

HYPERDIRE
HYPERgeometric functions Differential REduction:
MATHEMATICA based packages for differential reduction of
generalized hypergeometric functions
 ${}_pF_{p-1}, F_1, F_2, F_3, F_4$

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Abstract

HYPERDIRE is a project devoted to the creation of a set of Mathematica based programs for the differential reduction of hypergeometric functions. The current version includes two parts: one, **pfq**, is relevant for manipulations of hypergeometric functions ${}_pF_p$, and the second one, **AppellF1F4**, for manipulations with Appell hypergeometric functions F_1, F_2, F_3, F_4 of two variables.

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Keywords: Hypergeometric functions; Differential reduction

PROGRAM SUMMARY

Title of program: HYPERDIRE

Version: 1.0.0 *Release:* 1.0.0 *Catalogue number:*

Program obtained from <https://sites.google.com/site/loopcalculations/home>:

E-mail: bvv@jinr.ru

Licensing terms: GNU General Public Licence

Computers: all computers running Mathematica

Operating systems: operating systems running Mathematica

Programming language: Mathematica

Keywords: Generalized Hypergeometric functions, Appell functions, Feynman integrals.

Nature of the problem: Reduction of hypergeometric functions ${}_pF_{p-1}$, F_1 , F_2 , F_3 , F_4 to sets of basis functions.

Method of solution: Differential reduction

Restriction on the complexity of the problem: none

Typical running time: Depending on the complexity of problem.

LONG WRITE-UP

1 Introduction

Multiple hypergeometric functions [1–4] play an important role in many branches of science. In particular, a large class of Feynman diagrams are expressed in terms of Horn-type hypergeometric functions [5].

Let us consider a multiple series:

$$H(\vec{\gamma}; \vec{\sigma}; \vec{x}) = \sum_{m_1, m_2, \dots, m_r=0}^{\infty} \left(\frac{\prod_{j=1}^K \Gamma(\sum_{a=1}^r \mu_{ja} m_a + \gamma_j) \Gamma^{-1}(\gamma_j)}{\prod_{k=1}^L \Gamma(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k) \Gamma^{-1}(\sigma_k)} \right) x_1^{m_1} \cdots x_r^{m_r}, \quad (1)$$

with $\mu_{ab}, \nu_{ab} \in \mathbb{Z}$, $\gamma_j, \sigma_k \in \mathbb{C}$. The sequences $\vec{\gamma} = (\gamma_1, \dots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \dots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively.

Let $\vec{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ denote the unit vector with unity in its j^{th} entry, and let us define $\vec{x}^{\vec{m}} = x_1^{m_1} \cdots x_r^{m_r}$ for any integer multi-index $\vec{m} = (m_1, \dots, m_r)$. Two functions of type (1) with sets of parameters shifted by unity, $H(\vec{\gamma} + \vec{e}_c; \vec{\sigma}; \vec{x})$ and $H(\vec{\gamma}; \vec{\sigma}; \vec{x})$, are related by a linear differential operator:

$$H(\vec{\gamma} + \vec{e}_c; \vec{\sigma}; \vec{x}) = \frac{1}{\gamma_c} \left(\sum_{a=1}^r \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \equiv U_{[\gamma_c \rightarrow \gamma_c + 1]}^+ H(\vec{\gamma}, \vec{\sigma}, \vec{x}). \quad (2)$$

Similar relations also exist for the lower parameters:

$$H(\vec{\gamma}; \vec{\sigma} - \vec{e}_c; \vec{x}) = \frac{1}{\sigma_c - 1} \left(\sum_{b=1}^r \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c - 1 \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \equiv L_{[\sigma_c \rightarrow \sigma_c - 1]}^- H(\vec{\gamma}; \vec{\sigma}; \vec{x}). \quad (3)$$

The linear differential operators $U_{\gamma_c \rightarrow \gamma_c + 1}^+$, $L_{\sigma_c \rightarrow \sigma_c - 1}^-$ are called the *step-up* and *step-down* operators for the upper and lower indices, respectively. If additional step-down and step-up operators $U_{\gamma_c}^-, L_{\sigma_c}^+$ satisfying

$$U_{[\gamma_c + 1 \rightarrow \gamma_c]}^- U_{[\gamma_c \rightarrow \gamma_c + 1]}^+ H(\vec{\gamma}, \vec{\sigma}, \vec{x}) = L_{[\sigma_c - 1 \rightarrow \sigma_c]}^+ L_{[\sigma_c \rightarrow \sigma_c - 1]}^- H(\vec{\gamma}, \vec{\sigma}, \vec{x}) = H(\vec{\gamma}, \vec{\sigma}, \vec{x})$$

(i.e., the inverses of $U_{\gamma_c}^+, L_{\sigma_c}^-$) are constructed, we can combine these operators to shift the parameters of the hypergeometric function by any integer. This process of applying $U_{\gamma_c}^{\pm}, L_{\sigma_c}^{\pm}$ to shift the parameters by integers is called **differential reduction** of a hypergeometric function.

In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts:

$$P_0(\vec{x}) H(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{x}) = \sum_{m_1, \dots, m_p=0}^{\sum |k_i| + \sum |l_i|} P_{m_1, \dots, m_r}(\vec{x}) \left(\frac{\partial}{\partial x_1} \right)^{m_1} \cdots \left(\frac{\partial}{\partial x_r} \right)^{m_r} H(\vec{\gamma}; \vec{\sigma}; \vec{x}), \quad (4)$$

where $P_0(\vec{x})$ and $P_{m_1, \dots, m_p}(\vec{x})$ are polynomials with respect to $\vec{\gamma}, \vec{\sigma}$, and \vec{x} , and \vec{k}, \vec{l} are lists of integers.

Algebraic relations between the functions $H(\vec{\gamma}, \vec{\sigma}; \vec{x})$ with parameters shifted by integers are called **contiguous relations**. The development of systematic techniques for the solution of contiguous relations has a long history. It was started by Gauss, who described the reduction for the ${}_2F_1$ hypergeometric function in 1823 [1]. Numerous papers have since been published [6–8] on this problem. An algorithmic solution was found by Takayama in Ref. [9], and those methods have been extended later in a series of publications [10, 11] (see also Refs. [12–14]).

Let us recall that any hypergeometric function can be considered to be the solution of a proper system of partial differential equations (PDEs). In particular, for a Horn-type hypergeometric function, the system of PDEs can be derived from the coefficients of the series

$$H = \sum_{\vec{m}} C(\vec{m}) \vec{x}^{\vec{m}}.$$

In this case, the ratio of two coefficients can be represented as a ratio of two polynomials,

$$\frac{C(\vec{m} + e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} = \prod_{j=1}^K \frac{\Gamma(\sum_{a=1}^r \mu_{ja} m_a + \mu_{ja} \delta_{ai} + \gamma_j)}{\Gamma(\sum_{a=1}^r \mu_{ja} m_a + \gamma_j)} \prod_{k=1}^L \frac{\Gamma(\sum_{b=1}^r \nu_{kb} m_b + \sigma_k)}{\Gamma(\sum_{b=1}^r \nu_{kb} m_b + \nu_{kb} \delta_{bi} + \sigma_k)}, \quad (5)$$

so that the Horn-type hypergeometric function satisfies the following system of differential equations:

$$0 = D_j(\vec{\gamma}, \vec{\sigma}, \vec{x}) H(\vec{\gamma}, \vec{\sigma}, \vec{x}) = \left[Q_j \left(\sum_{k=1}^r x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} - P_j \left(\sum_{k=1}^r x_k \frac{\partial}{\partial x_k} \right) \right] H(\vec{\gamma}, \vec{\sigma}, \vec{x}), \quad (6)$$

where $j = 1, \dots, r$. It was pointed out in several publications [15–17] that (i) the differential reduction algorithm, Eq. (4), can be applied to the reduction of Feynman diagrams to some subsets of basis hypergeometric functions with well-known analytical properties [15, 16]; (ii) the system of differential equations, Eq. (6), can be also used for the construction of so-called ε expansions of hypergeometric functions about rational values of parameters via the direct solution of the systems of differential equations [17]. This is another motivation for creating a package for the manipulation of the parameters of Horn-type hypergeometric functions.

The aim of this paper is to present the *Mathematica* [18] based package **HYPERDIRE** for the differential reduction of the Horn-type hypergeometric function with arbitrary values of parameters to a set of basis functions. The current version consists of two parts: one, **pfq**, for the manipulation of hypergeometric functions, ${}_p F_p$, and the second one, **AppellF1F4**, for the manipulation of Appell functions, F_1, F_2, F_3, F_4 . The algorithm of differential reduction for other functions can be implemented as an additive module.

In contrast to the recent programs written by members of computational particles physics community [19–24], the aim of our package is the manipulation of hypergeometric functions without the construction of ε expansions [25, 26].

The preliminary version of **pfq** was presented in Ref. [27] and is available in Ref. [28]. The latest version is available in Ref. [34].

2 Differential-reduction algorithm for generalized hypergeometric function ${}_{p+1}F_p$

2.1 General consideration

Let us consider the generalized hypergeometric function, ${}_pF_q(a; b; z)$, defined around $z = 0$ by a series

$${}_pF_q(\vec{a}; \vec{b}; z) \equiv {}_pF_q \left(\begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^p (a_i)_k}{\prod_{j=1}^q (b_j)_k}, \quad (7)$$

where $(a)_k$ is a Pochhammer symbol, $(a)_k = \Gamma(a+k)/\Gamma(a)$. The sequences $\vec{a} = (a_1, \dots, a_p)$ and $\vec{b} = (b_1, \dots, b_q)$ are called the upper and lower parameters of hypergeometric functions, respectively. In terms of the operator θ :

$$\theta = z \frac{d}{dz}, \quad (8)$$

the differential equation for the hypergeometric function ${}_pF_q$ can be written as

$$[z \prod_{i=1}^p (\theta + a_i) - \theta \prod_{i=1}^q (\theta + b_i - 1)] {}_pF_q(\vec{a}; \vec{b}; z) = 0. \quad (9)$$

2.2 Differential reduction

The differential reduction for these functions was analyzed in details in Ref. [16]. Here we recall some of the main relations relevant to our program.

