^2 + \left[8H_{0,1} - 8\zeta_2 \right] H_1^2 \right. \\
& \left. + \frac{10}{3}H_0^3H_1 + \left[-136\zeta_3 + 48H_{0,0,-1} + 64H_{0,0,1} \right] H_1 + H_0 \left[H_1 \left[-40\zeta_2 - 24H_{0,-1} - 16H_{0,1} \right] \right. \right. \\
& \left. \left. - \frac{20}{3}H_1^3 \right] - 24H_{0,-1}H_{0,1} - 8H_{0,1,1,1} \right] + \frac{128(z - 1)}{3z} \left[-\frac{8(30z^3 + 278z^2 - 7z - 8)H_{0,1,1,1}}{3z} \right. \\
& \left. + z \left[\left[128\zeta_3 + 64H_{0,1} \right] H_0^2 + 384H_{0,1,1}H_0 - 384H_{0,1}\zeta_2 \right. \right. \\
& \left. \left. + 768H_{0,1,1,1} \right] + (2z^2 - 6z + 3) \left[-16H_{0,0,-1}H_0^2 - 96H_{0,-1,-1}\zeta_2 + 64H_{0,-1,0,1,1} \right] \right. \\
& \left. + (z^2 - z + 1) \left[\frac{32}{3}H_{0,-1}H_0^3 + \left[128H_{0,-1,0,1} - 64H_{0,-1}^2 \right] H_0 + H_{0,-1} \left[-288\zeta_3 + 256H_{0,-1,-1} \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +128H_{0,0,-1}] + 256H_{0,-1,-1,0,1} - 512H_{0,-1,0,-1,-1} - 256H_{0,-1,0,1,1} - 1024H_{0,0,-1,-1,-1} \\
& -128H_{0,0,-1,0,-1}] - 32(2z^2 + 18z - 3)H_{0,0,-1,0,1} - 32(2z^2 - 22z + 7)H_{0,0,0,-1,-1} \\
& -32(2z^2 - 14z + 5)H_{0,0,0,0,-1} - 32(40z^2 + 54z - 1)H_{0,0,0,0,1} + z^2 [32H_{0,-1}^2 + 1280H_{0,0,0,0,1}] \\
& + (4z^2 + 16z - 1) [-64H_{0,0,0,-1,1} - 64H_{0,0,0,1,-1}] + (8z^2 - 4z + 7) [64H_{0,0,0,-1,1} \\
& + 64H_{0,0,0,1,-1}] - 128z(11z + 7)H_{0,0,0,1,1} + 64(35z^2 + 97z + 32)H_{0,0,0,1,1} \\
& + (6z^2 + 14z + 1) [32H_{0,-1}H_{0,0,1} - 32H_{0,0,1,0,-1}] + (4z^2 + 3) [-64H_{0,-1}H_{0,0,1} \\
& - 128H_0H_{0,-1,-1,-1} - 128H_{0,-1,-1,0,1} + 64H_{0,0,1,0,-1}] + 64(5z^2 + 29z + 11)H_{0,0,1,0,1} \\
& - 128(18z^2 - 2z + 7)H_{0,0,1,1,1} + z(z + 1) [64H_{0,0,1}H_0^2 + 128H_{-1}H_{0,1}H_0 + (-128H_{0,-1,1} \\
& - 128H_{0,1,-1} + 384H_{0,0,1,1})H_0 - 384H_{0,0,1}\zeta_2 + 768H_{0,0,1,1,1}] \\
& - 64(9z^2 - 7z + 5)H_{0,1,0,1,1} + (2z^2 + 2z + 1) [-16H_0H_{-1}^4 + [\frac{64}{3}H_0^2 - \frac{160}{3}\zeta_2 \\
& + 64H_{0,-1} - \frac{64}{3}H_{0,1}]H_{-1}^3 + [-\frac{8}{3}H_0^3 + [40\zeta_2 - 64H_{0,-1} \\
& + 32H_{0,1}]H_0 + 160\zeta_3 - 192H_{0,-1,-1} - 64H_{0,-1,1} - 96H_{0,0,1} - 64H_{0,1,-1}]H_{-1}^2 + [-\frac{4}{3}H_0^4 \\
& + [36\zeta_2 - 16H_{0,-1} - 32H_{0,1}]H_0^2 + [-64\zeta_3 + 128H_{0,-1,-1} - 64H_{0,-1,1} + 96H_{0,0,-1} \\
& + 64H_{0,0,1} - 64H_{0,1,-1} - 64H_{0,1,1}]H_0 - \frac{376}{5}\zeta_2^2 - 16H_{0,-1}\zeta_2 + 96H_{0,1}\zeta_2 + 384H_{0,-1,-1,-1} \\
& + 128H_{0,-1,-1,1} + 128H_{0,-1,0,1} + 128H_{0,-1,1,-1} + 192H_{0,0,-1,1} - 160H_{0,0,0,-1} \\
& - 32H_{0,0,0,1} + 192H_{0,0,1,-1} + 128H_{0,0,1,1} + 128H_{0,1,-1,-1}]H_{-1} - 96H_{0,-1,1}\zeta_2 - 96H_{0,1,-1}\zeta_2 \\
& + H_0^2 [32H_{0,-1,1} + 32H_{0,1,-1}] + H_0 [64H_{0,-1,-1,1} - 64H_{0,-1,0,1} \\
& + 64H_{0,-1,1,-1} + 64H_{0,-1,1,1} - 96H_{0,0,-1,-1} - 64H_{0,0,-1,1} \\
& - 64H_{0,0,1,-1} + 64H_{0,1,-1,-1} + 64H_{0,1,-1,1} + 64H_{0,1,1,-1}] - 384H_{0,-1,-1,-1,-1} \\
& - 128H_{0,-1,-1,-1,1} - 128H_{0,-1,-1,1,-1} - 128H_{0,-1,0,-1,1} - 128H_{0,-1,0,1,-1} - 128H_{0,-1,1,-1,-1} \\
& - 192H_{0,0,-1,-1,1} + 64H_{0,0,-1,0,1} - 192H_{0,0,-1,1,-1} - 128H_{0,0,-1,1,1} - 192H_{0,0,1,-1,-1} \\
& - 128H_{0,0,1,-1,1} - 128H_{0,0,1,1,-1} - 128H_{0,1,-1,-1,-1}] - 64(z^2 + z + 1)H_{0,1,1,1,1} - \frac{640(z-1)}{9z}] \\
& + C_F^2 T_F \left[\frac{1}{3}(20z^2 - 7z - 1)H_0^4 - \frac{1}{3}(32z^2 + 2z - 13)H_0^3 + \frac{16}{3}z(z+1)\zeta_2H_0^3 \right. \\
& + \frac{4}{3}(20z^2 - 28z + 3)H_1H_0^3 + (2z - 1) \left[\frac{2\zeta_2}{3} + 8H_{0,1} \right] H_0^3 + 4(10z^2 - 12z + 1)H_1^2H_0^2 \\
& + 3z(176z - 67)H_0^2 + 8z(3z + 4)\zeta_2H_0^2 - 2(24z^2 - 19z - 4)\zeta_2H_0^2 \\
& - \frac{4}{3}(112z^2 - 34z + 17)\zeta_3H_0^2 - 2(16z^2 - 46z - 1)H_1H_0^2 + 4(6z - 7)H_{0,1}H_0^2 \\
& - 8(12z^2 + 2z - 1)H_{0,0,1}H_0^2 + 32(5z^2 - 4z + 2)H_{0,1,1}H_0^2 \\
& - 8(4z^2 + 2z + 1)\zeta_2^2H_0 - \frac{48}{5}(44z^2 - 14z + 7)\zeta_2^2H_0 - 16(z - 1)(2z - 3)H_1^2H_0 \\
& - 32(4z^2 - 2z + 1)H_{0,1}^2H_0 - 2(160z^2 + 105z + 59)H_0 + 64z\zeta_2H_0 - 4z(11z - 4)\zeta_2H_0 \\
& \left. - 4(24z^2 - 20z + 1)H_1\zeta_2H_0 + 8(36z^2 - 38z + 19)H_{0,1}\zeta_2H_0 - 8(32z^2 - 19z + 1)\zeta_3H_0 \right]
\end{aligned}$$

$$\begin{aligned}
& +4(264z^2 - 233z - 6)H_1H_0 - 8(14z + 11)H_{0,1}H_0 - 8(20z^2 - 14z - 9)H_{0,0,1}H_0 \\
& +32(8z^2 - 18z - 1)H_{0,1,1}H_0 + z^2 \left[64H_{0,0,1,1} - 32H_{0,-1}\zeta_2 \right] H_0 + (16z^2 - 12z - 5)H_1^4 \\
& + \frac{8}{3}(24z^2 - 40z + 7)H_1^3 + 562z^2 - \frac{8}{5}(255z^2 - 140z + 13)\zeta_2^2 + 2(152z^2 - 137z + 36)H_1^2 \\
& - 8(10z^2 - 16z + 1)H_{0,1}^2 - 718z + 4(4z^2 - 20z + 7)H_1^2\zeta_2 - \frac{1}{2}(264z^2 + 1078z - 311)\zeta_2 \\
& - 4(11z^2 + 36z - 21)H_1\zeta_2 + 4(32z^2 + 22z + 23)H_{0,1}\zeta_2 + 16(6z^2 + 2z + 1)H_{0,0,-1}\zeta_2 \\
& - 40(4z^2 - 10z + 5)H_{0,0,1}\zeta_2 - 32(5z^2 - 8z + 4)H_{0,1,1}\zeta_2 + (z + 1)(3z + 2) \left[-8\zeta_2^2 - 16H_{-1}H_0\zeta_2 \right. \\
& \left. + 16H_{0,-1}\zeta_2 \right] + (2z^2 + 2z + 1) \left[16H_0\zeta_2H_{-1}^2 + \left[16\zeta_2^2 - 8H_0^2\zeta_2 - 32H_{0,-1}\zeta_2 \right] H_{-1} + 32H_{0,-1,-1}\zeta_2 \right] \\
& - \frac{2}{3}(384z^2 - 862z + 277)\zeta_3 - 16(5z^2 + 2z + 1)\zeta_2\zeta_3 - 4(112z^2 - 271z - 24)H_{0,1} \\
& + \frac{16}{3}(19z^2 - 8z + 4)\zeta_2\zeta_3 - \frac{32}{3}(24z^2 - 34z + 1)H_1\zeta_3 + \frac{32}{3}(14z^2 - 30z + 15)H_{0,1}\zeta_3 \\
& - 4(80z^2 - 111z + 41)H_1 + (z - 1)(3z + 1) \left[\frac{16}{3}H_0H_1^3 - 16\zeta_2 \right] \\
& + L_M^3 \left[-\frac{64}{3}H_{0,1}z^2 + 8H_0z - \frac{4}{3}(8z^2 - 2z + 1)H_0^2 + \frac{2}{3}(2z - 11) + \frac{32}{3}(4z^2 - 2z + 1)\zeta_2 \right. \\
& \left. + \frac{16}{3}(4z - 1)H_1 + \gamma_{ag}^0 \left[\frac{8}{3}H_1^2 + \frac{8}{3}H_0H_1 \right] \right] + 4(4z + 39)H_{0,0,1} + (5z^2 - 8z + 1) \left[64H_1H_{0,0,1} \right. \\
& \left. - 32H_0H_1H_{0,1} \right] + (6z^2 + 2z - 1) \left[32H_{0,1}H_{0,0,1} - 64\zeta_5 \right] + 8(56z^2 - 95z + 29)H_{0,1,1} \\
& + 32(4z^2 - 6z + 3)H_{0,1}H_{0,1,1} + L_M^2 \left[-\frac{32}{3}z(z + 1)H_0^3 - 16z(3z + 4)H_0^2 - 4(8z^2 - 2z + 1)H_0^2 \right. \\
& - 128zH_0 - 2(100z^2 - 90z + 27)H_0 + 16(4z^2 + 2z + 1)\zeta_2H_0 - 16(4z^2 - 8z + 1)H_1H_0 \\
& + 64z^2H_{0,-1}H_0 + 8z^2 - 4(16z^2 - 20z + 1)H_1^2 + 20z + 32(z - 1)(3z + 1) \\
& + 16z(5z - 2)\zeta_2 + 32(5z^2 + 2z + 1)\zeta_3 - 8(25z^2 - 37z + 17)H_1 \\
& + \gamma_{ag}^0 \left[4H_1^3 + 12H_0H_1^2 + 4H_0^2H_1 - 16\zeta_2H_1 - 4\zeta_3 \right] + (z + 1)(3z + 2) \left[16\zeta_2 \right. \\
& \left. + 32H_{-1}H_0 - 32H_{0,-1} \right] + (4z^2 - 2z + 1) \left[H_0 \left[32\zeta_2 - 32H_{0,1} \right] - \frac{8}{3}H_0^3 \right] \\
& - 16(4z^2 + z + 1)H_{0,1} + (2z^2 + 2z + 1) \left[-32H_0H_{-1}^2 + \left[16H_0^2 - 32\zeta_2 \right. \right. \\
& \left. \left. + 64H_{0,-1} \right] H_{-1} - 64H_{0,-1,-1} \right] - 32(6z^2 + 2z + 1)H_{0,0,-1} + 16(8z^2 - 6z + 3)H_{0,0,1} \\
& - 16(2z - 1)H_{0,1,1} + 5 \left] + (4z - 1) \left[8H_1^2H_{0,1} - 32H_1H_{0,1,1} \right] + 16(30z^2 - 22z - 5)H_{0,0,0,1} \\
& + (8z^2 - 2z + 1) \left[\frac{1}{15}H_0^5 + 48H_{0,0,0,1}H_0 \right] - 32(8z^2 - 23z - 2)H_{0,0,1,1} + 8(36z^2 - 44z \\
& - 17)H_{0,1,1,1} + L_M \left[-\frac{1}{3}(24z^2 + 2z + 3)H_0^4 - \frac{4}{3}(40z^2 + 23z + 4)H_0^3 - 4(112z^2 \right. \\
& \left. + 32z + 25)H_0^2 + 16(2z - 1)(4z - 1)\zeta_2H_0^2 + 32(2z^2 + 1)\zeta_2H_0^2 + 32(z + 1)(5z + 4)H_{-1}H_0^2 \right. \\
& - 4(16z - 11)H_1H_0^2 + 16(6z^2 + 2z + 3)H_{0,-1}H_0^2 - 8(4z^2 - 6z - 1)H_{0,1}H_0^2 \\
& - 8(6z^2 + 2z - 7)H_1^2H_0 - 4(94z^2 + 79z + 67)H_0 + 16(8z^2 + 2z + 11)\zeta_2H_0 \\
& + 16(10z^2 - 18z - 7)\zeta_2H_0 + 32(8z^2 - 8z - 1)\zeta_3H_0 + 16(12z^2 - 6z + 7)\zeta_3H_0 \\
& \left. - 4(184z^2 - 214z + 63)H_1H_0 - 32(2z^2 + 16z - 3)H_{0,-1}H_0 - 8(12z^2 - 46z + 17)H_{0,1}H_0 \right]
\end{aligned}$$

