Planar undulator motion excited by a fixed traveling wave: Quasiperiodic Averaging, normal forms and the FEL Pendulum

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Abstract

We present a mathematical analysis of planar motion of energetic electrons moving through a planar dipole undulator, excited by a fixed planar polarized plane wave Maxwell field in the X-Ray FEL regime. Our starting point is the 6D Lorentz system, which allows planar motions, and we examine this dynamical system as the wave length λ of the traveling wave varies. By scalings and transformations the 6D system is reduced, without approximation, to a 2D system in a form for a rigorous asymptotic analysis using the Method of Averaging (MoA), a long time perturbation theory. The two dependent variables are a scaled energy deviation and a generalization of the so-called ponderomotive phase. As λ varies the system passes through resonant and nonresonant (NR) zones and we develop NR and near-to-resonant (NtoR) MoA normal form approximations. The NtoR normal forms contain a parameter which measures the distance from a resonance. For a special initial condition, for the planar motion and on resonance, the NtoR normal form reduces to the well known FEL pendulum system. We then state and prove NR and NtoR first-order averaging theorems which give explicit error bounds for the normal form approximations. We prove the theorems in great detail, giving the interested reader a tutorial on mathematically rigorous perturbation theory in a context where the proofs are easily understood. The proofs are novel in that they do not use a near identity transformation and they use a system of differential inequalities. The NR case is an example of quasiperiodic averaging where the small divisor problem enters in the simplest possible way. To our knowledge the planar problem has not been analyzed with the generality we aspire to here nor has the standard FEL pendulum system been derived with associated error bounds as we do here. We briefly discuss the low gain theory in light of our NtoR normal form. Our mathematical treatment of the *noncollective* FEL beam dynamics problem in the framework of *dynamical systems theory* sets the stage for our mathematical investigation of the *collective* high gain regime.

Contents

1 Introduction

We present a normal form analysis of the three-degree-of-freedom Lorentz force system of six ODE's (ordinary differential equations) governing the planar $(x, y = 0, z)$ motion of relativistic electrons moving through a planar dipole undulator along the z-axis perturbed by a traveling wave radiation field along the z direction. We are interested in the parameter range for an X-Ray FEL.

Our normal form analysis is based on the Method of Averaging (MoA) at first order. The method has four steps. The first step is to put the ODE's into a standard form. The second step is to identify the normal form approximations. The third step is the derivation of error bounds relating the exact and normal form solutions. The final step is the transformation back to the original variables of the Lorentz force system. In the first step new variables are typically introduced using scalings and transformations. In this process we discover that the exact problem can be formulated, without approximation, in terms of two ODE's for the normalized energy deviation and a generalized ponderomotive phase. Important in this process is the identification of an appropriate small dimensionless parameter, often denoted by ε, so that the system can be written as $\dot{u} = \varepsilon f(u, t) + O(\varepsilon^2)$. In the present context this is the most complicated step. The normal form approximation is obtained by dropping the $O(\varepsilon^2)$ term and replacing f by its $t-$average. The third step is often the most difficult, however here the system in standard form is fairly simple and we use this opportunity to give very detailed proofs of two averaging theorems, partly as a tutorial on the methods of proof, rather than applying general theorems from the literature. The latter allows us to obtain quite explicit error bounds which are likely near optimal.

An electron, as a member of an electron bunch, will enter the undulator with a given angle in the $y = 0$ plane and a given Lorentz factor. Here the normalized angle will be given by ΔP_{x0} and the Lorentz factor will be written $\gamma = \gamma_c(1 + \eta)$ where γ_c is a characteristic value of γ for the electron bunch, e.g. the mean, and η is the so-called normalized energy deviation. We will replace η by χ via the relation $\eta = \varepsilon\chi$, where a posteriori ε will be a measure of the spread of η values which lead to an FEL pendulum type behavior. We let B_u, k_u denote the undulator field strength and wave number and let $E_r, \nu k_r$ denote the Maxwell field strength and wave number of the fixed traveling wave radiation field. Thus our basic parameters are eight, namely $\Delta P_{x0}, \gamma_c, \varepsilon, B_u, k_u, E_r, k_r, \nu$. We will study the electron response to the radiation field as $\nu = O(1)$ varies. The choice of the parameter k_r will be discussed below.

For an X-Ray FEL, ε is small, γ_c is large and the undulator parameter,

$$K := \frac{eB_u}{mck_u} = .934\lambda_u[cm]B_u[T] \; , \tag{1.1}$$

is $O(1)$. Also $k_r = O(k_u\gamma_c^2)$ and we define the O(1) constant K_r by

$$K_r := \frac{k_r}{k_u\gamma_c^2} \; . \tag{1.2}$$

In §2.3 we will fix K_r (and thus k_r) by setting

$$K_r = 2[1 + \frac{1}{2}K^2 + K^2(\Delta P_{x0})^2]^{-1} \; . \tag{1.3}$$

For those familiar with FEL theory, k_r is, for $\Delta P_{x0} = 0$, the usual so-called resonant wave number (See e.g., [1]). The dependence of K_r on ΔP_{x0} will be a consequence of our analysis.

For the LCLS (Linac Coherent Light Source) $\lambda_u = 3\text{cm}$, $mc^2\gamma_c = 15\text{GeV}$ and $B_u = 1.32\text{T}$ so that $K = 3.70$ (see `http://www-ssrl.slac.stanford.edu/lcls/lcls_parms.html`).

Mathematically then, we are interested in an asymptotic analysis of the electron motion for ε small and γ_c large as ν varies. In particular we are interested in the (ε, γ_c) regime that gives rise to the pendulum type behavior important for the functioning of an X-Ray FEL. We find that in order to obtain this behavior, in the MoA at first-order, there must be a relation between ε and γ_c. Introducing the normalized field strength

$$\mathcal{E} := \frac{E_r}{cB_u} , \tag{1.4}$$

we show a pendulum type behavior emerges when $\varepsilon = O(\sqrt{\mathcal{E}}/\gamma_c)$ for $\gamma_c \gg 1$. Without loss of generality we will take the order constant to be 1, and choose

$$\varepsilon = \sqrt{\mathcal{E}}\frac{1}{\gamma_c} . \tag{1.5}$$

We also show that, for ε small, the system associated with (1.5) has a resonance structure, such that as ν varies the system goes through a sequence of nonresonant (NR) and near-to-resonant (NtoR) zones. The associated NtoR approximating normal forms are pendulum like and reduce to the standard FEL pendulum system for $\Delta P_{x0} = 0$ and ν an odd integer. This behavior is not present for $\varepsilon \ll 1/\gamma_c$ or $\varepsilon \gg 1/\gamma_c$ and so we refer to (1.5) as a *distinguished case*. This turns out to be a very simple example of the concept of a "distinguished limit" in the singular perturbation literature. This can be seen in action in the context of our equations (2.56) and (2.57).

In summary, for the distinguished case of (1.5), our basic nondimensional parameters are $K, \Delta P_{x0}, \mathcal{E}, \varepsilon, \nu$. For ε small we will obtain a sequence of nonresonant (NR) and near-to-resonant (NtoR) normal form approximations as ν varies. The NtoR normal forms can be understood in terms of the simple pendulum system and reduce to the usual FEL pendulum equations for $\Delta P_{x0} = 0$ and ν an odd integer (See Sections 3.4.2 and 3.4.3). The NtoR normal form allows us to study the effect of ν being slightly off resonance. This completes the first two steps in the MoA. In the third step we prove two theorems which give error bounds, relating the exact and normal form solutions, which go to zero as $\varepsilon \to 0+$. Our goal is to present a mathematically rigorous analysis that is self contained.

Standard derivations of the FEL pendulum equations can be found in [2],[3],[4],[5]. They differ from our approach in that they start from the ODE for the normalized energy deviation, η, and use physical reasoning to introduce approximations leading to the FEL pendulum normal form for $\Delta P_{x0} = 0$. In contrast, our starting point is the three-degree-of-freedom Lorentz force ODE's which are clearly more general and we make no approximation in going to the standard form for the MoA. Thus our only approximation is in going from the averaging standard form to the normal form approximations. Furthermore we obtain error bounds which do not appear to be possible in the standard derivations and these bounds are covered by our averaging theorems. Our definition of resonance is intimately linked to the derivation of our averaging normal forms, whereas in the standard derivations resonance is introduced in the context of maximizing energy exchange. We emphasize that we obtain more than the pendulum normal form; we also obtain the more general NtoR normal form as well as the NR normal forms.

We do not intend to minimize the importance of the standard derivations, the physical derivations are certainly important and as is often the case show great physical insight. Here we want to show what can be done in a mathematically rigorous way in the context of dynamical systems theory, but in that we have been guided by and are indebted to the work of e.g., [2],[3],[4],[5].

For ODE's, the MoA is the most robust of the longtime perturbation theories which include e.g., Lindstedt series [6], multiple scales [6], renormalization group methods [7] and Hamiltonian perturbation theory [8]. For example, Hamiltonian perturbation theory has the advantage that one is transforming a scalar function, however the MoA is more robust in that transformations and scalings are not restricted to canonical transformations. Central to the MoA, and in contrast to those just mentioned, is the derivation of error bounds. We emphasize these are true bounds and not just estimates. The MoA is a mature subject and there are several good books, see [6, 9, 10] for example as well as the Scholarpedia articles [11, 12]. We refer to the MoA approximation as a normal form. Generally, a normal form of a mathematical object is a simplified form of the object obtained with the aid of, for example, scalings and transformations such that the essential features of the object are preserved. Here we not only preserve the essential features of the exact ODE's but bound the errors in the approximation with a bound proportional to the small parameter ε. See [11] for the use of normal form in a similar context.

This paper has a pedagogical aspect, giving the reader, who may not be familiar with modern long time perturbation theory, an introduction in a context where the proofs are easily understood. In addition, we hope that both newcomers to the field and mathematical scientists will find this a good introduction to the noncollective case of an FEL. We also hope that experts will find something of interest. The reader does not need to be familiar with averaging theory as we give complete proofs including detailed error bounds. Furthermore we obtain better results as our theorems are tuned to the problem at hand. In addition, to our knowledge, the treatment of the undulator problem in the mathematically rigorous and self-contained way that we do here has not been done before. Our mathematical analysis is not deep, using only undergraduate mathematics as commonly taught in advanced calculus courses, however it is complicated and somewhat intricate in spots. Finally, for us, it sets the stage for our more serious goal of a deep mathematical understanding of the collective high gain FEL theory.

We proceed as follows. In §2 we start with the three-degree-of-freedom Lorentz equations with a general traveling wave field in (2.7)-(2.10) and then introduce z as the independent variable. The system has planar solutions where $0 = y = p_y$ and using a conservation law we arrive at a system of two ODE's (2.33),(2.34) for the energy deviation and a precursor to a generalization of the so-called ponderomotive phase. By scalings and transformations we discover the distinguished case of (1.5) which then leads to a standard form for the method of averaging in (2.62),(2.63). The two dependent variables are now a scaled energy deviation and a generalization of the so-called ponderomotive phase.

In §3 we present our main results. We begin by introducing the monochromatic traveling wave field, the case of main physical interest. The system is carefully defined in §3.1. In §3.2 we define nonresonant, Δ-nonresonant, resonant, and near-to-resonant ν in the MoA context. We emphasize that as ν varies the system passes through resonant and nonresonant zones. The NR case, its first-order averaging normal form and associated solutions are presented in §3.3 along with a proposition giving an appropriate domain for the associated vector field. §3.3 sets the stage for the more interesting NtoR case of §3.4. The NtoR system is carefully defined along with a proposition giving an appropriate domain for the associated vector field. The first-order averaging normal form is derived and solutions written in terms of solutions of the simple pendulum system. It is unlikely that all ν values are covered accurately by our normal forms, however we are able to argue in §3.4.4 that there is a sense in which the NR case emerges from the NtoR case. The third and fourth steps of the MoA are performed in §3.5 and §3.6. In fact, the statements of our first-order averaging theorems, which give an order ε bound on the error for long times, i.e., intervals of $O(1/\varepsilon)$, are presented in §3.5 and applied to the phase space variables in §3.6. By taking special initial conditions ($\Delta P_{x0} = 0$) we recover the result of standard approaches which focus on the energy transfer equations alone and do not consider the phase space variables. Finally in §3.7 we use our results in a low gain calculation and compare the result with [2].

The proofs of the two averaging theorems are presented in §4 and they are based on an idea of Besjes (see [13, 14, 15]) which leads to proofs without using a near-identity transformation, as in usual treatments of, e.g., [6, 9, 10]. The NR case is an example of quasiperiodic averaging with a rigorous treatment of a small divisor problem in what is surely the simplest setting. The NtoR case is an example of periodic averaging. A novelty of our approach is that we use a *system* of differential inequalities, rather than the usual Gronwall inequality, to obtain better error bounds.

The appendices contain calculations needed in the main text. Appendix A provides properties of the Bessel expansion of the function jj which is introduced in Section 3.2. In Appendices B,C we study the next-to-leading order terms g_1, g_2 used in Theorem 1 and in Appendices D,E we study the next-to-leading order terms g_1^R, g_2^R used in Theorem 2. Appendix F gives an outline of a rigorous approach to regular perturbation theory which could be made into a theorem at the level of our averaging theorems. It is applied in §3.4.4. Appendix G provides some formulas used in Section 3.7. In Appendix H we discuss $\mathcal{E} = E_r/cB_u$ in the high gain regime and obtain a crude upper bound estimate of it. Finally, in Appendix I we show that the solution of the system of differential inequalities that is used in the proof of both averaging theorems (as well as in Appendix F) is indeed a solution.

2 General Planar Undulator model

2.1 Lorentz force equations

Using SI units, the Lorentz equations for motion of a relativistic electron in an electromagnetic field, $(\mathbf{E}, \mathbf{B})$, are

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{p}) \ , \tag{2.1}$$

$$\dot{\mathbf{p}} = -e(\mathbf{E} + \mathbf{v}(\mathbf{p}) \times \mathbf{B}), \tag{2.2}$$

with $\dot{} = d/dt$ and where

$$\mathbf{v}(\mathbf{p}) = \frac{\mathbf{p}}{m\gamma} \ , \tag{2.3}$$

is the velocity, γ is the Lorentz factor defined by

$$\gamma^2 = 1 + \mathbf{p}\cdot\mathbf{p}/m^2c^2 \ , \tag{2.4}$$

and m and $-e$ are the electron mass and charge respectively. We introduce Cartesian coordinates as follows:

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \ , \tag{2.5}$$

$$\mathbf{p} = p_x\mathbf{e}_x + p_y\mathbf{e}_y + p_z\mathbf{e}_z \ , \tag{2.6}$$

where $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ are the standard unit vectors. Using (2.1)-(2.6) the system in Cartesian coordinates is

$$\dot{x} = \frac{p_x}{m\gamma} \ , \quad \dot{y} = \frac{p_y}{m\gamma} \ , \quad \dot{z} = \frac{p_z}{m\gamma} \ , \tag{2.7}$$

$$\dot{p}_x = -e[E_x + v_y B_z - v_z B_y] \ , \tag{2.8}$$

$$\dot{p}_y = -e[E_y + v_z B_x - v_x B_z] \ , \tag{2.9}$$

$$\dot{p}_z = -e[E_z + v_x B_y - v_y B_x] \ . \tag{2.10}$$

We denote the undulator magnetic field by $\mathbf{B}_u$ and the radiation field by $(\mathbf{E}_r, \mathbf{B}_r)$ whence

$$\mathbf{E} = \mathbf{E}_r \ , \quad \mathbf{B} = \mathbf{B}_r + \mathbf{B}_u \ . \tag{2.11}$$

A simple planar undulator model magnetic field which satisfies the Maxwell equations, $\nabla \cdot \mathbf{B}_u = 0$ and $\nabla \times \mathbf{B}_u = 0$, as in [3], is

$$\mathbf{B}_u = -B_u[\cosh(k_u y)\sin(k_u z)\mathbf{e}_y + \sinh(k_u y)\cos(k_u z)\mathbf{e}_z] \ , \tag{2.12}$$

where $B_u > 0$. Since $\nabla \times \mathbf{B}_u = 0$ there is a scalar potential ϕ such that $\mathbf{B}_u = \nabla\phi$. To satisfy $\nabla \cdot \mathbf{B}_u = 0$, ϕ must satisfy Laplace's equation. The field (2.12) is easily constructed by separation of variables and requiring periodicity in z with period λ_u and then taking the first eigen-mode (See, e.g., [16, p. 145]). The scalar field is $\phi = -(B_u/k_u)\sinh(k_u y)\sin(k_u z)$.

The traveling wave radiation field we choose is also a Maxwell field and is given by

$$\mathbf{E}_r = E_r h(\check{\alpha})\mathbf{e}_x \ , \quad \mathbf{B}_r = \frac{1}{c}(\mathbf{e}_z \times \mathbf{E}_r) = \frac{E_r}{c} h(\check{\alpha})\mathbf{e}_y \ , \tag{2.13}$$

where E_r is a constant, h is a real valued function on $\mathbb{R}$ and

$$\check{\alpha}(z,t) = k_r(z - ct) \ , \tag{2.14}$$

and k_r is the parameter mentioned in the Introduction.

Our primary emphasis is on the standard monochromatic example where

$$H(\check{\alpha}) = (1/\nu)\sin(\nu\check{\alpha}) \ , \quad h(\check{\alpha}) = H'(\check{\alpha}) = \cos(\nu\check{\alpha}) \ , \tag{2.15}$$

and $\nu \geq 1/2$ thus $h(\check{\alpha}(z,t)) = \cos(\nu k_r(z - ct))$. Note that the prime $'$ always indicates a derivative. Thus from §3 onwards we will use (2.15). However it is easy to carry through the first part of the analysis with general H and we do want to make a comment on the more general case. In this monochromatic case k_r will be defined by (1.2),(1.3) and the ν will allow for a variable wave number for the traveling wave; it will be shown that $\nu = 1$ gives the primary resonance with the concomitant pendulum normal form. The extension to a sum of monochromatic waves is trivial and won't be discussed.

Using (2.3),(2.12),(2.13) one can write (2.8)-(2.10) as

$$\begin{aligned}\dot{p}_x = -e[&\frac{p_z}{m\gamma}B_u\cosh(k_u y)\sin(k_u z) - \frac{p_y}{m\gamma}B_u\sinh(k_u y)\cos(k_u z) \\ &+E_r(1 - \frac{p_z}{m\gamma c})h(\check{\alpha}(z,t))] \ ,\end{aligned} \tag{2.16}$$

$$\dot{p}_y = -e\frac{p_x}{m\gamma}B_u\sinh(k_u y)\cos(k_u z) \ , \tag{2.17}$$

$$\dot{p}_z = -e[-\frac{p_x}{m\gamma}B_u\cosh(k_uy)\sin(k_uz) + E_r\frac{p_x}{m\gamma c}h(\breve{\alpha}(z,t))] \ . \tag{2.18}$$

It is easy to check that (2.7),(2.16)-(2.18) is a Hamiltonian system with Hamiltonian $\mathcal{H}$:

$$\mathcal{H} = c\sqrt{(\mathbf{P}_c + e\mathbf{A}(\mathbf{r},t))^2 + m^2c^2} = mc^2\gamma \ , \tag{2.19}$$

where the canonical momentum vector $\mathbf{P}_c$ is related to $\mathbf{p}$ by $\mathbf{p} = \mathbf{P}_c + e\mathbf{A}$ and the vector potential $\mathbf{A}$ is given by

$$\mathbf{A}(y,z,t) = [\frac{B_u}{k_u}\cosh(k_uy)\cos(k_uz) + \frac{E_r}{k_rc}H(\breve{\alpha}(z,t))]\mathbf{e}_x \ . \tag{2.20}$$

Since $\mathbf{A}$ is independent of x the x-component, $P_{c,x}$, of the canonical momentum vector $\mathbf{P}_c$ is conserved, i.e.,

$$p_x - eA_x(y,z,t) \ , \tag{2.21}$$

is constant along solutions of (2.7),(2.16)-(2.18) as is easily confirmed directly. We will not make explicit use of the Hamiltonian structure in the following. The MoA does not rely on a Hamiltonian structure and this frees us from having to deal only with canonical transformations as we proceed to put (2.7),(2.16)-(2.18) in an averaging standard form.

2.2 Motion in $y = 0$ plane with z as the independent variable

It is common to take the distance z along the undulator as the independent variable, rather than the time t. In fact after unsuccessfully trying to stay with t we decided to follow the common procedure. With the usual abuse of notation, we write, from now on $x(z), y(z), p_x(z), p_y(z), p_z(z)$ instead of $x(t(z)), y(t(z)), p_x(t(z)), p_y(t(z)), p_z(t(z))$ whence the ODE's (2.7),(2.16)-(2.18) become

$$\frac{dx}{dz} = \frac{p_x}{p_z} \ , \quad \frac{dy}{dz} = \frac{p_y}{p_z} \ , \quad \frac{dt}{dz} = \frac{m\gamma}{p_z} \ , \tag{2.22}$$

$$\begin{aligned}\frac{dp_x}{dz} &= -\frac{e}{c}[cB_u\cosh(k_uy)\sin(k_uz) - \frac{p_y}{p_z}cB_u\sinh(k_uy)\cos(k_uz) \\ &\qquad +E_r(\frac{m\gamma c}{p_z} - 1)h(\breve{\alpha}(z,t))] \ ,\end{aligned} \tag{2.23}$$

$$\frac{dp_y}{dz} = -\frac{e}{c}\frac{p_x}{p_z}cB_u\sinh(k_uy)\cos(k_uz) \ , \tag{2.24}$$

$$\frac{dp_z}{dz} = -\frac{e}{c}[-\frac{p_x}{p_z}cB_u\cosh(k_uy)\sin(k_uz) + E_r\frac{p_x}{p_z}h(\breve{\alpha}(z,t))] \ . \tag{2.25}$$

The initial conditions at $z = 0$ will be denoted by a subscript 0, e.g., $t(0) = t_0$. Clearly t_0 is the arrival time of an electron at the entrance, $z = 0$, of the undulator.

Here and in the rest of the paper we consider the initial value problem (IVP) with $y_0 = p_{y0} = 0$. It follows, with no approximation, that $y(z) = p_y(z) = 0$ for all z and the six ODE's (2.22)-(2.25) reduce to four. The righthand sides (rhs's) of (2.22)-(2.25) are independent of x and so we do not need to consider the x equation until §3.6. It is standard, and also quite convenient, to replace p_z by the energy variable γ. With $\gamma(z)$ defined in terms of $p_x(z)$ and $p_z(z)$ by (2.4) and using (2.23) and (2.25), we obtain $\gamma' = (p_x p_x' + p_z p_z')/m^2c^2\gamma = -(eE_r/mc^2)(p_x/p_z)h(\breve{\alpha}(z,t))$. Finally, we take $\breve{\alpha}$ as a dependent variable in place of t and we define

$$\alpha(z) := \breve{\alpha}(z, t(z)) = k_r(z - ct(z)) \; . \tag{2.26}$$

Later it will be seen that α is a precursor to a generalization of the so-called ponderomotive phase which emerges naturally as we put the ODE's in a standard form for averaging.

With the above four changes the ODE's for t, p_x, p_z in (2.22),(2.23),(2.25) become

$$\frac{d\alpha}{dz} = k_r(1 - \frac{m\gamma c}{p_z}) \; , \tag{2.27}$$

$$\frac{dp_x}{dz} = -\frac{e}{c}[cB_u \sin(k_u z) + E_r(\frac{m\gamma c}{p_z} - 1)h(\alpha)] \; , \tag{2.28}$$

$$\frac{d\gamma}{dz} = -\frac{eE_r}{mc^2}\frac{p_x}{p_z}h(\alpha) \; , \tag{2.29}$$

where the initial conditions are $\alpha(0) = \alpha_0 := -k_r c t_0, p_x(0) =: p_{x0}, \gamma(0) =: \gamma_0$. Here p_z must be replaced by

$$p_z = \sqrt{m^2c^2(\gamma^2 - 1) - p_x^2} \; , \tag{2.30}$$

and it is easy to see that (2.27)-(2.29) are then self contained. From now on we restrict p_z to be positive:

$$p_z > 0 \; . \tag{2.31}$$

Note that, by (2.27), α is a strictly decreasing function whence, as one expects, $z < c(t(z) - t_0)$. It is also easy to check that

$$\frac{p_x}{mcK} - \cos(k_u z) - \frac{E_r}{cB_u}\frac{k_u}{k_r}H(\alpha) \; , \tag{2.32}$$

is conserved along solutions of (2.27)-(2.29). This conservation law is identical to (2.21) with $y = 0$. Recall that K was defined by (1.1).

In summary, the solution of the IVP for (2.22)-(2.25) with $y_0 = p_{y0} = 0$, which entails $y = p_y = 0$, is given in terms of the solution of (2.27),(2.29), i.e., of

$$\frac{d\alpha}{dz} = k_r(1 - \frac{m\gamma c}{p_z}) \; , \quad \alpha(0) = \alpha_0 \; , \tag{2.33}$$

$$\frac{d\gamma}{dz} = -\frac{eE_r}{mc^2}\frac{p_x}{p_z}h(\alpha) \; , \quad \gamma(0) = \gamma_0 \; , \tag{2.34}$$

with

$$p_x = p_{x0} + mcK\left(\cos(k_u z) - 1 + \frac{E_r}{cB_u}\frac{k_u}{k_r}[H(\alpha) - H(\alpha_0)]\right) \; , \tag{2.35}$$

and p_z in (2.30). To complete the solution of (2.22)-(2.25) it suffices to note that $t(z)$ is determined from (2.26) in terms of $\alpha(z)$ and $x(z)$ is determined from (2.22) by integration.

2.3 Standard form for Method of Averaging

We begin by introducing the normalized energy deviation η and its $O(1)$ counterpart χ via

$$\gamma = \gamma_c(1+\eta) = \gamma_c(1+\varepsilon\chi) \ , \tag{2.36}$$

as mentioned in the Introduction. Here γ_c is a characteristic value of γ, e.g., its mean and ε is a characteristic spread of η so that χ becomes the new $O(1)$ dependent variable replacing γ in (2.33),(2.34). We are interested in an asymptotic analysis for γ_c large and η small as in an X-Ray FEL. Here we determine a relation between ε and γ_c which leads to a standard form for the MoA and which will contain the FEL pendulum system at first order in the case of (2.15).

As a first step we introduce new variables, in addition to χ, as follows. From the conservation law in (2.32) we anticipate that the order of magnitude of p_x will be mcK. In addition $\beta_z := p_z/mc\gamma$ will be near 1 and so $p_z \approx mc\gamma$. Thus we define dimensionless momenta by

$$p_x = mcKP_x \ , \quad p_z = mc\gamma P_z \ . \tag{2.37}$$

Of course, by (2.31),

$$P_z > 0 \ . \tag{2.38}$$

A natural scaling for z is

$$z = \zeta/k_u \ , \tag{2.39}$$

so that the undulator period is 2π in ζ.

Abbreviating

$$\theta_{aux}(\zeta) := \alpha(\zeta/k_u) \ , \tag{2.40}$$

and with (1.2) the system (2.33),(2.34) becomes

$$\theta'_{aux} = K_r\gamma_c^2(1-\frac{1}{P_z}) \ , \tag{2.41}$$

$$\chi' = -K^2\frac{\mathcal{E}}{\varepsilon\gamma_c^2}\frac{1}{1+\varepsilon\chi}\frac{P_x}{P_z}h(\theta_{aux}) \ , \tag{2.42}$$

where $' = d/d\zeta$ and $\mathcal{E}$ is defined in (1.4). The initial conditions are $\theta_{aux}(0,\varepsilon) = \theta_0 := \alpha_0$, $\chi(0,\varepsilon) = \chi_0$. Moreover P_z must be replaced, due to (2.30), by

$$P_z = \sqrt{1-\frac{1}{\gamma^2}(1+K^2P_x^2)} \quad \text{with} \quad \gamma = \gamma_c(1+\varepsilon\chi) \ , \tag{2.43}$$

and P_x must be replaced, due to (2.35), by

$$P_x = \cos\zeta + \Delta P_{x0} + \frac{\mathcal{E}}{K_r\gamma_c^2}[H(\theta_{aux}) - H(\theta_0)] \ , \tag{2.44}$$

where

$$\Delta P_{x0} := P_{x0} - 1 \ , \quad P_{x0} := P_x(0) = \frac{p_{x0}}{mcK} \ . \tag{2.45}$$

Since $p_z > 0$ we have $0 < P_z < 1$. We note that most derivations of the FEL pendulum take $\Delta P_{x0} = 0$, see [2, 3, 4, 5].

To expand P_z we need

$$\begin{aligned}
&1 + K^2 P_x^2 = 1 + K^2(\cos\zeta + \Delta P_{x0})^2 \\
&+ \frac{2K^2\mathcal{E}}{K_r\gamma_c^2}(\cos\zeta + \Delta P_{x0})(H(\theta_{aux}) - H(\theta_0)) + \frac{K^2\mathcal{E}^2}{K_r^2\gamma_c^4}(H(\theta_{aux}) - H(\theta_0))^2 \ ,
\end{aligned} \tag{2.46}$$

and it is convenient to define

$$q(\zeta) := 1 + K^2(\cos\zeta + \Delta P_{x0})^2 = \bar{q} + 2K^2\Delta P_{x0}\cos\zeta + \frac{K^2}{2}\cos 2\zeta \ , \tag{2.47}$$

$$\bar{q} := 1 + \frac{1}{2}K^2 + K^2(\Delta P_{x0})^2 \ . \tag{2.48}$$

Clearly $\bar{q}$ is the average of $q(\zeta)$ over ζ. Now P_x is $O(1)$ so, by (2.43),

$$\begin{aligned}
\frac{1}{P_z} &= 1 + \frac{1 + K^2 P_x^2}{2\gamma_c^2(1 + \varepsilon\chi)^2} + O(\frac{1}{\gamma_c^4}) \\
&= 1 + \frac{q(\zeta)}{2\gamma_c^2}(1 - 2\varepsilon\chi + O(\varepsilon^2)) + O(\frac{1}{\gamma_c^4}) \\
&= 1 + \frac{q(\zeta)}{2\gamma_c^2}(1 - 2\varepsilon\chi) + O(\frac{1}{\gamma_c^4}) + O(\frac{\varepsilon^2}{\gamma_c^2}) \ .
\end{aligned} \tag{2.49}$$

Thus using (2.44) and (2.49), eq.'s (2.41) and (2.42) become

$$\theta'_{aux} = -\frac{K_r q(\zeta)}{2} + \varepsilon K_r q(\zeta)\chi + O(\frac{1}{\gamma_c^2}) + O(\varepsilon^2) \ , \tag{2.50}$$

$$\chi' = -K^2\frac{\mathcal{E}}{\varepsilon\gamma_c^2}(\cos\zeta + \Delta P_{x0})h(\theta_{aux}) + O(1/\gamma_c^2) + O(1/\varepsilon\gamma_c^4) \ . \tag{2.51}$$

To transform (2.50),(2.51) into a standard form for the MoA we need to introduce dependent variables that are slowly varying. We anticipate that χ will be slowly varying, i.e., $\frac{\mathcal{E}}{\varepsilon\gamma_c^2}$ will be small. To remove the $O(1)$ in (2.50) we define

$$\theta := \theta_{aux} + Q(\zeta) \ , \tag{2.52}$$

where

$$Q(\zeta) := \zeta + \Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta \ , \tag{2.53}$$

$$\Upsilon_0 := \frac{2K^2\Delta P_{x0}}{\bar{q}}, \quad \Upsilon_1 := \frac{K^2}{4\bar{q}} \ . \tag{2.54}$$

Note that Υ_0 and Υ_1 depend only on K and ΔP_{x0} and that

$$Q'(\zeta) = \frac{K_r q(\zeta)}{2} \ . \tag{2.55}$$

Thus the system (2.50),(2.51) becomes

$$\theta' = \varepsilon K_r q(\zeta)\chi + O(1/\gamma_c^2) + O(\varepsilon^2) \ , \tag{2.56}$$

$$\chi' = -K^2 \frac{\mathcal{E}}{\varepsilon\gamma_c^2}(\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta)) + O(1/\gamma_c^2) + O(1/\varepsilon\gamma_c^4) \, . \tag{2.57}$$

The initial conditions are $\theta(0,\varepsilon) = \theta_0, \chi(0,\varepsilon) = \chi_0$. To obtain a system where θ and χ interact with each other in first-order averaging we must balance the $O(\varepsilon)$ term in (2.56) with the $O(\mathcal{E}/\varepsilon\gamma_c^2)$ in (2.57). In this spirit we relate ε and γ_c by choosing

$$\varepsilon = \frac{\mathcal{E}}{\varepsilon\gamma_c^2} \, , \tag{2.58}$$

and so we obtain (1.5). It is this balance that will lead to the FEL pendulum equations in §3. This is the distinguished case mentioned in the Introduction and the system (2.56),(2.57) can be written

$$\theta' = \varepsilon K_r q(\zeta)\chi + O(\varepsilon^2) \, , \tag{2.59}$$
$$\chi' = -\varepsilon K^2(\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta)) + O(\varepsilon^2) \, , \tag{2.60}$$

which are now in standard form. Up to this point K_r has not been fixed but now it is convenient to take

$$K_r = 2/\bar{q} \, , \tag{2.61}$$

which we do from now on. Using (2.48), (2.61) is identical to (1.3). Furthermore in the monochromatic case of (2.15) and §3, we will see that, with (2.61), the primary resonance appears at $\nu = 1$.