The universal differential operators, Eqs. (2) and (3), have the following form:

$${}_pF_q(a_1 + 1, \vec{a}; \vec{b}; z) = B_{a_1}^+ {}_pF_q(a_1, \vec{a}; \vec{b}; z) = \frac{1}{a_1} (\theta + a_1) {}_pF_q(a_1, \vec{a}; \vec{b}; z), \quad (10)$$

$${}_pF_q(\vec{a}; b_1 - 1, \vec{b}; z) = H_{b_1}^- {}_pF_q(\vec{a}; b_1, \vec{b}; z) = \frac{1}{b_1 - 1} (\theta + b_1 - 1) {}_pF_q(\vec{a}; b_1, \vec{b}; z), \quad (11)$$

where the operators $B_{a_1}^+$ ($H_{b_1}^-$) are called the step-up (step-down) operators for the upper (lower) parameters of hypergeometric functions. This type of operators were explicitly constructed for the hypergeometric function ${}_{p+1}F_p$ by Takayama in Ref. [10]. For completeness, we reproduce his result here:

$$\begin{aligned} {}_{p+1}F_p(a_i - 1, \vec{a}; \vec{b}; z) &= B_{a_i}^- {}_{p+1}F_p(a_i, \vec{a}; \vec{b}; z), \\ {}_{p+1}F_p(\vec{a}; b_i + 1, \vec{b}; z) &= H_{b_i}^+ {}_{p+1}F_p(\vec{a}; b_i, \vec{b}; z), \end{aligned} \quad (12)$$

where

$$\begin{aligned}
B_{a_i}^- &= -\frac{a_i}{c_i} [t_i(\theta) - z \prod_{j \neq i} (\theta + a_j)] \Big|_{a_i \rightarrow a_i - 1} , \\
c_i &= -a_i \prod_{j=1}^p (b_j - 1 - a_i) , \\
t_i(x) &= \frac{x \prod_{j=1}^p (x + b_j - 1) - c_i}{x + a_i} = \sum_{j=0}^p P_{p-j}^{(p)}(\{b_r - 1\}) \frac{[x^{j+1} - (-a_i)^{j+1}]}{x + a_i} \\
&= \sum_{j=0}^p P_{p-j}^{(p)}(\{b_r - 1\}) \sum_{k=0}^j x^{j-k} (-a_i)^k , \tag{13}
\end{aligned}$$

$$\begin{aligned}
H_{a_i}^+ &= \frac{b_i - 1}{d_i} \left[\frac{d}{dz} \prod_{j \neq i} (\theta + b_j - 1) - s_i(\theta) \right] \Big|_{b_i \rightarrow b_i + 1} , \\
d_i &= \prod_{j=1}^{p+1} (1 + a_j - b_i) , \\
s_i(x) &= \frac{\prod_{j=1}^{p+1} (x + a_j) - d_i}{x + b_i - 1} \sum_{j=0}^{p+1} P_{p+1-j}^{(p+1)}(\{a_r\}) \frac{[x^j - (1 - b_i)^j]}{x - (1 - b_i)} \\
&= \sum_{j=0}^p P_{p-j}^{(p+1)}(\{a_r\}) \sum_{k=0}^j x^{j-k} (1 - b_i)^k . \tag{14}
\end{aligned}$$

There $|_{a \rightarrow a+1}$ means substitution of a by $a + 1$, and the polynomials $P_j^{(p)}(r_1, \dots, r_p)$ are defined as

$$\prod_{k=1}^p (z + r_k) = \sum_{j=0}^p P_{p-j}^{(p)}(r_1, \dots, r_p) z^j \equiv \sum_{j=0}^p P_{p-j}^{(p)}(\vec{r}) z^j \equiv \sum_{j=0}^p P_j^{(p)}(\vec{r}) z^{p-j} . \tag{15}$$

$P_s^{(p)}(\vec{r})$ is a polynomial of order s with respect to the variables r , and

$$P_0^{(p)}(\vec{r}) = 1 , \quad P_j^{(p)}(\vec{r}) = \sum_{i_1, \dots, i_r=1}^p \prod_{i_1 < \dots < i_j} r_{i_1} \cdots r_{i_j} , \quad j = 1, \dots, p .$$

For example, $P_1^{(p)}(\vec{r}) = \sum_{j=1}^p r_j$ and $P_p^{(p)}(\vec{r}) = \prod_{j=1}^p r_j$. Keeping in mind that

$$\prod_{i=1}^p (z + r_i) \prod_{j=p+1}^{p+k} (z + r_j) = \prod_{l=1}^{p+k} (z + r_l) ,$$

we find that these polynomials satisfy the following relations:

$$P_{p+k-j}^{(p+k)}(r_1, \dots, r_p, q_1, \dots, q_k) = \sum_{n=0}^k P_{p+1-j-n}^{(p)}(r_1, \dots, r_p) P_n^{(k)}(q_1, \dots, q_k) , \tag{16}$$

where $j = 1, \dots, p - k$ and $P_{p+k}^{(p)}(\vec{r}) = 0$. In particular,

$$\begin{aligned} P_{p+1-j}^{(p+1)}(\vec{r}, f) &= P_{p+1-j}^{(p)}(\vec{r}) + f P_{p-j}^{(p)}(\vec{r}), \quad j = 1, \dots, p, \\ P_{p+1-j}^{(p+1)}(\vec{r}_{p-1}, q_1, q_2) &= \sum_{k=0}^2 P_{p+1-j-k}^{(p-1)}(\vec{r}) P_k^{(2)}(\vec{q}), \quad j = 1, \dots, p-1, \\ P_{p+1-j}^{(p+1)}(\vec{r}_{p-2}, q_1, q_2, q_3) &= \sum_{k=0}^3 P_{p+1-j-k}^{(p-2)}(\vec{r}) P_k^{(3)}(\vec{q}), \quad j = 1, \dots, p-2. \end{aligned}$$

The differential reduction has the form of a product of several differential step-up/step-down operators, $H_{b_k}^{\pm}$ and $B_{a_k}^{\pm}$, respectively:

$$F(\vec{a} + \vec{m}; \vec{b} + \vec{n}; z) = \left(H_{\{a\}}^{\pm} \right)^{\sum_i m_i} \left(B_{\{b\}}^{\pm} \right)^{\sum_j n_j} F(\vec{a}; \vec{b}; z), \quad (17)$$

so that the maximal power of θ in this expression is equal to $r \equiv \sum_i m_i + \sum_j n_j$. Since the hypergeometric function ${}_{p+1}F_p(\vec{a}; \vec{b}; z)$ satisfies the differential equation of order $p+1$ (see Eq. (6)):

$$\begin{aligned} &(1 - z)\theta^{p+1} {}_{p+1}F_p(\vec{a}; \vec{b}; z) \\ &= \left\{ \sum_{r=1}^p \left[z P_{p+1-r}^{(p+1)}(\{a_j\}) - P_{p+1-r}^{(p)}(\{b_j - 1\}) \right] \theta^r + z \prod_{k=1}^{p+1} a_k \right\} {}_{p+1}F_p(\vec{a}; \vec{b}; z), \quad (18) \end{aligned}$$

it is possible to express all terms containing higher powers of the operator θ^k , where $k \geq p+1$, in terms of product of θ^j with $j \leq p$ and rational functions of parameters and argument z .

In this way, any function ${}_{p+1}F_p(\vec{a} + \vec{m}; \vec{b} + \vec{k}; z)$ is expressible in terms of the basic function and its first p -derivative:

$$\begin{aligned} &{}_{p+1}F_p(\vec{a} + \vec{m}; \vec{b} + \vec{k}; z) = \\ &\frac{1}{S(a_i, b_j, z)} \left\{ R_1(a_i, b_j, z) + R_2(a_i, b_j, z)\theta + \dots + R_{p+1}(a_i, b_j, z)\theta^p \right\} {}_{p+1}F_p(\vec{a}; \vec{b}; z), \quad (19) \end{aligned}$$

where m, k is the set of integer numbers, and S and R_i are polynomials in the parameters $\{a_i\}, \{b_j\}$, and z .

From Eq. (13) it follows that if one of the upper parameters a_j is equal to unity, then the application of the step-down operator $B_{a_j}^-$ to the hypergeometric function ${}_{p+1}F_p$ will produce unity, $B_{1-p+1}^- {}_{p+1}F_p(1, \vec{a}; \vec{b}; z) \equiv 1$. Taking into account the explicit form of the step-down operator B_1^- ,

$$B_1^- = \frac{1}{\prod_{k=1}^p (b_k - 1)} \left[\prod_{j=1}^p (b_j - 1) + \sum_{j=1}^p P_{p-j}^{(p)}(\{b_k - 1\})\theta^j - z \prod_{j=1}^p (\theta + a_j) \right], \quad (20)$$

we get the differential identity

$$\begin{aligned} & \left\{ \Pi_{j=1}^p (b_j - 1) - z \Pi_{j=1}^p a_j + (1-z)\theta^p \right\}_{p+1} F_p(1, \vec{a}; \vec{b}; z) \\ & + \left\{ \sum_{j=1}^{p-1} \left[P_{p-j}^{(p)}(\{b_k - 1\}) - z P_{p-j}^{(p)}(\{a_k\}) \right] \theta^j \right\}_{p+1} F_p(1, \vec{a}; \vec{b}; z) = \Pi_{k=1}^p (b_k - 1). \end{aligned} \quad (21)$$

The case when two or more upper parameters are equal to unity, $a_1 = a_2 = 1$, does not generate any new identities.

