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Extended global symmetries for 4d $\mathcal{N} = 1$ SQCD theories

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Abstract

In arXiv:0811.1909 Spiridonov and Vartanov, using the superconformal index technique, found that 4–dimensional $\mathcal{N} = 1$ SQCD theory with $SU(2)$ gauge group and four flavors has 72 dual representations. Recently in arXiv:1209.1404 the authors showed that these dual theories, when coupled to $5d$ hypermultiplets with specific boundary conditions have an extended E_7 global symmetry. In this work we find that for a reduced theory with 3 flavors the explicit $SU(6)$ global symmetry is enhanced to an E_6 symmetry in the presence of $5d$ hypermultiplets. We also show connections between indices of different theories in 3 and 4 dimensions.

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1   Introduction and conclusion

In recent years considerable progress has been made in the study of rigid supersymmetric field theories in nontrivial spacetimes. In particular the superconformal index has been a primary goal of the recent interest in these theories. The index is a powerful tool to test Seiberg–like dualities in $\mathcal{N} = 1$ [1, 2, 3, 4, 5], S–dualities in $\mathcal{N} = 2$ [6, 7] and $\mathcal{N} = 4$ [6, 8] supersymmetric theories and has an elegant mathematical structure described by the theory of elliptic hypergeometric integrals [9, 10].

The superconformal index was introduced [1, 11, 12] as a nontrivial generalization of the Witten index [13], which counts BPS states in superconformal field theories in curved spacetime [14]. We give a short outline of a superconformal index and refer the reader to [3, 4, 15] for more details.

Let us consider the $\mathcal{N} = 1$ superconformal theory in four dimensions. The symmetry group of this theory is $SU(2,2|1)$, which has the following generators: Lorentz rotations J_i, $\bar{J}_i$, super-translations P_μ, Q_α, $\bar{Q}_{\dot\alpha}$ with $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$, special superconformal transformations K_μ, S_α, $S_{\dot\alpha}$ with $\{\bar{S}^{\dot\alpha}, S^\alpha\} = 2K^{\dot\alpha\alpha}$, dilatations H and $U(1)_R$–rotations R. To construct the super-conformal index let us consider, for example, the supercharges $\bar{Q}_1$ and $\bar{S}^1$, which satisfy the following relation

$$\{\bar{Q}_1, \bar{S}^1\} = -2(H - 2\bar{J}_3 - \frac{3}{2}R) \ . \tag{1}$$

Then one defines the superconformal index in the following way

$$\mathrm{ind}(t, x, g, f) \ = \ Tr\left((-1)^{\mathrm{F}} x^{2J_3} t^{\mathcal{R}} e^{\sum_{a=1}^{rank\,G} g_a G^a} e^{\sum_{j=1}^{rank\,F} f_j F^j}\right) \ . \tag{2}$$

Here $(-1)^F$ is the fermion number operator, $t^{\mathcal{R}}$ and x^{2J_3} are additional regulators with $|t| < 1$ and $|x| < 1$, g_a and f_j are the chemical potentials for groups G and F respectively, where G is a non-abelian gauge group with maximal torus generators G_a, $a = 1, \ldots$, rank G, and F is a flavor group with maximal torus generators F_j, $j = 1, \ldots$, rank F.

According to the Romelsberger prescription [1] for $\mathcal{N} = 1$ superconformal theories one can write the full index via a "plethystic" exponential [16] and integrate over the gauge group[3]

$$I(p, q, \underline{y}) = \int_{G_c} d\mu(g) \, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{ind}\big(p^n, q^n, \underline{z}^n, \underline{y}^n\big)\right), \tag{3}$$

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[3]Because we are interested in gauge invariant physical observables.

