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The universal Racah-Wigner symbol for $U_q(osp(1|2))$

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Abstract

We propose a new and elegant formula for the Racah-Wigner symbol of self-dual continuous series of representations of $U_q(osp(1|2))$. It describes the entire fusing matrix for both NS and R sector of N=1 supersymmetric Liouville field theory. In the NS sector, our formula is related to an expression derived in [1]. Through analytic continuation in the spin variables, our universal expression reproduces known formulas for the Racah-Wigner coefficients of finite dimensional representations.

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1 Introduction

The Racah-Wigner coefficients of Lie (super)algebras and their deformations play an important role in modern mathematical physics. Up to some normalization dependent prefactors, they coincide with the so-called fusing matrix of 2-dimensional Wess-Zumino-Novikov-Witten (WZNW) models and hence feature very prominently in the conformal bootstrap of these models and many descendants thereof. In fact, they do not only provide the coefficients in the bootstrap equations but also furnish some of their famous solutions e.g. for the bulk and boundary operator product coefficients. This dual purpose of the Racah-Wigner coefficients is based on a number of identities they satisfy, most importantly the well-known pentagon equation. The same identities are also exploited in the construction of state-sum models for topological 3-manifold invariants. These provide another important area in which Racah-Wigner symbols appear.

Recently, two of the authors and Leszek Hadasz constructed the Racah-Wigner symbol for a series of self-dual representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ [1] for $q = \exp(i\pi b^2)$ and real b^2. They also verified that the resulting expressions agree with the fusing matrix of N=1 Liouville field theory in the Neveu-Schwarz (NS) sector [2, 3]. A central goal of the present work is to extend the previous expression to include both NS and Ramond (R) sector fields. The way in which we shall achieve our goal is quite interesting in its own right.

Let us recall that the expression for the Racah-Wigner symbol found in [1] generalized previous formulas by Ponsot and Teschner for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$ [4, 5]. In a remarkable recent paper [6], Teschner and Vartanov found an alternative and much more natural way to express the same Racah-Wigner symbol. In particular, the new formulation is very closely modeled after the famous expressions for the Racah-Wigner coefficients of finite dimensional $\mathrm{U}_q(\mathrm{sl}(2))$ representations [7, 8], only that an integral appears instead of the usual summation and q-factorials are replaced by double Gamma functions.

Our strategy here is to extend the Teschner-Vartanov expressions for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$ to the supersymmetric case. Up to certain sign factors, this step is relatively straight-forward, taking into account some of the properties of the formula derived in [1]. The resulting expression is so natural that its extension to the R sector is rather easy to guess. Only the sign factors are a bit tricky to extend. We shall come up with a concrete proposal. In order to test our prescription for both NS and R sector labels we shall continue the integral formulas from spins $\alpha \in Q/2 + i\mathbb{R}$ to the discrete set $j = -\alpha/b \in \mathbb{N}/2$ at which the integrals can be evaluated by summing over certain residues.

When j is integer, the result of this evaluation gives the known 6J symbols for finite dimensional spin j representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ [9, 10] This limit only uses information from the NS sector, but can be considered a very strong test of our proposal for the universal Racah-Wigner symbol, including the sign factors we prescribe in the NS sector.

In order to probe the R sector of the theory we make use of a remarkable observation in [11, 12]. These authors found that the 6J symbols for finite dimensional integer spin representations of $\mathrm{U}_{q'}(\mathrm{sl}(2))$ and $\mathrm{U}_q(\mathrm{osp}(1|2))$ actually coincide when $q' = i\sqrt{q}$. Because of the usual relation between the deformation parameter $q = \exp(i\pi/(2k+3)$ and the level k, the deformation parameter q' actually tends to $q' = i$ in the semiclassical limit $k \to \infty$ of $\mathrm{U}_q(\mathrm{osp}(1|2))$, i.e. it is associated to a point $q' = \exp(i\pi/(k+2)$ with $k = 0$, deeply in the quantum region of $\mathrm{U}_{q'}(\mathrm{sl}(2))$. In this sense, the numerical coincidences between 6J symbols of finite dimensional representations observed in [11, 12] can be thought of as a non-perturbative duality. [1] In our context we will find that the limiting $\mathrm{U}_q(\mathrm{osp}(1|2))$ Racah-Wigner symbols with discrete weights, including those corresponding to half-integer spin j, coincide with the 6J symbols of finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$. Thereby, we provide highly non-trivial evidence for our choice of sign factors in the R sector of the theory.

The plan of this paper is as follows. In the next section we shall re-address the case of $\mathrm{U}_q(\mathrm{sl}(2))$ and show how to recover the Racah-Wigner coefficients of finite dimensional representations from the formula of Teschner and Vartanov. After this warm-up, we can turn to the supersymmetric case in section 3. There we propose a new expression for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{osp}(1|2))$. The comparison with the 6J symbols for integer spin representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ and with finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$ is performed in section 4. We conclude this work with a number of comments on open problems, including some speculations about the extension of the duality between $\mathrm{U}_q(\mathrm{sl}(2))$ and $\mathrm{U}_q(\mathrm{osp}(1|2))$ to infinite dimensional self-dual representations.

2 The Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$

In this section we will start from a recent integral formula for the Racah-Wigner symbol of a self-dual series representations of $\mathrm{U}_q(\mathrm{sl}(2))$ with $q = e^{i\pi b^2}$ which was presented in [6]. This symbol turns out to simplify when the representation labels $\alpha = Q/2 + i\mathbb{R}, Q = b + b^{-1}$, assume a value $-2\alpha b^{-1} \in \mathbb{N}$. In fact, it can be written as a sum over finitely many pole contributions. We compare the resulting expressions with the formulas for Racah-Wigner coefficients of finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$ and find complete agreement, at least up to some normalization dependent prefactors.

[1] We thank Edward Witten for stressing this aspect of the duality in a private conversation.

Let us begin our discussion by reviewing the formulas for the universal Racah-Wigner coefficients of $\mathrm{U}_q(\mathrm{sl}(2))$ which were proposed by Teschner and Vartanov [6]

$$\left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{array} \right\} = \Delta(\alpha_1,\alpha_2,\alpha_s)\Delta(\alpha_s,\alpha_3,\alpha_4)\Delta(\alpha_t,\alpha_3,\alpha_2)\Delta(\alpha_4,\alpha_t,\alpha_1) \tag{2.1}$$
$$\times \int_{\mathcal{C}} du\, S_b(u-\alpha_{12s})\, S_b(u-\alpha_{s34})\, S_b(u-\alpha_{23t})\, S_b(u-\alpha_{1t4})$$
$$S_b(\alpha_{1234}-u)\, S_b(\alpha_{st13}-u)\, S_b(\alpha_{st24}-u)\, S_b(2Q-u)$$

where

$$\Delta(\alpha_3,\alpha_2,\alpha_1) = \left(\frac{S_b(\alpha_{123}-Q}{S_b(\alpha_{12}-\alpha_3)\, S_b(\alpha_{23}-\alpha_1)\, S_b(\alpha_{31}-\alpha_2)}\right)^{\frac{1}{2}} \tag{2.2}$$

and the multi-index of α denotes summation, e.g. $\alpha_{ij} = \alpha_i + \alpha_j$. The contour $\mathcal{C}$ crosses the real axis in the interval $(\frac{3Q}{2}, 2Q)$ and approaches $2Q + i\mathbb{R}$ near infinity. The double sine function $S_b(x)$ is given in terms of Barnes' double Gamma function. Its definition and some relevant properties are listed in appendix (A.1). Let us note that Teschner and Vartanov were able to show that the expression (2.1) agrees with an earlier formula for the Racah-Wigner symbol of $\mathrm{U}_q(\mathrm{sl}(2))$ that was established by Teschner and Ponsot [4, 5].

Let us begin our analysis of the Racah-Wigner symbols (2.1) with the prefactor of the integral in the first line. Insertion of the definition (2.2) gives

$$\Delta(\alpha_1,\alpha_2,\alpha_s)\Delta(\alpha_s,\alpha_3,\alpha_4)\Delta(\alpha_t,\alpha_3,\alpha_2)\Delta(\alpha_4,\alpha_t,\alpha_1) \tag{2.3}$$
$$= \left(\frac{S_b(\alpha_{12s}-Q)S_b(\alpha_{s34}-Q)}{S_b(\alpha_{12}-\alpha_s)\, S_b(\alpha_{2s}-\alpha_1)\, S_b(\alpha_{1s}-\alpha_2)\, S_b(\alpha_{34}-\alpha_s)\, S_b(\alpha_{3s}-\alpha_4)\, S_b(\alpha_{4s}-\alpha_3)}\right)^{\frac{1}{2}}$$
$$\times \left(\frac{S_b(\alpha_{23t}-Q)S_b(\alpha_{1t4}-Q)}{S_b(\alpha_{23}-\alpha_t)\, S_b(\alpha_{2t}-\alpha_3)\, S_b(\alpha_{3t}-\alpha_2)\, S_b(\alpha_{14}-\alpha_t)\, S_b(\alpha_{1t}-\alpha_4)\, S_b(\alpha_{4t}-\alpha_1)}\right)^{\frac{1}{2}}$$

We observe that the prefactor vanishes each time one of the external weights approaches a degenerate value $\alpha_i \to -\frac{nb}{2} - \frac{n'}{2b}$ where $n, n' \in \mathbb{Z}_{\geq 0}$, and one of the intermediate weights may be obtained by fusion of α_j with the degenerate weight, i.e.

$$\alpha_s \to \alpha_j - \frac{sb}{2} - \frac{s'}{2b}\,, \qquad \text{or} \qquad \alpha_t \to \alpha_k - \frac{tb}{2} - \frac{t'}{2b}\,, \tag{2.4}$$
$$s,t \in \{-n, -n+2, \ldots, n\}\,, \qquad s',t' \in \{-n', -n'+2, \ldots, n'\}\,.$$

In Liouville theory, fields with degenerate weights satisfy additional null vector decoupling equations which restrict the possible operator products to a finite set of terms which are labeled by the weights in eq. (2.4). As we shall show below, the full Racah-Wigner symbol does not vanish for these special values because the integral in eq. (2.1) contributes singular terms such that the limit of the Racah-Wigner symbols is finite and non-zero.

