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Self-dual Continuous Series of Representations for $\mathcal{U}_q(sl(2))$ and $\mathcal{U}_q(osp(1|2))$

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Abstract

We determine the Clebsch-Gordan and Racah-Wigner coefficients for continuous series of representations of the quantum deformed algebras $\mathcal{U}_q(sl(2))$ and $\mathcal{U}_q(osp(1|2))$. While our results for the former algebra reproduce formulas by Ponsot and Teschner, the expressions for the orthosymplectic algebra are new. Up to some normalization factors, the associated Racah-Wigner coefficients are shown to agree with the fusing matrix in the Neveu-Schwarz sector of $N = 1$ supersymmetric Liouville field theory.

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Contents

1 Introduction

Quantum deformed Lie (super-)algebras are being studied partly because of their intimate relation with 2-dimensional conformal field theory and 3-dimensional Chern-Simons theory. Through the investigation of finite dimensional representations of q-deformed Lie algebras, for example, one can obtain solutions of the Moore-Seiberg polynomial equations which describe the fusing and braiding structure of Wess-Zumino-Witten (WZW) models with compact target group G. Because of the deep relation between WZW models and Chern-Simons theory, the same data also appear as building blocks for expectation values of Wilson loops in 3-dimensional topological theories.

The quantum deformed superalgebra $\mathcal{U}_q(sl(2))$ is known to possess a very interesting self-dual series of infinite dimensional representations, see e.g. [1]. As shown by Ponsot and Teschner [2, 3], this series furnishes a solution of the Moore-Seiberg relations which is relevant for Liouville conformal field theory. In fact, the Racah-Wigner coefficients (6j symbols) for these representations are known to agree with the fusing matrix of Liouville theory, up to some convention dependent normalization. The same data also appear in the context of SL(2) Chern-Simons theory or quantum Teichmüller theory [4, 5, 6, 7, 8]. The basic building block in this case is Faddeev's quantum dilogarithm [9],

$$\Phi_{\mathrm{b}}^{+}(z) = \Phi_{\mathrm{b}}(z) = \exp\left(\int_C \frac{e^{-2izw}}{\sinh(w\mathrm{b})\sinh(w/\mathrm{b})}\frac{dw}{4w}\right) . \tag{1.1}$$

This special function plays an important role in mathematical physics. It has many beautiful properties, in particular it satisfies a five term (pentagon) relation [9]. It may be considered as a quantization of Roger's five-term identity for the ordinary dilogarithm and can be formulated as an integral identity. The latter is also known as Ramanujan's identity.

Our goal here is to extend the results described in the previous paragraph to the Lie superalgebra $\mathcal{U}_q(osp(1|2))$. This algebra was first introduced in [10]. Finite dimensional representations and their Racah-Wigner coefficients were studied in [11] and other subsequent papers. The series of representations we are about to analyse below allows us to obtain the fusing matrix of N=1 Liouville field theory. In building the basic representation theoretic data, and in particular the Racah-Wigner coefficients, Faddeev's quantum dilogarithm gets accompanied by a second function

$$\Phi_{\mathrm{b}}^{-}(z) = \exp\left(\int_C \frac{e^{-2izw}}{\cosh(w\mathrm{b})\cosh(w/\mathrm{b})}\frac{dw}{4w}\right) . \tag{1.2}$$

From $\Phi_b^\pm$ we shall pass to the functions $\Phi_b^\nu, \nu = 0, 1$, which are obtained through

$$\Phi_b^\nu(z) = \log \Phi_b^-(z) - (-1)^\nu \log \Phi_b^+(z) \ . \tag{1.3}$$

The precise relation of Φ_b^ν to the functions S_ν that appear below is spelled out in Appendix A.2. Our functions Φ_b^ν share many features with Faddeev's quantum dilogarithm. In particular, they satisfy the same set of integral identities with an additional sum over the ν index wherever the standard identities involve an integration over x. These identities include a new variant of the pentagon relation for Faddeev's quantum dilogarithm. Clearly, the pair Φ_b^ν should play a central role for the $N = 1$ supersymmetric extension of quantum Teichmüller theory.

The plan of this paper is as follows. The first part is devoted to $\mathcal{U}_q(sl(2))$. The main purpose here is to review (and correct) the known results and to explain how they are proved. This will help us later to analyse representations of $\mathcal{U}_q(osp(1|2))$. In fact, with the right notations, most formulas for the supersymmetric algebra resemble those for $\mathcal{U}_q(sl(2))$. In addition, the strategy of the proofs can essentially be carried over form the non-supersymmetric theory. The first part commences with a short description of $\mathcal{U}_q(sl(2))$ and its self-dual representations. Then we construct the Clebsch-Gordan maps for the decomposition of products of self-dual representations and prove their intertwining and orthonormality property. Section 4 contains the Racah-Wigner coefficients for $\mathcal{U}_q(sl(2))$. The same steps are then taken in the second part on the quantum deformed algebra $\mathcal{U}_q(osp(1|2))$. Once more, we describe the algebra and a series of self-dual representations whose Clebsch-Gordan coefficients are determined in section 6. Our main result for the associated Racah-Wigner coefficients is contained in section 7. Finally, we compare our expressions for the Racah-Wigner coefficients of $\mathcal{U}_q(osp(1|2))$ with the known formulas for the fusion matrix of $N = 1$ Liouville field theory [12, 13, 14] and find agreement. The paper concludes with a list of open problems and further directions to explore.

2 Self-dual continuous series for $\mathcal{U}_q(sl(2))$

The goal of this section is to introduce the quantum deformed enveloping algebra $\mathcal{U}_q(sl(2))$ along with a continuous series of representations that was first discussed by Schmuedgen in [1] and further analysed by Ponsot and Teschner [2, 3]. It is self-dual under the replacement $b \to b^{-1}$ in a sense to be made more precise below.

The q-deformed universal enveloping algebra $\mathcal{U}_q(sl(2))$ of the Lie algebra $sl(2)$ is generated by the elements $K, K^{-1}, E^{\pm}$, with relations

$$KE^{\pm} = q^{\pm 1} E^{\pm} K,$$
$$[E^{+}, E^{-}] = -\frac{K^2 - K^{-2}}{q - q^{-1}},$$

where $q = e^{i\pi \mathrm{b}^2}$ is the deformation parameter. We shall parametrize the deformation through a real number b so that q takes values on the unit circle. Given such a choice, the deformed algebra comes equipped with the following *-structure

$$K^{*} \; = \; K \quad , \quad (E^{\pm})^{*} \; = \; E^{\pm} \; . \tag{2.1}$$

The tensor product of any two representations can be built with the help of the following co-product

$$\Delta(K) \; = \; K \otimes K \quad , \quad \Delta(E^{\pm}) \; = \; E^{\pm} \otimes K + K^{-1} \otimes E^{\pm} \; . \tag{2.2}$$

Finally, there is one more object we shall need below, namely the quadratic Casimir element C of $\mathcal{U}_q(sl(2))$ which reads

$$C \; = \; E^{-}E^{+} - \frac{qK^2 + q^{-1}K^{-2} + 2}{(q - q^{-1})^2} \; .$$

Having collected the most important formulas concerning the algebraic structure, we now want to introduce the series of representations we are going to analyse in this work. It is parametrized by a label α that takes values in $\alpha \in \frac{Q}{2} + i\mathbb{R}$, where Q is related to the deformation parameter through $Q = \mathrm{b} + \frac{1}{\mathrm{b}}$. The carrier spaces $\mathcal{P}_\alpha$ of the associated representations consist of entire analytic functions $f(x)$ in one variable x whose Fourier transform $\hat{f}(\omega)$ is meromorphic in the complex plane with possible poles in

$$\mathcal{S}_\alpha := \{ \; \omega = \pm i(\alpha - Q - n\mathrm{b} - m\mathrm{b}^{-1}); n, m \in \mathbb{Z}^{\leq 0} \; \} \; . \tag{2.3}$$

On this space, we represent the element K through a shift operator in the imaginary direction,

$$\pi_\alpha(K) \; = \; e^{\frac{i\mathrm{b}}{2}\partial_x} \; =: \; T_x^{\frac{i\mathrm{b}}{2}} \; . \tag{2.4}$$

By construction, the operator T_x^{ia} defined in the previous equation acts on functions $f \in \mathcal{P}_\alpha$ as

$$T_x^a f(x) := f(x + a) \; . \tag{2.5}$$

The expressions for the remaining two generators $E^{\pm}$ are linear combinations of two shift operators in opposite directions

$$\pi_\alpha(E^{\pm}) \; = \; e^{\pm 2\pi \mathrm{b} x} \, \frac{e^{\pm i\pi\mathrm{b}\alpha} \, T^{\frac{i\mathrm{b}}{2}} - e^{\mp i\pi\mathrm{b}\alpha} \, T^{-\frac{i\mathrm{b}}{2}}}{q-q^{-1}} \; =: \; e^{\pm 2\pi\mathrm{b}x}[(2\pi)^{-1}\partial_x \pm \bar{\alpha}]_{\mathrm{b}} \; . \tag{2.6}$$

Here and in the following we shall use the symbol $\bar{\alpha}$ to denote $\bar{\alpha} = Q - \alpha$ and we introduced the following notation

$$[x]_{\mathrm{b}} \; = \; \frac{\sin(\pi\mathrm{b}x)}{\sin(\pi\mathrm{b}^2)}. \tag{2.7}$$

We claimed before that the representations π_α are self-dual in a certain sense. Now we can make this statement more precise. To this end, let us define a second action $\tilde{\pi}_\alpha$ of $\mathcal{U}_{\tilde{q}}(sl(2))$ with $\tilde{q} = \exp(i\pi/\mathrm{b}^2)$ on the space $\mathcal{P}_\alpha$ through the formulas (2.4) and (2.6) with b replaced by b^{-1}. Remarkably, the two actions π_α and $\tilde{\pi}_\alpha$ commute with each other. This is the self-duality property we were referring to.

3 The Clebsch-Gordan coefficients for $\mathcal{U}_q(sl(2))$

The action $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$ of the quantum universal enveloping algebra $\mathcal{U}_q(sl(2))$ on the tensor product of any two representations π_{α_1} and π_{α_2} is defined in terms of the coproduct, as usual. Such a tensor product is reducible and its decomposition into a direct sum of irreducibles is what defines the Clebsh-Gordan coefficients. In this case at hand, one has the following decomposition,

$$\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \; \simeq \; \int_{\frac{Q}{2}+i\mathbb{R}^+}^{\otimes} \mathrm{d}\alpha_3 \, \mathcal{P}_{\alpha_3}.$$

We are going to spell out and prove an explicit formula for the homomorphism

$$f(x_2,x_1) \to F_f(\alpha_3,x_3) \; = \; \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} f(x_2,x_1) \; .$$

Here, $f(x_2,x_1)$ denotes an element in $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ and $F_f(\alpha_3,x_3)$ is its image in $\mathcal{P}_{\alpha_3}$. In order state a formula for the Clebsch-Gordan map, we introduce

$$D(z;\alpha) = \frac{S_{\mathrm{b}}(z)}{S_{\mathrm{b}}(z+\alpha)}, \tag{3.1}$$

and

$$z_{21} = ix_{12} - Q + \frac{1}{2}(2\bar{\alpha}_3 + \bar{\alpha}_1 + \bar{\alpha}_2),$$
$$z_{31} = ix_{13} + \frac{1}{2}(\bar{\alpha}_1 - \bar{\alpha}_3),$$
$$z_{32} = ix_{32} + \frac{1}{2}(\bar{\alpha}_2 - \bar{\alpha}_3),$$

where $\bar{\alpha}_i \in Q/2 + i\mathbb{R}$ is defined as before through $\bar{\alpha}_i = Q - \alpha_i$ and we used $x_{ij} = x_i - x_j$. The symbols α_{ij} stand for

$$\alpha_{21} = \alpha_1 + \alpha_2 + \alpha_3 - Q \quad , \quad \alpha_{31} = Q + \alpha_1 - \alpha_2 - \alpha_3 \quad , \quad \alpha_{32} = Q - \alpha_1 + \alpha_2 - \alpha_3 \ .$$

With all these notations, we are finally able to spell out the relevant Clebsh-Gordan coefficients [3],

$$\begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = \mathcal{N} D(z_{21}; \alpha_{21}) D(z_{23}; \alpha_{23}) D(z_{13}; \alpha_{13}) \ , \tag{3.2}$$

where

$$\mathcal{N} = \exp\left[-\frac{i\pi}{2}(\bar{\alpha}_3\alpha_3 - \bar{\alpha}_2\alpha_2 - \bar{\alpha}_1\alpha_1)\right] \ . \tag{3.3}$$

Let us note that this product form of the Clebsch-Gordan coefficients is familiar e.g. from the 3-point functions in conformal field theory which may be written as a product. Although the representations we study here are not obtained by deforming discrete series representations of $sl(2)$, i.e. of those representations that fields of a conformal field theory transform in, the familiar product structure of the Clebsch-Gordan coefficients survives.

3.1 The intertwining property

The fundamental intertwining property of the Clebsch-Gordan coefficients takes the following form

$$\pi_{\alpha_3}(X) \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} (\pi_{\alpha_2} \otimes \pi_{\alpha_1}) \Delta(X) \tag{3.4}$$

for $X = K, E^{\pm}$. The equation should be interpreted as an identity of operators in the representation space $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. While the operators K and $E^{\pm}$ may be expressed through multiplication and shift operators, the Clebsch-Gordan map itself provides the kernel of an integral transform. With the help of partial integration, we can re-write the intertwining relation as an identity for the integral kernel,

$$\pi_{\alpha_3}(X) \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix} = (\pi_{\alpha_2} \otimes \pi_{\alpha_1}) \Delta^t(X) \begin{bmatrix} \alpha_3 \, \alpha_2 \, \alpha_1 \\ x_3 \, x_2 \, x_1 \end{bmatrix}, \tag{3.5}$$

where the superscript ^{t} means that we should replace all shift operators by shifts in the opposite direction, i.e. $(T_x^{ia})^t = T_x^{-ia}$ and exchange the order between multiplication and shifts, i.e $(f(x)T_x^{ia})^t = T_x^{-ia} f(x)$. In this new form, the intertwining property is simply an identity of functions in the variables x_i.