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where $d\mu(g)$ is the G_c–invariant measure and the single particle states index is

$$\text{ind}(p, q, \underline{z}, \underline{y}) = \frac{2pq - p - q}{(1 - p)(1 - q)}\chi_{adj}(\underline{z})$$
$$+ \sum_j \frac{(pq)^{R_j/2}\chi_{R_F,j}(\underline{y})\chi_{R_G,j}(\underline{z}) - (pq)^{1-R_j/2}\chi_{\bar{R}_F,j}(\underline{y})\chi_{\bar{R}_G,j}(\underline{z})}{(1 - p)(1 - q)}. \quad (4)$$

Here we introduced the new parameters $p = tx$ and $q = tx^{-1}$. The first term in (4) represents the contribution of the gauge superfields lying in the adjoint representation of the gauge group G_c. The sum over j corresponds to the contribution of chiral matter superfields φ_j transforming in the gauge group representations $R_{G,j}$ and flavor group representations $R_{F,j}$ where R_j are the field R-charges. The functions $\chi_{adj}(\underline{z})$, $\chi_{R_F,j}(\underline{y})$ and $\chi_{R_G,j}(\underline{z})$ are the characters of the corresponding representations, where $\underline{z}$ and $\underline{y}$ are the set of complex eigenvalues of matrices realizing G and F, respectively.

Dolan and Osborn realized [2] that the exponential sum in (3) can be evaluated using elliptic Gamma function

$$\Gamma(z; p, q) = \prod_{i,j=0}^{\infty} \frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^iq^j}, \quad |p|, |q| < 1 , \quad (5)$$

and as a result the superconformal index can be expressed in terms of Spiridonov's elliptic hypergeometric integrals. For a detailed discussion, see [4] and also [17] for mathematical aspects of these integrals. Note that in the rest of the paper we will use the following standard shorthands

$$\Gamma(z, w; p, q) \equiv \Gamma(z; p, q)\Gamma(w; p, q), \quad (6)$$
$$\Gamma(z^{\pm k}; p, q) \equiv \Gamma(z^k; p, q)\Gamma(z^{-k}; p, q). \quad (7)$$

In [3] authors established multiple dualities based on the so–called V–function

$$I(t_1, \ldots, t_8; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi iz}, \quad (8)$$

where t_j, $j = 1, \ldots, 8$ are complex parameters with the balancing condition $\prod_{j=1}^8 t_j = (pq)^2$ and the q-Pochhammer symbol $(z; q)_\infty = \prod_{i=0}^\infty (1 - zq^i)$. They speculated on existence of E_7 global symmetry of the V–function from the fact that it has $W(E_7)$ Weyl symmetry group for integral transformation. In fact this symmetry was realized explicitly, based on $4d/5d$ system, by Dimofte and Gaiotto in [18].

In [3] the authors reduced $4d$ $\mathcal{N} = 1$ SYM with $SU(2)$ gauge group with 8 quarks to 6 quarks and found that the index of the reduced theory has $W(E_6)$ symmetry. After this reduction in the dual theories they realized additional $SU(2)$ global symmetries, the appearance of which was unclear to the authors. In this work we give the explanation of this extended symmetry by coupling of original $N_f = 3$ theory to free $5d$ hypermultiplets[4]. This coupling bring us to E_6 global symmetry. At the same time this E_6 symmetry can be obtained by restricting two parameters in combined $4d/5d$ index considered by Dimofte and Gaiotto [18].

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[4]Note that we use the subscript F for the flavor and the subscript f for the number of quarks.

We have E_6 global symmetry group and in different phases it produces us additional $SU(2)$ or $U(1)$ groups in dualities found in [3].

Our aim is to show connections between indices of different theories. The following "commutative diagram" demonstrates the plan of the paper pictorially. In section 2 we describe the reduction procedure from $4d$ $N_F = 4$ theory to $3d$ $N_f = 4$, in the diagram it follows to the anticlockwise direction. In section 3 we do further reduction from $3d$ $N_f = 6$ theory, which is right down arrow in the diagram.
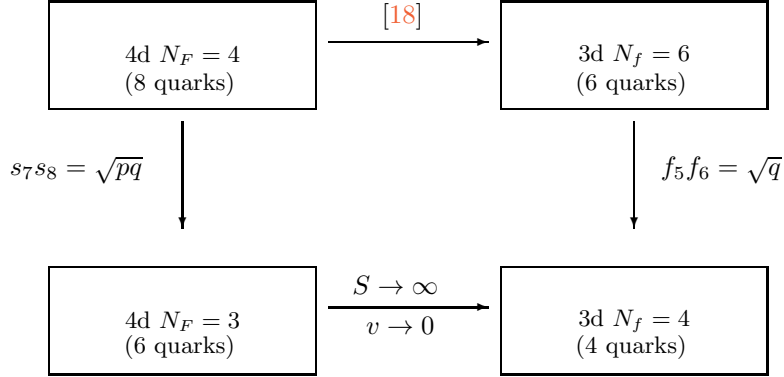
In this diagram s_i and f_i are the chemical potentials. The limit $v \to 0$ corresponds to dimensional reduction on the S^1 and $S \to \infty$ corresponds to the sending mass of quark supermultiplet to infinity.