3 Differential reduction of Appell hypergeometric functions

3.1 Appell hypergeometric functions: system of differential equations

Let us consider the system of linear differential equations of second order for the functions $\omega(\vec{z})$:

$$\theta_{11}\omega(\vec{z}) = \left\{ P_0(\vec{z})\theta_{12} + P_1(\vec{z})\theta_1 + P_2(\vec{z})\theta_2 + P_3(\vec{z}) \right\} \omega(\vec{z}), \quad (22)$$

$$\theta_{22}\omega(\vec{z}) = \left\{ R_0(\vec{z})\theta_{12} + R_1(\vec{z})\theta_1 + R_2(\vec{z})\theta_2 + R_3(\vec{z}) \right\} \omega(\vec{z}), \quad (23)$$

where $\vec{z} = (z_1, z_2)$ with z_1, z_2 being variables, $\{P_j, R_j\}$ are rational functions, $\theta_j = z_j \partial_{z_j}$ for $j = 1, 2$, and $\theta_{i_1 \dots i_k} = \theta_{i_1} \dots \theta_{i_k}$. Using θ_j instead of the standard ∂_j is explained by our applications. Taking the derivative of Eq.(22) with respect to θ_2 , using the well-known property $\partial_2 \theta_{11} \omega(\vec{z}) = \partial_1 \theta_{12} \omega(\vec{z})$ and applying Eq. (23), we rewrite Eq. (22) as follows:

$$\begin{aligned} & [\theta_1 - P_0 \theta_2] \theta_{12} \omega(\vec{z}) \\ & = \left\{ [\theta_2 P_0 + P_1 + P_2 R_0] \theta_{12} + [P_2 R_1 + \theta_2 P_1] \theta_1 + [P_2 R_2 + \theta_2 P_2 + P_3] \theta_2 + P_2 R_3 + \theta_2 P_3 \right\} \omega(\vec{z}). \end{aligned} \quad (24)$$

Applying a similar operation to Eq. (23), we get

$$\begin{aligned} & [-R_0 \theta_1 + \theta_2] \theta_{12} \omega(\vec{z}) \\ & = \left\{ [\theta_1 R_0 + R_2 + R_1 P_0] \theta_{12} + [P_1 R_1 + \theta_1 R_1 + R_3] \theta_1 + [P_2 R_1 + \theta_1 R_2] \theta_2 + R_1 P_3 + \theta_1 R_3 \right\} \omega(\vec{z}). \end{aligned} \quad (25)$$

It is well known [2] that under the condition

$$1 - P_0 R_0 \neq 0, \quad (26)$$

there are four independent solutions of the system of Eqs. (22) and (23). In this case, Eqs. (24) and (25) can be solved, so that

$$(1 - P_0 R_0) \theta_{112} \omega(\vec{z}) = \left\{ \begin{aligned} & [P_0 (\theta_1 R_0 + R_2 + R_1 P_0) + \theta_2 P_0 + P_1 + P_2 R_0] \theta_{12} \\ & + [P_2 R_1 + \theta_2 P_1 + P_0 (P_1 R_1 + \theta_1 R_1 + R_3)] \theta_1 \\ & + [P_2 R_2 + \theta_2 P_2 + P_3 + P_0 (P_2 R_1 + \theta_1 R_2)] \theta_2 \\ & + P_2 R_3 + \theta_2 P_3 + P_0 (R_1 P_3 + \theta_1 R_3) \end{aligned} \right\} \omega(\vec{z}), \quad (27)$$

$$(1 - P_0 R_0) \theta_{122} \omega(\vec{z}) = \left\{ \begin{aligned} & [\theta_1 R_0 + R_2 + R_1 P_0 + R_0 (\theta_2 P_0 + P_1 + P_2 R_0)] \theta_{12} \\ & + [R_0 (P_2 R_1 + \theta_2 P_1) + P_1 R_1 + \theta_1 R_1 + R_3] \theta_1 \\ & + [P_2 R_1 + \theta_1 R_2 + R_0 (P_2 R_2 + \theta_2 P_2 + P_3)] \theta_2 \\ & + R_0 (P_2 R_3 + \theta_2 P_3) + R_1 P_3 + \theta_1 R_3 \end{aligned} \right\} \omega(\vec{z}). \quad (28)$$

The condition of complete integrability is defined via the relation $\partial_1 (\partial_{122} \omega(\vec{z})) = \partial_2 (\partial_{112} \omega(\vec{z}))$. In terms of the quantities in Eqs. (27) and (28), it has the following form:

$$\theta_2 \left(\frac{1}{1 - P_0 R_0} [\text{r.h.s. Eq. (27)}] \right) - \theta_1 \left(\frac{1}{1 - P_0 R_0} [\text{r.h.s. Eq. (28)}] \right) = 0, \quad (29)$$

where we have used the relation $\theta_{jj} = z_j^2 \partial_j^2 + z_j \partial_j$. If the condition of Eq. (29) is valid, then Eqs. (22) and (23) can be reduced to the Pfaff system of four differential equations

$$d\vec{f} = R\vec{f}, \quad (30)$$

where $\vec{f} = (\omega(\vec{z}), \theta_1 \omega(\vec{z}), \theta_2 \omega(\vec{z}), \theta_{12} \omega(\vec{z}))$.

In the case

$$1 - P_0 R_0 = 0, \quad (31)$$

$\theta_{12} \omega(\vec{z})$ is expressible in terms of three other elements, $\omega(\vec{z})$, $\theta_1 \omega(\vec{z})$ and $\theta_2 \omega(\vec{z})$. In particular, using the notation of Eqs. (24) and (25), we have

$$\begin{aligned} & [\theta_2 P_0 + P_1 + P_2 R_0 + P_0 (\theta_1 R_0 + R_2 + R_1 P_0)] \theta_{12} \omega(\vec{z}) \\ = & \left\{ \begin{aligned} & - [P_0 (\theta_1 R_1 + R_3 + R_1 P_1) + P_2 R_1 + \theta_2 P_1] \theta_1 \\ & - [P_0 (\theta_1 R_2 + R_1 P_2) + P_2 R_2 + \theta_2 P_2 + P_3] \theta_2 \\ & - [P_0 (\theta_1 R_3 + R_1 P_3) + P_2 R_3 + \theta_2 P_3] \end{aligned} \right\} \omega(\vec{z}). \quad (32) \end{aligned}$$

Table 1: Values of the coefficients in Eqs. (22) and (23) for the Appell hypergeometric functions F_1, F_2, F_3 and F_4 .

	F_1	F_2	F_3	F_4
P_0	- 1	$\frac{z_1}{1-z_1}$	$-\frac{1}{(1-z_1)}$	$\frac{2z_1}{(1-z_1-z_2)}$
R_0	- 1	$\frac{z_2}{1-z_2}$	$-\frac{1}{(1-z_2)}$	$\frac{2z_2}{(1-z_1-z_2)}$
P_1	$\frac{(a+b_1)z_1-(c-1)}{(1-z_1)}$	$\frac{(a+b_1)z_1-(c_1-1)}{(1-z_1)}$	$\frac{(a_1+b_1)z_1-(c-1)}{(1-z_1)}$	$\frac{(a+b)z_1-(c_1-1)(1-z_2)}{(1-z_1-z_2)}$
R_1	$\frac{b_2z_2}{(1-z_2)}$	$\frac{b_2z_2}{(1-z_2)}$	0	$\frac{(a+b+1-c_1)z_2}{(1-z_1-z_2)}$
P_2	$\frac{b_1z_1}{(1-z_1)}$	$\frac{b_1z_1}{(1-z_1)}$	0	$\frac{(a+b+1-c_2)z_1}{(1-z_1-z_2)}$
R_2	$\frac{(a+b_2)z_2-(c-1)}{(1-z_2)}$	$\frac{(a+b_2)z_2-(c_2-1)}{(1-z_2)}$	$\frac{(a_2+b_2)z_2-(c-1)}{(1-z_2)}$	$\frac{(a+b)z_2-(c_2-1)(1-z_1)}{(1-z_1-z_2)}$
P_3	$\frac{ab_1z_1}{(1-z_1)}$	$\frac{ab_1z_1}{(1-z_1)}$	$\frac{a_1b_1z_1}{(1-z_1)}$	$\frac{abz_1}{(1-z_1-z_2)}$
R_3	$\frac{ab_2z_2}{(1-z_2)}$	$\frac{ab_2z_2}{(1-z_2)}$	$\frac{a_2b_2z_2}{(1-z_2)}$	$\frac{abz_2}{(1-z_1-z_2)}$

In this case, the integrability conditions are valid, Eqs. (22) and (23) can be reduced to the Pfaff system of Eq. (30) of three differential equations $\vec{f} = (\omega(\vec{z}), \theta_1\omega(\vec{z}), \theta_2\omega(\vec{z}))$, and the system has three solutions.

For the Appell hypergeometric functions F_1, F_2, F_3 , and F_4 , the values of the coefficients in Eqs. (22) and (23) are collected in Table 1.

3.2 Appell hypergeometric function F_1

3.2.1 General consideration

Let us consider the Appell hypergeometric function F_1 defined around $x = y = 0$ as

$$\omega \equiv F_1(a, b_1, b_2, c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c)_{m+n}} \frac{x^m y^n}{m! n!}. \quad (33)$$

In this case, Eqs. (22) and (23) have the following form:

$$\theta_{xx}\omega = \left[\frac{(a+b_1)x-(c-1)}{1-x} - b_2 \frac{y}{x-y} \right] \theta_x\omega + \frac{b_1x(1-y)}{(1-x)(x-y)} \theta_y\omega + \frac{x}{1-x} ab_1\omega, \quad (34)$$

$$\theta_{yy}\omega = \left[\frac{(a+b_2)y-(c-1)}{1-y} + b_1 \frac{x}{x-y} \right] \theta_y\omega - \frac{b_2y(1-x)}{(1-y)(x-y)} \theta_x\omega + \frac{y}{1-y} ab_2\omega. \quad (35)$$

Eq. (31) is fulfilled, and Eq. (32) has the following form:

$$(x-y) \frac{\partial^2 \omega}{\partial x \partial y} - b_2 \frac{\partial \omega}{\partial x} + b_1 \frac{\partial \omega}{\partial y} = 0, \quad (36)$$

or in terms of the operators θ_x, θ_y :

$$\theta_{xy}\omega = \frac{b_2y}{x-y} \theta_x\omega - \frac{b_1x}{x-y} \theta_y\omega. \quad (37)$$

3.2.2 Differential reduction of F_1

The direct differential expressions follow from Eqs. (2) and (3),

$$aF_1(a + \mathbf{1}, b_1, b_2, c; x, y) = (\theta_x + \theta_y + a)F_1(a, b_1, b_2, c; x, y), \quad (38)$$

$$b_1F_1(a, b_1 + \mathbf{1}, b_2, c; x, y) = (\theta_x + b_1)F_1(a, b_1, b_2, c; x, y), \quad (39)$$

$$(c - 1)F_1(a, b_1, b_2, c - \mathbf{1}; x, y) = (\theta_x + \theta_y + c - 1)F_1(a, b_1, b_2, c; x, y). \quad (40)$$