One can check eq. (3.5) by direct computation. This is particularly easy for the element K for which eq. (3.5) reads

$$T_{x_3}^{\frac{i\mathrm{b}}{2}} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} = T_{x_2}^{-\frac{i\mathrm{b}}{2}} T_{x_1}^{-\frac{i\mathrm{b}}{2}} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} . \tag{3.6}$$

Since the Clebsch-Gordan maps depend only in the differences x_{ij} we can replace $T_{x_1} = T_{12}T_{13}$ etc. where T_{ij} denotes a shift operator acting on x_{ij}. Consequently, the intertwining property for K becomes

$$T_{13}^{-\frac{i\mathrm{b}}{2}} T_{23}^{-\frac{i\mathrm{b}}{2}} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} = T_{12}^{\frac{i\mathrm{b}}{2}} T_{23}^{-\frac{i\mathrm{b}}{2}} T_{12}^{-\frac{i\mathrm{b}}{2}} T_{13}^{-\frac{i\mathrm{b}}{2}} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix}, \tag{3.7}$$

which is trivially satisfied since all shifts commute. This concludes the proof of the intertwining property (3.5) for $X = K$.

For $X = E^+$ the check is a bit more elaborate. Using the anti-symmetry $[-x]_\mathrm{b} = -[x]_\mathrm{b}$ of the function (2.7) and the property $\partial_x^t = -\partial_x$ of derivatives, we obtain

$$\begin{aligned} & e^{2\pi \mathrm{b} x_3} [\delta_{x_3} + \bar\alpha_3]_\mathrm{b} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} = \\ & - [\delta_{x_2} - \bar\alpha_2]_\mathrm{b} \, e^{2\pi b x_2} \, T_{x_1}^{\frac{i\mathrm{b}}{2}} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} - [\delta_{x_1} - \bar\alpha_1]_\mathrm{b} \, e^{2\pi \mathrm{b} x_1} \, T_{x_2}^{-\frac{i\mathrm{b}}{2}} \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} . \end{aligned}$$

where $\delta_x = (2\pi)^{-1}\partial_x$. After a bit of rewriting we find

$$\begin{aligned} \Big[& e^{i\pi\mathrm{b}(\bar\alpha_1 - \bar\alpha_2)/2} [-ix_{21} + Q - \frac{1}{2}(\bar\alpha_2 + \bar\alpha_1)]_\mathrm{b} T_{21}^{i\mathrm{b}} T_{23}^{ib} \\ & + e^{-\pi \mathrm{b} x_{23}} e^{-i\pi\mathrm{b}(\bar\alpha_3 + \bar\alpha_1)/2} [-ix_{13} + Q + \frac{1}{2}(\bar\alpha_3 - \bar\alpha_1)]_\mathrm{b} T_{13}^{i\mathrm{b}} T_{23}^{i\mathrm{b}} \\ & - e^{-i\pi\mathrm{b}Q} e^{-\pi \mathrm{b} x_{13}} e^{i\pi\mathrm{b}(\bar\alpha_2 + \bar\alpha_3)/2} [-ix_{23} + \frac{1}{2}(\bar\alpha_2 - \bar\alpha_3)]_\mathrm{b} \Big] \begin{bmatrix} \alpha_3 \; \alpha_2 \; \alpha_1 \\ x_3 \; x_2 \; x_1 \end{bmatrix} = 0. \end{aligned}$$

Now, because of the shift properties of the function S_b, see Appendix A.1, we have

$$T_x^{i\mathrm{b}} \frac{S_\mathrm{b}(-ix + a_1)}{S_\mathrm{b}(-ix + a_2)} = \frac{[-ix + a_1]_\mathrm{b}}{[-ix + a_2]_\mathrm{b}} \frac{S_\mathrm{b}(-ix + a_1)}{S_\mathrm{b}(-ix + a_2)} T_x^{i\mathrm{b}}.$$

With the help of this equation it is easy to check that our Clebsch-Gordan coefficients obey the desired intertwining relation with E^+. For the intertwining property involving $X = E^-$ one proceeds in a similar way.

3.2 Orthogonality and Completeness

The Clebsh-Gordan coefficients for the self-dual series of $\mathcal{U}_q(sl(2))$ satisfy the following orthogonality and completeness relation

$$\int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_3\,\alpha_2\,\alpha_1 \\ x_3\,x_2\,x_1 \end{bmatrix}^* \begin{bmatrix} \beta_3\,\alpha_2\,\alpha_1 \\ y_3\,x_2\,x_1 \end{bmatrix} = |S_{\mathrm{b}}(2\alpha_3)|^{-2}\delta(\alpha_3 - \beta_3)\delta(x_3 - y_3), \tag{3.8}$$

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} \mathrm{d}\alpha_3 \int_{\mathbb{R}} \mathrm{d}x_3 |S_{\mathrm{b}}(2\alpha_3)|^2 \begin{bmatrix} \alpha_3\,\alpha_2\,\alpha_1 \\ x_3\,x_2\,x_1 \end{bmatrix}^* \begin{bmatrix} \alpha_3\,\alpha_2\,\alpha_1 \\ x_3\,y_2\,y_1 \end{bmatrix} = \delta(x_2 - y_2)\delta(x_1 - y_1)\ . \tag{3.9}$$

Except for the normalizing factor on the right hand side of eq. (3.8), these relations follow from the intertwining properties of Clebsch-Gordan maps. Since the complete proof of eqs. (3.8) and (3.9) has not been spelled out in the literature we will discuss the derivation of eq. (3.8) in some detail here. This will allow us to skip over some details later when we discuss the corresponding issues for the deformed superalgebra. In order to compute the integral on the left hand side of eq. (3.8) we shall employ a star-triangle relation for the functions S_{b} along with several of its corollaries. All necessary integral formulas are collected in Appendix B.1.

Before we proceed proving the orthogonality relations, let us point out that the equations (3.8) involve products of the Clebsch-Gordan kernels. Since these are distributional kernels, one must take some care when multiplying two of them. Following [3], the strategy is to regularize the Clebsch-Gordan maps through some ϵ prescription, then to multiply the regularized kernels before we send the parameter ϵ to zero in the very end of the computation. For the problem at hand, one appropriate regularization takes the form

$$\begin{bmatrix} \alpha_3\;\alpha_2\;\alpha_1 \\ x_3\;x_2\;x_1 \end{bmatrix}_\epsilon = \mathcal{N} D(z_{21}+\epsilon;\alpha_{21}-\epsilon) D(z_{32}+\epsilon;\alpha_{32}-\epsilon) D(z_{31};\alpha_{31}+\epsilon), \tag{3.10}$$

with the same normalization (3.3) as above. Our prescription is different from the one used in [3].

Inserting the regularized Clebsch-Gordan maps into the orthogonality relation (3.8), we obtain

$$\int \mathrm{d}x_2\mathrm{d}x_1 \begin{bmatrix} \alpha_3\,\alpha_2\,\alpha_1 \\ x_3\;x_2\;x_1 \end{bmatrix}^*_\epsilon \begin{bmatrix} \beta_3\,\alpha_2\,\alpha_1 \\ y_3\;x_2\;x_1 \end{bmatrix}_\epsilon = \eta \int \mathrm{d}x_2\mathrm{d}x_1 \frac{S_\mathrm{b}(-ix_{21}+Q-\frac{1}{2}(\bar{\alpha}_2+\bar{\alpha}_1))}{S_\mathrm{b}(-ix_{21}+Q-\frac{1}{2}(\bar{\alpha}_2+\bar{\alpha}_1))} \times$$
$$\times\, S_\mathrm{b}(-ix_{21}-Q+\frac{1}{2}(2\bar{\beta}_3+\bar{\alpha}_1+\bar{\alpha}_2)+\epsilon)\; S_\mathrm{b}(ix_{21}+Q-\frac{1}{2}(2\bar{\alpha}_3+\bar{\alpha}_1+\bar{\alpha}_2)+\epsilon)\times$$
$$\times\, S_\mathrm{b}(-i(x_1-y_3)+Q+\frac{1}{2}(\bar{\alpha}_1-\bar{\beta}_3-2\bar{\alpha}_2)-\epsilon) S_\mathrm{b}(ix_{13}-\frac{1}{2}(\bar{\alpha}_1-\bar{\alpha}_3-2\bar{\alpha}_2)-\epsilon)\times$$
$$\times\, S_\mathrm{b}(-ix_{13}+\frac{1}{2}(\bar{\alpha}_3-\bar{\alpha}_1))\; S_\mathrm{b}(-i(x_1-y_3)-\frac{1}{2}(\bar{\beta}_3-\bar{\alpha}_1))\times$$
$$\times\;\; D^*(z_{32}+\epsilon,\alpha_{32}-\epsilon)\; D(\tilde{z}_{32}+\epsilon,\tilde{\alpha}_{32}-\epsilon) \;\; =: \;\; I_1^\epsilon\;.$$

In writing this expression we have expressed all the D-functions that contain some dependence on the variable x_1 in terms of S_b, see eq. (3.1). We brought all but three of the S_b functions to the numerator with the help of the property $S_\mathrm{b}^{-1}(x) = S_\mathrm{b}(Q-x)$. In taking the complex conjugate, we used that the variables x_i, y_3 and our regulator ϵ are real. The labels α_i and β_3, on the other hand, satisfy $\alpha_i^* = \bar{\alpha}_i = Q - \alpha_i$ and $\beta_3^* = \bar{\beta}_3 = Q - \beta_3$. Finally, we introduced $\tilde{\mathcal{N}}, \tilde{z}_{32}$ and $\tilde{\alpha}_{32}$. These are obtained from $\mathcal{N}, z_{32}$ and α_{32} by the substitution $x_3 \to y_3$ and $\alpha_3 \to \beta_3$. The constant prefactor η is given by $\eta = \mathcal{N}\tilde{\mathcal{N}}$.

Before we continue our evaluation of the integrals we note that the fraction of S-functions in the first line of the previous equation cancels out. Hence, we are left with a product of six S-functions that contain all the x_1 dependence of the integrand. It turns out that we can actually evaluate the x_1 integral with the help of the following star-triangle equation, see e.g. [15],

$$\int \mathrm{d}x_1 \prod_{i=1}^{3} S_\mathrm{b}(ix_1+\gamma_i) S_\mathrm{b}(-ix_1+\delta_i) = \prod_{i,j=1}^{3} S_\mathrm{b}(\gamma_i+\delta_j) \qquad (3.11)$$

which holds as long as the arguments on the left hand side add up to Q, i.e. if

$$\sum_{i=1}^{3}(\gamma_i+\delta_i) = Q,$$

It is not difficult to check that the arguments which appear in our formula for I_1^ϵ above satisfy this condition. Hence, we can perform the integral over x_1 to obtain

$$I_1^\epsilon = \eta S_{\rm b}(\bar\beta_3 - \bar\alpha_3 + 2\epsilon)\frac{S_{\rm b}(-i(y_3-x_3)+\frac{1}{2}(\bar\alpha_3-\bar\beta_3))}{S_{\rm b}(-i(y_3-x_3)-\frac{1}{2}(\bar\alpha_3-\bar\beta_3)+2\epsilon)}\frac{S_{\rm b}(\bar\alpha_3+\bar\alpha_2-\bar\alpha_1-\epsilon)}{S_{\rm b}(\bar\beta_3+\bar\alpha_2-\bar\alpha_1+\epsilon)}\times$$

$$\times\int {\rm d}x_2 \frac{S_{\rm b}(-ix_{23}-Q+\frac{1}{2}(2\bar\beta_3+\bar\alpha_2+\bar\alpha_3)+\epsilon)}{S_{\rm b}(-i(x_2-y_3)+\frac{1}{2}(2\bar\alpha_3+\bar\alpha_2+\bar\beta_3)-\epsilon))}\frac{S_{\rm b}(-i(x_2-y_3)+\frac{1}{2}(\bar\alpha_2-\bar\beta_3)+\epsilon)}{S_{\rm b}(-ix_{23}+Q+\frac{1}{2}(\bar\alpha_2-\bar\alpha_3)-\epsilon)} =$$

$$= \eta S_{\rm b}(\bar\beta_3 - \bar\alpha_3 + 2\epsilon)\frac{S_{\rm b}(-i(y_3-x_3)+\frac{1}{2}(\bar\alpha_3-\bar\beta_3))}{S_{\rm b}(-i(y_3-x_3)-\frac{1}{2}(\bar\alpha_3-\bar\beta_3)+2\epsilon)}\frac{S_{\rm b}(\bar\alpha_3+\bar\alpha_2-\bar\alpha_1-\epsilon)}{S_{\rm b}(\bar\beta_3+\bar\alpha_2-\bar\alpha_1+\epsilon)}\times$$

$$\times\int \frac{{\rm d}\tau}{i}\frac{S_{\rm b}(\tau+\xi_1+\epsilon)}{S_{\rm b}(Q+\tau+\xi_2-\epsilon)}\frac{S_{\rm b}(\tau-\xi_1+\epsilon)}{S_{\rm b}(Q+\tau-\xi_2-\epsilon)} =: I_2^\epsilon\,.$$

In the first step we evaluated the right hand side of the star-triangle relation (3.11) and we expressed the remaining two D-functions that appear in I_1^ϵ through the functions $S_{\rm b}$. After these two steps, the formula for I_1^ϵ should contain a total number of $9+4=13$ functions $S_{\rm b}$. It turns out that four of them cancel against each other so that we are left with the nine factors in the first two lines of the previous formula. In passing to the lower lines we simply performed the substitutions

$$\tau = -ix_2 + \bar\alpha_2/2 - i(x_3+y_3)/2 - Q/2 + (\bar\beta_3+\bar\alpha_3)/4,$$
$$\xi_1 = -\frac{i}{2}(y_3-x_3)+\frac{1}{4}(3\bar\beta_3+\bar\alpha_3)-\frac{Q}{2},$$
$$\xi_2 = \frac{i}{2}(y_3-x_3)+\frac{1}{4}(\bar\beta_3+3\bar\alpha_3)-\frac{Q}{2}.$$

In this form we can now also carry out the integral of the variable τ using a limiting case of the Saalschütz formula, see Appendix B.1, to find

$$I_2^\epsilon = \eta S_{\rm b}(\gamma+2\epsilon)\frac{S_{\rm b}(-\xi_- - \gamma)}{S_{\rm b}(2\epsilon-\xi_-)}\frac{S_{\rm b}(\bar\alpha_3+\bar\alpha_2-\bar\alpha_1-\epsilon)}{S_{\rm b}(\gamma+\bar\alpha_3+\bar\alpha_2-\bar\alpha_1+\epsilon)}\times$$
$$\times e^{-i\pi\xi_-\xi_+}\frac{S_{\rm b}(2\epsilon-\xi_-)S_{\rm b}(2\epsilon+\xi_-)S_{\rm b}(2\epsilon-\xi_+)S_{\rm b}(2\epsilon+\xi_+)}{S_{\rm b}(4\epsilon)}$$

where $\gamma = \bar\beta_3 - \bar\alpha_3 \in i\mathbb{R}$ and

$$\xi_- = \xi_2 - \xi_1 = i(y_3-x_3) - \frac{1}{2}\gamma \in i\mathbb{R},$$
$$\xi_+ = \xi_2 + \xi_1 = \bar\beta_3 + \bar\alpha_3 - Q \in i\mathbb{R}\setminus\{0\}.$$