In order to see how this works in detail, let us consider the limit of degenerate weight $\alpha_2 \to -\frac{nb}{2}$ $(n > 0)$ and $\alpha_s \to \alpha_1 - \frac{sb}{2}$. The zero in the prefactor comes from the first two terms in the denominator of eq. (2.3)

$$\lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \left(S_b(\alpha_{12} - \alpha_s)\, S_b(\alpha_{2s} - \alpha_1)\right)^{-\frac{1}{2}} = \left(S_b\left(\frac{s-n}{2}b\right) S_b\left(-\frac{s+n}{2}b\right)\right)^{-\frac{1}{2}}$$

$$= \left(-2\sin(\pi b^2)\right)^{\frac{n}{2}} \left(\left[\frac{n-s}{2}\right]! \left[\frac{n+s}{2}\right]!\right)^{\frac{1}{2}} S_b(0)^{-1}$$

where we used the shift relation (A.4) for the double sine function and the notation

$$[x] = \frac{\sin(\pi b^2 x)}{\sin \pi b^2} \ . \tag{2.5}$$

For integer x the factorial $[x]!$ is defined as,

$$[x]! = \prod_{a=1}^{x} [a] = \left(\sin \pi b^2\right)^{-x} \prod_{a=1}^{x} \sin(\pi b^2 a) \ . \tag{2.6}$$

In order to obtain a finite non-zero limit for the full Racah-Wigner symbol, the integral must contribute a divergent factor $S_b(0)$ to cancel the corresponding term from the prefactor. Let us therefore have a closer look at the integral

$$\int_{\mathcal{C}} du \quad S_b(u - \alpha_{12s})\, S_b(u - \alpha_{s34})\, S_b(u - \alpha_{23t})\, S_b(u - \alpha_{1t4}) \tag{2.7}$$
$$S_b(\alpha_{1234} - u)\, S_b(\alpha_{st13} - u)\, S_b(\alpha_{st24} - u)\, S_b(2Q - u) \ .$$

The first contribution to a singular result comes from two terms of the integrant $S_b(u - \alpha_{s34})\, S_b(\alpha_{1234} - u)$. The points $u = \alpha_{s34}$ and $u = \alpha_{1234}$ are situated on the left and right sides of the contour, respectively, see figure 1. Taking the limit $\alpha_2 \to -\frac{nb}{2}$ and $\alpha_s \to \alpha_1 - \frac{sb}{2}$

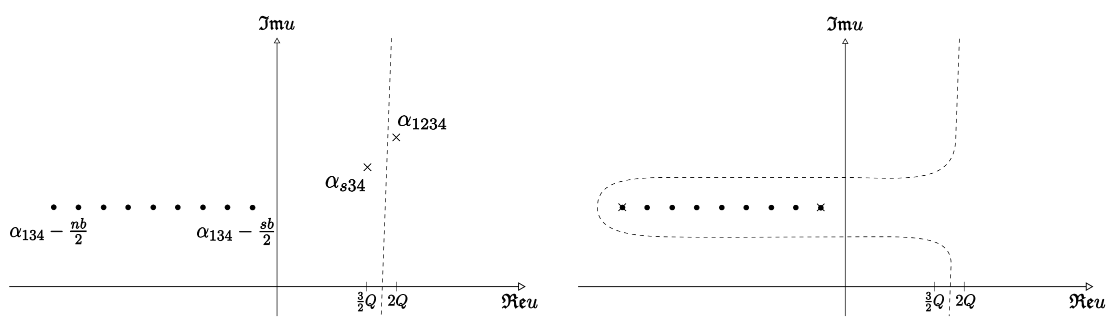

Figure 1: The original integration contour passes between the points $u = \alpha_{s34}$ and $u = \alpha_{1234}$. As we move the point α_{1234} to its limiting value, the shown poles contribute to the integral due to the pinching mechanism.

requires a certain deformation of the contour. Let us first consider the case of $s \geq 0$. Then α_{s34} can reach the point $\alpha_{134} - \frac{sb}{2}$ without crossing through the contour. On the other hand the point α_{1234} meets the contour on the way to $\alpha_{134} - \frac{nb}{2}$. We can deform the contour as long as it does not pass through one of the double poles of $S_b(u - \alpha_{134} + \frac{sb}{2})\, S_b(\alpha_{134} - \frac{nb}{2} - u)$ in $u = \alpha_{134} - \frac{sb}{2} - pb$ ($0 \leq p \leq \frac{n-s}{2}$). From each pole we get a singular term due to the so called "pinching mechanism", see e.g. [5], Lemma 3 and [13, 2] for similar calculations. This is illustrated on the right hand side of figure 1. In the end we obtain the following sum

$$\sum_{p=0}^{\frac{n-s}{2}} \left(\frac{\left(-2\sin(\pi b^2)\right)^{\frac{s-n}{2}}\, S_b(0)}{[p]!\, \left[\frac{n-s}{2} - p\right]!}\, S_b(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb)\, S_b(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb) \right. \tag{2.8}$$
$$\left. S_b(\alpha_3 - \alpha_t - \frac{sb}{2} - pb)\, S_b(\alpha_t - \alpha_3 - \frac{nb}{2} + pb)\, S_b(\alpha_{1t} - \alpha_4 + pb)\, S_b(2Q - \alpha_{134} + \frac{sb}{2} + pb) \right).$$

When $s < 0$ the poles $u = \alpha_{134} - \frac{sb}{2} - pb$ for $0 \leq p < -\frac{s}{2}$ and $-\frac{s}{2} \leq p \leq \frac{n-s}{2}$ are located on the right and left side of the contour, respectively, see figure 2. Taking the limit $\alpha_s \to \alpha_1 - \frac{sb}{2}$ we have to deform contour such that it passes the poles with $0 \leq p < -\frac{s}{2}$. By taking $\alpha_2 \to -\frac{nb}{2}$ we get contributions from the rest of the poles ($-\frac{s}{2} \leq p \leq \frac{n-s}{2}$), see figure 2. The final result will be the same as in the case of $s \geq 0$ (2.8).

The second contribution to the singular result of the integral (2.7) comes from the functions $S_b(u - \alpha_{1t4})\, S_b(\alpha_{st24} - u)$ with common poles in $u = \alpha_{1t4} - p'b$ for $0 \leq p' \leq \frac{n+s}{2}$. Since $s > -n$, all the poles lie on the left side of the contour independently of the sign of the parameter s. The point α_{st24} lies on the right side of the contour and before reaching $\alpha_{1t4} - \frac{(s+n)b}{2}$ one needs

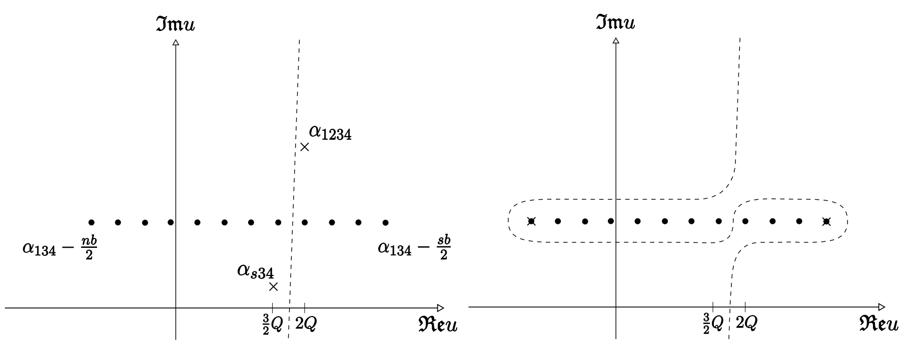

Figure 2: When $s < 0$ poles appear to both sides of the integration contour. While moving α_s and α_2 to their final values, we need to deform the contour such that it picks up contributions from all these poles.

to pass with the contour thought all the double poles obtaining the sum of singular terms,

$$\sum_{p'=0}^{\frac{n+s}{2}} \left(\frac{\left(-2\sin(\pi b^2)\right)^{-\frac{n+s}{2}} S_b(0)}{[p']!\,\left[\frac{n+s}{2}-p'\right]!} S_b(\alpha_{t4}-\alpha_1+\frac{(s+n)b}{2}-p'b)\, S_b(\alpha_{14}-\alpha_3+\frac{nb}{2}-p'b) \right. \tag{2.9}$$
$$\left. S_b(\alpha_t-\alpha_3+\frac{sb}{2}-p'b)\, S_b(\alpha_3-\alpha_t-\frac{nb}{2}+p'b)\, S_b(\alpha_{13}-\alpha_4-\frac{sb}{2}+p'b)\, S_b(2Q-\alpha_{1t4}+p'b) \right).$$