2 Reduction of $4d$ SCI to $3d$ partition function

In this section we will discuss $4d$ $SU(2)$ $\mathcal{N} = 1$ SQCD theories with $N_F = 4$. Let us consider first the electric theory with the flavor symmetry group $SU(8)$. The superconformal index for this theory is[5]

$$I_{4d,\ N_F=4} = \frac{(p;p)_\infty (q;q)_\infty}{2} \oint \frac{dz}{2\pi i z} \frac{\prod_{i=1}^{8} \Gamma(\sqrt[4]{pq} s_i z^{\pm}; p, q)}{\Gamma(z^{\pm 2}; p, q)}, \tag{9}$$

where the path of the contour is taken to be the unit circle with positive orientation. The chemical potentials of $SU(8)$ group s_i obey the balancing condition $\prod_{i=1}^{8} s_i = 1$. In this theory we have a chiral scalar multiplet in the fundamental representations of $SU(2)$ and $SU(8)$.

In the paper [3], Spiridonov and the second author established that there exist 71 dual magnetic theories in addition to the above electric theory. They classified these 71 theories in three groups.

The first type of dual magnetic theory is the theory which was found by Csaki et al. in [19]. There are 35 dual theories of this type and all of them are $4d$ $SU(2)$ $\mathcal{N} = 1$ theories with $SU(4)_l \times SU(4)_r \times U(1)_B$ flavor group, two scalar multiplets in the fundamental representation, a

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[5]This is the so–called V–function.

gauge field in the adjoint representation of the gauge group, and two singlets in the antisymmetric tensor representations of $SU(4)$ group. The index for this type of theory is

$$I_M^{(1)} = \frac{(p;p)_\infty (q;q)_\infty}{2} \prod_{1 \le i < j \le 4} \Gamma((pq)^{1/2} s_i s_j; p, q) \prod_{5 \le i < j \le 8} \Gamma((pq)^{1/2} s_i s_j; p, q)$$

$$\times \oint \frac{\prod_{i=1}^4 \Gamma((pq)^{1/4} v^{-2} s_i z^{\pm 1}; p, q) \prod_{i=5}^8 \Gamma((pq)^{1/4} v^2 s_i z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}, \qquad (10)$$

where v is a chemical potential of $U(1)_B$

$$v = \sqrt[4]{s_1 s_2 s_3 s_4}, \qquad v^{-1} = \sqrt[4]{s_5 s_6 s_7 s_8} . \qquad (11)$$

The second type is the original Seiberg dual theory [20] with $SU(2)$ gauge group and $SU(4) \times SU(4) \times U(1)_B \times U(1)_R$ flavor group, one singlet in the fundamental representation of $SU(4)$ and all other matter content is the same as the theory above. The superconformal index for this theory is

$$I_M^{(2)} = \frac{(p;p)_\infty (q;q)_\infty}{2} \prod_{i=1}^4 \prod_{j=5}^8 \Gamma((pq)^{1/2} s_i s_j; p, q)$$

$$\times \oint \frac{\prod_{i=1}^4 \Gamma((pq)^{1/4} v^2 s_i^{-1} z^{\pm 1}; p, q) \prod_{i=5}^8 \Gamma((pq)^{1/4} v^{-2} s_i^{-1} z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z} . \qquad (12)$$