Having performed both integrations, it remains to remove our regulator ϵ. The most nontrivial part of this computation is to show that

$$\lim_{\epsilon\to 0} \frac{S_{\rm b}(2\epsilon+\gamma)S_{\rm b}(-\xi_- - \gamma)S_{\rm b}(2\epsilon+\xi_-)}{S_{\rm b}(4\epsilon)} = \delta(i\gamma)\delta(i\xi_-) \ . \tag{3.12}$$

A full proof is given in Appendix C. The remaining factors in I_2^ϵ possess a regular limit. In particular we find

$$\lim_{\epsilon\to 0} S_{\rm b}(2\epsilon-\xi_+)S_{\rm b}(2\epsilon+\xi_+) = \frac{1}{S_{\rm b}(Q+\xi_+)S_{\rm b}(Q-\xi_+)} = |S_{\rm b}(\bar\beta_3+\bar\alpha_3)|^{-2}. \tag{3.13}$$

Finally, for the normalization factor $\eta = \mathcal{N}\tilde{\mathcal{N}}$ we obtain

$$\eta = e^{-\frac{i\pi}{2}(\bar\alpha_3\alpha_3 - \bar\alpha_2\alpha_2 - \bar\alpha_1\alpha_1)} e^{\frac{i\pi}{2}(\bar\beta_3\beta_3 - \bar\alpha_2\alpha_2 - \bar\alpha_1\alpha_1)} = e^{-\frac{i\pi}{2}\gamma(\gamma+2\bar\alpha_3 - Q)},$$

Putting all these results together we have shown that

$$\begin{aligned}\lim_{\epsilon\to 0}\int {\rm d}x_2{\rm d}x_1 \begin{bmatrix}\alpha_3\ \alpha_2\ \alpha_1\\ x_3\ x_2\ x_1\end{bmatrix}_\epsilon^* \begin{bmatrix}\beta_3\ \alpha_2\ \alpha_1\\ y_3\ x_2\ x_1\end{bmatrix}_\epsilon &= \\ = e^{-i\pi\xi_-\xi_+}\frac{e^{-\frac{i\pi}{2}\gamma(\gamma+2\bar\alpha_3-Q)}}{|S_{\rm b}(\bar\beta_3+\bar\alpha_3)|^2}&\frac{S_{\rm b}(\bar\alpha_3+\bar\alpha_2-\bar\alpha_1)}{S_{\rm b}(\gamma+\bar\alpha_3+\bar\alpha_2-\bar\alpha_1)}\delta(i\gamma)\delta(i\xi_-) \\ &= |S_{\rm b}(\bar\beta_3+\bar\alpha_3)|^{-2}\delta(i(\bar\beta_3-\bar\alpha_3)\delta(y_3-x_3) \ .\end{aligned}$$

This is the orthonormality relation we set out to prove. The proof of eq. (3.9) is left as an exercise, see [3] for some helpful comments. An alternative proof of the orthonormality relation was published recently in [16], [17].

4 The Racah-Wigner coefficients for $\mathcal{U}_q(sl(2))$

The Racah-Wigner coefficients describe a change of basis in the 3-fold tensor product of representations. Let us denote these three representations by $\pi_{\alpha_i}, i = 1,2,3$. In decomposing their product into irreducibles π_{α_4} there exists two possible fusion paths, denoted by t and s, which are described by the following combination of Clebsch-Gordan coefficients

$$\Phi^t_{\alpha_t}\begin{bmatrix}\alpha_3\ \alpha_2\\ \alpha_4\ \alpha_1\end{bmatrix}_\epsilon (x_4;x_i) = \int {\rm d}x_t \begin{bmatrix}\alpha_4\ \alpha_t\ \alpha_1\\ x_4\ x_t\ x_1\end{bmatrix}_\epsilon \begin{bmatrix}\alpha_t\ \alpha_3\ \alpha_2\\ x_t\ x_3\ x_2\end{bmatrix}_\epsilon , \tag{4.1}$$

$$\Phi^s_{\alpha_s}\begin{bmatrix}\alpha_3\ \alpha_2\\ \alpha_4\ \alpha_1\end{bmatrix}_\epsilon (x_4;x_i) = \int {\rm d}x_s \begin{bmatrix}\alpha_4\ \alpha_3\ \alpha_s\\ x_4\ x_3\ x_s\end{bmatrix}_\epsilon \begin{bmatrix}\alpha_s\ \alpha_2\ \alpha_1\\ x_s\ x_2\ x_1\end{bmatrix}_\epsilon . \tag{4.2}$$

The regularization we use here is the same as in the previous section. From the two objects Φ^s and Φ^t we obtain the Racah-Wigner coefficients as

$$\left\{\begin{matrix}\alpha_1 \; \alpha_3 \\ \alpha_2 \; \alpha_4\end{matrix}\Big|\begin{matrix}\alpha_s \\ \alpha_t\end{matrix}\right\}_{\mathrm{b}} = \lim_{\epsilon\to 0}\int d\alpha_4' dx_4' \int d^3x_i \Phi^t_{\alpha_t}\begin{bmatrix}\alpha_3 \; \alpha_2 \\ \alpha_4' \; \alpha_1\end{bmatrix}^*_\epsilon (x_4'; x_i)\Phi^s_{\alpha_s}\begin{bmatrix}\alpha_3 \; \alpha_2 \\ \alpha_4 \; \alpha_1\end{bmatrix}_\epsilon (x_4; x_i). \tag{4.3}$$

After inserting the concrete expressions (3.10) for the regularised Clebsch-Gordan coefficients one may evaluate the integrals to obtain [3]

$$\begin{aligned}\left\{\begin{matrix}\alpha_1 \; \alpha_3 \\ \alpha_2 \; \alpha_4\end{matrix}\Big|\begin{matrix}\alpha_s \\ \alpha_t\end{matrix}\right\}_{\mathrm{b}} &= |S_{\mathrm{b}}(2\alpha_4)|^2 \frac{S_{\mathrm{b}}(a_4)S_{\mathrm{b}}(a_1)}{S_{\mathrm{b}}(a_2)S_{\mathrm{b}}(a_3)} \times \\ &\times \int_{i\mathbb{R}} dt \frac{S_{\mathrm{b}}(u_4+t)S_{\mathrm{b}}(\tilde{u}_4+t)S_{\mathrm{b}}(u_3+t)S_{\mathrm{b}}(\tilde{u}_3+t)}{S_{\mathrm{b}}(u_{23}+t)S_{\mathrm{b}}(\tilde{u}_{23}+t)S_{\mathrm{b}}(2\alpha_s+t)S_{\mathrm{b}}(Q+t)}\end{aligned} \tag{4.4}$$

where the four variables a_i are associated with the four Clebsch-Gordan maps that appear in eqs. (4.1) and (4.2)

$$a_1 = \alpha_1 - \bar{\alpha}_t + \alpha_4 \quad , \qquad a_2 = \alpha_2 - \alpha_3 + \alpha_t$$
$$a_3 = \alpha_3 - \bar{\alpha}_s + \alpha_4 \quad , \qquad a_4 = \alpha_2 - \alpha_1 + \alpha_s$$

and similarly for the remaining set of variables,

$$\begin{aligned} u_4 &= \alpha_s + \alpha_1 - \alpha_2 \quad , & \tilde{u}_4 &= \alpha_s + \bar{\alpha}_1 - \alpha_2, \\ u_3 &= \alpha_s + \alpha_4 - \alpha_3 \quad , & \tilde{u}_3 &= \alpha_s + \alpha_4 - \bar{\alpha}_3, \\ u_{23} &= \alpha_s + \alpha_t + \alpha_4 - \alpha_2 \quad , & \tilde{u}_{23} &= \alpha_s + \bar{\alpha}_t + \alpha_4 - \alpha_2 \; . \end{aligned} \tag{4.5}$$

Note that the first factor in our formula (4.4) for the Racah-Wigner symbols differs from [3]. Our expression is a result of carefully analyzing the integrals the definition (4.3) of the Racah-Wigner symbols. The same normalization was found independently by Nidaiev in [18].

The derivation of eq. (4.4) from eq. (4.3) is in principle straightforward, though a bit cumbersome. One simply has to evaluate the integrals. The integrals over the variables $x_i, i = 1, 2, 3$, are performed with the help of Cauchy's integral formula. The resulting integral expression involves delta functions in both the difference $\alpha_4 - \alpha_4'$ and $x_4 - x_4'$. Hence the integrals over x_4' and α_4' are easy to perform at the end of the computation. So, let us get back to the integrals over $x_i, i = 1, 2, 3$. It is convenient so start with x_1. In order to perform the integration, one needs to keep track of all the poles in the integrand along with their residues. Since the functions Φ^s and Φ^t are ultimately built from $S_{\rm b}$ through equations (3.1), (3.10), (4.1) and (4.2), this step only requires knowledge of the poles and residues of $S_{\rm b}$. All this information on $S_{\rm b}$ can be found in Appendix A.1. Once the integration over x_1 has been performed, one focuses on the variable x_3. There are a few poles that have been around before we integrated over x_1. In addition, the integration over x_1 brought in some new poles through the usual pole collisions (pinching). These must all be accounted for before we can apply Cauchy's formula again to perform the integration over x_3. Similar comments apply to the final integral over x_2. Many more details on this computations can be found in section 5 of the paper by Ponsot and Teschner [3]. Let us stress again that no fancy identities are needed at any stage of the calculation.

After these comments on the derivation of eq. (4.4), let us list a few more properties of the Racah-Wigner symbols. To begin with, they can be shown to satisfy the following orthogonality relations

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} d\alpha_s |S_{\rm b}(2\alpha_s)|^2 \left(\begin{Bmatrix} \alpha_1 & \alpha_2 & | & \alpha_s \\ \alpha_3 & \alpha_4 & | & \beta_t \end{Bmatrix}_{\rm b} \right)^* \begin{Bmatrix} \alpha_1 & \alpha_2 & | & \alpha_s \\ \alpha_3 & \alpha_4 & | & \alpha_t \end{Bmatrix}_{\rm b} = |S_{\rm b}(2\alpha_t)|^2 \delta(\alpha_t - \beta_t).$$

As a consequence of their very definition, the Racah-Wigner symbols must also satisfy the pentagon equation

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} d\delta_1 \begin{Bmatrix} \alpha_1 & \alpha_2 & | & \beta_1 \\ \alpha_3 & \alpha_4 & | & \delta_1 \end{Bmatrix}_{\rm b} \begin{Bmatrix} \alpha_1 & \delta_1 & | & \beta_2 \\ \alpha_4 & \alpha_5 & | & \gamma_2 \end{Bmatrix}_{\rm b} \begin{Bmatrix} \alpha_2 & \alpha_3 & | & \delta_1 \\ \alpha_4 & \gamma_2 & | & \gamma_1 \end{Bmatrix}_{\rm b} = \begin{Bmatrix} \beta_1 & \alpha_3 & | & \beta_2 \\ \alpha_4 & \alpha_5 & | & \gamma_1 \end{Bmatrix}_{\rm b} \begin{Bmatrix} \alpha_1 & \alpha_2 & | & \beta_1 \\ \gamma_1 & \alpha_5 & | & \gamma_2 \end{Bmatrix}_{\rm b} .$$

More recently, Teschner and Vartanov found an interesting alternative expression for the Racah-Wigner coefficients [19]. We will discuss this representation along with its extension to the supersymmetric case in an accompanying paper.

5 Self-dual continuous series for $\mathcal{U}_q(osp(1|2))$

After this extended review of the quantum deformed enveloping algebra $\mathcal{U}_q(sl(2))$ and its self-conjugate series of representations we are now well prepared to turn to the algebra $\mathcal{U}_q(osp(1|2))$. We shall restate its definition here before we describe a 1-parameter series of self-conjugate representations.

Following [10], the quantum deformed superalgebra $\mathcal{U}_q(osp(1|2))$ is generated by the bosonic generators K, K^{-1} along with two fermionic (odd) ones $v^{(\pm)}$. These satisfy the relations

$$Kv^{(\pm)} = q^{\pm 1} v^{(\pm)} K,$$
$$\{v^{(+)}, v^{(-)}\} = -\frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}},$$

where $q = e^{i\pi \mathrm{b}^2}$, as before. The similarity with the defining relations of $\mathcal{U}_q(sl(2))$ is striking, except that the elements $v^{\pm}$ are fermionic (odd) so that we prescribe the anti-commutator $\{.,.\}$ of $v^{(+)}$ with $v^{(-)}$ instead of the commutator. The algebraic relations are compatible with the following star structure

$$K^* \;=\; K \quad , \quad v^{(\pm)*} \;=\; v^{(\pm)}, \tag{5.1}$$

and with the coproduct

$$\Delta(K) \;=\; K \otimes K \quad , \quad \Delta(v^{(\pm)}) \;=\; v^{(\pm)} \otimes K + K^{-1} \otimes v^{(\pm)}, \tag{5.2}$$

that can be used to define tensor products of representations. It is easy to verify that the following even element of $\mathcal{U}_q(osp(1|2))$ commutes with all generators,

$$C = -\frac{qK^4 + q^{-1}K^{-4} + 2}{(q - q^{-1})^2} + \frac{(qK^2 + q^{-1}K^{-2})v^{(-)}v^{(+)}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} + v^{(-)2}v^{(+)2},$$

i.e. C is a Casimir element. In addition, the algebra $\mathcal{U}_q(osp(1|2))$ also contains an element Q which is defined as

$$Q = \frac{1}{2}(v^{(-)}v^{(+)} - v^{(+)}v^{(-)}) + \frac{1}{2}\frac{K^2 + K^{-2}}{q^{1/2} + q^{-1/2}}.$$

Up to some shift, the element Q may be considered as the square root of the quadratic Casimir element C,

$$C = -\left(Q + \frac{2i}{q - q^{-1}}\right)\left(Q - \frac{2i}{q - q^{-1}}\right).$$