Combining the two above sums (2.8, 2.9) with the prefactor (2.3) we get a finite result for the limit:

$$\left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_1-\frac{sb}{2} \\ -\frac{nb}{2} & \alpha_4 & \alpha_t \end{array} \right\} \equiv \lim_{\substack{\alpha_2\to-\frac{nb}{2} \\ \alpha_s\to\alpha_1-\frac{sb}{2}}} \left\{ \begin{array}{cc|c} \alpha_1 & \alpha_3 & \alpha_s \\ \alpha_2 & \alpha_4 & \alpha_t \end{array} \right\}$$

$$= \left(\frac{S_b(\alpha_{14}+\alpha_t-Q)S_b(\alpha_3+\alpha_t-\frac{nb}{2}-Q)}{S_b(\alpha_3-\alpha_t-\frac{nb}{2})S_b(\alpha_t-\alpha_3-\frac{nb}{2})S_b(\alpha_{14}-\alpha_t)\,S_b(\alpha_{1t}-\alpha_4)\,S_b(\alpha_{4t}-\alpha_1)\,S_b(\alpha_{3t}+\frac{nb}{2})} \right)^{\frac{1}{2}}$$
$$\left(\frac{\left[\frac{n-s}{2}\right]!\,\left[\frac{n+s}{2}\right]!\,S_b(2\alpha_1-\frac{(s+n)b}{2}-Q)S_b(2\alpha_{134}-\frac{sb}{2}-Q)}{S_b(2\alpha_1+\frac{(n-s)b}{2})S_b(\alpha_{34}-\alpha_1+\frac{sb}{2})\,S_b(\alpha_{13}-\alpha_4-\frac{sb}{2})S_b(\alpha_{14}-\alpha_3-\frac{sb}{2})} \right)^{\frac{1}{2}}$$
$$\left\{ \sum_{q=0}^{\frac{n-s}{2}} \frac{\left(-2\sin(\pi b^2)\right)^{\frac{s}{2}}}{[q]!\,\left[\frac{n-s}{2}-q\right]!} S_b(\alpha_{34}-\alpha_1+\frac{nb}{2}-qb)\, S_b(\alpha_{14}-\alpha_t+\frac{(n-s)b}{2}-qb) \right.$$
$$S_b(\alpha_3-\alpha_t-\frac{sb}{2}-qb)\, S_b(\alpha_t-\alpha_3-\frac{nb}{2}+qb)\, S_b(\alpha_{1t}-\alpha_4+qb)\, S_b(2Q-\alpha_{134}+\frac{sb}{2}+qb)$$
$$+\sum_{p'=0}^{\frac{n+s}{2}} \frac{\left(-2\sin(\pi b^2)\right)^{-\frac{s}{2}}}{[p']!\,\left[\frac{n+s}{2}-p'\right]!} S_b(\alpha_{t4}-\alpha_1+\frac{(s+n)b}{2}-p'b)\, S_b(\alpha_{14}-\alpha_3+\frac{nb}{2}-p'b)$$
$$\left. S_b(\alpha_t-\alpha_3+\frac{sb}{2}-p'b)\, S_b(\alpha_3-\alpha_t-\frac{nb}{2}+p'b)\, S_b(\alpha_{13}-\alpha_4-\frac{sb}{2}+p'b)\, S_b(2Q-\alpha_{1t4}+p'b) \right\}.$$

Let us now consider the case when the other intermediate weight α_t also satisfies fusion rules (2.4) i.e. $\alpha_t \to \alpha_3 - \frac{tb}{2}$. Then prefactor in the formula above gives zero. On the other hand in each term of the sums there are double poles for $t \in \{-n+2p, -n+2p+2, \dots, s+2p\}$ and $t \in \{s-2p', s-2p'+2, \dots, n-2p'\}$ coming from $S_b(\alpha_3-\alpha_t-\frac{sb}{2}-pb)\, S_b(\alpha_t-\alpha_3-\frac{nb}{2}+pb)$ and $S_b(\alpha_t-\alpha_3-p'b+\frac{sb}{2})\, S_b(\alpha_3-\alpha_t+\alpha_2-p'b)$, respectively. The residue for a given $\alpha_t \to \alpha_3 - \frac{tb}{2}$

takes the form

$$\underset{\alpha_t\to\alpha_3-\frac{tb}{2}}{Res}\left\{\begin{array}{cc|c}\alpha_1 & \alpha_3 & \alpha_1-\frac{sb}{2}\\ -\frac{nb}{2} & \alpha_4 & \alpha_t\end{array}\right\}=2\left(\frac{S_b(2\alpha_1-\frac{(s+n)b}{2}-Q)S_b(2\alpha_3-\frac{(t+n)b}{2}-Q)}{S_b(2\alpha_1+\frac{(n-s)b}{2})\,S_b(2\alpha_3+\frac{(n-t)b}{2})}\right)^{\frac{1}{2}} \tag{2.10}$$

$$\sum_{p=max\{0,\frac{t-s}{2}\}}^{min\{\frac{n-s}{2},\frac{n+t}{2}\}}\frac{\left(\left[\frac{n-s}{2}\right]!\left[\frac{n+s}{2}\right]!\left[\frac{n-t}{2}\right]!\left[\frac{n+t}{2}\right]!\right)^{\frac{1}{2}}}{[p]!\left[\frac{n-s}{2}-p\right]!\left[\frac{s-t}{2}+p\right]!\left[\frac{n+t}{2}-p\right]!}\frac{S_b(\alpha_{13}-\alpha_4+pb-\frac{tb}{2})}{\left(S_b(\alpha_{13}-\alpha_4-\frac{sb}{2})S_b(\alpha_{13}-\alpha_4-\frac{tb}{2})\right)^{\frac{1}{2}}}$$

$$\frac{S_b(\alpha_{34}-\alpha_1-pb+\frac{nb}{2})}{\left(S_b(\alpha_{34}-\alpha_1+\frac{sb}{2})\,S_b(\alpha_{34}-\alpha_1-\frac{tb}{2})\right)^{\frac{1}{2}}}\frac{S_b(\alpha_{14}-\alpha_3-pb+\frac{(n+t-s)b}{2})}{\left(S_b(\alpha_{14}-\alpha_3-\frac{sb}{2})\,S_b(\alpha_{14}-\alpha_3+\frac{tb}{2})\right)^{\frac{1}{2}}}$$

$$\frac{\left(S_b(\alpha_{134}-\frac{sb}{2}-Q)S_b(\alpha_{134}-\frac{tb}{2}-Q)\right)^{\frac{1}{2}}}{S_b(\alpha_{134}-\frac{sb}{2}-pb-Q)}$$

where we redefined the second summation parameter $p'=p-\frac{t-s}{2}$ in order to obtain two identical sums. Now one can take a limit where all external weights have degenerate values $\alpha_i\to -j_ib$, $2j_i\in\mathbb{Z}_{\geq 0}$. We will denote this limit as

$$\left\{\begin{array}{cc|c}-j_1b & -j_3b & -j_sb\\ -j_2b & -j_4b & -j_tb\end{array}\right\} \tag{2.11}$$

remembering that it is a residue of the Racah-Wigner symbol with one degenerate external weight and both intermediate weights satisfying fusion rules.

Assuming that $\frac{n}{2}-\frac{\alpha_{134}}{b}=j_{134}+\frac{n}{2}$ in eq. (2.10) takes integer values one can write the S_b functions in terms of the [.]-factorials (2.6)

$$\left\{\begin{array}{cc|c}-j_1b & -j_3b & -j_1b-\frac{sb}{2}\\ -\frac{nb}{2} & -j_4b & -j_3b-\frac{t}{2}b\end{array}\right\}=2\left(\frac{[2j_1+\frac{s-n}{2}]!}{[2j_1+\frac{n+s}{2}+1]!}\frac{[2j_3+\frac{t-n}{2}]!}{[2j_3+\frac{n+t}{2}+1]!}\right)^{\frac{1}{2}}$$

$$\sum_{p=max\{0,\frac{t-s}{2}\}}^{min\{\frac{n-s}{2},\frac{t+n}{2}\}}(-1)^{j_1+j_3-p+\frac{n+t}{2}}\frac{\left(\left[\frac{n-s}{2}\right]!\left[\frac{n+s}{2}\right]!\left[\frac{n-t}{2}\right]!\left[\frac{n+t}{2}\right]!\right)^{\frac{1}{2}}}{[p]!\left[p+\frac{s-t}{2}\right]!\left[\frac{n-s}{2}-p\right]!\left[\frac{n+t}{2}-p\right]!}$$

$$\frac{[j_{134}+p+\frac{s}{2}+1]!}{\left([j_{134}+\frac{s}{2}+1]!\,[j_{134}+\frac{t}{2}+1]!\right)^{\frac{1}{2}}}\frac{\left([j_{13}-j_4+\frac{s}{2}]!\,[j_{13}-j_4+\frac{t}{2}]!\right)^{\frac{1}{2}}}{[j_{13}-j_4-p+\frac{t}{2}]!}$$

$$\frac{\left([j_{34}-j_1-\frac{s}{2}]!\,[j_{34}-j_1+\frac{t}{2}]!\right)^{\frac{1}{2}}}{[j_{34}-j_1+p-\frac{n}{2}]!}\frac{\left([j_{14}-j_3+\frac{s}{2}]!\,[j_{14}-j_3-\frac{t}{2}]!\right)^{\frac{1}{2}}}{[j_{14}-j_3+p-\frac{t+n-s}{2}]!}$$

where the minus sign comes from the difference in the shift relations (A.4) concerning $S_b(-xb)$ and $S_b(-xb+Q)$. Denoting $j_2=\frac{n}{2}, j_s=j_1+\frac{s}{2}, j_t=j_3+\frac{t}{2}$ and shifting the summation parameter to $z=p+j_{s34}$, one obtains the 6J symbol of the finite dimensional representations of the quantum deformed algebra $\mathrm{U}_q(\mathrm{sl}(2))$,

$$\left\{\begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array}\right\} = \frac{(-1)^{j_s+j_t}([2j_s+1]_q[2j_t+1]_q)^{-\frac{1}{2}}}{2\sin(\pi b^2)\sin(-\pi b^{-2})} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_q \tag{2.12}$$

where the deformation parameter q is given in terms of b as $q = e^{i\pi b^2}$ and the quantum numbers $[.]_q$ of $\mathrm{U}_q(\mathrm{sl}(2))$ are equal those defined in eq. (2.5), i.e.

