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Superconformal indices and partition functions for supersymmetric field theories

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Recently there was a substantial progress in understanding of supersymmetric theories (in particular, their BPS spectrum) in space-times of different dimensions due to the exact computation of superconformal indices and partition functions using localization method. Here we discuss a connection of $4d$ superconformal indices and $3d$ partition functions using a particular example of supersymmetric theories with matter in antisymmetric representation.

Keywords: Supersymmetric Dualities; Superconformal Index; Elliptic Hypergeometric Integrals.

1. Introduction

In a remarkable paper[1] Dolan and Osborn recognized the fact that the superconformal indices (SCIs) of $4d$ supersymmetric gauge theories[2,3] are expressed in terms of elliptic hypergeometric integrals (EHI).[4] This observation provides currently the most rigorous mathematical confirmation of $\mathcal{N}=1$ Seiberg electro-magnetic duality[5] through the equality of dual indices. The interrelation between SCIs and EHIs was systematically studied[6–8] and there were found many new $\mathcal{N}=1$ physical dualities and also conjectured new identities for EHIs. In particular, it was shown[9] that all 't Hooft anomaly matching conditions for Seiberg dual theories can be derived from $SL(3,\mathbb{Z})$-modular transformation properties of the kernels of dual indices. The theory of EHIs was applied also to a description of the S-duality conjecture for $\mathcal{N}=2,4$ extended supersymmetric field theories.[10] Several modifications of SCIs have been considered recently such as the inclusion of charge conjugation,[11] indices on lens spaces,[12] inclusion of surface operators[13] or line operators.[14,15]

By definition the SCI counts the BPS states protected by one supersymmetry which can not be combined to form long multiplets. The $SU(2,2|1)$ space-time symmetry group of $\mathcal{N}=1$ superconformal algebra consists of $J_i, \overline{J}_i$, the generators of

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two $SU(2)$ subgroups forming the Lorentz group, translations P_μ, special conformal transformations K_μ, $\mu = 1, 2, 3, 4$, the dilatations H and also the $U(1)_R$ generator R. Apart from the bosonic generators there are supercharges $Q_\alpha, \overline{Q}_{\dot\alpha}$ and their superconformal partners $S_\alpha, \overline{S}_{\dot\alpha}$. Distinguishing a pair of supercharges,[3] for example, $Q = \overline{Q}_1$ and $Q^\dagger = -\overline{S}_1$, one has $\{Q, Q^\dagger\} = 2\mathcal{H}, \quad \mathcal{H} = H - 2\overline{J}_3 - 3R/2$, and then the superconformal index is defined by the matrix integral

$$I(p, q, f_k) = \mathrm{Tr}\Big((-1)^{\mathcal{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} e^{\sum_k f_k F^k} e^{-\beta\mathcal{H}}\Big), \quad \mathcal{R} = R + 2\overline{J}_3, \tag{1}$$

where $\mathcal{F}$ is the fermion number operator. Only zero modes of $\mathcal{H}$ contribute to the trace because the commutation relation for the supercharges is preserved by the operators used in (1). The chemical potentials f_k are the group parameters of the flavor symmetry group with the maximal torus generators F^k; p and q are group parameters for operators $\mathcal{R}/2 \pm J_3$ commuting with Q and $Q^\dagger$.

According to the Römelsberger prescription[3] for $\mathcal{N} = 1$ superconformal theories one can write the full index via a "plethystic" exponential

$$I(p, q, \underline{y}) = \int_{G_c} d\mu(g)\, \exp\Big(\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{ind}\big(p^n, q^n, \underline{z}^n, \underline{y}^n\big)\Big), \tag{2}$$

where $d\mu(g)$ is the G_c-invariant measure and single particle states index

$$\begin{aligned}\mathrm{ind}(p, q, \underline{z}, \underline{y}) = {} & \frac{2pq - p - q}{(1-p)(1-q)} \chi_{adj}(\underline{z}) \\ & + \sum_j \frac{(pq)^{R_j/2} \chi_{R_F,j}(\underline{y}) \chi_{R_G,j}(\underline{z}) - (pq)^{1-R_j/2} \chi_{\bar{R}_F,j}(\underline{y}) \chi_{\bar{R}_G,j}(\underline{z})}{(1-p)(1-q)},\end{aligned}$$

where the first term represents contributions of the gauge superfields lying in the adjoint representation of the gauge group G_c. The sum over j corresponds to the contribution of chiral matter superfields φ_j transforming as the gauge group representations $R_{G,j}$ and flavor symmetry group representations $R_{F,j}$ with R_j being the field R-charges. The functions $\chi_{adj}(\underline{z})$, $\chi_{R_F,j}(\underline{y})$ and $\chi_{R_G,j}(\underline{z})$ are the corresponding characters.

