DESY 11-191

BFKL equation for the adjoint representation of the gauge group in the next-to-leading approximation at $N = 4$ SUSY †

V.S. Fadin *, L.N. Lipatov **
Universität Hamburg,
II. Institut für Theoretische Physik,
Luruper Chaussee, 149, D-22761 Hamburg,
* Budker Nuclear Physics Institute
and Novosibirsk State University,
630090, Novosibirsk, Russia
** Petersburg Nuclear Physics Institute
and St.Petersburg State University,
Gatchina, 188300, St.Petersburg, Russia

Abstract

We calculate the eigenvalues of the next-to-leading kernel for the BFKL equation in the adjoint representation of the gauge group $SU(N_c)$ in the N=4 supersymmetric Yang-Mills model. These eigenvalues are used to obtain the high energy behavior of the remainder function for the 6-point scattering amplitude with the maximal helicity violation in the kinematical regions containing the Mandelstam cut contribution. The leading and next-to-leading singularities of the corresponding collinear anomalous dimension are calculated in all orders of perturbation theory. We compare our result with the known collinear limit and with the recently suggested ansatz for the remainder function in three loops and obtain the full agreement providing that the numerical parameters in this anzatz are chosen in an appropriate way.

† *The work was supported in part by grant 14.740.11.0082 of Federal Program "Personnel of Innovational Russia," in part by RFBR grants 10-02-01238, 10-02-01338-a.*

1 Introduction

In the Regge pole model scattering amplitudes at large energies $\sqrt{s}$ and fixed momentum transfers $\sqrt{-t}$ have the form [1]

$$A^{p}_{Regge}(s,t) = \xi_p(t)\, s^{1+\omega_p(t)}\, \gamma^2(t)\,,\ \xi_p(t) = e^{-i\pi\omega_p(t)} - p\,, \tag{1}$$

where $p = \pm 1$ is the signature of the reggeon with the trajectory $\omega_p(t)$ and $\gamma^2(t)$ represents the product of reggeon vertices. The Pomeron is the Regge pole of the t-channel partial wave $f_\omega(t)$ with vacuum quantum numbers and the positive signature describing an approximately constant behaviour of total cross-sections for the hadron-hadron scattering. S. Mandelstam demonstrated, that the Regge poles generate cut singularities in the ω-plane [2].

In the leading logarithmic approximation (LLA) the scattering amplitude at high energies in QCD has the Regge form [3]

$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\, s^{\omega(t)}\,, \tag{2}$$

where M_{Born} is the Born amplitude and the gluon Regge trajectory is given below

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\, \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\ln\frac{|q^2|}{\lambda^2}\,. \tag{3}$$

Here λ is the infrared cut-off. In the multi-Regge kinematics, where the pair energies $\sqrt{s_k}$ of the produced gluons are large in comparison with momentum transfers $|q_i|$, the production amplitudes in LLA are constructed from products of the Regge factors $s_k^{\omega(t_k)}$ and effective reggeon-reggeon-gluon vertices $C_\mu(q_r, q_{r+1})$ [3]. The amplitudes satisfy the Steinmann relations and the s-channel unitarity incorporated in bootstrap equations [4].

The knowledge of $M_{2\to 2+n}$ allows one to construct the BFKL equation for the Pomeron wave function using analyticity, unitarity, renormalizability and crossing symmetry [3]. The integral kernel of this equation has the property of the holomorphic separability [5] and is invariant under the Möbius transformations [6]. The generalization of this equation to a composite state of several gluons [7] in the multi-color QCD leads to an integrable XXX model [8] having a duality symmetry [9].

The next-to-leading correction to the color singlet kernel in QCD is also calculated [10]. Its eigenvalue contains non-analytic terms proportional to $\delta_{n,0}$ and $\delta_{n,2}$, where n is the conformal spin of the Möbius group. But in the case of the $N = 4$ extended supersymmetric gauge model these Kronecker symbols are canceled leading to an expression having the properties of the hermitian separability [11] and maximal transcendentality [12]. The last property allowed to calculate the anomalous dimensions of twist-two operators up to three loops [13, 14]. It turns out, that evolution equations for the so-called quasi-partonic operators are integrable in $N = 4$ SUSY at the multi-color limit [15]. The $N = 4$ four-dimensional conformal field theory according to the Maldacena guess is equivalent to the superstring model living on the anti-de-Sitter 10-dimensional space [16, 17, 18]. Therefore the Pomeron in N=4 SUSY is equivalent to the reggeized graviton in this space. The equivalence gives a possibility to calculate the intercept of the BFKL Pomeron at large coupling constants [14, 19]. The Möbius invariance of the BFKL kernel was demonstrated also in two loops [20]. For next-to-leading calculations one can use the effective field theory for reggeized gluons [21].

The generalized bootstrap equation gives a possibility to prove the multi-Regge form of production amplitudes in the next-to-leading approximation [22].

Another application of the BFKL approach is a verification of the BDS ansatz [23] for the inelastic amplitudes in $N = 4$ SUSY. It was demonstrated [24, 25], that the BDS amplitude $M^{BDS}_{2\to 4}$ should be multiplied by the factor containing the contribution of the Mandelstam cuts [2] in LLA. In the two-loop approximation this factor can be found also from properties of analyticity and factorization [26] or directly from recently obtained exact result [27] for $M_{2\to 4}$ (see [28]). In a general case the wave function in LLA for the composite n-gluon state in the adjoint representation satisfies the Schrödinger equation for an open integrable Heisenberg spin chain [29].