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} = [x] \ . \tag{2.13}$$

The 6J symbol of finite dimensional representations of $\mathrm{U}_q(\mathrm{sl}(2))$ is given by the following sum [7, 8, 14]

$$\begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_q = \sqrt{[2j_s+1]_t[2j_t+1]_q}\,(-1)^{j_{12}-j_{34}-2j_s} \tag{2.14}$$
$$\times \sum_{z\geq 0}(-1)^z \frac{\Delta_q(j_s,j_2,j_1)\Delta_q(j_s,j_3,j_4)\Delta_q(j_t,j_3,j_2)\Delta_q(j_4,j_t,j_1)\,[z+1]_q!}{[z-j_{12s}]_q!\,[z-j_{34s}]_q!\,[z-j_{14t}]_q!\,[z-j_{23t}]_q![j_{1234}-z]_q!\,[j_{13st}-z]_q!\,[j_{24st}-z]_q!} \ .$$

Here, the summation extend over those values of z for which all arguments of the quantum number $[.]_q$ are non-negative. In addition we used the shorthand

$$\Delta_q(a,b,c) = \sqrt{[-a+b+c]_q!\,[a-b+c]_q!\,[a+b-c]_q!/[a+b+c+1]_q!}\,.$$

It is worth pointing out the similarities between the expression (2.14) and the original formula (2.1). In passing to eq. (2.14), the four factors Δ got replaced by Δ_q while the eight functions S_b have contributed the same number of quantum factorials. In addition, the integration over u became a summation over z.

Let as finally note that it is also possible to consider a limit of all weights approaching general degenerate values $\alpha_i \to -j_i b - j'_i b^{-1}$. In that case the limit is proportional to product of two 6J symbols of finite dimensional representations of the quantum deformed algebra $\mathrm{U}_q(\mathrm{sl}(2))$

$$\left\{\begin{array}{cc|c} -j_1 b - j'_1 b^{-1} & -j_3 b - j'_3 b^{-1} & -j_s b - j'_s b^{-1} \\ -j_2 b - j'_2 b^{-1} & -j_4 b - j'_4 b^{-1} & -j_t b - j'_t b^{-1} \end{array}\right\} = (-1)^{j_{st}+j'_{st}+3j_{1234st}j'_{1234st}-j_{13}j'_{13}-j_{24}j'_{24}-j_{st}j'_{st}}$$
$$\times \frac{([2j_s+1]_q[2j_t+1]_q[2j'_s+1]_{q'}[2j'_t+1]_{q'})^{-\frac{1}{2}}}{2\sin(\pi b^2)\sin(-\pi b^{-2})} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_q \begin{pmatrix} j'_1 & j'_2 & j'_s \\ j'_3 & j'_4 & j'_t \end{pmatrix}_{q'} \tag{2.15}$$

where deformation parameters are $q = e^{i\pi b^2}$ and $q' = e^{i\pi b^{-2}}$.

3 The supersymmetric Racah-Wigner symbol

After our warmup with the Racah-Wigner symbol of the $U_q(\mathrm{sl}(2))$, we are now prepared to study its extension to the supersymmetric case. We shall define the supersymmetric Racah-Wigner symbol in the next few paragraphs and comment a bit on its relation with N=1 Liouville field theory and the Racah-Wigner symbol for self-dual representations of $U_q(\mathrm{osp}(1|2))$. Then we perform an analysis along the lines of section 2, i.e. we compute the limit of the Racah-Wigner symbol for a discrete set of representation labels. The interpretation of the results is a bit more subtle than in the example of $U_q(\mathrm{sl}(2))$. It has to wait until section 4.

As a supersymmetric extension of the Racah-Wigner symbol (2.1) we propose the following integral formula

$$\left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = a_s + a_t \, mod\, 2} \Delta_{\nu_4}(\alpha_s,\alpha_2,\alpha_1)\Delta_{\nu_3}(\alpha_s,\alpha_3,\alpha_4)\Delta_{\nu_2}(\alpha_t,\alpha_3,\alpha_2) \tag{3.1}$$

$$\times \Delta_{\nu_1}(\alpha_4,\alpha_t,\alpha_1)\int_{\mathcal{C}} du \sum_{\nu=0}^{1}\Big((-1)^X S_{1+\nu+\nu_4+a_s}(u-\alpha_{12s})\, S_{1+\nu+\nu_3+a_s}(u-\alpha_{s34})$$
$$S_{1+\nu+\nu_2+a_t}(u-\alpha_{23t})\, S_{1+\nu+\nu_1+a_t}(u-\alpha_{1t4})\, S_{\nu+\nu_1+\nu_2+a_t}(\alpha_{1234}-u)$$
$$S_{\nu+\nu_1+\nu_3+a_2}(\alpha_{st13}-u)\, S_{\nu+\nu_1+\nu_4+a_3}(\alpha_{st24}-u)\, S_\nu(2Q-u)\Big)$$

where

$$\Delta_\nu(\alpha_3,\alpha_2,\alpha_1) = \left(\frac{S_{\nu+\frac{1}{2}a_{123}}(\alpha_{123}-Q)}{S_{\nu+\frac{1}{2}(a_{12}-a_3)}(\alpha_{12}-\alpha_3)\, S_{\nu+\frac{1}{2}(a_{23}-a_1)}(\alpha_{23}-\alpha_1)\, S_{\nu+\frac{1}{2}(a_{31}-a_2)}(\alpha_{31}-\alpha_2)}\right)^{\frac{1}{2}}$$

and the contour $\mathcal{C}$, as in the bosonic case, crosses the real axis in the interval $(\frac{3Q}{2}, 2Q)$ and approaches $2Q + i\mathbb{R}$ near infinity. Note that the arguments α^a of the Racah-Wigner symbol contain a continuous quantum number $\alpha \in Q/2 + i\mathbb{R}$ along with a superscript a that can take the values $a = 0$ and $a = 1$. The discrete label a keeps track on whether the corresponding representation is taken from the NS or R sector, respectively. We will comment a bit more on this below. Let us agree that the Racah-Wigner symbol is zero unless the discrete labels a_i satisfy the following conditions

$$a_s = a_1 + a_2 = a_3 + a_4 \, mod\, 2, \qquad a_t = a_1 + a_4 = a_2 + a_3 \, mod\, 2, \qquad \sum_{i=1}^{4} a_i = 0 \, mod\, 2. \tag{3.2}$$

The sign factor

$$(-1)^X = (-1)^{\nu(a_s\nu_1 + a_1\nu_3 + a_4\nu_4 + a_1 a_s + a_2 a_4 + a_s + a_t)} \tag{3.3}$$

kicks in as soon as some of the discrete labels a_i are nonzero. The supersymmetric double sine functions $S_\nu(x)$ with $\nu = 0, 1$ are defined in the appendix (A.5).

Before we continue our analysis, let us make a few comments on the status of the definition (3.1), its relation with $U_q(\mathrm{osp}(1|2))$ and with N=1 Liouville field theory. In recent work, two of the authors and Leszek Hadasz computed the Racah Wigner symbols for a certain series of self-dual representations of the quantum enveloping superalgebra $U_q(\mathrm{osp}(1|2))$. The arguments of this symbol assume values $\alpha \in Q/2 + i\mathbb{R}$. Furthermore, the symbol defined in [1] was shown to coincide with the fusing matrix of N=1 Liouville field theory when all field labels are taken from the NS sector of the model. The expression in [1] extends the one found by Teschner and Ponsot for $U_q(\mathrm{sl}(2))$. The latter has been rewritten by Teschner and Vartanov using some highly non-trivial integral identities. Our symbol (3.1) with $a_i = 0$ was defined to extend the Teschner-Vartanov version of the non-supersymmetric symbol to $U_q(\mathrm{osp}(1|2))$. At the moment we cannot prove that the expression (3.1), $a_i = 0$, agrees with the formula derived in [1] simply because we are missing certain supersymmetric analogues of the integral identities employed in [6]. On the other hand our results below make it seem highly plausible that both formulas agree. In [1] no attempt was made to extend the constructions to the R sector of $N = 1$ Liouville field theory. It is likely that $U_q(\mathrm{osp}(1|2))$ indeed possesses another self-dual series of representations which can mimic the R sector and that the fusing matrix involving R sector fields may be obtained from the Racah-Wigner symbol in an extended class of self-dual representations, but the details have not been worked out. Here we just make a bold proposal for the extension of the Racah-Wigner symbol cases with some $a_i \neq 0$. Our results below strongly support a relation with the R sector if N=1 Liouville field theory.