With (2.61) the ODE's (2.59), (2.60) become

$$\theta' = \varepsilon \frac{2q(\zeta)}{\bar{q}}\chi + O(\varepsilon^2) \, , \tag{2.62}$$
$$\chi' = -\varepsilon K^2(\cos\zeta + \Delta P_{x0})h(\theta - Q(\zeta)) + O(\varepsilon^2) \, . \tag{2.63}$$

We now relate θ to the so-called ponderomotive phase. We have, from (2.26),(2.40), (2.52) and (2.53),

$$\theta(\zeta,\varepsilon) = \frac{k_r}{k_u}(\zeta - k_u ct(\zeta/k_u)) + [\zeta + \Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta] \, . \tag{2.64}$$

Using (2.39) and (2.64) we obtain

$$\theta(k_u z,\varepsilon) = k_r(z - ct(z)) + k_u z + \Upsilon_0 \sin k_u z + \Upsilon_1 \sin(2k_u z) \, . \tag{2.65}$$

For $\Delta P_{x0} = 0$ the variable θ is the so-called ponderomotive phase, i.e.,

$$\theta(k_u z,\varepsilon) = (k_u + k_r)z - k_r ct(z) + \Upsilon_1 \sin(2k_u z) \, , \tag{2.66}$$

where, for $\Delta P_{x0} = 0$,

$$\Upsilon_1 = \frac{k_r K^2}{8k_u\gamma_c^2} = \frac{K_r K^2}{8} = \frac{K^2}{4\bar{q}} = \frac{K^2}{4 + 2K^2} \, . \tag{2.67}$$

Thus in our context the ponderomotive phase arises naturally in the process of finding the distinguished relation between ε and γ_c and transforming to slowly varying coordinates. In standard treatments it is introduced heuristically to maximize energy transfer.

To make the $O(\varepsilon^2)$ terms in (2.62),(2.63) explicit we first rewrite (2.41),(2.42) in terms of ε, K and $\mathcal{E}$ as

$$\theta'_{aux} = \frac{2\mathcal{E}}{\bar{q}\varepsilon^2}(1 - \frac{1}{P_z}) \ , \tag{2.68}$$

$$\chi' = -K^2\varepsilon\frac{1}{1+\varepsilon\chi}\frac{P_x}{P_z}h(\theta_{aux}) \ , \tag{2.69}$$

where

$$P_z^2 = 1 - \frac{\varepsilon^2}{\mathcal{E}}(1+\varepsilon\chi)^{-2}(1+K^2P_x^2) \ , \tag{2.70}$$

$$P_x = \cos\zeta + \Delta P_{x0} + \frac{\varepsilon^2\bar{q}}{2}[H(\theta_{aux}) - H(\theta_0)] \ . \tag{2.71}$$

The initial conditions are $\theta_{aux}(0,\varepsilon) = \theta_0, \chi(0,\varepsilon) = \chi_0$. Under (2.52),(2.61), the system becomes (2.68),(2.69) becomes

$$\theta' = \frac{2\mathcal{E}}{\varepsilon^2\bar{q}}(1 - \frac{1}{P_z}) + \frac{q(\zeta)}{\bar{q}} \ , \tag{2.72}$$

$$\chi' = -\varepsilon K^2\frac{1}{1+\varepsilon\chi}\frac{P_x}{P_z}h(\theta - Q(\zeta)) \ , \tag{2.73}$$

where

$$P_x = \cos\zeta + \Delta P_{x0} + \frac{\varepsilon^2\bar{q}}{2}[H(\theta - Q(\zeta)) - H(\theta_0)] \ . \tag{2.74}$$

The $O(\varepsilon^2)$ terms in (2.62),(2.63) can now be determined by comparison with (2.72),(2.73). We will do this in the monochromatic case of §3.

Remarks:

(1) Note that, by (1.4),(1.5), $\gamma_c = \sqrt{\mathcal{E}}/\varepsilon$, in particular $\gamma_c > 0$ and, by (2.36),

$$\gamma = \gamma_c(1+\varepsilon\chi) = \sqrt{\mathcal{E}}(\frac{1}{\varepsilon} + \chi) \ . \tag{2.75}$$

Since, by (2.31), we have the restriction $\gamma > 1$ we also have, by (2.75),

$$1 + \varepsilon\chi > 0 \ . \tag{2.76}$$

Because, by (2.38), $P_z > 0$, Eq. (2.70) gives $\frac{\varepsilon}{\sqrt{\mathcal{E}}}\sqrt{1+K^2P_x^2} < |1+\varepsilon\chi|$ and (2.76) gives

$$\chi > -\frac{1}{\varepsilon} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2P_x^2} \ . \tag{2.77}$$

Note that (2.77) defines our maximal domain of points (θ, χ, ζ), in particular it entails (2.38),(2.76). We will in §3.1 further restrict this domain.

Of course always $\gamma \geq 1$ and, in fact, in applications $\gamma_c, \gamma \gg 1$. However for our purposes it is convenient to base our work on the maximal domain (2.77).

(2) The transformation to the slowly varying θ in (2.52) works nicely because ζ (equivalently z) is the independent variable. If we had stayed with t as the independent variable this step wouldn't work.

(3) Equations (2.62),(2.63) are in the standard form for the MoA. However we did not prove that the $O(\varepsilon^2)$ are actually bounded by an ε-independent constant times ε^2. In the monochromatic case in §3 we will show that the two $O(\varepsilon^2)$ terms are truly bounded by $\mathcal{C}\varepsilon^2$ on an appropriate domain for appropriate constants $\mathcal{C}$.

(4) For the results of this paper the normalized field strength $\mathcal{E}$ cannot be too big (or ε won't be small) and it cannot be too small or another distinguished case will come into play. Of course for a seeded FEL, $\mathcal{E}$ will be set by the seeding field. In Appendix H we present two very crude bounds that have some relevance to the beginning stages of a High Gain FEL. Here we simply note that for $\mathcal{E} = 1000$, ε is approximately 0.001.

In an early approach to this problem we built a normal form analysis assuming $\mathcal{E}$ small, so that the radiation field was a small perturbation of the undulator motion. We thus considered $\mathcal{E}$ as a small parameter in addition to $1/\gamma_c$. This led to another distinguished case, which also had a resonant structure but with a different pendulum type behavior. Later we realized that $\mathcal{E}$ is not necessarily small for cases of interest and we were led to the current case of (1.5).

(5) As will become clear in §3 the normal form for (2.62) is $\theta' = \varepsilon 2\chi$. The normal form of (2.63) depends on h. In the monochromatic case $h(\theta - Q(\zeta)) = \cos(\nu[\theta - Q(\zeta)])$ and the nonresonant, resonant and near-to-resonant structure will appear as ν varies. In particular the primary resonance will appear at $\nu = 1$. However it is curious that if

$$h(\alpha) = \int_{-\infty}^{\infty} \tilde{h}(\xi) \exp(-i\xi\alpha) d\xi \ , \tag{2.78}$$

with $\tilde{h}(\xi)$ smooth and localized near $\xi = \pm 1$ the resonance effect is washed out in first-order averaging. We will explore this briefly in §5. We are studying the consequence of this in the collective case.

3 Special Planar Undulator Model and averaging theorems

We have the planar undulator in a standard form for the MoA in (2.62),(2.63) where the $O(\varepsilon^2)$ terms can be determined from (2.72),(2.73). We now specialize to a monochromatic radiation traveling wave, write the system in Fourier form, discuss resonance as a normal form phenomenon, develop the NR and NtoR normal forms and state two theorems giving precise bounds on the normal form approximations. Thus from now on the radiation field in (2.13) is monochromatic, i.e., h, H have the form (2.15) with $\nu \geq 1/2$.

3.1 The basic ODE's for the monochromatic radiation field

In this section we introduce the notation which will allow us to state and prove our three propositions and two theorems. With (2.15),(2.70), (2.74) we show the dependencies of P_x and P_z on $(\theta, \chi, \zeta, \varepsilon, \nu)$ by the replacement

$$P_x = \Pi_x \ , \qquad P_z = \Pi_z \ , \tag{3.1}$$

where

$$\Pi_x(\theta,\zeta,\varepsilon,\nu) := \cos\zeta + \Delta P_{x0} + \frac{\varepsilon^2\bar{q}}{2\nu}[\sin(\nu[\theta - Q(\zeta)]) - \sin(\nu\theta_0)] \,, \tag{3.2}$$

$$\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu) := \sqrt{1 - \frac{\varepsilon^2}{\mathcal{E}}(1+\varepsilon\chi)^{-2}(1+K^2\Pi_x^2(\theta,\zeta,\varepsilon,\nu)} \,. \tag{3.3}$$

Note that, by (2.77),(3.1),

$$\chi > -\frac{1}{\varepsilon} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2\Pi_x^2(\theta,\zeta,\varepsilon,\nu)} \,. \tag{3.4}$$

From now on, we restrict ε to a finite interval $(0,\varepsilon_0]$. We are of course interested in ε small, i.e., $0<\varepsilon\ll 1$, and so, without loss of generality, we take

$$0 < \varepsilon \le \varepsilon_0 \,, \quad 0 < \varepsilon_0 \le 1 \,. \tag{3.5}$$

Using (3.4),(3.5) we define the open set $\mathcal{D}(\varepsilon,\nu)$, for $0<\varepsilon\le\varepsilon_0, \nu\ge 1/2$, by

$$\mathcal{D}(\varepsilon,\nu) := \{(\theta,\chi,\zeta)\in\mathbb{R}^3 : \chi > -\frac{1}{\varepsilon} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2\Pi_x^2(\theta,\zeta,\varepsilon,\nu)}\} \,, \tag{3.6}$$

which is our maximal domain in extended phase space. Accordingly we define the domain of Π_x to be $\{(\theta,\zeta,\varepsilon,\nu)\in\mathbb{R}^4 : 0<\varepsilon\le\varepsilon_0, \nu\ge 1/2\}$ and the domain of Π_z to be $\{(\theta,\chi,\zeta,\varepsilon,\nu)\in(\mathcal{D}(\varepsilon,\nu)\times\mathbb{R}^2) : 0<\varepsilon\le\varepsilon_0, , \nu\ge 1/2\}$. It is easy to check that on the domain of Π_z the argument of the square root in (3.3) is positive and, for $(\theta,\chi,\zeta)\in\mathcal{D}(\varepsilon,\nu)$, we have (2.76) and

$$0 < \Pi_z(\theta,\chi,\zeta,\varepsilon,\nu) < 1 \,. \tag{3.7}$$

Moreover with (2.15) the ODE's (2.72),(2.73) become

$$\theta' = \frac{2\mathcal{E}}{\varepsilon^2\bar{q}}(1 - \frac{1}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)}) + \frac{q(\zeta)}{\bar{q}} \,, \tag{3.8}$$

$$\chi' = -\varepsilon K^2 \frac{1}{1+\varepsilon\chi}\frac{\Pi_x(\theta,\zeta,\varepsilon,\nu)}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)}\cos(\nu[\theta - Q(\zeta)]) \,, \tag{3.9}$$

where q and Q are defined in (2.47),(2.53). Of course the initial conditions are $\theta(0,\varepsilon) = \theta_0, \chi(0,\varepsilon) = \chi_0$.

As suggested by (2.62), (2.63) we now write (3.8),(3.9) as

$$\theta' = \varepsilon f_1(\chi,\zeta) + \varepsilon^2 g_1(\theta,\chi,\zeta;\varepsilon,\nu) \,, \tag{3.10}$$

$$\chi' = \varepsilon f_2(\theta,\zeta;\nu) + \varepsilon^2 g_2(\theta,\chi,\zeta;\varepsilon,\nu) \,, \tag{3.11}$$

where f_1, f_2 are given by

$$f_1(\chi,\zeta) := \frac{2q(\zeta)\chi}{\bar{q}} \,, \tag{3.12}$$

$$f_2(\theta,\zeta;\nu) := -K^2(\cos\zeta + \Delta P_{x0})\cos(\nu[\theta - Q(\zeta)]) \,, \tag{3.13}$$

so that g_1, g_2 are given by

$$\varepsilon^2 g_1(\theta,\chi,\zeta;\varepsilon,\nu) := \frac{2\mathcal{E}}{\varepsilon^2\bar{q}}(1 - \frac{1}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)}) + \frac{q(\zeta)}{\bar{q}}(1 - 2\varepsilon\chi) \,, \tag{3.14}$$

$$\varepsilon^2 g_2(\theta,\chi,\zeta;\varepsilon,\nu) := \varepsilon K^2\cos(\nu[\theta-Q(\zeta)])[\cos\zeta+\Delta P_{x0} \\ -\frac{1}{1+\varepsilon\chi}\frac{\Pi_x(\theta,\zeta,\varepsilon,\nu)}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)}]\ . \tag{3.15}$$

The ODE's (3.8),(3.9) and their equivalent form, (3.10),(3.11), will be the subject of Theorem 1, i.e., the averaging theorem for the NR case (see also Definition 1 in §3.2). They will also be the basis for the NtoR case.

We need an appropriate domain for the vector field in (3.10),(3.11) when it comes to averaging theorems. There are two types of singularities in (3.10),(3.11). The first involves the ε dependence of g_1, g_2 as $\varepsilon\to 0+$. On the surface it appears that the first term on the rhs of (3.14) is $O(1/\varepsilon^2)$, however it is $O(1)$. In fact, when combined with the second term the rhs is $O(\varepsilon^2)$ so that g_1 is $O(1)$. Similarly, g_2 appears to be $O(1/\varepsilon)$, however again there is a cancellation so that $g_2 = O(1)$. This should not come as a surprise since the construction of the distinguished case (see the remarks before (2.59)) Proposition 1 makes this precise by finding the limits of g_1, g_2 as $\varepsilon\to 0+$. Thus the $\varepsilon=0$ singularity is removable. There are also singularities for $\Pi_z=0, \varepsilon\chi=-1$ which are not removable. This is reflected in the fact that even though f_1, f_2 are nice, g_1, g_2 have these singularities. However these singularities are excluded from our maximal domain $\mathcal{D}(\varepsilon,\nu)$ (see (2.76),(3.7)) and so the vector field in (3.10),(3.11) is of class C^∞ on $\mathcal{D}(\varepsilon,\nu)$ for $0<\varepsilon\le\varepsilon_0\le 1, \nu\ge 1/2$. Nevertheless since $\mathcal{D}(\varepsilon,\nu)$ is dependent on ε it is inconvenient to use it in an averaging theorem. Thus we now restrict $\mathcal{D}(\varepsilon,\nu)$ to an ε-independent domain $W(\varepsilon_0)\times\mathbb{R}$.

To motivate W we note that, by (3.2) and since $\nu\ge 1/2$,

$$|\Pi_x(\theta,\zeta,\varepsilon,\nu)| \le \Pi_{x,ub}(\varepsilon)\ , \tag{3.16}$$

where

$$\Pi_{x,ub}(\varepsilon) := 1+|\Delta P_{x0}|+2\varepsilon^2\bar{q}\ . \tag{3.17}$$

Clearly, by (3.16),(3.17),

$$-\frac{1}{\varepsilon}+\frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2\Pi_x^2(\theta,\zeta,\varepsilon,\nu)} \le -\frac{1}{\varepsilon}+\frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2\Pi_{x,ub}^2(\varepsilon)} \\ \le -\frac{1}{\varepsilon_0}+\frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2\Pi_{x,ub}^2(\varepsilon_0)}\ , \tag{3.18}$$

whence, by (3.6), we can "shrink" the maximal domain $\mathcal{D}(\varepsilon,\nu)$ to the ε-independent domain $W(\varepsilon_0)\times\mathbb{R}$ where

$$W(\varepsilon) := \mathbb{R}\times(\chi_{lb}(\varepsilon),\infty)\ , \tag{3.19}$$

with

$$\chi_{lb}(\varepsilon) := -\frac{1}{\varepsilon}+\frac{1}{\sqrt{\mathcal{E}}}\sqrt{1+K^2\Pi_{x,ub}^2(\varepsilon)}\ . \tag{3.20}$$

3.2 Resonant, nonresonant, Δ-nonresonant, near-to-resonant

Now that the structure of the g_i have been characterized at the level needed for the averaging theorems, we discuss the structure of the f_i defined in (3.12),(3.13). Clearly f_1 is 2π periodic in ζ. We write, by (2.53),(3.13),

$$f_2(\theta,\zeta;\nu) = -K^2(\cos\zeta+\Delta P_{x0})\cos\Big(\nu\theta-\nu\zeta-\nu\Upsilon_0\sin\zeta-\nu\Upsilon_1\sin 2\zeta\Big)$$

$$=: \check{f}_2(\theta, \zeta, \nu\zeta; \nu) \ , \tag{3.21}$$

where $\check{f}_2(\theta, \zeta_1, \zeta_2; \nu) := -K^2(\cos\zeta_1 + \Delta P_{x0})$
$\times \cos\Big(\nu\theta - \zeta_2 - \nu\Upsilon_0 \sin\zeta_1 - \nu\Upsilon_1 \sin 2\zeta_1\Big)$. Since $\check{f}_2(\theta, \zeta_1, \zeta_2; \nu)$ is of class C^∞ in (ζ_1, ζ_2) and 2π-periodic in ζ_1 and ζ_2 we conclude from (3.21) that f_2 is a quasiperiodic function of ζ with two base frequencies 1 and ν (for the definition of quasiperiodic functions, see, e.g., [9]). To make the resonant structure explicit we write f_2 as

$$f_2(\theta, \zeta; \nu) = -\frac{K^2}{2} \exp(i\nu(\theta - \zeta)) jj(\zeta; \nu, \Delta P_{x0}) + cc \ , \tag{3.22}$$

where

$$jj(\zeta; \nu, \Delta P_{x0}) := (\cos\zeta + \Delta P_{x0}) \exp(-i\nu[\Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta]) \ , \tag{3.23}$$

is 2π-periodic in ζ. The Fourier series of jj is

$$jj(\zeta; \nu, \Delta P_{x0}) \sim \sum_{n\in\mathbb{Z}} \widehat{jj}(n; \nu, \Delta P_{x0}) e^{in\zeta} \ , \tag{3.24}$$

with

$$\widehat{jj}(n; \nu, \Delta P_{x0}) := \frac{1}{2\pi} \int_{[0,2\pi]} d\zeta jj(\zeta; \nu, \Delta P_{x0}) \, \mathrm{e}^{-in\zeta} \ , \tag{3.25}$$

and $\mathbb{Z}$ being the set of integers. Since $jj(\cdot; \nu, \Delta P_{x0})$ is a 2π-periodic C^∞ function its Fourier series (3.24) is absolutely convergent, i.e.,
$\sum_{n\in\mathbb{Z}} |\widehat{jj}(n; \nu, \Delta P_{x0})| < \infty$ whence $\sim$ in (3.24) can replaced by $=$. The f_2 in Eq. (3.11) can now be written

$$f_2(\theta, \zeta; \nu) = -\frac{K^2}{2} e^{i\nu\theta} \sum_{n\in\mathbb{Z}} \widehat{jj}(n; \nu, \Delta P_{x0}) e^{i(n-\nu)\zeta} + cc \ , \tag{3.26}$$

which clearly shows the resonant structure in that the ζ average of f_2 is zero for $\nu \neq$ integer. In Appendix A we find

$$\begin{aligned} \widehat{jj}(n; \nu, \Delta P_{x0}) = &\frac{1}{2}\mathcal{J}(n, 1, \nu, \Upsilon_0, \Upsilon_1) + \frac{1}{2}\mathcal{J}(n, -1, \nu, \Upsilon_0, \Upsilon_1) \\ &+ \Delta P_{x0} \mathcal{J}(n, 0, \nu, \Upsilon_0, \Upsilon_1) \ , \end{aligned} \tag{3.27}$$

where

$$\mathcal{J}(n, m, \nu, \Upsilon_0, \Upsilon_1) := \sum_{l\in\mathbb{Z}} J_{m-n-2l}(\nu\Upsilon_0) J_l(\nu\Upsilon_1) \ , \tag{3.28}$$

and J_k is the k-th-order Bessel function of the first kind. Note that
$jj(-\zeta; \nu, \Delta P_{x0}) = jj(\zeta; \nu, \Delta P_{x0})^*$ which implies $\widehat{jj}(n; \nu, \Delta P_{x0})$ is real. This is confirmed in the explicit form of (3.27),(3.28) since the J_k are real valued.

The time average of f_1 in (3.12) is clearly

$$\bar{f}_1(\chi) := \lim_{T\to\infty} [\frac{1}{T} \int_0^T f_1(\chi, \zeta) d\zeta] = 2\chi \ . \tag{3.29}$$

Since the series in (3.26) converges uniformly in ζ and since $\overline{\exp(i(n-\nu)\zeta)} = \delta_{n,\nu}$, the time average of the quasiperiodic f_2 is

$$\bar{f}_2(\theta;\nu) := \lim_{T\to\infty}[\frac{1}{T}\int_0^T f_2(\theta,\zeta;\nu)d\zeta] \\ = \begin{cases} 0 & \text{if } \nu \notin \mathbb{N} \\ -K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(k\theta) & \text{if } \nu = k \in \mathbb{N}\ , \end{cases} \tag{3.30}$$

where $\mathbb{N}$ denotes the set of positive integers and where we have used the fact that $\widehat{jj}$ is real. This forms the basis of our definitions of resonant, nonresonant and near-to-resonant frequencies ν.

Definition 1. *(Resonant, nonresonant, Δ-nonresonant, near-to-resonant)*
Let $\nu \geq 1/2$. We say ν is nonresonant (NR) if $\nu \notin \mathbb{N}$ and resonant otherwise. We also say that ν is Δ-nonresonant (Δ-NR) when $\nu \in [k+\Delta, k+1-\Delta]$ with $\Delta \in (0, 0.5)$ and $k \in \mathbb{N}$. Note that ν is NR if it is Δ-NR. We say that ν is near-to-resonant (NtoR) if $\nu = k + \varepsilon a$ where $k \in \mathbb{N}, a \in [-1/2, 1/2]$. Recall $0 < \varepsilon \leq \varepsilon_0 \leq 1$ and that we take $\mathbb{N}$ to denote the set of positive integers. □

Remark:
In our various estimates we need to keep ν away from zero but want to include $\nu = 1$ since it is the primary resonance. Thus we require $\nu \geq 1/2$ and since $\varepsilon \leq 1$ we require $|a| \leq 1/2$.

It follows from the Fourier form of (3.26) that it is only possible to have a nontrivial normal form, i.e., $\bar{f}_2 \neq 0$, if ν is an integer. Thus $\nu = 1$ is the primary resonance as discussed in the Introduction, justifying the choice of K_r in (1.3) and (2.61). The resonant normal form at $\nu = k$ is of the pendulum form with

$$\theta' = \varepsilon 2\chi\ , \quad \chi' = -\varepsilon K^2\widehat{jj}(k;k,\Delta P_{x0})\cos(k\theta)\ . \tag{3.31}$$

From Appendix A we have, for $\Delta P_{x0} = 0$,

$$\widehat{jj}(k;k,0) = \begin{cases} \frac{1}{2}(-1)^n[J_n(x_n) - J_{n+1}(x_n)] & \text{if } k = 2n+1 \\ 0 & \text{if } k \text{ even}\ , \end{cases} \tag{3.32}$$

where $x_n := (2n+1)\Upsilon_1$ and $n = 0, 1, ...$ with Υ_1 defined in (2.54). Thus, for $\Delta P_{x0} = 0$, (3.31) gives the standard FEL pendulum system (see also [2],[4],[5],[17]):

$$\theta' = \varepsilon 2\chi\ , \quad \chi' = -\varepsilon K^2\widehat{jj}(k;k,0)\cos(k\theta)\ . \tag{3.33}$$

For a general quasiperiodic function with base frequencies 1 and ν it is possible to have a nontrivial normal form for every rational ν and thus ν would be defined to be resonant if it were rational.

Since $\bar{f}_1(\chi)$ is independent of ν it plays no role in Definition 1. Clearly $\bar{f}_2(\theta;\nu) = 0$ if ν is NR. We state our NR theorem in Theorem 1 for the Δ-NR case. In fact because of a small divisor problem the theorem will require ν to stay away from neighborhoods of resonances in order to get an $o(1)$ error bound as $\varepsilon \to 0+$. We will obtain an $O(\varepsilon^{1-\beta})$ bound for $\beta \in (0, 1]$ depending on the distance from the resonance by letting $\Delta = O(\varepsilon^\beta)$. In the resonant case we will explore an $O(\varepsilon)$ neighborhood of the resonance. This will allow us to at least partially fill the gap between the Δ-NR νs in the NR theorem and the νs in the NtoR theorem. The way this occurs will be seen in the error analysis in the proofs of Theorems 1 and 2.

3.3 The nonresonant case and its normal form

The exact ODE's in the NR case are (3.10),(3.11). Clearly they are the same in the Δ-NR subcase. By definition, the NR normal form, i.e., the normal form with ν NR, is obtained from (3.10),(3.11) by dropping the $O(\varepsilon^2)$ terms and averaging the rhs over ζ holding θ, χ fixed whence, by (3.29),(3.30),

$$v_1' = \varepsilon \bar{f}_1(v_2) = \varepsilon 2 v_2 \ , \tag{3.34}$$
$$v_2' = \varepsilon \bar{f}_2(v_1; \nu) = 0 \ , \tag{3.35}$$

with the same initial conditions as in the exact ODE's, i.e., $v_1(0,\varepsilon) = \theta_0, v_2(0,\varepsilon) = \chi_0$ and solution

$$v_1(\zeta,\varepsilon) = 2\chi_0\varepsilon\zeta + \theta_0 \ , \quad v_2(\zeta,\varepsilon) = \chi_0 \ . \tag{3.36}$$

The solutions of (3.34),(3.35) with $\varepsilon = 1$ play an important role in the statement and proof of Theorem 1 and we refer to

$$\mathbf{v}(\cdot, 1) = (v_1(\cdot, 1), v_2(\cdot, 1)) \ , \tag{3.37}$$

as the guiding solution at (θ_0, χ_0). Note that the $\mathbf{v}$ in (3.37) should not be confused with the velocity vector $\mathbf{v}$ in (2.3).

Our basic result in the NR case will be that $|\theta(\zeta) - v_1(\zeta,\varepsilon)|$ and $|\chi(\zeta) - v_2(\zeta,\varepsilon)|$ are $O(\varepsilon/\Delta)$ in the Δ-NR subcase. If $\Delta = O(1)$ then the error is $O(\varepsilon)$. Putting Δ into the order symbol allows one to discuss Δ small, e.g., as a function of ε. The precise statement is given in §3.5.1 and its proof is given in §4.1.

Proposition 1. *Let $0 < \varepsilon \le \varepsilon_0 \le 1$ and let $\nu \ge 1/2$. Then*

$$W(\varepsilon_0) \times \mathbb{R} \subset W(\varepsilon) \times \mathbb{R} \subset \mathcal{D}(\varepsilon, \nu) \ . \tag{3.38}$$

Moreover $g_1(\cdot; \varepsilon, \nu), g_2(\cdot; \varepsilon, \nu)$ are C^∞ functions on $W(\varepsilon_0) \times \mathbb{R}$. Furthermore, for $(\theta, \chi, \zeta) \in W(\varepsilon_0) \times \mathbb{R}$,

$$\lim_{\varepsilon \to 0+} [g_1(\theta, \chi, \zeta; \varepsilon, \nu)] = -\frac{q(\zeta)}{4\bar{q}}(\frac{3q(\zeta)}{\mathcal{E}} + 12\chi^2)$$
$$-\frac{K^2}{2\nu}\Big(\sin(\nu[\theta - Q(\zeta)]) - \sin(\nu\theta_0)\Big)(\cos\zeta + \Delta P_{x0}) \ , \tag{3.39}$$
$$\lim_{\varepsilon \to 0+} [g_2(\theta, \chi, \zeta; \varepsilon, \nu)] = K^2 \chi \cos(\nu[\theta - Q(\zeta)])(\cos\zeta + \Delta P_{x0}) \ . \tag{3.40}$$

Remark:
Proposition 1 entails that the vector field on the rhs of (3.10),(3.11) is a C^∞ function on $W(\varepsilon_0) \times \mathbb{R}$ (whence the vector field on the rhs of (3.8),(3.9) is a C^∞ function on $W(\varepsilon_0) \times \mathbb{R}$, too). Proposition 1 will allow us to use, in Theorem 1, the domain $W(\varepsilon_0) \times \mathbb{R}$. Furthermore the domain is large enough to contain the χ of physical interest (see Proposition 3 in §3.5.3).

Proof of Proposition 1: Let $(\theta, \chi, \zeta) \in W(\varepsilon) \times \mathbb{R}$. Then, by (3.16),(3.19),(3.20),

$$\chi > -\frac{1}{\varepsilon} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1 + K^2 \Pi_x^2(\theta, \zeta, \varepsilon, \nu)} \ ,$$

whence, by (3.6), $(\theta, \chi, \zeta) \in \mathcal{D}(\varepsilon, \nu)$ which proves the second inclusion in (3.38). The first inclusion in (3.38) follows from (3.19) and from the fact that, by (3.20), $\chi_{lb}(\varepsilon)$ is increasing with ε. Moreover, by the remarks after (3.15), $g_1(\cdot; \varepsilon, \nu), g_2(\cdot; \varepsilon, \nu)$ are C^∞ functions on $\mathcal{D}(\varepsilon, \nu)$ whence, by (3.38), they are C^∞ functions on $W(\varepsilon_0) \times \mathbb{R}$. Finally, (3.39),(3.40) are proven in Appendix B (see (B.8),(B.13)). □

3.4 The Near-to-Resonant case and its normal form

3.4.1 The Near-to-Resonant system

According to Definition 1 we have, in the NtoR case,

$$\nu = k + \varepsilon a, \tag{3.41}$$

where $k \in \mathbb{N}$ and $a \in [-1/2, 1/2]$ is a measure of the distance of ν from k. The $O(\varepsilon)$ neighborhood of k is natural in first-order averaging. If $|\nu - k|$ is too small then the normal form will be close to the resonant normal form and if $|\nu - k|$ is too big, then ν will be in the NR regime. Eq. (3.41) clearly includes the resonant case for $a = 0$. We start from (3.10),(3.11),(3.13) use (3.41) and obtain

$$\theta' = \varepsilon f_1(\chi, \zeta) + \varepsilon^2 g_1(\theta, \chi, \zeta; \varepsilon, k + \varepsilon a) , \tag{3.42}$$
$$\chi' = \varepsilon f_2(\theta, \zeta; k + \varepsilon a) + \varepsilon^2 g_2(\theta, \chi, \zeta; \varepsilon, k + \varepsilon a) , \tag{3.43}$$

with initial conditions $\theta(0, \varepsilon) = \theta_0, \chi(0, \varepsilon) = \chi_0$.