In this paper we shall calculate the eigenvalues $\omega(t)$ of the kernel K for the BFKL equation in the adjoint representation of the gauge group at $N = 4$ SUSY in the next-to-leading approximation. The Green function of this equation allows one to find the asymptotic behavior of the inelastic amplitude in the Regge kinematics. There is a hypothesis [30, 31], that the inelastic amplitude with the maximal helicity violation in a planar approximation is factorized in the product of the BDS amplitude M^{BDS}, containing in crossing channels the Regge factors with corresponding infrared divergencies, and the remainder function R depending on the anharmonic ratios

$$M = R\, M^{BDS} \,. \tag{4}$$

In an accordance with this hypothesis the q^2-dependence of the eigenvalues of the octet BFKL equation is given by the expression (cf. [25] in LLA)

$$\omega(-q^2) = \omega_g(-q^2) + \omega_0 \,, \tag{5}$$

where $\omega_g(t)$ is the gluon Regge trajectory, which can be expressed in all orders of the perturbation theory of $N = 4$ SUSY in terms of two functions entering in the expression for the BDS amplitude [24]. The "intercept" ω_0 does not depend on q^2 due to the conformal invariance of $N = 4$ SUSY and can be written in terms of the "energy" $E = -\omega_0$ being the eigenvalue of the BFKL kernel discussed in the next section.

2 Integral kernel in the adjoint representation

The homogeneous BFKL equation can be written in the form

$$\omega_0 \phi = \hat{K} \phi \,, \tag{6}$$

where $\hat{K}$ is the integral operator from which the gluon Regge trajectory is subtracted. In the momentum representation it has the form

$$\hat{K}\phi(\vec{q}_1, \vec{q}_2) = \int \frac{d^2 q_1'}{|q_1'|^2 |q_2'|^2}\, K(\vec{q}_1, \vec{q}_1\,'; \vec{q})\, \phi(\vec{q}_1\,', \vec{q}_2\,') \,, \;\; \vec{q} = \vec{q}_1 + \vec{q}_2 = \vec{q}_1\,' + \vec{q}_2\,' \,. \tag{7}$$

The integral kernel for $N = 4$ SUSY can be presented as follows (cf. [3, 22] for the QCD case)

$$K(\vec{q}_1, \vec{q}_1\,'; \vec{q}) = \delta^2(\vec{q}_1 - \vec{q}_1\,') \vec{q}_1^{\,2} \vec{q}_2^{\,2} \left(\omega_g(-\vec{q}_1^{\,2}) + \omega_g(-\vec{q}_2^{\,2}) - \omega_g(-\vec{q}^{\,2}) \right) + K_r(\vec{q}_1, \vec{q}_1\,'; \vec{q}) \,, \tag{8}$$

where the first term corresponds to virtual corrections with the gluon regge trajectory subtraction (see (5)) and the second term appears from the real intermediate states in the s-channel. The total contribution does not contain infrared divergencies. Using results of Refs. [32] it can be written in the form

$$K(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = \frac{1}{2}\delta^2(\vec{q}_1 - \vec{q}_1^{\,\prime})\vec{q}_1^{\,2}\vec{q}_2^{\,2}\left(\omega_g(-\vec{q}_1^{\,2}) + \omega_g(-\vec{q}_2^{\,2}) - 2\omega_g(-\vec{q}^{\,2})\right) + K^{ns}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}), \quad (9)$$

where

$$\omega_g(-\vec{q}_1^{\,2}) + \omega_g(-\vec{q}_2^{\,2}) - 2\omega_g(-\vec{q}^{\,2}) = -\frac{\alpha\, N_c}{2\pi}\left(1 - \zeta(2)\,\frac{\alpha\, N_c}{2\pi}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}{\vec{q}^{\,4}}\right) \quad (10)$$

and

$$\begin{aligned} K^{ns}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}) = & -\delta^2(\vec{q}_1 - \vec{q}_1^{\,\prime})\vec{q}_1^{\,2}\vec{q}_2^{\,2}\frac{\alpha\, N_c}{8\pi^2}\Bigg(\left(1 - \zeta(2)\,\frac{\alpha\, N_c}{2\pi}\right)\int d^2k\,\left(\frac{2}{\vec{k}^{\,2}} + 2\frac{\vec{k}(\vec{q}_1 - \vec{k})}{\vec{k}^{\,2}(\vec{q}_1 - \vec{k})^2}\right) \\ & -3\alpha\, N_c\zeta(3)\Bigg) + \frac{\alpha\, N_c}{8\pi^2}\left\{\left(1 - \zeta(2)\,\frac{\alpha\, N_c}{2\pi}\right)\left(\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} + \vec{q}_1^{\,\prime\,2}\vec{q}_2^{\,2}}{\vec{k}^{\,2}} - \vec{q}^{\,2}\right) + \right. \\ & \frac{\alpha\, N_c}{4\pi}\left[\frac{\vec{q}^{\,2}}{2}\left(\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,2}}{\vec{q}^{\,2}}\right) + \ln\left(\frac{\vec{q}_1^{\,\prime\,2}}{\vec{q}^{\,2}}\right)\ln\left(\frac{\vec{q}_2^{\,\prime 2}}{\vec{q}^{\,2}}\right) + \ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)\right) - \frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} + \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}}\ln^2\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)\right. \\ & -\frac{1}{2}\,\frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} - \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}}\ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_1^{\,\prime\,2}}\right)\ln\left(\frac{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,4}}\right) + \Big[\vec{q}^{\,2}(\vec{k}^{\,2} - \vec{q}_1^{\,2} - \vec{q}_1^{\,\prime\,2}) \\ & \left.\left. +2\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2} - \vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} - \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2} + \frac{\vec{q}_1^{\,2}\vec{q}_2^{\,\prime\,2} - \vec{q}_2^{\,2}\vec{q}_1^{\,\prime\,2}}{\vec{k}^{\,2}}(\vec{q}_1^{\,2} - \vec{q}_1^{\,\prime\,2})\Big]\, I(\vec{q}_1^{\,2}, \vec{q}_1^{\,\prime\,2}, \vec{k}^{\,2})\right]\right\} \\ & + (\vec{q}_1 \leftrightarrow \vec{q}_2,\;\; \vec{q}_1^{\,\prime} \leftrightarrow \vec{q}_2^{\,\prime})\ , \end{aligned} \quad (11)$$