After these comments on the Racah-Wigner symbol (3.1), we would like to repeat the analysis we have performed in section 2. By analogy with the bosonic case we expect that the prefactor vanishes each time one of the external weights approaches a degenerate value $\alpha_{n,n'} = -\frac{nb}{2} - \frac{n'}{2b}$. Weights $\alpha_{n,n'}$ with even $n + n'$ are degenerate in the NS sector while those with $n + n'$ odd degenerate in the R sector of the theory. Suppose now that the external weight α_i degenerates. Fusion with a generic weight α_j gives the following finite set of intermediate weights

$$\alpha_s \to \alpha_j - \frac{sb}{2} - \frac{s'}{2b}\,, \qquad \text{or} \qquad \alpha_t \to \alpha_j - \frac{tb}{2} - \frac{t'}{2b}\,, \tag{3.4}$$
$$s, t \in \{-n, -n+2, \ldots, n\}\,, \qquad s', t' \in \{-n', -n'+2, \ldots, n'\}\,.$$

Let us now take a closer look at the prefactor of our Racah-Wigner symbols. When written in terms of the double sine function, it takes the from

$$\begin{aligned}
&\Delta_{\nu_4}(\alpha_s,\alpha_2,\alpha_1)\Delta_{\nu_3}(\alpha_s,\alpha_3,\alpha_4)\Delta_{\nu_2}(\alpha_t,\alpha_3,\alpha_2)\Delta_{\nu_1}(\alpha_4,\alpha_t,\alpha_1) \\
&= (S_{\nu_4+a_s}(\alpha_{12s}-Q)S_{\nu_3+a_s}(\alpha_{s34}-Q)S_{\nu_2+a_t}(\alpha_{23t}-Q)S_{\nu_1+a_t}(\alpha_{14t}-Q))^{\frac{1}{2}} \\
&\qquad \Big(S_{\nu_4}(\alpha_{12}-\alpha_s)\,S_{\nu_4+a_1}(\alpha_{1s}-\alpha_2)\,S_{\nu_4+a_2}(\alpha_{2s}-\alpha_1) \\
&\qquad S_{\nu_3}(\alpha_{34}-\alpha_s)\,S_{\nu_3+a_4}(\alpha_{s4}-\alpha_3)\,S_{\nu_3+a_3}(\alpha_{3s}-\alpha_4) \\
&\qquad S_{\nu_2}(\alpha_{23}-\alpha_t)\,S_{\nu_2+a_2}(\alpha_{t2}-\alpha_3)\,S_{\nu_2+a_3}(\alpha_{3t}-\alpha_2) \\
&\qquad S_{\nu_1}(\alpha_{14}-\alpha_t)\,S_{\nu_1+a_1}(\alpha_{1t}-\alpha_4)\,S_{\nu_1+a_4}(\alpha_{4t}-\alpha_1)\Big)^{-\frac{1}{2}}
\end{aligned} \tag{3.5}$$

This factor vanishes every time we hit a pole of the denominator. As one can easily see, there are four different cases in which this can happen, each corresponding to the consecutive products of three double sine functions,

$$\begin{aligned}
&\frac{n-s}{2}+\frac{n'-s'}{2}\in 2\mathbb{N}+1+\begin{cases}\nu_4, & \text{degenerate}\quad \alpha_i,\quad i=1,2\\ \nu_3, & \text{degenerate}\quad \alpha_i,\quad i=3,4\end{cases}\\
&\frac{n+s}{2}+\frac{n'+s'}{2}\in 2\mathbb{N}+1+\begin{cases}\nu_4+a_i, & \text{degenerate}\quad \alpha_i,\quad i=1,2\\ \nu_3+a_i, & \text{degenerate}\quad \alpha_i,\quad i=3,4\end{cases}
\end{aligned} \tag{3.6}$$

As one example, let us discuss the first line and suppose that $\alpha_i=\alpha_1$ for definiteness. It follows that $\alpha_j=\alpha_2$ because α_1 and α_s appear only in combination with α_2 in the arguments of the double sine functions. According to the properties listed in Appendix A the first double sine function $S_{\nu_4}(\alpha_{12}-\alpha_s)$ runs into a pole provided that its argument $\alpha_{12}-\alpha_s=\frac{s-n}{2}b+\frac{s'-n'}{2}b^{-1}$ satisfies $\frac{n-s}{2}+\frac{n'-s'}{2}\in 2\mathbb{N}+1+\nu_4$. This gives the first line above. The analysis for the other cases is similar.

The analysis for the t-channel, i.e. for terms involving α_t, is performed in exactly the same way and it leads to

$$\begin{aligned}
&\frac{n-t}{2}+\frac{n'-t'}{2}\in 2\mathbb{N}+1+\begin{cases}\nu_1, & \text{degenerate}\quad \alpha_i,\quad i=1,4\\ \nu_2, & \text{degenerate}\quad \alpha_i,\quad i=2,3\end{cases}\\
&\frac{n+t}{2}+\frac{n'+t'}{2}\in 2\mathbb{N}+1+\begin{cases}\nu_1+a_i, & \text{degenerate}\quad \alpha_i,\quad i=1,4\\ \nu_2+a_i, & \text{degenerate}\quad \alpha_i,\quad i=2,3\end{cases}
\end{aligned} \tag{3.7}$$

This implies a relation between the sets of parameters $\{\nu_i\}$, $\{a_i\}$ $(i=1,\dots,4)$ and the type of fusion rules satisfied by α_s,α_t in the limit of a degenerate weight. We see that the prefactor has the fusion rules of $N=1$ Liouville field theory built in. This provides a first non-trivial test for our proposal.

Let us anticipate that the numerator of the normalization factor furnishes residues that can be related to $S_\nu(0)$ by shift equations, see Appendix A. All four double sine functions satisfy this condition simultaneously, provided that the external weights obey

$$j_{1234} + j'_{1234} \in 2\mathbb{N} + \nu_3 + \nu_4 + a_s, \quad \text{and} \quad j_{1234} + j'_{1234} \in 2\mathbb{N} + \nu_1 + \nu_2 + a_t \tag{3.8}$$

at the same time. This is guaranteed by the Kronecker δ we have put into our definition of the Racah-Wigner symbol (3.1).

We plan to test our proposal (3.1) by evaluating it for degenerate weights, as in the previous section. To this end, let us consider the limit where $\alpha_2 \to -\frac{nb}{2}$. Before talking the limit of the Racah-Wigner symbol it is useful to pass from the summation over ν to a new summation index $\nu' = \nu + \nu_3 + a_s$. The Racah-Wigner symbol then reads,

$$\left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = a_s + a_t \, mod\, 2} \Delta_{\nu_4}(\alpha_s, \alpha_2, \alpha_1) \Delta_{\nu_3}(\alpha_s, \alpha_3, \alpha_4) \Delta_{\nu_2}(\alpha_t, \alpha_3, \alpha_2) \tag{3.9}$$

$$\Delta_{\nu_1}(\alpha_4, \alpha_t, \alpha_1) \int_{\mathcal{C}} du \sum_{\nu'=0}^{1} \Big((-1)^X \, S_{1+\nu_3+\nu_4+\nu'}(u - \alpha_{12s}) \, S_{1+\nu'}(u - \alpha_{s34})$$
$$S_{1+\nu_1+\nu_4+\nu'}(u - \alpha_{23t}) S_{1+\nu_2+\nu_4+\nu'}(u - \alpha_{1t4}) \, S_{\nu_4+\nu'}(\alpha_{1234} - u) \, S_{\nu_1+\nu'+a_1}(\alpha_{st13} - u)$$
$$S_{\nu_2+\nu'+a_2}(\alpha_{st24} - u) \, S_{\nu_3+\nu'+a_s}(2Q - u) \Big) \, .$$

As in the previous section, we need to determine the singular contributions from the integral. Note that the product

$$S_{1+\nu'}(u - \alpha_{s34}) S_{\nu_4+\nu'}(\alpha_{1234} - u)$$

has poles in the positions $u = \alpha_{134} - \frac{sb}{2} - pb$ for $p \in \{\nu', \nu'+2, \ldots, \leq \frac{n-s}{2} - \nu'\}$ (ν' keeps track of the parity of p). Due to the "pinching mechanism" each pole contributes a sum of singular terms. Once we include the summation over $\nu' = 0, 1$, the sum of singular terms runs through all values of $p \in \{0, 1, \ldots, \frac{n-s}{2}\}$,

$$\sum_{p=0}^{\frac{n-s}{2}} (-1)^X \frac{\left(2\cos(\frac{\pi b^2}{2})\right)^{\frac{s-n}{2}} S_{1+p}(0)}{[p]_+! \, \left[\frac{n-s}{2} - p\right]_+!} \, S_{1+\nu_3+\nu_4+\nu'}(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb) \tag{3.10}$$
$$S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb) S_{1+\nu_2+\nu_4+\nu'}(\alpha_3 - \alpha_t - \frac{sb}{2} - pb)$$
$$S_{\nu_1+\nu'+a_1}(\alpha_{1t} - \alpha_4 + pb) S_{\nu_2+\nu'+a_2}(\alpha_t - \alpha_3 - \frac{nb}{2} + pb) S_{\nu_3+\nu'+a_s}(2Q - \alpha_{134} + \frac{sb}{2} + pb),$$

where we used the shift relations for the supersymmetric double sine function (A.7) and the notation

$$[n]_+! = \begin{cases} \prod_{j=1\,mod\,2}^{n-1} \cos(j\frac{\pi b^2}{2}) \prod_{j=2\,mod\,2}^{n} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, \text{ for } n \in 2\mathbb{N} \\ \prod_{j=1\,mod\,2}^{n} \cos(j\frac{\pi b^2}{2}) \prod_{j=2\,mod\,2}^{n-1} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, \text{ for } n \in 2\mathbb{N}+1 \ . \end{cases} \tag{3.11}$$

