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Progress on double-logarithmic large-x and small-x resummations for (semi-)inclusive hard processes

A. Vogt[*], **C.H. Kom**, **N.A. Lo Presti**[†], **G. Soar**[‡]

Department of Mathematical Sciences, University of Liverpool Liverpool L69 3BX, UK
E-mail: Andreas.Vogt@liv.ac.uk

A.A. Almasy, **S. Moch**[§]

Deutsches Elektronensynchrotron DESY, Platanenallee 6, D–15738 Zeuthen, Germany

J.A.M. Vermaseren

NIKHEF Theory Group, Science Park 105, 1098 XG Amsterdam, The Netherlands

K. Yeats

Department of Mathematics, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada

Over the past few years considerable progress has been made on the resummation of double-logarithmically enhanced threshold (large-x) and high-energy (small-x) higher-order contributions to the splitting functions for parton and fragmentation distributions and to the coefficient functions for inclusive deep-inelastic scattering and semi-inclusive e^+e^- annihilation. We present an overview of the methods which allow, in many cases, to derive the coefficients of the highest three logarithms at all orders in the strong coupling from next-to-next-to-leading order results in massless perturbative QCD. Some representative analytical and numerical results are shown, and the present limitations of these resummations are discussed.

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[*] Speaker.
[†] Present address: Institut de Physique Théorique, CEA-Saclay, F-91191, Gif-sur-Yvette cedex, France
[‡] Address until September 2011
[§] Present address: II. Institut für Theoretische Physik, Universität Hamburg, D-22761 Hamburg, Germany

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1. Introduction: splitting and coefficient functions and their endpoint behaviour

(Semi-)inclusive lepton-hadron processes, see Refs. [1,2], provide benchmark observables in high-energy lepton-nucleon, electron-positron and proton-(anti-)proton collisions. We mainly report on the first two cases here, and specifically consider structure functions in inclusive deep-inelastic scattering (DIS) and fragmentation functions in single-hadron inclusive (semi-inclusive) electron-positron annihilation (SIA), $e^+e^- \to \gamma^*, Z, H(q) \to h(p)+X$, in massless perturbative QCD.

Disregarding contributions suppressed by powers of Q, these one-scale quantities are given by

$$F_a^h(x,Q^2) \;=\; \left[C_{a,i}(\alpha_s(\mu^2),\mu^2/Q^2) \otimes f_i^h(\mu^2)\right](x) \tag{1.1}$$

in terms of their *coefficient functions* $C_{a,i}$ and the parton or fragmentation distributions f_i^h of the hadron h. Here Q^2 is the physical hard scale, $Q^2 = \sigma q^2$ with $\sigma = -1$ for DIS and $\sigma = 1$ for SIA, where q is the momentum of the exchanged gauge or Higgs boson, x is the corresponding scaling variable, $x = [(2p\cdot q)/Q^2]^\sigma$, and $\otimes$ abbreviates the Mellin convolution. The dependence of f_i^h on the renormalization and factorization scale μ is given by the renormalization-group equations

$$\frac{d}{d\ln\mu^2}\, f_i(x,\mu^2) \;=\; [P_{ik,ki}^{S,T}(\alpha_s(\mu^2)) \otimes f_k(\mu^2)](x)\;, \tag{1.2}$$

where $P^{S,T}$ are the 'spacelike' ($\sigma = -1$) and 'timelike' ($\sigma = 1$) *splitting functions* and x now represents momentum fractions. Appropriate summations over i in Eq. (1.1) and k in Eq. (1.2) are understood. Choosing $\mu^2 = Q^2$ without loss of information, the α_s-expansions of C_a and P read

$$C_a(x,\alpha_s) \;=\; \sum_{n=0} a_s^{n+n_a}\, c_a^{(n)}(x)\;, \quad P(x,\alpha_s) \;=\; \sum_{n=0} a_s^{n+1} P^{(n)}(x) \quad \text{with} \quad 0<x<1\;. \tag{1.3}$$

We normalize the strong coupling as $a_s = \alpha_s(Q^2)/(4\pi)$. The contributions up to $c_a^{(\ell)}$ and $P^{(\ell)}$ form the N$^\ell$LO (leading order, next-to-leading order, ...) 'fixed-order' approximation for F_a.

The initial-state splitting functions P^S and the coefficient functions for the DIS structure functions $F_{L,2,3,\phi}$, where F_ϕ denotes the Higgs-exchange structure function in the heavy top-quark limit, are known to order α_s^3 from the diagram calculations in Refs. [3–7]. The corresponding results for the final-state quantities P^T have been derived only by indirect means so far [8–10]; in fact, the latter results still include an uncertainty which needs to be addressed by future calculations (but is not relevant in the present context). The coefficient functions for the fragmentation functions $F_{L,T,A,\phi}$, cf. Ref. [2], are completely known only at order α_s^2 [10–13].