$$\begin{aligned}
& -128H_{0,-1,-1}H_0 - 16(8z^2 + 2z + 7)H_{0,0,1}H_0 + 32(14z^2 - 10z + 5)H_{0,1,1}H_0 \\
& -16(z-1)(5z+1)H_1^3 + \frac{16}{5}(12z^2 - 30z + 11)\zeta_2^2 - \frac{8}{5}(156z^2 - 154z + 85)\zeta_2^2 \\
& -4(109z^2 - 128z + 29)H_1^2 - 32(2z^2 + 2z - 1)H_{0,-1}^2 - 32(4z^2 - 2z + 1)H_{0,1}^2 \\
& + \frac{1}{2}(-324z^2 + 1134z - 959) + 8(99z^2 - 56z - 14)\zeta_2 + 16(10z^2 - 2z - 7)H_1\zeta_2 \\
& -64H_{0,-1}\zeta_2 - 64H_{0,1}\zeta_2 + 16(11z^2 - 17z + 23)\zeta_3 \\
& + 16(49z^2 - 16z - 20)\zeta_3 - 8(47z^2 - 20z - 29)H_1 + (5z^2 + 23z + 20) \left[16\zeta_2 \right. \\
& \left. + 32H_{-1}H_0 - 32H_{0,-1} \right] - 4(34z^2 + 10z - 11)H_{0,1} - 64(z-1)(3z-2)H_1H_{0,1} \\
& + (z+1)^2 \left[-128H_0H_{-1}^2 + \left[256H_{0,-1} - 128\zeta_2 \right] H_{-1} - 256H_{0,-1,-1} \right] \\
& -64(3z^2 - 7z + 7)H_{0,0,-1} + (2z^2 + 2z + 1) \left[-32H_{-1}^2H_0^2 - 96H_{0,0,-1}H_0 \right. \\
& \left. + H_{-1} \left[\frac{16}{3}H_0^3 + 64H_{0,-1}H_0 \right] \right] - 24(4z^2 + 12z - 5)H_{0,0,1} + 16(z+10)H_{0,1,1} \\
& -32(2z^2 - 10z + 1)H_{0,0,0,-1} + 64(5z^2 - 2z + 4)H_{0,0,0,1} - 16(16z^2 - 10z + 5)H_{0,0,1,1} \\
& + \gamma_{qg}^0 \left[2H_1^4 + 6H_0^2H_1^2 + \left[16H_{0,1} - 24\zeta_2 \right] H_1^2 + \frac{8}{3}H_0^3H_1 + \left[-56\zeta_3 + 32H_{0,0,-1} - 24H_{0,0,1} \right. \right. \\
& \left. \left. - 24H_{0,1,1} \right] H_1 - 16H_{0,-1}H_{0,1} + H_0 \left[\frac{8}{3}H_1^3 + (-48\zeta_2 - 16H_{0,-1} + 24H_{0,1})H_1 + 16H_{0,-1,1} \right. \right. \\
& \left. \left. + 16H_{0,1,-1} \right] + 16H_{0,-1,0,1} + 8H_{0,1,1,1} \right] + \gamma_{qg}^0 \left[-\frac{1}{3}H_1^5 + \left[\frac{16}{3}H_{0,1} - 8\zeta_2 \right] H_1^3 - \frac{2}{3}H_0^3H_1^2 \right. \\
& \left. + \left[\frac{28\zeta_3}{3} - 24H_{0,0,1} - 16H_{0,1,1} \right] H_1^2 - \frac{1}{3}H_0^4H_1 + 12H_0^2H_{0,1}H_1 + \left[\frac{264}{5}\zeta_2^2 - 16H_{0,1}\zeta_2 \right. \right. \\
& \left. \left. + 16H_{0,1}^2 \right] H_1 + 96 \ln(2)\zeta_2 + H_0 \left[-\frac{1}{3}H_1^4 + \left[16H_{0,1} - 18\zeta_2 \right] H_1^2 + \left[\frac{112\zeta_3}{3} - 24H_{0,0,1} \right. \right. \\
& \left. \left. - 48H_{0,1,1} \right] H_1 + 40H_{0,1,1,1} \right] \right] - 16(24z^2 - 14z + 7)H_{0,0,0,0,1} - 32(26z^2 + 10z - 5)H_{0,0,0,1,1} \\
& -96(4z^2 + 2z - 1)H_{0,0,1,0,1} - 16(28z^2 - 54z + 27)H_{0,0,1,1,1} - 32(8z^2 - 14z + 7)H_{0,1,0,1,1} \\
& -16(16z^2 - 18z + 9)H_{0,1,1,1,1} + 133 \left] + C_F T_F^2 \left[\frac{2}{9}(8z^2 + 32z - 13)H_0^4 + \frac{8}{9}(20z^2 + 93z \right. \right. \\
& \left. \left. - 43)H_0^3 - \frac{16}{3}(16z^2 - 135z + 58)H_0^2 + \frac{4}{3}(32z^2 + 40z - 29)\zeta_2H_0^2 + \frac{16}{3}(10z^2 - 12z + 1)H_1H_0^2 \right. \right. \\
& \left. \left. - \frac{64}{3}(z-1)^2H_{0,1}H_0^2 + \frac{64}{3}(29z^2 + 9z - 69)H_0 + \frac{4}{9}(24z^2 - 136z - 559)\zeta_2H_0 \right. \right. \\
& \left. \left. - \frac{16}{9}(72z^2 - 40z + 29)\zeta_3H_0 - \frac{32}{3}(16z^2 - 27z + 10)H_1H_0 - \frac{64}{3}(5z^2 - 4z + 2)H_{0,1}H_0 \right. \right. \\
& \left. \left. - \frac{128}{3}(z^2 + 2z - 1)H_{0,0,1}H_0 + \frac{32}{9}(z-1)(3z+1)H_1^3 + L_M^3 \left[-\frac{16}{3}(2z-1)H_0^2 \right. \right. \right. \\
& \left. \left. - \frac{16}{9}(24z^2 - 8z - 5)H_0 + \frac{16(62z^3 - 111z^2 + 75z - 8)}{27z} + \frac{32}{9}\gamma_{qg}^0H_1 \right] \right. \\
& \left. - \frac{16}{15}(218z^2 - 122z + 61)\zeta_2^2 + \frac{16}{3}(12z^2 - 17z + 4)H_1^2 - \frac{2(9794z^3 - 20814z^2 + 12015z + 160)}{9z} \right. \\
& \left. - \frac{2(608z^3 - 6636z^2 + 4467z + 256)\zeta_2}{27z} + \frac{80}{9}(4z^2 - 4z - 1)H_1\zeta_2 - \frac{16(170z^3 - 183z^2 + 147z - 8)\zeta_3}{27z} \right. \\
& \left. + (2z-1) \left[\frac{4}{15}H_0^5 + 8\zeta_2H_0^3 + \frac{16}{3}\zeta_3H_0^2 \right] + \frac{32}{3}(20z^2 - 17z - 1)H_1 + \frac{32}{3}(28z^2 - 49z + 16)H_{0,1} \right.
\end{aligned}$$