With the help of conditions (3.6) one can verify that the product

$$S_{1+\nu_2+\nu_4+\nu'}(u-\alpha_{1t4})S_{\nu_2+\nu'+a_2}(\alpha_{st24}-u)$$

has common poles which are located in $u = \alpha_{1t4} - p'b$, where $p' \in \{\nu', \nu'+2, \ldots, \leq \frac{n+s}{2} - \nu'\}$. They lead to a second sum of singular terms. Once the two singular contributions from the integral are multiplied by the vanishing prefactor, they give a finite result for the limit of the Racah-Wigner symbol,

$$\left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & (\alpha_1 - \frac{sb}{2})^{a_s} \\ (-\frac{nb}{2})^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} \equiv \lim_{\substack{\alpha_2 \to -\frac{nb}{2} \\ \alpha_s \to \alpha_1 - \frac{sb}{2}}} \left\{ \begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & \alpha_s^{a_s} \\ \alpha_2^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t} \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = a_s + a_t\, mod\, 2}$$

$$\left(\frac{\left[\frac{n-s}{2}\right]_+! \left[\frac{n+s}{2}\right]_+! \, S_{\nu_4+a_s}(2\alpha_1 - \frac{(s+n)b}{2} - Q) S_{\nu_3+a_s}(2\alpha_{134} - \frac{sb}{2} - Q)}{S_{\nu_4+a_1}(2\alpha_1 + \frac{(n-s)b}{2}) S_{\nu_3}(\alpha_{34} - \alpha_1 + \frac{sb}{2}) \, S_{\nu_3+a_3}(\alpha_{13} - \alpha_4 - \frac{sb}{2}) S_{\nu_3+a_4}(\alpha_{14} - \alpha_3 - \frac{sb}{2})} \right)^{\frac{1}{2}}$$

$$\left(\frac{S_{\nu_2+a_t}(\alpha_3 + \alpha_t - \frac{nb}{2} - Q) S_{\nu_1+a_t}(\alpha_{14} + \alpha_t - Q) \left(S_{\nu_1+a_4}(\alpha_{4t} - \alpha_1) \, S_{\nu_2+a_3}(\alpha_{3t} + \frac{nb}{2}) \right)^{-1}}{S_{\nu_2}(\alpha_3 - \alpha_t - \frac{nb}{2}) S_{\nu_2+a_2}(\alpha_t - \alpha_3 - \frac{nb}{2}) S_{\nu_1}(\alpha_{14} - \alpha_t) \, S_{\nu_1+a_1}(\alpha_{1t} - \alpha_4)} \right)^{\frac{1}{2}}$$

$$\Bigg\{ \sum_{p=0}^{\frac{n-s}{2}} (-1)^X \frac{\left(2\cos(\frac{\pi b^2}{2})\right)^{\frac{s}{2}}}{[p]_+! \left[\frac{n-s}{2} - p\right]_+!} S_{1+\nu_3+\nu_4+\nu'}(\alpha_{34} - \alpha_1 + \frac{nb}{2} - pb) \tag{3.12}$$

$$S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_t + \frac{(n-s)b}{2} - pb) S_{1+\nu_2+\nu_4+\nu'}(\alpha_3 - \alpha_t - \frac{sb}{2} - pb)$$

$$S_{\nu_1+\nu'+a_1}(\alpha_{1t} - \alpha_4 + pb) S_{\nu_2+\nu'+a_2}(\alpha_t - \alpha_3 - \frac{nb}{2} + pb) \, S_{\nu_3+\nu'+a_s}(2Q - \alpha_{134} + \frac{sb}{2} + pb)$$

$$+ \sum_{p'=0}^{\frac{n+s}{2}} (-1)^X \frac{\left(2\cos(\frac{\pi b^2}{2})\right)^{-\frac{s}{2}}}{[p']_+! \left[\frac{n+s}{2} - p'\right]_+!} S_{\nu_4+\nu'}(\alpha_3 - \alpha_t - \frac{nb}{2} + p'b) S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14} - \alpha_3 + \frac{nb}{2} - p'b)$$

$$\frac{S_{1+\nu'}(\alpha_t - \alpha_3 + \frac{sb}{2} - p'b) S_{1+\nu_3+\nu_4+\nu'}(\alpha_{t4} - \alpha_1 + \frac{(n+s)b}{2} - p'b) S_{\nu_1+\nu'+a_1}(\alpha_{13} - \alpha_4 + p'b - \frac{sb}{2})}{S_{\nu_3+\nu'+a_s}(\alpha_{1t4} - p'b - Q)} \Bigg\}.$$

This expression has simple poles when the second intermediate weight approach a degenerate value $\alpha_t \to \alpha_3 - \frac{tb}{2}$. The residua are given by the following formula,

$$\underset{\alpha_t\to\alpha_3-\frac{tb}{2}}{Res}\left\{\begin{array}{cc|c} \alpha_1^{a_1} & \alpha_3^{a_3} & \left(\alpha_1-\frac{sb}{2}\right)^{a_s} \\ \left(-\frac{nb}{2}\right)^{a_2} & \alpha_4^{a_4} & \alpha_t^{a_t}\end{array}\right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = a_s+a_t \, mod\, 2} \tag{3.13}$$

$$2\left(\frac{S_{\nu_4+a_s}(2\alpha_1-\frac{(s+n)b}{2}-Q)S_{\nu_2+a_t}(2\alpha_3-\frac{(t+n)b}{2}-Q)}{S_{\nu_4+a_1}(2\alpha_1+\frac{(n-s)b}{2})\,S_{\nu_2+a_3}(2\alpha_3+\frac{(n-t)b}{2})}\right)^{\frac{1}{2}}$$

$$\sum_{p=max\{0,\frac{t-s}{2}\}}^{min\{\frac{n-s}{2},\frac{n+t}{2}\}}\left\{(-1)^X\,\frac{\left(S_{\nu_3+a_s}(\alpha_{134}-\frac{sb}{2}-Q)S_{\nu_1+a_t}(\alpha_{134}-\frac{tb}{2}-Q)\right)^{\frac{1}{2}}}{S_{\nu_3+\nu'+a_s}(\alpha_{134}-\frac{sb}{2}-pb-Q)}\right.$$

$$\frac{\left(\left[\frac{n-s}{2}\right]_+!\left[\frac{n+s}{2}\right]_+!\left[\frac{n-t}{2}\right]_+!\left[\frac{n+t}{2}\right]_+!\right)^{\frac{1}{2}}}{[p]_+!\left[\frac{n-s}{2}-p\right]_+!\left[p+\frac{s-t}{2}\right]_+!\left[\frac{t+n}{2}-p\right]_+!}\,\frac{S_{\nu_1+\nu'+a_1}(\alpha_{13}-\alpha_4+pb-\frac{tb}{2})}{\left(S_{\nu_3+a_3}(\alpha_{13}-\alpha_4-\frac{sb}{2})S_{\nu_1+a_1}(\alpha_{13}-\alpha_4-\frac{tb}{2})\right)^{\frac{1}{2}}}$$

$$\left.\frac{S_{1+\nu_3+\nu_4+\nu'}(\alpha_{34}-\alpha_1-pb+\frac{nb}{2})}{\left(S_{\nu_3}(\alpha_{34}-\alpha_1+\frac{sb}{2})\,S_{\nu_1+a_4}(\alpha_{34}-\alpha_1-\frac{tb}{2})\right)^{\frac{1}{2}}}\,\frac{S_{1+\nu_1+\nu_4+\nu'}(\alpha_{14}-\alpha_3-pb+\frac{(n+t-s)b}{2})}{\left(S_{\nu_3+a_4}(\alpha_{14}-\alpha_3-\frac{sb}{2})\,S_{\nu_1}(\alpha_{14}-\alpha_3+\frac{tb}{2})\right)^{\frac{1}{2}}}\right\}.$$

Now we can send all the external weights to degenerate values,

$$\alpha_i \to -j_i b, \qquad j_i \in \mathbb{Z}_{\geq 0} + \frac{a_i}{2}.$$

In complete analogy to the bosonic case, see eq. (2.11), we shall denote the limit by

$$\left\{\begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b\end{array}\right\}_{\nu_1\nu_2}^{\nu_3\nu_4},$$

remembering that it denotes a residue of the Racah-Wigner symbol (3.1) with one degenerate external weight and both intermediate weights satisfying fusion rules. Assuming that

$$\frac{n}{2} = j_2, \qquad \frac{s}{2} = j_s - j_1, \qquad \frac{t}{2} = j_t - j_3$$

satisfy the conditions (3.6) to (3.8), one can use the shift relations (A.7) for $S_\nu(x)$ to obtain

$$\left\{\begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b\end{array}\right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\, mod\, 2}\,\frac{(-1)^{A(j_i)}}{2\cos\left(\frac{\pi b^2}{2}\right)\cos\left(\frac{\pi}{2b^2}\right)} \tag{3.14}$$

$$\Delta_+(j_s,j_2,j_1)\Delta_+(j_s,j_3,j_4)\Delta_+(j_t,j_3,j_2)\Delta_+(j_4,j_t,j_1)\sum_{z\geq 0}\left\{(-1)^X(-1)^{\frac{1}{2}z(z-1)}[z+1]_+!\right.$$

$$\left.\left([z-j_{12s}]_+!\,[z-j_{34s}]_+!\,[z-j_{14t}]_+![z-j_{23t}]_+![j_{1234}-z]_+!\,[j_{13st}-z]_+!\,[j_{24st}-z]_+!\right)^{-1}\right\}$$

where the sum is over $z = p + j_{s34}$ such that all arguments $[.]_+$ are non-negative, and

$$\Delta_+(a,b,c) = \sqrt{[-a+b+c]_+!\,[a-b+c]_+!\,[a+b-c]_+!/[a+b+c+1]_+!}.$$

The powers of (-1) come from the relation (A.7) applied to the terms $S\nu(-xb-Q)$, in particular:

$$A(j_i) \;=\; \frac{1}{4}j_{12s}(j_{12s}-1)+\frac{1}{4}j_{s34}(j_{s34}-1)+\frac{1}{4}j_{23t}(j_{23t}-1)+\frac{1}{4}j_{14t}(j_{14t}-1)+1\;.$$

This concludes our computation of the Racah-Wigner symbol (3.1) for degenerate labels. The final formula looks somewhat similar to the corresponding equation in section 2. We are now going to see that it indeed vert closely related.