By the remarks after (3.15), the vector field in (3.42),(3.43) is of class C^∞ on the maximal domain $\mathcal{D}(\varepsilon, k + \varepsilon a)$. Since f_1 in (3.42) is independent of ε the normal form associated with it will be the same as in the NR case. We now need to study the ε dependence of f_2 in (3.43). From (3.22),

$$\begin{aligned} f_2(\theta, \zeta; k + \varepsilon a) &= -\frac{K^2}{2} \exp(i(k + \varepsilon a)(\theta - \zeta)) jj(\zeta; k + \varepsilon a, \Delta P_{x0}) + cc \\ &= -\frac{K^2}{2} \exp(i[k\theta - \varepsilon a \zeta]) \exp(-ik\zeta) jj(\zeta; k, \Delta P_{x0}) \\ &\quad \times \exp(i\varepsilon a[\theta - \Upsilon_0 \sin\zeta - \Upsilon_1 \sin 2\zeta]) + cc , \end{aligned} \tag{3.44}$$

where we have used from (3.23) that

$$\begin{aligned} jj(\zeta; k + \varepsilon a, \Delta P_{x0}) &= (\cos\zeta + \Delta P_{x0}) \exp(-i(k + \varepsilon a)[\Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta]) \\ &= jj(\zeta; k, \Delta P_{x0}) \exp(-i\varepsilon a[\Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta]) . \end{aligned} \tag{3.45}$$

For $a = 0$ the resonant normal form of (3.30) is obtained in (3.44). For $a \neq 0$ (3.44) displays two ε dependencies. The first is the $\varepsilon a \zeta$ one which cannot be expanded since it is $O(1)$ for $\zeta = O(1/\varepsilon)$ the upper range of our averaging theorem. The second is the εa factor in the final exponential which can be expanded and makes an $O(1)$ contribution to g_2 in (3.43) for all ζ. Therefore we rewrite f_2 as

$$f_2(\theta, \zeta; k + \varepsilon a) = f_2^R(\theta, \varepsilon\zeta, \zeta; k, a) + O(\varepsilon) , \tag{3.46}$$

where

$$f_2^R(\theta, \tau, \zeta; k, a) := -\frac{K^2}{2} \exp(i[k\theta - a\tau]) \exp(-ik\zeta) jj(\zeta; k, \Delta P_{x0}) + cc$$

$$= -\frac{K^2}{2}\exp(i[k\theta - a\tau])\sum_{n\in\mathbb{Z}}\widehat{j}j(n;k,\Delta P_{x0})e^{i\zeta[n-k]} + cc\,. \tag{3.47}$$

We can now write the basic system for the MoA, in this NtoR case. From (3.42)-(3.47) we obtain

$$\theta' = \varepsilon f_1^R(\chi,\zeta) + \varepsilon^2 g_1^R(\theta,\chi,\zeta,\varepsilon,k,a)\,, \tag{3.48}$$
$$\chi' = \varepsilon f_2^R(\theta,\varepsilon\zeta,\zeta;k,a) + \varepsilon^2 g_2^R(\theta,\chi,\zeta,\varepsilon,k,a)\,, \tag{3.49}$$

where

$$f_1^R(\chi,\zeta) := f_1(\chi,\zeta) = \frac{2q(\zeta)\chi}{\bar{q}}\,, \tag{3.50}$$
$$g_1^R(\theta,\chi,\zeta,\varepsilon,k,a) := g_1(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)\,, \tag{3.51}$$
$$\begin{aligned} g_2^R(\theta,\chi,\zeta,\varepsilon,k,a) &:= g_2(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)\\ &+\frac{1}{\varepsilon}[f_2(\theta,\zeta;k+\varepsilon a) - f_2^R(\theta,\varepsilon\zeta,\zeta;k,a)]\,, \end{aligned} \tag{3.52}$$

and where g_2^R can be rewritten as follows. By (3.21) we have

$$\begin{aligned} f_2(\theta,\zeta;k+\varepsilon a) &= -K^2(\cos\zeta + \Delta P_{x0})\\ &\cos\Big((k+\varepsilon a)[\theta - \zeta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta]\Big)\,, \end{aligned} \tag{3.53}$$

and, by (3.23),(3.47),

$$\begin{aligned} f_2^R(\theta,\varepsilon\zeta,\zeta;k,a) &= -\frac{K^2}{2}\exp(i[k\theta - \varepsilon a\zeta])\exp(-ik\zeta)(\cos\zeta + \Delta P_{x0})\\ &\times\exp(-ik[\Upsilon_0\sin\zeta + \Upsilon_1\sin 2\zeta]) + cc\\ &= -K^2(\cos\zeta + \Delta P_{x0})\cos\Big(k[\theta - \zeta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta] - \varepsilon a\zeta\Big)\,. \end{aligned} \tag{3.54}$$

Using (3.53),(3.54) we can write (3.52) as

$$\begin{aligned} g_2^R(\theta,\chi,\zeta,\varepsilon,k,a) &= g_2(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)\\ &-\frac{K^2}{\varepsilon}(\cos\zeta + \Delta P_{x0})\Bigg(\cos\Big((k+\varepsilon a)[\theta - \zeta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta]\Big)\\ &-\cos\Big(k[\theta - \zeta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta] - \varepsilon a\zeta\Big)\Bigg)\,, \end{aligned} \tag{3.55}$$

which will be useful in obtaining bounds for g_2^R in Appendix E.

The following proposition is the analogue of Proposition 1 for the NtoR case.

Proposition 2. *Let $0 < \varepsilon \le \varepsilon_0 \le 1$ and let $a \in [-1/2, 1/2], k \in \mathbb{N}$. Then $g_1^R(\cdot;\varepsilon,k,a), g_2^R(\cdot;\varepsilon,k,a)$ are C^∞ functions on $W(\varepsilon_0)\times\mathbb{R}$. Furthermore for $(\theta,\chi,\zeta) \in W(\varepsilon_0)\times\mathbb{R}$*

$$\lim_{\varepsilon\to 0+}[g_1^R(\theta,\chi,\zeta,\varepsilon,k,a)] = -\frac{q(\zeta)}{4\bar{q}}(\frac{3}{\mathcal{E}}q(\zeta) + 12\chi^2)$$

$$
-\frac{K^2}{2k}\Big(\sin(k[\theta - Q(\zeta)]) - \sin(k\theta_0)\Big)(\cos\zeta + \Delta P_{x0}) \ , \tag{3.56}
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0+}\ [g_2^R(\theta,\chi,\zeta,\varepsilon,k,a)] &= \chi K^2\cos(k[\theta - Q(\zeta)])(\cos\zeta + \Delta P_{x0}) \\
&\quad + K^2 a(\theta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta) \\
&\quad \times \sin(k[\theta - \zeta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta])(\cos\zeta + \Delta P_{x0}) \ .
\end{aligned} \tag{3.57}
$$

Remark: Proposition 2 entails that the vector field on the rhs of (3.48),(3.49) is a C^∞ function on $W(\varepsilon_0)\times\mathbb{R}$. Proposition 2 will allow us to use, in Theorem 2, the domain $W(\varepsilon_0)\times\mathbb{R}$.

Proof of Proposition 2: The C^∞ property of $g_1^R(\cdot;\varepsilon,k,a), g_2^R(\cdot;\varepsilon,k,a)$ follows from Proposition 1 and (3.51),(3.52). Moreover (3.56),(3.57) are proven in Appendix D (see (D.2),(D.11)). □

3.4.2 The NtoR normal form

The NtoR normal form ODE's are obtained from (3.48),(3.49) by dropping the $O(\varepsilon^2)$ terms and averaging the rhs over ζ holding the slowly varying quantities $\theta,\chi,\varepsilon a\zeta$ fixed. We thus obtain from (3.47),(3.48),(3.49), (3.50) that

$$
v_1' = \varepsilon\bar{f}_1^R(v_2) = 2\varepsilon v_2 \ , \tag{3.58}
$$

$$
v_2' = \varepsilon\bar{f}_2^R(v_1,\varepsilon\zeta;k) = -\varepsilon K_0(k)\cos(kv_1 - \varepsilon a\zeta) \ , \tag{3.59}
$$

where

$$
K_0(k) := K^2\widehat{jj}(k;k,\Delta P_{x0}) \ , \tag{3.60}
$$

and the same initial conditions as in the exact ODE's, i.e., $v_1(0,\varepsilon)=\theta_0, v_2(0,\varepsilon)=\chi_0$. For $a=0$, eq.'s (3.58),(3.59) become the resonant normal form (3.31). For $\Delta P_{x0}=a=0$, eq.'s (3.58),(3.59) are the standard FEL pendulum equations, given by (3.32),(3.33). In the special case when $K_0(k)=0$ the ODE's (3.58),(3.59) are the same as NR equations (3.34),(3.35) and so this case needs no further comment. Note that the special case $K_0(k)=0$ occurs, e.g., when $\Delta P_{x0}=0$ and k even (see the remark after (A.11)).

The ultimate justification for the normal form (3.58),(3.59) comes from the averaging theorem itself. However, if we replace $\varepsilon\zeta$ in (3.49) by τ and add the equation $\tau'=\varepsilon$ then this, together with (3.48),(3.49), is in a standard form for "periodic averaging" (=averaging over a periodic function) and the normal form (3.58),(3.59) is obtained by averaging over ζ holding θ,χ,τ fixed. In this θ,χ,τ formulation standard periodic averaging theorems apply for the 3D system of θ,χ,τ, see, e.g., [6, 13] and Section 3.3 in [10]. We will however prove an averaging theorem directly tuned to (3.48),(3.49) both to show the reader a proof in a simple context and in the process we obtain nearly optimal error bounds which are stronger than in those standard theorems.

3.4.3 Structure of the NtoR normal form solutions

Here we write the solution of the IVP for the normal form system (3.58),(3.59) in terms of solutions of the simple pendulum system and discuss their behavior. Therefore in this Section we exclude the simple subcase where $K_0=0$. Let $\mathbf{v}=(v_1,v_2)$, then it is easy to see that

$$
\mathbf{v}(\zeta,\varepsilon) = \mathbf{v}(\varepsilon\zeta,1) \ . \tag{3.61}
$$

We first make the transformation $\mathbf{v}(\tau,1) \to \hat{\mathbf{v}}(\tau)$ via

$$\hat{\mathbf{v}}(\tau) = \begin{pmatrix} \hat{v}_1(\tau) \\ \hat{v}_2(\tau) \end{pmatrix} := \begin{pmatrix} kv_1(\tau,1) - a\tau \\ v_2(\tau,1) \end{pmatrix} , \tag{3.62}$$

which gives

$$\frac{d\hat{v}_1}{d\tau} = 2k\hat{v}_2 - a , \quad \hat{v}_1(0) = k\theta_0 , \tag{3.63}$$

$$\frac{d\hat{v}_2}{d\tau} = -K_0(k)\cos\hat{v}_1 , \quad \hat{v}_2(0) = \chi_0 . \tag{3.64}$$

Thus we have scaled away the ε and made the transformed system autonomous. Solution properties of (3.63),(3.64) are easily understood in terms of its phase plane portrait (PPP). However it is more convenient to transform it to the simple pendulum system

$$X' = Y, \quad Y' = -\sin X , \tag{3.65}$$

$$X(0;Z_0) =: X_0 , \quad Y(0;Z_0) =: Y_0 , \quad Z_0 := \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} . \tag{3.66}$$

The required transformation is

$$\hat{v}_1(\tau) = X(\Omega\tau;Z_0) - \mathrm{sgn}(K_0)\frac{\pi}{2} , \tag{3.67}$$

$$\hat{v}_2(\tau) = \frac{\Omega Y(\Omega\tau;Z_0) + a}{2k} , \tag{3.68}$$

where

$$\Omega = \Omega(k) := \sqrt{2k|K_0(k)|} . \tag{3.69}$$

From (3.61),(3.62) (3.67) and (3.68), the solutions of (3.58),(3.59) are represented by

$$v_1(\zeta,\varepsilon) = \frac{X(\Omega\varepsilon\zeta;Z_0) - \mathrm{sgn}(K_0)\frac{\pi}{2} + \varepsilon a\zeta}{k} , \tag{3.70}$$

$$v_2(\zeta,\varepsilon) = \frac{\Omega Y(\Omega\varepsilon\zeta;Z_0) + a}{2k} , \tag{3.71}$$

where

$$Z_0(\theta_0,\chi_0,k,a) = \begin{pmatrix} X_0(\theta_0,k) \\ Y_0(\chi_0,k,a) \end{pmatrix} = \begin{pmatrix} k\theta_0 + \mathrm{sgn}(K_0(k))\frac{\pi}{2} \\ (2k\chi_0 - a)/\Omega(k) \end{pmatrix} . \tag{3.72}$$

We now discuss the solution properties of (3.58),(3.59) in terms of the simple pendulum PPP, [18], for (3.65) using (3.70) and (3.71). The equilibria of (3.65) are at $(X,Y) = (\pi l, 0)$ with integer l.

The systems obtained by linearizing about these equilibria are centers for l even and saddle points for l odd. From the theory of Almost Linear Systems (see, e.g., [19]), it follows that the equilibria are centers and saddle points for the nonlinear system. A conservation law for the simple pendulum system is easily derived by first noting that the direction field is given by

$$\frac{dY}{dX} = -\frac{\sin X}{Y} . \tag{3.73}$$

This equation is separable and has solutions given implicitly by $\frac{1}{2}Y^2 + 1 - \cos X = const$. Thus

$$\mathcal{E}_{Pen}(X,Y) := \frac{1}{2}Y^2 + U(X) \ , \quad U(X) = 1 - \cos X \tag{3.74}$$

is a constant of the motion which is easily checked directly. Incidentally $\mathcal{E}_{Pen}$ is also a Hamiltonian for the ODE's (3.58),(3.59) but this plays no role here. The PPP is easily constructed from the so-called potential plane which is simply a plot of the potential $U(X)$ vs. X, see [20]. The PPP shows that the solutions of the simple pendulum system has four types of behavior, the equilibria mentioned above, libration, rotation and separatrix motion. These can be characterized in terms of $\mathcal{E}_{Pen}$. Clearly, $\mathcal{E}_{Pen}$ is nonnegative, the centers correspond to $\mathcal{E}_{Pen}(X,Y) = 0$ and the saddle points and separatrices to $\mathcal{E}_{Pen}(X,Y) = 2$. The motion is libration for $0 < \mathcal{E}_{Pen}(X,Y) < 2$, rotation for $\mathcal{E}_{Pen}(X,Y) > 2$ and separatrix motion for $\mathcal{E}_{Pen}(X,Y) = 2$ with $Y \neq 0$. In the libration case the solutions are periodic, which is easy to show, and the period as a function of amplitude, [21], is given by

$$T(A) = 2\sqrt{2}\int_0^A \frac{dt}{[\cos t - \cos A]^{1/2}} \ , \quad (0 < A < \pi) \tag{3.75}$$

where $T(A)$ is the period associated with the initial conditions $X_0 = A, Y_0 = 0$. It is easy to show that $\lim_{A\to 0} T(A) = 2\pi$.

We denote by $\mathcal{B}_n$ the n-th pendulum bucket which is defined by

$$\mathcal{B}_n := \{(X,Y) \in \mathbb{R}^2 : \mathcal{E}_{Pen}(X,Y) < 2, |X - 2\pi n| < \pi\} \ , \tag{3.76}$$

with $n \in \mathbb{Z}$. Note that, by (3.72),(3.74),

$$\begin{aligned}\mathcal{E}_{Pen}(Z_0(\theta_0,\chi_0,k,a)) = \mathcal{E}_R(\theta_0,\chi_0,k,a) &:= \frac{1}{2}[\frac{2k\chi_0 - a}{\Omega(k)}]^2 \\ &+1 + \mathrm{sgn}(K_0)\sin(k\theta_0) \ . \end{aligned}\tag{3.77}$$

Note also that, by (3.70),(3.71),(3.72),

$$|v_1(\zeta,\varepsilon) - \theta_0| = \Big|\frac{X(\Omega\varepsilon\zeta;Z_0) - X_0 + \varepsilon a\zeta}{k}\Big| \le \frac{|X(\Omega\varepsilon\zeta;Z_0) - X_0| + \varepsilon|a|\zeta}{k} \ , \tag{3.78}$$

$$|v_2(\zeta,\varepsilon) - \chi_0| = \frac{\Omega}{2k}|Y(\Omega\varepsilon\zeta;Z_0) - Y_0| \ , \tag{3.79}$$

$$|v_2(\zeta,\varepsilon)| \le \frac{\Omega|Y(\Omega\varepsilon\zeta;Z_0)| + |a|}{2k} \ . \tag{3.80}$$

We can now discuss the four cases of equilibria, libration, rotation and separatrix motion. In each case, using (3.78),(3.79), (3.80), we will find $d_1^{min}, d_2^{min}, \chi_\infty \ge 0$ such that, for all $\zeta \ge 0$,

$$|v_1(\zeta,\varepsilon) - \theta_0| \le d_1^{min}(\theta_0,\chi_0,\varepsilon\zeta,k,a) \ , \quad |v_2(\zeta,\varepsilon) - \chi_0| \le d_2^{min}(\theta_0,\chi_0,k,a) \ , \tag{3.81}$$

$$|v_2(\zeta,\varepsilon)| \le \chi_\infty(\theta_0,\chi_0,k,a) \ , \tag{3.82}$$

and we will at the same time observe that $d_1^{min}(\theta_0,\chi_0,\tau,k,a)$ is increasing w.r.t. τ.

(I) Equilibria regime: $Y_0 = 0$ and either $\mathcal{E}_{Pen}(X_0, Y_0) = 0$ or 2.

Clearly $X_0 = \pi l$ where $l \in \mathbb{Z}$ and, by (3.72),

$$\begin{pmatrix} k\theta_0 + \operatorname{sgn}(K_0(k))\frac{\pi}{2} \\ (2k\chi_0 - a)/\Omega(k) \end{pmatrix} = Z_0(\theta_0, \chi_0, k, a) = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \pi l \\ 0 \end{pmatrix} , \tag{3.83}$$

so that $\theta_0 = (\pi l - \operatorname{sgn}(K_0(k))\frac{\pi}{2})/k$ and $\chi_0 = a/2k$. Thus, by (3.70),(3.71),

$$v_1(\zeta, \varepsilon) = \theta_0 + \frac{\varepsilon a \zeta}{k} , \tag{3.84}$$

$$v_2(\zeta, \varepsilon) = \chi_0 . \tag{3.85}$$

Clearly, by direct substitution, these are solutions of (3.58),(3.59). Incidentally these solutions are stable for l even and unstable for l odd.

Clearly, due to (3.81),(3.82),(3.84), (3.85), we can choose

$$d_1^{min}(\theta_0, \chi_0, \varepsilon\zeta, k, a) := \frac{\varepsilon|a|\zeta}{k} , \quad d_2^{min}(\theta_0, \chi_0, k, a) := 0 , \tag{3.86}$$

$$\chi_\infty(\theta_0, \chi_0, k, a) := |\chi_0| . \tag{3.87}$$

(II) Libration regime: $0 < \mathcal{E}_{Pen}(X_0, Y_0) < 2$.

In this case $Z_0(\theta_0, \chi_0, k, a) \in \mathcal{B}_{n(\theta_0,k)}$ where the integer $n = n(\theta_0, k)$ is determined by the condition $|X_0(\theta_0, k) - 2\pi n(\theta_0, k)| < \pi$. From (3.70),(3.71) we see that

$$\mathbf{v}(\zeta, \varepsilon) = \mathbf{v}_{per}(\zeta, \varepsilon) + \mathbf{v}_{lin}(\varepsilon\zeta) , \tag{3.88}$$

and it is easy to show that the periodic part has amplitude determined by the max and min values of X and Y and the linear growth term is

$$\mathbf{v}_{lin}(\varepsilon\zeta) = \begin{pmatrix} \varepsilon a \zeta/k \\ 0 \end{pmatrix} . \tag{3.89}$$

The maximum values X_{max} and Y_{max} of X and Y satisfy, by (3.74),

$$\mathcal{E}_{Pen}(Z_0) = \frac{1}{2}Y_0^2 + 1 - \cos X_0 = \frac{1}{2}Y_{max}^2 = 1 - \cos X_{max} , \tag{3.90}$$

whence

$$\begin{aligned} X_{max}(\theta_0, \chi_0, k, a) &= 2\pi n(\theta_0, k) + \arccos(\cos X_0 - \frac{1}{2}Y_0^2) , \\ &= 2\pi n(\theta_0, k) + \arccos\Big(1 - \mathcal{E}_R(\theta_0, \chi_0, k, a)\Big) , \\ Y_{max}(\theta_0, \chi_0, k, a) &:= \sqrt{2\mathcal{E}_{Pen}(Z_0(\theta_0, \chi_0, k, a))} \\ &= \sqrt{2\mathcal{E}_R(\theta_0, \chi_0, k, a)} , \end{aligned} \tag{3.91}$$

and the minimum values X_{min} and Y_{min} of X and Y are given by

$$X_{min} := 4\pi n - X_{max} , \quad Y_{min} := -Y_{max} . \tag{3.92}$$

Here arccos is the principle branch of the inverse cos mapping $[-1,1] \to [0,\pi]$.

We now determine d_1^{min}, d_2^{min} and χ_∞. It follows from (3.78),(3.79),(3.80), (3.81),(3.82),(3.91),(3.92) that

$$\begin{aligned}
|v_1(\zeta,\varepsilon) - \theta_0| &\le \frac{|X(\Omega\varepsilon\zeta; Z_0) - X_0| + \varepsilon|a|\zeta}{k} \\
&\le \frac{2X_{max}(\theta_0,\chi_0,k,a) - 4\pi n(\theta_0,k) + \varepsilon|a|\zeta}{k} \\
&= \frac{2\arccos\Big(1 - \mathcal{E}_R(\theta_0,\chi_0,k,a)\Big) + \varepsilon|a|\zeta}{k} =: d_1^{min}(\theta_0,\chi_0,\varepsilon\zeta,k,a)\ ,
\end{aligned} \tag{3.93}$$

$$\begin{aligned}
|v_2(\zeta,\varepsilon) - \chi_0| &= \frac{\Omega}{2k}|Y(\Omega\varepsilon\zeta; Z_0) - Y_0| \le \frac{\Omega}{k}Y_{max}(\theta_0,\chi_0,k,a) \\
&= \frac{\Omega(k)}{k}\sqrt{2\mathcal{E}_R(\theta_0,\chi_0,k,a)} =: d_2^{min}(\theta_0,\chi_0,k,a)\ ,
\end{aligned} \tag{3.94}$$

$$\begin{aligned}
|v_2(\zeta,\varepsilon)| &\le \frac{\Omega|Y(\Omega\varepsilon\zeta; Z_0)| + |a|}{2k} \le \frac{\Omega Y_{max}(\theta_0,\chi_0,k,a) + |a|}{2k} \\
&= \frac{\Omega(k)\sqrt{2\mathcal{E}_R(\theta_0,\chi_0,k,a)} + |a|}{2k} =: \chi_\infty(\theta_0,\chi_0,k,a)\ .
\end{aligned} \tag{3.95}$$

(III) Separatrix regime: $Y_0 \neq 0$ and $\mathcal{E}_{Pen}(X_0,Y_0) = 2$.

In this case $(X,Y) \in \overline{\mathcal{B}_{n(\theta_0,k)}}$ where the integer $n = n(\theta_0,k)$ is determined such that $|X_0(\theta_0,k) - 2\pi n(\theta_0,k)| < \pi$. Clearly

$$|X - X_0| \le 2\pi\ , \quad |Y - Y_0| \le \sqrt{2\mathcal{E}_{Pen}(X_0,Y_0)} = 2\ , \quad |Y| \le 2\ . \tag{3.96}$$

For $Y_0 > 0$, $(X(t),Y(t)) \to ((2n+1)\pi, 0)$ as $t \to \infty$ and, for $Y_0 < 0$, $(X(t),Y(t)) \to ((2n-1)\pi, 0)$ as $t \to \infty$. Thus for large ζ

$$v(\varepsilon\zeta) \approx \frac{1}{k}\begin{pmatrix} (2n\pm 1)\pi - \mathrm{sgn}(K_0(k))\frac{\pi}{2} + \varepsilon a\zeta \\ a/2 \end{pmatrix}\ , \tag{3.97}$$

which is the odd l solution in case I.

We now determine d_1^{min}, d_2^{min} and χ_∞. By (3.78),(3.79),(3.80), (3.81), (3.82),(3.96)

$$\begin{aligned}
|v_1(\zeta,\varepsilon) - \theta_0| &\le \frac{|X(\Omega\varepsilon\zeta; Z_0) - X_0| + \varepsilon|a|\zeta}{k} \\
&\le \frac{2\pi + \varepsilon|a|\zeta}{k} =: d_1^{min}(\theta_0,\chi_0,\varepsilon\zeta,k,a)\ ,
\end{aligned} \tag{3.98}$$

$$\begin{aligned}
|v_2(\zeta,\varepsilon) - \chi_0| &= \frac{\Omega}{2k}|Y(\Omega\varepsilon\zeta; Z_0) - Y_0| \le \frac{\Omega(k)}{k} \\
&=: d_2^{min}(\theta_0,\chi_0,k,a)\ ,
\end{aligned} \tag{3.99}$$

$$\begin{aligned}
|v_2(\zeta,\varepsilon)| &\le \frac{\Omega|Y(\Omega\varepsilon\zeta; Z_0)| + |a|}{2k} \\
&\le \frac{2\Omega(k) + |a|}{2k} =: \chi_\infty(\theta_0,\chi_0,k,a)\ .
\end{aligned} \tag{3.100}$$

(IV) Rotation regime: $\mathcal{E}_{Pen}(X_0, Y_0) > 2$.

For $Y_0 > 0$, X is increasing and Y is periodic such that

$$\sqrt{2}\sqrt{\mathcal{E}_{Pen}(X_0, Y_0) - 2} \leq Y \leq \sqrt{2}\sqrt{\mathcal{E}_{Pen}(X_0, Y_0)}\ , \tag{3.101}$$

and for $Y_0 < 0$, X is decreasing and Y is periodic such that

$$-\sqrt{2}\sqrt{\mathcal{E}_{Pen}(X_0, Y_0)} \leq Y \leq -\sqrt{2}\sqrt{\mathcal{E}_{Pen}(X_0, Y_0) - 2}\ . \tag{3.102}$$

Clearly $v_2(\cdot, \varepsilon)$ is periodic. We now determine d_1^{min}, d_2^{min} and χ_∞. It follows from (3.101),(3.102) that for any choice of Y_0

$$\begin{aligned} |Y - Y_0| &\leq \sqrt{2}\sqrt{\mathcal{E}_R(\theta_0, \chi_0, k, a)} - \sqrt{2}\sqrt{\mathcal{E}_R(\theta_0, \chi_0, k, a) - 2}\ , && (3.103)\\ |Y| &\leq \sqrt{2\mathcal{E}_R(\theta_0, \chi_0, k, a)}\ . && (3.104) \end{aligned}$$

It follows from (3.79),(3.80),(3.103), (3.104) that

$$\begin{aligned} |v_2(\zeta, \varepsilon) - \chi_0| &= \frac{\Omega}{2k}|Y(\Omega\varepsilon\zeta; Z_0) - Y_0| \\ &\leq \frac{\Omega}{2k}\left(\sqrt{2}\sqrt{\mathcal{E}_R(\theta_0, \chi_0, k, a)} - \sqrt{2}\sqrt{\mathcal{E}_R(\theta_0, \chi_0, k, a) - 2}\right) \\ &=: d_2^{min}(\theta_0, \chi_0, k, a)\ , && (3.105)\\ |v_2(\zeta, \varepsilon)| &\leq \frac{\Omega|Y(\Omega\varepsilon\zeta; Z_0)| + |a|}{2k} \leq \frac{\Omega(k)\sqrt{2\mathcal{E}_R(\theta_0, \chi_0, k, a)} + |a|}{2k} \\ &=: \chi_\infty(\theta_0, \chi_0, k, a)\ . && (3.106) \end{aligned}$$

It follows from (3.65),(3.104) that

$$\begin{aligned} |X(\Omega\varepsilon\zeta; Z_0) - X_0| &= |\int_0^{\Omega\varepsilon\zeta} X'(s)ds| = |\int_0^{\Omega\varepsilon\zeta} Y(s)ds| \\ &\leq \int_0^{\Omega\varepsilon\zeta} |Y(s)|ds \leq \sqrt{2}\int_0^{\Omega\varepsilon\zeta} \sqrt{\mathcal{E}_{Pen}(X(s), Y(s))}ds \\ &= \sqrt{2}\Omega\varepsilon\zeta\sqrt{\mathcal{E}_{Pen}(X_0, Y_0)} = \sqrt{2}\Omega\varepsilon\zeta\sqrt{\mathcal{E}_R(\theta_0, \chi_0, k, a)}\ , && (3.107) \end{aligned}$$

whence, by (3.78),

$$\begin{aligned} |v_1(\zeta, \varepsilon) - \theta_0| &\leq \frac{|X(\Omega\varepsilon\zeta; Z_0) - X_0| + \varepsilon|a|\zeta}{k} \\ &\leq \frac{\sqrt{2}\Omega(k)\varepsilon\zeta\sqrt{\mathcal{E}_R(\theta_0, \chi_0, k, a)} + \varepsilon|a|\zeta}{k} \\ &=: d_1^{min}(\theta_0, \chi_0, \varepsilon\zeta, k, a)\ . && (3.108) \end{aligned}$$

Clearly the simple pendulum system is central to our NtoR normal form approximation. Every student who has taken a course in ODE's or Classical Mechanics has studied the pendulum equation at some level. However, not every reader of this paper may know the general settings of the equation. So, as an aside, we thought some might be interested in knowing how it fits in a broader context. First, the pendulum equation is a special case of the nonlinear oscillator $\ddot{x} + g(x) = 0$ and second, the nonlinear oscillator is an important subclass of the class of second-order autonomous systems $\dot{x} = f(x, y), \dot{y} = g(x, y)$. The nonlinear oscillator is discussed in many texts, and here we mention [19] and [22]. Its PPP is easily constructed from the potential plane as mentioned above and in [20]. After the class of linear systems, the class of second-order autonomous systems has the most well developed theory [23]. Here the qualitative behavior is completely captured in the PPP's. What's missing from a PPP is the time it takes to go from one point on an orbit to another, but this is easily determined using a good ODE solver. The limiting behavior of all solutions bounded in forward time is given by the celebrated Poincaré-Bendixson theorem and as a consequence existence of periodic solutions can be inferred and the possibility of chaotic behavior is eliminated. It also follows that a closed orbit in the phase plane corresponds to a periodic solution.

3.4.4 NR limit far away from the pendulum buckets

Even though for small ε there will be gaps in ν between the Δ-NR and NtoR cases, as we will discuss in the context of Theorems 1,2, we show here that far away from the pendulum buckets the NR normal form emerges. While not a rigorous argument since we do not quantify "large" it is a consistency check. As in Section 3.4.3 we exclude the simple subcase where $K_0 = 0$.

For Z_0 far away from the pendulum buckets in the sense that $|Y_0| = |2k\chi_0 - a|/\Omega \gg 2$, we are in the rotation regime. Letting $X(\tilde{s}) = \hat{X}(s), Y(\tilde{s}) = Y_0\hat{Y}(s), s = Y_0\tilde{s}$, (3.65),(3.66) become

$$\frac{d\hat{X}}{ds} = \hat{Y} \ , \quad \frac{d\hat{Y}}{ds} = -\epsilon \sin \hat{X} \ , \quad \hat{X}(0) = X_0 \ , \quad \hat{Y}(0) = Y_0 = 1 \ , \tag{3.109}$$

where $\epsilon = 1/Y_0^2$. A regular perturbation expansion yields $\hat{X}(s) = s + X_0 + O(\epsilon), \hat{Y}(s) = 1 + O(\epsilon)$ as we show in Appendix F therefore $X(\tilde{s}) = Y_0\tilde{s} + X_0 + O(1/Y_0^2), Y(\tilde{s}) = Y_0 + O(1/Y_0)$ and thus from (3.70),(3.71),(3.72)

$$\begin{aligned} v_1(\zeta, \varepsilon) &= \frac{Y_0\Omega\varepsilon\zeta + X_0 + O(1/Y_0^2) - \mathrm{sgn}(K_0(k))\pi/2 + \varepsilon a\zeta}{k} \\ &= \theta_0 + \frac{Y_0\Omega + a}{k}\varepsilon\zeta + O(1/Y_0^2) = 2\chi_0\varepsilon\zeta + \theta_0 + O(1/Y_0^2) \ , \end{aligned} \tag{3.110}$$

$$v_2(\zeta, \varepsilon) = \frac{\Omega Y_0 + a}{2k} + O(1/Y_0) = \chi_0 + O(1/Y_0) \ , \tag{3.111}$$

consistent with (3.36).

3.5 Averaging theorems

Recall that we have gone from our basic Lorentz system, (2.22)-(2.25), to (3.10),(3.11) with no approximations. We have also derived two related normal forms for $\nu \geq 1/2$ in the NR (§3.3) and NtoR (§3.4) cases. Here we state theorems which conclude that the solutions of these normal form systems yield good approximations to the solutions of (2.22)-(2.25) in the appropriate ν domains.

Our NR theorem in §3.5.1 will cover the Δ-NR case, i.e., closed subintervals $[k+\Delta, k+1-\Delta]$ of $(k, k+1)$, where $k = 0, 1, ..., 0 < \Delta < 0.5$, and we will obtain error bounds of $O(\varepsilon/\Delta)$ (Here Δ can be small as mentioned in §3.2 and §3.3). Our NtoR theorem in §3.5.2 will cover the case where $\nu = k + \varepsilon a$ which includes the resonant $\nu = k$ case and we will obtain error bounds of $O(\varepsilon)$.

3.5.1 Δ-nonresonant case: $\nu \in [k+\Delta, k+1-\Delta]$ (Quasiperiodic Averaging)

The exact ODE's to be analyzed are (3.10),(3.11) with the initial conditions $\theta(0,\varepsilon) = \theta_0, \chi(0,\varepsilon) = \chi_0$ and where f_1, f_2 are defined by (3.12),(3.13) and where $\widehat{jj}(n;\nu,\Delta P_{x0})$ is defined by (3.25) and g_1, g_2 by (3.14),(3.15). The normal form ODE's are (3.34),(3.35) with initial conditions $v_1(0,\varepsilon) = \theta_0, v_2(0,\varepsilon) = \chi_0$ and solution (3.36). Note that $v_i(\zeta,\varepsilon) = v_i(\varepsilon\zeta,1)$.

We are now ready to state the NR theorem which roughly concludes that $|\theta(\zeta,\varepsilon) - 2\chi_0\varepsilon\zeta - \theta_0| = O(\varepsilon/\Delta)$ and $|\chi(\zeta,\varepsilon) - \chi_0| = O(\varepsilon/\Delta)$ for $0 \leq \zeta \leq O(1/\varepsilon)$ with ε sufficiently small. To make the statement of the theorem concise, we now set up the theorem in nine steps.