The theory considered by Intriligator and Pouliot in [21] corresponds to the third type. There is only a single model of this type and it has $SU(8)$ flavor group and $SU(2)$ gauge group, one chiral scalar multiplet in the fundamental representation of the gauge group and antisymmetric representation of the flavor group, a gauge field in the adjoint representation of the gauge group and one singlet in the antisymmetric tensor representation of flavor group. The superconformal index is

$$I_M^{(3)} = \frac{(p;p)_\infty (q;q)_\infty}{2} \prod_{1 \le i < j \le 8} \Gamma((pq)^{1/2} s_i s_j; p, q) \oint \frac{\prod_{i=1}^8 \Gamma((pq)^{1/4} s_i^{-1} z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z}. \qquad (13)$$

More detailed explanations about these dual theories can be found in the original paper [3] and also in [22]. The equality of all four indices follows from the following identity [10]

$$I(t_1, \ldots, t_8; p, q) = \prod_{1 \le j < k \le 4} \Gamma(t_j t_k; p, q) \Gamma(t_{j+4} t_{k+4}; p, q) \, I(s_1, \ldots, s_8; p, q), \qquad (14)$$

where the complex variables s_j, $|s_j| < 1$, are given in terms of t_j $(j = 1, \ldots, 8)$,

$$s_j = \rho^{-1} t_j, \; j = 1, 2, 3, 4, \quad s_j = \rho t_j, \; j = 5, 6, 7, 8, \qquad (15)$$

$$\rho = \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}} .$$

All 72 dual theories are associated with the orbit of the $W(E_7)$ Weyl group. Using this fact Spiridonov and the second author speculated in [3], that the index may have global symmetry

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group E_7. In fact, Dimofte and Gaiotto explicitly showed in [18] that the theories in question, when coupled to $5d$ hypermultiplet, have an enhanced symmetry group E_7. In order to show this, they added the $5d$ hypermultiplet contributions with a specific boundary condition to the index

$$I_{4d/5d,\, N_F=4} = \prod_{1\le i<j\le 8} \frac{1}{\left(\sqrt{pq}(s_is_j)^{-1};p,q\right)_\infty} \frac{(p;p)_\infty(q;q)_\infty}{2} \oint \frac{dz}{2\pi i z} \frac{\prod_{i=1}^8 \Gamma(\sqrt[4]{pq}s_i z^\pm;p,q)}{\Gamma(z^{\pm 2};p,q)}. \quad (16)$$

where the term

$$\prod_{1\le i<j\le 8} \frac{1}{\left(\sqrt{pq}(s_is_j)^{-1};p,q\right)_\infty} \quad (17)$$

corresponds to a $5d$ hypermultiplet [23]. By setting all chemical potentials to 1 and redefining $p = t^3 y$, $q = t^3 y^{-1}$ one can easily read off the E_7 symmetry of the index by expanding the last expression in powers of t and y

$$I_{4d/5d,\, N_F=4} = 1 + 56t^3 + 1463t^6 + 3002t^9 y + \dots, \quad (18)$$

where the coefficients 56 and 1463 are the dimensions of the irreducible representations of E_7 with highest weight $[0,0,0,0,0,0,1]$ and $[0,0,0,0,0,0,2]$, respectively and $3002 = 1539_{[0,0,0,0,0,1,0]} + 1463_{[0,0,0,0,0,0,2]}$[6].

Remarkably, the new index is invariant under the transformation of the chemical potentials to their duals and the expression (14) becomes [18]

$$I(t_1,\dots,t_8;p,q) = I(s_1,\dots,s_8;p,q). \quad (19)$$

If we set $s_7 s_8 = \sqrt{pq}$ in (9) one gets the reduction[7] of the index from $N_F = 4$ to $N_F = 3$. When we apply this reduction for the integrals $I_M^{(1)}$ and $I_M^{(2)}$, setting $s_4 s_5 = \sqrt{pq}$ and $s_7 s_8 = \sqrt{pq}$, respectively, we end up with the flavor group $SU(3)_l \times SU(3)_r \times U(1)_B \times U(1)_{add}$ for $I_M^{(1)}$ and the flavor group $SU(4) \times SU(2) \times SU(2)_{add} \times U(1)_B$ for $I_M^{(2)}$. The observation that one gets additional symmetries such as $SU(2)_{add}$ and $U(1)_{add}$ in the reduced theories, suggests that the reduced theories may also have larger symmetry than $SU(6)$, in fact E_6 flavor symmetry. Indeed it is possible to show this by adding the $5d$ hypermultiplet contribution to the index and apply reduction procedure. The new reduced index is