Let us consider the initial Seiberg duality[5] for SQCD. Namely, we take a $4d$ $\mathcal{N} = 1$ SYM theory with $G_c = SU(N_c)$ gauge group and N_f flavors with $SU(N_f)_l \times SU(N_f)_r \times U(1)_B$ flavor symmetry group. The original (electric) theory has N_f left and N_f right quarks Q and $\widetilde{Q}$ lying in fundamental and anti–fundamental representation of the gauge group $SU(N_c)$ and having $+1$ and -1 baryonic charges, $R = (N_f - N_c)/N_f$ is their R-charge[a]. The field content of the described theory is summarized in the following table

	$SU(N_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
Q	f	f	1	1	$\tilde{N}_c/N_f$
$\widetilde{Q}$	$\overline{f}$	1	$\overline{f}$	-1	$\tilde{N}_c/N_f$
V	adj	1	1	0	1

[a]This is the R-charge for the scalar component, the R-charge of the fermion component is $R - 1$.

The corresponding SCI is given by the following elliptic hypergeometric integral[1]

$$I_E = \kappa_{N_c} \int_{\mathbb{T}^{N_c-1}} \frac{\prod_{i=1}^{N_f}\prod_{j=1}^{N_c}\Gamma(s_i z_j, t_i^{-1}z_j^{-1};p,q)}{\prod_{1\le i<j\le N_c}\Gamma(z_i z_j^{-1}, z_i^{-1}z_j;p,q)} \prod_{j=1}^{N_c-1}\frac{dz_j}{2\pi \mathrm{i} z_j}, \qquad (3)$$

where $\prod_{j=1}^{N_c} z_j = 1$. The balancing condition reads $ST^{-1} = (pq)^{N_f-N_c}$ with $S = \prod_{i=1}^{N_f} s_i$, $T = \prod_{i=1}^{N_f} t_i$. We introduced the parameters s_i and t_i as

$$s_i = (pq)^{R/2} v x_i, \qquad t_i = (pq)^{-R/2} v y_i, \qquad (4)$$

where x_i, y_i are chemical potentials for $SU(N_f)_l$ and $SU(N_f)_r$ groups satisfying the constraints $\prod_{i=1}^{N_f} x_i = \prod_{i=1}^{N_f} y_i = 1$, v is the chemical potential for $U(1)_B$-group, and

$$\kappa_{N_c} = \frac{(p;p)_\infty^{N_c-1}(q;q)_\infty^{N_c-1}}{N_c!}, \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Here $\mathbb{T}$ denotes the unit circle with positive orientation and we use conventions $\Gamma(a,b;p,q) := \Gamma(a;p,q)\Gamma(b;p,q)$, $\Gamma(az^{\pm 1};p,q) := \Gamma(az;p,q)\Gamma(az^{-1};p,q)$, where

$$\Gamma(z;p,q) = \prod_{i,j=0}^{\infty} \frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j}, \quad |p|,|q|<1, \qquad (5)$$

is the elliptic gamma function.

The dual (magnetic) theory is described by a $4d$ $\mathcal{N}=1$ SYM theory with the gauge group $\widetilde{G}_c = SU(N_f - N_c)$ sharing the same flavor symmetry.[5] Here one has dual quarks q and $\widetilde{q}$ lying in the fundamental/anti–fundamental representation of $\widetilde{G}_c$, which have $U(1)_B$–charges $N_c/(N_f-N_c)$, $-N_c/(N_f-N_c)$ and the R-charge N_c/N_f, and additional mesons – singlets of $\widetilde{G}_c$ lying in the fundamental representation of $SU(N_f)_l$ and anti–fundamental representation of $SU(N_f)_r$ ($M_i^j = Q_i\widetilde{Q}^j, i,j = 1,\ldots,N_f$). It is convenient to collect again all field data in one table

	$SU(\tilde{N}_c)$	$SU(N_f)_l$	$SU(N_f)_r$	$U(1)_B$	$U(1)_R$
M	1	f	$\overline{f}$	0	$2\tilde{N}_c/N_f$
q	f	$\overline{f}$	1	$N_c/\tilde{N}_c$	N_c/N_f
$\widetilde{q}$	$\overline{f}$	1	f	$-N_c/\tilde{N}_c$	N_c/N_f
V	adj	1	1	0	1