This concludes our short description of the algebraic setup so that we can begin to discuss the representations we are about to analyse. Following the intuition developed from the non-supersymmetric case, we shall introduce representation on carrier spaces $\mathcal{Q}_\alpha$ which are parametrized by a single parameter α of the form $\alpha \in \frac{Q}{2} + i\mathbb{R}$ where $Q = \mathrm{b} + \frac{1}{\mathrm{b}}$ and the relation between b and q is the same as before. The spaces $\mathcal{Q}_\alpha$ are graded vector spaces. By definition, they consist of pairs $(f^0(x), f^1(x))$ of functions f^j which are entire analytic and whose Fourier transform $\hat{f}^j(\omega)$ is allowed to possess poles in the set $\mathcal{S}_\alpha$ that was defined in eq. (2.3). The upper index j indicates the parity of the element, i.e. vectors of the form $(f^0, 0)$ are considered even while we declare elements of the form $(0, f^1)$ to be odd. On these carrier spaces, we represent the elements K and $v^\pm$ through

$$\pi_\alpha(K) = T_x^{\frac{i\mathrm{b}}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \pi_\alpha\left(v^{(\pm)}\right) = ie^{\pm\pi\mathrm{b}x} \begin{pmatrix} 0 & [\delta_x \pm \bar{\alpha}]_- \\ [\delta_x \pm \bar{\alpha}]_+ & 0 \end{pmatrix} \tag{5.3}$$

where T_x^{ia} denotes the shift operator that was defined in eq. (2.5) and we introduced

$$[x]_- = \frac{\sin(\frac{\pi\mathrm{b}x}{2})}{\sin(\frac{\pi\mathrm{b}^2}{2})} \quad , \quad [x]_+ = \frac{\cos(\frac{\pi\mathrm{b}x}{2})}{\cos(\frac{\pi\mathrm{b}^2}{2})}. \tag{5.4}$$

Consequently, the matrix elements in our expression for $\pi_\alpha(v^\pm)$ are given by

$$[\delta_x + \bar{\alpha}]_- = \frac{e^{\frac{i\pi\mathrm{b}\alpha}{2}} T^{\frac{i\mathrm{b}}{2}} - e^{\frac{-i\pi\mathrm{b}\alpha}{2}} T^{-\frac{i\mathrm{b}}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

$$[\delta_x + \bar{\alpha}]_+ = \frac{e^{\frac{i\pi\mathrm{b}\alpha}{2}} T^{\frac{i\mathrm{b}}{2}} + e^{\frac{-i\pi\mathrm{b}\alpha}{2}} T^{-\frac{i\mathrm{b}}{2}}}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}.$$

It is not difficult to check that our prescription respects the algebraic relations in the universal enveloping algebra $\mathcal{U}_q(osp(1|2))$ and hence provides a family of representations. In these representations, we can evaluate the Casimir element C and its square root Q,

$$\pi_\alpha(C) = [\frac{Q}{2} - \alpha]_-^2 [\frac{Q}{2} - \alpha]_+^2 \sigma_0 \quad , \quad \pi_\alpha(Q) = [\bar{\alpha} - \frac{\mathrm{b}}{2}]_- [\bar{\alpha} - \frac{\mathrm{b}}{2}]_+ \sigma_3,$$

where σ_0 denotes the 2-dimensional identity matrix and σ_3 is the Pauli matrix that is diagonal in our basis. Note that the value of the Casimir element C is the same in the representations π_α and $\pi_{\bar{\alpha}}$. This is because the representations are actually equivalent. In fact, one may easily check that the following unitary operator

$$\mathcal{I}_\alpha = \begin{pmatrix} 0 & \frac{S_1(\alpha - i\omega)}{S_1(\bar{\alpha} - i\omega)} \\ \frac{S_0(\alpha - i\omega)}{S_0(\bar{\alpha} - i\omega)} & 0 \end{pmatrix},$$

involving the special functions S_ν defined in Appendix A.2, satisfies

$$\pi_{\bar{\alpha}}(X)\mathcal{I}_\alpha = \mathcal{I}_\alpha \pi_\alpha(X) \quad , \quad \text{for} \quad X = K, v^{(\pm)}.$$

In order to discuss the reality properties of our representation, we need to introduce the following matrix

$$\lambda^2 = \begin{pmatrix} \varrho & 0 \\ 0 & \varrho^{-1} \end{pmatrix} \quad \text{where} \quad \varrho = i\frac{q^{1/2} - q^{-1/2}}{q^{1/2} + q^{-1/2}}\ . \tag{5.5}$$

A square root λ of the matrix λ^2 appears in the definition of the scalar product

$$\langle g, f\rangle = \sum_{i,j} \int \mathrm{d}x g^i(x)^* \lambda_{ij} f^j(x) \ . \tag{5.6}$$

A short calculation shows that the adjoint with respect to this scalar product implements the $*$ operation we defined above, i.e. $\langle g, Xf\rangle = \langle Xg, f\rangle$ for all three generators of $\mathcal{U}_q(osp(1|2))$. Once again one can check that the representations π_α admit duality $\mathrm{b} \to \mathrm{b}^{-1}$ in the same sense as above. More concretely, our formulas can be used to define a representation of $\mathcal{U}_{\tilde{q}}(osp(1|2))$ with $\tilde{q} = e^{i\pi \mathrm{b}^{-2}}$ on $\mathcal{Q}_\alpha$ such that the corresponding operators (anti-)commute with the representation operators for the original action of $\mathcal{U}_q(osp(1|2))$ on $\mathcal{Q}_\alpha$. Let us mention that a similar series of representations was recently discussed in [20], though the precise relation to the ones we consider here not clear to us.

6 The Clebsch-Gordan coefficients for $\mathcal{U}_q(osp(1|2))$

As in the case of $\mathcal{U}_q(sl(2))$, we are interested in the Clebsch-Gordan decomposition of the representation $\pi_{\alpha_2} \otimes \pi_{\alpha_1}$,

$$\pi_{\alpha_2} \otimes \pi_{\alpha_1} \simeq \int_{\frac{\mathbb{Q}}{2}+i\mathbb{R}^+}^{\otimes} \mathrm{d}\alpha_3 \, \pi_{\alpha_3} \ .$$

We shall show below that there exist two independent intertwiners for any given choice of α_1, α_2 and α_3. We shall label these by an index ν. The corresponding Clebsch-Gordan coefficients are defined as

$$F_f^{(\nu)\, j_3}(\alpha_3, x_3) =: \sum_{j_2, j_1} \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \begin{bmatrix} \alpha_3 \ \alpha_2 \ \alpha_1 \\ x_3 \ x_2 \ x_1 \end{bmatrix}^{(\nu)\, j_3}_{j_2 j_1} f^{j_2 j_1}(x_2, x_1) \ .$$

In order to construct these coefficients, we introduce the following products

$$D^{(\nu);\epsilon}_{\tau\sigma}(x_i;\alpha_i) = (-1)^{\nu(\tau+1)(\sigma+1)} \frac{S_{\tau+\nu+1}(z_{21}+\epsilon)}{S_\tau(z_{21}+\epsilon+\alpha_{21})} \frac{S_{\nu+\sigma+1}(z_{23}+\epsilon)}{S_\sigma(z_{23}+\epsilon+\alpha_{23})} \frac{S_{\tau+\sigma+\nu}(z_{13}-\epsilon)}{S_{\tau+\sigma+1}(z_{13}-\epsilon+\alpha_{13})}, \tag{6.1}$$

where z_{ij} and α_{ij} in the same way as in the $\mathcal{U}_q(sl(2))$ case. In addition, we have introduced a parameter ϵ that will serve as a regulator in products of Clebsch-Gordan coefficients later on, just as in the case of $\mathcal{U}_q(sl(2))$. The Clebsch-Gordan maps are now obtained as

$$\begin{bmatrix} \alpha_3 \ \alpha_2 \ \alpha_1 \\ x_3 \ x_2 \ x_1 \end{bmatrix}^{(\nu)\, j_3}_{j_2 j_1} = \sum_{\tau,\sigma} (-1)^{(j_1+\sigma)(j_2+\tau+\sigma)} (-|\rho|)^{\nu(1-(j_1-j_2)^2)-j_1 j_2} \delta_{j_1+j_2+\nu, j_3} \mathcal{N}^{1/2} D^{(\nu)}_{\tau\sigma}(x_i;\alpha_i) \ . \tag{6.2}$$

The normalizing factor $\mathcal{N}^{1/2}$ is the square root of the $\mathcal{N}$ we defined in eq. (3.3). Regularization is understood whenever it is necessary. If we remove the regulator ϵ, we obtain the Clebsch-Gordan coefficients

$$\begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(\nu)\, j_3}_{j_2 j_1} = \lim_{\epsilon\to 0} \left(\begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}_\epsilon \right)^{(\nu)\, j_3}_{j_2 j_1} .$$

The intertwiner properties and orthogonality relations for these Clebsch-Gordan coefficients are established following the same steps as in the case of $\mathcal{U}_q(sl(2))$. Our discussion in the subsequent section will therefore focus on equations containing additional signs and on the final results.

6.1 Intertwiner property

The Clebsh-Gordan coefficients satisfy the intertwining properties for $X = K, v^{(\pm)}$

$$\pi_{\alpha_3}(X)^i{}_{j_3} \delta^{j_2}_j \delta^{j_1}_k \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{j_3}_{j_2 j_1} = \delta^i_{j_3} (\pi_{\alpha_2} \otimes \pi_{\alpha_1}) \Delta^t(X)^{j_2}{}_j{}^{j_1}{}_k \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{j_3}_{j_2 j_1} .$$

The transpose on the right hand side is defined with respect to the scalar product (5.6). All these equations may be checked by direct computations. For $X = K$ the analysis is identical to the one outlined in section 3.1. So, let us proceed to $X = v^{(+)}$ right away. When written out in components, our basic intertwining relation reads

$$(v^{(+)}_{\alpha_3})^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 0}_{00} = \Delta^t_{10}(v^{(+)})^0{}_0{}^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 1}_{01} + \Delta^t_{10}(v^{(+)})^1{}_0{}^0{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 1}_{10} ,$$

$$(v^{(+)}_{\alpha_3})^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 1}_{01} = \Delta^t_{10}(v^{(+)})^0{}_0{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 0}_{00} + \Delta^t_{10}(v^{(+)})^1{}_0{}^1{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 0}_{11} ,$$

$$(v^{(+)}_{\alpha_3})^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 1}_{10} = -\Delta^t_{10}(v^{(+)})^1{}_1{}^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 0}_{11} + \Delta^t_{10}(v^{(+)})^0{}_1{}^0{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 0}_{00} ,$$

$$(v^{(+)}_{\alpha_3})^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 0}_{11} = \Delta^t_{10}(v^{(+)})^0{}_1{}^1{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 1}_{01} - \Delta^t_{10}(v^{(+)})^1{}_1{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(0)\, 1}_{10} .$$

For the second set of Clebsch-Gordan coefficients we find,

$$(v^{(+)}_{\alpha_3})^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,1}_{11} = \Delta^t_{10}(v^{(+)})^0{}_1{}^1{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,0}_{01} - \Delta^t_{10}(v^{(+)})^1{}_1{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,0}_{10},$$

$$(v^{(+)}_{\alpha_3})^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,1}_{00} = \Delta^t_{10}(v^{(+)})^0{}_0{}^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,0}_{01} + \Delta^t_{10}(v^{(+)})^1{}_0{}^0{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,0}_{10},$$

$$(v^{(+)}_{\alpha_3})^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,0}_{01} = \Delta^t_{10}(v^{(+)})^0{}_0{}^0{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,1}_{00} + \Delta^t_{10}(v^{(+)})^1{}_0{}^1{}_1 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,1}_{11},$$

$$(v^{(+)}_{\alpha_3})^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,0}_{10} = -\Delta^t_{10}(v^{(+)})^1{}_1{}^1{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,1}_{11} + \Delta^t_{10}(v^{(+)})^0{}_1{}^0{}_0 \begin{bmatrix} \alpha_3\ \alpha_2\ \alpha_1 \\ x_3\ x_2\ x_1 \end{bmatrix}^{(1)\,1}_{00},$$

As in the nonsupersymetric case one may employ the identities

$$T^{\mathrm{ib}}_x \frac{S_1(-ix+a_1)}{S_1(-ix+a_2)} = \frac{[-ix+a_1]_1}{[-ix+a_2]_1} \frac{S_0(-ix+a_1)}{S_0(-ix+a_2)} T^{\mathrm{ib}}_x,$$

$$T^{\mathrm{ib}}_x \frac{S_0(-ix+a_1)}{S_0(-ix+a_2)} = \frac{[-ix+a_1]_0}{[-ix+a_2]_0} \frac{S_1(-ix+a_1)}{S_1(-ix+a_2)} T^{\mathrm{ib}}_x,$$

$$T^{\mathrm{ib}}_x \frac{S_0(-ix+a_1)}{S_1(-ix+a_2)} = -i\frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{q^{\frac{1}{2}}+q^{-\frac{1}{2}}} \frac{[-ix+a_1]_0}{[-ix+a_2]_1} \frac{S_1(-ix+a_1)}{S_0(-ix+a_2)} T^{\mathrm{ib}}_x.$$

to check that all the intertwining relation for $X = v^{(+)}$ is satisfied. The same steps are carried out to discuss $X = v^{(-)}$. Details are left to the reader.

6.2 Orthogonality and Completeness

The most difficult part in the analysis of the Clebsch-Gordan coefficients is once again concerning their orthonormality relations. The intertwining relations we have established in the previous subsection guarantee that

$$\sum_{j_2,j_2} \int_{\mathbb{R}} \mathrm{d}x_2 \mathrm{d}x_1 \overline{\left[\begin{smallmatrix} \alpha_3\,\alpha_2\,\alpha_1 \\ x_3\,x_2\,x_1 \end{smallmatrix}\right]^{(\nu),j_3}_{j_2 j_1}} \left[\begin{smallmatrix} \beta_3\,\alpha_2\,\alpha_1 \\ y_3\,x_2\,x_1 \end{smallmatrix}\right]^{(\mu),i_3}_{k_2 k_1} \lambda^{j_2 k_2} \lambda^{j_1 k_1} =$$

$$= 32\sqrt{\rho}^{(-1)^{(\nu+1)}} \sum_{\sigma} (-1)^{(\nu+1)(\sigma+1)} |S_\sigma(2\alpha_3)|^{-2} \delta_{\nu,\mu} \lambda^{j_3,i_3} \delta(\bar\beta_3 - \bar\alpha_3)\delta(y_3 - x_3), \quad (6.3)$$

up to an overall factor. This normalization is established with the help of a set of integral identities which follow from a supersymmetric version of the star-triangle identity, see Appendix B.2. In particular one uses

$$\sum_{\tau,\sigma} \int dx_2 dx_1 (-1)^{(\rho+\nu+\mu)\tau} \left(\tilde D^{(\mu)\,\epsilon}_{\tau\sigma}\right)^* D^{(\nu)\,\epsilon}_{\tau(\sigma+\rho)} = \frac{16(-1)^{\rho\nu}}{|S_{\rho+1}(2\bar\alpha_3)|^2} \delta_{\mu,\nu} \delta(\bar\beta_3 - \bar\alpha_3)\delta(y_3 - x_3),$$

where we introduced

$$D^{(\nu)\,\epsilon}_{\tau\sigma} = D^{(\nu)\,\epsilon}_{\tau\sigma}(x_3, x_2, x_1; \alpha_3, \alpha_2, \alpha_1) \quad , \quad \tilde{D}^{(\nu)\,\epsilon}_{\tau\sigma} = D^{(\nu)\,\epsilon}_{\tau\sigma}(y_3, x_2, x_1; \beta_3, \alpha_2, \alpha_1) \; .$$

Since the analogous computation for $\mathcal{U}_q(sl(2))$ was described in great detail in section 3.2 we can leave the derivation of eq. (6.3) as an exercise.