Generically, the α_s-coefficients in Eq. (1.3) can be expanded around $x=1$ and $x=0$ in the form

$$\left\{c_a^{(n)}, P^{(n)}\right\} \;=\; \sum_{r=-1}^{\infty} X^r \left(A_{r,0}\ln^{2n-n_0} X + A_{r,1}\ln^{2n-n_0-1} X + \ldots\right) \tag{1.4}$$

with $X = 1-x$ or $X = x$, where $X \ll 1$ respectively represents the threshold (large-x) and high-energy (small-x) limits. The important exceptions to this *double-logarithmic enhancement* (i.e., two additional powers of $\ln X$ occur per order in α_s) are the diagonal splitting functions $P_{qq}^{S,T}$ and $P_{gg}^{S,T}$, which show no large-x enhancement at $r=-1$ and $r=0$ in the standard $\overline{\rm MS}$ scheme [14,15], and the flavour-singlet splitting functions P^S and DIS coefficient functions, which are only single-log ('BFKL') enhanced in the $r=-1$ terms dominating the small-x behaviour [16,17]. The dominant $r=-1$ large-x contributions to the coefficient functions can be resummed in the framework of the soft-gluon exponentiation, see, e.g., Refs. [18,19] and for the present status Refs. [20–22].

In the remainder of this contribution we briefly summarize results derived for all other cases in Refs. [23–33] with some emphasis on the phenomenologically directly relevant cases of $r = 0$ at large x in DIS and $r = -1$ at small x in SIA. This relevance is illustrated in Fig. 1: the left panel shows, after transformation to Mellin-N space, that the soft and virtual contributions do not sufficiently dominate the N^{3}LO non-singlet quark coefficient function for F_2 at 'large' N; the right panel illustrates the dramatic small-x instability of the known fixed-order approximations for $P_{\rm gg}^T$.

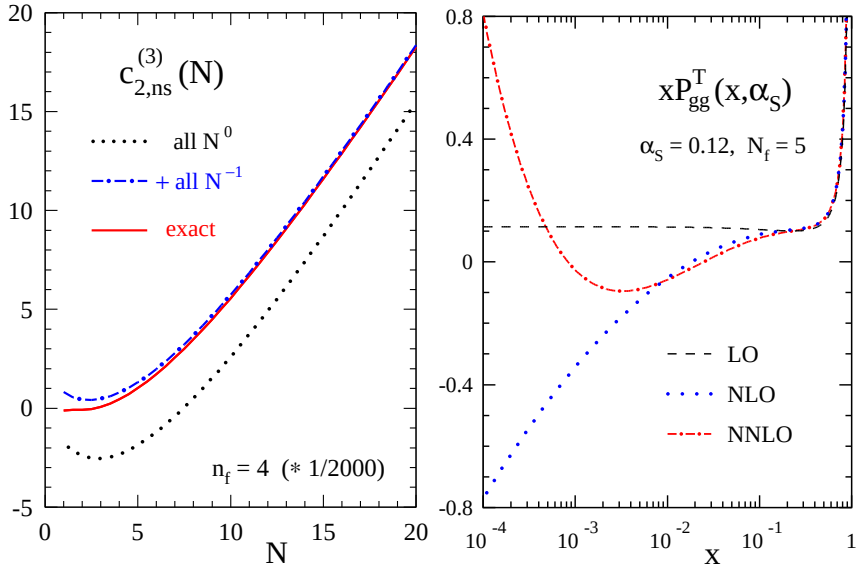

Figure 1: Left: the third-order coefficient function for $F_{2,\rm ns}$ [5] compared to large-N approximations by only the soft+virtual N^0 terms and by those plus the $(r = 0)\, N^{-1}$ contributions. Right: the LO, NLO and NNLO [9] approximations to the timelike splitting function $P_{\rm gg}^T$ for five flavours at a typical scale $Q^2 \simeq M_Z^2$.

2. Fourth- and all-order ln^k(1-x) predictions from physical kernels

It can be useful to eliminate the parton or fragmentation distributions, and the associated choice of a factorization scheme and scale, from the description of the dependence of observables on the physical scale Q^2. This leads to *physical evolution kernels* K_a, which generically can be written as

$$\frac{dF_a^h}{d\ln Q^2} \;=\; \frac{d}{d\ln Q^2}\left(C_a \otimes f^h\right) \;=\; \left(\beta(a_s)\,\frac{dC_a}{da_s} + C_a \otimes P\right) \otimes C_a^{-1} \otimes F_a^h \;\equiv\; K_a \otimes F_a^h\,, \qquad (2.1)$$

where $\beta(a_s) = -\beta_0\, a_s^2 - \beta_1 a_s^3 - \ldots$, with $\beta_0 = \frac{11}{3}\,C_A - \frac{2}{3}\,n_f$ in our normalization, is the (four-dimensional) beta function of QCD. $C_A = 3$ and $C_F = \frac{4}{3}$ are the usual SU($n_{colours} = 3$) group factors, and n_f denotes the number of effectively massless flavours.

For **flavour non-singlet cases** such as, e.g., the structure function F_3, (2.1) is a scalar equation. The first term then represents a logarithmic derivative in N-space, and the second is reduced to a combination of quark-(anti-)quark splitting functions. The resummations discussed here were initiated by observing that the $K_{a,\rm ns}$ in DIS are *single-log enhanced* not only for the $(1-x)^{-1}$ terms, as guaranteed by the soft-gluon exponentiation, cf. Ref. [34], but at all powers in $(1-x)$,

$$K_a\big|_{a_s^{n+1}} \;=\; \sum_{r=-1}^{\infty} (1-x)^r \left(K_a^{(r,0)} \ln^n(1-x) + K_a^{(r,1)} \ln^{n-1}(1-x) + \ldots\right)\,. \qquad (2.2)$$

If this behaviour holds also beyond the orders covered by Refs. [4–6], as suggested by the all-order leading large-n_f result of Refs. [35], it implies a *resummation of the double-logarithmic contributions* to the coefficient functions $C_{a,\rm ns}$ due to the non-enhancement of $P_{\rm qq}$ mentioned above.