$$\begin{aligned}
& +L_M^2 \left[-\frac{32}{3}(2z-1)H_0^3 - \frac{64}{3}(z+1)(3z-2)H_0^2 + \frac{8}{9}(144z^2 + 166z + 325)H_0 \right. \\
& - \frac{4(2296z^3 - 24z^2 - 1677z - 208)}{27z} - \frac{32}{9}(8z^2 - 8z - 5)H_1 + \gamma_{qg}^0 \left[4H_1^2 + \frac{16}{3}H_0H_1 - 8\zeta_2 \right. \\
& \left. \left. + \frac{8}{3}H_{0,1} \right] \right] + \frac{32}{3}(10z^2 - 4z + 7)H_{0,0,1} + \frac{64}{3}(3z^2 - 6z - 1)H_{0,1,1} \\
& +L_M \left[-\frac{20}{3}(2z-1)H_0^4 - \frac{8}{9}(16z^2 + 80z - 55)H_0^3 + \frac{4}{9}(296z^2 - 260z + 931)H_0^2 \right. \\
& - \frac{8}{27}(240z^2 - 2203z - 4228)H_0 + \frac{64}{9}(23z^2 - 26z + 4)H_1H_0 - \frac{64}{3}(4z^2 - 6z + 3)H_{0,1}H_0 \\
& - \frac{32}{9}(2z^2 + z - 8)H_1^2 + \frac{2(6892z^3 - 85110z^2 + 85137z - 1600)}{81z} + \frac{64}{9}(2z^2 - 11z - 8)\zeta_2 \\
& + \frac{32}{3}(6z^2 + 2z - 1)\zeta_3 - \frac{16}{27}(244z^2 - 271z + 56)H_1 - \frac{64}{9}(25z^2 - 37z - 4)H_{0,1} \\
& \left. + \frac{256}{3}(z-1)^2H_{0,0,1} + \gamma_{qg}^0 \left[\frac{4}{3}H_1^3 - \frac{16}{3}H_0^2H_1 + 8H_{0,1,1} \right] \right] + \frac{64}{3}(10z^2 + 2z - 1)H_{0,0,0,1} \\
& + \gamma_{qg}^0 \left[-\frac{2}{9}H_1^4 + \left[\frac{16}{3}H_{0,1} - \frac{26\zeta_2}{3} \right] H_1^2 - \frac{8}{9}H_0^3H_1 + \left[\frac{160\zeta_3}{9} - \frac{64}{3}H_{0,0,1} - \frac{64}{3}H_{0,1,1} \right] H_1 \right. \\
& \left. + \frac{32}{3}H_{0,1}^2 - 12H_{0,1}\zeta_2 + H_0 \left[H_1 \left[\frac{32}{3}H_{0,1} - \frac{16\zeta_2}{3} \right] - \frac{64}{3}H_{0,1,1} \right] + \frac{80}{3}H_{0,1,1,1} \right] \\
& +C_F T_F^2 N_F \left[-\frac{4}{9}(z-2)(6z+1)H_0^4 + \frac{4}{27}(404z^2 - 54z + 69)H_0^3 \right. \\
& - \frac{4}{27}(3112z^2 - 1329z - 210)H_0^2 - \frac{8(94z^3 - 234z^2 + 159z - 16)H_1H_0^2}{9z} \\
& + \frac{16}{3}(6z^2 + 4z - 11)H_{0,1}H_0^2 - \frac{8}{3}(14z^2 - 5z - 11)\zeta_2H_0^2 \\
& + \frac{8}{81}(27824z^2 + 4929z + 2631)H_0 + \frac{16(1556z^3 - 1539z^2 + 72z - 80)H_1H_0}{27z} \\
& - \frac{32(101z^3 + 222z^2 - 39z + 8)H_{0,1}H_0}{9z} - \frac{32}{3}(2z^2 + 28z - 17)H_{0,0,1}H_0 \\
& + \frac{8}{9}(240z^2 + 166z + 193)\zeta_2H_0 + \frac{32}{9}(78z^2 - 133z - 28)\zeta_3H_0 + \frac{16}{9}(z-1)(3z+1)H_1^3 \\
& \left. + L_M^3 \left[-\frac{16}{3}(2z-1)H_0^2 - \frac{32}{9}(6z^2 - z - 4)H_0 + \frac{8(124z^3 - 258z^2 + 159z - 16)}{27z} + \frac{8}{9}\gamma_{qg}^0H_1 \right] \right] \\
& + \frac{8}{3}(12z^2 - 17z + 4)H_1^2 - \frac{64}{15}(2z^2 + 58z - 11)\zeta_2^2 - \frac{2}{27}(7280z^3 - 5646z^2 - 555z - 368)\frac{\zeta_2}{z} \\
& - \frac{4(221158z^3 - 226026z^2 + 17163z - 5248)}{243z} - \frac{8}{27}(3784z^3 + 2046z^2 + 1095z - 16)\frac{\zeta_3}{z} \\
& + \frac{16}{3}(20z^2 - 17z - 1)H_1 - \frac{16(1448z^3 - 1341z^2 + 18z - 80)H_{0,1}}{27z} \\
& + \frac{16(498z^3 + 654z^2 + 3z + 16)H_{0,0,1}}{9z} + \frac{32}{3}(3z^2 - 6z - 1)H_{0,1,1} \\
& - \frac{32}{3}(14z^2 - 74z + 19)H_{0,0,0,1} + L_M^2 \left[\frac{32}{3}(2z-1)H_0^3 + \frac{16}{3}(2z+3)(4z-3)H_0^2 \right. \\
& \left. - \frac{8}{9}(160z^2 + 146z + 305)H_0 + \frac{4(1000z^3 + 1356z^2 - 2247z - 208)}{27z} + \frac{32}{9}(4z^2 - 4z + 5)H_1 \right]
\end{aligned}$$

$$\begin{aligned}
& +\gamma_{ag}^0 \left[\frac{4}{3}H_1^2 + \frac{8}{3}H_{0,1} - \frac{8\zeta_2}{3} \right] + \frac{16}{9}(2z^2 - 2z - 5)H_1\zeta_2 + \gamma_{ag}^0 \left[-\frac{1}{9}H_1^4 + \frac{8}{3}H_{0,1}H_1^2 \right. \\
& - \frac{4}{9}H_0^3H_1 + \left[-\frac{32}{3}H_{0,0,1} - \frac{32}{3}H_{0,1,1} \right]H_1 + \frac{88}{9}\zeta_3H_1 + \frac{16}{3}H_{0,1}^2 + H_0 \left[\frac{16}{3}H_1H_{0,1} - \frac{32}{3}H_{0,1,1} \right] \\
& + \frac{40}{3}H_{0,1,1,1} + \left[-4H_1^2 - \frac{4}{3}H_0H_1 - \frac{20}{3}H_{0,1} \right]\zeta_2 \left. \right] + L_M \left[-\frac{16}{3}(6z^2 + 3z - 7)H_0^3 \right. \\
& + \frac{8}{9}(290z^2 + 194z + 377)H_0^2 - \frac{16}{27}(1244z^2 - 1226z - 1319)H_0 + \frac{32}{3}(2z^2 + 8z - 13)H_{0,1}H_0 \\
& - \frac{16(62z^3 - 202z^2 + 149z - 16)H_1H_0}{9z} + \frac{2(20960z^3 - 77478z^2 + 62367z - 4256)}{81z} \\
& + \frac{16}{9}(5z^2 - 8z + 13)H_1^2 - \frac{32}{27}(68z^2 - 77z + 28)H_1 + \frac{16(72z^3 - 194z^2 + 175z - 16)H_{0,1}}{9z} \\
& - \frac{32}{3}(10z^2 + 10z - 23)H_{0,0,1} + \gamma_{ag}^0 \left[\frac{8}{9}H_1^3 - 4H_0^2H_1 + \frac{16}{3}H_{0,1,1} \right] - \frac{32}{9}(5z^2 + 4z + 13)\zeta_2 \\
& + \frac{64}{3}(10z^2 - 9)\zeta_3 + (2z - 1) \left[-\frac{20}{3}H_0^4 + 64H_{0,0,1}H_0 + 128\zeta_3H_0 + \frac{384\zeta_2^2}{5} - 192H_{0,0,0,1} \right] \\
& + (2z - 1) \left[-\frac{4}{15}H_0^5 - 8\zeta_2H_0^3 + 32H_{0,0,1}H_0^2 + \frac{208}{3}\zeta_3H_0^2 - 128H_{0,0,0,1}H_0 + 128H_{0,0,0,0,1} - 128\zeta_5 \right] \\
& + C_{ACFTF} \left[-\frac{1}{18}(76z^2 - 178z + 11)H_0^4 - \frac{1}{27}(4496z^2 + 1656z + 993)H_0^3 - \frac{8}{3}z(9z + 5)\zeta_2H_0^3 \right. \\
& + \frac{8}{3}(13z^2 + 8z + 3)\zeta_2H_0^3 - \frac{4}{3}(8z^2 + 4z - 1)H_{-1}H_0^3 - \frac{4(274z^3 - 238z^2 - 7z - 24)H_1H_0^3}{9z} \\
& + \frac{16}{3}(2z^2 + 8z + 5)H_{0,1}H_0^3 + 4(12z^2 + 16z + 1)H_{-1}^2H_0^2 - \frac{2(166z^3 - 156z^2 - 9z - 16)H_1^2H_0^2}{3z} \\
& + \frac{1}{27}(36352z^2 + 3459z - 48)H_0^2 + 4z(13z + 4)\zeta_2H_0^2 - \frac{1}{3}(76z^2 + 110z - 1)\zeta_2H_0^2 \\
& - 32z(2z + 1)\zeta_3H_0^2 + \frac{8}{3}(36z^2 + 2z + 11)\zeta_3H_0^2 - 8(5z^2 + 7z - 4)H_{-1}H_0^2 \\
& + \frac{2(952z^3 - 1740z^2 + 969z - 184)H_1H_0^2}{9z} - 4(4z + 1)H_{0,-1}H_0^2 \\
& - \frac{4(32z^3 + 122z^2 - 25z + 24)H_{0,1}H_0^2}{3z} - 32(z + 1)^2H_{0,-1,-1}H_0^2 + 96(z^2 - 2z - 1)H_{0,0,1}H_0^2 \\
& - 8(36z^2 - 46z + 11)H_{0,1,1}H_0^2 - \frac{8}{3}(z + 2)(3z - 2)H_1^3H_0 - \frac{32}{5}(4z^2 - 8z - 1)\zeta_2^2H_0 + \frac{48}{5}(12z^2 \\
& - 49z - 3)\zeta_2^2H_0 + \frac{8(229z^3 - 297z^2 + 63z - 40)H_1^2H_0}{9z} + 8(2z - 5)(8z + 1)H_{0,1}^2H_0 \\
& - \frac{2}{81}(122248z^2 - 67419z + 15057)H_0 - 96z(z + 2)\zeta_2H_0 - \frac{2}{9}(724z^2 + 1757z + 629)\zeta_2H_0 \\
& - 8z(13z + 19)H_{-1}\zeta_2H_0 + 8(32z^2 + 36z + 1)H_{-1}\zeta_2H_0 + \frac{4(138z^3 - 122z^2 - 23z - 16)H_1\zeta_2H_0}{3z} \\
& + 16(z^2 + 4z + 2)H_{0,-1}\zeta_2H_0 - 8(26z^2 - 18z + 21)H_{0,1}\zeta_2H_0 - \frac{4}{9}(736z^2 + 1826z + 149)\zeta_3H_0 \\
& + 16z(13z + 9)\zeta_3H_0 - \frac{4(888z^3 - 695z^2 + 380z - 396)H_1H_0}{9z} - 16(7z^2 - z + 4)H_{0,-1}H_0 \\
& + \frac{32(9z^3 + 231z^2 - 99z + 23)H_{0,1}H_0}{9z} + \frac{8(382z^3 - 368z^2 + 7z - 32)H_1H_{0,1}H_0}{3z} \\
& - 64z(2z - 3)H_{0,0,1}H_0 + \frac{16(189z^3 - 8z^2 - 35z + 12)H_{0,0,1}H_0}{3z}
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(792z^3 - 700z^2 + 5z - 4)H_{0,1,1}H_0}{3z} + 16(8z^2 + 2z + 1)H_{0,0,0,-1}H_0 \\
& + 64z(6z + 7)H_{0,0,0,1}H_0 - 64(11z^2 - 3z - 2)H_{0,0,0,1}H_0 + 32(2z + 1)(8z + 9)H_{0,0,1,1}H_0 \\
& + 32(20z^2 - 18z + 9)H_{0,1,1,1}H_0 - \frac{2(174z^3 - 137z^2 - 32z - 4)H_1^4}{9z} \\
& - \frac{4(221z^3 - 262z^2 + 13z + 4)H_1^3}{3z} - \frac{2}{5}(26z^2 - 102z - 23)\zeta_2^2 + \frac{2}{15}(7754z^2 - 9554z + 31)\zeta_2^2 \\
& - \frac{4(990z^3 - 1029z^2 + 151z - 17)H_1^2}{3z} + \frac{8(275z^3 - 200z^2 + 19z - 16)H_{0,1}^2}{3z} \\
& - \frac{96828z^3 - 132146z^2 - 16891z + 8388}{81z} - \frac{8(47z^3 - 45z^2 + 6z - 4)H_1^2\zeta_2}{z} \\
& - 8(141z^2 + 46z + 9)\zeta_2 + \frac{(5072z^3 - 69594z^2 + 5289z - 4320)\zeta_2}{54z} \\
& - \frac{4(2302z^3 - 2680z^2 + 293z - 216)H_1\zeta_2}{9z} - \frac{4(126z^3 - 218z^2 + 121z - 16)H_{0,1}\zeta_2}{3z} \\
& + 16(6z^2 - 2z - 1)H_{0,-1,-1}\zeta_2 + 8(10z^2 - 2z - 1)H_{0,0,-1}\zeta_2 \\
& - 8(22z^2 + 46z - 35)H_{0,0,1}\zeta_2 + 16(48z^2 - 26z + 31)H_{0,1,1}\zeta_2 \\
& + (4z^2 + 2z + 1)\left[16H_0H_{0,-1}\zeta_2 - 32H_{0,0,-1}\zeta_2\right] - 4(59z^2 + 65z + 20)\zeta_3 \\
& + \frac{2}{3}(3526z^2 - 3590z + 323)\zeta_3 - 8(7z^2 - 12z + 2)\zeta_2\zeta_3 + \frac{4}{3}(134z^2 - 374z - 77)\zeta_2\zeta_3 \\
& + 12(36z^2 + 56z + 5)H_{-1}\zeta_3 - 4(52z^2 + 88z + 9)H_{-1}\zeta_3 + \frac{8(1666z^3 - 1454z^2 - 17z - 128)H_1\zeta_3}{9z} \\
& + 8(8z^2 - 22z - 11)H_{0,-1}\zeta_3 - 32(12z^2 + 6z + 13)H_{0,1}\zeta_3 - 4(12z^2 + 6z - 5)\zeta_5 \\
& - 4(28z^2 - 34z - 17)\zeta_5 - \frac{4(76964z^3 - 85257z^2 + 9162z - 2381)H_1}{81z} \\
& + (z + 1)(2z + 5)\left[16H_{0,-1} - 16H_{-1}H_0\right] + \frac{8(84z^3 - 70z^2 + 5z - 8)H_1^2H_{0,1}}{3z} \\
& + 32z(16z + 23)H_{0,1} + \frac{4(2084z^3 + 5661z^2 + 1260z - 1188)H_{0,1}}{27z} + \frac{16}{3}(z - 1)(15z + 1)H_1H_{0,1} \\
& + L_M^3\left[-\frac{16}{3}(4z + 1)H_0^2 + \frac{4}{9}(292z^2 - 94z + 11)H_0 + \frac{2}{3}(40z^2 - 20z - 31)\right. \\
& \left. - \frac{32}{3}(4z^2 - 10z - 1)\zeta_2 + \frac{8(146z^3 - 142z^2 + 5z - 16)H_1}{9z} + \gamma_{ag}^0\left[-\frac{16}{3}H_1^2 - \frac{16}{3}H_0H_1\right]\right. \\
& \left. - 32(2z + 1)H_{0,1}\right] + 16(19z^2 + 5z + 4)H_{0,0,-1} - 32(z - 17)zH_{0,0,1} \\
& - \frac{4(2012z^3 + 1014z^2 - 807z + 184)H_{0,0,1}}{9z} - \frac{8(764z^3 - 724z^2 + 11z - 64)H_1H_{0,0,1}}{3z} \\
& - 16(4z^2 - 50z - 23)H_{0,1}H_{0,0,1} + (z^2 - z + 4)\left[-16H_0H_{-1}^2 + \left[-48\zeta_2 + 32H_{0,-1} + 32H_{0,1}\right]H_{-1}\right. \\
& \left. - 32H_{0,-1,-1} - 32H_{0,-1,1} - 32H_{0,1,-1}\right] + (7z^2 + 5z + 4)\left[H_{-1}\left[32\zeta_2 - 32H_{0,1}\right]\right. \\
& \left. + 32H_{0,-1,1} + 32H_{0,1,-1}\right] - \frac{8(2105z^3 - 3075z^2 + 216z - 80)H_{0,1,1}}{9z} \\
& - \frac{8(336z^3 - 292z^2 + 23z - 32)H_1H_{0,1,1}}{3z} - 32(18z^2 - 10z + 11)H_{0,1}H_{0,1,1} \\
& + L_M^2\left[-\frac{8}{3}(13z^2 + 21z + 6)H_0^2 - 8z(5z - 6)H_0 + \frac{2}{9}(2468z^2 - 3050z - 587)H_0\right.
\end{aligned}$$