where $\vec{k} = \vec{q}_1 - \vec{q}_1^{\,\prime}$ and the function I is given below

$$I(\vec{q}_1^{\,2}, \vec{q}_1^{\,\prime\,2}, \vec{k}^{\,2}) = \int_0^1 \frac{dx}{\vec{q}_1^{\,2}(1-x) + \vec{q}_1^{\,\prime\,2}x - \vec{k}^{\,2}x(1-x)}\ln\left(\frac{\vec{q}_1^{\,2}(1-x) + \vec{q}_1^{\,\prime\,2}x}{\vec{k}^{\,2}x(1-x)}\right)\ . \quad (12)$$

Note that $I(a,b,c)$ is a totally symmetric function of the variables a, b and c.

One could expect, that the BFKL kernel in $N=4$ SUSY is Möbius invariant in the momentum representation, which would lead to the following simple form of its eigenfunctions (cf. [25])

$$\phi_{\nu n}(\vec{q}_1, \vec{q}_2) = \left|\frac{q_1}{q_2}\right|^{2i\nu} e^{i\,n\,\phi}\,, \quad (13)$$

where ϕ is the azimuthal angle of the complex number constructed from transverse components of the vectors $\vec{q}_1$ and $\vec{q}_2$

$$\frac{q_1}{q_2} = \left|\frac{q_1}{q_2}\right|\, e^{i\phi}\,. \quad (14)$$

However, in the existing form the kernel is not Möbius invariant and in future one should construct the similarity transformation to the invariant form (cf. [20]). Such transformation exists because the remainder function R, corresponding to the correction factor for the BDS expression, should be invariant under four-dimensional dual conformal transformations and the Green function obtained from the BFKL equation in the adjoint representation allows to find the asymptotic behavior of the remainder function in the Mandelstam kinematical regions [25].

3 Eigenvalues of the kernel

It is important, that the eigenvalues of the BFKL kernel do not depend on its representation and can be found from our expression (8). To calculate these eigenvalues we consider the BFKL equation in the limit (cf. [25])

$$|q_1| \sim |q_1'| \ll |q| \approx |q_2| \approx |q_2'| \,. \tag{15}$$

Denoting the two dimensional vectors $\vec{q}_1$ and $\vec{q}_1^{\,\prime}$ by $\vec{p}$ and $\vec{p}^{\,\prime}$, respectively, we write the BFKL equation in the form

$$\int \frac{d^2p\,'}{|p\,'|^2}\, K(\vec{p},\vec{p}^{\,\prime})\, \Phi(\vec{p}^{\,\prime}) = \omega_0\, \Phi(\vec{p}) \,. \tag{16}$$

Its kernel is given below

$$K(\vec{p},\vec{p}^{\,\prime}) = -\delta^2(\vec{p}-\vec{p}^{\,\prime})\, |p|^2\, \frac{\alpha N_c}{4\pi^2} \left(\left(1-\frac{\alpha N_c}{2\pi}\zeta(2)\right) \int d^2p\,' \left(\frac{2}{|p\,'|^2} + \frac{2(p\,',p-p\,')}{|p\,'|^2|p-p\,'|^2} \right) - 3\alpha\,\zeta(3) \right)$$

$$+\frac{\alpha N_c}{4\pi^2} \left(1-\frac{\alpha N_c}{2\pi}\zeta(2)\right) \left(\frac{|p|^2+|p\,'|^2}{|p-p\,'|^2} - 1 \right) + \frac{\alpha^2 N_c^2}{32\,\pi^3}\, R(\vec{p},\vec{p}^{\,\prime}) \,. \tag{17}$$

Here $\vec{p}$ and $\vec{p}^{\,\prime}$ are momenta of the same reggeized gluon before and after its scattering in the t_2-channel (momenta of another gluon tend to infinity together with q). The reduced kernel $R(\vec{p},\vec{p}^{\,\prime})$ is given below

$$R(\vec{p},\vec{p}^{\,\prime}) = \left(\frac{1}{2} - \frac{|p|^2+|p\,'|^2}{|p-p\,'|^2} \right) \ln^2 \frac{|p|^2}{|p\,'|^2} - \frac{|p|^2-|p\,'|^2}{2|p-p\,'|^2} \ln \frac{|p|^2}{|p\,'|^2} \ln \frac{|p|^2|p\,'|^2}{|p-p\,'|^4} \tag{18}$$

$$+\left(-|p+p\,'|^2 + \frac{(|p|^2-|p\,'|^2)^2}{|p-p\,'|^2} \right) \int_0^1 dx\, \frac{1}{|(1-x)p+xp\,'|^2} \ln \frac{|(1-x)p+xp\,'|^2}{x(1-x)|p-p\,'|^2} \,. \tag{19}$$

From the rotational and dilatational invariance of the kernel we obtain its eigenfunctions in the simple form

$$\Phi_{\nu n}(\vec{p}) = |p|^{2i\nu} e^{i\phi n} \,, \tag{20}$$

where ϕ is the angle of the transverse vector $\overrightarrow{p}$ with respect to the axis x. Note, that ν is real and n is integer.