A closer look, see also Ref. [36] for an introductory overview, reveals that the third-order coefficient functions are sufficient to fix the coefficients of the *highest three logarithms* in Eq. (1.4) for (the non-singlet components of) $C_{a,\rm q}$. The resulting all-order N^{-1} [$r = 0$ in Eq. (1.4)] predictions have been presented in Ref. [23] for $a = L$ [with $n_L = 1$ and $n_0 = 0$ in Eqs. (1.3) and (1.4)] and in section 5 of Ref. [24] for $a = 1, 2, 3$ [where $n_a = 0$ and $n_0 = 1$]. In the latter cases, only one parameter, denoted by $\xi_{\rm DIS_4}$, is missing for the fourth logarithms. Section 5 of Ref. [24] also includes the highest three logarithms at order α_s^4 for all these cases at all powers in $1-x$, written down in a closed form using suitably modified harmonic polylogarithms [37] up to weight 3.

Completely analogous results for the fragmentation functions $F_{L,T,I,A}$ in SIA can be found in section 6 of Ref. [24]. As mentioned above, the third-order coefficient functions are not fully known for these quantities. However, it was possible to derive the highest three large-x logarithms to all orders in $1-x$ using the (generally non-trivial) analytic-continuation ($\mathscr{AC}$) or Drell-Yan-Levy relation between inclusive DIS and SIA, cf. also Ref. [38] and references therein; the results are given in section 3 of Ref. [24]. Finally the same approach can be applied to the quark-antiquark annihilation contribution to the total cross section for *Drell-Yan lepton-pair production*, $pp/p\bar{p} \to l^+l^- + X$. In this case the fixed-order information is limited to order α_s^2 [39], hence only the coefficients of the *highest two logarithms* in Eq. (1.4) are completely determined by the single-logarithmic enhancement of the corresponding physical kernel. The resulting all-order N^{-1} and third- and fourth-order all-r predictions can also be found in section 6 of Ref. [24].

The single-log enhancement (2.2) also holds for the 2×2 matrices of **flavour-singlet kernels** for combinations such as $F_{2\phi} = (F_2, F_\phi)$ and $F_{2L} = (F_2, F_L)$ in DIS and their counterparts in SIA. In these cases both the coefficient and splitting functions contribute double-log enhanced terms on the r.h.s. of Eq. (2.1), hence Eq. (2.2) does not imply an all-order resummation. It is, however, possible to use this equation for $F_{2\phi}$ at $n = 3$ to derive the highest three large-x logarithms of the (otherwise unknown) N^{3}LO *singlet splitting functions* $P_{ik}^{(3)S}$ from the N^{3}LO coefficient functions computed in Refs. [5, 7]. This calculation has been carried out in Ref. [25] to all orders in r. A surprising, at the time, outcome was that the coefficient of all four leading logarithms, $\ln^6(1-x)$ for $P_{i\neq k}^{(3)}(x)$ and $\ln^5(1-x)$ for $P_{i=k}^{(3)}(x)$, were found to vanish to all orders in $1-x$.

With the highest three logarithms determined for $P_{ik}^{(3)S}$, it became possible to employ Eq. (2.2) for F_{2L} to derive corresponding results for the [four-loop, due to $n_a = 1$ in Eq. (1.3)] N^{3}LO singlet *coefficient functions for* F_L. This has been done in Ref. [26], but only for the leading-r contributions, with new results for $c_{L,\rm g}^{(3)}$ at $r = 1$. The all-r generalization, which would also yield results beyond those of Refs. [23, 24] also for the quark coefficient function, is not available in the literature yet.

No new large-x results have been derived (only) from physical evolution kernels after Ref. [26]. The limitations of this approach – the need to conjecture the all-n validity of Eq. (2.2) and lack of all-order predictions for the singlet splitting and coefficient functions – have been overcome since then by starting from the unfactorized structure and fragmentation functions as discussed below. However, the physical kernels still represent the easiest route to all-r results at order α_s^4, and they can provide invaluable hints for the functional forms of the all-order large-x coefficient functions.

3. Double logarithmic endpoint resummations via unfactorized quantities

The splitting functions and the coefficient functions for the (combinations of) DIS and SIA observables discussed above originate in the *unfactorized expressions* in $D = 4 - 2\bar{\varepsilon}$ dimensions,

$$\widehat{F}_a \;=\; \widetilde{C}_a \otimes Z \quad \text{with} \quad P \;=\; \beta_D(a_s)\,\frac{dZ}{da_s} \otimes Z^{-1}\,. \tag{3.1}$$