7 The Racah-Wigner coefficients for $\mathcal{U}_q(osp(1|2))$

The definition and computation of the Racah-Wigner coefficients for $\mathcal{U}_q(osp(1|2))$ proceeds very much along the same lines as for $\mathcal{U}_q(sl(2))$, see section 4. After giving a precise definition in the Racah-Wigner coefficients, we will state an explicit formula. It resembles the one for $\mathcal{U}_q(sl(2))$, see eq. (4.4), except that all special functions carry an additional label $\nu \in \{0, 1\}$.

As in the case of $\mathcal{U}_q(sl(2))$ be begin by defining the following two maps for the decomposition of triple tensor products,

$$\left(\Phi^t_{\alpha_t}\begin{bmatrix}\alpha_3\;\alpha_2\\ \alpha_4\;\alpha_1\end{bmatrix}^{\nu_1\nu_2}_{\epsilon}\right)^{i}_{jkl}(x_4; x_i) \;=\; \sum_n \int \mathrm{d}x_t \begin{bmatrix}\alpha_4\;\alpha_t\;\alpha_1\\ x_4\;x_t\;x_1\end{bmatrix}^{(\nu_1)\,i}_{\epsilon\;nl} \begin{bmatrix}\alpha_t\;\alpha_3\;\alpha_2\\ x_t\;x_3\;x_2\end{bmatrix}^{(\nu_2)\,n}_{\epsilon\;\;jk} \tag{7.1}$$

$$\left(\Phi^s_{\alpha_s}\begin{bmatrix}\alpha_3\;\alpha_2\\ \alpha_4\;\alpha_1\end{bmatrix}^{\nu_3\nu_4}_{\epsilon}\right)^{i}_{jkl}(x_4; x_i) \;=\; \sum_m \int \mathrm{d}x_s \begin{bmatrix}\alpha_4\;\alpha_3\;\alpha_s\\ x_4\;x_3\;x_s\end{bmatrix}^{(\nu_3)\,i}_{\epsilon\;jm} \begin{bmatrix}\alpha_s\;\alpha_2\;\alpha_1\\ x_s\;x_2\;x_1\end{bmatrix}^{(\nu_4)\,m}_{\epsilon\;\;kl} \; . \tag{7.2}$$

From these two maps we can compute the Racah-Wigner coefficients through the usual prescription

$$\left(\begin{Bmatrix}\alpha_1\;\alpha_3 & \alpha_s\\ \alpha_2\;\alpha_4 & \alpha_t\end{Bmatrix}_{\mathrm{b}}\right)^{\nu_3\nu_4}_{\nu_1\nu_2} = \tag{7.3}$$

$$= \lim_{\epsilon\to 0}\sum_{jklm}\int \mathrm{d}\alpha_4 \int \mathrm{d}^4x \left(\left(\Phi^t_{\alpha_t}\begin{bmatrix}\alpha_3\;\alpha_2\\ \alpha_4'\;\alpha_1\end{bmatrix}^{\nu_1\nu_2}_{\epsilon}\right)^{m}_{jkl}(x_4'; x_i)\right)^{*} \left(\Phi^s_{\alpha_s}\begin{bmatrix}\alpha_3\;\alpha_2\\ \alpha_4\;\alpha_1\end{bmatrix}^{\nu_3\nu_4}_{\epsilon}\right)^{n}_{jkl}(x_4; x_i)$$

where the integration measure is $d^4x = \prod_{i=1}^4 dx_i$. After integration and summation, the right hand side turns out to be independent of α_4', x_4' and n. Using the explicit formulas for the regularized Clebsch-Gordan maps along with our knowledge of poles and residues of the special functions S_ν, see Appendix B.2, we can perform the integrations with the help of Cauchy's integral formula to obtain,

$$\left(\begin{Bmatrix} \alpha_1\ \alpha_3 \,|\, \alpha_s \\ \alpha_2\ \alpha_4 \,|\, \alpha_t \end{Bmatrix}_{\mathrm{b}}\right)^{\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i=0\ mod\ 2}\, (-1)^{\nu_2\nu_3+\nu_4}\, \frac{S_{\nu_4}(a_4)S_{\nu_1}(a_1)}{S_{\nu_2}(a_2)S_{\nu_3}(a_3)} \times \tag{7.4}$$

$$\times \int_{i\mathbb{R}} \mathrm{d}t \sum_{\nu=0}^{1} (-1)^{\nu(\nu_2+\nu_4)} \frac{S_{1+\nu+\nu_4}(u_4+t)S_{1+\nu+\nu_4}(\tilde{u}_4+t)S_{1+\nu+\nu_3}(u_3+t)S_{1+\nu+\nu_3}(\tilde{u}_3+t)}{S_{\nu+\nu_2+\nu_3}(u_{23}+t)S_{\nu+\nu_2+\nu_3}(\tilde{u}_{23}+t)S_{\nu}(2\alpha_s+t)S_{\nu}(Q+t)}\,.$$

All the variables a_i and $u_i, \tilde{u}_i$ etc. where defined in section 4. Note that they are associated to the four vertices which in turn correspond to the indices ν_i. In this form, our result appears as a natural extension of the expression (4.4) for the Racah-Wigner coefficients of $\mathcal{U}_q(sl(2))$. The sum over ν accompanies the integral over t. The shift $\nu \to \nu+1$ in the index of S_ν appears for those S_ν that we decided to write into the numerator. The parameters ν_i are placed such that they mimic the arguments of the S_ν. Unfortunately, we do not have a simple rule to explain the sign factor, but of course it comes out of the calculation as stated.

8 Comparison with fusing matrix of Liouville theory

As we outlined in the introduction, the Racah-Wigner coefficients for the self-dual series of representations we considered here coincide with the Fusing matrices of (super-symmetric) Liouville theory, at least up to some normalization dependent prefactors. For the case of $\mathcal{U}_q(sl(2))$ this was shown by Teschner in [21, 22]. Entries of the fusing matrix of $N=1$ supersymmetric Liouville theory were computed in [12] and we are now going to compare these with the Racah-Wigner symbols (7.4) of $\mathcal{U}_q(osp(1|2))$, after a short review of the non-supersymmetric theory.

8.1 Liouville field theory and $\mathcal{U}_q(sl(2))$

The fusion matrix of Liouville field theory can be obtained by calculating the exchange relations for the chiral operators in the scalar field representation [21, 22] (see also [23] for an earlier construction). To spell out this result we need to introduce some notation.

The Verma module $\mathcal{V}_\Delta$ of the Virasoro algebra of the highest weight Δ and the central charge c is defined as a free vector space generated by all vectors of the form

$$\nu_{\Delta,MK} = L_{-m_j}\ldots L_{-m_1}\nu_\Delta\,, \tag{8.1}$$

where $m_j \geqslant \ldots \geqslant m_2 \geqslant m_1, \quad m_r \in \mathbb{N}$ and ν_Δ is the highest weight state with respect to the Virasoro algebra,

$$L_0 \nu_\Delta = \Delta \nu_\Delta, \quad L_m \nu_\Delta = 0, \quad m > 0. \tag{8.2}$$

The chiral vertex operator,

$$V\big[{}^{\Delta_2}_{\Delta_3 \Delta_1}\big](z) \;:\; \mathcal{V}_{\Delta_1} \;\to\; \mathcal{V}_{\Delta_3},$$

is a linear map parameterized by the conformal weight of the "intermediate" module $\mathcal{V}_{\Delta_2}$ and defined by the commutation relations

$$\big[L_n, V\big[{}^{\Delta_2}_{\Delta_3 \Delta_1}\big](z)\big] \;=\; z^n\big(z\partial_z + (n+1)\Delta_2\big) V\big[{}^{\Delta_2}_{\Delta_3 \Delta_1}\big](z), \tag{8.3}$$

the form of the Virasoro algebra and a normalization

$$V\big[{}^{\Delta_2}_{\Delta_3 \Delta_1}\big](z)\nu_{\Delta_1} \;=\; z^{\Delta_3 - \Delta_2 - \Delta_1}\left(\mathbf{1} + \mathcal{O}(z)\right)\nu_{\Delta_3}, \qquad z \to 0. \tag{8.4}$$

With the notion of the chiral vertex operator at hand we can define a four-point conformal block,

$$\mathcal{F}_{\Delta_s}\big[{}^{\Delta_3\ \Delta_2}_{\Delta_4\ \Delta_1}\big](z) \;=\; \Big\langle \nu_{\Delta_4}, V\big[{}^{\Delta_3}_{\Delta_4 \Delta_s}\big](1) V\big[{}^{\Delta_2}_{\Delta_s \Delta_1}\big](z)\nu_{\Delta_1}\Big\rangle, \tag{8.5}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the usual hermitian, bilinear form on $\mathcal{V}_\Delta$, which is uniquely characterized by the conditions $L_n^\dagger = L_{-n}$, and $\langle \nu_\Delta, \nu_\Delta\rangle = 1$. Finally, the Liouville fusion matrix (or monodromy of conformal block) is defined as an integral kernel appearing in the relation

$$\mathcal{F}_{\Delta_s}\big[{}^{\Delta_3\ \Delta_2}_{\Delta_4\ \Delta_1}\big](z) \;=\; \int_{\frac{Q}{2}+i\mathbb{R}^+} \mathrm{d}\alpha_t\; F_{\alpha_s\,\alpha_t}\big[{}^{\alpha_3\ \alpha_2}_{\alpha_4\ \alpha_1}\big]\, \mathcal{F}_{\Delta_t}\big[{}^{\Delta_1\ \Delta_2}_{\Delta_4\ \Delta_3}\big](1-z), \tag{8.6}$$

with $\Delta_i = \alpha_i(Q - \alpha_i)$. It appears to be difficult to derive the form of the fusion matrix directly from its definition. However, there exists a simple relation (which we shall formulate explicitly below) between the fusion matrix and the braiding matrix of the Virasoro chiral vertex operators, i.e. the integral kernel appearing in the formula

$$V\big[{}^{\Delta_3}_{\Delta_4 \Delta_s}\big](z_2) V\big[{}^{\Delta_2}_{\Delta_s \Delta_1}\big](z_1) \;=\; \int_{\frac{Q}{2}+i\mathbb{R}^+} \mathrm{d}\alpha_u\; B_{\alpha_s\,\alpha_u}\big[{}^{\alpha_3\ \alpha_2}_{\alpha_4\ \alpha_1}\big]\, V\big[{}^{\Delta_2}_{\Delta_4 \Delta_u}\big](z_1) V\big[{}^{\Delta_3}_{\Delta_u \Delta_1}\big](z_2). \tag{8.7}$$

The latter can be derived by direct calculations of the exchange relation of chiral vertex operators in the free field representation [21, 22].

Let the hermitian operators p, q, with $[\mathsf{p}, \mathsf{q}] = -\frac{i}{2}$, act on the Hilbert space $L^2(\mathbb{R})$. Denote by $\mathcal{F}$ the Fock space generated by action of negative modes of the algebra

$$[\mathsf{a}_m, \mathsf{a}_n] = \frac{1}{2} m \delta_{m,-n}, \quad m, n \in \mathbb{Z} \setminus \{0\}, \qquad \mathsf{a}_m^\dagger = \mathsf{a}_{-m},$$

on the vacuum Ω, where $\mathsf{a}_m\Omega = 0,\ m > 0$. On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{F}$ there exists a well known representation of the Virasoro algebra with the central charge $c = 1 + 6Q^2$, given by,

$$\begin{aligned} L_m(\mathsf{p}) &= \sum_{n\neq 0,m} \mathsf{a}_n\mathsf{a}_{m-n} + (2\mathsf{p} + imQ)\mathsf{a}_m, \qquad m \neq 0, \\ L_0(\mathsf{p}) &= 2\sum_{n=1}^{\infty} \mathsf{a}_n\mathsf{a}_{-n} + \mathsf{p}^2 + \frac{1}{4}Q^2. \end{aligned} \tag{8.8}$$

Each state of the form $|\nu_p\rangle \equiv |p\rangle \otimes \Omega$, with $\mathsf{p}|p\rangle = p|p\rangle$, is of highest weight with respect to the algebra (8.8) and satisfies

$$L_0(\mathsf{p})|\nu_p\rangle = \Delta(p)|\nu_p\rangle, \qquad \Delta(p) = p^2 + \frac{1}{4}Q^2 = \alpha(Q-\alpha),\ \ \alpha = \frac{Q}{2} + ip.$$

Acting on such a state with $L_{-n}(\mathsf{p})$ one generates a Virasoro Verma module $\mathcal{V}_{\Delta(p)}$.

The normal ordered exponentials

$$\mathsf{E}^{\alpha}(x) = \mathrm{e}^{\alpha\mathsf{q}}\, \mathrm{e}^{2\alpha\varphi_<(x)}\, \mathrm{e}^{2\alpha x\mathsf{p}}\, \mathrm{e}^{2\alpha\varphi_>(x)}\, \mathrm{e}^{\alpha\mathsf{q}} \tag{8.9}$$

where

$$\varphi_<(x) = -i\sum_{n=1}^{\infty}\frac{\mathsf{a}_{-n}}{n}\,\mathrm{e}^{inx}, \qquad \varphi_<(x) = i\sum_{n=0}^{\infty}\frac{\mathsf{a}_n}{n}\,\mathrm{e}^{-inx},$$

together with the screening charge

$$\mathsf{Q}(x) = \int_x^{x+2\pi} \mathsf{E}^{\mathrm{b}}(y)dy$$

serve as building block for a chiral field

$$\mathsf{g}_s^{\alpha}(x) = \mathsf{E}^{\alpha}(x)\big(\mathsf{Q}(x)\big)^s. \tag{8.10}$$

Commutation relations of the field (8.10) with the Virasoro algebra generators are of the form

$$[L_n(\mathsf{p}), \mathsf{g}_s^{\alpha}(x)] = \mathrm{e}^{inx}\left(-i\frac{d}{dx} + n\alpha(Q-\alpha)\right)\mathsf{g}_s^{\alpha}(x) \tag{8.11}$$

and coincide with (8.3) (with $\Delta(p)$ substituted for Δ_2) for w chiral vertex operator transformed from the complex z plane to the zero-time slice of an infinite cylinder. It is also easy to check that $\mathsf{g}_s^{\alpha}(x)$ maps vectors form $\mathcal{V}_{p_1}$ to $\mathcal{V}_q$ with $q = p_1 - i(\alpha + bs)$. The field (8.10) therefore yields a model for a family of (unnormalized) vertex operators with

$\Delta_2 = \Delta(p)$ and arbitrary (thanks to a possibility of adjusting the value of s) $\Delta_1 = \Delta(p_1)$ and $\Delta_3 = \Delta(q)$.