$$I_{4d/5d,\, N_F=3} = \prod_{1\le i<j\le 6} \frac{1}{\left((pq)^{\frac{2}{3}} s_i^{-1} s_j^{-1};p,q\right)_\infty} \prod_{i=1}^6 \frac{1}{\left((pq)^{\frac{1}{3}} s_i^{-1} w^{\pm 1};p,q\right)_\infty}$$
$$\times \frac{(p,p)_\infty(q,q)_\infty}{2} \oint \frac{dz}{2\pi i z} \frac{\prod_{i=1}^6 \Gamma(\sqrt[6]{pq}s_i z^\pm;p,q)}{\Gamma(z^{\pm 2};p,q)}. \quad (20)$$

Note that we have redefined the chemical potentials $s_i \to (pq)^{-1/12} s_i$. The balancing condition is $\prod_{i=1}^6 s_i = 1$. Now by setting all chemical potentials to 1 and redefining $p = t^3 y$ and $q = t^3 y^{-1}$ one can read off the E_6 symmetry of the index

$$I_{4d/5d,\, N_F=3} = 1 + 27t^2 + 378t^4 + 3653t^6 + 27t^5(y^{-1} + y) + \dots \quad (21)$$

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[6]To find dimensions of irreducible representations of Lie algebras one can use
http://www-math.univ-poitiers.fr/~maavl/LiE/form.html

[7]We have used the reflection identity for an elliptic Gamma function $\Gamma(z;p,q)\Gamma(pqz^{-1};p,q) = 1$.

The coefficient 27 is the dimension of the irreducible representation of E_6 with highest weight $[1,0,0,0,0,0]$ and

$$378 = 351_{[0,0,1,0,0,0]} + 27_{[1,0,0,0,0,0]},\tag{22}$$

$$3653 = 3003_{[3,0,0,0,0,0]} + 650_{[1,0,0,0,0,1]}.\tag{23}$$

There is a reduction scheme [24] (also see [25, 26]) of the superconformal index for a $4d$ supersymmetric theory to the partition function for a $3d$ theory. Actually from the mathematical point of view this reduction nothing but a special limit that brings elliptic gamma functions to the hyperbolic level. Let us do this procedure for the index (20), following [24]. First we reparameterize

$$p = e^{2\pi i v\omega_1},\quad q = e^{2\pi i v\omega_2},\quad z = e^{2\pi i v u},\quad s_i = e^{2\pi i v\alpha_i},\quad w = e^{2\pi i v\alpha_7},\tag{24}$$

and use the asymptotic formula for the elliptic Γ-functions. Recall that in the limit $v \to 0$ the elliptic Γ-function reduce to hyperbolic $\gamma^{(2)}(z)$-function

$$\Gamma(e^{2\pi i vz}; e^{2\pi i v\omega_1}, e^{2\pi i v\omega_2}) \underset{v\to 0}{=} e^{-\pi i(2z-(\omega_1+\omega_2))/24v\omega_1\omega_2}\gamma^{(2)}(z;\omega_1,\omega_2),\tag{25}$$

where

$$\gamma^{(2)}(u;\omega_1,\omega_2) = e^{-\pi iB_{2,2}(u;\omega)/2}\frac{(e^{2\pi iu/\omega_1}\tilde{q};\tilde{q})}{(e^{2\pi iu/\omega_1};q)}\quad\text{with}\quad q = e^{2\pi i\omega_1/\omega_2},\quad \tilde{q} = e^{-2\pi i\omega_2/\omega_1},\tag{26}$$

and $B_{2,2}(u;\omega)$ is the second order Bernoulli polynomial,

$$B_{2,2}(u;\omega) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.\tag{27}$$

In the limit $v \to 0$ we also have

$$(z; p, q)_\infty \underset{v\to 0}{\to} \frac{1}{\Gamma_2(u;\omega_1,\omega_2)},\tag{28}$$

where $\Gamma_2(u;\omega_1,\omega_2)$ is the Barnes double Gamma function (see Appendix A).