Here $\widetilde{C}_a$, the D-dimensional coefficient functions, are given by Taylor series in ε (which differs from $\bar{\varepsilon}$ by some 'artefacts' of dimensional regularization) with the ε^0 coefficient leading to the 'physical' ($\overline{\rm MS}$-scheme) coefficient function in Eqs. (1.1) and (1.3). $\beta_D(a_s)$ is the D-dimensional beta function, $\beta_D(a_s) = -\varepsilon a_s + \beta(a_s)$ with $\beta(a_s)$ defined below Eq. (2.1). For non-singlet cases all quantities in Eq. (3.1) are scalars, for flavour-singlet combinations $\widehat{F}_a$, $\widetilde{C}_a$, P and Z are 2×2 matrices [unlike in Eq. (1.1), the quark and gluon contributions to F_a are considered separately in Eq. (3.1)]. Z consists of $1/\varepsilon$ *poles up to* ε^{-n} at order a_s^n, and its $\alpha_s^n \varepsilon^{-n+\ell}$ coefficients include endpoint logarithms up to $\ln^{n+\ell-1} X$ with $X = 1-x$ and $X = x$. Hence we have, generically,

$$\widehat{F}_a\big|_{a_s^n \varepsilon^{-n+\ell}} \;=\; \sum_{r=-1}^{\infty} X^r \left(\mathscr{F}^{(0)}_{a,n,\ell,r} \ln^{n+\ell-1} X + \mathscr{F}^{(1)}_{a,n,\ell,r} \ln^{n+\ell-2} X + \dots \right)\,. \tag{3.2}$$

The second equation in (3.1) can be iteratively inverted to yield Z in terms of the expansion coefficient of $P(x,\alpha_s)$ and $\beta(a_s)$. The resulting dependence on $P^{(n)}$ and β_n can be summarized as

$$a_s^n \varepsilon^{-n} :\, P^{(0)}, \beta_0\,, \quad a_s^n \varepsilon^{-n+1} :\, +P^{(1)}, \beta_1\,, \quad a_s^n \varepsilon^{-n+2} :\, +P^{(2)}, \beta_2\,, \quad \dots\,, \quad a_s^n \varepsilon^{-1} :\, P^{(n-1)}\,. \tag{3.3}$$

Explicit all-order expressions for $r = 0$, recall Eq. (1.4), in the large-x case have been presented in section 2 of Ref. [28]. Beyond this accuracy and for the small-x case, $Z(\alpha_s,\varepsilon)$ has been evaluated iteratively to a high but finite order in α_s. Eq. (3.3) implies that a fixed-order knowledge at N^{m}LO, recall Eq. (1.3), determine the first $m+1$ non-vanishing coefficients in the ε expansion of $\widehat{F}_a$ at all orders in a_s. If this determination can be extended, at a certain logarithmic accuracy, to all orders in ε, then the *all-order mass factorization* can be performed at this accuracy, yielding a resummation of the splitting and coefficient functions. In practice this is done order by order in α_s; we have employed recent developments in FORM [40] to reach as high an order as possible.

One way to obtain all-n and all-ε expressions for coefficients $\mathscr{F}^{(k)}$ in Eq. (3.2) is by expressing the coefficients $\widehat{F}_a^{(n)}$ in terms of known lower-order results, i.e., by finding an all-order **iteration of the unfactorized expressions** $\widehat{F}_a^{(n)}$ at a given logarithmic accuracy. This approach has been used for the $r = 0$ *large-x structure functions* in Refs. [27, 28] at $k = 0$ and $k = 1$. For example, the unfactorized gluon contributions to the structure function F_2 can be expressed as

$$\widehat{F}_{2,\rm g}^{(n)} \;=\; \frac{1}{n}\,\widehat{F}_{2,\rm g}^{(1)} \left\{ \sum_{i=0}^{n-1} f_{2,\rm g}(n,i,\varepsilon)\,\widehat{F}_{\phi,\rm g}^{(i)}\,\widehat{F}_{2,\rm q}^{(n-i-1)} \;-\; \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} g_{2,\rm g}(n,i)\,\widehat{F}_{\phi,\rm g}^{(i)}\,\widehat{F}_{2,\rm q}^{(n-i-2)} \right\} \tag{3.4}$$

with

$$f_{2,\rm g}(n,i,\varepsilon) \;=\; \binom{n-1}{i}^{-1} \left[1 + \varepsilon f_{2,\rm g}^{(1)}(n,i)\right] \quad,\quad g_{2,\rm g}(n,i) \;=\; \binom{n}{i+1}^{-1} \tag{3.5}$$

at next-to-leading logarithmic (NLL) accuracy. The $r=-1$ 'diagonal' quantities $\widehat{F}_{2,\rm q}^{(k)}$, $\widehat{F}_{\phi,\rm g}^{(k)}$ required in Eq. (3.4) are known to a high accuracy from the soft-gluon exponentiation, see Ref. [28].

Especially the non-LL parts of this result (the function $f_{2,\mathrm{g}}^{(1)}(n,i)$ can be found in section 3 of Ref. [28]) have rather been 'engineered' and verified – using results in Section 2 the ensuing system of linear equations is overconstrained by two equations per order in α_s – than derived from first principles. Since a second approach turned out to be clearer and more convenient, we have not attempted to generalize Eq. (3.4) to the third (NNL) logarithms. Still, the iteration of unfactorized observables might offer access to resummations of quantities outside the scope of Refs. [28–33].