These two SQCD-type theories are dual to each other in their infrared fixed points when the magnetic theory has dynamically generated superpotential,[5] $W_{dyn} = M_i^j q^i \widetilde{q}_j$. The SCI of the magnetic theory is

$$I_M = \kappa_{N_{\widetilde{N}_c}} \prod_{1\le i,j\le N_f}\Gamma(s_i t_j^{-1};p,q) \int_{\mathbb{T}^*} \prod_{j=1}^{\widetilde{N}_c-1}\frac{d\widetilde{z}_j}{2\pi\mathrm{i}\widetilde{z}_j} \frac{\prod_{i=1}^{N_f}\prod_{j=1}^{\widetilde{N}_c}\Gamma(S^{1/\widetilde{N}_c}s_i^{-1}\widetilde{z}_j, T^{-1/\widetilde{N}_c}t_i\widetilde{z}_j^{-1};p,q)}{\prod_{1\le i<j\le\widetilde{N}_c}\Gamma(\widetilde{z}_i\widetilde{z}_j^{-1},\widetilde{z}_i^{-1}\widetilde{z}_j;p,q)}, \qquad (6)$$

where $\widetilde{N}_c = N_f - N_c$, $\prod_{j=1}^{\widetilde{N}_c}\widetilde{z}_j = 1$ and $\mathbb{T}^* = \mathbb{T}^{\widetilde{N}_c-1}$.

As discovered by Dolan and Osborn,[1] the equality of SCIs $I_E = I_M$ coincides with a mathematical identity established for $N=2, N_f = 3, 4$[4] and for arbitrary parameters.[16]

2. The anti-symmetric tensor matter field

Recently the connection of $4d$ SCIs and $3d$ PFs was found[17] and the simplest example of SQCD type theory with $SP(2N)$ gauge group was considered. Here we would like to consider more complicated cases with additional matter content. We start from the duality for $4d$ supersymmetric theory with the $SP(2N)$ group introduced by Intriligator.[18] The matter content of electric and magnetic theories are given below in tables, respectively:

	$SP(2N)$	$SU(2N_f)$	$U(1)_R$
Q	f	f	$2r = 1 - \frac{2(N+K)}{(K+1)N_f}$
X	T_A	1	$2s = \frac{2}{K+1}$

	$SP(2\widetilde{N})$	$SU(2N_f)$	$U(1)_R$
q	f	$\overline{f}$	$2\widetilde{r} = 1 - \frac{2(\widetilde{N}+K)}{(K+1)N_f}$
Y	T_A	1	$2s = \frac{2}{K+1}$
M_j	1	T_A	$2r_j = 2\frac{K+j}{K+1} - 4\frac{\widetilde{N}+K}{(K+1)N_f}$

where $j = 1, \dots, K$, and $\widetilde{N} \ = \ K(N_f - 2) - N, \quad K = 1, 2, \dots$.

Defining $U = (pq)^s = (pq)^{\frac{1}{K+1}}$, we find the following indices for these theories[7]

$$I_E = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \Gamma(U;p,q)^{N-1} \tag{7}$$
$$\times \int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\Gamma(U z_i^{\pm1} z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1} z_j^{\pm1};p,q)} \prod_{j=1}^N \frac{\prod_{i=1}^{2N_f} \Gamma(s_i z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)} \prod_{j=1}^N \frac{dz_j}{2\pi \mathrm{i} z_j},$$

$$I_M = \frac{(p;p)_\infty^{\widetilde{N}} (q;q)_\infty^{\widetilde{N}}}{2^{\widetilde{N}} \widetilde{N}!} \Gamma(U;p,q)^{\widetilde{N}-1} \prod_{l=1}^K \prod_{1\le i<j\le 2N_f} \Gamma(U^{l-1} s_i s_j;p,q) \tag{8}$$
$$\times \int_{\mathbb{T}^{\widetilde{N}}} \prod_{1\le i<j\le \widetilde{N}} \frac{\Gamma(U z_i^{\pm1} z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1} z_j^{\pm1};p,q)} \prod_{j=1}^{\widetilde{N}} \frac{\prod_{i=1}^{2N_f} \Gamma(U s_i^{-1} z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)} \prod_{j=1}^{\widetilde{N}} \frac{dz_j}{2\pi \mathrm{i} z_j},$$

where the balancing condition reads $U^{2(N+K)} \prod_{i=1}^{2N_f} s_i = (pq)^{N_f}$.