(1) (Basic parameters)
Let $0 < \varepsilon \leq \varepsilon_0 \leq 1$, fix $0 < \Delta < 0.5$ and let $\nu \in [k+\Delta, k+1-\Delta]$ where k is a nonnegative integer.

(2) (Initial data)
Choose θ_0, χ_0 such that $(\theta_0, \chi_0) \in (\mathbb{R} \times [-\chi_M, \chi_M])$ where $\chi_M > 0$ is chosen such that $-\chi_M > \chi_{lb}(\varepsilon_0)$ where χ_{lb} is defined by (3.20). Clearly $(\mathbb{R} \times [-\chi_M, \chi_M]) \subset W(\varepsilon_0)$ where $W(\varepsilon_0)$ is defined by (3.19). Note also that, by (3.36), the corresponding guiding solution $\mathbf{v}(\zeta,1) = (2\chi_0\zeta + \theta_0, \chi_0)$ belongs to $(\mathbb{R} \times [-\chi_M, \chi_M])$ for all $\zeta \in [0,\infty)$.

(3) (Guiding solution)
Choose $T > 0$ and define the compact (=closed and bounded) subset

$$S := \{\mathbf{v}(\tau,1) : \tau \in [0,T]\} = \{(2\chi_0\tau + \theta_0, \chi_0) : \tau \in [0,T]\} \tag{3.112}$$

of $(\mathbb{R} \times [-\chi_M, \chi_M]) \subset W(\varepsilon_0)$. Recall that $\mathbf{v}(\zeta,\varepsilon) = \mathbf{v}(\varepsilon\zeta,1)$.

(4) (Rectangle around initial value (θ_0, χ_0): the basic domain for averaging theorem)
Let $\hat{W}(\theta_0,\chi_0,d_1,d_2)$ be the following open rectangle around S where

$$\hat{W}(\theta_0,\chi_0,d_1,d_2) := (\theta_0 - d_1, \theta_0 + d_1) \times (\chi_0 - d_2, \chi_0 + d_2) , \tag{3.113}$$

where

$$2|\chi_0|T < d_1 , \quad 0 < d_2 < \chi_0 - \chi_{lb}(\varepsilon_0) . \tag{3.114}$$

Note that the closure, $\overline{\hat{W}(\theta_0,\chi_0,d_1,d_2)} = [\theta_0 - d_1, \theta_0 + d_1] \times [\chi_0 - d_2, \chi_0 + d_2]$, of $\hat{W}(\theta_0,\chi_0,d_1,d_2)$ is compact and that, by (3.19),(3.112), (3.113),(3.114), $(\theta_0,\chi_0) \in S \subset \hat{W}(\theta_0,\chi_0,d_1,d_2) \subset \overline{\hat{W}(\theta_0,\chi_0,d_1,d_2)} \subset W(\varepsilon_0)$. Thus, by Proposition 1 in §3.3, the vector field of the ODE's (3.10),(3.11) is C^∞ on $\hat{W}(\theta_0,\chi_0,d_1,d_2) \times \mathbb{R}$.

(5) (Restriction on ε_0)
Choose ε_0 so small that $\chi_{lb}(\varepsilon_0) < -\chi_M - d_2$. Note that this is made possible since, by (3.20),

$$\chi_{lb}(\varepsilon_0) \leq -\frac{1}{\varepsilon_0} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1 + K^2\Pi^2_{x,ub}(1)} ,$$

whence $\chi_{lb}(\varepsilon_0) < -\chi_M - d_2$ if

$$\varepsilon_0 < \left(\frac{1}{\sqrt{\mathcal{E}}}\sqrt{1 + K^2\Pi^2_{x,ub}(1)} + \chi_M + d_2\right)^{-1} . \tag{3.115}$$

Since the RHS of (3.115) is positive ε_0 can indeed be chosen sufficiently small.

(6) (Exact solution in rectangle)
Since the vector fields in (3.10),(3.11) are C^∞, solutions in $\hat{W}(\theta_0, \chi_0, d_1, d_2)$ with initial condition $\theta(0, \varepsilon) = \theta_0, \chi(0, \varepsilon) = \chi_0$ exist uniquely in $\hat{W}(\theta_0, \chi_0, d_1, d_2)$ on a maximum forward interval of existence $[0, \beta(\varepsilon))$. Here d_1, d_2 satisfy (3.114). Either $\beta(\varepsilon) = \infty$ or the solution approaches the boundary of $\hat{W}$ as $\zeta \to \beta(\varepsilon)-$. See Chapter 1 of [24] for a discussion of existence, uniqueness and continuation to a maximum forward interval of existence.

For convenience we define $I(\varepsilon, T) := [0, T/\varepsilon] \cap [0, \beta(\varepsilon))$.

(7) (Lipschitz constants for f_1, f_2 on rectangle)
Let L_1, L_2 be defined by

$$L_1 := \frac{2}{\bar{q}} max_{\zeta \in [0,2\pi]} \, |q(\zeta)| = 2[1 + \frac{2K^2}{\bar{q}}|\Delta P_{x0}| + \frac{K^2}{2\bar{q}}] \,, \tag{3.116}$$

$$L_2 := \nu K^2 (1 + |\Delta P_{x0}|) \,. \tag{3.117}$$

It follows by (3.12),(3.13), (3.116),(3.117) and for $\theta_1, \theta_2, \chi_1, \chi_2, \zeta \in \mathbb{R}$, that

$$|f_1(\chi_2, \zeta) - f_1(\chi_1, \zeta)| \leq \frac{2|q(\zeta)|}{\bar{q}} |\chi_2 - \chi_1| \leq L_1 |\chi_2 - \chi_1| \,, \tag{3.118}$$

$$\begin{aligned} &|f_2(\theta_2, \zeta; \nu) - f_2(\theta_1, \zeta; \nu)| \\ &\quad = K^2 |\cos\zeta + \Delta P_{x0}| \, |\cos(\nu[\theta_2 - Q(\zeta)]) - \cos(\nu[\theta_1 - Q(\zeta)])| \\ &\quad \leq K^2 (1 + |\Delta P_{x0}|) \, |\nu[\theta_2 - Q(\zeta)] - \nu[\theta_1 - Q(\zeta)]| \\ &\quad = \nu K^2 (1 + |\Delta P_{x0}|) \, |\theta_2 - \theta_1| = L_2 |\theta_2 - \theta_1| \,, \end{aligned} \tag{3.119}$$

where we have also used the fact that $|\cos x - \cos y| \leq |x - y|$. Thus L_1, L_2 are Lipschitz constants for f_1, f_2 on $\hat{W}(\theta_0, \chi_0, d_1, d_2)$ respectively (in fact even on $\mathbb{R}^2$).

(8) (Bounds for g_1, g_2 on rectangle)
Appendix C gives a very detailed derivation of quite explicit minimal bounds for g_1 and g_2. There we show, for (θ, χ, ζ) in $\hat{W}(\theta_0, \chi_0, d_1, d_2) \times \mathbb{R}$,

$$|g_i(\theta, \chi, \zeta, \varepsilon, \nu)| \leq C_i(\chi_0, \varepsilon_0, \nu, d_2) \,, \tag{3.120}$$

where $i = 1, 2$ and d_1, d_2 satisfy (3.114) and where the finite C_1 and C_2 are defined by (C.27),(C.30).

(9) (Besjes terms)
Let B_1, B_2 be defined by

$$\begin{aligned} B_1(\zeta) &:= |\int_0^\zeta \tilde{f}_1(v_2(s, \varepsilon), s) \, ds| = |\int_0^\zeta \tilde{f}_1(\chi_0, s) \, ds| \,, \\ B_2(\zeta) &:= |\int_0^\zeta \tilde{f}_2(v_1(s, \varepsilon), s; \nu) ds| = |\int_0^\zeta \tilde{f}_2(2\chi_0 \varepsilon s + \theta_0, s; \nu) ds| \,, \end{aligned} \tag{3.121}$$

where

$$\tilde{f}_1(v_2, s) := f_1(v_2, s) - \bar{f}_1(v_2) = 2(\frac{q(s)}{\bar{q}} - 1)v_2 \,,$$

$$\tilde{f}_2(v_1, s; \nu) := f_2(v_1, s; \nu) - \bar{f}_2(v_1; \nu) = f_2(v_1, s; \nu) \; . \tag{3.122}$$

In (3.121) we have used (3.36). We will also need $B_{1,\infty}, B_{2,\infty}$ defined by

$$B_{i,\infty}(\zeta) := \sup_{s \in [0,\zeta)} B_i(s) \; , \tag{3.123}$$

for $i = 1, 2$.

We refer to B_1, B_2 as "Besjes terms" and their importance will be seen both in the bounds presented in Theorem 1 and in the proof of the theorem where they eliminate the need for a near identity transformation (for the latter, see [6, 9, 10, 11, 12]).

With this setup we can now state the NR approximation theorem.

Theorem 1. *(Averaging theorem in Δ-NR case: $\nu \in [k+\Delta, k+1-\Delta]$, $k = 0, 1, ..., 0 < \Delta < 0.5$)*

With the setup given by items 1-9 of the above preamble we obtain, for $\zeta \in I(\varepsilon, T)$, that

$$|\theta(\zeta, \varepsilon) - 2\chi_0\varepsilon\zeta - \theta_0| = O(\varepsilon/\Delta) \; , \quad |\chi(\zeta, \varepsilon) - \chi_0| = O(\varepsilon/\Delta) \; . \tag{3.124}$$

More precisely

$$|\theta(\zeta, \varepsilon) - 2\chi_0\varepsilon\zeta - \theta_0| \le \varepsilon\Big([B_{1,\infty}(T/\varepsilon) + C_1 T] \cosh(T\sqrt{L_1 L_2})$$
$$+ [B_{2,\infty}(T/\varepsilon) + C_2 T] \sqrt{\frac{L_1}{L_2}} \sinh(T\sqrt{L_1 L_2}) \Big) \; , \tag{3.125}$$

$$|\chi(\zeta, \varepsilon) - \chi_0| \le \varepsilon\Big([B_{1,\infty}(T/\varepsilon) + C_1 T] \sqrt{\frac{L_2}{L_1}} \sinh(T\sqrt{L_1 L_2})$$
$$+ [B_{2,\infty}(T/\varepsilon) + C_2 T] \cosh(T\sqrt{L_1 L_2}) \Big) \; . \tag{3.126}$$

Moreover

$$B_{1,\infty}(T/\varepsilon) \le \check{B}_1 \; , \quad B_{2,\infty}(T/\varepsilon) \le \check{B}_2(T, \Delta) \; , \tag{3.127}$$

where $i = 1, 2$ and the $\check{B}_1, \check{B}_2(T, \Delta) \in [0, \infty)$ are finite, ε-independent and are defined in terms of our basic parameters and initial conditions by

$$\check{B}_1 := \frac{2K^2|\chi_0|}{\bar{q}}(2|\Delta P_{x0}| + \frac{1}{4}) \; , \tag{3.128}$$

$$\check{B}_2(T, \Delta) := \frac{1}{\Delta}\check{B}_{21}(T) + \check{B}_{22}(T) \; , \tag{3.129}$$

$$\check{B}_{21}(T) := 2K^2[1 + (k+1)|\chi_0|T] \Big(|\widehat{jj}(k; \nu, \Delta P_{x0})| + |\widehat{jj}(k+1; \nu, \Delta P_{x0})| \Big) \; , \tag{3.130}$$

$$\check{B}_{22}(T) := 2K^2 \Big(1 + (k+1)|\chi_0|T \Big) \sum_{n \in (\mathbb{Z} \setminus \{k, k+1\})} |\widehat{jj}(n; \nu, \Delta P_{x0})| \; . \tag{3.131}$$

Furthermore, for ε_0 sufficiently small, $(\theta(\zeta, \varepsilon), \chi(\zeta, \varepsilon))$ stays away from the boundary of the rectangle $\hat{W}(\theta_0, \chi_0, d_1, d_2)$ for $\zeta \in I(\varepsilon, T)$. Thus the ODE continuation theorem (see [24, Section 1.2]) gives $\beta(\varepsilon) > T/\varepsilon$, hence $I(\varepsilon, T) = [0, T/\varepsilon]$.

The proof of Theorem 1 is presented in §4.1. Note that the symbol $O(\varepsilon/\Delta)$ conveys that the error contains the factor $\frac{1}{\Delta}$.

3.5.2 NtoR case: $\nu = k + \varepsilon a$ (Periodic Averaging)

The NtoR case was defined in §3.2. The exact ODE's to be analyzed in this case were derived in §3.4 and are given by (3.48),(3.49) with initial conditions $\theta(0,\varepsilon) = \theta_0, \chi(0,\varepsilon) = \chi_0$ and where g_1^R, g_2^R are defined by (3.51),(3.52) and f_1^R, f_2^R by (3.47),(3.50). The normal form ODE's are (3.58),(3.59) with initial conditions $v_1(0,\varepsilon) = \theta_0, v_2(0,\varepsilon) = \chi_0$ solved by (3.70),(3.71). where X, Y satisfy the standard pendulum equations (3.65) with the initial conditions (3.72).

The setup for the theorem is as follows.

(1) (Basic parameters)
Let $0 < \varepsilon \le \varepsilon_0 \le 1$, $a \in [-1/2, 1/2]$ and k be a positive integer.

(2) (Initial data)
Choose θ_0, χ_0 such that $(\theta_0, \chi_0) \in (\mathbb{R} \times [-\chi_M, \chi_M])$ where $\chi_M > 0$ is chosen such that $-\chi_M > \chi_{lb}(\varepsilon_0)$. Clearly $(\mathbb{R} \times [-\chi_M, \chi_M]) \subset W(\varepsilon_0)$.

(3) (Guiding solution)
Choose $T > 0$ and define the compact subset $S_R := \{\mathbf{v}(\tau, 1) : \tau \in [0, T]\}$ of $W(\varepsilon_0)$ where $\mathbf{v} = (v_1, v_2)$ with v_1, v_2 given by (3.70),(3.71). Note that $S_R \subset W(\varepsilon_0)$ holds for arbitrary $T > 0$ if

$$\chi_{lb}(\varepsilon_0) < \chi_0 - d_2^{min}(\theta_0, \chi_0, k, a) \tag{3.132}$$

since $|v_2(\tau, 1) - \chi_0| \le d_2^{min}(\theta_0, \chi_0, k, a)$ where d_2^{min} is defined in §3.4.3.

(4) (Rectangle around initial value (θ_0, χ_0): the basic domain for averaging theorem)
Define an open rectangle $\hat{W}_R(\theta_0, \chi_0, d_1, d_2)$ around S_R by

$$\hat{W}_R(\theta_0, \chi_0, d_1, d_2) := (\theta_0 - d_1, \theta_0 + d_1) \times (\chi_0 - d_2, \chi_0 + d_2) \ , \tag{3.133}$$

where d_1, d_2 satisfy

$$0 \le d_1^{min}(\theta_0, \chi_0, T, k, a) < d_1 \ , \tag{3.134}$$
$$0 \le d_2^{min}(\theta_0, \chi_0, k, a) < d_2 < \chi_0 - \chi_{lb}(\varepsilon_0) \ , \tag{3.135}$$

with d_1^{min}, d_2^{min} defined in §3.4.3. Note that (3.135) entails (3.132). Note also that, by (3.81),(3.134), (3.135),

$$\begin{aligned} &|v_1(\tau, 1) - \theta_0| \le d_1^{min}(\theta_0, \chi_0, \tau, k, a) \le d_1^{min}(\theta_0, \chi_0, T, k, a) < d_1 \ , \\ &|v_2(\tau, 1) - \chi_0| \le d_2^{min}(\theta_0, \chi_0, k, a) < d_2 \ , \end{aligned} \tag{3.136}$$

where we also used that $d_1^{min}(\theta_0, \chi_0, \tau, k, a)$ is increasing w.r.t. τ. It follows from (3.133),(3.136) that $(\theta_0, \chi_0) \in S_R \subset \hat{W}_R(\theta_0, \chi_0, d_1, d_2)$ and, by (3.19),(3.132) that $\overline{\hat{W}_R(\theta_0, \chi_0, d_1, d_2)} \subset W(\varepsilon_0)$. Thus, by Proposition 2 in §3.4, the vector field of the ODE's (3.48),(3.49) is of class C^∞ on $\hat{W}_R(\theta_0, \chi_0, d_1, d_2) \times \mathbb{R}$. Note that the closure, $\overline{\hat{W}_R(\theta_0, \chi_0, d_1, d_2)} = [\theta_0 - d_1, \theta_0 + d_1] \times [\chi_0 - d_2, \chi_0 + d_2]$, of $\hat{W}_R(\theta_0, \chi_0, d_1, d_2)$ is compact.

(5) (Restriction on ε_0)
Choose ε_0 so small that $\chi_{lb}(\varepsilon_0) < -\chi_M - d_2$. Recall from item 5 of the preamble to Theorem 1 that such a choice is always possible.

(6) (Exact solution in rectangle)
Since the vector fields in (3.48),(3.49) are C^∞, solutions in $\hat{W}(\theta_0, \chi_0, d_1, d_2)$ with initial condition $\theta(0, \varepsilon) = \theta_0, \chi(0, \varepsilon) = \chi_0$ exist uniquely on a maximum forward interval of existence $[0, \beta(\varepsilon))$. Here d_1, d_2 satisfy (3.134),(3.135). Either $\beta(\varepsilon) = \infty$ or the solution approaches the boundary of $\hat{W}$ as $\zeta \to \beta(\varepsilon)-$. See Chapter 1 of [24] for a discussion of existence, uniqueness and continuation to a maximum forward interval of existence.

It is convenient to introduce $I(\varepsilon, T) := [0, T/\varepsilon] \cap [0, \beta(\varepsilon))$.

(7) (Lipschitz constants for f_1^R, f_2^R on rectangle)
Let L_1^R, L_2^R be defined by

$$L_1^R := L_1 = 2[1 + \frac{2K^2}{\bar{q}}|\Delta P_{x0}| + \frac{K^2}{2\bar{q}}] \; , \tag{3.137}$$

$$L_2^R := K^2 k(1 + |\Delta P_{x0}|) \; , \tag{3.138}$$

where we have also used (3.116) and where d_1, d_2 satisfy (3.134),(3.135). It follows by (3.50),(3.54), (3.118),(3.137),(3.138) and, for $\theta_1, \theta_2, \chi_1, \chi_2, \zeta \in \mathbb{R}$,

$$\begin{aligned}
&|f_1^R(\chi_2, \zeta) - f_1^R(\chi_1, \zeta)| = |f_1(\chi_2, \zeta) - f_1(\chi_1, \zeta)| \\
&\quad \leq L_1|\chi_2 - \chi_1| = L_1^R|\chi_2 - \chi_1| \; , \qquad (3.139)\\
&|f_2^R(\theta_2, \varepsilon\zeta, \zeta; k, a) - f_2^R(\theta_1, \varepsilon\zeta, \zeta; k, a)| \\
&\quad = K^2|\cos\zeta + \Delta P_{x0}|\Big|\cos\Big(k[\theta_2 - \zeta - \Upsilon_0 \sin\zeta - \Upsilon_1 \sin 2\zeta] - \varepsilon a\zeta\Big) \\
&\qquad - \cos\Big(k[\theta_1 - \zeta - \Upsilon_0 \sin\zeta - \Upsilon_1 \sin 2\zeta] - \varepsilon a\zeta\Big)\Big| \\
&\quad \leq kK^2(1 + |\Delta P_{x0}|)|\theta_2 - \theta_1| = L_2^R|\theta_2 - \theta_1| \; , \qquad (3.140)
\end{aligned}$$

where we have also used the fact that $|\cos x - \cos y| \leq |x - y|$. Thus L_1^R, L_2^R are Lipschitz constants for f_1^R, f_2^R on $\hat{W}_R(\theta_0, \chi_0, d_1, d_2)$ (in fact even on $\mathbb{R}^2$).

(8) (Bounds for g_1^R, g_2^R on rectangle)
Appendix E gives a very detailed derivation of quite explicit minimal bounds for g_1^R and g_2^R. There we show that, for $(\theta, \chi, \zeta) \in \hat{W}_R(\theta_0, \chi_0, d_1, d_2) \times \mathbb{R}$,

$$\begin{aligned}
&|g_1^R(\theta, \chi, \zeta, \varepsilon, k, a)| \leq C_1^R(\chi_0, \varepsilon_0, d_2) \; , \\
&|g_2^R(\theta, \chi, \zeta, \varepsilon, k, a)| \leq C_2^R(\theta_0, \chi_0, \varepsilon_0, a, d_1, d_2) \; ,
\end{aligned} \tag{3.141}$$

where $i = 1, 2$ and d_1, d_2 satisfy (3.134),(3.135) and where the finite C_1^R and C_2^R are defined by (E.5),(E.14).

(9) (Besjes terms)
Let B_1^R, B_2^R be defined by

$$\begin{aligned}
B_1^R(\zeta) &:= |\int_0^\zeta \tilde{f}_1^R(v_2(s, \varepsilon), s)\, ds| \; , \\
B_2^R(\zeta) &:= |\int_0^\zeta \tilde{f}_2^R(v_1(s, \varepsilon), \varepsilon s, s; k, a) ds| \; ,
\end{aligned}$$

$$(3.142)$$

where

$$\tilde{f}_1^R(\chi, s) := f_1^R(\chi, s) - \bar{f}_1^R(\chi) ,$$
$$\tilde{f}_2^R(\theta, \varepsilon s, s; k, a) := f_2^R(\theta, \varepsilon s, s; k, a) - \bar{f}_2^R(\theta, \varepsilon s; k) . \tag{3.143}$$

We will also need $B_{1,\infty}^R, B_{2,\infty}^R$ defined by

$$B_{i,\infty}^R(\zeta) := \sup_{s\in[0,\zeta)} B_i^R(s) , \tag{3.144}$$

where $i = 1, 2$.

We refer to B_1^R, B_2^R as “Besjes terms” and their importance will be seen both in the bounds presented in Theorem 2 and in the proof of the theorem where they eliminate the need for a near identity transformation.

With this setup we can now state the NtoR approximation theorem.

Theorem 2. *(Averaging theorem in NtoR case: $\nu = k + \varepsilon a, 0 < \varepsilon \le \varepsilon_0, k \in \mathbb{N}, |a| \le 0.5$)*

With the setup given by items 1-9 of the above preamble we obtain, for $\zeta \in I(\varepsilon, T)$, that

$$|\theta(\zeta, \varepsilon) - v_1(\zeta, \varepsilon)| = O(\varepsilon) , \qquad |\chi(\zeta, \varepsilon) - v_2(\zeta, \varepsilon)| = O(\varepsilon) .$$

More precisely

$$|\theta(\zeta) - v_1(\zeta, \varepsilon)| \le \varepsilon\Bigg([B_{1,\infty}^R(T/\varepsilon) + C_1^R T] \cosh(T\sqrt{L_1^R L_2^R}) + [B_{2,\infty}^R(T/\varepsilon) + C_2^R T]\sqrt{\frac{L_1^R}{L_2^R}} \sinh(T\sqrt{L_1^R L_2^R})\Bigg) , \tag{3.145}$$

$$|\chi(\zeta) - v_2(\zeta, \varepsilon)| \le \varepsilon\Bigg([B_{1,\infty}^R(T/\varepsilon) + C_1^R T]\sqrt{\frac{L_2^R}{L_1^R}} \sinh(T\sqrt{L_1^R L_2^R}) + [B_{2,\infty}^R(T/\varepsilon) + C_2^R T] \cosh(T\sqrt{L_1^R L_2^R})\Bigg) . \tag{3.146}$$

Moreover

$$B_{i,\infty}^R(T/\varepsilon) \le \check{B}_i^R(T) , \tag{3.147}$$

where $i = 1, 2$ and $\check{B}_i^R(T) \in [0, \infty)$ are independent of ε and defined by

$$\check{B}_1^R(T) := \frac{2K^2}{\bar{q}}[2|\Delta P_{x0}| + \frac{1}{4}]\Bigg(\chi_\infty(\theta_0, \chi_0, k, a) + K^2 T|\widehat{\bar{j}j}(k; k, \Delta P_{x0})|\Bigg) , \tag{3.148}$$

$$\check{B}_2^R(T) := K^2\Bigg(2 + T\ [|a| + 2k\chi_\infty(\theta_0, \chi_0, k, a)]\Bigg)$$

$$\times \sum_{n\in\mathbb{Z}\setminus\{k\}} \frac{|\widehat{jj}(n;k,\Delta P_{x0})|}{|n-k|} \,. \tag{3.149}$$

Furthermore, there exists an $0 < \varepsilon_0 \le 1$ such that for $0 < \varepsilon \le \varepsilon_0$, $(\theta(\zeta,\varepsilon),\chi(\zeta,\varepsilon))$ stays away from the boundary of the rectangle $\hat{W}_R(\theta_0,\chi_0,d_1,d_2)$ for $\zeta \in I(\varepsilon,T)$. Thus the ODE continuation theorem (see [24, Section 1.2]) gives $\beta(\varepsilon) > T/\varepsilon$, hence $I(\varepsilon,T) = [0,T/\varepsilon]$.

The proof of Theorem 2 is presented in §4.2.

3.5.3 Remarks on the averaging theorems

(1) We have now explored the θ,χ dynamics as a function of ν in the Δ-NR case and $\nu = k+\varepsilon a$ in the NtoR case. However asymptotically there are gaps for $\nu \in (k+\varepsilon a, k+\Delta)$ when ε is small. For $\Delta = O(\varepsilon)$ the NR normal form breaks down because the error is $O(1)$, however we can come close to the NtoR neighborhood by letting $\Delta = O(\varepsilon^\beta)$ with β near 1 however the error in the NR normal form does deteriorate to $O(\varepsilon^{1-\beta})$. It could be interesting to explore the dynamics in these gaps.

(2) Important for the functioning of the FEL is knowledge of the fraction of the bunch that occupies a bucket. From the analysis in §3.4.3 this occurs for ICs in the libration case, i.e., $0 < \mathcal{E}_{Pen}(Z_0) < 2$ where Z_0 is given in (3.66). One can thus determine the set of (θ_0,χ_0) for which Z_0 occupies the pendulum buckets. For more details on the pendulum motion and its impact on the low gain theory see §3.7.

(3) Mathematically we want to make sure the buckets are covered by our domain $W(\varepsilon_0)\times\mathbb{R}$ for physically reasonable χ_0. From (3.71) the range of the v_2-values in the buckets for the NtoR normal form is the interval $(-\frac{\Omega}{k}+\frac{a}{2k},\frac{\Omega}{k}+\frac{a}{2k})$. Now $a \ge -1/2$ so, for every k, the smallest v_2 in a bucket is $-\frac{\Omega}{k}-\frac{1}{4k}$ whence, since $k\ge 1$, the very smallest v_2 in a bucket is $-\Omega - 1/4$. Thus requiring

$$\chi_b := -\Omega - \frac{1}{4} < 0 \,, \tag{3.150}$$

entails that χ_b is smaller than any χ-value inside the buckets and smaller than any χ-value on the separatrix. It is plausible to restrict the physically interesting χ-values to be greater than, say $3\chi_b$. The condition that $(\theta,3\chi_b)\in W(\varepsilon_0)$ entails that the buckets are covered by $W(\varepsilon_0)$ and that ε_0 satisfies the constraint $3\chi_b > \chi_{lb}(\varepsilon_0)$. The following proposition is a corollary to Propositions 1,2.

Proposition 3. *Let $0 < \varepsilon \le \varepsilon_0$ where $0 < \varepsilon_0 \le 1$ and $\nu \in [1/2,\infty)$. Let also $\Delta\gamma$ be a positive constant and let*

$$\varepsilon_0 < \sqrt{\mathcal{E}}\left(\Delta\gamma + \sqrt{1+K^2\Pi^2_{x,ub}(1)}\right)^{-1} \,. \tag{3.151}$$

If $\chi\in\mathbb{R}$ satisfies the condition:

$$1 \le \gamma_c - \Delta\gamma \le \gamma_c(1+\varepsilon\chi) \le \gamma_c + \Delta\gamma \,, \tag{3.152}$$

then

$$\chi > \chi_{lb}(\varepsilon_0) \,. \tag{3.153}$$

In other words if ε_0 satisfies (3.151) then the γ values in $[\gamma_c-\Delta\gamma,\gamma_c+\Delta\gamma]$ are covered by $W(\varepsilon_0)$.

The proposition guarantees, by choosing a sufficiently small ε_0, that the domain $W(\varepsilon_0)\times\mathbb{R}$ is large enough to contain the physical relevant values of θ, χ, ζ.

Proof of Proposition 3: Let $\chi \in \mathbb{R}$ satisfy (3.152). Then, by (1.5), $\chi \in [-\frac{1}{\sqrt{\mathcal{E}}}\Delta\gamma, \frac{1}{\sqrt{\mathcal{E}}}\Delta\gamma]$ whence, by (3.17),(3.20),(3.151),

$$\begin{aligned}\chi_{lb}(\varepsilon_0) &= -\frac{1}{\varepsilon_0} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1 + K^2\Pi^2_{x,ub}(\varepsilon_0)} \\ &\leq -\frac{1}{\varepsilon_0} + \frac{1}{\sqrt{\mathcal{E}}}\sqrt{1 + K^2\Pi^2_{x,ub}(1)} < -\frac{1}{\sqrt{\mathcal{E}}}\Delta\gamma \leq \chi \;,\end{aligned}$$

which entails (3.153). □

Note that the condition: $1 \leq \gamma_c - \Delta\gamma$ in (3.152) is not used in the proof of Proposition 3 but serves to guarantee that χ satisfies the physical condition: $\gamma \geq 1$, i.e., $1 \leq \gamma_c(1+\varepsilon\chi)$.

(4) In applications of Theorems 1,2, T should be chosen so that $z \in [0, T/\varepsilon k_u]$ is the domain of interest, e.g., so that $T/(\varepsilon k_u)$ is the length of the undulator.

(5) In many discussions of this nature, researchers often just assert the existence of bounds, for example by using the well known fact that a continuous function on a compact set is bounded, or bounds are obtained which are crude. Here we wanted to do more. By using, in the proofs of Theorems 1 and 2, a system of differential inequalities instead of the Gronwall inequality we have been able to use two Lipschitz constants in each proof instead of their maximum and in a similar manner can treat the two Besjes' terms independently as well as the components of g and g^R. Furthermore, we believe the Besjes bounds and the bounds on g_1, g_2, g_1^R, g_2^R are nearly optimal.

We also note that there are only 3 restrictions on the size of ε_0 and thus ε. The first is that we require $\varepsilon_0 \leq 1$. But this is only a matter of convenience and is really no restriction at all since the averaging theorems are only useful for ε small. The second restriction is in item 5 of the preambles to the two theorems, however as indicated there this is not a significant restriction. Thus the only real restriction is keeping the solution away from the boundary of $\hat{W}, \hat{W}_R$ in order to obtain $I(\varepsilon, T) = [0, T/\varepsilon]$. This is an optimization problem; by making $\hat{W}, \hat{W}_R$ larger, ε can be larger, however this is compensated to some extent in the Lipschitz constants as well as the bounds on g_1, g_2, g_1^R, g_2^R which would become larger. Nonetheless, the situation is quite good in comparison to say KAM or Nekhoroshev theorems (see e.g., [8]), where the restrictions on ε are quite severe and it is with great effort that the restrictions on ε have been improved in some applications, e.g., solar system problems.