That second approach is to **decompose the n^{th}-order unfactorized expressions** at $x < 1$, without any reference to lower-order counterparts, in n terms of D-dimensional exponentials, e.g.,

$$\left(A_{n,k}\,\varepsilon^{\,-2n+1} + B_{n,k}\,\varepsilon^{\,-2n+2} + C_{n,k}\,\varepsilon^{\,-2n+3} + \ldots\right) X^{r-(\eta_0+k\,\eta_1)\,\varepsilon} \;, \quad k = 1, \ldots, n \;. \tag{3.6}$$

For the *large-x logarithms*, $X = 1-x$, this structure arises at all r in inclusive DIS and SIA from the phase-space integrations for the undetected final-state particles and the loop integrals of the virtual corrections [12, 41] with $\eta_0 = 0$ and $\eta_1 = 1$. The decomposition (3.6) is related, but not identical to that into contributions with $1, \ldots, n$ undetected partons in the final state. In the soft-gluon limit, $r = -1$, it has been employed before to 'reverse-engineer' the $\alpha_\mathrm{s}^3\varepsilon^{\,-n}$ pole terms of the $\gamma^* qq$ and Hgg form factors (and the $n_f\alpha_\mathrm{s}^3\varepsilon^{\,0}$ contributions in the former case) from the calculations for Refs. [5, 7] in Refs. [42] and to extend the soft-gluon resummation to N^{3}LL accuracy for SIA, Drell-Yan lepton-pair production and the total cross section for Higgs production via gluon-gluon fusion [23, 43]. It may be worthwhile to note that the results of Refs. [42] were confirmed (and substantially extended, to order $\alpha_\mathrm{s}^3\varepsilon^{\,2}$) by direct diagram calculations in Refs. [44, 45].

The situation is far more complicated for the *small-x logarithms*, $X = x$. Here we focus on contributions that do not vanish for $x \to 0$, i.e., $r = 0$ and $r = -1$. In the former case Eq. (3.6) is found to hold with $\eta_0 = 0$ and $\eta_1 = -1$ in DIS and $\eta_0 = 1$ and $\eta_1 = 1$ in SIA, as one might have 'naively' expected from Refs. [12, 41], but only for 'even-N' quantities such as $F_{L,1,2,\phi}$ in DIS and their SIA counterparts. For g_1 in polarized DIS and F_3, where the operator-product expansion provides the odd Mellin moments, and the corresponding asymmetric fragmentation function F_A already the colour structure of the leading logarithms [6, 46, 47] excludes such a decomposition. On the other hand, Eq. (3.6) is applicable all orders at $r = -1$ in SIA, but with $\eta_0 = 0$ and $\eta_1 = 2$, another structure that has been discovered and verified but still lacks a proper explanation.

After expanding in ε, the $\varepsilon^{\,-2n+1}, \ldots, \varepsilon^{\,-n-1}$ terms in Eq. (3.6) have to cancel in the sum (3.1). This implies $n-1$ relations between the LL coefficients $A_{n,k}$ which lead to the constants $\mathscr{F}_{n,\ell}^{(0)}$ in Eq. (3.2), $n-2$ relations between the NLL coefficients $B_{n,k}$ determining $\mathscr{F}_{n,\ell}^{(1)}$ etc. As discussed above, a N^{m}LO calculation fixes the (non-vanishing) coefficients of $\varepsilon^{\,-n}, \ldots, \varepsilon^{\,-n+m}$ at all orders n, adding $m+1$ more relations between the coefficients in Eq. (3.6). Consequently the highest $m+1$ double logarithms, i.e., *the* N^{m}LL *approximation, can be determined from the* N^{m}LO *results* wherever an equation like (3.6) (the expression for F_L, e.g., is slightly different) is applicable.

Therefore the present fixed-order results allow the determination, to all orders in α_s, of the highest four $r = 0$ large-x logarithms for the coefficient functions in non-singlet DIS (where the accuracy effectively is N^{3}LO due to the non-enhancement of the splitting functions) and SIA (due to its close relation to the DIS case, see Ref. [24]), and of the corresponding highest three large-x and small-x logarithms in the flavour singlet splitting and coefficient functions, including the phenomenologically most important $x^{\,-1}$ terms in SIA illustrated above.

4. Selected recent results on large-x and small-x double logarithms

We now show a small subset of the results derived by the methods described in Section 3. In the non-singlet large-x cases the physical-kernel results have been verified, and the previously missing parameter has been determined as $\xi_{\rm DIS_4} = \xi_{\rm SIA_4} = \frac{100}{3}$, a result that has been obtained independently in Ref. [49]. Except for the numerical impact of the thus known four $r = 0$ logs on the fourth-order coefficient function $C^{(4)}_{2,\rm ns}$, which will be illustrated in Fig. 2 below, we will focus on the large-x singlet and small-x cases, where progress has been made since the Wernigerode workshop.