Using the asymptotic formula for the elliptic gamma function

$$\Gamma(e^{2\pi \mathrm{i} rz}; e^{2\pi \mathrm{i} r\omega_1}, e^{2\pi \mathrm{i} r\omega_2}) \underset{r\to 0}{=} e^{-\pi \mathrm{i}(2z-(\omega_1+\omega_2))/24r\omega_1\omega_2} \gamma^{(2)}(z;\omega_1,\omega_2), \tag{9}$$

where $\gamma^{(2)}(z)$ is a hyperbolic gamma function, one can proceed with the reduction of SCIs for a dual pair presented above. Let us reparameterize the variables in (7) and (8) in the following way $p = e^{2\pi \mathrm{i} v\omega_1}, \ q = e^{2\pi \mathrm{i} v\omega_2}, \ s_i = e^{2\pi \mathrm{i} v\alpha_i}, \ z_j = e^{2\pi \mathrm{i} v u_j}$,, $i = 1, \dots, 2N_f, \ j = 1, \dots, N$. Then after limit $v \to 0$ one gets[b]

$$I_E^{red} = \frac{1}{2^N N!} \gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{N-1} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \prod_{1\le i<j\le N} \frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1} \pm u_i \pm u_j\big)}{\gamma\big(\pm u_i \pm u_j\big)} \prod_{j=1}^N \frac{\prod_{i=1}^{2N_f} \gamma\big(\alpha_i \pm u_j\big)}{\gamma\big(\pm 2u_j\big)} \frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}}, \tag{10}$$

[b] Omitting the same divergent coefficients $\exp\left(\frac{-2\pi i(-1+K-6KN-4N^2)(\omega_1+\omega_2)}{24v\omega_1\omega_2(1+K)}\right)$.

$$I_M^{red} = \frac{1}{2^{\widetilde{N}}\widetilde{N}!}\gamma\left(\tfrac{\omega_1+\omega_2}{K+1}\right)^{\widetilde{N}-1}\prod_{l=1}^{K}\prod_{1\le i<j\le 2N_f}\gamma\left((l-1)\tfrac{\omega_1+\omega_2}{K+1}+\alpha_i+\alpha_j\right) \tag{11}$$

$$\times\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\le i<j\le\widetilde{N}}\frac{\gamma\left(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\right)}{\gamma(\pm u_i\pm u_j)}\prod_{j=1}^{\widetilde{N}}\frac{\prod_{i=1}^{2N_f}\gamma\left(\frac{\omega_1+\omega_2}{K+1}-\alpha_i\pm u_j\right)}{\gamma(\pm 2u_j)}\prod_{j=1}^{\widetilde{N}}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}},$$

where the balancing condition reads $(\omega_1+\omega_2)\frac{2(N+K)}{(K+1)}+\sum_{i=1}^{2N_f}\alpha_i = N_f(\omega_1+\omega_2)$. Above and in the rest of the paper, we use the following notation $\gamma(z)\equiv\gamma^{(2)}(z;\omega_1,\omega_2)$ and conventions $\gamma(a,b)\equiv\gamma(a)\gamma(b)$, $\gamma(a\pm u)\equiv\gamma(a+u)\gamma(a-u)$.

2.1. *Dualities for $SP(2N)$ gauge group*

Let us consider now $\alpha_{2N_f}=\xi_1+aS$, $\alpha_{2N_f-1}=\xi_2-aS$ and take the limit $S\to\infty$, then I_E^{red} and I_M^{red} become

$$Z_E = \frac{1}{2^N N!}\gamma\left(\tfrac{\omega_1+\omega_2}{K+1}\right)^{N-1}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\le i<j\le N}\frac{\gamma\left(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\right)}{\gamma(\pm u_i\pm u_j)}\prod_{j=1}^{N}\frac{\prod_{i=1}^{2(N_f-1)}\gamma(\alpha_i\pm u_j)}{\gamma(\pm 2u_j)}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}} \tag{12}$$

$$Z_M = \frac{1}{2^{\widetilde{N}}\widetilde{N}!}\gamma\left(\tfrac{\omega_1+\omega_2}{K+1}\right)^{\widetilde{N}-1}\prod_{l=1}^{K}\gamma\left((\omega_1+\omega_2)\left(N_f-\tfrac{2N+2K-l+1}{K+1}\right)-\textstyle\sum_{i=1}^{2(N_f-1)}\alpha_i\right) \tag{13}$$