(6) We here clarify the contributions of $\widehat{jj}$ to the error bounds of Theorems 1 and 2 by finding simple upper bounds for $\check{B}_{21}(T), \check{B}_1^R(T), \check{B}_{22}(T)$ and $\check{B}_2^R(T)$. First of all we note from (3.23) and (3.25) that

$$|\widehat{jj}(n;\nu,\Delta P_{x0})| \leq 1 + |\Delta P_{x0}| \;, \tag{3.154}$$

where $\nu \geq 1/2$. Clearly (3.154) gives upper bounds for $\check{B}_{21}(T), \check{B}_1^R(T)$ in (3.130),(3.148). Secondly, we obtain from the Cauchy-Schwarz inequality that

$$\sum_{0\neq n\in\mathbb{Z}} |\widehat{jj}(n;\nu,\Delta P_{x0})| = \sum_{0\neq n\in\mathbb{Z}} \frac{1}{|n|}|n|\;|\widehat{jj}(n;\nu,\Delta P_{x0})|$$

$$\leq \Big(\sum_{0 \neq n \in \mathbb{Z}} n^2 |\widehat{jj}(n; \nu, \Delta P_{x0})|^2 \Big)^{1/2} \Big(\sum_{0 \neq n \in \mathbb{Z}} \frac{1}{n^2} \Big)^{1/2}$$

$$= \frac{\pi}{\sqrt{3}} \Big(\sum_{0 \neq n \in \mathbb{Z}} n^2 |\widehat{jj}(n; \nu, \Delta P_{x0})|^2 \Big)^{1/2} , \tag{3.155}$$

where the finiteness of the rhs follows from the fact that the function $jj(\cdot; \nu, \Delta P_{x0})$ is of class C^∞. Since $jj(\cdot; \nu, \Delta P_{x0})$ is also 2π-periodic we can apply Parseval's theorem to get

$$\frac{1}{2\pi} \int_{[0,2\pi]} d\zeta |\frac{d}{d\zeta} jj(\zeta; \nu, \Delta P_{x0})|^2 = \sum_{0 \neq n \in \mathbb{Z}} n^2 |\widehat{jj}(n; \nu, \Delta P_{x0})|^2 . \tag{3.156}$$

It also follows from (3.23) that

$$\frac{d}{d\zeta} jj(\zeta; \nu, \Delta P_{x0}) = -\exp(-i\nu[\Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta]) \Big(\sin\zeta$$
$$+ i\nu(\cos\zeta + \Delta P_{x0})[\Upsilon_0 \cos\zeta + 2\Upsilon_1 \cos 2\zeta] \Big) ,$$

whence

$$|\frac{d}{d\zeta} jj(\zeta; \nu, \Delta P_{x0})|^2 \leq 1 + \nu^2 (1 + |\Delta P_{x0}|)^2 [|\Upsilon_0| + 2\Upsilon_1]^2 ,$$

so that, by (3.155),(3.156),

$$\sum_{0 \neq n \in \mathbb{Z}} |\widehat{jj}(n; \nu, \Delta P_{x0})| \leq \frac{\pi}{\sqrt{3}} \Big(1 + \nu^2 (1 + |\Delta P_{x0}|)^2 [|\Upsilon_0| + 2\Upsilon_1]^2 \Big)^{1/2} , \tag{3.157}$$

which entails, by (3.154),

$$\sum_{n \in (\mathbb{Z} \setminus \{k, k+1\})} |\widehat{jj}(n; \nu, \Delta P_{x0})| \leq 1 + |\Delta P_{x0}| + \sum_{0 \neq n \in \mathbb{Z}} |\widehat{jj}(n; \nu, \Delta P_{x0})|$$
$$\leq 1 + |\Delta P_{x0}| + \frac{\pi}{\sqrt{3}} \Big(1 + \nu^2 (1 + |\Delta P_{x0}|)^2 [|\Upsilon_0| + 2\Upsilon_1]^2 \Big)^{1/2} \Big) . \tag{3.158}$$

Clearly (3.158) gives an upper bound for $\check{B}_{22}(T)$ in (3.131). Moreover, by (3.154),(3.157),

$$\sum_{n \in \mathbb{Z} \setminus \{k\}} \frac{|\widehat{jj}(n; k, \Delta P_{x0})|}{|n-k|} \leq |\widehat{jj}(0; k, \Delta P_{x0})| + \sum_{0 \neq n \in \mathbb{Z}} |\widehat{jj}(n; k, \Delta P_{x0})|$$
$$\leq 1 + |\Delta P_{x0}| + \frac{\pi}{\sqrt{3}} \Big(1 + \nu^2 (1 + |\Delta P_{x0}|)^2 [|\Upsilon_0| + 2\Upsilon_1]^2 \Big)^{1/2} ,$$

which gives an upper bound for $\check{B}_2^R(T)$ in (3.149).

3.6 Approximation for the phase space variables in (2.22)-(2.25)

Here we discuss the approximate solutions of (2.22)-(2.25) and (2.29) in terms of the normal form approximations given in (3.36),(3.70),(3.71), namely

$$\theta_{NF}(\tau) := \begin{cases} 2\chi_0\tau + \theta_0 & \text{NR case} \\ \Big(X(\Omega\tau; Z_0) - \mathrm{sgn}(K_0(k))\pi/2 + a\tau\Big)/k & \text{NtoR case}\ , \end{cases} \tag{3.159}$$

and

$$\chi_{NF}(\tau) := \begin{cases} \chi_0 & \text{NR case} \\ \Big(\Omega Y(\Omega\tau; Z_0) + a\Big)/2k & \text{NtoR case}\ , \end{cases} \tag{3.160}$$

where K_0 is given in (3.60) and Ω in (3.69). Recall from Theorems 1 and 2 that

$$\theta(\zeta,\varepsilon) = \theta_{NF}(\varepsilon\zeta) + O(\varepsilon)\ , \tag{3.161}$$

$$\chi(\zeta,\varepsilon) = \chi_{NF}(\varepsilon\zeta) + O(\varepsilon)\ , \tag{3.162}$$

for $\zeta \in I(\varepsilon, T)$. From (1.2),(2.53),(2.61),(2.64)

$$\theta(\zeta,\varepsilon) = \frac{2\mathcal{E}}{\varepsilon^2\bar{q}}\Big(\zeta - k_u ct(\zeta/k_u)\Big) + Q(\zeta)\ , \tag{3.163}$$

and from (2.36)

$$\gamma(\zeta/k_u) = \gamma_c(1 + \varepsilon\chi(\zeta,\varepsilon))\ . \tag{3.164}$$

Now we can determine the approximate solution of (2.22)-(2.25) and (2.29). From (3.161),(3.163) the arrival time, $t(z)$, of a particle at z is given by

$$t(z) = \frac{z}{c} - \frac{\varepsilon^2\bar{q}}{2\mathcal{E}k_u c}\Big(\theta_{NF}(\varepsilon k_u z) - Q(k_u z) + O(\varepsilon)\Big)\ . \tag{3.165}$$

Furthermore from (1.5),(3.162),(3.164) the energy in (2.29) is given by

$$\gamma(z) = \sqrt{\mathcal{E}}(\frac{1}{\varepsilon} + \chi_{NF}(\varepsilon k_u z) + O(\varepsilon))\ , \tag{3.166}$$

and is clearly slowly varying. From (2.37),(3.1),(3.2) we have

$$p_x(z) = mcK[\cos(k_u z) + \Delta P_{x0} + O(\varepsilon^2)]\ . \tag{3.167}$$

It is tedious but straightforward to derive from (1.5),(2.37),(3.1),(3.2), (3.166)

$$p_z(z) = mc\sqrt{\mathcal{E}}\Big(\frac{1}{\varepsilon} + \chi_{NF}(\varepsilon k_u z) + O(\varepsilon)\Big)\ . \tag{3.168}$$

Finally we can now determine $x(z)$. From (2.22),(3.167) and (3.168)

$$\frac{d}{dz}x(z) = \frac{p_x(z)}{p_z(z)}$$

$$
\begin{aligned}
&= \left(mcK[\cos(k_uz) + \Delta P_{x0} + O(\varepsilon^2)]\right) \Big/ \left(mc\sqrt{\mathcal{E}}\left(\frac{1}{\varepsilon} + \chi_{NF}(\varepsilon k_u z) + O(\varepsilon)\right)\right) \\
&= \varepsilon \frac{(K/\sqrt{\mathcal{E}})[\cos(k_uz) + \Delta P_{x0} + O(\varepsilon^2)]}{1 + \varepsilon\chi_{NF}(\varepsilon k_u z) + O(\varepsilon^2)} \\
&= \frac{\varepsilon K}{\sqrt{\mathcal{E}}}\left(\cos(k_uz) + \Delta P_{x0} + O(\varepsilon^2)\right)\left(1 - \varepsilon\chi_{NF}(\varepsilon k_u z) + O(\varepsilon^2)\right) \\
&= \frac{\varepsilon K}{\sqrt{\mathcal{E}}}[\cos(k_uz) + \Delta P_{x0}][1 - \varepsilon\chi_{NF}(\varepsilon k_u z)] + O(\varepsilon^3) \; . \qquad (3.169)
\end{aligned}
$$

Integrating (3.169) gives

$$
\begin{aligned}
x(z) &= x(0) \\
&\quad + \frac{\varepsilon K}{\sqrt{\mathcal{E}}}\left(\frac{\sin(k_uz)}{k_u} + z\Delta P_{x0} - \varepsilon \int_0^z \; [\cos(k_us) + \Delta P_{x0}]\chi_{NF}(\varepsilon k_u s)ds\right) \\
&\quad + O(\varepsilon^3 z) \; . \qquad (3.170)
\end{aligned}
$$

For ε sufficiently small, $I(\varepsilon, T) = [0, T/\varepsilon]$ and then (3.165)-(3.168) and (3.170) hold for $0 \le k_u z \le T/\varepsilon$.

3.7 Low Gain Calculation in the NtoR regime

Low gain theories in [2, 3, 4] are done in the context of the pendulum equations, i.e., (3.58),(3.59) with $a = 0, \Delta P_{x0} = 0$, and $k = 1$. Here we will not make those assumptions and we define the gain by

$$
G(\zeta, \varepsilon) := \varepsilon\overline{(v_2(\zeta, \varepsilon) - \chi_0)}_{\theta_0} = \varepsilon\overline{(v_2(\varepsilon\zeta, 1) - \chi_0)}_{\theta_0} \; , \qquad (3.171)
$$

where v_2 is given in (3.71) and $\overline{(\;)}_{\theta_0}$ denotes the average over θ_0. This is consistent with [2, 3, 4].

The gain G could be calculated numerically using a quadrature formula and an ODE solver, however standard treatments calculate it perturbatively using a regular (and thus short time) perturbation expansion. We could do a regular perturbation expansion in (3.58),(3.59) by letting $v_i = \sum_{k=0}^4 \; \varepsilon^k A_{ik} + O(\varepsilon^5)$ and using Grownwall techniques to make the $O(\varepsilon^5)$ error rigorous (see [25, p.594] for an example of a regular perturbation theorem at first order and its proof). However at the fourth order needed here this would be quite cumbersome. Because of the special scaling structure in (3.58),(3.59) as given in (3.61) we can use a Taylor expansion. For $\varepsilon = 1$ we get from (3.58),(3.59)

$$
\begin{aligned}
&v_1'(\cdot, 1) = 2v_2(\cdot, 1) \; , \quad v_1(0, 1) = \theta_0 \; , \\
&v_2'(\cdot, 1) = -K_0(k)\cos(kv_1(\cdot, 1) - a\tau) \; , \quad v_2(0, 1) = \chi_0 \; ,
\end{aligned}
\qquad (3.172)
$$

and we expand $v_2(\cdot, 1)$ about $\tau = 0$ so that

$$
v_2(\tau, 1) = \chi_0 + \sum_{k=1}^4 \; \frac{1}{k!}v_2^{(k)}(0, 1)\tau^k + \frac{\tau^5}{4!} \int_0^1 \; (1 - t)^4 v_2^{(5)}(t\tau, 1)dt \; . \qquad (3.173)
$$

From (G.6) in Appendix G we have

$$
v_2'(0, 1) = -K_0(k)\cos(k\theta_0) \; ,
$$

$$
\begin{aligned}
v_2''(0,1) &= K_0(k)(2k\chi_0 - a)\sin(k\theta_0)\ , \\
v_2'''(0,1) &= K_0(k)\Big(-kK_0(k)\sin(2k\theta_0) + [2k\chi_0 - a]^2\cos(k\theta_0)\Big)\ , \\
v_2''''(0,1) &= K_0(k)\Big(2kK_0(k)(2k\chi_0 - a)[\sin^2(k\theta_0) - 3\cos^2(k\theta_0)] \\
&\quad -[2k\chi_0 - a]^3\sin(k\theta_0)\Big)\ . \qquad (3.174)
\end{aligned}
$$

It follows from (3.173),(3.174) that the average over θ_0 leads to

$$
\overline{(v_2(\tau,1)-\chi_0)}_{\theta_0} = \frac{\tau^4}{4!}\overline{v_2''''(0,1)}_{\theta_0} + O(\tau^5) = -\frac{\tau^4}{12}kK_0^2(k)[2k\chi_0 - a] + O(\tau^5)\ , \qquad (3.175)
$$

which gives, by (3.171),

$$
G(\zeta,\varepsilon) = \varepsilon\overline{(v_2(\varepsilon\zeta,1)-\chi_0)}_{\theta_0} = -\frac{\varepsilon^5\zeta^4}{12}kK_0^2(k)[2k\chi_0 - a] + O(\varepsilon^6)\ . \qquad (3.176)
$$

This shows the effect of a and k on the gain.

We now compare our gain formula in (3.176) with the corresponding calculation in [2], where $a = 0, \Delta P_{x0} = 0$, and $k = 1$. From our NtoR normal form system (3.58),(3.59) and letting $\theta = v_1$ and $\eta = \varepsilon v_2$ we obtain the IVP

$$
\theta' = 2\eta\ , \quad \theta(0) = \theta_0\ , \qquad (3.177)
$$

$$
\eta' = -\epsilon\cos\theta\ , \quad \eta(0) = \varepsilon\chi_0 =: \eta_0\ , \qquad (3.178)
$$

where $\epsilon = \varepsilon^2 K_0(1)$. The procedure in [2] is a regular perturbation expansion in ϵ that does not assume that η_0 is small. Proceeding as they do, we write

$$
\theta(\zeta,\epsilon) = \theta^0(\zeta) + \epsilon\theta^1(\zeta) + \epsilon^2\theta^2(\zeta) + O(\epsilon^3)\ , \qquad (3.179)
$$

$$
\eta(\zeta,\epsilon) = \eta^0(\zeta) + \epsilon\eta^1(\zeta) + \epsilon^2\eta^2(\zeta) + O(\epsilon^3)\ . \qquad (3.180)
$$

We find

$$
\eta^0(\zeta) = \eta_0\ , \qquad (3.181)
$$

$$
\theta^0(\zeta) = 2\eta_0\zeta + \theta_0\ , \qquad (3.182)
$$

$$
\eta^1(\zeta) = \frac{1}{2\eta_0}[\sin\theta_0 - \sin(2\eta_0\zeta + \theta_0)]\ , \qquad (3.183)
$$

$$
\theta^1(\zeta) = \frac{1}{\eta_0}\{\zeta\sin\theta_0 + \frac{1}{2\eta_0}[\cos(2\eta_0\zeta + \theta_0) - \cos\theta_0]\}\ , \qquad (3.184)
$$

$$
\begin{aligned}
\eta^2(\zeta) &= \frac{1}{\eta_0}\int_0^\zeta dt\,\sin(2\eta_0 t + \theta_0)\{t\sin\theta_0 \\
&+\frac{1}{2\eta_0}[\cos(2\eta_0 t + \theta_0) - \cos\theta_0]\}\ . \qquad (3.185)
\end{aligned}
$$

It follows that $\overline{\eta^1(\zeta)}_{\theta_0} = 0$ and

$$
\overline{\eta^2(\zeta)}_{\theta_0} = \frac{1}{2\eta_0}\int_0^\zeta (t\cos 2\eta_0 t - \frac{1}{2\eta_0}\sin 2\eta_0 t)dt\ . \qquad (3.186)
$$

We can rewrite (3.186) as

$$\overline{\eta^2(\zeta)}_{\theta_0} = \frac{\zeta^3}{4}\frac{d}{d\tau}\Big(\frac{\sin\tau}{\tau}\Big)^2 \,, \quad \tau := \eta_0\zeta \,, \tag{3.187}$$

and the gain becomes

$$G(\zeta,\varepsilon) = \epsilon^2\overline{\eta^2(\zeta)}_{\theta_0} = \varepsilon^4 K_0^2(1)\frac{1}{4}\zeta^3\frac{d}{d\tau}\Big(\frac{\sin\tau}{\tau}\Big)^2 \,, \tag{3.188}$$

consistent with [2]. For η_0 small, which is required by our averaging approximation (since $\eta_0 = \varepsilon\chi_0$ and $\chi_0 = O(1)$), we obtain from (3.186) that

$$\overline{\eta^2(\zeta)}_{\theta_0} = \frac{1}{2\eta_0}\int_0^\zeta [-\frac{4}{3}\eta_0^2 t^3 + O(\eta_0 t)^4]dt \approx -\frac{1}{6}\eta_0\zeta^4 \,. \tag{3.189}$$

It follows from (3.188),(3.189) that

$$G(\zeta,\varepsilon) \approx -\epsilon^2\frac{1}{6}\eta_0\zeta^4 = -\frac{\varepsilon^5\zeta^4}{6}K_0^2(1)\chi_0 \,, \tag{3.190}$$

as in (3.176) with $a = 0$ and $k = 1$.

Thus we see that (3.176) is consistent with the standard gain formula for $\tau = \eta_0\zeta$ small. The $O(\varepsilon^6)$ error in (3.176) can be made precise by estimating the remainder term in (3.173). However, we cannot justify the gain formula either in (3.176) or in (3.188) in the context of our Lorentz system in (2.22) - (2.25), because our NtoR normal form approximation only gives an approximation to $O(\varepsilon)$. Thus a justification of the gain formulas, based on our Lorentz system, would need to come from elsewhere, e.g., a numerical calculation based on (3.8) and (3.9).

4 Proof of averaging theorems

In §4.1 we prove the NR theorem, Theorem 1 of §3.5.1, and in §4.2 we prove the NtoR theorem, Theorem 2 of §3.5.2.

4.1 Proof of Theorem 1 (Averaging theorem in Δ-NR case)

Here we compare solutions of the exact IVP (3.10),(3.11):

$$\theta' = \varepsilon f_1(\chi,\zeta) + \varepsilon^2 g_1(\theta,\chi,\zeta;\varepsilon,\nu) \,, \quad \theta(0,\varepsilon) = \theta_0 \,, \tag{4.1}$$
$$\chi' = \varepsilon f_2(\theta,\zeta;\nu) + \varepsilon^2 g_2(\theta,\chi,\zeta;\varepsilon,\nu) \,, \quad \chi(0,\varepsilon) = \chi_0 \,, \tag{4.2}$$

where

$$f_1(\chi,\zeta) = \frac{2q(\zeta)\chi}{\bar{q}} \,, \tag{4.3}$$
$$\begin{aligned} f_2(\theta,\zeta;\nu) &= -K^2(\cos\zeta + \Delta P_{x0})\cos(\nu[\theta - Q(\zeta)]) \\ &= -\frac{K^2}{2}e^{i\nu\theta}\sum_{n\in\mathbb{Z}}\widehat{jj}(n;\nu,\Delta P_{x0})e^{i(n-\nu)\zeta} + cc \,, \end{aligned} \tag{4.4}$$

with the normal form IVP of (3.34),(3.35):

$$v_1' = \varepsilon\bar{f}_1(v_2) \,, \quad v_1(0,\varepsilon) = \theta_0 \,, \tag{4.5}$$

$$v_2' = \varepsilon \bar{f}_2(v_1;\nu)\ , \quad v_2(0,\varepsilon) = \chi_0\ , \tag{4.6}$$

where

$$\bar{f}_1(v_2) = 2v_2\ , \quad \bar{f}_2(v_1;\nu) = 0\ , \tag{4.7}$$

for $\nu \in [k+\Delta, k+1-\Delta]$.

Subtracting and integrating, we obtain from (3.122), (4.1),(4.2),(4.5),(4.6) that

$$\begin{aligned}
\theta(\zeta,\varepsilon) - v_1(\zeta,\varepsilon) &= \varepsilon \int_0^\zeta \big[f_1(\chi(s,\varepsilon),s) - f_1(v_2(s,\varepsilon),s) \\
&\quad + f_1(v_2(s,\varepsilon),s) - \bar{f}_1(v_2(s,\varepsilon)) + \varepsilon g_1(\theta(s,\varepsilon),\chi(s,\varepsilon),s;\varepsilon,\nu)\big]\, ds \\
&= \varepsilon \int_0^\zeta \big[f_1(\chi(s,\varepsilon),s) - f_1(v_2(s,\varepsilon),s) \\
&\quad + \tilde{f}_1(\chi_0,s) + \varepsilon g_1(\theta(s,\varepsilon),\chi(s,\varepsilon),s;\varepsilon,\nu)\big]\, ds\ ,
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\chi(\zeta,\varepsilon) - v_2(\zeta,\varepsilon) &= \varepsilon \int_0^\zeta \big[f_2(\theta(s,\varepsilon),s;\nu) - f_2(v_1(s,\varepsilon),s;\nu) \\
&\quad + f_2(v_1(s,\varepsilon),s;\nu) + \varepsilon g_2(\theta(s,\varepsilon),\chi(s,\varepsilon),s;\varepsilon,\nu)\big]\, ds \\
&= \varepsilon \int_0^\zeta \big[f_2(\theta(s,\varepsilon),s;\nu) - f_2(v_1(s,\varepsilon),s;\nu) \\
&\quad + \tilde{f}_2(v_1(s,\varepsilon),s;\nu) + \varepsilon g_2(\theta(s,\varepsilon),\chi(s,\varepsilon),s;\varepsilon,\nu)\big]\, ds\ ,
\end{aligned} \tag{4.9}$$

for $\zeta \in I(\varepsilon,T) = [0,T/\varepsilon] \cap [0,\beta(\varepsilon))$. Important for our analysis below is that the points $(\theta(\zeta,\varepsilon),\chi(s,\varepsilon))$ and $(v_1(s,\varepsilon),v_2(s,\varepsilon))$ belong to the rectangle $\hat{W}(\theta_0,\chi_0,d_1,d_2)$ for $\zeta \in I(\varepsilon,T)$. Note that we have added and subtracted $f_1(v_2(s,\varepsilon),s)$ in (4.8) and $f_2(v_1(s,\varepsilon),s;\nu)$ in (4.9), an idea introduced by Besjes [15] (see also [13]).

Taking absolute values, applying the Lipschitz condition on $\hat{W}(\theta_0,\chi_0,d_1,d_2)$ and defining

$$e_1(s) := |\theta(s,\varepsilon) - v_1(s,\varepsilon)|\ , \tag{4.10}$$

$$e_2(s) := |\chi(s,\varepsilon) - v_2(s,\varepsilon)|\ , \tag{4.11}$$

gives, by (3.116),(3.117),(3.120), (3.121),(3.123), (4.8),(4.9) for $\zeta \in I(\varepsilon,T)$,

$$\begin{aligned}
0 \le e_1(\zeta) &\le \varepsilon[L_1 \int_0^\zeta e_2(s)ds + |\int_0^\zeta \tilde{f}_1(\chi_0,s)ds| \\
&\quad + \varepsilon \int_0^\zeta |g_1(\theta(s,\varepsilon),\chi(s,\varepsilon),s;\varepsilon,\nu)|] \le \varepsilon[L_1 \int_0^\zeta e_2(s)ds + B_1(\zeta) + TC_1] \\
&\le \varepsilon[L_1 \int_0^\zeta e_2(s)ds + B_{1,\infty}(T/\varepsilon) + TC_1] =: R_1(\zeta)\ ,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
0 \le e_2(\zeta) &\le \varepsilon[L_2 \int_0^\zeta e_1(s)ds + |\int_0^\zeta \tilde{f}_2(2\chi_0\varepsilon s + \theta_0, s;\nu)ds| \\
&\quad + \varepsilon \int_0^\zeta |g_2(\theta(s,\varepsilon),\chi(s,\varepsilon),s;\varepsilon,\nu)|] \le \varepsilon[L_2 \int_0^\zeta e_1(s)ds + B_2(\zeta) + TC_2]
\end{aligned}$$

$$\leq \varepsilon[L_2 \int_0^\zeta e_1(s)ds + B_{2,\infty}(T/\varepsilon) + TC_2] =: R_2(\zeta) \ , \tag{4.13}$$

where we also used that $I(\varepsilon, T) \subset [0, T/\varepsilon]$ and where we have introduced the R_i as in the proof of the Gronwall inequality for a single integral inequality (the Gronwall inequality is discussed in many ODE books, see, e.g., [24, p.36] and [26, p.310 and 317]). $\zeta \in I(\varepsilon, T)$.

Recall that $L_1, L_2, C_1, C_2, B_1, B_2$ are defined in items 7,8 and 9 of the preamble to the theorem. For convenience we have suppressed the ε dependence of e_1 and e_2.

Before we proceed with the proof, several comments are in order.

1. We refer to the terms $B_1(\zeta), B_2(\zeta)$ in (3.121) as Besjes terms since they were introduced by him in order to prove an averaging theorem without a near identity transformation; a simplification. Standard proofs use the near identity transformation (see e.g., [6, 9, 10]).

   One may fear that the Besjes terms could grow as large as $O(1/\varepsilon)$ for $\zeta \in [0, T/\varepsilon]$, i.e., that $B_{i,\infty}(T/\varepsilon) = O(1/\varepsilon)$. However this doesn't happen here since, by (3.127), $\check{B}_1, \check{B}_2(T, \Delta)$ are upper bounds for $B_{i,\infty}(T/\varepsilon)$ and are ε independent. Two facts are mainly responsible for this: (a) the fact that for fixed v_1 and v_2 the integrands have zero mean, i.e., the quantities in (3.122) have zero mean in s, and (b) the fact that $v_1(s, \varepsilon)$ and $v_2(s, \varepsilon)$ are slowly varying.

2. We maintain the system form in (4.12),(4.13). We could add these two inequalities and obtain an error estimate using a Gronwall inequality. That is, let $L_\infty = max(L_1, L_2)$, $B_\infty = B_{1,\infty} + B_{2,\infty}$, $C_\infty = C_1 + C_2$, then adding gives

$$0 \leq e_\infty(\zeta) \leq \varepsilon[L_\infty \int_0^\zeta e_\infty(s)ds + B_\infty(T/\varepsilon) + C_\infty T] \ , \tag{4.14}$$

   where $e_\infty = e_1 + e_2$. The Gronwall inequality gives $e_\infty(\zeta) \leq \varepsilon[B_\infty(T/\varepsilon) + C_\infty T] \exp(\varepsilon L_\infty \zeta)$. However our system approach gives better bounds.

3. We have a draft of a general paper on quasiperiodic averaging which uses the Besjes idea and deals with the small divisor problem (See [14]). However the proof we are presenting here is simple, the small divisor problem is trivial and the error bounds are quite explicit. Thus we feel it is good to give complete proofs here rather than appealing to a more general theory. Also it serves the pedagogical purpose of showing how an averaging theorem is proved in a simple context; here the context of (3.10), (3.11) and (3.48), (3.49). We have incorporated the Besjes idea in much of our previous averaging work, see [13, 25, 27, 28, 29].

We now proceed with the proof. It follows from (4.12),(4.13) that

$$R_1' = \varepsilon L_1 e_2(\zeta) \leq \varepsilon L_1 R_2(\zeta) \ , \quad R_1(0) = \varepsilon[B_{1,\infty}(T/\varepsilon) + C_1 T] \ , \tag{4.15}$$
$$R_2' = \varepsilon L_2 e_1(\zeta) \leq \varepsilon L_2 R_1(\zeta) \ , \quad R_2(0) = \varepsilon[B_{2,\infty}(T/\varepsilon) + C_2 T] \ , \tag{4.16}$$

whence, by Appendix I for $\zeta \in I(\varepsilon, T)$,

$$R_1(\zeta) \leq \varepsilon w_1(\varepsilon\zeta) \ , \quad R_2(\zeta) \leq \varepsilon w_2(\varepsilon\zeta) \ , \tag{4.17}$$

where

$$w_1' = L_1 w_2 \ , \quad w_1(0) = B_{1,\infty}(T/\varepsilon) + C_1 T \ , \tag{4.18}$$

$$w_2' = L_2 w_1\ , \quad w_2(0) = B_{2,\infty}(T/\varepsilon) + C_2 T\ . \tag{4.19}$$

Note that in Appendix I we use the fact that R_1, R_2 are of class C^1.

Solving (4.18),(4.19) we find

$$\begin{pmatrix} w_1(s) \\ w_2(s) \end{pmatrix} = \begin{pmatrix} \cosh(s\sqrt{L_1L_2}) & \sqrt{\frac{L_1}{L_2}}\sinh(s\sqrt{L_1L_2}) \\ \sqrt{\frac{L_2}{L_1}}\sinh(s\sqrt{L_1L_2}) & \cosh(s\sqrt{L_1L_2}) \end{pmatrix} \begin{pmatrix} B_{1,\infty}(T/\varepsilon) + C_1T \\ B_{2,\infty}(T/\varepsilon) + C_2T \end{pmatrix}\ , \tag{4.20}$$

whence, by (4.12),(4.13),(4.17),

$$\begin{aligned} e_1(\zeta) \le \varepsilon w_1(\varepsilon\zeta) \le \varepsilon w_1(T) = \varepsilon\Big(& [B_{1,\infty}(T/\varepsilon) + C_1T]\cosh(T\sqrt{L_1L_2}) \\ & + [B_{2,\infty}(T/\varepsilon) + C_2T]\sqrt{\frac{L_1}{L_2}}\sinh(T\sqrt{L_1L_2})\Big)\ , \end{aligned} \tag{4.21}$$

$$\begin{aligned} e_2(\zeta) \le \varepsilon w_2(\varepsilon\zeta) \le \varepsilon w_2(T) = \varepsilon\Big(& [B_{1,\infty}(T/\varepsilon) + C_1T]\sqrt{\frac{L_2}{L_1}}\sinh(T\sqrt{L_1L_2}) \\ & + [B_{2,\infty}(T/\varepsilon) + C_2T]\cosh(T\sqrt{L_1L_2})\Big)\ , \end{aligned} \tag{4.22}$$

for $\zeta \in I(\varepsilon, T)$, where, at the second inequalities, we have used the fact that w_1 and w_2 are increasing (the latter follows from (4.18),(4.19),(4.20)). We thus have proven (3.125),(3.126) in Theorem 1.

We note that $\check{B}_1$ and $\check{B}_{2,1}(T)$ are finite. Also, since the Fourier series of $jj(\cdot;\nu,\Delta P_{x0})$ is absolutely convergent, we conclude from (3.131) that $\check{B}_{22}(T)$ is finite whence, by (3.129), $\check{B}_2(T,\Delta)$ is finite.

By restricting ε_0, and thus ε in (4.21),(4.22), we can keep $(\theta(\zeta,\varepsilon),\chi(\zeta,\varepsilon))$ away from the boundary of $\hat{W}(\theta_0,\chi_0,d_1,d_2)$ for $\zeta \in I(\varepsilon,T)$. In this case T/ε must be less than $\beta(\varepsilon)$ thus $I(\varepsilon,T) = [0, T/\varepsilon]$.

To complete the proof we have to show (3.127) which is the heart of the proof. Thus we have to estimate B_1, B_2. From (2.47),(3.36),(3.122) we obtain

$$\tilde{f}_1(v_2(s,\varepsilon),s) = 2\frac{q(s)-\bar{q}}{\bar{q}}v_2(s,\varepsilon) = \frac{2K^2}{\bar{q}}[2\Delta P_{x0}\cos s + \frac{1}{2}\cos(2s)]\chi_0\ ,$$

and thus, by (3.121),(3.128),

$$\begin{aligned} B_1(\zeta) &= \frac{2K^2}{\bar{q}}\Big|\int_0^\zeta [2\Delta P_{x0}\cos s + \frac{1}{2}\cos(2s)]\chi_0\, ds\Big| \\ &= \frac{2K^2|\chi_0|}{\bar{q}}|2\Delta P_{x0}\sin\zeta + \frac{1}{4}\sin(2\zeta)| \le \frac{2K^2|\chi_0|}{\bar{q}}(2|\Delta P_{x0}| + \frac{1}{4}) \\ &= \check{B}\ , \end{aligned} \tag{4.23}$$

so that, by (3.123), $B_{1,\infty}(T/\varepsilon) \le \check{B}_1$. From (3.36),(3.122),(4.4) we obtain

$$\tilde{f}_2(v_1(s,\varepsilon),s;\nu) = -\frac{K^2}{2}e^{i\nu[2\varepsilon\chi_0 s+\theta_0]}\sum_{n\in\mathbb{Z}}\widehat{jj}(n;\nu,\Delta P_{x0})e^{i(n-\nu)s} + cc\ ,$$

whence, by (3.121) and for $\zeta \in \mathbb{R}$,

$$
\begin{aligned}
B_2(\zeta) &= \frac{K^2}{2} \Big| \int_0^\zeta e^{i\nu[2\varepsilon\chi_0 s+\theta_0]} \sum_{n\in\mathbb{Z}} \widehat{jj}(n;\nu,\Delta P_{x0}) e^{i(n-\nu)s} ds + cc \Big| \\
&= \frac{K^2}{2} \Big| \sum_{n\in\mathbb{Z}} \widehat{jj}(n;\nu,\Delta P_{x0}) \int_0^\zeta e^{i\nu[2\varepsilon\chi_0 s+\theta_0]} e^{i(n-\nu)s} ds + cc \Big| \\
&\le K^2 \sum_{n\in\mathbb{Z}} |\widehat{jj}(n;\nu,\Delta P_{x0})| \, \Big| \int_0^\zeta e^{i2\varepsilon\nu\chi_0 s} e^{i(n-\nu)s} ds \Big| \ ,
\end{aligned}
\tag{4.24}
$$

where in the second equality we used the fact that the Fourier series of $jj(\cdot;\nu,\Delta P_{x0})$ is uniformly convergent. Integrating by parts gives, for $0 \le \zeta \le T/\varepsilon$,

$$
\begin{aligned}
&\Big| \int_0^\zeta e^{i2\varepsilon\nu\chi_0 s} e^{i(n-\nu)s} ds \Big| = \Big| \frac{e^{i(n-\nu+2\varepsilon\nu\chi_0)\zeta} - 1 - i2\varepsilon\nu\chi_0 \int_0^\zeta e^{i(n-\nu+2\varepsilon\nu\chi_0)s} ds}{i(n-\nu)} \Big| \\
&\le \frac{2 + 2\varepsilon\nu|\chi_0|\zeta}{|n-\nu|} \le \frac{2 + 2(k+1)|\chi_0|T}{|n-\nu|} \ ,
\end{aligned}
$$

whence, by (4.24), for $0 \le \zeta \le T/\varepsilon$,

$$
B_2(\zeta) \le 2K^2[1 + (k+1)|\chi_0|T] \sum_{n\in\mathbb{Z}} \Big| \frac{\widehat{jj}(n;\nu,\Delta P_{x0})}{n-\nu} \Big| \ . \tag{4.25}
$$

The $n-\nu$ in the denominator is the so-called small divisor problem in this context. It is easily resolved in this Δ-NR case. In fact, for ν Δ-NR, i.e., $k + \Delta \le \nu \le k + 1 - \Delta$, we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \Big| \frac{\widehat{jj}(n;\nu,\Delta P_{x0})}{n-\nu} \Big| = \frac{|\widehat{jj}(k;\nu,\Delta P_{x0})|}{|k-\nu|} \\
&+ \frac{|\widehat{jj}(k+1;\nu,\Delta P_{x0})|}{|k+1-\nu|} + \sum_{n\in(\mathbb{Z}\setminus\{k,k+1\})} \frac{|\widehat{jj}(n;\nu,\Delta P_{x0})|}{|n-\nu|} \le \frac{|\widehat{jj}(k;\nu,\Delta P_{x0})|}{\Delta} \\
&+ \frac{|\widehat{jj}(k+1;\nu,\Delta P_{x0})|}{\Delta} + \sum_{n\in(\mathbb{Z}\setminus\{k,k+1\})} |\widehat{jj}(n;\nu,\Delta P_{x0})| \ ,
\end{aligned}
$$

whence, by (3.129), (3.130),(3.131),(4.25),

$$
\begin{aligned}
B_2(\zeta) &\le 2K^2\{1 + (k+1)|\chi_0|T\}\Big\{ \frac{|\widehat{jj}(k;\nu,\Delta P_{x0})| + |\widehat{jj}(k+1;\nu,\Delta P_{x0})|}{\Delta} \\
&+ \sum_{n\in(\mathbb{Z}\setminus\{k,k+1\})} |\widehat{jj}(n;\nu,\Delta P_{x0})| \Big\} = \frac{1}{\Delta}\check{B}_{21}(T) + \check{B}_{22}(T) = \check{B}_2(T,\Delta) \ ,
\end{aligned}
\tag{4.26}
$$

so that, by (3.123), $B_{2,\infty}(T/\varepsilon) \le \check{B}_2(T,\Delta)$.