$$\begin{aligned}
& -16(4z^2 - 6z + 1)\zeta_2 H_0 + \frac{16(z-1)(55z^2 + 7z + 4)H_1 H_0}{3z} + 16(2z^2 - 10z - 1)H_{0,1}H_0 \\
& + \frac{8(90z^3 - 89z^2 + 4z - 4)H_1^2}{3z} - 4(z-1)(3z-1) + \frac{2380z^3 - 1636z^2 - 533z - 48}{3z} \\
& - \frac{4}{3}(70z^2 - 278z + 31)\zeta_2 - 8(18z^2 - 2z + 9)\zeta_3 + \frac{8(842z^3 - 937z^2 + 83z - 28)H_1}{9z} \\
& + 24z(z+1)H_0^2 + \gamma_{gg}^0 \left[-8H_1^3 - 12H_0H_1^2 - 2H_0^2H_1 + 24\zeta_2H_1 \right] - \frac{16(33z^3 + 26z^2 - 13z - 4)H_{0,1}}{3z} \\
& - 32(z^2 - 8z - 1)H_{0,0,1} + (2z^2 + 2z + 1) \left[-12\zeta_2 + 8\zeta_3 + 24H_{0,-1} - 16H_0H_{0,-1} \right. \\
& \left. + H_{-1} \left[8H_0^2 - 24H_0 - 32\zeta_2 + 32H_{0,1} \right] - 32H_{0,-1,1} + 16H_{0,0,-1} - 32H_{0,1,-1} \right] - 48(2z+1)H_{0,1,1} \\
& - 8(16z^2 + 20z + 1)H_{0,0,0,-1} - \frac{8(730z^3 - 260z^2 - 137z + 24)H_{0,0,0,1}}{3z} + (z-1)z \left[128H_1^3 \right. \\
& \left. + 512H_1^2 + 64H_0^2H_1 + \left[-384\zeta_2 - 224 \right] H_1 - 24H_{0,-1}\zeta_2 + H_0 \left[192H_1^2 + 512H_1 \right] + 256H_{0,0,0,1} \right] \\
& + (2z-1)(2z+1) \left[H_{-1} \left[16H_{0,0,1} - 16H_{0,0,-1} \right] - 16H_{0,-1,0,1} + 16H_{0,0,-1,-1} \right. \\
& \left. - 16H_{0,0,-1,1} - 16H_{0,0,1,-1} \right] + \gamma_{gg}^0 \left[\frac{2}{3}H_1^5 + \left[\frac{32\zeta_2}{3} - \frac{16}{3}H_{0,1} \right] H_1^3 \right. \\
& \left. + \frac{4}{3}H_0^3H_1^2 + \left[-\frac{80}{3}\zeta_3 + 68H_{0,0,1} + 20H_{0,1,1} \right] H_1^2 + \frac{1}{3}H_0^4H_1 + H_0^2 \left[-\zeta_2 - 20H_{0,1} \right] H_1 \right. \\
& \left. + \left[-\frac{494}{5}\zeta_2^2 + 48H_{0,1}\zeta_2 - 36H_{0,1}^2 + 16H_{0,0,0,1} - 24H_{0,0,1,1} \right] H_1 - 48\ln(2)\zeta_2 \right. \\
& \left. + H_0 \left[-\frac{4}{3}H_1^4 + \left[16\zeta_2 - 40H_{0,1} \right] H_1^2 + \left[-\frac{152}{3}\zeta_3 + 32H_{0,0,1} + 104H_{0,1,1} \right] H_1 \right] \right] \\
& - 128z(2z-5)H_{0,0,1,1} + \frac{8(740z^3 - 546z^2 - 93z - 48)H_{0,0,1,1}}{3z} \\
& + (4z^2 + 8z + 1) \left[(16\zeta_2 - 16H_{0,1})H_{-1}^2 + \left[-16H_0H_{0,-1} + 32H_{0,-1,1} + 32H_{0,1,-1} \right] H_{-1} \right. \\
& \left. + 16H_0H_{0,-1,-1} - 32H_{0,-1,-1,1} - 32H_{0,-1,1,-1} - 32H_{0,1,-1,-1} \right] + (8z^2 + 12z + 1) \left[-\frac{16}{3}H_0H_{-1}^3 \right. \\
& \left. + (-24\zeta_2 + 16H_{0,-1} + 16H_{0,1})H_{-1}^2 + \left[16H_0H_{0,1} - 32H_{0,-1,-1} - 32H_{0,-1,1} - 16H_{0,0,1} \right. \right. \\
& \left. \left. - 32H_{0,1,-1} \right] H_{-1} - 8H_{0,-1}\zeta_2 + H_0 \left[-16H_{0,-1,1} + 8H_{0,0,-1} - 16H_{0,1,-1} \right] \right. \\
& \left. + 32H_{0,-1,-1,-1} + 32H_{0,-1,-1,1} + 32H_{0,-1,1,-1} + 16H_{0,0,-1,1} + 16H_{0,0,1,-1} + 32H_{0,1,-1,-1} \right] \\
& + L_M \left[\frac{4}{3}(5z+2)H_0^4 - \frac{2}{3}(236z^2 - 2z + 11)H_0^3 - \frac{2}{9}(1578z^2 - 1883z - 353)H_0^2 \right. \\
& \left. + 16(2z^2 - 4z - 3)\zeta_2H_0^2 - 8(8z^2 + 2z + 3)\zeta_2H_0^2 - 8(6z^2 + 8z + 5)H_{-1}H_0^2 \right. \\
& \left. + 4(8z^2 + 12z + 1)H_{-1}H_0^2 - \frac{4(274z^3 - 360z^2 + 111z - 16)H_1H_0^2}{3z} \right. \\
& \left. + 8(2z^2 - 10z - 3)H_{0,-1}H_0^2 - 16(7z^2 - 10z + 2)H_{0,1}H_0^2 - \frac{8(97z^3 - 99z^2 + 6z - 16)H_1^2H_0}{3z} \right. \\
& \left. + \frac{4}{27}(4322z^2 - 10579z + 200)H_0 - 16(5z^2 + 13)\zeta_2H_0 + 8(128z^2 + 124z + 29)\zeta_2H_0 \right. \\
& \left. - 4(52z^2 + 10z + 33)\zeta_3H_0 + 4(52z^2 + 370z + 97)\zeta_3H_0 \right. \\
& \left. + \frac{4(450z^3 - 812z^2 + 415z - 208)H_1H_0}{9z} - 8(4z+1)H_{0,-1}H_0 - 16(4z^2 - 16z + 21)H_{0,-1}H_0 \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{8(108z^3 + 394z^2 - 101z + 16)H_{0,1}H_0}{3z} - 64(6z^2 + 4z + 1)H_{0,-1,-1}H_0 \\
& -16(26z^2 - 2z + 11)H_{0,0,-1}H_0 + 16(18z^2 - 22z + 15)H_{0,0,1}H_0 \\
& + (2z^2 - 10z + 1) \left[32H_{0,-1,1} + 32H_{0,1,-1} \right] H_0 - 16(8z^2 - 30z - 9)H_{0,1,1}H_0 \\
& + \frac{8}{9}(193z^2 - 166z - 16)H_1^3 - \frac{12}{5}(88z^2 + 6z + 75)\zeta_2^2 + \frac{12}{5}(88z^2 + 106z + 113)\zeta_2^2 \\
& + \frac{4(2365z^3 - 2510z^2 + 121z - 56)H_1^2}{9z} + 16(6z^2 + 2z - 1)H_{0,-1}^2 - 32(2z^2 + 3)H_{0,1}^2 \\
& + \frac{122748z^3 - 97910z^2 - 30697z - 1440}{54z} + 8(50z^2 - 19z - 37)\zeta_2 \\
& - \frac{8}{9}(1207z^2 - 1190z - 233)\zeta_2 - 16(5z^2 + 6z + 4)H_{-1}\zeta_2 \\
& + \frac{8(52z^3 - 78z^2 + 45z - 16)H_1\zeta_2}{3z} + 32(8z + 1)H_{0,-1}\zeta_2 \\
& + 16(4z^2 + 10z + 5)H_{0,-1}\zeta_2 + 32(6z^2 - 16z + 3)H_{0,1}\zeta_2 - 2(44z^2 - 184z + 235)\zeta_3 \\
& - \frac{2(3428z^3 - 3120z^2 - 825z - 192)\zeta_3}{3z} + \frac{8(8065z^3 - 9010z^2 + 1094z + 102)H_1}{27z} \\
& + (z - 1)z \left[-128H_1^2 - 128H_0H_1 \right] + (z + 1)(5z + 2) \left[16H_{-1}H_0 - 16H_{0,-1} \right] \\
& + (z + 1)(18z - 37) \left[16H_{-1}H_0 - 16H_{0,-1} \right] + (2z + 1) \left[-8H_{0,-1}H_0^2 - 16H_{0,-1}^2 \right] \\
& + \frac{4(1640z^3 - 1658z^2 - 215z + 208)H_{0,1}}{9z} + \frac{16(80z^3 - 90z^2 + 9z - 8)H_1H_{0,1}}{3z} \\
& + (z + 1)(5z + 9) \left[16H_0H_{-1}^2 - 32H_{0,-1}H_{-1} + 32H_{0,-1,-1} \right] \\
& + z^2 \left[H_0 \left[-64H_{0,-1,-1} - 64H_{0,0,-1} \right] - 64H_0^2 \right] - 8(8z^2 + 4z - 1)H_{0,0,-1} \\
& + 16(14z^2 - 24z + 47)H_{0,0,-1} + \frac{32(17z - 4)(2z^2 + 2z - 1)H_{0,0,1}}{3z} \\
& + (4z^2 + 8z + 1) \left[-8H_0H_{-1}^2 + \left[-24\zeta_2 + 16H_{0,-1} + 16H_{0,1} \right] H_{-1} - 16H_{0,-1,-1} - 16H_{0,-1,1} \right. \\
& \left. - 16H_{0,1,-1} \right] + (10z^2 + 20z + 13) \left[16H_{-1}H_{0,1} - 16H_{0,-1,1} - 16H_{0,1,-1} \right] \\
& - \frac{8}{3}(72z^2 + 290z + 11)H_{0,1,1} + z \left[128H_0H_{0,1} - 192H_{0,0,1} + 256H_{0,1,1} \right] \\
& + \gamma_{qg}^0 \left[-4H_1^4 + 6H_0^2H_1^2 + \left[32\zeta_2 - 24H_{0,1} \right] H_1^2 - \frac{2}{3}H_0^3H_1 + \left[168\zeta_3 - 48H_{0,0,-1} - 40H_{0,0,1} \right. \right. \\
& \left. \left. + 24H_{0,1,1} \right] H_1 + H_0 \left[4H_1^3 + \left[64\zeta_2 + 24H_{0,-1} - 8H_{0,1} \right] H_1 \right] + 24H_{0,-1}H_{0,1} \right] \\
& - 32(6z - 1)H_{0,-1,0,1} + 32(8z^2 - 6z + 3)H_{0,-1,0,1} + 16(8z^2 + 2z + 1)H_{0,0,0,-1} \\
& + 16(50z^2 + 2z + 31)H_{0,0,0,-1} - 64z(4z + 5)H_{0,0,0,1} - 16(18z^2 - 18z + 25)H_{0,0,0,1} \\
& - 32(10z^2 + 14z + 5)H_{0,0,1,1} + z(z + 1) \left[-128H_{0,1} + 128H_0H_{0,0,1} + 256H_{0,0,1,1} \right] \\
& + (2z^2 + 2z + 1) \left[\frac{160}{3}H_0H_{-1}^3 + (-72H_0^2 + 208\zeta_2 - 160H_{0,-1} - 128H_{0,1})H_{-1}^2 + \left[\frac{104}{3}H_0^3 \right. \right. \\
& \left. \left. + \left[-352\zeta_2 + 128H_{0,-1} + 64H_{0,1} \right] H_0 - 384\zeta_3 + 320H_{0,-1,-1} + 256H_{0,-1,1} + 32H_{0,0,-1} + 96H_{0,0,1} \right. \right. \\
& \left. \left. + 256H_{0,1,-1} + 64H_{0,1,1} \right] H_{-1} - 320H_{0,-1,-1,-1} - 256H_{0,-1,-1,1} - 256H_{0,-1,1,-1} - 64H_{0,-1,1,1} \right. \\
& \left. \left. - 32H_{0,0,-1,-1} - 96H_{0,0,-1,1} - 96H_{0,0,1,-1} - 256H_{0,1,-1,-1} - 64H_{0,1,-1,1} - 64H_{0,1,1,-1} \right] \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{8(282z^3 + 160z^2 + 25z - 48)H_{0,1,1,1}}{3z} + z \left[\frac{4}{15}H_0^5 - 64H_{0,1}H_0^2 + [-128H_{0,1} - 384H_{0,1,1}]H_0 \right. \\
& + 384H_{0,1}\zeta_2 - 768H_{0,1,1,1} \left. \right] + z^2 \left[-\frac{64}{3}H_0^3 + [256 - 32H_{0,0,-1}]H_0^2 - 224H_0 - 192H_{0,-1,-1}\zeta_2 \right. \\
& + 128H_{0,-1,0,1,1} \left. \right] + (2z + 1) \left[-\frac{8}{3}H_{0,-1}H_0^3 + [16H_{0,-1}^2 - 32H_{0,-1,0,1}]H_0 \right. \\
& + H_{0,-1} \left[72\zeta_3 - 64H_{0,-1,-1} - 32H_{0,0,-1} \right] - 64H_{0,-1,-1,0,1} + 128H_{0,-1,0,-1,-1} \\
& + 64H_{0,-1,0,1,1} + 256H_{0,0,-1,-1,-1} + 32H_{0,0,-1,0,-1} \left. \right] + 32(10z^2 + 6z + 3)H_{0,0,-1,0,1} \\
& - 64(5z^2 + 2z + 1)H_{0,0,0,-1,-1} - 32(6z^2 + 2z + 1)H_{0,0,0,0,-1} - 320z(2z + 3)H_{0,0,0,0,1} \\
& + 16(80z^2 + 22z - 1)H_{0,0,0,0,1} + (20z^2 + 14z + 7) \left[32H_{0,0,0,-1,1} + 32H_{0,0,0,1,-1} \right] \\
& + (4z^2 + 10z + 5) \left[-32H_{0,0,0,-1,1} - 32H_{0,0,0,1,-1} \right] + 128z(9z + 10)H_{0,0,0,1,1} \\
& - 16(60z^2 + 374z + 129)H_{0,0,0,1,1} + (5z^2 + 4z + 2) \left[64H_{0,0,1,0,-1} - 64H_{0,-1}H_{0,0,1} \right] \\
& + (4z^2 + 6z + 3) \left[32H_{0,-1}H_{0,0,1} + 64H_0H_{0,-1,-1,-1} + 64H_{0,-1,-1,0,1} - 32H_{0,0,1,0,-1} \right] \\
& - 16(110z + 53)H_{0,0,1,0,1} + 32(104z^2 - 8z + 55)H_{0,0,1,1,1} + z(z + 1) \left[-64H_{0,0,1}H_0^2 \right. \\
& - 128H_{-1}H_{0,1}H_0 + [128H_{0,-1,1} + 128H_{0,1,-1} - 384H_{0,0,1,1}]H_0 + 384H_{0,0,1}\zeta_2 \\
& - 128H_{0,1,1} + 128H_{0,-1,0,1} - 768H_{0,0,1,1,1} \left. \right] + 16(72z^2 - 26z + 49)H_{0,1,0,1,1} \\
& + (2z^2 + 2z + 1) \left[16H_0H_{-1}^4 + \left[-\frac{64}{3}H_0^2 - \frac{-160}{3}\zeta_2 - 64H_{0,-1} \right. \right. \\
& \left. \left. - \frac{64}{3}H_{0,1} \right] H_{-1}^3 + \left[\frac{8}{3}H_0^3 + [-56\zeta_2 + 64H_{0,-1} - 32H_{0,1}]H_0 - 160\zeta_3 + 192H_{0,-1,-1} \right. \right. \\
& + 64H_{0,-1,1} + 96H_{0,0,1} + 64H_{0,1,-1} \left. \right] H_{-1}^2 + \left[\frac{4}{3}H_0^4 + [-28\zeta_2 + 16H_{0,-1} + 32H_{0,1}]H_0^2 \right. \\
& + [64\zeta_3 - 128 - 128H_{0,-1,-1} + 64H_{0,-1,1} - 96H_{0,0,-1} - 64H_{0,0,1} + 64H_{0,1,-1} + 64H_{0,1,1}]H_0 \\
& + \frac{296}{5}\zeta_2^2 + 48H_{0,-1}\zeta_2 - 96H_{0,1}\zeta_2 - 384H_{0,-1,-1,-1} - 128H_{0,-1,-1,1} - 128H_{0,-1,0,1} - 128H_{0,-1,1,-1} \\
& - 192H_{0,0,-1,1} + 160H_{0,0,0,-1} + 32H_{0,0,0,1} - 192H_{0,0,1,-1} - 128H_{0,0,1,1} - 128H_{0,1,-1,-1} \left. \right] H_{-1} \\
& + 96H_{0,-1,1}\zeta_2 + 96H_{0,1,-1}\zeta_2 + 128H_{0,-1} + H_0^2 \left[-32H_{0,-1,1} - 32H_{0,1,-1} \right] + H_0 \left[-64H_{0,-1,-1,1} \right. \\
& + 64H_{0,-1,0,1} - 64H_{0,-1,1,-1} - 64H_{0,-1,1,1} + 96H_{0,0,-1,-1} + 64H_{0,0,-1,1} + 64H_{0,0,1,-1} - 64H_{0,1,-1,-1} \\
& - 64H_{0,1,-1,1} - 64H_{0,1,1,-1} \left. \right] + 384H_{0,-1,-1,-1,-1} + 128H_{0,-1,-1,-1,1} \\
& + 128H_{0,-1,-1,1,-1} + 128H_{0,-1,0,-1,1} + 128H_{0,-1,0,1,-1} + 128H_{0,-1,1,-1,-1} + 192H_{0,0,-1,-1,1} \\
& - 64H_{0,0,-1,0,1} + 192H_{0,0,-1,1,-1} + 128H_{0,0,-1,1,1} + 192H_{0,0,1,-1,-1} \\
& \left. + 128H_{0,0,1,-1,1} + 128H_{0,0,1,1,-1} + 128H_{0,1,-1,-1,-1} \right] + 16(20z^2 - 14z + 13)H_{0,1,1,1,1} \left. \right\}. \quad (607)
\end{aligned}$$