Suppose there exists an exchange relation

$$\mathsf{g}_{s_2}^{\alpha_2}(x_2)\mathsf{g}_{s_1}^{\alpha_1}(x_1) = \int dt_1 dt_2 \; B(\alpha_i|s_i,t_i)\, \mathsf{g}_{t_1}^{\alpha_1}(x_1)\mathsf{g}_{t_2}^{\alpha_2}(x_2). \tag{8.12}$$

Using

$$\mathsf{E}^{\alpha}(x)\mathsf{E}^{\beta}(y) = \mathrm{e}^{-2\pi i\,\mathrm{sign}(x-y)}\,\mathsf{E}^{\beta}(y)\mathsf{E}^{\alpha}(x),$$

and a clever representation of the screening charges in term of a Weyl type operators it is possible to normal order both sides of (8.12). Schematically

$$\begin{aligned}
\mathsf{g}_{s_2}^{\alpha_2}(x_2)\mathsf{g}_{s_1}^{\alpha_1}(x_1) &= \mathsf{E}^{\alpha_2}(x_2)\mathsf{E}^{\alpha_1}(x_1)A(\mathsf{x})P_{2,1}(\mathsf{p},\mathsf{t}),\\
\mathsf{g}_{t_1}^{\alpha_1}(x_1)\mathsf{g}_{t_2}^{\alpha_2}(x_2) &= \mathsf{E}^{\alpha_2}(x_2)\mathsf{E}^{\alpha_1}(x_1)A(\mathsf{x})P_{1,2}(\mathsf{p},\mathsf{t}),
\end{aligned}$$

where the functions $A, P_{1,2}$ and $P_{2,1}$ are explicitly known and the operators $\mathsf{x},\mathsf{p},\mathsf{t}$ satisfy commutation relations

$$[\mathsf{p},\mathsf{x}] = -\frac{i}{2}, \qquad [\mathsf{p},\mathsf{t}] = [\mathsf{t},\mathsf{x}] = 0.$$

Upon acting on a common eigenstate of p and t the formula (8.12) thus boils down to a relation involving just functions of eigenvalues of these operators and the integral kernel B we are after. The special function G_{b} appears in this formula thanks to identities of the form

$$\mathrm{e}^{\mathrm{b}\mathsf{x}}\left(1+\mathrm{e}^{-2\pi\mathrm{b}\mathsf{p}}\right)\mathrm{e}^{-\mathrm{b}\mathsf{x}} = G_{\mathrm{b}}\left(\tfrac{Q}{2}+i\mathsf{p}\right)\mathrm{e}^{2\mathrm{b}\mathsf{x}}G_{\mathrm{b}}^{-1}\left(\tfrac{Q}{2}+i\mathsf{p}\right) = \mathrm{e}^{2\mathrm{b}\mathsf{x}}G_{\mathrm{b}}\left(\tfrac{Q}{2}+i\mathsf{p}+\mathrm{b}\right)G_{\mathrm{b}}^{-1}\left(\tfrac{Q}{2}+i\mathsf{p}\right)$$

which also allow to calculate an arbitrary power of the screening charge Q.

To calculate the braiding matrix of the normalized vertex operators, appearing in eq. (8.7), one needs to determine the matrix element of the chiral field $\mathsf{g}_s^{\alpha}(1)$ between highest weight states of the Verma modules $\mathcal{V}_{\Delta(p_1)}$ and $\mathcal{V}_{\Delta(q)}$. This was achieved in [21, 22] by deriving and solving a pair of difference equations for this matrix element, which follow from considering the four-point correlation function involving degenerate field $\mathsf{E}^{-\frac{\mathrm{b}}{2}}(x)$.

Finally, a relation between braiding matrix and the fusion matrix can be derived by considering a sequence of "moves" including braiding of generic vertices, "elementary" braiding of a generic vertex with the vertex acting on the vacuum Verma module (with $\Delta = 0$) and use of a state-operator correspondence [22] (see also [24] for a more detailed explanation). It reads

$$F_{\alpha_s\,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = B_{\alpha_s\,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_1\\ \alpha_4 & \alpha_2\end{bmatrix}. \tag{8.13}$$

The resulting form of the Liouville fusion matrix, to be found for instance in [25] section 3.5, coincides with eq. (4.4) up to a factor due to a different normalizations of chiral vertex operators and Clebsh-Gordan coefficients.

8.2 Neveu-Schwarz sector of the $\mathcal{N} = 1$ superconformal theory

The Neveu-Schwarz (or NS for short) supermodule $\mathcal{V}_\Delta$ of the highest weight Δ and the central charge c is a free vector space spanned by vectors of the form

$$\nu_{\Delta,MK} \;=\; L_{-M}G_{-K}\nu_\Delta \;\equiv\; L_{-m_j}\ldots L_{-m_1}G_{-k_i}\ldots G_{-k_1}\nu_\Delta\,, \tag{8.14}$$

where $K = \{k_1, k_2, \ldots, k_i\}$ and $M = \{m_1, m_2, \ldots, m_j\}$ are arbitrary ordered sets of indices

$$k_i > \ldots > k_2 > k_1, \quad k_s \in \mathbb{N} - \frac{1}{2}, \qquad m_j \geqslant \ldots \geqslant m_2 \geqslant m_1, \quad m_r \in \mathbb{N}.$$

Here ν_Δ is the highest weight state of to the NS algebra,

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n}, \\
[L_m, G_k] &= \frac{m-2k}{2}G_{m+k}, \\
\{G_k, G_l\} &= 2L_{k+l} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}, \qquad c = \frac{3}{2} + 3Q^2,
\end{aligned} \tag{8.15}$$

even with respect to the the fermion parity operator $(-1)^F$ defined by relations

$$[(-1)^F, L_m] \;=\; \{(-1)^F, G_k\} = 0.$$

The NS module is thus a direct sum of an even and an odd (with respect to $(-1)^F$) subspaces

$$\mathcal{V}_\Delta = \mathcal{V}^{\rm e}_\Delta \oplus \mathcal{V}^{\rm o}_\Delta.$$

This $\mathbb{Z}_2$ grading reflects itself in a parity structure of the chiral vertex operators: we define them as two families of even,

$$V^{\rm e}\big[{}^{\;\Delta_2}_{\Delta_3\Delta_1}\big](z) :\; \mathcal{V}^\eta_{\Delta_1} \;\to\; \mathcal{V}^\eta_{\Delta_3}, \qquad V^{\rm e}\big[{}^{\;*\Delta_2}_{\Delta_3\Delta_1}\big](z) :\; \mathcal{V}^\eta_{\Delta_1} \;\to\; \mathcal{V}^\eta_{\Delta_3}, \qquad \eta = {\rm e}, {\rm o}$$

and two families of odd linear maps

$$V^{\rm o}\big[{}^{\;\Delta_2}_{\Delta_3\Delta_1}\big](z) :\; \mathcal{V}^\eta_{\Delta_1} \;\to\; \mathcal{V}^{\bar\eta}_{\Delta_3}, \qquad V^{\rm o}\big[{}^{\;*\Delta_2}_{\Delta_3\Delta_1}\big](z) :\; \mathcal{V}^\eta_{\Delta_1} \;\to\; \mathcal{V}^{\bar\eta}_{\Delta_3}, \qquad \bar{\rm e} = {\rm o},\; \bar{\rm o} = {\rm e},$$

uniquely specified by the (anti)commutation relations (here $\_\Delta_2$ stands either for Δ_2 or for $*\Delta_2$)

$$\begin{aligned}
\big[L_m, V^\eta\big[{}^{\Delta_2}_{\Delta_3\Delta_1}\big](z)\big] &= z^m \left(z\partial_z + (m+1)\Delta_2\right) V^\eta\big[{}^{\Delta_2}_{\Delta_3\Delta_1}\big](z),\\
\big[L_m, V^\eta\big[{}^{*\Delta_2}_{\Delta_3\Delta_1}\big](z)\big] &= z^m \left(z\partial_z + (m+1)\left(\Delta_2 + \tfrac{1}{2}\right)\right) V^\eta\big[{}^{*\Delta_2}_{\Delta_3\Delta_1}\big](z),\\
\big[G_k, V^{\rm e}\big[{}^{_\Delta_2}_{\Delta_3\Delta_1}\big](z)\big] &= z^{k+\frac{1}{2}}\, V^{\rm o}\big[{}^{_\Delta_2}_{\Delta_3\Delta_1}\big](z),\\
\big\{G_k, V^{\rm o}\big[{}^{_\Delta_2}_{\Delta_3\Delta_1}\big](z)\big\} &= z^{k-\frac{1}{2}} \left(z\partial_z + \Delta_2(2k+1)\right) V^{\rm e}\big[{}^{_\Delta_2}_{\Delta_3\Delta_1}\big](z),
\end{aligned} \tag{8.16}$$

and appropriate normalization conditions. We are thus in a position to define four even

$$\mathcal{F}^{\rm e}_{\Delta_s}\left[{}^{\_\Delta_3}_{\;\Delta_4}\;{}^{\_\Delta_2}_{\;\Delta_1}\right](z) = \left\langle \nu_{\Delta_4}, V^{\rm e}\big[{}^{\_\Delta_3}_{\Delta_4\Delta_s}\big](1) V^{\rm e}\big[{}^{\_\Delta_2}_{\Delta_s\Delta_1}\big](z)\nu_{\Delta_1}\right\rangle \tag{8.17}$$

and four odd conformal blocks,

$$\mathcal{F}^{\rm o}_{\Delta_s}\left[{}^{\_\Delta_3}_{\;\Delta_4}\;{}^{\_\Delta_2}_{\;\Delta_1}\right](z) = \left\langle \nu_{\Delta_4}, V^{\rm o}\big[{}^{\_\Delta_3}_{\Delta_4\Delta_s}\big](1) V^{\rm o}\big[{}^{\_\Delta_2}_{\Delta_s\Delta_1}\big](z)\nu_{\Delta_1}\right\rangle. \tag{8.18}$$

As in the Liouville case we define the fusion matrices F by the relation

$$\mathcal{F}^{\eta}_{\Delta_s}\left[{}^{\_\Delta_3}_{\;\Delta_4}\;{}^{\_\Delta_2}_{\;\Delta_1}\right](z) = \int\limits_{\frac{Q}{2}+i\mathbb{R}^+} {\rm d}\alpha_t \sum_{\rho={\rm e,o}} F_{\alpha_s\alpha_t}\left[{}^{\_\alpha_3}_{\;\alpha_4}\;{}^{\_\alpha_2}_{\;\alpha_1}\right]^{\eta}{}_{\rho}\; \mathcal{F}^{\rho}_{\Delta_s}\left[{}^{\_\Delta_1}_{\;\Delta_4}\;{}^{\_\Delta_2}_{\;\Delta_3}\right](1-z). \tag{8.19}$$

Calculation of the braiding matrices above, given in [13], is parallel to the calculation in the Liouville case and we shall not present any of its details here, referring the interested reader to the original paper. To relate the findings of that paper to the form of the 6j symbols given in (7.4) let us start by observing that formulae (5.10), (4.61) and (4.53) from [13] give

$$\begin{aligned}
&F_{\alpha_s\alpha_t}\left[{}^{\alpha_3}_{\alpha_4}\;{}^{\alpha_2}_{\alpha_1}\right]^{\rm e}{}_{\rm e}\\
&\quad= \frac{\Gamma_1(\alpha_t+\alpha_4-\alpha_1)\Gamma_1(\bar\alpha_t+\alpha_4-\alpha_1)\Gamma_1(\alpha_t+\bar\alpha_4-\alpha_1)\Gamma_1(\bar\alpha_t+\bar\alpha_4-\alpha_1)}{\Gamma_1(\alpha_s+\alpha_4-\alpha_3)\Gamma_1(\bar\alpha_s+\alpha_4-\alpha_3)\Gamma_1(\alpha_s+\bar\alpha_4-\alpha_3)\Gamma_1(\bar\alpha_s+\bar\alpha_4-\alpha_3)}\\
&\quad\times \frac{\Gamma_1(\alpha_t+\alpha_2-\alpha_3)\Gamma_1(\bar\alpha_t+\alpha_2-\alpha_3)\Gamma_1(\alpha_t+\bar\alpha_2-\alpha_3)\Gamma_1(\bar\alpha_t+\bar\alpha_2-\alpha_3)}{\Gamma_1(\alpha_s+\alpha_2-\alpha_1)\Gamma_1(\bar\alpha_s+\alpha_2-\alpha_1)\Gamma_1(\alpha_s+\bar\alpha_2-\alpha_1)\Gamma_1(\bar\alpha_s+\bar\alpha_2-\alpha_1)}\\
&\quad\times \frac{1}{4}\frac{\Gamma_1(2\alpha_s)\Gamma_1(2\bar\alpha_s)}{\Gamma_1(Q-2\alpha_t)\Gamma_1(2\alpha_t-Q)} \int\limits_{i\mathbb{R}} \frac{dt}{i}\; J_{\alpha_s\alpha_t}\left[{}^{\alpha_3}_{\alpha_4}\;{}^{\alpha_1}_{\alpha_2}\right]
\end{aligned}$$

where

$$J_{\alpha_s\alpha_t}\left[{}^{\alpha_3}_{\alpha_4}\;{}^{\alpha_1}_{\alpha_2}\right] = \sum_{\nu=0}^{1} \frac{S_\nu(\alpha_2+t)S_\nu(\bar\alpha_2+t)S_\nu(\bar\alpha_4-\alpha_3+\alpha_1+t)S_\nu(\alpha_4-\alpha_3+\alpha_1+t)}{S_\nu(\alpha_t+\bar\alpha_3+t)S_\nu(\bar\alpha_t+\bar\alpha_3+t)S_\nu(\alpha_s+\alpha_1+t)S_\nu(\bar\alpha_s+\alpha_1+t)}.$$