The inverse differential relations were considered in Refs. [2, 7]:

$$(c - a)F_1(a - \mathbf{1}, b_1, b_2, c; x, y) = [c - a - b_1x - b_2y + (1 - x)\theta_x + (1 - y)\theta_y]F_1(a, b_1, b_2, c; x, y), \quad (41)$$

$$(c - b_1 - b_2)F_1(a, b_1 - \mathbf{1}, b_2, c; x, y) = \left[c - b_1 - b_2 - ax + (1 - x)\theta_x - x \left(1 - \frac{1}{y} \right) \theta_y \right] F_1(a, b_1, b_2, c; x, y), \quad (42)$$

$$(c - a)(c - b_1 - b_2)F_1(a, b_1, b_2, c + \mathbf{1}; x, y) = c \left[(c - a - b_1 - b_2) - \left(1 - \frac{1}{x} \right) \theta_x - \left(1 - \frac{1}{y} \right) \theta_y \right] F_1(a, b_1, b_2, c; x, y). \quad (43)$$

The differential reduction for the parameter b_2 follows from Eqs. (39) and (42), and the symmetry property of the function F_1 , $F_1(a, b_1, b_2, c; x, y) = F_1(a, b_2, b_1, c; y, x)$, i.e.

$$b_1 \Leftrightarrow b_2, \quad x \Leftrightarrow y.$$

3.3 Appell hypergeometric function F_2

3.3.1 General consideration

Let us consider the Appell hypergeometric function F_2 defined around $x = y = 0$ as

$$\omega \equiv F_2(a, b_1, b_2, c_1, c_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_m (b_2)_n}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!}, \quad (44)$$

In this case Eqs. (22) and (23) have the following form:

$$(1 - x)\theta_{xx}\omega = x\theta_{xy}\omega + [(a + b_1)x - (c_1 - 1)]\theta_x\omega + b_1x\theta_y\omega + ab_1x\omega, \quad (45)$$

$$(1 - y)\theta_{yy}\omega = y\theta_{xy}\omega + [(a + b_2)y - (c_2 - 1)]\theta_y\omega + b_2y\theta_x\omega + ab_2y\omega. \quad (46)$$

The condition of Eq. (26) is fulfilled, and Eqs. (27) and (28) have the following form:

$$\begin{aligned}
(1-x-y)\theta_{xxy}\omega &= \left[(a+b_1+1-c_2)x - (c_1-1)(1-y) + \frac{b_2xy}{1-x} \right] \theta_{xy}\omega \\
&+ \frac{b_2xy}{1-x} [a+b_1+1-c_1] \theta_x\omega \\
&+ \left[(a+1-c_2)b_1x + \frac{b_1b_2xy}{1-x} \right] \theta_y\omega \\
&+ \frac{ab_1b_2xy}{1-x} \omega, \tag{47}
\end{aligned}$$

$$\begin{aligned}
(1-x-y)\theta_{yyx}\omega &= \left[(a+b_2+1-c_1)y - (c_2-1)(1-x) + \frac{b_1xy}{1-y} \right] \theta_{xy}\omega \\
&+ \frac{b_1xy}{1-y} [a+b_2+1-c_2] \theta_y\omega \\
&+ \left[(a+1-c_1)b_2y + \frac{b_1b_2xy}{1-y} \right] \theta_x\omega \\
&+ \frac{ab_1b_2xy}{1-y} \omega. \tag{48}
\end{aligned}$$

3.3.2 Differential reduction of F_2

The direct differential expressions follow from Eqs. (2) and (3),

$$aF_2(a + \mathbf{1}, b_1, b_2, c_1, c_2; x, y) = (a + \theta_x + \theta_y)F_2(a, b_1, b_2, c_1, c_2; x, y), \tag{49}$$

$$b_1F_2(a, b_1 + \mathbf{1}, b_2, c_1, c_2; x, y) = (b_1 + \theta_x)F_2(a, b_1, b_2, c_1, c_2; x, y), \tag{50}$$

$$(c_1-1)F_2(a, b_1, b_2, c_1 - \mathbf{1}, c_2; x, y) = (c_1-1 + \theta_x)F_2(a, b_1, b_2, c_1, c_2; x, y), \tag{51}$$

The inverse differential relations were considered in Ref. [7]:

$$\begin{aligned}
F_2(a - \mathbf{1}, b_1, b_2, c_1, c_2; x, y) &= \left\{ 1 - \frac{xb_1 - (1-x)\theta_x}{c_1 - a} - \frac{yb_2 - (1-y)\theta_y}{c_2 - a} \right. \\
&+ \left. \frac{1}{c_1 + c_2 - a - 1} \left[\frac{1}{c_1 - a} + \frac{1}{c_2 - a} \right] [(1-x-y)\theta_{xy} - b_1x\theta_y - b_2y\theta_x] \right\} \\
&\times F_2(a, b_1, b_2, c_1, c_2; x, y), \tag{52}
\end{aligned}$$

$$F_2(a, b_1 - \mathbf{1}, b_2, c_1, c_2; x, y) = \left\{ 1 + \frac{(1-x)\theta_x - x(a + \theta_y)}{c_1 - b_1} \right\} F_2(a, b_1, b_2, c_1, c_2; x, y), \tag{53}$$

$$\begin{aligned}
F_2(a, b_1, b_2, c_1 + \mathbf{1}, c_2; x, y) &= \frac{c_1}{(c_1 - a)(c_1 - b_1)} \left\{ c_1 - a - b_1 - \left(1 - \frac{1}{x} \right) \theta_x \right. \\
&\left. \frac{1}{x(c_1 + c_2 - a - 1)} [xb_1\theta_y + yb_2\theta_x - (1-x-y)\theta_{xy}] \right\} F_2(a, b_1, b_2, c_1, c_2; x, y). \tag{54}
\end{aligned}$$

The differential reductions for the parameters b_2 and c_2 follow from Eqs. (50),(53), and Eqs. (51),(54), respectively, and the symmetry property of the function F_2 , $F_2(a, b_1, b_2, c; x, y) = F_2(a, b_2, b_1, c; y, x)$, i.e.

$$b_1 \Leftrightarrow b_2, \quad c_1 \Leftrightarrow c_2, \quad x \Leftrightarrow y.$$

3.4 Appell hypergeometric function F_3

3.4.1 General consideration

Let us consider the Appell hypergeometric function F_3 defined around $x = y = 0$ as

$$\omega \equiv F_3(a_1, a_2, b_1, b_2, c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n x^m y^n}{(c)_{m+n} m! n!}, \quad (55)$$

In this case, Eqs. (22) and (23) have the following form:

$$(1-x)\theta_{xx}\omega = -\theta_{xy}\omega + [(a_1+b_1)x - (c-1)]\theta_x\omega + xa_1b_1\omega, \quad (56)$$

$$(1-y)\theta_{yy}\omega = -\theta_{xy}\omega + [(a_2+b_2)y - (c-1)]\theta_y\omega + ya_2b_2\omega. \quad (57)$$

The condition of Eq. (26) is fulfilled, and Eq. (27) and (28) have the following form:

$$(xy-x-y)\theta_{xxy}\omega = [(1-y)(a_1+b_1)x - y(a_2+b_2+1-c)]\theta_{xy}\omega + (1-y)xa_1b_1\theta_y\omega - ya_2b_2\theta_x\omega, \quad (58)$$

$$(xy-x-y)\theta_{xyy}\omega = [(1-x)(a_2+b_2)y - x(a_1+b_1+1-c)]\theta_{xy}\omega + (1-x)ya_2b_2\theta_x\omega - xa_1b_1\theta_y\omega. \quad (59)$$

3.4.2 Differential reduction of F_3

The direct differential expressions follow from Eqs. (2) and (3),

$$a_1F_3(a_1 + \mathbf{1}, a_2, b_1, b_2, c; x, y) = (a_1 + \theta_x)F_3(a_1, a_2, b_1, b_2, c; x, y), \quad (60)$$

$$b_1F_3(a_1, a_2, b_1 + \mathbf{1}, b_2, c; x, y) = (b_1 + \theta_x)F_3(a_1, a_2, b_1, b_2, c; x, y), \quad (61)$$

$$(c-1)F_3(a_1, a_2, b_1, b_2, c - \mathbf{1}; x, y) = (c-1 + \theta_x + \theta_y)F_3(a, b_1, b_2, c_1, c_2; x, y). \quad (62)$$

The inverse differential relations were considered in Ref. [8]:

$$F_3(a_1 - \mathbf{1}, a_2, b_1, b_2, c; x, y) = 1 + \frac{1}{(c-a_1-a_2)(c-b_2-a_1)} \left\{ (c-b_2-a_1-a_2) [(1-x)\theta_x - xb_1] + b_1x \left(1 - \frac{1}{y}\right) \theta_y - \left(1-x + \frac{x}{y}\right) \theta_{xy} \right\} \times F_3(a, b_1, b_2, c_1, c_2; x, y), \quad (63)$$

$$F_3(a_1, a_2, b_1 - \mathbf{1}, b_2, c; x, y) = 1 + \frac{1}{(c-b_1-b_2)(c-a_2-b_1)} \left\{ (c-a_2-b_1-b_2) [(1-x)\theta_x - xa_1] + a_1x \left(1 - \frac{1}{y}\right) \theta_y - \left(1-x + \frac{x}{y}\right) \theta_{xy} \right\} \times F_3(a, b_1, b_2, c_1, c_2; x, y), \quad (64)$$

$$\Delta F_3(a_1, a_2, b_1, b_2, c + \mathbf{1}; x, y) = c \left\{ A - D_1 \left(1 - \frac{1}{x}\right) \theta_x - D_2 \left(1 - \frac{1}{y}\right) \theta_y + B \left(1 - \frac{1}{x} - \frac{1}{y}\right) \theta_{xy} \right\} \times F_3(a, b_1, b_2, c_1, c_2; x, y), \quad (65)$$

where in the last expression,

$$\begin{aligned} \delta_1 &= c - a_1 - b_1, & \delta_2 &= c - a_2 - b_2, & F &= c - a_1 - a_2 - b_1 - b_2, \\ A &= \delta_1 \delta_2 F + a_1 b_1 \delta_1 + a_2 b_2 \delta_2, \\ D_1 &= \delta_2 F + a_1 b_1 - a_2 b_2, & D_2 &= \delta_1 F + a_2 b_2 - a_1 b_1, & B &= \delta_1 + \delta_2, \\ \Delta &= (c - b_1 - b_2)(c - a_1 - a_2)(c - a_2 - b_1)(c - a_1 - b_2). \end{aligned} \quad (66)$$