This completes the proof.

4.2 Proof of Theorem 2 (Averaging theorem in NtoR case where $\nu = k + \varepsilon a$)

The proof goes analogously to the proof of Theorem 1 in §4.1 and so we omit some details.

Thus we begin by comparing solutions of the exact IVP (3.48),(3.49)

$$\theta' = \varepsilon f_1^R(\chi, \zeta) + \varepsilon^2 g_1^R(\theta, \chi, \zeta, \varepsilon, k, a) \ , \quad \theta(0, \varepsilon) = \theta_0 \ , \tag{4.27}$$

$$\chi' = \varepsilon f_2^R(\theta, \varepsilon\zeta, \zeta; k, a) + \varepsilon^2 g_2^R(\theta, \chi, \zeta, \varepsilon, k, a) \ , \quad \chi(0, \varepsilon) = \chi_0 \ , \tag{4.28}$$

where, by (3.47),(3.50),(3.54),

$$f_1^R(\chi, \zeta) = \frac{2q(\zeta)\chi}{\bar{q}} \ , \tag{4.29}$$

$$f_2^R(\theta, \varepsilon\zeta, \zeta; k, a) = -\frac{K^2}{2} \exp(i[k\theta - a\varepsilon\zeta]) \sum_{n\in\mathbb{Z}} \widehat{\jmath}(n; k, \Delta P_{x0}) e^{i\zeta[n-k]} + cc \ , \tag{4.30}$$

with the normal form IVP of (3.58),(3.59)

$$v_1' = \varepsilon \bar{f}_1^R(v_2) \ , \quad v_1(0, \varepsilon) = \theta_0 \ , \tag{4.31}$$

$$v_2' = \varepsilon \bar{f}_2^R(v_1, \varepsilon\zeta; k) \ , \quad v_2(0, \varepsilon) = \chi_0 \ , \tag{4.32}$$

where

$$\bar{f}_1^R(v_2) = 2v_2 \ , \tag{4.33}$$

$$\bar{f}_2^R(v_1, \varepsilon\zeta; k) = -\frac{K^2}{2} \exp(i[kv_1 - a\varepsilon\zeta]) \widehat{\jmath}(k; k, \Delta P_{x0}) + cc \ . \tag{4.34}$$

Subtracting and integrating, we obtain from (3.143), (4.27),(4.28),(4.31),(4.32) that

$$\begin{aligned} \theta(\zeta) - v_1(\zeta, \varepsilon) &= \varepsilon \int_0^\zeta \big[f_1^R(\chi(s), s) - f_1^R(v_2(s, \varepsilon), s) \\ &\quad + f_1^R(v_2(s, \varepsilon), s) - \bar{f}_1^R(v_2(s, \varepsilon)) + \varepsilon g_1^R(\theta(s), \chi(s), s, \varepsilon, k, a) \big] \, ds \\ &= \varepsilon \int_0^\zeta \big[f_1^R(\chi(s), s) - f_1^R(v_2(s, \varepsilon), s) \\ &\quad + \tilde{f}_1^R(v_2(s, \varepsilon), s) + \varepsilon g_1^R(\theta(s), \chi(s), s, \varepsilon, k, a) \big] \, ds \ , \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \chi(\zeta) - v_2(\zeta, \varepsilon) &= \varepsilon \int_0^\zeta \big[f_2^R(\theta(s), \varepsilon s, s; k, a) - f_2^R(v_1(s, \varepsilon), \varepsilon s, s; k, a) \\ &\quad + f_2^R(v_1(s, \varepsilon), \varepsilon s, s; k, a) - \bar{f}_2^R(v_1(s, \varepsilon), \varepsilon s; k) + \varepsilon g_2^R(\theta(s), \chi(s), s, \varepsilon, k, a) \big] \, ds \\ &= \varepsilon \int_0^\zeta \big[f_2^R(\theta(s), \varepsilon s, s; k, a) - f_2^R(v_1(s, \varepsilon), \varepsilon s, s; k, a) \\ &\quad + \tilde{f}_2^R(v_1(s, \varepsilon), \varepsilon s, s; k, a) + \varepsilon g_2^R(\theta(s), \chi(s), s, \varepsilon, k, a) \big] \, ds \ , \end{aligned} \tag{4.36}$$

for $\zeta \in I(\varepsilon, T) = [0, T/\varepsilon] \cap [0, \beta(\varepsilon))$. Taking absolute values, applying the Lipschitz condition and defining

$$e_1(s) := |\theta(s) - v_1(s, \varepsilon)| \,, \tag{4.37}$$
$$e_2(s) := |\chi(s) - v_2(s, \varepsilon)| \,, \tag{4.38}$$

gives, by (3.139),(3.140),(3.141), (3.142),(3.144),(4.35),(4.36) for $\zeta \in I(\varepsilon, T)$,

$$\begin{aligned} 0 \le e_1(\zeta) &\le \varepsilon[L_1^R \int_0^\zeta e_2(s)ds + |\int_0^\zeta \tilde{f}_1^R(v_2(s,\varepsilon), s)ds| \\ &+\varepsilon \int_0^\zeta |g_1^R(\theta(s), \chi(s), s, \varepsilon, k, a)|ds] \le \varepsilon[L_1^R \int_0^\zeta e_2(s)ds + B_1^R(\zeta) + TC_1^R] \\ &\le \varepsilon[L_1^R \int_0^\zeta e_2(s)ds + B_{1,\infty}^R(T/\varepsilon) + TC_1^R] \,, \end{aligned} \tag{4.39}$$

$$\begin{aligned} 0 \le e_2(\zeta) &\le \varepsilon[L_2^R \int_0^\zeta e_1(s)ds + |\int_0^\zeta \tilde{f}_2^R(v_1(s,\varepsilon), \varepsilon s, s; k, a)ds| \\ &+\varepsilon \int_0^\zeta |g_2^R(\theta(s), \chi(s), s, \varepsilon, k, a)|ds] \le \varepsilon[L_2^R \int_0^\zeta e_1(s)ds + B_2^R(\zeta) + TC_2^R] \\ &\le \varepsilon[L_2^R \int_0^\zeta e_1(s)ds + B_{2,\infty}^R(T/\varepsilon) + TC_2^R] \,, \end{aligned} \tag{4.40}$$

where we also used that $I(\varepsilon, T) \subset [0, T/\varepsilon]$. Recall that L_i^R, C_i^R, B_i^R are defined in items 7,8 and 9 of the preamble to the theorem.

We are now in the same situation as in the proof of Theorem 1 since replacing L_i, C_i, B_i in (4.12),(4.13) by L_i^R, C_i^R, B_i^R results in (4.39),(4.40). Since, as shown in the proof of Theorem 1, (4.12),(4.13) entail (4.21),(4.22) we thus conclude here that (4.39),(4.40) entail:

$$\begin{aligned} e_1(\zeta) \le \varepsilon\Big(& [B_{1,\infty}^R(T/\varepsilon) + C_1 T] \cosh(T\sqrt{L_1^R L_2^R}) \\ & +[B_{2,\infty}^R(T/\varepsilon) + C_2 T]\sqrt{\frac{L_1^R}{L_2^R}} \sinh(T\sqrt{L_1^R L_2^R})\Big) \,, \end{aligned} \tag{4.41}$$

$$\begin{aligned} e_2(\zeta) \le \varepsilon\Big(& [B_{1,\infty}^R(T/\varepsilon) + C_1 T]\sqrt{\frac{L_2^R}{L_1^R}} \sinh(T\sqrt{L_1^R L_2^R}) \\ & +[B_{2,\infty}^R(T/\varepsilon) + C_2 T] \cosh(T\sqrt{L_1^R L_2^R})\Big) \,, \end{aligned} \tag{4.42}$$

for $\zeta \in I(\varepsilon, T)$. We thus have proven (3.145),(3.146).

Clearly, by (3.148), $\check{B}_1^R(T)$ is finite. Also, since $jj(\cdot; \nu, \Delta P_{x0})$ is a C^∞ function, the series on the rhs of (3.149) converges whence $\check{B}_2^R(T)$ is also finite.

By restricting ε_0, and thus ε in (4.41),(4.42), we can keep $(\theta(\zeta, \varepsilon), \chi(\zeta, \varepsilon))$ away from the boundary of $\hat{W}(\theta_0, \chi_0, d_1, d_2)$ for $\zeta \in I(\varepsilon, T)$. In this case T/ε must be less than $\beta(\varepsilon)$ thus $I(\varepsilon, T) = [0, T/\varepsilon]$.

To complete the proof we have to show (3.147). Thus we have to estimate B_1^R, B_2^R and beginning with B_1^R we conclude from (2.47),(3.143),(4.29), (4.33) that, for $\zeta \in \mathbb{R}$,

$$\tilde{f}_1^R(v_2(s,\varepsilon), s) = 2\frac{q(s) - \bar{q}}{\bar{q}} v_2(s,\varepsilon)$$

$$= \frac{2K^2}{\bar q}[2\Delta P_{x0}\cos s + \frac{1}{2}\cos(2s)]v_2(s,\varepsilon)\ ,$$

whence, by (3.82), (3.142),(3.148), (4.32),(4.34) for $0 \le \zeta \le T/\varepsilon$,

$$\begin{aligned}
B_1^R(\zeta) &= \frac{2K^2}{\bar q}\Big|\int_0^\zeta [2\Delta P_{x0}\cos s + \frac{1}{2}\cos(2s)]v_2(s,\varepsilon)\, ds\Big| \\
&= \frac{2K^2}{\bar q}\Big|[2\Delta P_{x0}\sin\zeta + \frac{1}{4}\sin(2\zeta)]v_2(\zeta,\varepsilon) \\
&\quad - \int_0^\zeta [2\Delta P_{x0}\sin s + \frac{1}{4}\sin(2s)]\frac{dv_2}{ds}(s,\varepsilon)ds\Big| \\
&= \frac{2K^2}{\bar q}\Big|[2\Delta P_{x0}\sin\zeta + \frac{1}{4}\sin(2\zeta)]v_2(\zeta,\varepsilon) \\
&\quad +\varepsilon K^2\widehat{jj}(k;k,\Delta P_{x0})\int_0^\zeta [2\Delta P_{x0}\sin s + \frac{1}{4}\sin(2s)]\cos\Big(kv_1(s,\varepsilon)-\varepsilon as\Big)ds\Big| \\
&\le \frac{2K^2}{\bar q}\Big([2|\Delta P_{x0}| + \frac{1}{4}]|v_2(\zeta,\varepsilon)| \\
&\quad +\varepsilon K^2\big|\widehat{jj}(k;k,\Delta P_{x0})\big|[2|\Delta P_{x0}| + \frac{1}{4}]\zeta\Big) \\
&\le \frac{2K^2}{\bar q}[2|\Delta P_{x0}| + \frac{1}{4}]\Big(|v_2(\zeta,\varepsilon)| + K^2\varepsilon\zeta|\widehat{jj}(k;k,\Delta P_{x0})|\Big) \\
&\le \frac{2K^2}{\bar q}[2|\Delta P_{x0}| + \frac{1}{4}]\Big(\chi_\infty(\theta_0,\chi_0,k,a) \\
&\quad +K^2T\big|\widehat{jj}(k;k,\Delta P_{x0})\big|\Big) = \check B_1^R(T)\ ,
\end{aligned} \tag{4.43}$$

so that, by (3.144), $B_{1,\infty}^R(T/\varepsilon) \le \check B_1^R(T)$ which proves (3.147) for $i=1$. The key step here is the integration by parts at the second equality which makes explicit the slowly varying nature of v_2 by pulling out the explicit ε after the third equality.

To prove (3.147) for $i=2$ we conclude from (3.143),(4.30), (4.34) that, for $\zeta \in \mathbb{R}$,

$$\tilde f_2^R(v_1(s,\varepsilon),\varepsilon s,s;k,a) = -\frac{K^2}{2}e^{i[kv_1(s,\varepsilon)-\varepsilon as]}\sum_{n\in\mathbb{Z}\setminus\{k\}}\widehat{jj}(n;k,\Delta P_{x0})e^{i(n-k)s} + cc\ ,$$

whence, by (3.142) for $\zeta \in \mathbb{R}$,

$$\begin{aligned}
B_2^R(\zeta) &= \frac{K^2}{2}\Big|\int_0^\zeta e^{i[kv_1(s,\varepsilon)-\varepsilon as]}\sum_{n\in\mathbb{Z}\setminus\{k\}}\widehat{jj}(n;k,\Delta P_{x0})e^{i(n-k)s}ds + cc\Big| \\
&\le K^2\sum_{n\in\mathbb{Z}\setminus\{k\}}\big|\widehat{jj}(n;k,\Delta P_{x0})\big|\ \Big|\int_0^\zeta e^{i[kv_1(s,\varepsilon)-\varepsilon as]}e^{i(n-k)s}ds\Big|\ ,
\end{aligned} \tag{4.44}$$

where in the inequality we used the fact that the Fourier series of $jj(\cdot;k,\Delta P_{x0})$ is uniformly convergent. Integrating by parts gives, by (3.82), (4.31),(4.33) for $0 \le \zeta \le T/\varepsilon$,

$$|\int_0^\zeta e^{i[kv_1(s,\varepsilon)-\varepsilon as]}e^{i(n-k)s}ds| = \Big|\frac{1}{i(n-k)}\Big[e^{i[kv_1(\zeta,\varepsilon)-\varepsilon a\zeta]}e^{i(n-k)\zeta} - e^{ik\theta_0}$$

$$
\begin{aligned}
&- \int_0^\zeta i(k\frac{dv_1}{ds}(s,\varepsilon) - \varepsilon a) e^{i[kv_1(s,\varepsilon)-\varepsilon as]} e^{i(n-k)s} ds \Big] \Big| \\
&\le \frac{1}{|n-k|} \Big[2 + \int_0^\zeta (k|\frac{dv_1}{ds}(s,\varepsilon)| + \varepsilon|a|) ds \Big] \\
&\le \frac{1}{|n-k|} \Big[2 + \varepsilon \int_0^\zeta (2k|v_2(s,\varepsilon)| + |a|) ds \Big] \\
&\le \frac{1}{|n-k|} \Big(2 + \varepsilon\zeta \Big[|a| + 2k\chi_\infty(\theta_0,\chi_0,k,a) \Big] \Big) \\
&\le \frac{1}{|n-k|} \Big(2 + T \Big[|a| + 2k\chi_\infty(\theta_0,\chi_0,k,a) \Big] \Big) ,
\end{aligned}
$$

whence, by (3.149),(4.44) for $0 \le \zeta \le T/\varepsilon$,

$$
\begin{aligned}
B_2^R(\zeta) &\le K^2 \Big(2 + T \left[|a| + 2k\chi_\infty(\theta_0,\chi_0,k,a) \right] \Big) \\
&\times \sum_{n\in\mathbb{Z}\setminus\{k\}} \frac{|\widehat{jj}(n;k,\Delta P_{x0})|}{|n-k|} = \check{B}_2^R(T) ,
\end{aligned} \tag{4.45}
$$

so that, by (3.144), $B_{2,\infty}^R(T/\varepsilon) \le \check{B}_2^R(T)$. This completes the proof.

5 Summary and future work

We started with the 6D Lorentz equations for a planar undulator in (2.7),(2.16)-(2.18) with time as the independent variable. In §2.2 we introduced z as the independent variable and considered the IVP at $z = 0$ with $y_0 = p_{y0} = 0$. Solutions of this system are completely determined by the solutions of our basic 2D system (2.33),(2.34) for α and γ. This basic 2D system is the starting point for the rest of the paper and the first step is to transform it into a form for first-order averaging; the subject of §2.3. We introduce $\zeta = k_u z$ as the new independent variable, and χ as a new dependent variable by $\gamma = \gamma_c(1 + \varepsilon\chi)$. Here we are thinking of electrons as part of an electron bunch with γ_c as a characteristic value of γ and ε as a measure of the energy spread so that χ is an $O(1)$ variable. We thus arrive at the system for (θ_{aux}, χ) given in (2.41),(2.42) and we are interested, in this FEL application, in an asymptotic analysis for ε and $1/\gamma_c$ small. Expanding the vector field for (2.41),(2.42) gives (2.50),(2.51). Here θ_{aux} is not slowly varying and we thus introduce the generalized ponderomotive phase, θ, in (2.52) which leads to the slowly varying form of (2.56),(2.57). Most importantly, we discover that in order for θ and χ to interact at first order we must have $\varepsilon = O(1/\gamma_c)$ and without loss of generality we take (1.5) as a result of (2.58). Finally we obtain (2.62),(2.63) which is in a standard form for the MoA. Consequently this will lead to a pendulum type behavior which is central to the operation of an FEL.

The MoA can be applied to (2.62),(2.63) after an appropriate h is defined and the rest of the paper, in Sections 3,4, focuses on the monochromatic case of (2.15).

Before continuing with the summary we note that in the collective case there is a continuous range of frequencies and so it is natural to ask, "what happens in the noncollective case considered in this paper if there is a continuous range of frequencies?". Here h can be modeled as in (2.78), i.e.,

$$h(\alpha) = \int_{-\infty}^{\infty} \tilde{h}(\xi) \exp(-i\xi\alpha) d\xi \ . \tag{5.1}$$

In the nonsmooth monochromatic case $\tilde{h}(\xi) = [\delta(\xi - \nu) + \delta(\xi + \nu)]/2$ and (5.1) gives $h(\alpha) = \cos(\nu\alpha)$ as in the monochromatic case of (2.15), and, as we have discussed in §3, there are resonances for integer ν. However we have found that in the smooth case the average of $(\cos\zeta + \Delta P_{x0}) h(\theta - Q(\zeta))$ is zero and so the averaging normal form for (2.62),(2.63) is just the NR normal form of §3.3. Thus a smooth $\tilde{h}(\xi)$, localized near the $\nu = 1$ monochromatic resonance, washes out the effect of that resonance in the first-order averaging normal form. This does not mean that there is no resonant behavior near $\nu = 1$ because it may not be possible to prove an averaging theorem. We are pursuing this. Furthermore even if an averaging theorem can be proven there might still be an effect in second-order averaging.

In §3 we begin by determining the $O(\varepsilon^2)$ terms of (2.62),(2.63) using (2.72),(2.73). Thus we obtain (3.10)-(3.15) as our basic system for θ, χ. Proposition 1 gives a domain, $W(\varepsilon_0) \times \mathbb{R}$, on which g_1, g_2 are well defined as well as their limits as $\varepsilon \to 0+$. In particular the vector field in (3.10),(3.11) is well defined on $W(\varepsilon_0) \times \mathbb{R}$.

Eq.'s (3.10),(3.11) are in a standard form for the MoA and for each ν the normal form is obtained by dropping the $O(\varepsilon^2)$ terms and averaging f_1, f_2 over ζ. However the average of f_2 is not clear from (3.13) and it is convenient to expand it in a Fourier series which is given in (3.26)-(3.28). The average is then easily obtained in (3.30) and leads to the definition of NR, Δ-NR, resonant and NtoR ν. The NR normal form equations are $\theta' = \varepsilon 2\chi$ and $\chi' = 0$ and the resonant normal form equations are given by (3.31). The NR case is stated precisely in §3.3. Instead of focusing on the resonant case of (3.31) we consider in §3.4 the more general NtoR case where we study the dynamics in neighborhoods of the $\nu = k$ resonances. If the neighborhood is too small then the resonant normal form of (3.31) will be dominant thus the natural neighborhood to study with first-order averaging is $O(\varepsilon)$ and this is the content of §3.4. Replacing ν by $k + \varepsilon a$, our basic equations (3.10),(3.11) are rewritten in (3.42),(3.43). The function f_2 in (3.43) has two ε dependencies one of which contributes to the $O(\varepsilon^2)$ term and we are led to the basic NtoR system (3.48)-(3.52). Proposition 2 is analogous to Proposition 1 by giving us the domain $W(\varepsilon_0) \times \mathbb{R}$ on which g_1^R, g_2^R are well behaved as well as their limits as $\varepsilon \to 0+$. In particular the vector field in (3.48),(3.49) is well defined on $W(\varepsilon_0) \times \mathbb{R}$. In §3.4.2 the NtoR normal form is presented in (3.58),(3.59). The solution structure is conveniently illuminated, in terms of the simple pendulum system, in §3.4.3. The simple pendulum exhibits four types of behavior and these are exploited to discuss the structure of solutions of (3.58),(3.59) in these four cases.

At this stage we have normal forms for $\nu \in [k + \Delta, k + 1 - \Delta]$ and $\nu = k + \varepsilon a$. However there may be gaps between the dynamics covered by the Δ-NR normal form and that of the NtoR normal form. So it is comforting to note that there is a link between the two dynamical behaviors in that the NtoR normal form is approximated by the NR normal form far away from the pendulum buckets as discussed in §3.4.4.

In §3.5 we state the two averaging theorems which relate the Δ-NR and NtoR normal form approximations to the corresponding exact systems. Each theorem has a detailed preamble which sets up a compact statement of the theorem. The theorems establish the main results of the paper. Namely that the normal form solutions give an $O(\varepsilon)$ approximation to the exact solutions on long time, $O(1/\varepsilon)$, intervals. In the Δ-NR case, the ν interval can be made larger by making Δ smaller but this is at the expense of increasing the error as discussed in Remark

(1) of §3.5.3.

The results of the theorems are applied in §3.6, where the normal form approximations are used to derive the approximate solutions of the Lorentz equations with z as the independent variable. In §3.7 we discuss the small gain theory for $\nu = k + \varepsilon a$ based on our NtoR normal form and compare it with the standard theory for $k = 1,\ a = 0$. We do point out however, that we have not justified the low gain theory in the context of our NtoR averaging theorem as we mention at the end of §3.7.

Finally the proofs are given in §4. It can be seen that the proofs themselves are quite simple. The proofs are somewhat novel in that they do not use a near identity transformation, due to the Besjes approach, and they use a system of differential inequalities in the calculation of the error bounds, rather than a Gronwall type inequality, which leads to better error bounds. Therefore a solution of the system of differential inequalities is presented and verified in Appendix I. The first theorem, which is stated for the Δ-NR case, is an example of a quasiperiodic averaging theorem with its concomitant small divisor problem. It's inherently interesting in that the small divisor problem arises in what must be the simplest possible way. We develop the general theory of quasiperiodic averaging in [14]. The second theorem, which is stated for the NtoR case, is an example of periodic averaging which has a vast literature, however as mentioned above our approach here is novel. While the proofs of Theorems 1 and 2 are simple the whole application of the MoA is not. There was considerable work to put the problem into the standard form and considerable effort to calculate the bounds on g_1, g_2 in Appendix C and g_1^R, g_2^R in Appendix E as well as their $\varepsilon = 0$ limits in Appendixes B and D.

We now comment on future work. First of all it would be interesting to include the y dynamics using (2.12) as we do, but not assuming the zero initial conditions in y, thus treating the full 3D dynamics.

Secondly, it would be interesting to study the helical undulator as we have done here for the planar undulator, i.e., via first-order averaging.

Thirdly, the work here sets the stage for a second-order averaging study of the NR case in (3.10),(3.11) using (3.39),(3.40) and the NtoR case in (3.48),(3.49) using (3.56),(3.57). In both cases we have systems of the form

$$\frac{dU}{dt} = \varepsilon F(U,t) + \varepsilon^2 G(U,t) + O(\varepsilon^3)\ , \tag{5.2}$$

with approximating normal form given by

$$\frac{dV}{dt} = \varepsilon \bar{F}(V) + \varepsilon^2 \hat{G}(V)\ , \tag{5.3}$$

where $\bar{F}$ is the t-average of F and $\hat{G}$ is a linear combination of the t-average of G and terms depending on F (See [25, Section 5, p.610] for a construction of the normal form, i.e., $\hat{G}$, and an associated theorem and proof). Such a study would include a computation of the averages from (3.39),(3.40) and (3.56),(3.57) and then a phase plane analysis of this second order normal form system including a comparison with our first-order normal form system. In addition averaging theorems could be proven which we anticipate will give an $O(\varepsilon^2)$ error on $[0, T/\varepsilon]$ as in [25]. Furthermore, it would be interesting to see what happens in the NR case, e.g., is the energy deviation χ still conserved. We note that generically second-order averaging gives a better error estimate but the interval of validity remains the same (See [25] for situations where the time interval can be extended). Finally it would be interesting to know if, in the NtoR case, there is a breakdown in the integrability of the NtoR normal form due to separatrix splitting, [30], with the concomitant chaotic behavior. This is a delicate issue, which cannot be studied with second-order averaging, since (5.3) is a second order autonomous system and as such it cannot exhibit chaos as pointed out at the end of §3.4.3. This work could be a possible future project, however it does not appear to be interesting from the application point of view since collective effects are surely more important than noncollective effects at second order.

Fourthly, we are therefore eager to move on to the collective case based in part on our understanding here. As a first step we are studying the consequence of (H.1)-(H.6). We have not seen this form of the solution of the 1D wave equation in the FEL literature although the first equality in (H.3) is derived in many elementary PDE books. In addition, we are pursuing the issue raised in the paragraph containing Eq. (5.1), concerning a smooth $\tilde{h}$.

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Table of notation

a	(3.41)
B_1, B_2	(3.121)
B_1^R, B_2^R	(3.142)
$\mathcal{D}(\varepsilon, \nu)$	(3.6)
$\mathcal{E}$	(1.4)
f_1, f_2	(3.12), (3.13)
f_1^R, f_2^R	(3.47), (3.50)
g_1, g_2	(3.14), (3.15)
g_1^R, g_2^R	(3.51), (3.52)
h, H	(2.15)
$jj, \hat{jj}$	(3.23), (3.25)
K	(1.1)
K_r	(1.3)
K_0	(3.60)
MoA	Method of Averaging
NR (nonresonant)	Definition 1 (§3.2)
NtoR (near − to − resonant)	Definition 1 (§3.2)
$\mathbb{N}$	Set of positive integers
P_x, P_z	(2.37)
$q, \bar{q}, Q$	(2.47), (2.48), (2.53)
$W(\varepsilon), \hat{W}, \hat{W}_R$	(3.19), (3.113), (3.133)
$\mathbb{Z}$	Set of integers
$\breve{\alpha}, \alpha$	(2.14), (2.26)
γ_c	(2.36)
Δ	Definition 1 (§3.2)
Δ − NR (Δ − nonresonant)	Definition 1 (§3.2)
ΔP_{x0}	(2.45)
ε	(1.5)
ζ	(2.39)
η	(2.36)
θ_{aux}, θ	(2.40), (2.52)
$\Pi_x, \Pi_z, \Pi_{x,ub}, \Pi_{z,lb}$	(3.2), (3.3), (3.17), (C.15)
Υ_0, Υ_1	(2.54)
$\chi, \chi_{lb}(\varepsilon)$	(2.36), (3.20)
Ω	(3.69)

Appendix

A The Bessel expansion

Here we derive the Bessel expansion (3.27) of $jj(\cdot; \nu, \Delta P_{x0})$. In fact by (3.23)

$$\begin{aligned} jj(\zeta; \nu, \Delta P_{x0}) &= (\cos\zeta + \Delta P_{x0}) \exp(-i\nu\Upsilon_0 \sin\zeta) \exp(-i\nu\Upsilon_1 \sin 2\zeta) \\ &= \frac{1}{2} jj_1(\zeta) + \frac{1}{2} jj_{-1}(\zeta) + \Delta P_{x0} jj_0(\zeta) \ , \end{aligned} \tag{A.1}$$

where

$$jj_m(\zeta) := \exp(im\zeta)\exp(-i\nu[\Upsilon_0 \sin\zeta + \Upsilon_1 \sin 2\zeta]) \ . \tag{A.2}$$

Now

$$\exp(ix\sin\theta) = \sum_{n\in\mathbb{Z}} J_n(x)\exp(in\theta) \ , \quad J_{-n}(x) = (-1)^n J_n(x) \ , \tag{A.3}$$

whence, by (A.2),

$$\begin{aligned} jj_m(\zeta) &= e^{im\zeta} e^{-i\nu\Upsilon_0 \sin\zeta} e^{-i\nu\Upsilon_1 \sin 2\zeta} \\ &= e^{im\zeta}[\sum_{k\in\mathbb{Z}} J_k(\nu\Upsilon_1) e^{-i2k\zeta}]\,[\sum_{l\in\mathbb{Z}} J_l(\nu\Upsilon_0) e^{-il\zeta}] \\ &= \sum_{k,l\in\mathbb{Z}} J_l(\nu\Upsilon_0) J_k(\nu\Upsilon_1) e^{i(m-l-2k)\zeta} \\ &= \sum_{n\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} J_{m-n-2k}(\nu\Upsilon_0) J_k(\nu\Upsilon_1) \Big) e^{in\zeta} \ . \end{aligned} \tag{A.4}$$

Let

$$\mathcal{J}(n,m,\nu,\Upsilon_0,\Upsilon_1) := \sum_{k\in\mathbb{Z}} J_{m-n-2k}(\nu\Upsilon_0) J_k(\nu\Upsilon_1) \ , \tag{A.5}$$

then, by (A.4),

$$jj_m(\zeta) = \sum_{n\in\mathbb{Z}} \mathcal{J}(n,m,\nu,\Upsilon_0,\Upsilon_1) e^{in\zeta} \ , \tag{A.6}$$

and thus, by (A.1),

$$\begin{aligned} jj(\zeta;\nu,\Delta P_{x0}) = \sum_{n\in\mathbb{Z}} \Big(&\frac{1}{2}\mathcal{J}(n,1,\nu,\Upsilon_0,\Upsilon_1) + \frac{1}{2}\mathcal{J}(n,-1,\nu,\Upsilon_0,\Upsilon_1) \\ &+\Delta P_{x0}\mathcal{J}(n,0,\nu,\Upsilon_0,\Upsilon_1) \Big) e^{in\zeta} \ , \end{aligned} \tag{A.7}$$

whence, by (3.25),

$$\begin{aligned} \widehat{jj}(n;\nu,\Delta P_{x0}) = &\frac{1}{2}\mathcal{J}(n,1,\nu,\Upsilon_0,\Upsilon_1) + \frac{1}{2}\mathcal{J}(n,-1,\nu,\Upsilon_0,\Upsilon_1) \\ &+\Delta P_{x0}\mathcal{J}(n,0,\nu,\Upsilon_0,\Upsilon_1) \ , \end{aligned} \tag{A.8}$$

so that indeed (3.27) holds.