To go further let us apply the limit $v \to 0$ to the index (20) and use the asymptotic formulae above. Finally we arrive at[8]

$$I_{4d/5d} \underset{v\to 0}{=} e^{\pi i(\omega_1+\omega_2)/12v\omega_1\omega_2}I_{4d/5d}^r,\tag{29}$$

where

$$I_{4d/5d}^r = \prod_{1\le i<j\le 6}\Gamma_2\Big(\frac{\omega_1+\omega_2}{2} - (\alpha_i+\alpha_j)\Big)\prod_{i=1}^{6}\Gamma_2\Big(-\frac{\omega_1+\omega_2}{2} - (\alpha_i\pm\alpha_7)\Big)$$

$$\times\frac{1}{2}\int\frac{du}{i\sqrt{\omega_1\omega_2}}\frac{\prod_{i=1}^{6}\gamma^{(2)}(\alpha_i\pm u + \frac{\omega_1+\omega_2}{4};;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u;\omega_1,\omega_2)}.\tag{30}$$

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[8]We have also used the reflection identity and some asymptotic formulas for $\gamma^{(2)}(z)$ function (see Appendix B). Here and below we will use the shorthand notations $\gamma^{(2)}(a,b;\omega_1,\omega_2) \equiv \gamma^{(2)}(a;\omega_1,\omega_2)\gamma^{(2)}(b;\omega_1,\omega_2)$, and $\gamma^{(2)}(a\pm u;\omega_1,\omega_2) \equiv \gamma^{(2)}(a+u;\omega_1,\omega_2)\gamma^{(2)}(a-u;\omega_1,\omega_2)$.

If one considers

$$\alpha_5 = \xi_1 + aS, \qquad \alpha_6 = \xi_2 - aS, \tag{31}$$

and applies the additional limit $S \to \infty$, then the final answer gives an expression for the partition function of $3d$ $\mathcal{N} = 2$ SYM theory[9] [27, 28, 29]

$$Z_{3d/4d} \underset{S\to\infty}{\approx} F Z^r_{3d/4d} \,, \tag{32}$$

where

$$Z^r_{3d/4d} = \Gamma_2\Big(\frac{\omega_1 + \omega_2}{2} - \xi_1 - \xi_2\Big) \prod_{1 \leq i < j \leq 4} \Gamma_2\Big(\frac{\omega_1 + \omega_2}{2} - (\alpha_i + \alpha_j)\Big) \prod_{i=1}^{4} \Gamma_2\Big(-\frac{\omega_1 + \omega_2}{2} - (\alpha_i \pm \alpha_7)\Big)$$

$$\times \frac{1}{2} \int \frac{du}{i\sqrt{\omega_1\omega_2}} \frac{\prod_{i=1}^{4} \gamma^{(2)}(\alpha_i \pm u + \frac{\omega_1+\omega_2}{4}; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2u; \omega_1, \omega_2)}. \tag{33}$$

From the physical point of view this reduction corresponds to adding mass terms to two quark supermultiplets and then integrating them out by sending their masses to infinity. As one can see, this theory has 4 quarks, one chiral field in the antisymmetric representation of the gauge group, and contributions from a $5d$ hypermultiplet.

3 Reduction to $N_f = 4$

In [18] it was shown that $3d$ $\mathcal{N} = 2$ theory with $N_f = 6$ has $SO(12)$ symmetry. The authors obtained the index of the $3d$ theory by reduction from $4d$ $\mathcal{N} = 1$ theory with $N_F = 4$ inspired by [30]. We will now demonstrate that the index for the 3d $\mathcal{N} = 2$ SYM theory with 4 quarks has $SO(10)$ symmetry group.