4 Comparison with the finite dimensional 6J symbols

Our formula (3.14) for the limiting value of the proposed Racah-Wigner symbol could turn into a strong test of eq. (3.1) provided we were able to show that the expression (3.14) gives rise to a solution of the pentagon equation. In our discussion of the Racah-Wigner symbol for $\mathrm{U}_q(\mathrm{sl}(2))$ this followed from the comparison with the 6J symbol for finite dimensional representations. By construction, the latter are known to satisfy the pentagon equation. By analogy one might now hope that the coefficients (3.14) coincide with the 6J symbol for finite dimensional representations of the quantum universal enveloping algebra $\mathrm{U}_q(\mathrm{osp}(1|2))$. This, however, is not quite the case. To start the comparison, we quote an expression for the 6J symbols of $\mathrm{U}_q(\mathrm{osp}(1|2))$ from [9, 10],

$$\begin{aligned}\begin{bmatrix} l_1 & l_2 & l_s \\ l_3 & l_4 & l_t \end{bmatrix}_q \;=\;& (-1)^{\frac{1}{2}(l_{1234}+l_s+l_t)(l_{1234}+l_s+l_t+1)+\frac{1}{2}\left(\sum_{i=1}^4 l_i(l_i-1)+l_s(l_s-1)+l_t(l_t-1)\right)} \\ & \Delta'_q(l_s,l_2,l_1)\Delta'_q(l_s,l_3,l_4)\Delta'_q(l_t,l_3,l_2)\Delta'_q(l_4,l_t,l_1) \\ & \sum_{z\geq 0}(-1)^{\frac{1}{2}z(z-1)}[z+1]'_q!\Big([z-l_{12s}]'_q!\,[z-l_{34s}]'_q!\,[z-l_{14t}]'_q! \\ & \quad [z-l_{23t}]'_q![l_{1234}-z]'_q!\,[l_{13st}-z]'_q!\,[l_{24st}-z]'_q!\Big)^{-1}\end{aligned}$$

where

$$\Delta'_q(a,b,c)=\sqrt{[-a+b+c]'_q!\,[a-b+c]'_q!\,[a+b-c]'_q!/[a+b+c+1]'_q!}.$$

Let us stress that irreducible finite dimensional representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ are labeled by integers l. Hence all the arguments l_i in the above 6J symbols satisfy $l_i\in\mathbb{N}$. In the previous definition the q-number $[.]'_q$ is defined as

$$[n]'_q=\frac{q^{-\frac{n}{2}}-(-1)^n q^{\frac{n}{2}}}{q^{-\frac{1}{2}}+q^{\frac{1}{2}}}\;. \tag{4.1}$$

For $q = e^{i\pi b^2}$ the quantum factorial takes the form

$$[n]'_q! = \begin{cases} \prod_{j=1\,mod\,2}^{n-1} \cos(j\frac{\pi b^2}{2}) \prod_{j=2\,mod\,2}^{n} \left(i\,\sin(-j\frac{\pi b^2}{2})\right) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1\,mod\,2}^{n} \cos(j\frac{\pi b^2}{2}) \prod_{j=2\,mod\,2}^{n-1} \left(i\,\sin(-j\frac{\pi b^2}{2})\right) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N}+1\ . \end{cases} \tag{4.2}$$

It is related to the similar symbol $[.]_+!$ which we defined in eq. (3.11) through

$$[n]_+! = (-1)^{\frac{1}{12}n(n+1)(2n+1)}(-i)^n\ [n]'_q!\ . \tag{4.3}$$

In order to compare the limiting values (3.14) Racah-Wigner symbols (3.1) with the 6J symbols (4.1) we rewrite the latter in terms of the new symbol $[n]'_q$,

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\,mod\,2}\ \frac{(-1)^{A'(j_i)}\Delta'_q(j_s,j_2,j_1)\Delta'_q(j_s,j_3,j_4)}{2\cos(\frac{\pi b^2}{2})\cos(\frac{\pi}{2b^2})}$$
$$\Delta'_q(j_t,j_3,j_2)\Delta'_q(j_4,j_t,j_1)\sum_{z\geq 0}(-1)^X(-1)^{\frac{1}{2}z(z-1)+2z(j_{1234st}+j_1j_3+j_2j_4+j_sj_t)}[z+1]'_q! \tag{4.4}$$
$$\left([z-j_{12s}]'_q!\,[z-j_{34s}]'_q![z-j_{14t}]'_q!\,[z-j_{23t}]'_q![j_{1234}-z]'_q!\,[j_{13st}-z]'_q!\,[j_{24st}-z]'_q!\right)^{-1}$$

where

$$\begin{aligned} A'(j_i) = & -\frac{1}{2} - (j_{1234st}+1)(j_1j_3+j_2j_4+j_sj_t+1) \\ & +\frac{1}{2}j_{12s}(j_{12s}-1) + \frac{1}{2}j_{s34}(j_{s34}-1) + \frac{1}{2}j_{23t}(j_{23t}-1) + \frac{1}{2}j_{14t}(j_{14t}-1) \\ & -F(j_1,j_2,j_s) - F(j_3,j_4,j_s) - F(j_2,j_3,j_t) - F(j_1,j_4,j_t) \end{aligned}$$

and

$$F(j_1,j_2,j_3) = \frac{3}{4}j_{123}(j_{123}+1) + j_1j_2j_3 + j_1j_2 + j_1j_3 + j_2j_3\ .$$

For integer j_i the signs $(-1)^{2z(j_{1234st}+j_1j_3+j_2j_4+j_sj_t)}$ and $(-1)^X$ which were defined in eq. (3.3) vanish so that we can relate the limit of the Racah-Wigner symbol to the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J coefficients,

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}_{\nu_1\nu_2}^{\nu_3\nu_4} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\,mod\,2}\ \frac{(-1)^{A''(j_i)}}{2\cos(\frac{\pi b^2}{2})\cos(\frac{\pi}{2b^2})} \left[\begin{array}{ccc} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{array} \right]_q \tag{4.5}$$

where

$$\begin{aligned} A''(j_i) = & \frac{1}{2} - j_{1234st}(j_1j_3+j_2j_4+j_sj_t) \\ & -F(j_1,j_2,j_s) - F(j_3,j_4,j_s) - F(j_2,j_3,j_t) - F(j_1,j_4,j_t)\ . \end{aligned}$$

Let us emphasize that in arriving at the expressions (3.14) for the limiting values of the Racah-Wigner symbol, the parameters j_i were allowed to take either integer (NS weights) and half-integer (R weights) values. We have now shown that the limit is proportional to the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J coefficients, provided all arguments j_i are integer. In order to find an interpretation of the limit (3.14) in the case of half-integer j_i, we will have to bring in a different idea. It is related to an intriguing duality between the 6J symbol of $\mathrm{U}_q(\mathrm{osp}(1|2))$ and $\mathrm{U}_q(\mathrm{sl}(2))$.

As was originally noticed in [11], [12], the $\mathrm{U}_q(\mathrm{sl}(2))$ quantum numbers (2.13) with the deformation parameter $q' = i\sqrt{q}$ are related to the $\mathrm{U}_q(\mathrm{osp}(1|2))$ quantum numbers (4.1) through,

$$[x]_{q'} = (-1)^{\frac{1-x}{2}} [x]'_q \ . \tag{4.6}$$

This equation implies a relation between the quantum factorials,

$$[x]'_q! = (-1)^{\frac{x(x-1)}{4}} [x]_{q'}! \ . \tag{4.7}$$

With its help we can rewrite the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J symbol in terms of the $\mathrm{U}_q(\mathrm{sl}(2))$ quantum factorials,

$$\begin{aligned}
\begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q \ = \ & (-1)^{\sum_{i=1}^4 \frac{j_i}{2}(j_i-1)+\frac{j_s}{2}(j_s-1)+\frac{j_t}{2}(j_t-1)-\frac{1}{2}j_{st}j_{1234}-\frac{1}{2}j_{13}j_{24}} \\
& \Delta_{q'}(j_s,j_2,j_1)\Delta_{q'}(j_s,j_3,j_4)\Delta_{q'}(j_t,j_3,j_2)\Delta_{q'}(j_4,j_t,j_1) \\
& \sum_{z\geq 0}(-1)^{z+2zj_{1234st}}[z+1]_{q'}!\Big([z-j_{12s}]_{q'}!\,[z-j_{34s}]_{q'}!\,[z-j_{14t}]_{q'}! \\
& \quad [z-j_{23t}]_{q'}![j_{1234}-z]_{q'}!\,[j_{13st}-z]_{q'}!\,[j_{24st}-z]_{q'}!\Big)^{-1} \ .
\end{aligned}$$