The OME $A_{gg,Q}(z)$ as a diagonal element in the singlet-gluon matrix has distribution-valued (+, $\delta(1-z)$) and regular (reg) contributions :

$$A_{gg,Q}(z) = [A_{gg,Q,+}(z)]_+ + A_{gg,Q,\text{reg}}(z) + C_{gg,Q}\delta(1-z), \quad (608)$$

with

$$\int_0^1 dz f(z) [A_{gg,Q}(z)]_+ = \int_0^1 dz [f(z) - f(1)] A_{gg,Q,+}(z) \quad (609)$$

$$\int_0^1 dz C_{gg,Q} \delta(1-z) = C_{gg,Q}. \quad (610)$$

The different parts are given by :

$$\begin{aligned}
A_{gg,Q,+} = & \\
& a_s^2 \frac{1}{z-1} \left\{ C_{AT_F} \left[-\frac{8}{3} L_M^2 - \frac{80}{9} L_M - \frac{224}{27} \right] \right\} \\
& + a_s^3 \frac{1}{z-1} \left\{ C_{A^2 T_F} \left[L_M \left[H_0 \left[-\frac{64}{3} H_{0,-1} + \frac{32}{3} H_{0,1} - \frac{640}{9} H_1 - \frac{16}{3} \right] + \frac{128}{3} H_{0,0,-1} - \frac{64}{3} H_{0,0,1} \right. \right. \right. \\
& + \left. \left. \left[-\frac{16}{3} H_1 - \frac{160}{9} \right] H_0^2 + \frac{320 \zeta_2}{9} - \frac{256 \zeta_3}{3} - \frac{1240}{81} \right] \right. + L_M^2 \left[\left[-\frac{16}{3} H_0^2 - \frac{64}{3} H_1 H_0 + \frac{32 \zeta_2}{3} - \frac{184}{9} \right] \right. \\
& + \left. \left. \left[\zeta_2 \left[\frac{8}{3} H_0^2 + \frac{32}{3} H_1 H_0 + \frac{16}{27} \right] - \frac{88 H_0}{9} - \frac{16 \zeta_2^2}{3} - \frac{176 \zeta_3}{27} - \frac{22672}{243} \right] + L_M^3 \frac{176}{27} \right] \right. \\
& + C_{AC_F T_F} \left[L_M \left[64 \zeta_3 - \frac{40}{3} \right] - 8 L_M^2 + \left[-40 \zeta_2 - \frac{466}{9} \right] \right] \\
& + C_{AT_F^2} \left[\left[\frac{16 H_0}{3} + \frac{560 \zeta_2}{27} + \frac{224 \zeta_3}{27} + \frac{5248}{81} \right] - L_M^3 \frac{224}{27} - L_M^2 \frac{640}{27} - L_M \frac{320}{9} \right] \\
& + C_{AT_F^2 N_F} \left[\left[\frac{32 H_0}{9} + \frac{160 \zeta_2}{27} + \frac{64 \zeta_3}{27} + \frac{10496}{243} \right] - L_M^3 \frac{64}{27} - L_M \frac{2176}{81} \right] + a_{gg,Q,(+)}^{(3)} \left. \right\}, \quad (611)
\end{aligned}$$