The highest three $r = 0$ *large-x logarithms* in the off-diagonal splitting functions P^S and coefficient functions for $F_{L,2,\phi}$ in DIS have now been cast in closed all-order forms that supersede the tables in the appendix of Ref. [28]. Also the NNLL contributions can be expressed in N-space in terms of the apparently new *Bernoulli functions* $\mathscr{B}_n(x)$ introduced and discussed in Refs. [27, 28],

$$\mathscr{B}_k(x) \;=\; \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!}\, x^n \quad \text{and} \quad \mathscr{B}_{-k}(x) \;=\; \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!}\, x^n \,, \tag{4.1}$$

where B_n are the Bernoulli numbers in the normalization of Ref. [48]. As an example, we show the NNLL approximation to the spacelike gluon-quark splitting function which can be written as

$$\begin{aligned}
NP^S_{\rm qg}(N,\alpha_{\rm s}) \;=\;& 2a_{\rm s}\, n_f\, \mathscr{B}_0(\tilde a_{\rm s}) \\
&+ a_{\rm s}^2 \ln\widetilde{N}\, n_f \Big\{ (6C_F - \beta_0)\, \big(\mathscr{B}_1(\tilde a_{\rm s}) + 2\,\tilde a_{\rm s}^{-1}\mathscr{B}_{-1}(\tilde a_{\rm s})\big) + \beta_0\, \tilde a_{\rm s}^{-1}\mathscr{B}_{-2}(\tilde a_{\rm s}) \Big\} \\
&+ \frac{a_{\rm s}^2 n_f}{48 C_{AF}} \Big\{ \beta_0^2 \Big[2\tilde a_{\rm s}\mathscr{B}_2(\tilde a_{\rm s}) - 12\mathscr{B}_1(\tilde a_{\rm s}) + 12\mathscr{B}_0(\tilde a_{\rm s}) - 6\mathscr{B}_{-1}(\tilde a_{\rm s}) - 12\tilde a_{\rm s}^{-1}\mathscr{B}_{-2}(\tilde a_{\rm s}) \\
&\quad - 4\tilde a_{\rm s}^{-1}\mathscr{B}_{-3}(\tilde a_{\rm s}) + 3\tilde a_{\rm s}^{-1}\mathscr{B}_{-4}(\tilde a_{\rm s}) \Big] - 36\beta_0 C_F \Big[\tilde a_{\rm s}\mathscr{B}_2(\tilde a_{\rm s}) - 3\mathscr{B}_1(\tilde a_{\rm s}) + 4\mathscr{B}_0(\tilde a_{\rm s}) - \mathscr{B}_{-1}(\tilde a_{\rm s}) \\
&\quad + \tilde a_{\rm s}^{-1}\big(2\mathscr{B}_{-1}(\tilde a_{\rm s}) + \mathscr{B}_{-2}(\tilde a_{\rm s}) - 2\mathscr{B}_{-3}(\tilde a_{\rm s})\big) \Big] + 108 C_F^2 \Big[2\tilde a_{\rm s}\mathscr{B}_2(\tilde a_{\rm s}) - 4\mathscr{B}_1(\tilde a_{\rm s}) + 5\mathscr{B}_0(\tilde a_{\rm s}) \\
&\quad + \tilde a_{\rm s}^{-1}\big(2\mathscr{B}_{-1}(\tilde a_{\rm s}) + 4\mathscr{B}_{-2}(\tilde a_{\rm s})\big) \Big] + 80 C_{AF}\beta_0 \Big[\tilde a_{\rm s}\mathscr{B}_2(\tilde a_{\rm s}) - 4\mathscr{B}_1(\tilde a_{\rm s}) + 4\mathscr{B}_0(\tilde a_{\rm s}) + \mathscr{B}_{-1}(\tilde a_{\rm s}) \Big] \\
&\quad - 32 C_{AF} C_F \Big[(19 - 3\zeta_2)\tilde a_{\rm s}\mathscr{B}_2(\tilde a_{\rm s}) - 34\mathscr{B}_1(\tilde a_{\rm s}) + (13 + 6\zeta_2)\mathscr{B}_0(\tilde a_{\rm s}) - (2 - 3\zeta_2)\mathscr{B}_{-1}(\tilde a_{\rm s}) \Big] \\
&\quad + 32 C_{AF}^2 \Big[(2 + 3\zeta_2)\tilde a_{\rm s}\mathscr{B}_2(\tilde a_{\rm s}) + (4 + 12\zeta_2)\mathscr{B}_1(\tilde a_{\rm s}) + (2 - 12\zeta_2)\mathscr{B}_0(\tilde a_{\rm s}) + (2 - 3\zeta_2)\mathscr{B}_{-1}(\tilde a_{\rm s}) \Big] \Big\}
\end{aligned} \tag{4.2}$$

with $\ln\widetilde{N} = \ln N + \gamma_{\rm e}$, $\tilde a_{\rm s} = 4a_{\rm s}C_{AF}\ln^2\widetilde{N}$ and $C_{AF} = C_A - C_F$, where $\gamma_{\rm e}$ is the Euler constant. Corresponding results for $P^S_{\rm gq}$, their timelike counterparts with, e.g., $C_F^{-1}P^T_{\rm gq} = n_f^{-1}P^S_{\rm qg}$ at LL accuracy, and the related coefficient functions will be presented in Ref. [32]. These results explain the vanishing LL coefficients at order $\alpha_{\rm s}^4$ as the start of an all-order pattern due to $B_{2n+1} = 0$ for $n \geq 1$.