The orthonormality condition for this set of functions is obvious

$$\frac{1}{2\pi^2} \int \frac{d^2p}{|p|^2}\, \Phi^*_{\mu m}(\vec{p})\, \Phi_{\nu n}(\vec{p}^{\,\prime}) = \delta(\mu-\nu)\, \delta_{m,n} \,. \tag{21}$$

The corresponding eigenvalues can be calculated with the action of the BFKL kernel on the eigenfunctions and are given below

$$\omega(\nu,n) = -a\left(E_{\nu n} + a\,\epsilon_{\nu n}\right)\,,\;\; a = \frac{\alpha N_c}{2\pi}\,, \tag{22}$$

where $E_{\nu n}$ is the "energy" in the leading approximation [25]

$$E_{\nu n} = -\frac{1}{2}\,\frac{|n|}{\nu^2+\frac{n^2}{4}} + \psi(1+i\nu+\frac{|n|}{2}) + \psi(1-i\nu+\frac{|n|}{2}) - 2\psi(1)\,,\;\; \psi(x) = (\ln\Gamma(x))' \tag{23}$$

and the next-to-leading correction $\epsilon_{\nu n}$ can be written as follows

$$\epsilon_{\nu n} = -\frac{1}{4}\left(\psi''(1+i\nu+\frac{|n|}{2}) + \psi''(1-i\nu+\frac{|n|}{2}) + \frac{2i\nu\left(\psi'(1-i\nu+\frac{|n|}{2}) - \psi'(1+i\nu+\frac{|n|}{2})\right)}{\nu^2+\frac{n^2}{4}}\right)$$

$$-\zeta(2)\,E_{\nu n} - 3\zeta(3) - \frac{1}{4}\,\frac{|n|\,\left(\nu^2-\frac{n^2}{4}\right)}{\left(\nu^2+\frac{n^2}{4}\right)^3}\,. \tag{24}$$

Here the ζ-functions are expressed in terms of polylogarithms

$$Li_n(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^n}\,,\;\; \zeta(n) = Li_n(1)\,. \tag{25}$$

Note, that $\omega(\nu,n)$ has the important property

$$\omega(0,0) = 0\,. \tag{26}$$

It is in an agreement with the existence of the eigenfunction $\Phi = 1$ with a vanishing eigenvalue, which is a consequence of the bootstrap relation [3, 22].

4 Corrections to the remainder function

One can easily construct the Green function for the conformally invariant BFKL kernel in terms of its eigenvalues. This Green function allows us to calculate the remainder functions R_n for an arbitrary number of external legs in the regions, where there are Mandelstam's cuts corresponding to the composite states of two reggeized gluons. For simplicity we consider the remainder function R_6 for the gluon transition $2 \to 4$ depending on three anharmonic ratios (cf. [28])

$$u_1 = \frac{s s_2}{s_{012}s_{123}}\,,\;\; u_2 = \frac{s_1 t_3}{s_{012}t_2}\,,\;\; u_3 = \frac{s_3 t_1}{s_{123}t_2}\,. \tag{27}$$

In the multi-regge kinematics one obtains

$$s \gg s_{012}, s_{123} \gg s_1, s_2, s_3 \gg t_1, t_2, t_3\,, \tag{28}$$

which corresponds to the following restrictions on the variables u_k

$$1-u_1 \to 0\,,\;\; \tilde{u}_2 = \frac{u_2}{1-u_1} \sim 1\,,\;\; \tilde{u}_3 = \frac{u_3}{1-u_1} \sim 1\,. \tag{29}$$

It is convenient also to introduce the complex variable w [28]

$$w = |w|e^{i\phi_{23}}\,,\ \ |w|^2 = \frac{u_2}{u_3}\,,\ \ \cos\phi_{23} = \frac{1-u_1-u_2-u_3}{2\sqrt{u_2u_3}} \tag{30}$$

expressed in terms of transverse momenta of produced particles k_1, k_2 and momentum transfers q_1, q_2, q_3

$$w = \frac{q_3k_1}{k_2q_1}\,. \tag{31}$$

In this case the remainder function R in the Mandelstam region, where

$$s, s_2 \to +\infty\,,\ \ s_1, s_3 \to -\infty\,, \tag{32}$$

can be presented in the form of a dispersion-like relation [26]

$$R\,e^{i\pi\delta} = cos\,\pi\omega_{ab} + i\,\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\,\Phi_{Reg}(\nu,n)\left(-\frac{1}{\sqrt{u_2u_3}}\right)^{\omega(\nu,n)}\,, \tag{33}$$

where

$$\delta = \frac{\gamma_K}{8}\,\ln(\tilde{u}_2\tilde{u}_3) = \frac{\gamma_K}{8}\,\ln\frac{|w|^2}{|1+w|^4}\,,\ \omega_{ab} = \frac{\gamma_K}{8}\,\ln\frac{\tilde{u}_2}{\tilde{u}_3} = \frac{\gamma_K}{8}\,\ln|w|^2 \tag{34}$$

and the cusp anomalous dimensions

$$\gamma_K = 4a - 4\,a^2\,\zeta(2) + 22\,\zeta(4)\,a^3 + ... \tag{35}$$

is known in all orders of perturbation theory [33].

Further, instead of the traditional variable $1/(1-u_1)$ (see [24, 25]) we used in eq. (33) the following energy invariant

$$\frac{1}{\sqrt{u_2u_3}} = s_2\,\frac{|q_2|^2}{\sqrt{|k_1|^2|q_1|^2\,|k_2|^2|q_3|^2}} = \frac{1}{1-u_1}\,\frac{|1+w|^2}{|w|}\,, \tag{36}$$

because according to the Regge theory the amplitude should be factorized in the t_2-channel. As a result, by expanding this expression for R in the perturbation theory

$$R = 1 + i\,a^2\left(b_1\ln\frac{1}{1-u_1}+b_2\right) + a^3\left(ic_1\ln^2\frac{1}{1-u_1} + (d_1+ic_2)\ln\frac{1}{1-u_1} + d_2 + ic_3\right) + ... =$$

$$1+i\,a^2\left(\widetilde{b}_1\ln\frac{1}{\sqrt{u_2u_3}}+\widetilde{b}_2\right)+a^3\left(i\widetilde{c}_1\ln^2\frac{1}{\sqrt{u_2u_3}} + (\widetilde{d}_1+i\widetilde{c}_2)\ln\frac{1}{\sqrt{u_2u_3}} + \widetilde{d}_2 + i\widetilde{c}_3\right) + ...\,, \tag{37}$$