$$\begin{aligned}
C_{gg,Q} = & \\
& \frac{4}{3} a_s T_F L_M + a_s^2 \left\{ C_{AT_F} \left[\frac{16}{3} L_M + \frac{10}{9} \right] + C_{FT_F} \left[4 L_M - 15 \right] + \frac{16}{9} T_F^2 L_M^2 \right\} \\
& + a_s^3 \left\{ C_{A^2 T_F} \left[\left[\frac{16 \zeta_3}{3} - \frac{2}{3} \right] L_M^2 + \left[\frac{16 \zeta_2^2}{3} + \frac{160 \zeta_3}{9} + \frac{277}{9} \right] L_M + \zeta_2 \left(4 - \frac{8 \zeta_3}{3} \right) - \frac{616}{27} \right] \right. \\
& + C_{FC_{AT_F}} \left\{ -\frac{22}{3} L_M^2 + \frac{736}{9} L_M + \frac{20 \zeta_2}{3} + 16 \zeta_3 - \frac{1045}{6} - 64 \zeta_2 \log(2) \right\} \\
& + C_{FT_F^2 N_F} \left\{ 28 \zeta_2 + \frac{118}{3} - \frac{268}{9} L_M \right\} + C_{FT_F^2} \left[\frac{40}{3} L_M^2 - \frac{584}{9} L_M + \frac{782}{9} - \frac{40 \zeta_2}{3} \right] \\
& + C_{AT_F^2 N_F} \left[\frac{224}{27} - \frac{4 \zeta_2}{3} - \frac{44}{3} L_M \right] + C_{AT_F^2} \left[\frac{56}{3} L_M^2 - 2 L_M - \frac{44 \zeta_2}{3} - \frac{8}{27} \right] \\
& + C_{F^2 T_F} \left[-2 L_M + -80 \zeta_2 - 32 \zeta_3 - 39 + 128 \zeta_2 \log(2) \right] + T_F^3 \left[\frac{64}{27} L_M^3 - \frac{64 \zeta_3}{27} \right] \\
& + a_{gg,Q,\delta}^{(3)} \left. \right\}, \quad (612)
\end{aligned}$$

and

$$A_{gg,Q,\text{reg}} = a_s^2 \left\{ C_{AT_F} \left[\frac{4}{3} (z+1) H_0^2 + \frac{4}{9} (22z+13) H_0 - \frac{8(z^3 - z^2 + 2z - 1) L_M^2}{3z} \right] \right\}$$

$$\begin{aligned}
& -\frac{4(175z^3 - 137z^2 + 157z - 139)}{27z} + \frac{4}{3}zH_1 + L_M \left[\frac{16}{3}(z+1)H_0 - \frac{8(23z^3 - 19z^2 + 29z - 23)}{9z} \right] \\
& + C_{FTF} \left[\frac{4}{3}(z+1)H_0^3 + 2(5z+3)H_0^2 + 16(3z+2)H_0 + L_M^2 \left[8(z+1)H_0 - \frac{4(z-1)(4z^2 + 7z + 4)}{3z} \right] \right. \\
& \left. - \frac{8(z-1)(3z^2 + 9z - 1)}{z} + L_M \left[8(z+1)H_0^2 + 8(5z+3)H_0 - \frac{16(z-1)(5z^2 + 11z - 1)}{3z} \right] \right] \Big\} \\
& + a_s^3 \left\{ C_{ATF}^2 \left[\frac{88}{27}(z+1)H_0^3 + \frac{44}{27}(22z+13)H_0^2 + \frac{8}{3}(2z^3 - 2z^2 - 4z - 1) \frac{\zeta_2}{z+1} H_0^2 \right. \right. \\
& + \frac{88}{81}(161z+62)H_0 + \frac{8}{9}(44z^2 + 11z + 47)\zeta_2 H_0 + \frac{176(z^3 - z^2 + 2z - 1)L_M^3}{27z} - \frac{44}{9}zH_1^2 \\
& + L_M^2 \left[\frac{(z^2 + z + 1)^2}{z(z+1)} \left[\frac{64}{3}H_{-1}H_0 - \frac{64}{3}H_{0,-1} \right] - \frac{16(2z^3 - 2z^2 - 4z - 1)H_0^2}{3(z+1)} \right. \\
& + \frac{8(208z^3 - 273z^2 + 204z - 208)}{27z} + \frac{32}{3}(2z^3 + 2z^2 + 4z + 3) \frac{\zeta_2}{z+1} - \frac{64}{9}(11z^2 + 9)H_0 \\
& \left. \left. - \frac{64(z^3 - z^2 + 2z - 1)H_0H_1}{3z} \right] - \frac{8(12283z^3 - 9665z^2 + 8143z - 7927)}{243z} \right. \\
& - \frac{4}{27}(416z^3 - 437z^2 + 433z - 416) \frac{\zeta_2}{z} - \frac{16}{3}(2z^3 + 2z^2 + 4z + 3) \frac{\zeta_2^2}{z+1} + \frac{176(4z^2 + 3z - 3)H_1}{27z} \\
& + (z^2 + z + 1)^2 \frac{\zeta_2}{z(z+1)} \left[\frac{32}{3}H_{0,-1} - \frac{32}{3}H_{-1}H_0 \right] + \frac{(z^3 - z^2 + 2z - 1)}{z} \left[\frac{32}{3}H_0H_1\zeta_2 - \frac{176\zeta_3}{27} \right] \\
& + L_M \left[\frac{(z+1)^3}{z} \left[-\frac{64}{3}H_{-1}H_{0,1} + \frac{64}{3}H_{0,-1,1} + \frac{64}{3}H_{0,1,-1} \right] + \frac{8(4z^2 - 31z + 4)H_{-1}H_0^2(z+1)}{3z} \right. \\
& - \frac{16}{3}(2z^2 - 35z + 2) \frac{\zeta_2}{z} H_{-1}(z+1) + \frac{(2z^2 - 9z + 2)(z+1)}{z} \left[-16H_0H_{-1}^2 + 32H_{0,-1}H_{-1} \right. \\
& \left. \left. - 32H_{0,-1,-1} \right] + \left[-24H_{0,1}H_0^2 + 96H_{0,0,1}H_0 - \frac{1832}{9}H_{0,1} + 192H_{0,0,0,-1} - 192H_{0,0,0,1} \right. \right. \\
& \left. \left. + 96H_{0,1}\zeta_2 \right] (z+1) + \frac{8}{9}(19z - 12)H_0^3 - \frac{24}{5}(13z + 23)\zeta_2^2 + \frac{4(13194z^3 - 13763z^2 + 12661z - 12402)}{81z} \right. \\
& + \frac{8}{9}(132z^4 - 137z^3 + 156z^2 + 481z + 96) \frac{\zeta_2}{z(z+1)} - \frac{32}{3}(8z^3 + 7z^2 + 31z - 11) \frac{\zeta_3}{z} \\
& - \frac{4(3918z^3 - 1495z^2 + 4022z + 568)H_0}{27z} - \frac{8}{3}(8z^4 + 43z^3 + 34z^2 - 5z - 8) \frac{\zeta_2}{z(z+1)} H_0 \\
& + \frac{4}{3}(25z - 29)H_0^2H_1 + \frac{4(410z^3 + 1965z^2 - 2031z - 410)H_1}{27z} - 16(z-1)(2z^2 + 9z + 2) \frac{\zeta_2}{z} H_1 \\
& - \frac{32(33z^3 - 29z^2 + 49z - 33)H_0H_1}{9z} + \frac{64}{3}(z^2 + 9z - 7)H_0H_{0,-1} - \frac{8}{3}(17z - 37)H_0H_{0,1} \\
& - \frac{16(12z^3 + 45z^2 - 83z + 4)H_{0,0,-1}}{3z} - 32(5z - 1)H_0H_{0,0,-1} + \frac{16}{3}(4z^2 + 31z - 23)H_{0,0,1} \\
& + 8(5z + 3)H_0^2\zeta_2 + (z-1) \left[48H_{0,-1}H_0^2 - 192H_{0,-1,-1}H_0 + 96H_{0,-1}^2 - 96H_{0,-1}\zeta_2 \right] \\
& + z(192H_0\zeta_3 - \frac{4}{3}H_0^4) - \frac{4(124z^3 - 459z^2 - 613z - 70)H_0^2}{9(z+1)} \\
& + \frac{(114z^4 - 77z^3 - 342z^2 - 77z + 114)}{z(z+1)} \left[\frac{16}{9}H_{-1}H_0 - \frac{16}{9}H_{0,-1} \right] \Big\}
\end{aligned}$$

$$\begin{aligned}
& + C_{AT_F^2} \left[-L_M^3 \frac{224(z^3 - z^2 + 2z - 1)}{27z} + L_M^2 \left[\frac{128}{9}(z+1)H_0 - \frac{64(23z^3 - 19z^2 + 29z - 23)}{27z} \right] \right] \\
& + L_M \left[-\frac{8(814z^3 - 633z^2 + 777z - 598)}{81z} - \frac{8}{27}(52z^2 - 89z - 2)H_0 - \frac{8(52z^3 - 51z^2 + 33z - 52)H_1}{27z} \right. \\
& + (z+1) \left[\frac{80}{9}H_0^2 + \frac{128}{9}H_{0,1} - \frac{128\zeta_2}{9} \right] - \frac{8}{9}(22z+13)H_0^2 + \frac{8}{3}zH_1^2 - \frac{16}{27}(161z+62)H_0 \\
& + \frac{16(1187z^3 - 949z^2 + 881z - 791)}{81z} + \frac{56}{27}(23z^3 - 19z^2 + 29z - 23)\frac{\zeta_2}{z} + \frac{224}{27}(z^3 - z^2 + 2z - 1)\frac{\zeta_3}{z} \\
& - \frac{32(4z^2 + 3z - 3)H_1}{9z} + (z+1) \left[-\frac{16}{9}H_0^3 - \frac{112}{9}\zeta_2H_0 \right] + C_{AN_{FT_F}^2} \left[-\frac{64(z^3 - z^2 + 2z - 1)L_M^3}{27z} \right. \\
& + L_M \left[-\frac{16(114z^3 - 85z^2 + 149z - 42)}{81z} - \frac{16}{27}(52z^2 - z + 50)H_0 - \frac{16(52z^3 - 39z^2 + 33z - 52)H_1}{27z} \right. \\
& + (z+1) \left[\frac{32}{3}H_0^2 + \frac{256}{9}H_{0,1} - \frac{256\zeta_2}{9} \right] - \frac{16}{27}(22z+13)H_0^2 + \frac{32(1187z^3 - 949z^2 + 881z - 791)}{243z} \\
& + \frac{16}{9}zH_1^2 + \frac{16}{27}(23z^3 - 19z^2 + 29z - 23)\frac{\zeta_2}{z} + \frac{64}{27}(z^3 - z^2 + 2z - 1)\frac{\zeta_3}{z} - \frac{32}{81}(161z+62)H_0 \\
& \left. - \frac{64(4z^2 + 3z - 3)H_1}{27z} + (z+1) \left[-\frac{32}{27}H_0^3 - \frac{32}{9}\zeta_2H_0 \right] \right] \\
& + C_{F^2T_F} \left[\frac{2}{9}(4z^2 - 3z + 3)H_0^4 + \frac{2}{9}(40z^2 + 149z + 115)H_0^3 - \frac{2}{3}(64z^2 + 23z - 69)H_0^2 \right. \\
& + \frac{4(z-1)(20z^2 + 41z - 4)H_1H_0^2}{3z} - \frac{8(4z^3 + 27z^2 + 3z - 4)H_{0,1}H_0^2}{3z} + \frac{8}{3}(4z^2 - 3z - 3)\zeta_2H_0^2 \\
& + \frac{4}{3}(80z^2 + 33z + 246)H_0 - \frac{8(z-1)(32z^2 + 127z - 18)H_1H_0}{3z} - \frac{16(10z^3 - 35z^2 - 49z + 2)H_{0,1}H_0}{3z} \\
& - \frac{16(4z^3 - 33z^2 - 15z + 4)H_{0,0,1}H_0}{3z} + \frac{2}{3}(80z^2 + 469z + 221)\zeta_2H_0 - \frac{16}{9}(44z^2 + 51z - 18)\zeta_3H_0 \\
& + \frac{8(z-1)(6z^2 - z - 6)H_1^3}{9z} + \frac{4(z-1)(24z^2 - 13z + 17)H_1^2}{3z} - \frac{64(z+1)^2(2z-1)H_{0,1}^2}{3z} \\
& - \frac{8}{15}(188z^2 - 27z - 105)\zeta_2^2 - \frac{4(z-1)(56z^2 + 418z - 5)}{3z} - \frac{4}{3}(84z^3 + 79z^2 + 75z - 60)\frac{\zeta_2}{z} \\
& + \frac{4(z-1)(80z^2 - 181z - 9)H_1}{3z} + \frac{4}{3}(z-1)(40z^2 + 33z + 4)\frac{\zeta_2}{z}H_1 \\
& + \frac{8(56z^3 + 136z^2 - 121z + 18)H_{0,1}}{3z} + \frac{8}{3}(12z^3 + 15z^2 + 9z + 8)\frac{\zeta_2}{z}H_{0,1} + \frac{32}{3}(z-1)H_1H_{0,1} \\
& + \frac{8(20z^3 - 239z^2 - 187z + 4)H_{0,0,1}}{3z} + \frac{8}{3}(12z^2 - 23z - 22)H_{0,1,1} \\
& + \frac{16(20z^3 - 21z^2 - 33z + 4)H_{0,0,0,1}}{3z} + \frac{32(3z^2 + 15z + 8)H_{0,0,1,1}}{3z} \\
& - \frac{16(20z^3 + 15z^2 - 27z - 24)H_{0,1,1,1}}{3z} + L_M^3 \left[-\frac{16(4z^2 + 7z + 4)H_1(z-1)}{9z} - \frac{92(z-1)}{9} \right. \\
& \left. - \frac{16}{9}z(4z+3)H_0 + (z+1) \left[\frac{8}{3}H_0^2 + \frac{32}{3}H_{0,1} - \frac{32\zeta_2}{3} \right] - \frac{4}{9}(72z^2 - 1673z - 1009)\zeta_3 \right]
\end{aligned}$$