It was proven in [12] that this function enjoys the following symmetry property

$$F_{\alpha_s\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{e}}{}_{\mathrm{e}} \quad = \quad F_{\alpha_s\alpha_t}\begin{bmatrix}\bar{\alpha}_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{bmatrix}^{\mathrm{e}}{}_{\mathrm{e}}$$

so that we have

$$\begin{aligned} & F_{\alpha_s\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{e}}{}_{\mathrm{e}} \\ & \quad = \frac{\Gamma_1(\alpha_t+\alpha_3-\alpha_2)\Gamma_1(\bar{\alpha}_t+\alpha_3-\alpha_2)\Gamma_1(\alpha_t+\bar{\alpha}_3-\alpha_2)\Gamma_1(\bar{\alpha}_t+\bar{\alpha}_3-\alpha_2)}{\Gamma_1(\alpha_s+\alpha_3-\bar{\alpha}_4)\Gamma_1(\bar{\alpha}_s+\alpha_3-\bar{\alpha}_4)\Gamma_1(\alpha_s+\bar{\alpha}_3-\bar{\alpha}_4)\Gamma_1(\bar{\alpha}_s+\bar{\alpha}_3-\bar{\alpha}_4)} \\ & \quad \times \frac{\Gamma_1(\alpha_t+\alpha_1-\bar{\alpha}_4)\Gamma_1(\bar{\alpha}_t+\alpha_1-\bar{\alpha}_4)\Gamma_1(\alpha_t+\bar{\alpha}_1-\bar{\alpha}_4)\Gamma_1(\bar{\alpha}_t+\bar{\alpha}_1-\bar{\alpha}_4)}{\Gamma_1(\alpha_s+\alpha_1-\alpha_2)\Gamma_1(\bar{\alpha}_s+\alpha_1-\alpha_2)\Gamma_1(\alpha_s+\bar{\alpha}_1-\alpha_2)\Gamma_1(\bar{\alpha}_s+\bar{\alpha}_1-\alpha_2)} \\ & \quad \times \frac{1}{4}\frac{\Gamma_1(2\alpha_s)\Gamma_1(2\bar{\alpha}_s)}{\Gamma_1(Q-2\alpha_t)\Gamma_1(2\alpha_t-Q)} \int_{i\mathbb{R}} \frac{dt}{i}\; J_{\alpha_s\alpha_t}\begin{bmatrix}\bar{\alpha}_4 & \alpha_2\\ \alpha_3 & \alpha_1\end{bmatrix} \end{aligned}$$

where, after a shift of the integration variable $t \to t+\alpha_s-\alpha_2$,

$$\int_{i\mathbb{R}} \frac{dt}{i}\; J_{\alpha_s\alpha_t}\begin{bmatrix}\bar{\alpha}_4 & \alpha_2\\ \alpha_3 & \alpha_1\end{bmatrix} = \int_{i\mathbb{R}} \frac{dt}{i} \sum_{\nu=0}^{1} \frac{S_\nu(u_4+t)S_\nu(\tilde{u}_4+t)S_\nu(u_3+t)S_\nu(\tilde{u}_3+t)}{S_\nu(u_{23}+t)S_\nu(\tilde{u}_{23}+t)S_\nu(2\alpha_s+t)S_\nu(Q+t)} \tag{8.20}$$

with the variables $u_i, \tilde{u}_i$ etc. defined in section 4. Comparing eq. (8.20) with eq. (7.4) we thus see that

$$F\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{e}}{}_{\mathrm{e}} \;\propto\; \left(\left\{\begin{matrix}\alpha_1\;\alpha_3\\ \alpha_2\;\alpha_4\end{matrix}\Big|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\}_{\mathrm{b}}\right)^{11}_{11},$$

with the proportionality constant again due to a different normalization of chiral vertex operators and 6j symbols.

Repeating the same calculation for the other components of the fusion matrix given by eq. (5.10) from [13] we obtain

$$F\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{e}}{}_{\mathrm{o}} \;\propto\; \left(\left\{\begin{matrix}\alpha_1\;\alpha_3\\ \alpha_2\;\alpha_4\end{matrix}\Big|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\}_{\mathrm{b}}\right)^{11}_{00}$$

and

$$F\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{o}}{}_{\mathrm{e}} \;\propto\; \left(\left\{\begin{matrix}\alpha_1\;\alpha_3\\ \alpha_2\;\alpha_4\end{matrix}\Big|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\}_{\mathrm{b}}\right)^{00}_{11}, \qquad F\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{o}}{}_{\mathrm{o}} \;\propto\; \left(\left\{\begin{matrix}\alpha_1\;\alpha_3\\ \alpha_2\;\alpha_4\end{matrix}\Big|\begin{matrix}\alpha_s\\ \alpha_t\end{matrix}\right\}_{\mathrm{b}}\right)^{00}_{00}.$$

Similarly, components of the fusion matrix given by formula (5.11) from [13], as well as the remaining eight components of the fusion matric appearing in (8.19) not explicitly given in that paper, are expressible in terms of the 6j symbols (7.4).

9 Conclusions

In this work we have constructed and studied a set of infinite dimensional self-dual representations of the quantum deformed algebra $\mathcal{U}_q(osp(1|2))$. In particular, we computed the Clebsch-Gordan coefficients (6.2) for the decomposition of tensor products of self-dual representations and the associated Racah-Wigner coefficients (7.4). All these data were built out of a pair of special functions $\Phi^{\pm}_{\mathrm{b}}(z)$. In our analysis we employed a number of beautiful integral identities for these functions, see section 6 and appendix B.2. These mimic corresponding identities satisfied by the quantum dilogarithm, only that all integrations are now accompanied by a summation of discrete indices $\nu = 0, 1$.

There are a number of issues that would be interesting to address. To begin with, we have only computed the fusing matrix for the Neveu-Schwarz sector of $N = 1$ Liouville field theory. It would certainly be important to include matrix elements when some of the fields are taken from the Ramond-sector. Representations from the Ramond sector of the field theory should be associated with another series of self-dual representations of $\mathcal{U}_q(osp(1|2))$. Once these representations have been identified one needs to extend the above analysis to products within such an enlarged class of representations. In field theory, at least some elements of the extended fusing matrix are known. Since their form is not very different from what we encountered in the Neveu-Schwarz sector, the entire analysis is likely to rest on the set of integral identities we stated and used above.

Another interesting direction is to extend the number of supersymmetries. In stepping from $N = 1$ to $N = 2$ supersymmetric Liouville theory, we must replace $\mathcal{U}_q(osp(1|2))$ by $\mathcal{U}_q(osp(2|2)) = \mathcal{U}_q(sl(1|2))$. Even though this is probably the most relevant case, it would be possible to continue and study the entire series $\mathcal{U}_q(osp(N|2))$. Finally, let us also mention that there exists an intriguing duality between the 6j symbols for finite dimensional representations of $\mathcal{U}_q(sl(2))$ and $\mathcal{U}_q(osp(1|2))$, see [11, 26]. It would be very interesting to extend this duality to the self-dual series. We shall return to these issues in forthcoming publications.

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A Some special functions

This appendix contains definitions and a short list of important properties for the special functions that appear in the main text. The first subsections deals with those functions that arise in the context of $\mathcal{U}_q(sl(2))$ while the second subsection is tailored towards out discussion of $\mathcal{U}_q(osp(1|2))$. Derivations of some of the identities can be found at many places in the literature.

A.1 Special functions for $\mathcal{U}_q(sl(2))$

The basic building block for all objects that appear in the context of the quantum algebra $\mathcal{U}_q(sl(2))$ is Barnes' double Gamma function. For $\mathfrak{Re}x > 0$ it admits an integral representation

$$\log \Gamma_{\mathrm{b}}(x) = \int_0^\infty \frac{\mathrm{d}t}{t} \left[\frac{e^{-xt} - e^{-\frac{Q}{2}t}}{(1-e^{-t\mathrm{b}})(1-e^{-\frac{t}{\mathrm{b}}})} - \frac{\left(\frac{Q}{2}-x\right)^2}{2e^t} - \frac{\frac{Q}{2}-x}{t} \right],$$

where $Q = \mathrm{b} + \frac{1}{\mathrm{b}}$. One can analytically continue Γ_{b} to a meromorphic function defined on the entire complex plane $\mathbb{C}$. The most important property of Γ_{b} is its behavior with respect to shifts by $\mathrm{b}^{\pm}$,

$$\Gamma_{\mathrm{b}}(x+\mathrm{b}) = \frac{\sqrt{2\pi}\mathrm{b}^{\mathrm{b}x-\frac{1}{2}}}{\Gamma_{\mathrm{b}}(bx)}\Gamma_{\mathrm{b}}(x) \quad , \quad \Gamma_{\mathrm{b}}(x+\mathrm{b}^{-1}) = \frac{\sqrt{2\pi}\mathrm{b}^{-\frac{\mathrm{b}}{x}+\frac{1}{2}}}{\Gamma_{\mathrm{b}}(\frac{x}{\mathrm{b}})}\Gamma_{\mathrm{b}}(x) \; . \tag{A.1}$$

These shift equation allows us to calculate residues of the poles of Γ_{b}. When $x \to 0$, for instance, one finds

$$\Gamma_{\mathrm{b}}(x) = \frac{\Gamma_{\mathrm{b}}(Q)}{2\pi x} + O(1). \tag{A.2}$$

From Barnes' double Gamma function we can build two other important special functions,

$$S_{\mathrm{b}}(x) = \frac{\Gamma_{\mathrm{b}}(x)}{\Gamma_{\mathrm{b}}(Q-x)}, \tag{A.3}$$

$$G_{\mathrm{b}}(x) = e^{-\frac{i\pi}{2}x(Q-x)}S_{\mathrm{b}}(x). \tag{A.4}$$

We shall often refer to the function S_{b} as double sine function. It is related to Faddeev's quantum dilogarithm through,

$$\Phi_{\mathrm{b}}(x) = \; AG_{\mathrm{b}}^{-1}(-ix+\frac{Q}{2}),$$

where

$$A = e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} \quad , \quad c_{\mathrm{b}} = iQ/2 \ . \tag{A.5}$$

The S_{b} function is meromorphic with poles and zeros in

$$S_{\mathrm{b}}(x) = 0 \Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0} \ ,$$
$$S_{\mathrm{b}}(x)^{-1} = 0 \Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0} \ .$$

From its definition and the shift property of Barnes' double Gamma function it is easy to derive the following shift and reflection properties of G_{b},

$$G_{\mathrm{b}}(x + \mathrm{b}) = (1 - e^{2\pi i \mathrm{b} x}) G_{\mathrm{b}}(x) \ , \tag{A.6}$$

$$G_{\mathrm{b}}(x) G_{\mathrm{b}}(Q - x) = e^{\pi i x (x - Q)} \ . \tag{A.7}$$

We also need to the asymptotic behavior of the function G_{b} along the imaginary axis,

$$\begin{aligned} G_{\mathrm{b}}(x) \ &\sim 1 \ , & \mathfrak{Im} x \to +\infty, \\ G_{\mathrm{b}}(x) &\sim e^{i\pi x(x-Q)} \ , & \mathfrak{Im} x \to -\infty. \end{aligned} \tag{A.8}$$

A.2 Special functions for $\mathcal{U}_q(osp(1|2))$

In discussing the representation theory of the quantum superalgebra $\mathcal{U}_q(osp(1|2))$ we need the following additional special functions

$$\Gamma_1(x) = \Gamma_{\mathrm{NS}}(x) = \Gamma_{\mathrm{b}}\left(\frac{x}{2}\right) \Gamma_{\mathrm{b}}\left(\frac{x + Q}{2}\right),$$
$$\Gamma_0(x) = \Gamma_{\mathrm{R}}(x) = \Gamma_{\mathrm{b}}\left(\frac{x + \mathrm{b}}{2}\right) \Gamma_{\mathrm{b}}\left(\frac{x + \mathrm{b}^{-1}}{2}\right).$$

Furthermore, let us define

$$\begin{aligned} S_1(x) \ &= S_{\mathrm{NS}}(x) = \tfrac{\Gamma_{\mathrm{NS}}(x)}{\Gamma_{\mathrm{NS}}(Q-x)}, \qquad & G_1(x) \ &= G_{\mathrm{NS}}(x) = \zeta_0 e^{-\frac{i\pi}{4} x(Q-x)} S_{\mathrm{NS}}(x), \\ S_0(x) \ &= S_{\mathrm{R}}(x) = \tfrac{\Gamma_{\mathrm{R}}(x)}{\Gamma_{\mathrm{R}}(Q-x)}, & G_0(x) \ &= G_{\mathrm{R}}(x) = e^{-\frac{i\pi}{4}} \zeta_0 e^{-\frac{i\pi}{4} x(Q-x)} S_{\mathrm{R}}(x) \end{aligned} \tag{A.9}$$

where $\zeta_0 = \exp(-i\pi Q^2/8)$. The functions S_ν are related to the supersymmetric analogue of Faddeev's quantum dilogarithm through

$$\Phi_{\mathrm{b}}^{\nu}(x) = \ A^2 G_{\nu}^{-1}(-ix + \frac{Q}{2}),$$

with a constant A as defined in eq. (A.5). As for $S_{\rm b}$, the functions $S_0(x)$ and $S_1(x)$ are meromorphic with poles and zeros in

$$\begin{aligned} S_0(x) = 0 &\Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}+1, \\ S_1(x) = 0 &\Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}, \\ S_0(x)^{-1} = 0 &\Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}+1, \\ S_1(x)^{-1} = 0 &\Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}. \end{aligned}$$

As in the previous subsection, we want to state the shift and reflection properties of the functions G_1 and G_0,

$$G_\nu(x + \beta^{\pm 1}) = (1 - (-1)^\nu e^{\pi i {\rm b}^{\pm 1} x}) G_{\nu+1}(x), \tag{A.10}$$

$$G_\nu(x) G_\nu(Q - x) = e^{\frac{i\pi}{2}(\nu-1)} \zeta_0^2 e^{\frac{\pi i}{2} x(x-Q)} \ . \tag{A.11}$$

Asymptotically, the functions G_1 and G_0 behave as

$$G_\nu(x) \sim 1 \ , \qquad \mathfrak{Im} x \to +\infty \ , \tag{A.12}$$

$$G_\nu(x) \sim e^{\frac{i\pi}{2}(\nu-1)} \zeta_0^2 e^{\frac{i\pi}{2} x(x-Q)} \ , \quad \mathfrak{Im} x \to -\infty \ . \tag{A.13}$$

B Integral identities

For our proof of the orthogonality relations of Clebsch Gordan coefficients we need a number of integral formulas for the special functions discussed in the previous section. We shall state these identities here. In both subsections we shall start with the star triangle relations and then deduce a number of simpler integral identities.