we obtain [25, 28]

$$\widetilde{b}_1 = b_1 = -\frac{\pi}{2}\,\ln|1+w|^2\ln\frac{|1+w|^2}{|w|^2}\,, \tag{38}$$

$$\widetilde{b}_2 = b_2 - b_1\ln\frac{|1+w|^2}{|w|}\,,\ \ \frac{1}{\pi}\,b_2 = \frac{1}{2}\ln|w|^2\ln^2|1+w|^2$$

$$-\frac{1}{3}\ln^3|1+w|^2 + \ln|w|^2\left(Li_2(-w)+Li_2(-w^*)\right) - 2\left(Li_3(-w)+L_3(-w^*)\right)\,, \tag{39}$$

and (see ref. [28])

$$\frac{4}{\pi}\,\widetilde{c}_1 = \frac{4}{\pi}\,c_1 = \ln|w|^2\,\ln^2|1+w|^2 - \frac{2}{3}\ln^3|1+w|^2 - \frac{1}{4}\ln^2|w|^2\ln|1+w|^2$$

$$+\frac{1}{2}\,\ln|w|^2\left(Li_2(-w)+Li_2(-w^*)\right) - Li_3(-w) - Li_3(-w^*)\,, \tag{40}$$

$$\frac{4}{\pi^2}\,\widetilde{d}_1 = \frac{4}{\pi^2}\,d_1 = -\ln|w|^2\,\ln^2|1+w|^2 + \frac{2}{3}\ln^3|1+w|^2 + \frac{1}{2}\ln^2|w|^2\ln|1+w|^2$$

$$+\ln|w|^2\left(Li_2(-w)+Li_2(-w^*)\right) - 2Li_3(-w) - 2Li_3(-w^*)\,. \tag{41}$$

Note, that in the second order the real contribution to R is absent [26].

The product of two impact factors $\Phi_{Reg}(\nu,n)$ can be obtained with the use of the Fourier transformation of the function $\widetilde{b}_2$

$$\Phi_{Reg}(\nu,n) = 1 + \Phi^{(1)}_{Reg}(\nu,n)\,a + \Phi^{(2)}_{Reg}(\nu,n)\,a^2 + ...\,, \tag{42}$$

$$\Phi^{(1)}_{Reg}(\nu,n) = \Phi^{(1)}(\nu,n) + \Delta\Phi(\nu,n) = -\frac{1}{2}E^2_{\nu n} - \frac{3}{8}\,\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2} - \zeta(2)\,, \tag{43}$$

where $\Delta\Phi(\nu,n)$ is the contribution of the term $-b_1\ln\frac{|1+w|^2}{|w|}$ in $\widetilde{b}_2$ (39) and the contribution $\Phi^{(1)}(\nu,n)$ appearing from the term b_2 was calculated in ref. [28] [1]

$$\Phi^{(1)}(\nu,n) = E^2_{\nu n} - \frac{1}{4}\,\frac{n^2}{\left(\nu^2+\frac{n^2}{4}\right)^2} - \zeta(2)\,. \tag{44}$$

The knowledge of eigenvalues (24) in the next-to-leading approximation gives a possibility to calculate the coefficients $\widetilde{c}_2$ and $\widetilde{d}_2$ from expression (33)

$$\frac{1}{\pi}\,\widetilde{c}_2 = -\frac{1}{4}\,\ln|w|^2\left(S_{1,2}(-w)+S_{1,2}(-w^*)+\ln(1+w)\,Li_2(-w)+\ln(1+w^*)\,Li_2(-w*)\right)$$

$$+\frac{\zeta(3)}{2}\,\ln|1+w|^2 - \ln\frac{|1+w|^2}{|w|}\left(Li_3(-w)+Li_3(-w^*)-\frac{1}{2}\ln|w|^2(Li_2(-w)+Li_2(-w^*))\right)$$

$$+\frac{1}{4}\,\ln|1+w|^2(Li_3(-w)+Li_3(-w^*)) + \frac{1}{16}\,\ln^2|w|^2\ln|1+w|^2\ln\frac{|1+w|^2}{|w|^2}$$

$$+\frac{1}{8}\,\ln^2|1+w|^2\ln^2\frac{|1+w|^2}{|w|^2} + \frac{1}{8}\ln^2|w|^2\ln(1+w)\,\ln(1+w^*) + \zeta(2)\,\ln|1+w|^2\ln\frac{|1+w|^2}{|w|^2}\,, \tag{45}$$

$$\widetilde{d}_2 = \pi\left(\widetilde{c}_2 - \ln\frac{|1+w|^2}{|w|}\,\widetilde{b}_2 + 2\,\zeta(2)\,\widetilde{b}_1\right)\,. \tag{46}$$

[1] In the reference [28] the quantity $\Phi^{(1)}(\nu,n)$ was found for the remainder function, but here we need it for the full amplitude. According to (33) they differ by the term appearing from the expansion of $\exp(i\pi\delta)$ and proportional to the second order contribution to the anomalous dimension γ_K (35).