We now turn to the $r = -1$ *small-x resummation in SIA*, where even more results were only known via tables of expansion coefficients in Ref. [29]. This situation changed dramatically as a consequence of discussions which, in all likelihood, would not have occurred without the 2012 Loops & Legs workshop. All results of Ref. [29], and more, are now known in closed form, e.g.,

$$\begin{aligned}
P^T_{\rm qq}(N) \;=\;& \frac{4}{3}\frac{C_F n_f}{C_A}\, a_{\rm s}\Big\{\frac{1}{2\xi}(S-1)(\mathscr{L}+1) + 1\Big\} \\
&+ \frac{1}{18}\frac{C_F n_f}{C_A^3}\, a_{\rm s}\bar N\Big\{ (-11C_A^2 + 6C_A n_f - 20C_F n_f)\,\frac{1}{2\xi}(S - 1 + 2\xi) + 10C_A^2\,\frac{1}{\xi}(S-1)\mathscr{L} \\
&\quad - (51C_A^2 - 6C_A n_f + 12C_F n_f)\,\frac{1}{2}(S-1) + (11C_A^2 + 2C_A n_f - 4C_F n_f)\,S^{-1}\mathscr{L} \\
&\quad + (5C_A^2 - 2C_A n_f + 6C_F n_f)\,\frac{1}{\xi}(S-1)\mathscr{L}^2 + (51C_A^2 - 14C_A n_f + 36C_F n_f)\mathscr{L}\Big\}\,,
\end{aligned} \tag{4.3}$$

$$
\begin{aligned}
P_{\rm gg}^T(N) \;=\; & \frac{1}{4}\bar{N}(S-1) \;-\; P_{\rm qq}^T(N) \;-\; \tfrac{1}{6C_A}\,a_{\rm s}\,(11C_A^2+2C_An_f-4C_Fn_f)\,(S^{-1}-1) \\
& + \tfrac{1}{576C_A^3}\,a_{\rm s}\bar{N}\Big\{\big([1193-576\,\zeta_2]C_A^4-140C_A^3n_f+4C_A^2n_f^2-56C_A^2C_Fn_f-48C_F^2n_f^2 \\
& \quad +16C_AC_Fn_f^2\big)\,(S-1)+\big([830-576\,\zeta_2]\,C_A^4+96C_A^3n_f-8C_A^2n_f^2-208C_A^2C_Fn_f \\
& \quad +64C_AC_Fn_f^2-96C_F^2n_f^2\big)\,(S^{-1}-1)+(11C_A^2+2C_An_f-4C_Fn_f)^2(S^{-3}-1)\Big\} \qquad (4.4)
\end{aligned}
$$

with $S = (1-4\xi)^{1/2}$, $\mathscr{L} = \ln\big(\frac{1}{2}(1+S)\big)$, $\xi = -8C_Aa_{\rm s}\bar{N}^{-2}$ and $\bar{N} \equiv N-1$. The first term in Eq. (4.4) is the well-known leading-logarithmic result of Refs. [50]. Beyond this accuracy, the splitting functions were unknown in the $\overline{\rm MS}$ scheme, see Refs. [51], before Ref. [29]. The crucial step towards the closed forms illustrated above was the derivation of the first line of Eq. (4.3) which made use of Ref. [52] as described in appendix A of Ref. [31].

Expressions such as Eq. (4.4) allow the evaluation of the oscillating combined NLO+NNLL splitting functions down to extremely small x, see Fig. 2, and at $N = 1$ in Mellin space, e.g.,

$$
\begin{aligned}
P_{\rm gg}^T(N=1) \;=\; & (2C_Aa_{\rm s})^{1/2} - \frac{1}{6C_A}(11C_A^2+2C_An_f+12C_Fn_f)\,a_{\rm s} \\
& + \frac{1}{144C_A^3}\,\big([1193-576\,\zeta_2]\,C_A^4-140C_A^3n_f+4C_A^2n_f^2+760C_A^2C_Fn_f \\
& \qquad -80C_AC_Fn_f^2+144C_F^2n_f^2\big)\,(2C_Aa_{\rm s}^3)^{1/2} \;+\; \mathscr{O}(a_{\rm s}^2) \\
\cong\; & 0.6910\,\alpha_{\rm s}^{1/2} - 0.9240\,\alpha_{\rm s} + 0.6490\,\alpha_{\rm s}^{3/2} + \mathscr{O}(\alpha_{\rm s}^2) \quad \text{for} \quad n_f = 5\,. \qquad (4.5)
\end{aligned}
$$

The latter results have already been used in an analysis of multiplicities in quark and gluon jets [53]. See Ref. [31] for more results on timelike splitting functions and SIA coefficient functions, including a first step towards a higher logarithmic accuracy for $P_{\rm gg}^T$ based on Ref. [15], see also Ref. [54].

Similar results have been derived for the *small-x logarithms of even-N DIS quantities*, e.g.,

$$
\begin{aligned}
P_{\rm qq}^{\rm ns+}(N) \;=\; & -\frac{1}{2}N(S-1)+\frac{1}{2}a_{\rm s}(2C_F-\beta_0)(S^{-1}-1) \\
& + \frac{1}{96C_F}\,a_{\rm s}N\Big\{\big([156-960\,\zeta_2]\,C_F^2-[80-1152\,\zeta_2]\,C_AC_F-360\,\zeta_2C_A^2-100\,\beta_0C_F \\
& \quad +3\beta_0^2\big)\,(S-1)+2\,\big([12-576\,\zeta_2]\,C_F^2+[40+576\,\zeta_2]\,C_AC_F-180\,\zeta_2C_A^2 \\
& \quad +56\,\beta_0C_F-3\beta_0^2\big)\,(S^{-1}-1)+3\,(2C_F-\beta_0)^2(S^{-3}-1)\Big\} \qquad (4.6)
\end{aligned}
$$

with $S = (1-4\bar{\xi})^{1/2}$ and $\bar{\xi} = 2C_Fa_{\rm s}N^{-2}$, of which only the leading-logarithmic first term was known before [46]. While Eq. (4.6) is formally analogous to Eq. (4.4), the different sign of $\bar{\xi}$ as compared to ξ leads to a qualitatively different x-space behaviour, see Ref. [33] where also the resummed coefficient functions $F_{2,\rm ns}$ and $F_{L,\rm ns}$ and the flavour-singlet results will be presented.