$$\times\prod_{l=1}^{K}\prod_{1\le i<j\le 2(N_f-1)}\gamma\left((l-1)\tfrac{\omega_1+\omega_2}{K+1}+\alpha_i+\alpha_j\right)$$

$$\times\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\le i<j\le\widetilde{N}}\frac{\gamma\left(\frac{\omega_1+\omega_2}{K+1}\pm u_i\pm u_j\right)}{\gamma(\pm u_i\pm u_j)}\prod_{j=1}^{\widetilde{N}}\frac{\prod_{i=1}^{2(N_f-1)}\gamma\left(\frac{\omega_1+\omega_2}{K+1}-\alpha_i\pm u_j\right)}{\gamma(\pm 2u_j)}\prod_{j=1}^{\widetilde{N}}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}}.$$

To obtain these expressions we used the inversion relation $\gamma(z,\omega_1+\omega_2-z)=1$ and the asymptotic formulas

$$\begin{aligned}\lim_{u\to\infty} e^{\frac{\pi\mathrm{i}}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma(u)=1,\quad &\text{for }\arg\omega_1<\arg u<\arg\omega_2+\pi,\\ \lim_{u\to\infty} e^{-\frac{\pi\mathrm{i}}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma(u)=1,\quad &\text{for }\arg\omega_1-\pi<\arg u<\arg\omega_2,\end{aligned} \tag{14}$$

where $B_{2,2}(u;\omega)$ is the second order Bernoulli polynomial,

$$B_{2,2}(u;\omega)=\frac{u^2}{\omega_1\omega_2}-\frac{u}{\omega_1}-\frac{u}{\omega_2}+\frac{\omega_1}{6\omega_2}+\frac{\omega_2}{6\omega_1}+\frac{1}{2}. \tag{15}$$

Note here, that the balancing condition is absent. Expressions (12) and (13) reproduces the partition functions of $3d$ $\mathcal{N}=2$ supersyemmtric field theories.[19,20] Equality of (12) and (13) gives us the duality for the $3d$ $\mathcal{N}=2$ SYM theories with the matter content presented in the below tables

	$SP(2N)$	$SU(2(N_f-1))$	$U(1)_A$	$U(1)_R$
Q	f	f	1	1
X	T_A	1	0	$2/(K+1)$

	$SP(2(K(N_f-2)-N))$	$SU(2(N_f-1))$	$U(1)_A$	$U(1)_R$ $(j=1,\ldots,K)$
q	f	$\overline{f}$	-1	$\frac{3-K}{K+1}$
x	T_A	1	0	$\frac{2}{K+1}$
Y_j	1	1	$-2(N_f-1)$	$4N_f-\frac{4N+6K-2j+4}{K+1}$
M_j	1	T_A	2	$1+2\frac{j-1}{K+1}$

One can proceed with the reduction of flavors and take the limit $\alpha_{2N_f-2}\to\infty$ after which one gets the equality for PFs of the Chern–Simons Theory (CS) theories. Let us set $N_f\to N_f-2$, then the electric theory is $3d$ $\mathcal{N}=2$ CS (!) theory with $k=1/2$ and the magnetic theory is $3d$ $\mathcal{N}=2$ CS theory with $k=-1/2$.

Now one can proceed further in integrating out the quarks by taking further limits $s_i\to\infty$. As the result one gets the extension for Kutasov-Schwimmer duality in three dimensions: the electric theory is $3d$ $\mathcal{N}=2$ CS theory with $SP(2N)$ gauge group and level k (such as N_f+k is even), N_f quarks (which can be also odd[21]), a chiral superfield X in adjoint representation, and the magnetic theory is $3d$ $\mathcal{N}=2$ CS theory with $SP(K(N_f+2(k-1))-2N)$ gauge group and level $-k$, N_f quarks,a chiral superfield in adjoint representation of the gauge group, mesons in T_A representation of $SU(N_f)$ global symmetry group.

2.2. *Dualities for $U(N)$ gauge groups*

We now consider different limit for the equality between (10) and (11). Let us reparameterize the parameters in the following way $\alpha_i\to\alpha_i+\mu,\ \alpha_{i+N_f}\to\alpha_{i+N_f}-\mu,\ i=1,\ldots,N_f$ and take the limit $\mu\to\infty$ after which one gets

$$I_E^{red,U(N)}=\frac{1}{N!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{N-1}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{N}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}} \tag{16}$$
$$\times\prod_{1\le i<j\le N}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm(u_i-u_j)\big)}{\gamma\big(\pm(u_i-u_j)\big)}\prod_{j=1}^{N}\prod_{i=1}^{N_f}\gamma\big(\alpha_i+u_j,\alpha_{i+N_f}-u_j\big)$$