In the above expression the function $S_{1,2}(-x)$ has the following representation

$$S_{1,2}(-x)=\int_0^x \frac{dx'}{2x'}\,\ln^2(1+x')=Li_3(\frac{x}{1+x})+Li_3(-x)-\ln(1+x)Li_2(-x)-\frac{1}{6}\ln^3(1+x)\,. \quad (47)$$

One can verify with the use of the known relations among polylogarithms $Li_n(x)$, that the coefficients $\widetilde{c}_2$ and $\widetilde{d}_2$ are single-valued functions on the two-dimensional plane $\overrightarrow{w}$ and are symmetric to the inversion $w\to 1/w$. We can calculate also the coefficients c_2 and d_2 in (37) using the relations

$$c_2=\widetilde{c}_2+2\widetilde{c}_1\ln\frac{|1+w|^2}{|w|}\,,\ d_2=\widetilde{d}_2+\widetilde{d}_1\,\ln\frac{|1+w|^2}{|w|}\,. \quad (48)$$

Note, that recently the authors of ref. [35] suggested an anzatz for the remainder function R_6 in three loops based on the theory of symbols. They calculated its high energy behavior in our Mandelstam region in the form of the polynomial expansion in $\log(1-u_1)$. It turns out, that up to three loops their results are completely coincides with our perturbative expansion (37). In particular, one can derive the expressions (58) and (66) from the paper [35] using the fact, that the corresponding functions $g_1^{(2)}(w,w^*)$ and $h_0^{(3)}(w,w^*)$ are related with our coefficients c_2 and d_2 in (37) as follows

$$g_1^{(2)}(w,w^*)=-\frac{c_2}{2\pi}\,,\ h_0^{(3)}(w,w^*)=-\frac{d_2}{(2\pi)^2}\,. \quad (49)$$

It gives a possibility to fix the parameters γ' and γ''' appearing in ref. [35] in the form

$$\gamma'=-\frac{9}{2}\,,\ \gamma'''=0\,. \quad (50)$$

In expression (63) of the paper [35] also the additional function $g_0^{(3)}(w,w^*)$ was calculated. This function contains three unknown parameters appearing in the last line of (63). Our coefficients c_3 and $\widetilde{c}_3$ in (37) can be expressed in terms of it

$$c_3=2\pi\,g_0^{(3)}(w,w^*)\,,\ \widetilde{c}_3=c_3-\ln\frac{|1+w|^2}{|w|}\,c_2+\ln^2\frac{|1+w|^2}{|w|}\,c_1\,. \quad (51)$$

It gives a possibility to construct the following function

$$\rho(w,w^*)=\frac{\widetilde{c}_3}{\pi}+\pi\,\widetilde{c}_1+\ln\frac{|1+w|^2}{|w|}\left(\zeta(2)\,\ln^2\frac{|1+w|^2}{|w|}-\frac{11}{2}\,\zeta(4)\right), \quad (52)$$

where the term proportional to $\zeta(4)$ appears from the third order contribution to γ_K (35) which was calculated firstly in ref. [13]. The important next-to-next-to-leading corrections to the product of impact-factors $\Phi_{Reg}(\nu,n)$ (42) can be expressed through $\rho(w,w^*)$

$$\Phi_{Reg}^{(2)}(\nu,n)=(-1)^n\left(\nu^2+\frac{n^2}{4}\right)\int\frac{d^2w}{\pi}\,\rho(w,w^*)\,|w|^{-2i\nu-2}\left(\frac{w^*}{w}\right)^{\frac{n}{2}}\,. \quad (53)$$

We are going to calculate $\Phi_{Reg}^{(2)}(\nu,n)$ in future.

Similar results can be obtained for the remainder function describing the $3\to 3$ transitions in the corresponding Mandelstam regions [34].

5 Collinear limit

It is well known, that the BFKL equation for the Pomeron wave function gives a possibility to predict the leading singularities of the anomalous dimensions γ of the twist-2 operators at $\omega \to 0$ in all orders of perturbation theory [3, 10]. In particular, for the case of $N = 4$ SUSY the predictions of Ref. [11] are in a full agreement with the direct calculations of γ up to 5 loops [13, 36, 37]. As it follows from the previous section, the BFKL kernel for the adjoint representation of the gauge group allows one to find the high energy corrections to the remainder functions. On the other hand, in the collinear limit the remainder functions obey the renormalization group-like equations [38, 39]. The analytic continuation of the collinear expressions for R to the Mandelstam regions was performed in Ref. [40]. The leading asymptotics corresponds to the unit conformal spin $|n| = 1$. The anomalous dimensions γ_{col} for the collinear limit in the Euclidean region were constructed [39] and the relation between the Regge and collinear limits was investigated [40].

To calculate γ_{col} in the Mandelstam region we present expression (33) in the following form with the use of the Fourier transformation

$$R\, e^{i\pi\delta} = cos\, \pi\omega_{ab} + i\, \frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{\infty} d\nu\, |w|^{2i\nu}\, L_{\nu n}\left(-\frac{1}{1-u_1}\right), \tag{54}$$

where

$$L_{\nu n}\left(-\frac{1}{1-u_1}\right) = \sum_{n'=-\infty}^{\infty} (-1)^{n'-n} \int_{-\infty}^{\infty} \frac{\Phi_{reg}(\nu', n')d\nu'}{\nu'^2 + \frac{n'^2}{4}}\, S_{\nu' n'}^{\nu n} \left(-\frac{1}{1-u_1}\right)^{\omega(\nu', n')} \tag{55}$$

and

$$S_{\nu' n'}^{\nu n} = \int \frac{d^2 w}{2\pi^2}\, |w|^{2i(\nu'-\nu)-2} \left(\frac{w}{w^*}\right)^{\frac{n'-n}{2}} \left(\frac{|1+w|^2}{|w|}\right)^{\omega(\nu' n')}. \tag{56}$$