B.1 Integral identities for $\mathcal{U}_q(sl(2))$

The most complex identity we need in the main text is the following star triangle relation for double sine function,

$$\int \frac{{\rm d}x}{i} \prod_{i=1}^{3} S_{\rm b}(x + a_i) S_{\rm b}(-x + b_i) = \prod_{i,j=1}^{3} S_{\rm b}(a_i + b_j) \ ,$$

which holds provided that the arguments satisfy the balancing condition

$$\sum_{i=1}^{3} (a_i + b_i) = Q.$$

A proof can be found e.g. in [15]. Here, we will only state the necessary results. The star triangle relation can be reduced to the Saalschütz summation formula

$$\frac{1}{i}\int_{-i\infty}^{i\infty} \mathrm{d}\tau e^{2\pi i\tau Q}\frac{G_{\mathrm{b}}(\tau+a)G_{\mathrm{b}}(\tau+b)G_{\mathrm{b}}(\tau+c)}{G_{\mathrm{b}}(\tau+a+b+c-d+Q)G_{\mathrm{b}}(\tau+Q)G_{\mathrm{b}}(\tau+d)} =$$

$$= e^{i\pi d(Q-d)}G_{\mathrm{b}}(a)G_{\mathrm{b}}(b)G_{\mathrm{b}}(c)\frac{G_{\mathrm{b}}(Q+b-d)G_{\mathrm{b}}(Q+c-d)G_{\mathrm{b}}(Q+a-d)}{G_{\mathrm{b}}(Q+b+c-d)G_{\mathrm{b}}(Q+a+c-d)G_{\mathrm{b}}(Q+a+b-d)}.$$

A useful consequence of the Saalschütz summation formula can be obtained by taking the limit $c \to i\infty$

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{2\pi i\tau Q}\frac{G_{\mathrm{b}}(\tau+a)G_{\mathrm{b}}(\tau+b)}{G_{\mathrm{b}}(\tau+d)G_{\mathrm{b}}(\tau+Q)} =$$

$$= e^{i\pi d(Q-d)}G_{\mathrm{b}}(a)G_{\mathrm{b}}(b)G_{\mathrm{b}}(Q+b-d)\frac{G_{\mathrm{b}}(Q+a-d)}{G_{\mathrm{b}}(Q+a+b-d)}.$$

Also, by taking the additional limits $a \to -i\infty$, $d \to -i\infty$ with $a - d + Q$ fixed one may derive the well known Ramanujan summation formula

$$\int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{2\pi i\tau\beta}\frac{G_{\mathrm{b}}(\tau+\alpha)}{G_{\mathrm{b}}(\tau+Q)} = \frac{G_{\mathrm{b}}(\alpha)G_{\mathrm{b}}(\beta)}{G_{\mathrm{b}}(\alpha+\beta)}, \tag{B.1}$$

which holds for arbitrary $\alpha = a - d + Q$ and $\beta = b$. Ramanujan's summation formula is a five-term (pentagon) identity. In may be considered a quantization of the familiar Rogers five-term identity satisfied by dilogarithms. In fact, the function G_{b} that was used throughout most of this test is closely related to Faddeev's quantum dilogarithm Φ_{b} which we introduced in the introduction, see eq. (1.1).

B.2 Integral identities for $\mathcal{U}_q(osp(1|2))$

In the supersymmetric case, the star triangle relations take the following form

$$\sum_{\nu=0,1}(-1)^{\nu(1+\sum_i(\nu_i+\mu_i))/2}\int\frac{\mathrm{d}x}{i}\prod_{i=1}^{3}S_{\nu+\nu_i}(x+a_i)S_{1+\nu+\mu_i}(-x+b_i) = 2\prod_{i,j=1}^{3}S_{\nu_i+\mu_i}(a_i+b_j),$$

with

$$\sum_i(\nu_i+\mu_i) = 1 \ mod \ 2 \tag{B.2}$$

and the balancing condition

$$\sum_{i=1}^{3}(a_i+b_i) = Q \ .$$

From these equations one can get 16 "supersymmetric" analogues of the Saalschütz summation formula, some of which are stated with proofs for instance in [12]. As in the non-supersymmetric case, taking the limit $d \to i\infty$ leads to the reduced formulae

$$\sum_{\sigma=0,1} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} e^{i\pi\tau Q} \frac{G_{\sigma+\rho_a}(\tau+a)G_{\sigma+\rho_\beta}(\tau+b)}{G_{\sigma+\rho_c}(\tau+c)G_{1+\sigma}(\tau+Q)} =$$
$$= 2i^{1-\rho_c}\zeta_0^{-3} e^{\frac{i\pi}{2}c(Q-c)} \frac{G_{\rho_a}(a)G_{\rho_b}(b)G_{1+\rho_a-\rho_c}(Q+a-c)G_{1+\rho_b-\rho_c}(Q+b-c)}{G_{\rho_a+\rho_b-\rho_c}(Q+a+b-c)}.$$

where $\zeta_0 = \exp(-i\pi Q^2/8)$ is the same constant factor as before. From these identities one can easily obtain a system of four equations that generalize Ramanujan's formula (B.1) to the supersymmetric case,

$$\sum_{\sigma=0,1} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} (-1)^{\rho_\beta\sigma} e^{\pi i\tau\beta} \frac{G_{\sigma+\rho_\alpha}(\tau+\alpha)}{G_{\sigma+1}(\tau+Q)} = 2\zeta_0^{-1} \frac{G_{\rho_\alpha}(\alpha)G_{1+\rho_\beta}(\beta)}{G_{\rho_\alpha+\rho_\beta}(\alpha+\beta)} \tag{B.3}$$

The notations are the same as in section B.1. The last identity is is supersymmetric version of the pentagon identity for Faddeev's quantum dilogarithm.

C Removing the regulator

Lets consider the distribution

$$D(x,\xi_-) = \lim_{\epsilon\to 0} \frac{S_\mathrm{b}(2\epsilon+x)S_\mathrm{b}(-\xi_- - x)S_\mathrm{b}(2\epsilon+\xi_-)}{S_\mathrm{b}(4\epsilon)}.$$

One wants to show that the following holds

$$D(x,\xi_-) = \delta(x)\delta(i\xi_-).$$

In order to do that, it is sufficient to independently integrate over the first or the second variable and establish that in both cases the result is proportional to the appropriate delta function.

For any test function $f(y,x)$, one can consider:

$$\int \frac{\mathrm{d}x}{i}\frac{\mathrm{d}y}{i} f(y,x)D(x,y) = \lim_{\epsilon\to 0} \int \frac{\mathrm{d}x}{i}\frac{\mathrm{d}y}{i} f(y,x) \frac{S_\mathrm{b}(2\epsilon+x)S_\mathrm{b}(-\xi_- - x)S_\mathrm{b}(2\epsilon+\xi_-)}{S_\mathrm{b}(4\epsilon)}$$
$$= \lim_{\epsilon\to 0} (2\pi)^{-2} \int \frac{\mathrm{d}x}{i}\frac{\mathrm{d}y}{i} f(y,x) \frac{\epsilon}{(\epsilon+x)(x+y)(\epsilon+y)} =$$
$$= \lim_{\epsilon\to 0} (2\pi)^{-2} \int \frac{\mathrm{d}x}{i}\frac{\mathrm{d}y}{i} f(\epsilon y,\epsilon x) \frac{1}{(1+x)(x+y)(1+y)} =$$
$$= \left((2\pi)^{-2} \int \frac{\mathrm{d}x}{i}\frac{\mathrm{d}y}{i} \frac{1}{(1+x)(x+y)(1+y)} \right) f(0,0),$$

and one only needs to fix the multiplicative constant.

Calculation for ξ_-

One wants to evaluate the expression

$$\int \frac{\mathrm{d}\xi_-}{i} f(\xi_-, x) D(x, \xi_-) = f(0,0)\delta(ix).$$

As a test function, lets choose $f(\xi, x) = \exp(-i\pi\xi(x+2a))$, for $a \in i\mathbb{R}\backslash\{0\}$ and $x \in i\mathbb{R}$. To begin with, let us define

$$\mu = -x + 2\epsilon,$$
$$\tau = -(\xi_- + 2\epsilon),$$

so that one has

$$\begin{aligned}
&\int \frac{\mathrm{d}\xi_-}{i} e^{-i\pi\xi_-(x+2a)} S_{\mathrm{b}}(2\epsilon + \xi_-) S_{\mathrm{b}}(-\xi_- - x) = \\
&= \int \frac{\mathrm{d}\tau}{i} e^{i\pi(\tau+2\epsilon)(x+2a)} \frac{S_{\mathrm{b}}(\tau+\mu)}{S_{\mathrm{b}}(\tau+Q)} = \\
&= e^{2i\pi\epsilon(x+2a)} \int \frac{\mathrm{d}\tau}{i} e^{i\pi\tau(x+2a)} e^{-i\pi\tau\mu} e^{\frac{i\pi}{2}(Q-\mu)\mu} \frac{G_{\mathrm{b}}(\tau+\mu)}{G_{\mathrm{b}}(\tau+Q)} = \\
&= e^{2i\pi\epsilon(x+2a)} e^{\frac{i\pi}{2}(Q-\mu)\mu} \int \frac{\mathrm{d}\tau}{i} e^{2i\pi\tau(\frac{x+2a-\mu}{2})} \frac{G_{\mathrm{b}}(\tau+\mu)}{G_{\mathrm{b}}(\tau+Q)} = \\
&= e^{2i\pi\epsilon(x+2a)} e^{\frac{i\pi}{2}(Q-\mu)\mu} \frac{G_{\mathrm{b}}(\mu) G_{\mathrm{b}}(\frac{x+2a-\mu}{2})}{G_{\mathrm{b}}(\frac{x+2a+\mu}{2})} = \\
&= e^{2i\pi\epsilon(x+2a)} e^{\frac{i\pi}{2}(Q-\mu)\mu} \frac{G_{\mathrm{b}}(2\epsilon - x) G_{\mathrm{b}}(x+a-\epsilon)}{G_{\mathrm{b}}(a+\epsilon)},
\end{aligned}$$

where one used Ramanujan summation formula. Therefore

$$\begin{aligned}
&\int \frac{\mathrm{d}\xi_-}{i} f(\xi_-, x) D(x, \xi_-) = \int \frac{\mathrm{d}\xi_-}{i} \lim_{\epsilon\to 0} e^{-i\pi\xi(x+2a)} \frac{S_{\mathrm{b}}(2\epsilon+x) S_{\mathrm{b}}(-\xi_- - x) S_{\mathrm{b}}(2\epsilon+\xi_-)}{S_{\mathrm{b}}(4\epsilon)} = \\
&= \lim_{\epsilon\to 0} \frac{S_{\mathrm{b}}(2\epsilon+x)}{S_{\mathrm{b}}(4\epsilon)} \int \frac{\mathrm{d}\xi_-}{i} e^{-i\pi\xi_-(x+2a)} S_{\mathrm{b}}(2\epsilon+\xi_-) S_{\mathrm{b}}(-\xi_- - x) = \\
&= e^{-\frac{i\pi}{2}(Q+x)x} \lim_{\epsilon\to 0} \frac{S_{\mathrm{b}}(2\epsilon+x)}{S_{\mathrm{b}}(4\epsilon)} \frac{G_{\mathrm{b}}(2\epsilon-x) G_{\mathrm{b}}(x+a-\epsilon)}{G_{\mathrm{b}}(a+\epsilon)} = \\
&= e^{-i\pi x^2} \frac{G_{\mathrm{b}}(x+a)}{G_{\mathrm{b}}(a)} \lim_{\epsilon\to 0} \frac{G_{\mathrm{b}}(2\epsilon+x) G_{\mathrm{b}}(2\epsilon-x)}{G_{\mathrm{b}}(4\epsilon)} = (*)
\end{aligned}$$

Since $\lim_{x\to 0} x G_{\mathrm{b}}(x) = \frac{1}{2\pi}$ and it is known that

$$\lim_{\epsilon\to 0} \frac{2\epsilon}{\pi(4\epsilon^2 - x^2)} = \delta(ix),$$

one obtains eventually

$$(*) = e^{-i\pi x^2} \frac{G_{\mathrm{b}}(x+a)}{G_{\mathrm{b}}(a)} \delta(ix) = \delta(ix).$$

Calculation for x

Now one can repeat the above procedure for integration over x, i.e. show that

$$\int \frac{\mathrm{d}x}{i} g(\xi_-, x) D(x, \xi_-) = g(0,0)\delta(i\xi_-).$$

Lets take a different test function g s.t.

$$g(x, \xi_-) = e^{-i\pi x(\xi_- + 2a)},$$

where $A \in i\mathbb{R}\backslash\{0\}$. Then define

$$\mu = -\xi_- + 2\epsilon,$$
$$\tau = -(x + 2\epsilon),$$

so that one has

$$\begin{aligned} &\int \frac{\mathrm{d}x}{i} e^{-i\pi x(\xi_- + 2a)} S_{\mathrm{b}}(2\epsilon + x) S_{\mathrm{b}}(2\epsilon - \xi_- - x) = \\ &= e^{2i\pi\epsilon(\xi_- + 2a)} e^{\frac{i\pi}{2}(Q-\mu)\mu} \int \frac{\mathrm{d}\tau}{i} e^{2i\pi\tau(\frac{\xi_- + 2a - \mu}{2})} \frac{G_{\mathrm{b}}(\tau + \mu)}{G_{\mathrm{b}}(\tau + Q)} = \\ &= e^{2i\pi\epsilon(\xi_- + 2a)} e^{\frac{i\pi}{2}(Q-\mu)\mu} \frac{G_{\mathrm{b}}(2\epsilon - \xi_-) G_{\mathrm{b}}(\xi_- + a - \epsilon)}{G_{\mathrm{b}}(a + \epsilon)}, \end{aligned}$$

where one again used Ramanujan summation formula. Therefore

$$\begin{aligned} &\int \frac{\mathrm{d}x}{i} g(\xi_-, x) D(x, \xi_-) = \lim_{\epsilon \to 0} \frac{S_{\mathrm{b}}(2\epsilon + \xi_-)}{S_{\mathrm{b}}(4\epsilon)} \int \frac{\mathrm{d}x}{i} e^{-i\pi x(\xi_- + 2a)} S_{\mathrm{b}}(2\epsilon + x) S_{\mathrm{b}}(-\xi_- - x) = \\ &= e^{-i\pi\xi_-^2} \lim_{\epsilon \to 0} \frac{G_{\mathrm{b}}(2\epsilon + \xi_-) G_{\mathrm{b}}(2\epsilon - \xi_-)}{G_{\mathrm{b}}(4\epsilon)} = e^{-i\pi\xi_-^2} \delta(i\xi_-) = \delta(\xi_-). \end{aligned}$$

Putting the two results together one concludes that

$$D(x, \xi_-) = \delta(ix)\delta(i\xi_-).$$

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