It is useful for the discussion after Definition 1 to have the following special case. We have, by (A.8),

$$\widehat{jj}(k;k,0) = \frac{1}{2}[\mathcal{J}(k,1,k,0,\Upsilon_1) + \mathcal{J}(k,-1,k,0,\Upsilon_1)] \ , \tag{A.9}$$

where

$$\mathcal{J}(k,1,k,0,\Upsilon_1) = \sum_{k'\in\mathbb{Z}} J_{1-k-2k'}(0) J_{k'}(k\Upsilon_1)$$

$$= \begin{cases} J_{(1-k)/2}(k\Upsilon_1) & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \,, \end{cases} \tag{A.10}$$

$$\mathcal{J}(k,-1,k,0,\Upsilon_1) = \sum_{k'\in\mathbb{Z}} J_{-1-k-2k'}(0)J_{k'}(k\Upsilon_1)$$

$$= \begin{cases} J_{-(1+k)/2}(k\Upsilon_1) & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \,. \end{cases} \tag{A.11}$$

Thus from (A.9) $\widehat{jj}(k;k,0) = 0$ for k even and, for $k = 2n+1$ with $n \in \mathbb{Z}$,

$$\widehat{jj}(2n+1;2n+1,0) = \frac{1}{2}[J_{-n}((2n+1)\Upsilon_1) + J_{-(n+1)}((2n+1)\Upsilon_1)]$$
$$= \frac{1}{2}(-1)^n[J_n((2n+1)\Upsilon_1) - J_{n+1}((2n+1)\Upsilon_1)] \,. \tag{A.12}$$

B Limit of g_1, g_2

Let $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon_0 \in (0,1]$, let $\nu \in [1/2,\infty)$ and let $(\theta,\chi,\zeta) \in W(\varepsilon_0)\times\mathbb{R}$. In this appendix we will prove the properties (B.5), (B.8),(B.12),(B.13) of g_1 and g_2. The properties (B.8),(B.13) are used in the proof of Proposition 1. Furthermore the properties (B.5),(B.12) will be used in Appendix C. Since all assumptions of this appendix are also satisfied in Appendix B, we can apply the results of Appendix B.

We first consider g_1. Note that, by (2.47),(3.2),

$$\begin{aligned} 1 + K^2\Pi_x^2(\theta,\zeta,\varepsilon,\nu) &= q(\zeta) \\ &+ \frac{\varepsilon^2K^2\bar{q}}{2\nu}\Big(\sin(\nu[\theta - Q(\zeta)]) - \sin(\nu\theta_0)\Big)\Big(2(\cos\zeta + \Delta P_{x0}) \\ &\quad + \frac{\varepsilon^2\bar{q}}{2\nu}(\sin(\nu[\theta - Q(\zeta)]) - \sin(\nu\theta_0))\Big) \,. \end{aligned} \tag{B.1}$$

We obtain from (3.14) that

$$\varepsilon^2 g_1(\theta,\chi,\zeta;\varepsilon,\nu) = \frac{2\mathcal{E}}{\varepsilon^2\bar{q}}(1 - \frac{1}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)}) + \frac{q(\zeta)}{\bar{q}}(1-2\varepsilon\chi) \,,$$

whence

$$\begin{aligned} \frac{1}{2\mathcal{E}}\bar{q}\Pi_z(\Pi_z+1)\varepsilon^4 g_1 &= \Pi_z^2 - 1 + \frac{1}{2\mathcal{E}}q\Pi_z(\Pi_z+1)\varepsilon^2(1-2\varepsilon\chi) \\ &= \frac{1}{(1+\varepsilon\chi)^2}\Big(-\frac{\varepsilon^2}{\mathcal{E}}(q+\varepsilon^2\kappa_1) + \frac{1}{2\mathcal{E}}q\Pi_z(\Pi_z+1)\varepsilon^2(1+\varepsilon\chi)^2(1-2\varepsilon\chi)\Big) \,, \end{aligned} \tag{B.2}$$

where we used from (3.3),(B.1) the fact that

$$\Pi_z^2(\theta,\chi,\zeta,\varepsilon,\nu) - 1 = -\frac{\varepsilon^2}{\mathcal{E}(1+\varepsilon\chi)^2}\Big(q(\zeta) + \varepsilon^2\kappa_1(\theta,\zeta,\varepsilon,\nu)\Big) \,, \tag{B.3}$$

with

$$\kappa_1(\theta,\zeta,\varepsilon,\nu) := \frac{K^2\bar{q}}{2\nu}\Big(\sin(\nu[\theta - Q(\zeta)]) - \sin(\nu\theta_0)\Big)\Big(2(\cos\zeta + \Delta P_{x0})$$

$$+\frac{\varepsilon^2\bar{q}}{2\nu}(\sin(\nu[\theta - Q(\zeta)]) - \sin(\nu\theta_0))\Big) \,. \tag{B.4}$$

Clearly, by (B.2),(B.3),

$$\begin{aligned}
&\frac{1}{2\mathcal{E}}\bar{q}\Pi_z(\Pi_z+1)\varepsilon^4 g_1 \\
&\quad = -\frac{\varepsilon^2 q}{\mathcal{E}(1+\varepsilon\chi)^2}\left(1 - \frac{1}{2}\Pi_z(\Pi_z+1)(1-3\varepsilon^2\chi^2 - 2\varepsilon^3\chi^3)\right) - \frac{\varepsilon^4\kappa_1}{\mathcal{E}(1+\varepsilon\chi)^2} \\
&\quad = -\frac{\varepsilon^2 q}{\mathcal{E}(1+\varepsilon\chi)^2}\left(-\frac{1}{2}(\Pi_z-1)(\Pi_z+2) + \frac{1}{2}\Pi_z(\Pi_z+1)(3\varepsilon^2\chi^2 + 2\varepsilon^3\chi^3)\right) \\
&\qquad -\frac{\varepsilon^4\kappa_1}{\mathcal{E}(1+\varepsilon\chi)^2} \,,
\end{aligned}$$

whence

$$\begin{aligned}
&\frac{1}{2\mathcal{E}}\bar{q}\Pi_z(\Pi_z+1)^2\varepsilon^4 g_1 \\
&\quad = -\frac{\varepsilon^2 q}{2\mathcal{E}(1+\varepsilon\chi)^2}\left(-(\Pi_z^2-1)(\Pi_z+2) + \varepsilon^2\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)\right) \\
&\qquad -\frac{\varepsilon^4\kappa_1}{\mathcal{E}(1+\varepsilon\chi)^2} = -\frac{\varepsilon^2 q}{2\mathcal{E}(1+\varepsilon\chi)^4}\left(\frac{\varepsilon^2}{\mathcal{E}}(q+\varepsilon^2\kappa_1)(\Pi_z+2)\right. \\
&\qquad \left.+\varepsilon^2\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\right) - \frac{\varepsilon^4\kappa_1}{\mathcal{E}(1+\varepsilon\chi)^2} \\
&\quad = -\frac{\varepsilon^2 q}{2\mathcal{E}(1+\varepsilon\chi)^4}\left(\frac{\varepsilon^2}{\mathcal{E}}q(\Pi_z+2) + \varepsilon^2\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\right) \\
&\qquad -\frac{\varepsilon^6 q(\Pi_z+2)\kappa_1}{2\mathcal{E}^2(1+\varepsilon\chi)^4} - \frac{\varepsilon^4\kappa_1}{\mathcal{E}(1+\varepsilon\chi)^2} \\
&\quad = -\frac{\varepsilon^2 q}{2\mathcal{E}(1+\varepsilon\chi)^4}\left(\frac{\varepsilon^2}{\mathcal{E}}q(\Pi_z+2) + \varepsilon^2\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\right) \\
&\qquad -\frac{\varepsilon^4\kappa_1}{2\mathcal{E}(1+\varepsilon\chi)^4}\left(2(1+\varepsilon\chi)^2 + \frac{\varepsilon^2}{\mathcal{E}}q(\Pi_z+2)\right) \,,
\end{aligned}$$

so that

$$\begin{aligned}
&\bar{q}\Pi_z(\Pi_z+1)^2 g_1 \\
&\quad = -\frac{q}{(1+\varepsilon\chi)^4}\left(\frac{q}{\mathcal{E}}(\Pi_z+2) + \Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\right) \\
&\qquad -\frac{\kappa_1}{(1+\varepsilon\chi)^4}\left(2(1+\varepsilon\chi)^2 + \frac{\varepsilon^2 q}{\mathcal{E}}(\Pi_z+2)\right) \,,
\end{aligned}$$

i.e.,

$$\begin{aligned}
g_1(\theta,\chi,\zeta;\varepsilon,\nu) &= -\frac{q}{\bar{q}\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\left(\frac{q}{\mathcal{E}}(\Pi_z+2)\right. \\
&\quad \left.+\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\right)
\end{aligned}$$

$$
-\frac{\kappa_1}{\bar{q}\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\Bigg(2(1+\varepsilon\chi)^2+\frac{\varepsilon^2 q}{\mathcal{E}}(\Pi_z+2)\Bigg)\ . \tag{B.5}
$$

Clearly, by (3.3),(B.4),

$$
\lim_{\varepsilon\to 0+}\ [\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)] = 1\ , \tag{B.6}
$$

$$
\lim_{\varepsilon\to 0+}\ [\kappa_1(\chi,\zeta,\varepsilon,\nu)] = \frac{K^2\bar{q}}{\nu}\Bigg(\sin(\nu[\theta-Q(\zeta)])-\sin(\nu\theta_0)\Bigg)(\cos\zeta+\Delta P_{x0})\ , \tag{B.7}
$$

whence, by (B.5),

$$
\begin{aligned}
\lim_{\varepsilon\to 0+}\ [g_1(\theta,\chi,\zeta;\varepsilon,\nu)] &= -\frac{q(\zeta)}{4\bar{q}}(\frac{3}{\mathcal{E}}q(\zeta)+12\chi^2) \\
&-\frac{K^2}{2\nu}\Bigg(\sin(\nu[\theta-Q(\zeta)])-\sin(\nu\theta_0)\Bigg)(\cos\zeta+\Delta P_{x0})\ .
\end{aligned} \tag{B.8}
$$

We now consider g_2 and we obtain from (3.15) that

$$
\begin{aligned}
\varepsilon^2 g_2(\theta,\chi,\zeta;\varepsilon,\nu) = \varepsilon K^2\cos(\nu[\theta-Q(\zeta)])\Bigg(&\cos\zeta+\Delta P_{x0} \\
&-\frac{1}{1+\varepsilon\chi}\frac{\Pi_x(\theta,\zeta,\varepsilon,\nu)}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)}\Bigg)\ ,
\end{aligned}
$$

whence

$$
\begin{aligned}
\Pi_z(1+\varepsilon\chi)\varepsilon g_2 &= K^2\cos(\nu[\theta-Q(\zeta)])\Bigg((1+\varepsilon\chi)\Pi_z(\cos\zeta+\Delta P_{x0})-\Pi_x\Bigg) \\
&= K^2\cos(\nu[\theta-Q(\zeta)])\Bigg((\cos\zeta+\Delta P_{x0})[(1+\varepsilon\chi)\Pi_z-1]-\varepsilon^2\kappa_2\Bigg)\ ,
\end{aligned} \tag{B.9}
$$

where we used from (3.2) the fact that

$$
\Pi_x(\theta,\zeta,\varepsilon,\nu) = \cos\zeta+\Delta P_{x0}+\varepsilon^2\kappa_2(\theta,\zeta,\nu)\ , \tag{B.10}
$$

with

$$
\kappa_2(\theta,\zeta,\nu) := \frac{\bar{q}}{2\nu}[\sin(\nu[\theta-Q(\zeta)])-\sin(\nu\theta_0)]\ . \tag{B.11}
$$

Clearly, by (B.9),

$$
\Pi_z(1+\varepsilon\chi)\varepsilon g_2 = K^2\cos(\nu[\theta-Q(\zeta)])\Bigg((\cos\zeta+\Delta P_{x0})[\Pi_z-1+\varepsilon\chi\Pi_z]-\varepsilon^2\kappa_2\Bigg)\ ,
$$

whence, by (B.3),

$$
\begin{aligned}
&(\Pi_z+1)\Pi_z(1+\varepsilon\chi)\varepsilon g_2 \\
&\quad = K^2\cos(\nu[\theta-Q(\zeta)])\Bigg((\cos\zeta+\Delta P_{x0})[\Pi_z^2-1+\varepsilon\chi\Pi_z(\Pi_z+1)]
\end{aligned}
$$

$$
\begin{aligned}
&-\varepsilon^2\kappa_2(\Pi_z+1)\Big)\\
&= K^2\cos(\nu[\theta-Q(\zeta)])\Big((\cos\zeta+\Delta P_{x0})[-\frac{\varepsilon^2}{\mathcal{E}(1+\varepsilon\chi)^2}(q+\varepsilon^2\kappa_1)\\
&\quad +\varepsilon\chi\Pi_z(\Pi_z+1)]-\varepsilon^2\kappa_2(\Pi_z+1)\Big)\ ,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^3\varepsilon g_2\\
&\quad = K^2\cos(\nu[\theta-Q(\zeta)])\Big((\cos\zeta+\Delta P_{x0})[-\frac{\varepsilon^2}{\mathcal{E}}(q+\varepsilon^2\kappa_1)\\
&\qquad +\varepsilon\chi\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^2]-\varepsilon^2\kappa_2(\Pi_z+1)(1+\varepsilon\chi)^2\Big)\ ,
\end{aligned}
$$

which entails that

$$
\begin{aligned}
&\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^3 g_2\\
&\quad = K^2\cos(\nu[\theta-Q(\zeta)])\Big((\cos\zeta+\Delta P_{x0})[-\frac{\varepsilon}{\mathcal{E}}(q+\varepsilon^2\kappa_1)\\
&\qquad +\chi\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^2]-\varepsilon\kappa_2(\Pi_z+1)(1+\varepsilon\chi)^2\Big)\ ,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
g_2(\theta,\chi,\zeta;\varepsilon,\nu) &= \frac{K^2\cos(\nu[\theta-Q(\zeta)])}{\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^3}\Big((\cos\zeta+\Delta P_{x0})[-\frac{\varepsilon}{\mathcal{E}}(q(\zeta)+\varepsilon^2\kappa_1)\\
&\quad +\chi\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^2]-\varepsilon\kappa_2(\Pi_z+1)(1+\varepsilon\chi)^2\Big)\ . \qquad \text{(B.12)}
\end{aligned}
$$

Clearly, by (B.6),(B.12),

$$
\lim_{\varepsilon\to 0+}[g_2(\theta,\chi,\zeta;\varepsilon,\nu)] = \chi K^2\cos(\nu[\theta-Q(\zeta)])(\cos\zeta+\Delta P_{x0})\ . \qquad \text{(B.13)}
$$

C Bounds on g_1, g_2

Let $\varepsilon\in(0,\varepsilon_0]$ with $\varepsilon_0\in(0,1]$, let $\nu\in[1/2,\infty)$ and let $(\theta_0,\chi_0)\in W(\varepsilon_0)$. Let also

$$
\chi_{lb}(\varepsilon_0) < -\chi_M\ , \qquad \text{(C.1)}
$$

where χ_M is the positive constant from Theorem 1 (see item 2 of the setup list for Theorem 1). We also assume that

$$
(\theta,\chi,\zeta)\in\mathbb{R}\times(\chi_0-d_2,\chi_0+d_2)\times\mathbb{R}\ , \qquad \text{(C.2)}
$$

where

$$
0 < d_2 < \chi_0-\chi_{lb}(\varepsilon_0)\ . \qquad \text{(C.3)}
$$

Note that, by (3.19),(3.38),(C.2),(C.3),

$$(\theta,\chi,\zeta)\in\Big(\mathbb{R}\times(\chi_0-d_2,\chi_0+d_2)\times\mathbb{R}\Big)\subset\Big(W(\varepsilon_0)\times\mathbb{R}\Big)\subset\mathcal{D}(\varepsilon,\nu)\,. \tag{C.4}$$

In this appendix we will prove the properties (C.27),(C.30) of g_1 and g_2. We thus show in this appendix that the properties (C.27),(C.30) hold in the situation of Theorem 1 (see item 8 of the setup of Theorem 1). Moreover the properties (C.27),(C.30) will be used in Appendix E.

We first consider g_1 and we obtain from (B.5)

$$\begin{aligned}|g_1|=\Big|&-\frac{q}{\bar q\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\bigg(\frac{q}{\mathcal{E}}(\Pi_z+2)\\&+\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\bigg)\\&-\frac{\kappa_1}{\bar q\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\bigg(2(1+\varepsilon\chi)^2+\frac{\varepsilon^2 q}{\mathcal{E}}(\Pi_z+2)\bigg)\Big|\,.\end{aligned} \tag{C.5}$$

It follows from (2.47),(2.48),(3.6), (3.7),(C.4) that

$$\begin{aligned}&q>0\,,\quad \bar q>0\,,\quad 1+\varepsilon\chi>0\,,\quad 0<\Pi_z<1\,,\\&3\chi^2+2\varepsilon\chi^3=\chi^2+2\chi^2(1+\varepsilon\chi)\geq 0\,,\end{aligned} \tag{C.6}$$

whence, by (C.5),

$$\begin{aligned}|g_1|\leq&\frac{q}{\bar q\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\bigg(\frac{q}{\mathcal{E}}(\Pi_z+2)\\&+\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\bigg)\\&+\frac{|\kappa_1|}{\bar q\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\bigg(2(1+\varepsilon\chi)^2+\frac{\varepsilon^2 q}{\mathcal{E}}(\Pi_z+2)\bigg)\\=&\frac{q}{\bar q(1+\varepsilon\chi)^2}\bigg(\frac{q(\Pi_z+2)}{\mathcal{E}\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^2}+3\chi^2+2\varepsilon\chi^3\bigg)\\&+\frac{|\kappa_1|}{\bar q\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^2}\bigg(2+\frac{\varepsilon^2 q(\Pi_z+2)}{\mathcal{E}(1+\varepsilon\chi)^2}\bigg)\,.\end{aligned} \tag{C.7}$$

Note also that, by (3.3), (3.16),

$$\begin{aligned}\Pi_z^2(\theta,\chi,\zeta,\varepsilon,\nu)&=1-\frac{\varepsilon^2}{\mathcal{E}}\frac{1+K^2\Pi_x^2(\theta,\zeta,\varepsilon,\nu)}{(1+\varepsilon\chi)^2}\\&\geq 1-\frac{\varepsilon^2}{\mathcal{E}}\frac{1+K^2\Pi_{x,ub}^2(\varepsilon)}{(1+\varepsilon\chi)^2}\,.\end{aligned} \tag{C.8}$$

Moreover $\varepsilon^2/(1+\varepsilon\chi)^2$ and $1+K^2\Pi_{x,ub}^2(\varepsilon,\nu)$ are increasing w.r.t. ε whence, by (C.8),

$$\Pi_z^2(\theta,\chi,\zeta,\varepsilon,\nu)\geq 1-\frac{\varepsilon_0^2}{\mathcal{E}}\frac{1+K^2\Pi_{x,ub}^2(\varepsilon_0)}{(1+\varepsilon_0\chi)^2}\,. \tag{C.9}$$

Since $0 < \varepsilon \le \varepsilon_0$ we have, by (C.2),

$$1 + \varepsilon\chi > 1 + \varepsilon(\chi_0 - d_2) \ge 1 + \inf_{\varepsilon \in (0,\varepsilon_0]} (\varepsilon(\chi_0 - d_2)) = 1 + min(0, \varepsilon_0(\chi_0 - d_2))$$
$$=: \kappa_3(\chi_0, \varepsilon_0, d_2) \ . \tag{C.10}$$

Note that, by (3.20), (C.3),

$$1 + \varepsilon_0(\chi_0 - d_2) > 1 + \varepsilon_0\chi_{lb}(\varepsilon_0) > 0 \ , \tag{C.11}$$

whence, by (C.10),

$$\kappa_3(\chi_0, \varepsilon_0, d_2) > 0 \ , \tag{C.12}$$

so that, for $n \in \mathbb{N}$ and by (C.10),

$$\frac{1}{(1 + \varepsilon\chi)^n} < \frac{1}{\kappa_3^n(\chi_0, \varepsilon_0, d_2)} \ . \tag{C.13}$$

It follows from (C.9),(C.13),

$$\Pi_z^2(\theta, \chi, \zeta, \varepsilon, \nu) > \check{\Pi}_{z,lb}(\varepsilon_0) \ , \tag{C.14}$$

where

$$\check{\Pi}_{z,lb}(\varepsilon) := 1 - \varepsilon^2 \frac{1 + K^2\Pi_{x,ub}^2(\varepsilon)}{\mathcal{E}\kappa_3^2(\chi_0, \varepsilon, d_2)} \ . \tag{C.15}$$

To show that $\check{\Pi}_{z,lb}(\varepsilon_0) > 0$ we compute, by using (3.20),

$$\varepsilon_0^2 \frac{1 + K^2\Pi_{x,ub}^2(\varepsilon_0)}{\mathcal{E}\kappa_3^2(\chi_0, \varepsilon_0, d_2)} = \left(\frac{1 + \varepsilon_0\chi_{lb}(\varepsilon_0)}{\kappa_3(\chi_0, \varepsilon_0, d_2)}\right)^2 \ . \tag{C.16}$$

If $\chi_0 \le 0$ then, by (C.10),(C.11),

$$\kappa_3(\chi_0, \varepsilon_0, d_2) = 1 + \varepsilon_0(\chi_0 - d_2) > 1 + \varepsilon_0\chi_{lb}(\varepsilon_0) > 0 \ , \tag{C.17}$$

whence

$$0 < \frac{1 + \varepsilon_0\chi_{lb}(\varepsilon_0)}{\kappa_3(\chi_0, \varepsilon_0, d_2)} < 1 \ , \tag{C.18}$$

so that, by (C.16),

$$\varepsilon_0^2 \frac{1 + K^2\Pi_{x,ub}^2(\varepsilon_0)}{\mathcal{E}\kappa_3^2(\chi_0, \varepsilon_0, d_2)} < 1 \ . \tag{C.19}$$

If $\chi_0 > 0$ then, by (3.20),(C.1),(C.10),

$$\kappa_3(\chi_0, \varepsilon_0, d_2) = 1 > 1 - \varepsilon_0\chi_M > 1 + \varepsilon_0\chi_{lb}(\varepsilon_0) > 0 \ , \tag{C.20}$$

whence again (C.18) holds which entails (C.19) by (C.16). Having thus proven (C.19) we conclude from (C.15) that

$$\check{\Pi}_{z,lb}(\varepsilon_0) > 0 \ , \tag{C.21}$$

whence, by (C.6),(C.14),

$$\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu) > \Pi_{z,lb}(\varepsilon_0)\ , \tag{C.22}$$

where

$$\Pi_{z,lb}(\varepsilon) := \sqrt{\check{\Pi}_{z,lb}(\varepsilon)} = \sqrt{1-\varepsilon^2\frac{1+K^2\Pi^2_{x,ub}(\varepsilon)}{\mathcal{E}\kappa_3^2(\chi_0,\varepsilon,d_2)}}\ . \tag{C.23}$$

Of course since $\Pi_z, \Pi_{z,lb} > 0$ we conclude from (C.22) that

$$\frac{1}{\Pi_z(\theta,\chi,\zeta,\varepsilon,\nu)} < \frac{1}{\Pi_{z,lb}(\varepsilon_0)}\ . \tag{C.24}$$

Inserting (C.6),(C.13),(C.24) into (C.7) yields to

$$\begin{aligned}|g_1| \le{}& \frac{q}{\bar{q}\kappa_3^2(\chi_0,\varepsilon_0,d_2)}\left(\frac{3q}{\mathcal{E}\Pi_{z,lb}(\varepsilon_0)\kappa_3^2(\chi_0,\varepsilon_0,d_2)}+3\chi^2+2\varepsilon_0|\chi|^3\right)\\ &+\frac{|\kappa_1|}{\bar{q}\Pi_{z,lb}(\varepsilon_0)\kappa_3^2(\chi_0,\varepsilon_0,d_2)}\left(2+\frac{3\varepsilon_0^2 q}{\mathcal{E}\kappa_3^2(\chi_0,\varepsilon_0,d_2)}\right)\ .\end{aligned} \tag{C.25}$$

Furthermore, by (2.47), (B.4), (C.2),(C.6),

$$\begin{aligned}&|\chi| = |\chi-\chi_0+\chi_0| \le |\chi-\chi_0|+|\chi_0| < d_2+|\chi_0|\ ,\\ &|\kappa_1(\theta,\zeta,\varepsilon,\nu)| \le \frac{K^2\bar{q}}{\nu}\left(2+2|\Delta P_{x0}|+\frac{\varepsilon^2\bar{q}}{\nu}\right) \le \frac{K^2\bar{q}}{\nu}\left(2+2|\Delta P_{x0}|+\frac{\varepsilon_0^2\bar{q}}{\nu}\right)\ ,\\ &q(\zeta) \le 1+K^2(1+|\Delta P_{x0}|)^2 =: q_{ub}\ .\end{aligned} \tag{C.26}$$

Inserting (C.26) into (C.25) yields to

$$\begin{aligned}|g_1(\theta,\chi,\zeta;\varepsilon,\nu)| \le{}& \frac{q_{ub}}{\bar{q}\kappa_3^2(\chi_0,\varepsilon_0,d_2)}\\ &\times\left(\frac{3q_{ub}}{\mathcal{E}\Pi_{z,lb}(\varepsilon_0)\kappa_3^2(\chi_0,\varepsilon_0,d_2)}+3(d_2+|\chi_0|)^2+2\varepsilon_0(d_2+|\chi_0|)^3\right)\\ &+\frac{K^2}{\nu\Pi_{z,lb}(\varepsilon_0)\kappa_3^2(\chi_0,\varepsilon_0,d_2)}\left(2+2|\Delta P_{x0}|+\frac{\varepsilon_0^2\bar{q}}{\nu}\right)\left(2+\frac{3\varepsilon_0^2 q_{ub}}{\mathcal{E}\kappa_3^2(\chi_0,\varepsilon_0,d_2)}\right)\\ &=: C_1(\chi_0,\varepsilon_0,\nu,d_2)\ .\end{aligned} \tag{C.27}$$

We now consider g_2 and we obtain from (B.12),(C.6)

$$\begin{aligned}|g_2| \le{}& \frac{K^2}{\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^3}\bigg((1+|\Delta P_{x0}|)[\frac{\varepsilon_0}{\mathcal{E}}(q+\varepsilon_0^2|\kappa_1|)\\ &+|\chi|\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^2]+\varepsilon_0|\kappa_2|(\Pi_z+1)(1+\varepsilon\chi)^2\bigg)\\ ={}& K^2\bigg(\frac{\varepsilon_0(1+|\Delta P_{x0}|)}{\mathcal{E}\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^3}(q+\varepsilon_0^2|\kappa_1|)+\frac{|\chi|(1+|\Delta P_{x0}|)}{1+\varepsilon\chi}\end{aligned}$$

$$+\frac{\varepsilon_0|\kappa_2|}{\Pi_z(1+\varepsilon\chi)}\Bigg) \; . \tag{C.28}$$

Note that, by (B.11),(C.6),

$$|\kappa_2(\theta,\zeta,\nu)| \le \frac{\bar{q}}{\nu} \; . \tag{C.29}$$

Inserting (C.6),(C.13),(C.24), (C.26),(C.29) into (C.28) yields to

$$\begin{aligned}
&|g_2(\theta,\chi,\zeta;\varepsilon,\nu)| \\
&\le K^2\Bigg(\frac{\varepsilon_0(1+|\Delta P_{x0}|)}{\mathcal{E}\Pi_{z,lb}(\varepsilon_0)\kappa_3^3(\chi_0,\varepsilon_0,d_2)}\Bigg(q_{ub}+\varepsilon_0^2\frac{K^2\bar{q}}{\nu}(2+2|\Delta P_{x0}|+\frac{\varepsilon_0^2\bar{q}}{\nu})\Bigg) \\
&\quad +\frac{(d_2+|\chi_0|)(1+|\Delta P_{x0}|)}{\kappa_3(\chi_0,\varepsilon_0,d_2)}+\frac{\varepsilon_0\bar{q}}{\nu\Pi_{z,lb}(\varepsilon_0)\kappa_3(\chi_0,\varepsilon_0,d_2)}\Bigg) \\
&\quad =: C_2(\chi_0,\varepsilon_0,\nu,d_2) \; ,
\end{aligned} \tag{C.30}$$

where $\kappa_3, \Pi_{z,lb}, q_{ub}$ are given by (C.10),(C.23),(C.26). With (C.27),(C.30) we have shown that $g_1(\cdot,\nu)$ and $g_2(\cdot,\nu)$ are bounded for $\nu \ge 1/2$ for the points

$$(\theta,\chi,\zeta,\varepsilon) \in \mathbb{R}\times(\chi_0-d_2,\chi_0+d_2)\times\mathbb{R}\times(0,\varepsilon_0] \; . \tag{C.31}$$

D Limit of g_1^R, g_2^R

Let $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon_0 \in (0,1]$ and $k \in \mathbb{N}, a \in [-1/2,1/2]$ and let $(\theta,\chi,\zeta) \in W(\varepsilon_0)\times\mathbb{R}$. In this appendix we will prove the properties (D.1),(D.2),(D.3),(D.5), (D.7),(D.11) of g_1^R and g_2^R. The properties (D.2),(D.11) are used in the proof of Proposition 2. Furthermore the properties (D.1),(D.3),(D.5), (D.7) will be used in Appendix E. Since all assumptions of this appendix are also satisfied in Appendix B, we can apply the results of Appendix B.