The collinear limit $w \to 0$ or $w \to \infty$ of the remainder function (54) should be performed at fixed $1 - u_1$ [39, 40]. Generally expressions (54) and (55) correspond to the collinear renormalization with an infinite number of the multiplicatively renormalizable operators (cf. [40]). But in the case, when we take into account only the asymptotic terms at $|w| \to \infty$ with the conformal spin $|n| = 1$, we can obtain for R the simple expression

$$R\, e^{i\pi\delta} \approx cos\, \pi\omega_{ab} - ia\, \cos\phi_{23}\, |w|^{-1} \int_{-i\infty}^{i\infty} d\omega\, \frac{\Phi^{Reg}(\nu, 1)}{\left(\nu^2 + \frac{1}{4}\right) \frac{d\omega}{d\nu}}\, |w|^{2\gamma_{col}(\omega)} \left(-\frac{1}{1-u_1}\right)^{\omega}, \tag{57}$$

where the contour of integration goes to the right of the BFKL singularity $\nu \sim \sqrt{\omega - \omega(0,1)}$ present in the integrand in an accordance with the fact, that the functions $\gamma = \gamma_{col}(\omega)$, $\nu = \nu(\omega)$ satisfy the set of equations [2]

$$\gamma = \frac{1}{2} + i\nu + \frac{\omega}{2}\,, \;\; \omega = \omega(\nu, 1)\,. \tag{58}$$

[2]Note, that our definition of the collinear anomalous dimension γ_{col} differs with the factor $-1/2$ from that used in ref. [39].

For finding γ_{col} in perturbation theory the function $\omega(\nu,1)$ (22) should be expanded near the point $\nu = i/2$

$$\lim_{\nu\to\frac{i}{2}} \omega(\nu,1) = \frac{a}{2}\, f_1\left(i\nu+\frac{1}{2}\right) - \frac{a^2}{8}\, f_2\left(i\nu+\frac{1}{2}\right)\,, \tag{59}$$

where

$$f_1(x) = \frac{1}{x} - 1 - x - x^2(1-4\zeta(3)) - x^3 + O(x^4)\,, \tag{60}$$

$$f_2(x) = \frac{1}{x^3} + \frac{1}{x^2} + \frac{4\zeta(2)}{x} - 8\zeta(3) - 4\zeta(2) - 2 + O(x)\,, \tag{61}$$

Thus, we obtain the following equation for $\gamma = \gamma_{col}(\omega)$

$$\omega = \frac{a}{2}\, f_1(\gamma) - \frac{a^2}{8}\, \left(f_1'(\gamma) f_1(\gamma) + f_2(\gamma)\right)\,. \tag{62}$$

Its perturbative solution is given below

$$\gamma_{col}(\omega) = \frac{a}{2}\left(\frac{1}{\omega}-1\right) - \frac{a^2}{4}\left(\frac{1}{\omega^2} + 2\,\frac{\zeta(2)}{\omega}\right) + \frac{a^3}{4\,\omega^2}\,(1+2\zeta(2)+\zeta(3)) + O(a^4)\,. \tag{63}$$

The above approach is similar to that for the singlet BFKL kernel, but in that case one obtained the main contribution to the Bjorken limit from $n = 0$ [10].

The collinear anomalous dimension $\gamma_{col}(\omega)$ in the Mandelstam region $s, s_2 > 0, s_1, s_3 < 0$ can be found in one loop using the results of the paper [40]. We start with the perturbative expansion of the remainder function in the collinear limit $|w| \to \infty$ in LLA of the Operator Product Expansion (OPE) [38]

$$R_{OPE} \approx a\cos\phi\, \frac{e^{-\sigma}}{2|w|} \sum_{k=0}^{\infty} \frac{(-a\ln|w|)^k}{k!}\, h_k(\sigma)\,,\ \ \sigma = \frac{1}{2}\,\ln\frac{u_1}{1-u_1}\,, \tag{64}$$

where we expressed the world sheet coordinates τ and σ in terms of our variables w and u_1 (see eqs (76)-(79) from ref. [40]) and included one loop contribution contained in the BDS amplitude. The analytic continuation of the two loop remainder function calculated in ref. [27] to the Mandelstam region $s, s_2 > 0,\ s_1, s_3 < 0$ gives the result (see eqs. (51), (C.12)-(C.16) from ref. [40])

$$\cos\phi\ \to \cos\phi_{23};\ \ h_k(\sigma) \to -h_k(\sigma) + \Delta_k(\sigma)\,,\ \frac{\Delta_0(\sigma)}{2\pi i} = -2\, e^{\sigma}\,,$$

$$\frac{\Delta_1(\sigma)}{2\pi i} = 4\left(\cosh\sigma\ln(1+e^{2\sigma}) - e^{\sigma}\right)\,. \tag{65}$$

Here the functions $h_k(\sigma)$ for $k = 0, 1$ in the right hand side of the first relation are known from ref. [39]. They are not essential for the calculation of γ_{col} because they are real and fall at large σ. The contributions $\Delta_k(\sigma)$ appear from the analytic continuation of the corresponding discontinuities of the functions $h_k(\sigma)$ on the cut $-1 < \widetilde{s}_2 < 0$, where $\widetilde{s}_2 = \exp(2\sigma)$ [40]. After the continuation we can write this discontinuity using the collinear renormalization group in the form

$$\Delta R_{OPE} = -\, a\, \cos\phi_{23}\, \frac{1}{|w|} \int_{-i\infty}^{i\infty} \frac{d\omega}{\omega(\omega+1)}\, |w|^{2\gamma_{col}(\omega)} e^{2\omega\sigma}\,, \tag{66}$$

where

$$\gamma_{col}(\omega) = a\,\omega\,(1+\omega)\int_0^\infty d(2\sigma)\,e^{-2\sigma\omega}\left(e^{-\sigma}\cosh(\sigma)\ln\left(1+e^{2\sigma}\right)-1\right) =$$

$$\frac{a}{2}\left(\frac{1}{\omega(\omega+1)} - 2\omega + (\omega+1)\left(\psi(\omega+1)-\psi(\frac{\omega+2}{2})\right) + \omega\,\psi(\omega+2) - \omega\,\psi(\frac{\omega+3}{2})\right). \quad (67)$$

As one can see from expression (63), the BFKL approach reproduces correctly the first two terms of γ_{col} at $\omega \to 0$.