The differential reductions for the parameters a_2 and b_2 follow from Eqs. (60), (63) and Eqs. (61), (64), respectively, and the symmetry property of the function F_3 ,

$$F_3(a_1, a_2, b_1, b_2, c; x, y) = F_3(a_2, a_1, b_2, b_1, c; y, x), \text{ i.e.}$$

$$a_1 \Leftrightarrow a_2, \quad b_1 \Leftrightarrow b_2, \quad x \Leftrightarrow y.$$

3.5 Appell hypergeometric function F_4

3.5.1 General consideration

Let us consider the Appell hypergeometric function F_4 defined around $x = y = 0$ as

$$\omega \equiv F_4(a, b, c_1, c_2; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c_1)_m (c_2)_n} \frac{x^m y^n}{m! n!}. \quad (67)$$

The condition of Eq. (26) is fulfilled, and Eqs. (27),(28) have the following form:

$$(1-x-y)\theta_{xx}\omega = 2x\theta_{xy}\omega + [(a+b)x - (c_1-1)(1-y)]\theta_x\omega + [a+b+1-c_2]x\theta_y\omega + abx\omega \quad (68)$$

$$(1-x-y)\theta_{yy}\omega = 2y\theta_{xy}\omega + [(a+b)y - (c_2-1)(1-x)]\theta_y\omega + [a+b+1-c_1]y\theta_x\omega + aby\omega \quad (69)$$

The condition of Eq. (26) is fulfilled. However, instead of Eqs. (27) and (28), which are very lengthy in this case, we present the results in the following form:

$$\begin{aligned}
& [(1-x-y)^2 - 4xy] \theta_{xxy} \omega = \\
& y [1-x-y + 2x(a+b+1-c_1)] \theta_{xx} \omega + x [2x + (1-x-y)(a+b+1-c_2)] \theta_{yy} \omega + \\
& \left[2(a+b+1-c_2)(1-x)x - (c_1-1)(1-x-y)^2 - x(1-x-y)(a+b+c_1-1) \right] \theta_{xy} \omega \\
& + x [(c_1-1)(1-x-y) + 2abx] \theta_x \omega + x [2x(c_2-1) + (1-x-y)ab] \theta_y \omega , \tag{70}
\end{aligned}$$

$$\begin{aligned}
& [(1-x-y)^2 - 4xy] \theta_{yyx} \omega = \\
& x [1-x-y + 2y(a+b+1-c_2)] \theta_{yy} \omega + y [2y + (1-x-y)(a+b+1-c_1)] \theta_{xx} \omega + \\
& \left[2(a+b+1-c_1)(1-y)y - (c_2-1)(1-x-y)^2 - y(1-x-y)(a+b+c_2-1) \right] \theta_{xy} \omega \\
& + x [(c_2-1)(1-x-y) + 2aby] \theta_y \omega + y [2y(c_1-1) + (1-x-y)ab] \theta_x \omega , \tag{71}
\end{aligned}$$

where the values of $\theta_{xx}\omega$ and $\theta_{yy}\omega$ are taken from Eqs. (68) and (69).

3.5.2 Differential reduction of F_4

The direct differential expressions follow from Eqs. (2) and (3),

$$aF_4(a+1, b, c_1, c_2; x, y) = (a+\theta_x+\theta_y)F_4(a, b, c_1, c_2; x, y) , \tag{72}$$

$$(c_1-1)F_4(a, b, c_1-1, c_2; x, y) = (c_1-1+\theta_x)F_4(a, b, c_1, c_2; x, y) . \tag{73}$$

The inverse differential relations were considered in Ref. [7]:

$$\begin{aligned}
F_4(a-1, b, c_1, c_2; x, y) = & \left\{ 1 \right. \\
& - \frac{x}{(c_1-a)} \left[\left(1 - \frac{1}{x}\right) \theta_x + \theta_y + b \right] - \frac{y}{(c_2-a)} \left[\left(1 - \frac{1}{y}\right) \theta_y + \theta_x + b \right] \\
& + \frac{1}{(1-x-y)(c_1+c_2-a-1)} \left[\frac{1}{c_1-a} + \frac{1}{c_2-a} \right] \left(\right. \\
& \quad [(1-x-y)^2 - 4xy]\theta_{xy} \\
& \quad - y[2x(a+b+1-c_1) + (1-x-y)(b+1-c_1)]\theta_x \\
& \quad - x[2y(a+b+1-c_2) + (1-x-y)(b+1-c_2)]\theta_y \\
& \quad \left. - 2abxy \right) \left. \right\} F_4(a, b, c_1, c_2; x, y), \tag{74}
\end{aligned}$$

$$\begin{aligned}
x(c_1-a)(c_1-b)F_4(a, b, c_1+1, c_2; x, y) = & \\
c_1 \left\{ x(c_1-a-b) + (1-x-y)\theta_x - \frac{x}{c_1}(a+b+1-c_2)\theta_y \right. & \\
+ \frac{(2c_1+c_2-a-b-1)}{(1-x-y)(c_1+c_2-a-1)(c_1+c_2-b-1)} \left(\right. & \\
& [(1-x-y)^2 - 4xy]\theta_{xy} \\
& - y[(1-x-y)(a+b-c_2-2c_1+2) + 2x(a+b-c_1+1)]\theta_x \\
& - \frac{x}{c_1}[2c_1y(a+b+1-c_2) + (1-x-y)(c_2-a-1)(c_2-b-1)]\theta_y \\
& \left. - 2abxy \right) \left. \right\} F_4(a, b, c_1, c_2; x, y). \tag{75}
\end{aligned}$$

The differential reduction for the parameters b and c_2 follows from Eqs. (72), (74) and Eqs. (73), (75), respectively, and the symmetry property of the function F_4 ,

- (i) $F_4(a, b, c_1, c_2, c; x, y) = F_4(b, a, c_1, c_2, c; x, y) : a \Leftrightarrow b$,
- (ii) $F_4(a, b, c_1, c_2, c; x, y) = F_4(a, b, c_2, c_1, c; y, x) : c_1 \Leftrightarrow c_2, x \Leftrightarrow y$.

Table 2: Exceptional set of parameters for the Appell hypergeometric functions F_1, F_2, F_3 and F_4 .

F_1	$\{a, b_1, b_2, c-a, c-b_1-b_2\} \in \mathbb{Z}$
F_2	$\{a, b_1, b_2, c_1-a, c_2-a, c_1+c_2-a, c_1-b_1, c_2-b_2\} \in \mathbb{Z}$
F_3	$\{a_1, a_2, b_1, b_2, c-a_1-a_2, c-b_1-b_2, c-a_2-b_1, c-a_1-b_2\} \in \mathbb{Z}$
F_4	$\{a, b, c_1-a, c_1-b, c_2-a, c_2-b, c_1+c_2-a, c_1+c_2-b\} \in \mathbb{Z}$

3.6 Appell hypergeometric functions: exceptional values of parameters

As was explained in Section 3.1, the differential-reduction algorithm as applied to Appell functions may be written symbolically as

$$R(x, y)F_1(\vec{A} + \vec{m}; x, y) = [P_0(x, y) + P_1(x, y)\theta_x + P_2(x, y)\theta_y] F_1(\vec{A}; x, y), \quad (76)$$

$$S(x, y)F_j(\vec{A} + \vec{m}; x, y) = [Q_0(x, y) + Q_1(x, y)\theta_x + Q_2(x, y)\theta_y + Q_3(x, y)\theta_{xy}] F_j(\vec{A}; x, y), \quad (77)$$

where $j = 2, 3, 4$, \vec{m} is a set of integers, \vec{A} is a set of parameters, R, S, P_i, Q_i are some polynomials, and $\theta_x = x\partial_x$ ($\theta_y = y\partial_y$).

However, there is a special subset of values of parameters for which the results of the differential reduction, Eqs. (76) and (77), have simpler forms. This set of exceptional values of parameters can be defined from (i) the condition that the hypergeometric function entering the l.h.s. of Eqs. (41)–(43), (52)–(54), (63)–(65), (74), and (75), is expressible in terms of simpler hypergeometric functions (e.g. Gauss hypergeometric functions); (ii) the condition that some of the coefficients entering the inverse differential relations are equal to zero(infinity).

For the Appell hypergeometric functions F_1, F_2, F_3 , and F_4 , the exceptional sets of parameters are listed in Table 2.

It is not surprising that the set of exceptional values of parameters coincides with the set of parameters defining the condition of irreducibility of the monodromy group of the corresponding hypergeometric functions (see Ref. [29] and references therein). The condition of irreducibility of Mellin-Barnes integrals [30] is related to the criterion of irreducibility of Feynman diagrams [31].

4 pfq - differential reduction of hypergeometric function ${}_pF_{p-1}$.

4.1 Non-exceptional values of parameters

In this section, we will present the Mathematica¹ based package **pfq** for the differential reduction of the hypergeometric function ${}_pF_{p-1}$. In contrast to the version presented in Ref. [27], the current version deals with non-exceptional and exceptional values of parameters. The Takayama algorithm is implemented in the file **pfq.m**, and an example of its application is given in the file **example-pfq.m**.

The program may be loaded in the standard way:

```
<< "pfq.m"
```

and includes two routines: **ToGroebnerBasis**[...] and **explicitForm**[...].