$$\begin{aligned}
& +L_M \left[\frac{16}{3}(5z+2)H_0^3 - \frac{2}{3}(72z^2 - 599z - 207)H_0^2 - \frac{8}{3}(78z^2 - 256z - 135)H_0 \right. \\
& - \frac{8(z-1)(36z^2 + 25z + 24)H_1H_0}{3z} + \frac{128(6z+1)H_{0,1}H_0}{3z} - 256zH_{0,0,-1}H_0 \\
& - 32(8z+3)\zeta_2H_0 + 64(z-3)\zeta_3H_0 - \frac{4(z-1)(28z^2 + 21z + 4)H_1^2}{3z} - \frac{32}{5}(13z+23)\zeta_2^2 \\
& - \frac{4(z-1)(268z^2 + 377z - 68)}{3z} + \frac{16}{3}(36z^3 - 51z^2 + 3z - 8)\frac{\zeta_3}{z} - \frac{16(z-1)(39z^2 - 45z - 13)H_1}{3z} \\
& + \frac{8(8z^3 - 23z^2 - 43z - 24)H_{0,1}}{3z} + \frac{(z-1)(4z^2 + 7z + 4)}{z} \left[\frac{8}{9}H_1^3 + \frac{16}{3}H_0H_1^2 - \frac{32}{3}H_{0,1}H_1 \right] \\
& - \frac{128(3z^3 + 3z^2 - 9z - 1)H_{0,0,-1}}{3z} - \frac{32(4z^3 - 12z^2 + 27z + 4)H_{0,0,1}}{3z} \\
& + \frac{16(12z^3 + 27z^2 + 3z - 8)H_{0,1,1}}{3z} + \frac{16}{3}(14z^2 - 128z + 21)\zeta_2 \\
& + \frac{(z+1)(z^2 - 4z + 1)}{z} \left[-\frac{128}{3}H_0H_{-1}^2 + \left(\frac{64}{3}H_0^2 + \frac{256}{3}H_{0,-1}\right)H_{-1} - \frac{128}{3}\zeta_2H_{-1} - \frac{256}{3}H_{0,-1,-1} \right] \\
& + \frac{(z-1)(z^2 + 4z + 1)}{z} \left[\frac{64}{3}H_1H_0^2 + \frac{128}{3}H_{0,-1}H_0 - \frac{128}{3}H_1\zeta_2 \right] + (z-1) \left[64H_{0,-1}H_0^2 \right. \\
& - 256H_{0,-1,-1}H_0 + 128H_{0,-1}^2 - 128H_{0,-1}\zeta_2 \left. \right] + (z+1) \left[2H_0^4 - 64H_{0,1}H_0^2 - \frac{2144}{3}H_{-1}H_0 \right. \\
& + \left[288H_{0,0,1} - 64H_{0,1,1} \right] H_0 + 32H_{0,1}^2 + \frac{2144}{3}H_{0,-1} + 384H_{0,0,0,-1} - 416H_{0,0,0,1} - 32H_{0,1,1,1} \\
& \left. + \left[128H_{0,1} - 32H_0^2 \right] \zeta_2 \right] + \frac{(z-1)(4z^2 + 7z + 4)}{z} \left[\frac{2}{9}H_1^4 - \frac{16}{3}H_{0,1}H_1^2 + \frac{8}{9}H_0^3H_1 \right. \\
& + \left[\frac{64}{3}H_{0,0,1} + \frac{64}{3}H_{0,1,1} \right] H_1 - \frac{176}{9}\zeta_3H_1 + H_0 \left[\frac{64}{3}H_{0,1,1} - \frac{32}{3}H_1H_{0,1} \right] + \left[\frac{20}{3}H_1^2 + \frac{16}{3}H_0H_1 \right] \zeta_2 \left. \right] \\
& + L_M^2 \left[-\frac{8}{3}(4z^2 - 9z - 3)H_0^2 - \frac{8}{3}(z+1)(32z - 31)H_0 - \frac{16(z-1)(4z^2 + 7z + 4)H_1H_0}{3z} \right. \\
& - \frac{4(z-1)(32z^2 + 81z + 12)}{3z} - \frac{8(z-1)(32z^2 + 35z + 8)H_1}{3z} + \frac{16(4z^3 + 21z^2 + 9z - 4)H_{0,1}}{3z} \\
& - 32(3z+2)\zeta_2 + (z+1) \left[\frac{16}{3}H_0^3 + 32H_{0,1}H_0 - 32\zeta_2H_0 - 32H_{0,0,1} + 32\zeta_3 \right] + (z+1) \left[-\frac{2}{15}H_0^5 \right. \\
& - \frac{16}{3}H_{0,1}H_0^3 + 48H_{0,0,1}H_0^2 + \frac{448}{5}\zeta_2^2H_0 + \left[32H_{0,1}^2 - 160H_{0,0,0,1} - 128H_{0,0,1,1} \right] H_0 \\
& + H_{0,1} \left[64H_{0,1,1} - 128H_{0,0,1} \right] + 192H_{0,0,0,0,1} + 768H_{0,0,0,1,1} + 320H_{0,0,1,0,1} - 416H_{0,0,1,1,1} \\
& - 192H_{0,1,0,1,1} - 32H_{0,1,1,1,1} + \left[\frac{88}{3}H_0^2 + \frac{352}{3}H_{0,1} \right] \zeta_3 + \zeta_2 \left[-4H_0^3 - 32H_{0,1}H_0 \right. \\
& \left. - 32H_{0,0,1} - 80H_{0,1,1} + \frac{80\zeta_3}{3} \right] - 160\zeta_5 \left. \right] \\
& + C_{FT_F}^2 \left[-\frac{8}{3}(5z+3)H_0^3 - 32(3z+2)H_0^2 - 64(7z+5)H_0 + \frac{8}{9}(8z^2 - 61z - 31)\zeta_2H_0 \right. \\
& + L_M^3 \left[\frac{160}{9}(z+1)H_0 - \frac{80(z-1)(4z^2 + 7z + 4)}{27z} \right] + \frac{32(z-1)(19z^2 + 64z - 5)}{3z} \\
& + \frac{16}{27}(z-1)(67z^2 + 184z - 41)\frac{\zeta_2}{z} + L_M^2 \left[-\frac{32(z-1)(22z^2 + 85z - 32)}{27z} \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{64}{9}(2z^2 - 4z - 1)H_0 - \frac{32(z-1)(4z^2 + 7z + 4)H_1}{9z} + (z+1)\left[\frac{64}{3}H_0^2 + \frac{64}{3}H_{0,1} - \frac{64\zeta_2}{3}\right] \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[\frac{16}{9}H_1\zeta_2 + \frac{80\zeta_3}{27}\right] + (z+1)\left[-\frac{4}{3}H_0^4 - \frac{160}{9}\zeta_3H_0 + \frac{32\zeta_2^2}{3}\right. \\
& + \left.[-\frac{56}{3}H_0^2 - \frac{32}{3}H_{0,1}]\zeta_2\right] + L_M\left[-\frac{8}{9}(16z^2 + 13z + 19)H_0^2 - \frac{64}{27}(11z^2 - 13z - 76)H_0\right. \\
& - \frac{32(z-1)(73z^2 + 631z + 73)}{81z} - \frac{32(z-1)(22z^2 + 85z - 32)H_1}{27z} \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[-\frac{16}{9}H_1^2 - \frac{64}{9}H_0H_1\right] + \frac{64(2z^3 + 7z^2 - 2z - 4)H_{0,1}}{9z} \\
& + \left.\frac{64}{9}(2z^2 - 4z - 1)\zeta_2 + (z+1)\left[\frac{80}{9}H_0^3 + \frac{128}{3}H_{0,1}H_0 - \frac{128}{3}\zeta_2H_0 - \frac{128}{3}H_{0,0,1} + \frac{64}{3}H_{0,1,1} + \frac{64\zeta_3}{3}\right]\right] \\
& + C_{FNFT_F}^2\left[-\frac{16}{9}(5z+3)H_0^3 - \frac{64}{3}(3z+2)H_0^2 - \frac{64}{81}(110z^2 + 755z + 518)H_0\right. \\
& - \frac{16}{9}(4z^2 + 37z + 25)\zeta_2H_0 + L_M^3\left[\frac{128}{9}(z+1)H_0 - \frac{64(z-1)(4z^2 + 7z + 4)}{27z}\right] \\
& + \frac{16(z-1)(38z^2 + 47z + 20)H_1^2}{27z} + \frac{64(z-1)(1138z^2 + 2470z + 193)}{243z} \\
& - \frac{16}{27}(8z^3 + 223z^2 + 271z + 14)\frac{\zeta_2}{z} + \frac{64}{27}(10z^3 + 36z^2 + 21z - 4)\frac{\zeta_3}{z} \\
& - \frac{128(z-1)(55z^2 + 64z + 28)H_1}{81z} + \frac{64}{27}(19z^2 + 67z + 43)H_{0,1} - \frac{64}{9}(2z^2 + 11z + 8)H_{0,1,1} \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[-\frac{16}{27}H_1^3 - \frac{16}{9}\zeta_2H_1\right] + (z+1)\left[-\frac{8}{9}H_0^4 - \frac{128}{9}\zeta_3H_0 - \frac{96\zeta_2^2}{5} + \frac{64}{3}H_{0,1,1,1}\right. \\
& + \left.\left[\frac{32}{3}H_{0,1} - \frac{16}{3}H_0^2\right]\zeta_2\right] + L_M\left[-\frac{16}{9}(16z^2 + 43z + 37)H_0^2 - \frac{64}{9}(z^2 + 5z - 41)H_0\right. \\
& - \frac{64(z-1)(7z^2 + 91z + 61)}{27z} - \frac{32(z-1)(2z^2 + 41z - 28)H_1}{9z} + \frac{64(2z^3 + 3z^2 - 12z - 8)H_{0,1}}{9z} \\
& + \frac{(z-1)(4z^2 + 7z + 4)}{z}\left[-\frac{16}{3}H_1^2 - \frac{128}{9}H_0H_1\right] + \frac{64}{3}(2z^2 + z + 2)\zeta_2 \\
& + (z+1)\left[\frac{32}{3}H_0^3 + \frac{256}{3}H_{0,1}H_0 - \frac{256}{3}\zeta_2H_0 - \frac{256}{3}H_{0,0,1} + 64H_{0,1,1} + \frac{64\zeta_3}{3}\right]\left. \right] \\
& + C_{ACFT_F}\left[\frac{2}{9}(26z+5)H_0^4 + \frac{4}{27}(20z^2 + 141z + 174)H_0^3 - \frac{4}{9}(300z^2 - 901z - 169)H_0^2\right. \\
& - \frac{8(z-1)(26z^2 - 32z + 23)H_1H_0^2}{3z} - \frac{8(3z^2 + 3z - 8)H_{0,1}H_0^2}{3z} + 32(2z-1)H_{0,0,1}H_0^2 \\
& + \frac{8}{3}(32z+5)\zeta_2H_0^2 + \frac{16}{3}(14z-13)\zeta_3H_0^2 + \frac{16(z-1)(19z^2 + 16z + 10)H_1^2H_0}{9z} - \frac{16}{5}(38z-5)\zeta_2^2H_0 \\
& + \frac{4(36604z^3 + 28737z^2 + 27060z + 2624)H_0}{81z} + \frac{4}{9}(298z^3 + 121z^2 + 427z + 80)\frac{\zeta_2}{z}H_0 \\
& - \frac{16}{9}(145z^2 - 83z - 4)\frac{\zeta_3}{z}H_0 + \frac{32(z-1)(461z^2 - 73z + 227)H_1H_0}{27z} \\
& + \frac{32(19z^3 - 24z^2 - 6z + 10)H_{0,-1}H_0}{9z} - \frac{16(10z^3 + 249z^2 - 48z + 109)H_{0,1}H_0}{9z}
\end{aligned}$$