The expression for the index of the electric 3d $\mathcal{N} = 2$ supersymmetric theory [31, 32] with an arbitrary number of flavors N_f and chemical potentials s_i, t_i, $(i = 1, ..., N_f)$ is [33]

$$I_{3d,N_f} = \prod_{a,b=1}^{N_f} \frac{1}{\big(q^{\frac{1}{2}} t_a^{-1} s_b^{-1}; q\big)_\infty} \sum_{k \in \mathbb{Z}} a^{N_f |k|/2}$$

$$\times \oint \frac{dz}{2\pi i z} \prod_{i=1}^{N_f} \frac{(a^{1/2} q^{1/2+|k|/2} t_i^{-1} z; q)_\infty}{(a^{-1/2} q^{1/2+|k|/2} t_i z^{-1}; q)_\infty} \frac{(a^{1/2} q^{1/2+|k|/2} s_i^{-1} z^{-1}; q)_\infty}{(a^{-1/2} q^{1/2+|k|/2} s_i z; q)_\infty}, \tag{34}$$

where $\prod_{a=1}^{N_f} t_a = 1$ and $\prod_{a=1}^{N_f} s_a = 1$. It is clear that by taking $a = q^{\frac{1}{2}}$ for $N_f = 4$ (8 quarks),

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[9]For $\omega_1 = \frac{1}{\omega_2}$ we obtained that

$$F = \Big(-\xi_1 - \frac{5i\pi\xi_1}{6} - \xi_2 - \frac{i\pi\xi_2}{6}\Big)(\omega + \frac{1}{\omega}) + (\frac{i\pi}{3} - \frac{4}{3})(\frac{1}{\omega} + \omega)^2 - i\pi$$

$$- \frac{5}{2}i\pi\xi_1^2 + \frac{15\xi_2^2}{2} + \Big(\frac{3}{2} - \frac{i\pi}{2}\Big)(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + 5\xi_1^2 - 2\xi_1\xi_2 + 8\alpha_7^2).$$

we obtain the following expression

$$I_{3d,N_f=6} = \prod_{a,b=1}^{4} \frac{1}{\left(q^{\frac{1}{2}}t_a^{-1}s_b^{-1};q\right)_\infty} \sum_{k\in\mathbb{Z}} q^{|k|}$$

$$\times \oint \frac{dz}{2\pi i z} \prod_{i=1}^{4} \frac{(q^{1/4}q^{1/2+|k|/2}t_i^{-1}z;q)_\infty}{(q^{-1/4}q^{1/2+|k|/2}t_i z^{-1};q)_\infty} \frac{(q^{1/4}q^{1/2+|k|/2}s_i^{-1}z^{-1};q)_\infty}{(q^{-1/4}q^{1/2+|k|/2}s_i z;q)_\infty}. \tag{35}$$

One can rewrite this index in the following form [18]

$$I_{3d,N_f=6} = \frac{1}{\left(q^{\frac{1}{2}}f_1 f_2 f_3 f_4 f_5 f_6;q\right)_\infty} \prod_{1\le i<j\le 6} \frac{1}{\left(q^{\frac{1}{2}}f_i^{-1}f_j^{-1};q\right)_\infty}$$

$$\times \frac{1}{2}\sum_{k\in\mathbb{Z}} \oint \frac{dz}{2\pi i z}(1-q^{|k|}z^{\pm 2}) \prod_{i=1}^{6} f_i^{-|k|} \frac{1-q^{r+\frac{1}{2}|k|+1}(q^{\frac{1}{4}}f_i z^{\pm 1})^{-1}}{1-q^{r+\frac{1}{2}|k|}q^{\frac{1}{4}}f_i z^{\pm 1}}, \tag{36}$$

where $f_i = t_i/\sqrt{t_1 t_2 t_3 s_4}$ and $f_{i+3} = s_i\sqrt{t_1 t_2 t_3 s_4}$ $(i=1,2,3)$. The reduction of superconformal indices in $3d$ is similar to the $4d$ case. For the result of this paper, we set $f_5 f_6 = q^{\frac{1}{2}}$ which reduces the index of the theory with 6 quarks to the index of the theory with 4 quarks

$$I_{3d,N_f=4} = \frac{(q^{1/3};q)_\infty}{(qf_1 f_2 f_3 f_4;q)_\infty} \prod_{1\le i<j\le 4} \frac{1}{\left(q^{\frac{1}{2}}f_i^{-1}f_j^{-1};q\right)_\infty} \prod_{i=1}^{4} \frac{1}{\left(q^{\frac{1}{2}}f_i^{-1}q^{-\frac{1}{4}}v^{\pm 1};q\right)_\infty}$$