The main routine,

$$\mathbf{ToGroebnerBasis}[\text{Argumentsvector}], \quad (78)$$

calculates explicitly the ratio of the functions $\{R_k\}$ and S of Eq. (19). The “Argumentsvector” in Eq. (78) is the set of parameters of the hypergeometric function on the l.h.s of Eq. (19) with an explicit set of integer numbers, by which each parameter should be shifted:

$$\text{Argumentsvector} = \{ \{\vec{a} + \vec{M}\}, \{\vec{b} + \vec{K}\}, \{x\} \}, \quad (79)$$

where \vec{M}, \vec{K} are integers, \vec{a}, \vec{b} are any symbols, and x denotes the argument of the hypergeometric function. The output of **ToGroebnerBasis**[...] has the following structure:

$$\{ \{Q_1, \dots, Q_{p+1}\}, \{ \{a_1, \dots, a_{p+1}\}, \{1 + b_1, \dots, 1 + b_p\}, x \}, \text{factor} \}, \quad (80)$$

where

- $\{ \{a_1, \dots, a_{p+1}\}, \{1 + b_1, \dots, 1 + b_p\}, x \}$ are the set of parameters and the argument of the resulting hypergeometric function entering in the r.h.s. of Eq. (19);
- $\{Q_1, \dots, Q_{p+1}\}$ are the rational functions whose numbering corresponds to the power of θ^i , $i = 0, \dots, p$;
- factor is an overall factor.

The explicit form of Eqs. (78)-(80) is the following:

$$\begin{aligned} & {}_{p+1}F_p \left(\begin{matrix} a_1 + M_1, \dots, a_{p+1} + M_{p+1} \\ b_1 + K_1, \dots, b_p + K_p \end{matrix} \middle| x \right) \\ &= \text{factor} \times \left(Q_1 + Q_2\theta + \dots + Q_{p+1}\theta^p \right) {}_{p+1}F_p \left(\begin{matrix} a_1, \dots, a_{p+1} \\ 1 + b_1, \dots, 1 + b_p \end{matrix} \middle| x \right). \end{aligned} \quad (81)$$

¹It was tested for Mathematica 7.0.

1. **Example 1**² : Reduction of ${}_3F_2$.

ToGroebnerBasis[$\{\{1+a_1, 2+a_2, a_3\}, \{1+b_1, b_2+2\}, x\}$],

IntegerPart= $\{1, 2, 0, 1, 2\}$ changeVector= $\{-1, -2, 0, 0, -1\}$,

$$\left\{ \left\{ -\frac{(a_2-a_3+1)(b_2+1)}{(a_2+1)(a_3-b_2-1)}, -\frac{(b_2+1)(xa_2^2-(a_3x-x+b_1)a_2+xa_1(a_2-a_3+1)+b_1b_2)}{xa_1a_2(a_2+1)(a_3-b_2-1)}, -\frac{(b_2+1)(-a_3x+x+(x-1)a_2+b_2)}{xa_1a_2(a_2+1)(a_3-b_2-1)} \right\}, \right. \\ \left. \left\{ \{a_1, a_2, a_3\}, \{b_1+1, b_2+1\}, x, 1 \right\} \right\}$$

corresponds to

$${}_3F_2 \left(\begin{matrix} a_1+1, a_2+2, a_3 \\ b_1+1, b_2+2 \end{matrix} \middle| x \right) = \left[-\frac{(a_2-a_3+1)(b_2+1)}{(a_2+1)(a_3-b_2-1)} \right. \\ \left. -\frac{(b_2+1)(xa_2^2-(a_3x-x+b_1)a_2+xa_1(a_2-a_3+1)+b_1b_2)}{xa_1a_2(a_2+1)(a_3-b_2-1)} \theta \right. \\ \left. -\frac{(b_2+1)(-a_3x+x+(x-1)a_2+b_2)}{xa_1a_2(a_2+1)(a_3-b_2-1)} \theta^2 \right] {}_3F_2 \left(\begin{matrix} a_1, a_2, a_3 \\ b_1+1, b_2+1 \end{matrix} \middle| x \right) . \quad (82)$$

2. **Example 2**: Reduction of ${}_4F_3$.

ToGroebnerBasis [$\{\{1+a_1, 1+a_2, a_3, a_4\}, \{1+b_1, b_2+1, b_3\}, x\}$] ,

IntegerPart= $\{1, 1, 0, 0, 1, 1, 0\}$ changeVector= $\{-1, -1, 0, 0, 0, 0, 1\}$,

$$\left\{ \left\{ 1, \frac{1}{a_2} + \frac{1}{b_3} + \frac{1}{a_1}, \frac{a_1+a_2+b_3}{a_1a_2b_3}, \frac{1}{a_1a_2b_3} \right\}, \left\{ \{a_1, a_2, a_3, a_4\}, \{b_1+1, b_2+1, b_3+1\}, x, 1 \right\} \right\}$$

corresponds to

$${}_4F_3 \left(\begin{matrix} 1+a_1, 1+a_2, a_3, a_4 \\ 1+b_1, 1+b_2, b_3 \end{matrix} \middle| x \right) = \left[1 + \left(\frac{1}{a_2} + \frac{1}{b_3} + \frac{1}{a_1} \right) \theta \right. \\ \left. + \frac{a_1+a_2+b_3}{a_1a_2b_3} \theta^2 + \frac{1}{a_1a_2b_3} \theta^3 \right] {}_4F_3 \left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1+1, b_2+1, b_3+1 \end{matrix} \middle| x \right) . \quad (83)$$

In both of these examples, IntegerPart = $\{\dots\}$ corresponds to the set of integer values of parameters in the original hypergeometric function, e.g. IntegerPart = $\{1, 2, 0, 1, 2\}$ in

²All functions in the package HYPERDIRE generate output without additional simplification. This is done for the maximum efficiency of the algorithm. To bring the output into a simpler form, we recommend to use in addition the command **Simplify**. In particular, all considered examples are treated with **FullSimplify**[**ToGroebnerBasis**[...]].

Example 1, and `changeVector = {...}` corresponds to the values of the parameters to be changed, e.g. `changeVector = {-1, -2, 0, 0, -1}` in **Example 1**.

Routine `explicitForm[...]` converts the results of the reduction, Eq. (81), to Mathematica-standard expressions for generalized hypergeometric functions.

Example 3 : Reduction of ${}_2F_1$.

```
answer=ToGroebnerBasis[{1+a1,1+a2},{1+b1},x],
```

```
IntegerPart={1,1,1} changeVector={-1,-1,0},
```

```
explicitForm[answer]
```

$$\frac{\text{HypergeometricPFQ}(\{a_1, a_2\}, \{b_1+1\}, x)}{1-x} + \frac{a_1 a_2 x (a_1 + a_2 - b_1) \text{HypergeometricPFQ}(\{a_1+1, a_2+1\}, \{b_1+2\}, x)}{(b_1+1)(a_1 a_2 - a_1 a_2 x)}.$$

For non-exceptional values of the parameters, the differential reduction is performed with the help of Eqs. (10), (11), (13), and (14). The higher powers of the operators θ^k are expressed with the help of the differential equation for the hypergeometric function, Eq. (9). Also, the following relation is used in some cases:

$${}_pF_q \left(\begin{matrix} \{a_i + m\}_p \\ \{b_k + m\}_q \end{matrix} \middle| z \right) = \frac{\prod_{k=1}^q \{(b_k)_m\}}{\prod_{j=1}^p \{(a_j)_m\}} \left(\frac{d}{dz} \right)^m {}_pF_q \left(\begin{matrix} \{a_i\}_p \\ \{b_k\}_q \end{matrix} \middle| z \right). \quad (84)$$

4.2 Exceptional values of parameters

When some of the upper parameters of the initial hypergeometric function are integer, then the higher powers of the differential operator can be excluded with the help of Eq. (21). In this case, the input of `ToGroebnerBasis[...]` does not change, and the output of `ToGroebnerBasis[...]` has the following structure (in the case when only one upper parameter is integer):

$$\{ \{ \{ Q_1, \dots, Q_p \}, \{ \{ 1+a_1, \dots, 1+a_p \}, \{ 2+b_1, \dots, 2+b_p \}, x \}, \text{factor1} \}, \{ Q_{p+1}, \{ \}, \text{factor2} \} \}, \quad (85)$$

where

- $\{ \{ 1+a_1, \dots, 1+a_{p+1} \}, \{ 2+b_1, \dots, 2+b_p \}, x \}$ are the set of parameters and argument of the resulting hypergeometric function;
- **if an upper parameter is integer, the appropriate a_j is equal to zero;**
- $\{ Q_1, \dots, Q_p \}$ are the rational functions whose numbering corresponds to the power of θ^i , $i = 0, \dots, p-1$;
- factor is an overall factor;
- Q_{p+1} is the resulting polynomial entering the r.h.s. of Eq. (21).

This has the explicit form:

$$\begin{aligned}
& {}_{p+1}F_p \left(\begin{matrix} M_1, \dots, M_r, a_{r+1} + M_{r+1}, \dots, a_{p+1} + M_{p+1} \\ b_1 + K_1, \dots, b_p + K_p \end{matrix} \middle| x \right) \\
&= \text{factor1} \times \left(Q_1 + Q_2\theta + \dots + Q_p\theta^{p-1} \right) {}_{p+1}F_p \left(\begin{matrix} 1, \dots, 1_r, 1 + a_{r+1}, \dots, 1 + a_{p+1} \\ 2 + b_1, \dots, 2 + b_p \end{matrix} \middle| x \right) \\
&+ \text{factor2} \times Q_{p+1}, \tag{86}
\end{aligned}$$

where \vec{M} is a set of integers.

Example 4 : Reduction of ${}_3F_2$ with an integer parameter.