We first consider g_1 and we obtain from (3.51), (B.5) that

$$\begin{aligned}
g_1^R(\theta,\chi,\zeta,\varepsilon,k,a) &= g_1(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a) \\
&= -\frac{q}{\bar{q}\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\Bigg(\frac{q}{\mathcal{E}}(\Pi_z+2) \\
&\quad +\Pi_z(\Pi_z+1)^2(3\chi^2+2\varepsilon\chi^3)(1+\varepsilon\chi)^2\Bigg) \\
&\quad -\frac{\kappa_1}{\bar{q}\Pi_z(\Pi_z+1)^2(1+\varepsilon\chi)^4}\Bigg(2(1+\varepsilon\chi)^2+\frac{\varepsilon^2 q}{\mathcal{E}}(\Pi_z+2)\Bigg) \; ,
\end{aligned} \tag{D.1}$$

where $\Pi_z = \Pi_z(\theta,\chi,\zeta,\varepsilon,k+\varepsilon a)$ and $\kappa_1 = \kappa_1(\theta,\zeta,\varepsilon,k+\varepsilon a)$ whence, by (B.5),(B.8),

$$\begin{aligned}
\lim_{\varepsilon\to 0+}[g_1^R(\theta,\chi,\zeta;\varepsilon,k,a)] &= \lim_{\varepsilon\to 0+}[g_1(\theta,\chi,\zeta;\varepsilon,k)] = -\frac{q(\zeta)}{4\bar{q}}(\frac{3}{\mathcal{E}}q(\zeta)+12\chi^2) \\
&\quad -\frac{K^2}{2k}\Bigg(\sin(k[\theta-Q(\zeta)])-\sin(k\theta_0)\Bigg)(\cos\zeta+\Delta P_{x0}) \; .
\end{aligned} \tag{D.2}$$

We now consider g_2^R and we conclude from (3.55) that

$$g_2^R(\theta,\chi,\zeta,\varepsilon,k,a) = g_{2,1}^R(\theta,\chi,\zeta;\varepsilon,k,a)+g_{2,2}^R(\theta,\chi,\zeta;\varepsilon,k,a) \; , \tag{D.3}$$

where

$$
\begin{aligned}
g^R_{2,1}(\theta,\chi,\zeta;\varepsilon,k,a) &:= g_2(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)\ , \qquad \text{(D.4)}\\
g^R_{2,2}(\theta,\chi,\zeta;\varepsilon,k,a) &:= -\frac{K^2}{\varepsilon}(\cos\zeta+\Delta P_{x0})\Big(\cos(\kappa_4+\kappa_5)-\cos(\kappa_4)\Big)\\
&= -\frac{K^2}{\varepsilon}(\cos\zeta+\Delta P_{x0})\Big(\cos(\kappa_4)[\cos(\kappa_5)-1]-\sin(\kappa_4)\sin(\kappa_5)\Big)\\
&= -\frac{K^2}{\varepsilon}(\cos\zeta+\Delta P_{x0})\Big(-2\cos(\kappa_4)\sin^2(\kappa_5/2)\\
&\quad -2\cos(\kappa_5/2)\sin(\kappa_5/2)\sin(\kappa_4)\Big)\\
&= \frac{2K^2}{\varepsilon}(\cos\zeta+\Delta P_{x0})\sin(\kappa_5/2)\Big(\cos(\kappa_4)\sin(\kappa_5/2)+\cos(\kappa_5/2)\sin(\kappa_4)\Big)\ ,
\end{aligned}
\tag{D.5}
$$

with

$$
\begin{aligned}
\kappa_4(\theta,\zeta,\varepsilon,k,a) &:= k(\theta-\zeta-\Upsilon_0\sin\zeta-\Upsilon_1\sin 2\zeta)-\varepsilon a\zeta\ ,\\
\kappa_5(\theta,\zeta,\varepsilon,a) &:= \varepsilon a(\theta-\Upsilon_0\sin\zeta-\Upsilon_1\sin 2\zeta)\ .
\end{aligned}
\tag{D.6}
$$

We obtain from (B.12),(D.4)

$$
\begin{aligned}
g^R_{2,1}(\theta,\chi,\zeta;\varepsilon,k,a) &= g_2(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)\\
&= \frac{K^2}{\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^3}\cos(\nu[\theta-Q(\zeta)])\Big((\cos\zeta+\Delta P_{x0})[-\frac{\varepsilon}{\mathcal{E}}(q+\varepsilon^2\kappa_1)\\
&\quad +\chi\Pi_z(\Pi_z+1)(1+\varepsilon\chi)^2]-\varepsilon\kappa_2(\Pi_z+1)(1+\varepsilon\chi)^2\Big)\ ,
\end{aligned}
\tag{D.7}
$$

where $\Pi_z=\Pi_z(\theta,\chi,\zeta,\varepsilon,k+\varepsilon a)$ and $\kappa_2=\kappa_2(\theta,\zeta,k+\varepsilon a)$ whence, by (B.12),(B.13),

$$
\begin{aligned}
\lim_{\varepsilon\to0+}[g^R_{2,1}(\theta,\chi,\zeta;\varepsilon,k,a)] &= \lim_{\varepsilon\to0+}[g_2(\theta,\chi,\zeta;\varepsilon,k)]\\
&= \chi K^2\cos(k[\theta-Q(\zeta)])(\cos\zeta+\Delta P_{x0})\ .
\end{aligned}
\tag{D.8}
$$

Clearly, by (D.6),

$$
\begin{aligned}
&\lim_{\varepsilon\to0+}[\frac{\sin(\kappa_5(\theta,\zeta,\varepsilon,a)/2)}{\varepsilon}] = \frac{a}{2}(\theta-\Upsilon_0\sin\zeta-\Upsilon_1\sin 2\zeta)\ ,\\
&\lim_{\varepsilon\to0+}[\kappa_5(\theta,\zeta,\varepsilon,a)] = 0\ ,\\
&\lim_{\varepsilon\to0+}[\kappa_4(\theta,\zeta,\varepsilon,k,a)] = k(\theta-\zeta-\Upsilon_0\sin\zeta-\Upsilon_1\sin 2\zeta)\ ,
\end{aligned}
\tag{D.9}
$$

whence, by (D.5),

$$
\lim_{\varepsilon\to0+}[g^R_{2,2}(\theta,\chi,\zeta;\varepsilon,k,a)] = K^2a(\theta-\Upsilon_0\sin\zeta-\Upsilon_1\sin 2\zeta)
$$

$$\times \sin(k[\theta - \zeta - \Upsilon_0 \sin\zeta - \Upsilon_1 \sin 2\zeta])(\cos\zeta + \Delta P_{x0}) \ , \tag{D.10}$$

so that, by (D.3),(D.8),

$$\begin{aligned}\lim_{\varepsilon\to 0+} [g_2^R(\theta,\chi,\zeta,\varepsilon,k,a)] &= \chi K^2 \cos(k[\theta - Q(\zeta)])(\cos\zeta + \Delta P_{x0}) \\ &+K^2 a(\theta - \Upsilon_0 \sin\zeta - \Upsilon_1 \sin 2\zeta) \\ &\quad \times \sin(k[\theta - \zeta - \Upsilon_0 \sin\zeta - \Upsilon_1 \sin 2\zeta])(\cos\zeta + \Delta P_{x0}) \ . \end{aligned} \tag{D.11}$$

E Bounds on g_1^R, g_2^R

Let $\varepsilon \in (0,\varepsilon_0]$ with $\varepsilon_0 \in (0,1]$ and let $k \in \mathbb{N}, a \in [-1/2, 1/2]$. Let also $(\theta_0,\chi_0) \in W(\varepsilon_0)$. Moreover let $\chi_{lb}(\varepsilon_0)$ satisfy the restriction (C.1) where χ_M is the positive constant from Theorem 2 (see item 2 of the setup list for Theorem 2). Furthermore we assume that

$$(\theta,\chi,\zeta) \in (\theta_0 - d_1, \theta_0 + d_1) \times (\chi_0 - d_2, \chi_0 + d_2) \times \mathbb{R} \ , \tag{E.1}$$

where χ_0, d_1, d_2 satisfy

$$0 < d_1 \ , \quad 0 < d_2 < \chi_0 - \chi_{lb}(\varepsilon_0) \ . \tag{E.2}$$

In this appendix we will prove the properties (E.6),(E.14) of g_1^R and g_2^R. We thus show in this appendix that the properties (E.6),(E.14) hold in the situation of Theorem 2 (see item 8 of the setup of Theorem 2). Since all assumptions of this appendix are also satisfied in Appendix C and Appendix D, we can apply the results of those appendices.

We first consider g_1^R and we obtain from (3.51) that

$$|g_1^R(\theta,\chi,\zeta,\varepsilon,k,a)| = |g_1(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)| \ , \tag{E.3}$$

whence, by (C.27),

$$|g_1^R(\theta,\chi,\zeta,\varepsilon,k,a)| \le C_1(\chi_0,\varepsilon_0,k+\varepsilon a,d_2) \ , \tag{E.4}$$

where C_1 is given by (C.27). Note that, by (C.27), $C_1(\chi_0,\varepsilon_0,\nu,d_2)$ is decreasing w.r.t. ν whence

$$C_1(\chi_0,\varepsilon_0,k+\varepsilon a,d_2) \le C_1(\chi_0,\varepsilon_0,1/2,d_2) =: C_1^R(\chi_0,\varepsilon_0,d_2) \ , \tag{E.5}$$

so that, by (E.4),

$$|g_1^R(\theta,\chi,\zeta,\varepsilon,k,a)| \le C_1^R(\chi_0,\varepsilon_0,d_2) \ , \tag{E.6}$$

where C_1^R is given by (E.5).

We now consider g_2^R and we obtain from (D.3) that

$$|g_2^R(\theta,\chi,\zeta,\varepsilon,k,a)| \le |g_{2,1}^R(\theta,\chi,\zeta;\varepsilon,k,a)| + |g_{2,2}^R(\theta,\chi,\zeta;\varepsilon,k,a)| \ . \tag{E.7}$$

Note that, by (C.30),(D.4),

$$|g_{2,1}^R(\theta,\chi,\zeta;\varepsilon,k,a)| = |g_2(\theta,\chi,\zeta;\varepsilon,k+\varepsilon a)| \le C_2(\chi_0,\varepsilon_0,k+\varepsilon a,d_2) \ , \tag{E.8}$$

where C_2 is given by (C.30). Note that, by (C.30), $C_2(\chi_0,\varepsilon_0,\nu,d_2)$ is decreasing w.r.t. ν whence

$$C_2(\chi_0,\varepsilon_0,k+\varepsilon a,d_2) \le C_2(\chi_0,\varepsilon_0,1/2,d_2) =: C_{2,1}^R(\chi_0,\varepsilon_0,d_2) \ , \tag{E.9}$$

so that, by (E.8),

$$|g^R_{2,1}(\theta,\chi,\zeta,\varepsilon,k,a)| \le C^R_{2,1}(\chi_0,\varepsilon_0,d_2)\ , \tag{E.10}$$

where $C^R_{2,1}$ is given by (E.9). We also have, by (D.5),

$$\begin{aligned}
&|g^R_{2,2}(\theta,\chi,\zeta;\varepsilon,k,a)| \\
&\quad = \Big|\frac{2K^2}{\varepsilon}(\cos\zeta + \Delta P_{x0})\sin(\kappa_5/2)\Big(\cos(\kappa_4)\sin(\kappa_5/2) + \cos(\kappa_5/2)\sin(\kappa_4)\Big)\Big| \\
&\quad \le \frac{4K^2}{\varepsilon}|\sin(\kappa_5/2)|(1+|\Delta P_{x0}|)\ .
\end{aligned} \tag{E.11}$$

Of course, by (D.6),(E.1),

$$\begin{aligned}
\frac{|\sin(\kappa_5/2(\theta,\zeta,\varepsilon,a))|}{\varepsilon} &= \frac{1}{\varepsilon}|\sin(\frac{\varepsilon a}{2}[\theta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta])| \\
\le \frac{|a|}{2}\Big|\theta - \Upsilon_0\sin\zeta - \Upsilon_1\sin 2\zeta\Big| &\le \frac{|a|}{2}(|\theta| + |\Upsilon_0| + |\Upsilon_1|) \\
\le \frac{|a|}{2}(|\theta_0| + d_1 + |\Upsilon_0| + |\Upsilon_1|)\ ,
\end{aligned} \tag{E.12}$$

whence, by (E.11),

$$\begin{aligned}
|g^R_{2,2}(\theta,\chi,\zeta;\varepsilon,k,a)| &\le 2K^2|a|(1+|\Delta P_{x0}|)(|\theta_0| + d_1 + |\Upsilon_0| + |\Upsilon_1|) \\
&=: C^R_{2,2}(\theta_0,a,d_1)\ .
\end{aligned} \tag{E.13}$$

We conclude from (E.7),(E.10),(E.13) that

$$\begin{aligned}
|g^R_2(\theta,\chi,\zeta,\varepsilon,k,a)| &\le C^R_{2,1}(\chi_0,\varepsilon_0,d_2) + C^R_{2,2}(\theta_0,a,d_1) \\
&=: C^R_2(\theta_0,\chi_0,\varepsilon_0,a,d_1,d_2)\ ,
\end{aligned} \tag{E.14}$$

where $C^R_{2,1}$ is given by (E.9) and $C^R_{2,2}$ is given by (E.13).

With (E.6),(E.14) we have shown that $g^R_1(\cdot,k,a)$ and $g^R_2(\cdot,k,a)$ are bounded for $k\in\mathbb{N}, |a|\le 1/2$ for the points

$$(\theta,\chi,\zeta,\varepsilon) \in (\theta_0-d_1,\theta_0+d_1)\times(\chi_0-d_2,\chi_0+d_2)\times\mathbb{R}\times(0,\varepsilon_0]\ . \tag{E.15}$$

F Error bounds in a regular perturbation problem

Here we outline a derivation of error bounds in a regular perturbation problem of relevance for §3.4.4. This could be made into a theorem and proof at the level of §3.5 and §4 but we leave this to the interested reader (see [25, §2] for a detailed discussion of regular perturbation theory relevant here, complete with a theorem and proof). We write the IVP in (3.109) as

$$x_1' = x_2\ ,\quad x_1(0) = \xi\ , \tag{F.1}$$

$$x_2' = -\epsilon\sin x_1\ ,\quad x_2(0) = 1\ . \tag{F.2}$$

Then the zeroth-order approximation is

$$u_1' = u_2\ ,\quad u_1(0) = \xi\ , \tag{F.3}$$

$$u_2' = 0 \ , \quad u_2(0) = 1 \ , \tag{F.4}$$

with solutions

$$u_1(s) = s + \xi \ , \quad u_2(s) = 1 \ . \tag{F.5}$$

Subtracting and integrating we obtain

$$e_1 := |x_1(s) - u_1(s)| \leq \int_0^s |x_2(\tau) - u_2(\tau)| d\tau \ , \tag{F.6}$$

$$\begin{aligned} e_2 &:= |x_2(s) - u_2(s)| = \epsilon \left| \int_0^s [\sin(x_1(\tau)) - \sin(u_1(\tau)) + \sin(\tau + \xi)] d\tau \right| \\ &\leq \epsilon \int_0^s |x_1(\tau) - u_1(\tau)| d\tau + \epsilon| - \cos(s + \xi) + \cos(\xi)| \\ &\leq \epsilon \int_0^s |x_1(\tau) - u_1(\tau)| d\tau + 2\epsilon \ . \end{aligned} \tag{F.7}$$

Introducing R_1 and R_2 as in §4, we have

$$e_1(s) \leq \int_0^s e_2(\tau) d\tau =: R_1(s) \ , \tag{F.8}$$

$$e_2(s) \leq \epsilon \int_0^s e_1(\tau) d\tau + 2\epsilon =: R_2(s) \ . \tag{F.9}$$

Differentiating gives the differential inequalities

$$R_1' = e_2 \leq R_2 \ , \quad R_1(0) = 0 \ , \tag{F.10}$$

$$R_2' = \epsilon e_1 \leq \epsilon R_1 \ , \quad R_2(0) = 2\epsilon \ . \tag{F.11}$$

Let

$$w_1' = w_2 \ , \quad w_1(0) = 0 \ , \tag{F.12}$$

$$w_2' = \epsilon w_1 \ , \quad w_2(0) = 2\epsilon \ . \tag{F.13}$$

Then

$$w_1 = \sqrt{\epsilon} 2 \sinh(\sqrt{\epsilon} s) \ , \tag{F.14}$$

$$w_2 = \epsilon 2 \cosh(\sqrt{\epsilon} s) \ . \tag{F.15}$$

Now as shown in Appendix I, $R_1(s) \leq w_1(s)$ and $R_2(s) \leq w_2(s)$ whence if $0 \leq s \leq T$,

$$e_1(s) \leq \sqrt{\epsilon} 2 \sinh(\sqrt{\epsilon} s) \leq \sqrt{\epsilon} 2 \sinh(\sqrt{\epsilon} T) = O(\epsilon) \ , \tag{F.16}$$

$$e_2(s) \leq \epsilon 2 \cosh(\sqrt{\epsilon} s) \leq \epsilon 2 \cosh(\sqrt{\epsilon} T) = O(\epsilon) \ . \tag{F.17}$$

In the context of §3.4.4 with $\epsilon = 1/Y_0^2, \xi = X_0, x_1 = \hat{X}, x_2 = \hat{Y}$ we obtain from (F.16),(F.17) that $\hat{X}(s) = s + X_0 + O(1/Y_0^2), \hat{Y}(s) = 1 + O(1/Y_0^2)$ whence $X(t) = Y_0 t + X_0 + O(1/Y_0^2), Y(t) = Y_0(1 + O(1/Y_0))$.

G Derivatives for Low Gain Problem

We here derive (G.6) which is needed in §3.7. By (3.172) we have

$$\begin{aligned} v_1'(\cdot,1) = 2v_2(\cdot,1)\ ,\quad v_1(0,1) = \theta_0\ ,\quad v_2'(\tau,1) = -K_0(k)\cos(kv_1(\tau,1) - a\tau) \\ = -\frac{K_0(k)}{2}\exp(u(\tau)) + cc\ ,\quad v_2(0,1) = \chi_0\ , \end{aligned} \tag{G.1}$$

where

$$u(\tau) := i[kv_1(\tau,1) - a\tau]\ . \tag{G.2}$$

It follows from (G.1) that

$$\begin{aligned} v_2''(\tau,1) &= K_0(k)(kv_1'(\tau,1) - a)\sin(kv_1(\tau,1) - a\tau) \\ &= K_0(k)(2kv_2(\tau,1) - a)\sin(kv_1(\tau,1) - a\tau) = -\frac{K_0(k)}{2}\exp(u(\tau))u'(\tau) + cc\ , \\ v_2'''(\cdot,1) &= -\frac{K_0(k)}{2}\exp(u)[u'' + (u')^2] + cc\ , \\ v_2''''(\cdot,1) &= -\frac{K_0(k)}{2}\exp(u)[u''' + 3u'u'' + (u')^3] + cc\ , \end{aligned} \tag{G.3}$$

and from (G.1),(G.2), (G.3) that

$$\begin{aligned} u'(\tau) &= i[kv_1'(\cdot,1) - a] = i[2kv_2(\cdot,1) - a]\ , \\ u''(\tau) &= i2kv_2'(\tau,1) = -i2kK_0(k)\cos(kv_1(\tau,1) - a\tau)\ , \\ u'''(\tau) &= i2kv_2''(\tau,1) = i2kK_0(k)(2kv_2(\tau,1) - a)\sin(kv_1(\tau,1) - a\tau)\ . \end{aligned} \tag{G.4}$$

We conclude from (G.1),(G.2), (G.4) that

$$\begin{aligned} u(0) &= ikv_1(0,1) = ik\theta_0\ , \\ u'(0) &= i[2kv_2(0,1) - a] = i[2k\chi_0 - a]\ , \\ u''(0) &= -i2kK_0(k)\cos(kv_1(0,1)) = -i2kK_0(k)\cos(k\theta_0)\ , \\ u'''(0) &= i2kK_0(k)(2kv_2(0,1) - a)\sin(kv_1(0,1)) \\ &= i2kK_0(k)(2k\chi_0 - a)\sin(k\theta_0)\ , \end{aligned} \tag{G.5}$$

whence, by (G.1),(G.3),

$$\begin{aligned} v_2'(0,1) &= -K_0(k)\cos(kv_1(0,1)) = -K_0(k)\cos(k\theta_0)\ , \\ v_2''(0,1) &= K_0(k)(2kv_2(0,1) - a)\sin(kv_1(0,1)) = K_0(k)(2k\chi_0 - a)\sin(k\theta_0)\ , \\ v_2'''(0,1) &= -\frac{K_0(k)}{2}\exp(u(0))[u''(0) + (u'(0))^2] + cc \\ &= -\frac{K_0(k)}{2}\exp(ik\theta_0)\Big(-i2kK_0(k)\cos(k\theta_0) - [2k\chi_0 - a]^2\Big) + cc \\ &= -K_0(k)\Big(2kK_0(k)\sin(k\theta_0)\cos(k\theta_0) - [2k\chi_0 - a]^2\cos(k\theta_0)\Big) \end{aligned}$$

$$= K_0(k)\bigg(-kK_0(k)\sin(2k\theta_0) + [2k\chi_0 - a]^2\cos(k\theta_0)\bigg) \ ,$$

$$v_2''''(0,1) = -\frac{K_0(k)}{2}\exp(u(0))[u'''(0) + 3u'(0)u''(0) + (u'(0))^3] + cc$$

$$= -\frac{K_0(k)}{2}\exp(ik\theta_0)\bigg(i2kK_0(k)(2k\chi_0 - a)\sin(k\theta_0)$$

$$+6kK_0(k)[2k\chi_0 - a]\cos(k\theta_0) - i[2k\chi_0 - a]^3\bigg) + cc$$

$$= -\frac{K_0(k)}{2}\bigg(-4kK_0(k)(2k\chi_0 - a)\sin^2(k\theta_0)$$

$$+12kK_0(k)[2k\chi_0 - a]\cos^2(k\theta_0) + 2[2k\chi_0 - a]^3\sin(k\theta_0)\bigg)$$

$$= K_0(k)\bigg(2kK_0(k)(2k\chi_0 - a)\sin^2(k\theta_0)$$

$$-6kK_0(k)[2k\chi_0 - a]\cos^2(k\theta_0) - [2k\chi_0 - a]^3\sin(k\theta_0)\bigg) \ . \tag{G.6}$$

H Calculation of E_r/cB_u in high gain regime

In this appendix we aim to estimate the magnitude of the electric field. The basic field equation is

$$(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2})E_x(z,t) = -cZ_{vac}\frac{\partial j}{\partial t}(z,t) \ , \tag{H.1}$$

where $Z_{vac} = 1/c\epsilon_0$ is the free space impedance and

$$j(z,t) := -\frac{ecK}{\Sigma_\perp}\cos(k_u z)\sum_{n=1}^{N}\frac{1}{\gamma_n(t)}\delta(z - z_n(t))$$

$$\approx -\frac{ecKN}{\gamma_c\Sigma_\perp}\cos(k_u z)\frac{1}{N}\sum_{n=1}^{N}\delta(z - z_n(t)) \ , \tag{H.2}$$

with $\Sigma_\perp$ being the transverse emittance, see [2] and [31]. We proceed in two ways. In the first we solve (H.1) and (H.2) directly and in the second we use Fourier tranforms.

The unique solution of the homogeneous IVP at $t = 0$ is

$$E_x(z,t) = -\frac{Z_{vac}}{2}\int_0^t ds\int_{z-ct+cs}^{z+ct-cs} dy\frac{\partial j}{\partial s}(y,s)$$

$$= -\frac{Z_{vac}}{2}[U_-(z,t) + U_+(z,t)] \ , \tag{H.3}$$

where

$$U_-(z,t) := \int_{z-ct}^{z} dy[j(y,t + \frac{1}{c}(y - z)) - j(y,0)] \ , \tag{H.4}$$

$$U_+(z,t) := \int_z^{z+ct} dy[j(y,t-\frac{1}{c}(y-z)) - j(y,0)] \ . \tag{H.5}$$

The first equality in (H.3) is often obtained using Duhamel's principle and d'Alembert's formula and the second equality is obtained after changing the order of integration. To obtain our estimate we consider $z_n(t) = \beta_c ct + z_n(0)$ which is quite crude (but may suffice for a rough estimate) and where the nonnegative β_c is determined by $\beta_c^2 = (\gamma_c^2 - 1)/\gamma_c^2$. We obtain [32] $U_+ \ll U_-$ and

$$U_-(z,t) \approx -\frac{2ecK\gamma_c N}{\Sigma_\perp}\frac{1}{N}\sum_{n=1}^N I_n(z,t)\cos(2k_u\gamma_c^2[z-ct-z_n(0)]) \ , \tag{H.6}$$

where

$$I_n(z,t) := \begin{cases} 1 & \text{if } z_n(t) < z < z_n(0)+ct \\ 0 & \text{if otherwise .} \end{cases} \tag{H.7}$$

So if all the particles contributed at z, which they don't, then $U_-(z,t) = O(\frac{2ecK\gamma_c N}{\Sigma_\perp})$ and $E_{r1} = \frac{Z_{vac}ecK\gamma_c N}{\Sigma_\perp}$ would be a typical value of the field E_x at (z,t).

We now give a second estimate, E_{r2}, of E_r. Following [31] which is based on [2] we Fourier transform (H.1) by defining

$$\hat{E}_x(z,\omega) := \frac{1}{2\pi}\int_{-\infty}^{\infty} dsE_x(z,\frac{z}{c}-\frac{s}{ck_r})\exp(-i\omega s) \ . \tag{H.8}$$

The Fourier inversion theorem gives

$$E_x(z,t) = \int_{-\infty}^{\infty} d\omega\hat{E}_x(z,\omega)\exp(i\omega k_r[z-ct]) \ . \tag{H.9}$$

We define $\hat{j}(z,\omega)$ in the same way as $\hat{E}_x(z,\omega)$ whence, in the slowly varying approximation, (H.1) reduces to

$$\frac{\partial\hat{E}_x}{\partial z}(z,\omega) = -\frac{Z_{vac}}{2}\hat{j}(z,\omega) \ , \tag{H.10}$$

and from (H.2) we obtain

$$\hat{j}(z,\omega) = -\frac{ecKNk_r}{2\pi\beta_c\gamma_c\Sigma_\perp}\check{j}(z,\omega) \ , \tag{H.11}$$

where

$$\check{j}(z,\omega) := \cos(k_u z)\exp(-i\omega k_r z)\frac{1}{N}\sum_{n=1}^N \exp(i\omega ck_r T_n(z)) \ . \tag{H.12}$$

Here the function T_n is the inverse of the function z_n. To obtain our estimate we note that $|\check{j}|$ is bounded by 1 and replace it by 1 which is quite crude but may suffice for a rough estimate. Inserting this into (H.10) and integrating we obtain

$$\hat{E}_x(z,\omega) = O\left(\frac{Z_{vac}}{2}\frac{ecKNk_r}{2\pi\beta_c\gamma_c\Sigma_\perp}\frac{1}{k_u}k_u z\right) \ , \tag{H.13}$$

and, for $k_u z = O(1)$,

$$\hat{E}_x = O(E_{r2}) \ , \quad E_{r2} := \frac{Z_{vac}}{4\pi}\frac{ecKN}{\Sigma_\perp}\frac{k_r}{k_u\beta_c\gamma_c^2}\gamma_c \ . \tag{H.14}$$

We now have, recalling that $K = 3.7$ in LCLS,

$$\frac{E_{r1}}{E_{r2}} = 4\pi\frac{k_u\gamma_c^2}{k_r} = 4\pi/K_r = 2\pi(1+\frac{K^2}{2}) \approx 2\pi(1+(3.7)^2/2) \approx 49 \ , \tag{H.15}$$

and we calculate E_{r2}/cB_u. From (H.14)

$$\frac{E_{r2}}{cB_u} = \frac{Z_{vac}c}{4\pi}\frac{eK}{cB_u}\frac{k_r}{k_u\gamma_c^2}\gamma_c\frac{N}{\Sigma_\perp} \ . \tag{H.16}$$

Now $K/cB_u = e/mc^2k_u$ and $k_r/k_u\gamma_c^2 = 2(1+K^2/2)^{-1}$ therefore

$$\frac{E_{r2}}{cB_u} = \frac{Z_{vac}c}{4\pi}\frac{e^2}{mc^2}\frac{1}{k_u}\frac{2}{(1+K^2/2)}\gamma_c\frac{N}{\Sigma_\perp} = r_e\frac{1}{k_u}\frac{2}{(1+K^2/2)}\gamma_c\frac{N}{\Sigma_\perp} \ , \tag{H.17}$$

where r_e denotes the classical electron radius. Furthermore

$$r_e \approx 2.82\cdot 10^{-15}m \ , \quad \frac{1}{k_u} = \frac{3cm}{2\pi} \ , \quad \frac{2}{(1+K^2/2)} \approx 0.255 \ , \quad \gamma_c = 10^4 \ ,$$

and so

$$\frac{E_{r2}}{cB_u} \approx 0.034\cdot 10^{-12}m^2\frac{N}{\Sigma_\perp} \approx 34 \ , \quad \frac{E_{r1}}{cB_u} = \frac{E_{r2}}{cB_u}\frac{E_{r1}}{E_{r2}} \approx 34\cdot 49 \approx 1700 \ ,$$

for $N = 10^9$ and $\Sigma_\perp = 1mm^2$.

I IVP for a system of differential inequalities

Here we present and verify a solution of the IVP for a system of differential inequalities which is used in §4.1,§4.2 and Appendix F. Consider the IVP for

$$R_1'(\zeta) \le a_1R_2(\zeta) \ , \tag{I.1}$$
$$R_2'(\zeta) \le a_2R_1(\zeta) \ , \tag{I.2}$$

where $a_1, a_2 > 0$ and R_1, R_2 are of class C^1. We want to show, for $\zeta \ge 0$, that

$$R_1(\zeta) \le r_1(\zeta) \ , \quad R_2(\zeta) \le r_2(\zeta) \ , \tag{I.3}$$

where

$$r_1' = a_1r_2 \ , \quad r_1(0) = R_1(0) \ , \tag{I.4}$$
$$r_2' = a_2r_1 \ , \quad r_2(0) = R_2(0) \ . \tag{I.5}$$

We do this in two ways. First we define $\hat{r}_j(\zeta) := R_j(\zeta) - r_j(\zeta)$ for $j = 1,2, \zeta \ge 0$ whence, by (I.1),(I.2),(I.4),
(I.5),

$$\hat{r}_1'(\zeta) \le a_1\hat{r}_2(\zeta) \ , \quad \hat{r}_2'(\zeta) \le a_2\hat{r}_1(\zeta) \ , \quad \hat{r}_1(0) = \hat{r}_2(0) = 0 \ . \tag{I.6}$$

Clearly we have to show that, for $j = 1, 2, \zeta \geq 0$,

$$\hat{r}_j(\zeta) \leq 0 \; . \tag{I.7}$$

It follows from (I.6) that

$$\hat{r}_1'(\zeta) \leq a_1 \int_0^\zeta ds \hat{r}_2'(s) \leq a_1 a_2 \int_0^\zeta ds \hat{r}_1(s) \; ,$$
$$\hat{r}_2'(\zeta) \leq a_2 \int_0^\zeta ds \hat{r}_1'(s) \leq a_1 a_2 \int_0^\zeta ds \hat{r}_2(s) \; ,$$

i.e.,

$$\hat{r}_j'(\zeta) \leq a_0^2 \int_0^\zeta ds \hat{r}_j(s) \; , \tag{I.8}$$

where $a_0 := \sqrt{a_1 a_2}$. It follows from (I.8) and by partial integration that

$$\begin{aligned} \exp(-a_0\zeta)\hat{r}_j(\zeta) + a_0 \int_0^\zeta ds \exp(-a_0 s)\hat{r}_j(s) = \int_0^\zeta ds \exp(-a_0 s)\hat{r}_j'(s) \\ \leq a_0^2 \int_0^\zeta ds \exp(-a_0 s) \int_0^s d\tilde{s}\hat{r}_j(\tilde{s}) \\ = -a_0 \exp(-a_0\zeta) \int_0^\zeta ds \hat{r}_j(s) + a_0 \int_0^\zeta ds \exp(-a_0 s)\hat{r}_j(s) \; , \end{aligned} \tag{I.9}$$

which entails

$$\hat{r}_j(\zeta) \leq -a_0 \int_0^\zeta ds \hat{r}_j(s) \; . \tag{I.10}$$

Abbreviating

$$\check{r}_j(\zeta) := \int_0^{\zeta/a_0} ds \hat{r}_j(s) \; , \tag{I.11}$$

we obtain from (I.10)

$$\check{r}_j'(\zeta) = \frac{1}{a_0}\hat{r}_j(\zeta/a_0) \leq -\int_0^{\zeta/a_0} ds \hat{r}_j(s) = -\check{r}_j(\zeta) \; , \tag{I.12}$$

whence

$$0 \geq \exp(\zeta)[\check{r}_j(\zeta) + \check{r}_j'(\zeta)] = [\exp(\zeta)\check{r}_j(\zeta)]' \; , \tag{I.13}$$

so that $\exp(\zeta)\check{r}_j(\zeta)$ is decreasing w.r.t. ζ which entails, by (I.11), that

$$0 = \exp(0)\check{r}_j(0) \geq \exp(\zeta)\check{r}_j(\zeta) \; , \tag{I.14}$$

i.e.,

$$\check{r}_j(\zeta) \leq 0 \; . \tag{I.15}$$

We conclude from (I.8),(I.11),(I.15) that

$$\hat{r}_j'(\zeta) \le a_0^2 \int_0^\zeta ds \hat{r}_j(s) = \check{r}_j(a_0\zeta) \le 0 \ , \tag{I.16}$$

whence $\hat{r}_j(\zeta)$ is decreasing w.r.t. ζ so that (I.7) follows from (I.6).

The result in (I.3) is a special case of a much more general theorem on pages 112-113 of [26]. That proof simplifies in the special case here and we present it for the interested reader. The proof proceeds by cleverly introducing a comparison function $\mathbf{h}$. Here

$$\mathbf{h}(\zeta) = \begin{pmatrix} h_1(\zeta) \\ h_2(\zeta) \end{pmatrix} := a_4 \exp(2a_3\zeta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \tag{I.17}$$

where $a_3 := max(a_1, a_2), a_4 > 0$. Then

$$h_1' = 2a_3 h_1 = 2a_3 h_2 > a_1 h_2 \ , \tag{I.18}$$
$$h_2' = 2a_3 h_2 = 2a_3 h_1 > a_2 h_1 \ , \tag{I.19}$$

and we have, by (I.6),

$$\hat{r}_1' - a_1 \hat{r}_2 \le 0 < h_1' - a_1 h_2 \ , \tag{I.20}$$
$$\hat{r}_2' - a_2 \hat{r}_1 \le 0 < h_2' - a_2 h_1 \ . \tag{I.21}$$

We now show that, for $j = 1, 2, \zeta \ge 0$,

$$\hat{r}_j(\zeta) \le h_j(\zeta) \ . \tag{I.22}$$

Suppose that (I.22) is wrong then there exists a smallest $\zeta_0 > 0$ such that an index j_0 exists with

$$\hat{r}_{j_0}(\zeta_0) = h_{j_0}(\zeta_0) \ , \tag{I.23}$$

where we used that, by (I.6),(I.17) and for $j = 1, 2$,

$$\hat{r}_j(0) = 0 < a_4 = h_j(0) \ . \tag{I.24}$$

Clearly, for $j = 1, 2, 0 \le \zeta < \zeta_0$,

$$\hat{r}_j(\zeta) < h_j(\zeta) \ . \tag{I.25}$$

Without loss of generality we take $j_0 = 1$ whence, for $0 \le \zeta \le \zeta_0$,

$$\hat{r}_2(\zeta) \le h_2(\zeta) \ . \tag{I.26}$$

It follows from (I.25) that at the first intersection

$$\hat{r}_1'(\zeta_0) \ge h_1'(\zeta_0) \ . \tag{I.27}$$

But by (I.20),(I.26)

$$\hat{r}_1'(\zeta_0) - h_1'(\zeta_0) < a_1(\hat{r}_2(\zeta_0) - h_2(\zeta_0)) \le 0 \ , \tag{I.28}$$

which is a contradiction.

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