5. Summary and Outlook

We have derived all-order results for the highest three (in a few cases four) large-x and small-x double-logarithmic contributions to spacelike and timelike splitting functions and to coefficient functions for inclusive DIS and SIA. These results have been obtained from NNLO (in a few cases effectively N^{3}LO) fixed-order results in a 'bottom-up' approach using the large-x behaviour of physical evolution kernels, large-x iterations of unfactorized structure and fragmentation functions, and KLN-related large-x and small-x decompositions of the unfactorized expressions in dimensional regularization combined with constraints imposed by the mass-factorization relations.

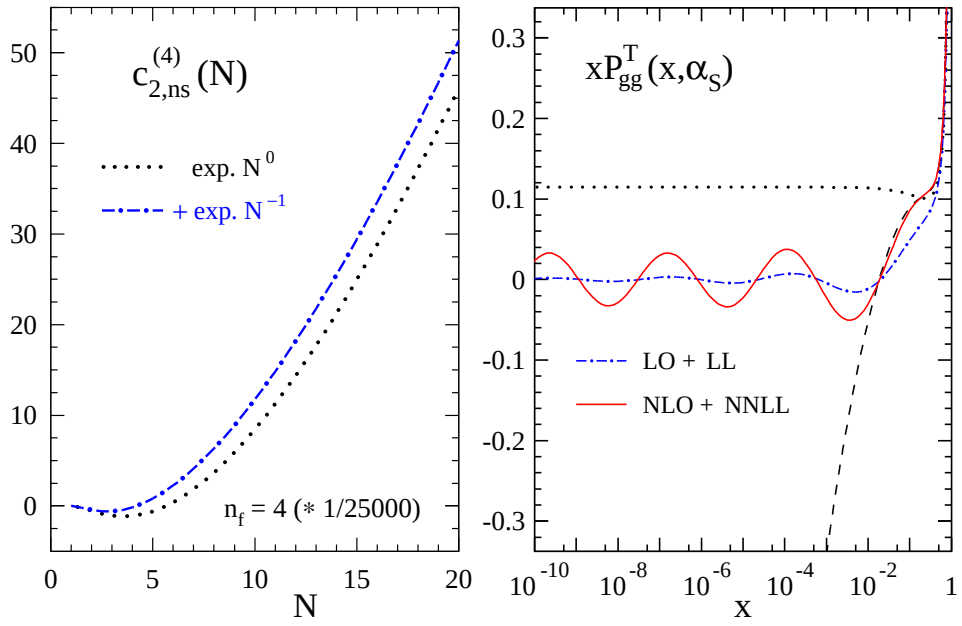

Figure 2: Left: the fourth-order coefficient function for the structure function $F_{2,\mathrm{ns}}$. Shown are the large N estimates by the known seven (of eight) $\ln^n N$ soft-gluon contributions and by adding the highest four (of seven) $N^{-1}\ln^n N$ corrections. Right: the timelike gluon-gluon splitting functions (multiplied by x) for a very wide range of the momentum fraction x at a value of α_s corresponding to $Q^2 \simeq M_Z^2$. The all-x ($N=1$ finite) LO + LL and NLO + NNLL results are compared to the LO and NLO approximations valid only at large x.

At least in the form presented above, the last and so far most powerful approach is applicable neither to inclusive lepton-pair and Higgs production at large x, nor to the odd Mellin-N based structure functions such as $F_3^{\nu+\bar{\nu}}$ in charge-current DIS and g_1 in polarized DIS. In the former cases, Eq. (3.6) holds in the $(1-x)^{-1}$ soft-gluon limit with $n_0 = 0$ and $n_1 = 2$ [41, 43] but, for example, additional odd-$k\eta_1$ terms which spoil most of the predictivity are found to be required in the qg channel of the Drell-Yan process [55]. Without additional theoretical insight, which may come from the more rigorous approaches pursued in Refs. [56], it will also not be possible to improve upon our present 'N^{n}LO implies N^{n}LL' accuracy in cases for which Eq. (3.6) is applicable.

Most of the results in Refs. [23–33] will not have a direct phenomenological impact, but will hopefully prove useful in conjunction with future developments such as, e.g., a computation of fourth-order moments of structure functions analogous to Refs. [57]. The exception are the $(1-x)^0$ contributions to the non-singlet DIS coefficient functions, which should be of interest for high-precision analyses of structure functions at large x, and the x^{-1} small-x timelike splitting functions as discussed above. In both cases the relative size of the known N^{n}LL contributions indicates that an improvement on the present accuracy is required for quantitatively reliable results.

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