ToGroebnerBasis [{ {3,1+a₂,1+a₃},{2+b₁,2+b₂},x }],

IntegerPart={3,1,1,2,2} changeVector={-2,0,0,0,0},

$$\begin{aligned}
& \left\{ \left\{ \frac{(b_1+1)(b_2+1)-x(a_2+1)(a_3+1)}{2(x-1)} + 1, \frac{1}{2} \left(\frac{-x(a_2+a_3+2)+b_1+b_2+2}{x-1} + 3 \right) \right\}, \right. \\
& \left. \left\{ \{1, a_2 + 1, a_3 + 1\}, \{b_1 + 2, b_2 + 2\}, x, 1 \right\}, \left\{ -\frac{(b_1+1)(b_2+1)}{2(x-1)}, \{\}, 1 \right\} \right\}.
\end{aligned}$$

This has the explicit form:

$$\begin{aligned}
& {}_3F_2 \left(\begin{matrix} 3, a_2+1, a_3+1 \\ b_1+2, b_2+2 \end{matrix} \middle| x \right) = \left[\frac{(b_1+1)(b_2+1)-x(a_2+1)(a_3+1)}{2(x-1)} + 1 \right. \\
& \left. + \frac{1}{2} \left(\frac{-x(a_2+a_3+2)+b_1+b_2+2}{x-1} + 3 \right) \theta \right] {}_3F_2 \left(\begin{matrix} 1, a_2+1, a_3+1 \\ b_1+2, b_2+2 \end{matrix} \middle| x \right) \\
& - \frac{(b_1+1)(b_2+1)}{2(x-1)}.
\end{aligned}$$

Other useful relations which are implemented in the function **ToGroebnerBasis**[...] include the relations derived in Ref. [32]:

$$\begin{aligned}
& {}_pF_q \left(\begin{matrix} b_1 + m_1, \dots, b_n + m_n, a_{n+1}, \dots, a_p \\ b_1, \dots, b_n, b_{n+1}, \dots, b_q \end{matrix} \middle| z \right) \\
&= \sum_{j_1=0}^{m_1} \dots \sum_{j_n=0}^{m_n} A(j_1, \dots, j_n) z^{J_n} {}_{p-n}F_{q-n} \left(\begin{matrix} a_{n+1} + J_n, \dots, a_p + J_n \\ b_{n+1} + J_n, \dots, b_q + J_n \end{matrix} \middle| z \right), \tag{87}
\end{aligned}$$

where m_j are positive integers, $J_n = j_1 + \dots + j_n$, and

$$A(j_1, \dots, j_n) = \binom{m_1}{j_1} \dots \binom{m_n}{j_n} \frac{(b_2+m_2)_{J_1} (b_3+m_3)_{J_2} \dots (b_n+m_n)_{J_{n-1}} (a_{n+1})_{J_n} \dots (a_p)_{J_n}}{(b_1)_{J_1} (b_2)_{J_2} \dots (b_n)_{J_n} (b_{n+1})_{J_n} \dots (b_q)_{J_n}}. \tag{88}$$

Example 5 : Reduction of ${}_3F_2$ with an integer difference of values of parameters.

ToGroebnerBasis [$\{\{3+b_1, 1+a_2, 1+a_3\}, \{2+b_1, 2+b_2\}, x\}$],

IntegerPart= $\{3, 1, 1, 2, 2\}$ changeVector= $\{-1, -1, -1\}$,

$$\left\{ \left\{ -\frac{b_2+1}{(x-1)(b_1+2)}, -\frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)} \right\}, \{\{a_2, a_3\}, \{b_2+1\}, x\}, 1 \right\}.$$

This has the explicit form:

$$\begin{aligned} & {}_3F_2 \left(\begin{array}{c} 3 + b_1, 1 + a_2, 1 + a_3 \\ 2 + b_1, 2 + b_2 \end{array} \middle| x \right) \\ &= \left[-\frac{b_2+1}{(x-1)(b_1+2)} - \frac{(a_2x+a_3x-b_1x-x+b_1-b_2+1)(b_2+1)}{(x-1)xa_2a_3(b_1+2)} \theta \right] {}_2F_1 \left(\begin{array}{c} a_2, a_3 \\ b_2+1 \end{array} \middle| x \right). \end{aligned} \quad (89)$$

The following relations are implemented in Mathematica 7.0:

$$\begin{aligned} \theta^k {}_{p+1}F_p \left(\begin{array}{c} A, \vec{a} \\ 1 + A, \vec{b} \end{array} \middle| z \right) &= (-A)^k {}_{p+1}F_p \left(\begin{array}{c} A, \vec{a} \\ 1 + A, \vec{b} \end{array} \middle| z \right) - \sum_{j=0}^{k-1} (-A)^{k-j} \theta^j {}_pF_{p-1} \left(\begin{array}{c} \vec{a} \\ \vec{b} \end{array} \middle| z \right), \\ \left(\frac{\theta}{A} \right)^q {}_{p+1}F_p \left(\begin{array}{c} \{A\}_r, \vec{a} \\ \{1 + A\}_r, \vec{b} \end{array} \middle| z \right) &= \sum_{j=0}^q (-1)^{(j+q)} \binom{q}{j} {}_{p+1-j}F_{p-j} \left(\begin{array}{c} \{A\}_{r-j}, \vec{a} \\ \{1 + A\}_{r-j}, \vec{b} \end{array} \middle| z \right), \\ {}_pF_q \left(\begin{array}{c} 1, \{a_i\}_{p-1} \\ 2, \{b_k\}_{q-1} \end{array} \middle| z \right) &= \frac{1}{z} \frac{\prod_{l=1}^{q-1} (b_l - 1)}{\prod_{j=1}^{p-1} (a_j - 1)} \left[{}_{p-1}F_{q-1} \left(\begin{array}{c} \{a_i - 1\}_{p-1} \\ \{b_k - 1\}_{q-1} \end{array} \middle| z \right) - 1 \right], \end{aligned} \quad (90)$$

where $q \leq r$ and $a_j, b_k \neq 1$.

5 AppellF1F4 - Mathematica-based program for differential reduction of Appell's functions F_1, F_2, F_3, F_4

In this section, we will present the MATHEMATICA-based³ program **AppellF1F4** for the differential reduction of the Appell hypergeometric functions F_1, F_2, F_3 , and F_4 . The program is available from Ref. [34]. The current version, 1.0, only deals with non-exceptional values of parameters.

The program may be loaded in the standard way:

<< "AppellF1F4.m"

³It was tested for MATHEMATICA 7.0.

The package includes the following basic routines:

$$\mathbf{F1IndexChange}[\text{changingVector}, \text{parameterVector}], \quad (91)$$

$$\mathbf{F2IndexChange}[\text{changingVector}, \text{parameterVector}], \quad (92)$$

$$\mathbf{F3IndexChange}[\text{changingVector}, \text{parameterVector}], \quad (93)$$

$$\mathbf{F4IndexChange}[\text{changingVector}, \text{parameterVector}], \quad (94)$$

$$\mathbf{explicitFormF1}[\dots]. \quad (95)$$

The "changingVector" in Eqs. (91)–(94) is the set of integers at which we wish to change the values of parameters of the Appell functions (the vector \vec{m} in Eqs. (76) and (77)). The set of parameters of the Appell function are defined in the list "parameterVector" (corresponding to the vector $\vec{A} + \vec{m}$ and the arguments x, y on the l.h.s. of Eqs. (76) and (77)). We wish to point out that the enumeration of the parameters and arguments in the list "parameterVector" corresponds one-to-one to Eqs. (33), (44), (55), and (67).

The output of $\mathbf{F1IndexChange}[]$ and $\mathbf{F}\{234\}\mathbf{IndexChange}[]$ has a somewhat different structure.

5.1 Appell function F_1

The structure of the output of $\mathbf{F1IndexChange}[]$ is the following:

$$\{\{A,B,C\}, \{\text{parameterVectorNew}\}, \{\text{AppellF1}\}\}, \quad (96)$$

where

1. parameterVectorNew is the set of new parameters of the Appell function F_1 ;
2. A, B, C are the rational functions corresponding to the ratios $P_0/R, P_1/R$, and P_2/R of the functions entering Eq. (76).

Example 6⁴: Reduction of F_1 .

$$\mathbf{F1IndexChange}[\{1, -1, 0, 0\}, \{a, b_1, b_2, c, z_1, z_2\}],$$

$$\left\{ \left\{ \frac{a(-z_1) + a + b_1 z_1 + b_2 z_2 - c - z_1 + 1}{a - c + 1}, -\frac{(z_1 - 1)(a - b_1 + 1)}{(b_1 - 1)(a - c + 1)}, \frac{z_2 - 1}{a - c + 1} \right\}, \{a + 1, b_1 - 1, b_2, c, z_1, z_2\}, \text{AppellF1} \right\}.$$

This has the explicit form:

$$\begin{aligned} F_1(a, b_1, b_2, c; z_1, z_2) = & \\ & \left[\frac{-az_1 + a + b_1 z_1 + b_2 z_2 - c - z_1 + 1}{a - c + 1} - \frac{(z_1 - 1)(a - b_1 + 1)}{(b_1 - 1)(a - c + 1)} \theta_1 + \frac{z_2 - 1}{a - c + 1} \theta_2 \right] \\ & \times F_1(a + 1, b_1 - 1, b_2, c; z_1, z_2). \end{aligned} \quad (97)$$

⁴ See footnote 2.

Routine **explicitFormF1**[\dots] converts the result of the reduction to a Mathematica standard expression for the Appell function F_1 .

Example 7 : Reduction of F_1 .

result = **F1IndexChange**[[1,-1,0,0], {a,b₁,b₂,c,z₁,z₂}],

explicitFormF1[result]

$$\begin{aligned} & -\frac{(a+1)(z_1-1)z_1(a-b_1+1)}{c(a-c+1)} \text{AppellF1}(a+2; b_1, b_2; c+1; z_1, z_2) \\ & + \frac{(a+1)b_2(z_2-1)z_2}{c(a-c+1)} \text{AppellF1}(a+2; b_1-1, b_2+1; c+1; z_1, z_2) \\ & + \frac{(a(-z_1)+a+b_1z_1+b_2z_2-c-z_1+1)}{a-c+1} \text{AppellF1}(a+1; b_1-1, b_2; c; z_1, z_2) . \end{aligned}$$

5.2 Appell functions F_2, F_3, F_4

The outputs of **F2IndexChange**[], **F3IndexChange**[], and **F4IndexChange**[] are similar and have the following structure:

$$\{\{A,B,C,D\}, \{\text{parameterVectorNew}\}, \{\text{NameOfFunction}\}, \quad (98)$$

where

1. NameOfFunction is the name of the Appell functions to be reduced: AppellF2, AppellF3, or AppellF4 ;
2. parameterVectorNew is the set of new parameters of the Appell function;
3. A, B, C, D are the rational functions corresponding to the ratios $Q_0/S, Q_1/S, Q_2/S,$ and Q_3/S of the functions entering Eq. (77).

Example 8 : Reduction of F_2 .