$$\begin{aligned}
& -\frac{32(9z^2 - 9z + 8)H_{0,0,1}H_0}{3z} - 64(5z - 2)H_{0,0,0,1}H_0 - \frac{8(z - 1)(2z + 1)(14z + 1)H_1^3}{27z} \\
& -\frac{4(z - 1)(328z^2 + 313z + 67)H_1^2}{27z} - \frac{2(417275z^3 - 288519z^2 - 19632z - 102833)}{243z} \\
& -\frac{16}{27}(845z^3 - 432z^2 + 255z - 182)\frac{\zeta_2}{z} - \frac{8}{15}(20z^3 + 340z^2 - 137z + 152)\frac{\zeta_2^2}{z} \\
& -\frac{8}{9}(416z^3 + 498z^2 + 555z + 76)\frac{\zeta_3}{z} - \frac{4(2500z^3 - 69z^2 - 984z - 1771)H_1}{81z} - \frac{4}{9} \\
& (z - 1)(154z^2 + 163z + 46)\frac{\zeta_2}{z}H_1 + \frac{(z + 1)(182z^2 - 122z + 47)}{z}\left[\frac{32}{27}H_{0,-1} - \frac{32}{27}H_{-1}H_0\right] \\
& -\frac{8(1732z^3 - 2205z^2 + 972z - 908)H_{0,1}}{27z} - \frac{32(19z^3 - 51z^2 - 6z + 10)H_{0,0,-1}}{9z} \\
& +\frac{16(98z^3 + 324z^2 + 69z + 149)H_{0,0,1}}{9z} - \frac{8(56z^3 - 105z^2 - 66z - 40)H_{0,1,1}}{9z} \\
& +\frac{16(21z^2 - 15z + 8)H_{0,0,0,1}}{z} + \frac{16(20z^3 + 18z^2 - 15z - 20)H_{0,0,1,1}}{3z} \\
& +\frac{32(z - 1)(z + 2)(2z + 1)H_{0,1,1,1}}{3z} + 128(4z - 1)H_{0,0,0,0,1} + \frac{8}{3}(23z + 14)H_{0,1}\zeta_2 \\
& +\frac{(z + 1)(19z^2 - 16z + 10)}{z}\left[\frac{32}{9}H_0H_{-1}^2 + \left[-\frac{16}{9}H_0^2 - \frac{64}{9}H_{0,-1}\right]H_{-1} + \frac{32}{9}\zeta_2H_{-1} + \frac{64}{9}H_{0,-1,-1}\right] \\
& +L_M^3\left[-\frac{16}{3}(2z - 1)H_0^2 - \frac{16(8z^2 + 11z + 4)H_0}{9z} + \frac{8(z - 1)(44z^2 - z + 44)}{9z}\right. \\
& \left. +\frac{16(z - 1)(4z^2 + 7z + 4)H_1}{9z} + (z + 1)\left[\frac{32\zeta_2}{3} - \frac{32}{3}H_{0,1}\right]\right] - \frac{8}{3}(43z + 37)\zeta_2\zeta_3 \\
& +L_M^2\left[-\frac{8}{3}(38z + 5)H_0^2 - \frac{8(184z^3 + 103z^2 + 205z + 40)H_0}{9z}\right. \\
& \left. -\frac{16(z - 1)(4z^2 + 7z + 4)H_1H_0}{3z} + \frac{8(517z^3 - 444z^2 + 45z - 127)}{9z} + \frac{16}{3}(4z^3 + 17z^2 + 11z + 4)\frac{\zeta_2}{z}\right. \\
& \left. +\frac{8(z - 1)(104z^2 + 119z + 32)H_1}{9z} + \frac{(z + 1)(4z^2 - 7z + 4)\left[\frac{32}{3}H_{-1}H_0 - \frac{32}{3}H_{0,-1}\right]}{z}\right. \\
& \left. -\frac{32}{3}(10z + 7)H_{0,1} + (z + 1)\left[32H_0H_{0,1} - 32H_{0,0,1}\right] + (z - 1)\left[\frac{32}{3}H_0^3 + 64H_{0,-1}H_0 + 32\zeta_2H_0\right.\right. \\
& \left. -128H_{0,0,-1}\right] + 64(2z - 1)\zeta_3\left. + \frac{(z + 1)(4z^2 - 7z + 4)\left[-\frac{32}{9}H_0H_{-1}^3 + \left[\frac{8}{3}H_0^2 + \frac{32}{3}H_{0,-1}\right]H_{-1}^2\right.\right. \\
& \left. +\left[\frac{8}{9}H_0^3 + \left[\frac{32}{3}H_{0,1} - \frac{32}{3}H_{0,-1}\right]H_0 - \frac{64}{3}H_{0,-1,-1} + \frac{32}{3}H_{0,0,-1} - \frac{64}{3}H_{0,0,1}\right]H_{-1} + 16\zeta_3H_{-1}\right. \\
& \left. -\frac{8}{3}H_0^2H_{0,-1} + H_0\left[\frac{32}{3}H_{0,-1,-1} - \frac{32}{3}H_{0,-1,1} + \frac{16}{3}H_{0,0,-1} - \frac{32}{3}H_{0,1,-1}\right] + \frac{64}{3}H_{0,-1,-1,-1}\right. \\
& \left. +\frac{32}{3}H_{0,-1,0,1} - \frac{32}{3}H_{0,0,-1,-1} + \frac{64}{3}H_{0,0,-1,1} - \frac{16}{3}H_{0,0,0,-1} + \frac{64}{3}H_{0,0,1,-1} + \left[-\frac{16}{3}H_{-1}^2 - 8H_0H_{-1}\right.\right. \\
& \left. +8H_{0,-1}\right]\zeta_2\left. + L_M\left[\frac{4}{3}(11z - 7)H_0^4 + \frac{16}{9}(37z - 23)H_0^3 - \frac{4}{9}(234z^2 + 1487z + 1031)H_0^2\right.\right. \\
& \left. +\frac{8(z + 1)(8z^2 + 43z + 8)H_{-1}H_0^2}{3z} - \frac{8(z - 1)(16z^2 + 55z + 16)H_1H_0^2}{3z}\right. \\
& \left. -\frac{8(2384z^3 + 4795z^2 + 2230z - 120)H_0}{27z} + \frac{64}{3}(3z^3 + 16z^2 - 2z - 2)\frac{\zeta_2}{z}H_0\right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{16(z-1)(115z^2-17z+52)H_1H_0}{9z} - \frac{16(32z^3+9z^2-27z+16)H_{0,-1}H_0}{3z} \\
& + \frac{16}{3}(12z^2+35z-40)H_{0,1}H_0 + 64(3z+1)H_{0,0,-1}H_0 - 192(z+2)H_{0,0,1}H_0 - 32(2z+3)\zeta_3H_0 \\
& + \frac{4(z-1)(20z^2+21z+2)H_1^2}{3z} + \frac{16}{5}(89z+73)\zeta_2^2 + \frac{8(2021z^3-1164z^2-1313z+441)}{9z} \\
& + \frac{16}{9}(30z^3+539z^2-334z-56)\frac{\zeta_2}{z} - \frac{16}{3}(4z^3-154z^2-49z+16)\frac{\zeta_3}{z} \\
& + \frac{16}{3}(z+1)(4z^2-79z+4)\frac{\zeta_2}{z}H_{-1} + \frac{16(z-1)(340z^2-1007z-254)H_1}{27z} \\
& + \frac{32}{3}(z-1)(8z^2+35z+8)\frac{\zeta_2}{z}H_1 + \frac{(z+1)(29z^2+259z-34)}{z}\left[\frac{32}{9}H_{-1}H_0 - \frac{32}{9}H_{0,-1}\right] \\
& + \frac{16(85z^3-95z^2+403z-64)H_{0,1}}{9z} + \frac{(z-1)(4z^2+7z+4)}{z}\left[-\frac{8}{9}H_1^3 - 8H_0H_1^2 + \frac{32}{3}H_{0,1}H_1\right] \\
& + \frac{(z+1)(4z^2-21z+4)}{z}\left[16H_0H_{-1}^2 - 32H_{0,-1}H_{-1} + 32H_{0,-1,-1}\right] \\
& + \frac{16(56z^3-33z^2-105z+24)H_{0,0,-1}}{3z} - \frac{16(20z^3+53z^2-49z+16)H_{0,0,1}}{3z} \\
& + \frac{(z+1)^3}{z}\left[\frac{128}{3}H_{-1}H_{0,1} - \frac{128}{3}H_{0,-1,1} - \frac{128}{3}H_{0,1,-1}\right] - \frac{16(8z^3+23z^2+5z-8)H_{0,1,1}}{3z} \\
& - 96(3z+5)H_{0,0,0,-1} + 96(z+7)H_{0,0,0,1} + (z-1)\left[-48H_{0,-1}H_0^2 + 448H_{0,-1,-1}H_0 - 224H_{0,-1}^2\right. \\
& \left.+ 224H_{0,-1}\zeta_2\right] + (z+1)\left[112H_{0,1}H_0^2 + 96H_{0,1,1}H_0 - 32H_{0,1}^2 - 32H_{0,0,1,1} + 32H_{0,1,1,1} + \left[-96H_0^2\right.\right. \\
& \left.\left.- 256H_{0,1}\right]\zeta_2\right] + \frac{(z-1)(4z^2+7z+4)}{z}\left[-\frac{2}{9}H_1^4 - \frac{4}{3}H_0^2H_1^2 + \left[-\frac{80}{3}H_{0,0,1} - \frac{16}{3}H_{0,1,1}\right]H_1 + \frac{80}{9}\zeta_3H_1\right. \\
& \left.+ \frac{16}{3}H_{0,1}^2 + H_0\left[\frac{16}{9}H_1^3 + 16H_{0,1}H_1 - \frac{80}{3}H_{0,1,1}\right] + \left[\frac{8}{3}H_0H_1 - \frac{4}{3}H_1^2\right]\zeta_2\right] + (z-1)\left[-\frac{4}{15}H_0^5\right. \\
& \left.+ \frac{16}{3}H_{0,-1}H_0^3 + \left[32H_{0,-1,-1} - 32H_{0,0,-1}\right]H_0^2 + \left[-32H_{0,-1}^2 - 128H_{0,-1,-1,-1} + 64H_{0,-1,0,1}\right.\right. \\
& \left.\left.+ 96H_{0,0,0,-1}\right]H_0 + H_{0,-1}\left[128H_{0,-1,-1} + 64H_{0,0,-1} - 128H_{0,0,1}\right] - 256H_{0,-1,0,-1,-1} \right. \\
& \left.- 512H_{0,0,-1,-1,-1} - 64H_{0,0,-1,0,-1} + 128H_{0,0,-1,0,1} - 192H_{0,0,0,-1,-1} + 384H_{0,0,0,-1,1} \right. \\
& \left.- 128H_{0,0,0,0,-1} + 384H_{0,0,0,1,-1} + 128H_{0,0,1,0,-1} + \left[-8H_0^3 - 48H_{0,-1}H_0 - 64H_{0,-1,-1}\right.\right. \\
& \left.\left.+ 96H_{0,0,-1}\right]\zeta_2 + 96H_{0,-1}\zeta_3\right] + (z+1)\left[16H_{0,1,1}H_0^2 + \left[-48H_{0,1}^2 + 128H_{0,0,1,1} - 64H_{0,1,1,1}\right]H_0\right. \\
& \left.+ 160H_{0,1}H_{0,0,1} - 864H_{0,0,0,1,1} - 288H_{0,0,1,0,1} + 128H_{0,0,1,1,1} + 32H_{0,1,0,1,1} + 32H_{0,1,1,1,1}\right. \\
& \left.+ \left[-16H_0H_{0,1} + 16H_{0,0,1} + 16H_{0,1,1}\right]\zeta_2 - \frac{160}{3}H_{0,1}\zeta_3\right] - 80(7z-3)\zeta_5\left. + a_{gg,Q,\text{reg}}^{(3)}\right\}. \quad (613)
\end{aligned}$$

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