$$\times \frac{1}{2}\sum_{k\in\mathbb{Z}} \oint \frac{dz}{2\pi i z}(1-q^{|k|}z^{\pm 2}) \prod_{i=1}^{4} f_i^{-|k|} \frac{1-q^{r+\frac{1}{2}|k|+\frac{3}{4}}f_i^{-1}z^{\pm 1}}{1-q^{r+\frac{1}{2}|k|+\frac{1}{4}}f_i z^{\pm 1}}, \tag{37}$$

where the term $(q^{\frac{1}{3}};q)_\infty$ is a monopole contribution. Note that we have chosen the representation (36) of the index because it is closely related to the $3d$ $\mathcal{N}=2$ partition function (33). This procedure can be repeated for the initial expression of the index (36) in a similar way. Now one can read off the $SO(10)$–invariant operator content of the theory by expanding the last expression in powers of q and setting all chemical potentials to 1

$$I = 1 + 16q^{1/3} + 136q^{2/3} + 816q + 3892q^{4/3} + \dots \tag{38}$$

The coefficients are related to the dimensions of irreducible representations of $SO(10)$

$$16 \text{ is the dimension of the spinor representation of } SO(10) \tag{39}$$

$$136 = 54_{[2,0,0,0,0]} + 45_{[0,1,0,0,0]} + 16_{[0,0,0,1,0]} + 10_{[1,0,0,0,0]} + 1_{[0,0,0,0,0]}, \tag{40}$$

$$816 = 320_{[1,1,0,0,0]} + 210_{[0,0,0,1,1]} + 144_{[1,0,0,1,0]} + 126_{[0,0,0,2,0]} + 16_{[0,0,0,1,0]}, \tag{41}$$

$$3892 = 2772_{[0,0,0,4,0]} + 945_{[1,0,1,0,0]} + 120_{[0,0,1,0,0]} + 54_{[2,0,0,0,0]} + 1_{[0,0,0,0,0]}. \tag{42}$$

Appendices

A Barnes double Gamma function

The Barnes double Gamma function $\Gamma_2(u; \omega_1, \omega_2)$ is defined as

$$\log \Gamma_2(x; a, b) = \zeta_2'(0; a, b, x) + \log \rho_2(a, b), \tag{43}$$

where

$$\zeta_2(s; a, b, x) = \sum_{m,n=0} (am + bn + x)^{-s} \tag{44}$$

$$\log \rho_2(a, b) = - \lim_{x \to 0} \left(\zeta_2'(0; a, b, x) + \log x \right) \tag{45}$$

There is also the integral representation of this function

$$\Gamma_2(x; a, b) = \exp \left(\frac{1}{2\pi i} \int_{C_H} \frac{e^{-xt}(\log(-t) + \gamma)}{t(1 - e^{-at})(1 - e^{-bt})} dt \right), \tag{46}$$

where γ is the Euler constant and the Hankel contour C_H starts and finishes near the point $+\infty$, turning around the half–axis $[0, \infty)$ anticlockwise.

Useful references for specific details are [17, 34].

B Hyperbolic gamma-function

The reflection identity for a hyperbolic gamma-function is as follows

$$\gamma^{(2)}(z, \omega_1 + \omega_2 - z; \omega_1, \omega_2) = 1, \tag{47}$$

and the asymptotic formulas are

$$\lim_{u \to \infty} e^{\frac{\pi i}{2} B_{2,2}(u; \omega_1, \omega_2)} \gamma^{(2)}(u; \omega_1, \omega_2) = 1, \quad \text{for } \arg \omega_1 < \arg u < \arg \omega_2 + \pi, \tag{48}$$

$$\lim_{u \to \infty} e^{-\frac{\pi i}{2} B_{2,2}(u; \omega_1, \omega_2)} \gamma^{(2)}(u; \omega_1, \omega_2) = 1, \quad \text{for } \arg \omega_1 - \pi < \arg u < \arg \omega_2. \tag{49}$$

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