F2IndexChange[[0,0,1,1,0], {a,b₁,b₂,c₁,c₂,z₁,z₂}],

$$\left\{ \left\{ \frac{az_2(c_1(z_1-1)-b_1z_1)}{c_1(z_1-1)(b_2-c_2+1)} + 1, \frac{1-\frac{z_2(a+z_1(b_1-c_1))}{(z_1-1)(b_2-c_2+1)}}{c_1}, \frac{c_1(z_1-1)(z_2-1)-b_1z_1z_2}{c_1(z_1-1)(b_2-c_2+1)}, -\frac{z_1+z_2-1}{c_1(z_1-1)(b_2-c_2+1)} \right\}, \right. \\ \left. \{a, b_1, b_2+1, c_1+1, c_2, z_1, z_2\}, \text{AppellF2} \right\}.$$

This has the explicit form:

$$\begin{aligned} & F_2(a, b_1, b_2, c_1, c_2, z_1, z_2) = \\ & \left[\frac{az_2(c_1(z_1-1)-b_1z_1)}{c_1(z_1-1)(b_2-c_2+1)} + 1 + \frac{1-\frac{z_2(a+z_1(b_1-c_1))}{(z_1-1)(b_2-c_2+1)}}{c_1} \theta_1 + \frac{c_1(z_1-1)(z_2-1)-b_1z_1z_2}{c_1(z_1-1)(b_2-c_2+1)} \theta_2 \right. \\ & \left. - \frac{z_1+z_2-1}{c_1(z_1-1)(b_2-c_2+1)} \theta_1 \theta_2 \right] F_2(a, b_1, b_2+1, c_1+1, c_2; z_1, z_2) . \quad (99) \end{aligned}$$

The Appell functions F_2 , F_3 , and F_4 are not built in the current version of Mathematica (version 7.0), so that the **AppellF1F4** package does not include a function similar to **explicitFormF1**[].

6 Conclusion

The differential-reduction algorithm [9] allows one to compare Horn-type hypergeometric functions with parameters whose values differ by integers. In this paper, we presented the Mathematica-based package **HYPERDIRE** for the differential reduction of the generalized hypergeometric functions ${}_pF_p$ and the Appell functions F_1, F_2, F_3 , and F_4 to sets of basis functions, defined by Eqs. (19), (76), (77), respectively. These functions are closely related to a large class of Feynman diagrams [5]. In contrast to existing packages, our package performs the reduction of hypergeometric functions before the ε expansion and works with arbitrary values of parameters. This package could be easily extended to other Horn-type hypergeometric functions by adding new modules.

As an illustration of our approach, we considered a few examples [15, 16] with arbitrary powers of propagators and space-time dimension [33].

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References

- [1] C.F. Gauss, *Gesammelte Werke*, vol. 3, Teubner, Leipzig, 1823, pp. 1866–1929.
- [2] P. Appell, J. Kampé de Fériet, *Fonctions Hypergeometriques et Hyperspheriques. Polynomes d’Hermite*, Gauthier-Villars, Paris, 1926.
- [3] W.N. Bailey, *Generalized Hypergeometric Series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, New York 1964;
 A. Erdelyi (Ed.), *Higher Transcendental Functions*, vol.I, McGraw-Hill, New York, 1953;
 L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge 1966;
 H. Exton, *Multiple hypergeometric functions and applications*, Ellis Horwood Ltd., Chichester; Halsted Press, New York-London-Sydney, 1976;

- H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian hypergeometric series*, Ellis Horwood Ltd., Chichester; Halsted Press, New York, 1985.
- [4] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, *Funct. Anal. i Priloz.* **23** (1989) 94;
 I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky, *Adv. Math.* **84** (1990) 255;
 I.M. Gel'fand, M.I. Graev, V.S. Retakh, *Russian Math. Surveys* **47** (1992) 1.
- [5] V.A. Smirnov, *Evaluating Feynman integrals*, Springer Tracts Mod. Phys. 211 (2004) 1;
 V.A. Smirnov, *Feynman Integral Calculus*, Springer, Berlin, 2006.
- [6] E.D. Rainville, *Bull. Amer. Math. Soc.* **51** (1945) 714;
 J. Wimp, *Math. Comp.* **22** (1968) 363;
 W. Miller Jr., *SIAM J. Math. Anal.*, **3** (1972) 31; **4** (1973) 638; **5** (1974) 309; *J. Math. Phys.* **13** (1972) 1393; *SIAM J. Appl. Math.*, **25** (1973) 226;
 T. Sasaki, *SIAM J. Math. Anal.* **22** (1991) 821;
 E. Horikawa, *J. Math. Sci. Univ. Tokyo* **1** (1994) 181;
 A. Adolphson, B. Dwork, *Trans. Amer. Math. Soc.* **347** (1995) 615;
 K. Roach, *Proc. of ISSAC'96*, 301, ACM, New York;
 K. Roach, *Proc. of ISSAC'99*, 205, ACM, New York;
 R.G. Buschman, *J. Comput. Appl. Math.* **107** (1999) 127.
- [7] J.A. Mullen, *SIAM J. Appl. Math.* **14** (1966) 1152.
- [8] R.P. Singal, *SIAM J. Math. Anal.* **11** (1980) 390.
- [9] N. Takayama, *Jpn. J. Appl. Math.* **6** (1989) 147.
- [10] N. Takayama, *J. Symbolic Comput.* **20** (1995) 637.
- [11] B. Sturmfels, N. Takayama, *Gröbner Bases and Hypergeometric Functions*, London Math. Soc. Lecture Note Ser., **251**, Cambridge Univ. Press, Cambridge, 1998, pp. 246–258;
 M. Saito, B. Sturmfels, N. Takayama, *Compositio Math.* **115** (1999) 185;
 M. Saito, B. Sturmfels, N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer, Berlin, 2000.
- [12] M. Bronstein, M. Petkovšek, *Theoret. Comput. Sci.* **157** (1996) 3;
 F. Chyzak, B. Salvy, *J. Symbolic Comput.* **26** (1998) 187;
 F. Chyzak, *Gröbner Bases, Symbolic Summation and Symbolic Integration*, London Math. Soc. Lecture Note Ser., **251**, Cambridge Univ. Press, Cambridge, 1998, pp. 32–60.
- [13] C. Krattenthaler, *J. Symbolic Comput.* **20** (1995) 737.
- [14] P. Paule, *Contiguous Relations and Creative Telescoping*, Technical Report, RISC, Austria, 2001;

- Ch. Koutschan, Ph.D. Thesis, *Advanced Applications of the Holonomic Systems Approach*, RISC, Johannes Kepler University, Linz, Austria, 2009.
- [15] M.Yu. Kalmykov, J. High Energy Phys. 04 (2006) 056;
M.Yu. Kalmykov, V.V. Bytev, B.A. Kniehl, B.F.L. Ward, S.A. Yost, PoS ACAT **08** (2008) 125;
V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, B.F.L. Ward, S.A. Yost, arXiv:0902.1352 [hep-th];
S.A. Yost, V.V. Bytev, M.Y. Kalmykov, B.A. Kniehl, B.F.L. Ward, arXiv:1110.0210 [math-ph];
M.Yu. Kalmykov, B.A. Kniehl, Phys. Lett. B **702** (2011) 268.
- [16] V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, Nucl. Phys. B **836** (2010) 129.
- [17] M.Yu. Kalmykov, B.F.L. Ward, S. Yost, J. High Energy Phys. 02 (2007) 040. J. High Energy Phys. 11 (2007) 009;
S.A. Yost, M.Yu. Kalmykov, B.F.L. Ward, Proc. of ICHEP 2008 [arXiv:0808.2605];
M.Yu. Kalmykov, B.A. Kniehl, B.F.L. Ward, S.A. Yost, arXiv:0810.3238 [hep-th];
M.Yu. Kalmykov, B. Kniehl, Nucl. Phys. B **809** (2009) 365;
M.Yu. Kalmykov, B. Kniehl, Phys. Part. Nucl. **41** (2010) 942.
- [18] <http://www.wolfram.com/>
- [19] S. Weinzierl, Comput. Phys. Commun. **145** (2002) 357;
S. Moch, P. Uwer, Comput. Phys. Commun. **174** (2006) 759.
- [20] T. Huber, D. Maître, Comput. Phys. Commun. **175** (2006) 122; **178** (2008) 755.
- [21] J. Blümlein, S. Klein, C. Schneider F. Stan, J. Symbolic Comput. **47** (2012) 1267.
- [22] Z.W. Huang, J. Liu, arXiv:1209.3971 [physics.comp-ph],
- [23] C. Anzai, Y. Sumino, arXiv:1211.5204 [hep-th].
- [24] D. Greynat, J. Sesma, arXiv:1302.2423 [math-ph] —
- [25] S. Moch, P. Uwer, S. Weinzierl, J. Math. Phys. **43** (2002) 3363;
S. Weinzierl, J. Math. Phys. **45** (2004) 2656.
- [26] A.I. Davydychev, M.Yu. Kalmykov, Nucl. Phys. B **699** (2004) 3;
M.Yu. Kalmykov, Nucl. Phys. Proc. Suppl. **135** (2004) 280;
F. Jegerlehner, M.Yu. Kalmykov, O. Veretin, Nucl. Phys. B **658** (2003) 49;
M.Yu. Kalmykov, B.F.L. Ward, S. Yost, JHEP **0710** (2007) 048.
- [27] V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, [arXiv:0904.0214v2], Chapter 4.
- [28] M.Yu. Kalmykov, <http://theor.jinr.ru/~kalmykov/hypergeom/hyper.html>

- [29] E. Bod, J. Differential Equations **252** (2012) 541.
- [30] F. Beukers, arXiv:1101.0493 [math.AG].
- [31] M.Yu. Kalmykov, B.A. Kniehl, Phys. Lett. B **714** (2012) 103.
- [32] P.W. Karlsson, J. Math. Phys. **12** (1971) 270.
- [33] G. 't Hooft, M. Veltman, Nucl. Phys. B **44** (1972) 189;
C.G. Bollini, J.J. Giambiagi, Nuovo Cim. B **12** (1972) 20;
J.F. Ashmore, Lett. Nuovo Cim. **4** (1972) 289;
G.M. Cicuta, E. Montaldi, Lett. Nuovo Cim. **4** (1972) 329.
- [34] V.V. Bytev, <https://sites.google.com/site/loopcalculations/home>.