and

$$I_M^{red,U(N)}=\frac{1}{\widetilde{N}!}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}\big)^{\widetilde{N}-1}\prod_{l=1}^{K}\prod_{i,j=1}^{N_f}\gamma\big((l-1)\tfrac{\omega_1+\omega_2}{K+1}+\alpha_i+\alpha_{j+N_f}\big)\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{j=1}^{\widetilde{N}}\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}}$$
$$\times\prod_{1\le i<j\le\widetilde{N}}\frac{\gamma\big(\frac{\omega_1+\omega_2}{K+1}\pm(u_i-u_j)\big)}{\gamma\big(\pm(u_i-u_j)\big)}\prod_{j=1}^{\widetilde{N}}\prod_{i=1}^{N_f}\gamma\big(\tfrac{\omega_1+\omega_2}{K+1}-\alpha_i-u_j,\tfrac{\omega_1+\omega_2}{K+1}-\alpha_{i+N_f}+u_j\big), \tag{17}$$

where the balancing condition reads $(\omega_1+\omega_2)2\frac{N+K}{K+1}+\sum_{i=1}^{N_f}(\alpha_i+\alpha_{i+N_f})=N_f(\omega_1+\omega_2)$. Considering the following reparametrization

$$\alpha_{N_f-1}=\xi_1+\mu,\quad \alpha_{N_f}=\xi_3-\nu,\quad \alpha_{2N_f-1}=\xi_2-\mu,\quad \alpha_{2N_f}=\xi_4+\nu \tag{18}$$

with the following limit $\mu\to\infty$ and $\nu\to\infty$ one obtains the following PFs. Since verifying dualities for $U(N)$ gauge groups is quite similar procedure to which was done above, we only comment briefly on matter content of these theories. More detailed explanations can be found in the original papers.[17]

The electric theory is $3d\,\mathcal{N}=2$ SYM theory with the matter content presented in the below table

	$U(N)$	$SU(2(N_f-2))$	$SU(2(N_f-2))$	$U(1)_A$	$U(1)_R$
Q	f	f	1	1	$1/2$
$\widetilde{Q}$	$\overline{f}$	1	f	1	$1/2$
X	adj	1	1	0	$2/(K+1)$

The magnetic theory is $3d\,\mathcal{N}=2$ SYM theory with the matter content presented in the below table

	$U(\tilde{N})$	$SU(2(N_f-2))$	$SU(2(N_f-2))$	$U(1)$	$U(1)_A$	$U(1)_R$
q	f	$\overline{f}$	1	0	-1	$\frac{3-K}{2(K+1)}$
$\widetilde{q}$	$\overline{f}$	1	$\overline{f}$	0	-1	$\frac{3-K}{2(K+1)}$
x	adj	1	1	0	0	$\frac{2}{K+1}$
$Y_j^{(1,2)}$	1	1	1	± 1	$-(N_f-2)$	R_{Y_j}
M_j	1	f	f	0	2	$1+2\frac{j-1}{K+1}$

where $\tilde{N}=K(N_f-2)-N$, $j=1,\dots,K$ and $R_{Y_j}=(N_f-2(N-j))/(K+1)$. For $K=1$ as in four-dimensional case the duality goes to the Aharony duality.[22]

The above duality is the duality between two $3d\,\mathcal{N}=2$ SYM (not CS) theories, namely between $3d\,\mathcal{N}=2$ SYM (electric) theory with $U(N)$ gauge group, N_f quarks in fundamental and anti–fundamental representation, a chiral superfield in adjoint representation and $3d\,\mathcal{N}=2$ SYM magnetic theory with $U(KN_f-N)$ gauge group N_f quarks in fundamental and anti–fundamental representation, a chiral superfield in adjoint representation, mesons in (f,f) representation of $SU(N_f)\times SU(N_f)$ global symmetry groups, chiral superfields $Y_j^{(1,2)}$, $j=1,\dots,K$, which coincide with the duality suggested by Niarchos.[23]

One can obtain CS theories by the following integrating out the matter fields. For example, integrating out a pair of quarks by taking the limit $\alpha_{N_f-3},\alpha_{2N_f-3}\to\infty$ one gets the following equality of PFs of the $3d\,\mathcal{N}=2$ CS electric theory with CS level equals to 1 and the $3d\,\mathcal{N}=2$ CS magnetic theory with CS level equals to -1 which coincides with the results of Kapustin et al.[24]

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