6 Conclusion

In this paper we solved the BFKL equation for the channel with color octet quantum numbers in the next-to-leading approximation. The eigenvalues of its integral kernel were used to calculate in the next-to-leading logarithmic approximation the remainder function for the production amplitude $2 \to 4$ in the multi-Regge kinematics at the Mandelstam channels. The obtained result in three loops is in an agreement with the recently suggested anzatz [35] for the remainder function. This anzatz allowed us to construct the product of corresponding impact-factors in the next-to-next-to-leading approximation. The collinear anomalous dimension in the Mandelstam region was calculated explicitly in one loop. Its leading and next-to-leading singularities are found in all loops.

Acknowledgments. We thank J. Bartels, A. Prygarin and G. Vacca for helpful discussions, the Hamburg University and DESY for the warm hospitality and support. This work was done in the framework of the program LEXI "Connecting Particles with the Cosmos".

References

[1] V. N. Gribov, Sov. Phys. JETP **14** (1962) 478.

[2] S. Mandelstam, Nuovo Cim. **30** (1963) 1148.

[3] L. N. Lipatov, Sov. J. Nucl. Phys. **23** (1976) 338;
V. S. Fadin, E. A. Kuraev and L. N. Lipatov, Phys. Lett. **B 60** (1975) 50;
E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **44** (1976) 443; **45** (1977) 199;
Ya. Ya. Balitskii and L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822.

[4] Ya. Ya. Balitskii, L. N. Lipatov and V. S. Fadin, in Materials of IV Winter School of LNPI (Leningrad, 1979) p.109.

[5] L. N. Lipatov, Phys. Lett. **B 309** (1993) 394.

[6] L. N. Lipatov, Sov. Phys. JETP **63** (1986) 904.

[7] J. Bartels, Nucl. Phys. **B 175** (1980) 365;
J. Kwiecinskii and M. Praszalowicz, Phys. Lett. **B 94** (1980) 413.

[8] L. N. Lipatov, High energy asymptotics of multi-colour QCD and exactly solvable lattice models, arXiv:9311037 [hep-th].

[9] L. N. Lipatov, Nucl. Phys. **B 548** (1999) 328.

[10] V. S. Fadin and L. N. Lipatov, Phys. Lett. **B 429** (1998) 127;
M. Ciafaloni and G. Camici, Phys. Lett. **B 430** (1998) 349.

[11] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. **B 582** (2000) 19.

[12] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. **B 661** (2003) 19.

[13] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, Phys. Lett. **B 595** (2004) 521;

[14] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, Phys. Lett. **B 632** (2006) 754.

[15] L. N. Lipatov, talk at "Perspectives in Hadronic Physics", Proc. of Conf. ICTP, Triest, Italy, May 1997.

[16] J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231.

[17] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. **B 428** (1998) 105.

[18] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253.

[19] R. C. Brower, J. Polchinski, M. J. Strassler, C. I. Tan, JHEP **0712** (2007) 005.

[20] I. Balitsky and G. A. Chirilli, Nucl. Phys. **B 822** (2009) 45;
V. S. Fadin, R. Fiore and A. V. Grabovsky, Nucl. Phys. **B 831** (2010) 248.

[21] L. N. Lipatov, Nucl. Phys. **B 452** (1995) 369; Phys. Rept. **286** (1997) 131.

[22] V. S. Fadin, R. Fiore, M. G. Kozlov and A. V. Reznichenko, Phys. Lett.**B 639** (2006) 74.

[23] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. **D 72** (2005) 085001.

[24] J. Bartels, L. N. Lipatov and A. Sabio Vera, Phys. Rev. **D 80** (2009) 045002.

[25] J. Bartels, L. N. Lipatov and A. Sabio Vera, Eur. Phys. J. **C 65** (2009) 587.

[26] L. N. Lipatov, arXiv:1008.1015 [hep-th], Proceedings of the conference "Quarks-2010".

[27] A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, arXiv:1006.5703 [hep-th].

[28] L. N. Lipatov and A. Prygarin, Phys. Rev. **D 83** (2011) 045020; **D 83** (2011) 125001.

[29] L. N. Lipatov, J. Phys. **A 42** (2009) 304020.

[30] L. F. Alday and J. M. Maldacena, IHEP **0706** (2007) 064.

[31] J. M. Drummond, J. Henn, G. P.Korchemsky and E. Sokatchev, Nucl. Phys. **B 826** (2010) 337 ; **B 828** (2010) 3179.

[32] V. S. Fadin, R. Fiore, A. V. Grabovsky and A. Papa, Nucl. Phys. **B 784** (2007) 49;
R. E. Gerasimov and V. S. Fadin, Phys. Atom. Nucl. **73** (2010) 1214;
V.S. Fadin, R. Fiore and A. Papa, Phys. Rev. **D 60** (1999) 074025.

[33] N. Beisert, B. Eden, M. Staudacher, J. Stat. Mech. **0701** (2007) P01021.

[34] J. Bartels, L. N. Lipatov and A. Prygarin, arXiv:1012.3178 [hep-th].

[35] L. J. Dixon, J. M. Drummond and J. M. Henn, arXiv:1108.4461 [hep-th].

[36] A. V. Kotikov, L. N. Lipatov, A. Rej, M. Staudacher and V. N. Velizhanin, J. Stat. Mech. **0710** (2007) P10003;
Z. Bajnok, R. A. Janik and T. Lukovski, Nucl. Phys. **B 816** (2009) 376.

[37] T. Lukovski, A. Rej and V. N. Velizhanin, Nucl. Phys. **B 831** (2010) 105.

[38] L. F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, IHEP **1104** (2011) 088.

[39] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, arXiv:1102.0062 [hep-th].

[40] J. Bartels, L. N. Lipatov and A. Prygarin, arXiv:1104.4709 [hep-th].