Due to the condition $j_i \in \mathbb{N}$ in the $\mathrm{U}_q(\mathrm{osp}(1|2))$ 6J symbol, the sign $(-1)^{2zj_{1234st}}$ vanishes and one arrives at the following relation between the 6J symbols (2.14) and (4.1)

$$\begin{aligned}
\begin{bmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{bmatrix}_q \ = \ & (-1)^{\sum_{i=1}^4 \frac{j_i}{2}(j_i-1)+\frac{j_s}{2}(j_s-1)+\frac{j_t}{2}(j_t-1)-\frac{1}{2}j_{st}j_{1234}-\frac{1}{2}j_{13}j_{24}} \\
& \frac{(-1)^{-j_{12}+j_{34}+2j_s}}{\sqrt{[2j_s+1]_{q'}[2j_t+1]_{q'}}} \begin{pmatrix} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{pmatrix}_{q'} \ .
\end{aligned}$$

In a similar way we can relate our limit of Racah-Wigner coefficients (4.4) to the 6J symbol of $\mathrm{U}_{q'}(\mathrm{sl}(2))$ even if some of the arguments j_i assume (half-)integer values. When written in terms of $[x]_{q'}$, the Racah-Wigner coefficients (4.4) take the following form,

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}^{\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = 2(j_s+j_t)\, mod\, 2} \frac{(-1)^{A'''(j_i)} \Delta_{q'}(j_s, j_2, j_1) \Delta_{q'}(j_s, j_3, j_4)}{2\cos(\frac{\pi b^2}{2})\cos(\frac{\pi}{2b^2})}$$
$$\Delta_{q'}(j_t, j_3, j_2)\Delta_{q'}(j_4, j_t, j_1) \sum_{z\geq 0} (-1)^X (-1)^{z+2(z+1)(j_1j_3+j_2j_4+j_sj_t)} [z+1]_{q'}! \tag{4.8}$$
$$\Big([z-j_{12s}]_{q'}!\, [z-j_{34s}]_{q'}! [z-j_{14t}]_{q'}!\, [z-j_{23t}]_{q'}! [j_{1234}-z]_{q'}!\, [j_{13st}-z]_{q'}!\, [j_{24st}-z]_{q'}! \Big)^{-1}$$

where

$$\begin{aligned} A'''(j_i) &= \frac{1}{2} - (j_{1234st}+2)(j_1j_3+j_2j_4+j_sj_t) \\ &\quad -F'(j_1,j_2,j_s) - F'(j_3,j_4,j_s) - F'(j_2,j_3,j_t) - F'(j_1,j_4,j_t) \\ F'(j_1,j_2,j_3) &= j_1j_2j_3 + \frac{1}{2}(j_1+j_2+j_3)\ . \end{aligned}$$

Using the relations (3.6, 3.7) and (3.8) one may check that

$$(-1)^{2j_1j_3+2j_2j_4+2j_sj_t} = (-1)^{a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t}\ . \tag{4.9}$$

Since the parameter z is related to the summation parameter p (3.10) as $z = p + j_{34s}$ and the parity of p is tracked by $\nu' = \nu + \nu_3 + a_s$, we may relate the sign under the sum in eq. (4.8) to the sign factor $(-1)^X$ that was defined in eq. (3.3),

$$\begin{aligned} (-1)^{2(z+1)(j_1j_3+j_2j_4+j_sj_t)} &= (-1)^{2(\nu+\nu_3+a_s+j_{34s}+1)(j_1j_3+j_2j_4+j_sj_t)} \\ &= (-1)^{\nu(a_s\nu_1+a_1\nu_3+a_4\nu_4+a_1a_s+a_2a_4+a_s+a_t)} = (-1)^X, \end{aligned} \tag{4.10}$$

where we used (3.8) to check that $\nu+\nu_3+a_s+j_{34s}+1 \in 2\mathbb{N} + 2(\nu+\nu_3+\nu_4+a_s)+\nu$. Thus the limit (4.8) is proportional to the 6J symbol of finite dimensional representations of $\mathrm{U}_{q'}(\mathrm{sl}(2))$,

$$\left\{ \begin{array}{cc|c} -j_1 b & -j_3 b & -j_s b \\ -j_2 b & -j_4 b & -j_t b \end{array} \right\}^{\nu_3\nu_4}_{\nu_1\nu_2} = \delta_{\sum_i \nu_i = 2j_s+2j_t\, mod\, 2} \frac{(-1)^{A'''(j_i)}}{2\cos(\frac{\pi b^2}{2})\cos(\frac{\pi}{2b^2})} \frac{(-1)^{-j_{12}+j_{34}+2j_s}}{\sqrt{[2j_s+1]_{q'}[2j_t+1]_{q'}}} \left(\begin{array}{ccc} j_1 & j_2 & j_s \\ j_3 & j_4 & j_t \end{array} \right)_{q'}\ . \tag{4.11}$$

This concludes our discussion of the limiting Racah-Wigner coefficients (3.14). Our analysis has shown that the expression we obtained from our proposal (3.1) is dual to the 6J symbol for finite dimensional representations of the quantum universal enveloping algebra $\mathrm{U}_q(\mathrm{sl}(2))$. By construction the latter satisfy the pentagon equation. Even though we have not demonstrated that the original symbol (3.1) solved the pentagon identity for arbitrary values of the weights

α, our results provide highly non-trivial evidence in favor of the proposal. Note in particular that our sign factors were rather crucial in making things work as soon as some of the weights were taken from the R sector. It is actually possible to carry things a bit further. In fact, the evaluation of the Racah-Wigner symbols (2.1) and (3.1) is possible for all degenerate weights, not just the one parameter series we have studied above. In that case, the limiting values of the Racah-Wigner symbol are no longer given by a single 6J symbol. On the other hand the coefficients obtained from the the symbol (2.14) are guaranteed to satisfy the pentagon relations, simply because the full symbol does [4, 5, 6]. We have checked in a few examples that the limiting values of the proposed Racah-Wigner symbol (3.1) are still related to those of the $\mathrm{U}_q(\mathrm{sl}(2))$ symbol even when $\alpha \sim -\frac{nb}{2} - \frac{n'}{2b}$ for $n' \neq 0$. With all these non-trivial test being performed, we trust that our formula (3.1) correctly describes the fusing matrix of $N = 1$ Liouville field theory for both NS and R sector fields.

5 Conclusions

In this work we proposed a formula (3.1) for the Racah-Wigner symbol of the non-compact quantum universal enveloping algebra $\mathrm{U}_q(\mathrm{osp}(1|2))$. In order to test our proposal we continued the symbol to a discrete set of momenta $j = -\alpha/b \in \mathbb{N}/2$. For integer $j \in \mathbb{N}$ we recovered the known expressions for Racah-Wigner coefficients of finite dimensional $\mathrm{U}_q(\mathrm{osp}(1|2))$ representations. Half integer values j are not related to the 6J symbols of $\mathrm{U}_q(\mathrm{osp}(1|2))$ but rather to those of $\mathrm{U}_q(\mathrm{sl}(2))$. The relation is furnished by a duality which extends the known correspondence between finite dimensional representations of $\mathrm{U}_q(\mathrm{osp}(1|2))$ and integer spin representations of $\mathrm{U}_q(\mathrm{sl}(2))$to the case of half-integer spins. A related extension was also uncovered by Mikhaylov and Witten [15]. There are a number of interesting open issues that merit further investigation.

As we stressed before, the Racah-Wigner symbol (3.1) should coincide with the complete fusing matrix of N=1 Liouville field theory in both the NS and the R sector [2, 3],[16]. For NS sector representations a related statement was established in [1]. Of course, it would be interesting to incorporate R sector representations into this comparison. Given such a reinterpretation of the Racah-Wigner symbol as a fusing matrix in N=1 Liouville theory, our expression (3.1), and special cases thereof, should then also describe various operator product coefficients in the bulk and boundary theory, and in particular the coefficients of boundary operator product expansion, see e.g. [17] for a review of the relation.

Recently, it has been observed that the operator product coefficients of N=1 Liouville field theory with central charge $c = 15/2 + 3(b^2 + b^{-2})$ can be factorized into a products of the coefficients in ordinary (non-supersymmetric) Liouville field theory and those of an imaginary (time-like) version thereof [18, 19, 20]. The central charges of the latter are given by $c_i = 13 + 6(b_i^2 + b_i^{-2})$ for $i = 1, 2$ with

$$b_1^2 = \frac{1}{2}(b^2 - 1) \quad , \quad b_2^2 = 2(b^{-2} - 1)^{-1} = -b_1^{-2} - 2 \ .$$

This suggest a relation between Racah-Wigner symbols of $\mathrm{U}_q(\mathrm{osp}(1|2))$ for $q = \exp i\pi b^2$ and those of $\mathrm{U}_{q_i}(\mathrm{sl}(2))$ for the two values $q_1 = \exp(i\pi b_1^2) = \sqrt{-q}$ and $q_2 = \tilde{q}_1$. Note that the latter is obtained from the former by modular transformation. The relation we have just argued for resembles the relation (4.11) between the Racah-Wigner symbols for finite dimensional representations. Indeed, the deformation parameters q and q_1 take the same values as in eq. (4.11). On the other hand, we saw no sign of the additional factor with the modular transformed deformation parameter $q_2 = \tilde{q}_1$. A reasonable explanation could be that the second factor is removed through the limiting process. If so, it should be possible the detect it before we specialize to discrete spin variables. We plan to investigate the extension of the duality between $\mathrm{U}_q(\mathrm{osp}(1|2))$ and $\mathrm{U}_q(\mathrm{sl}(2))$ to the continuous self-dual series of representations in future work. It should also be linked with a strong-weak coupling duality between the non-compact OSP(2|1)/U(1) cigar-like coset model and double Liouville theory that was described in [21].

As we recalled in the introduction, the fusing matrix of $N = 1$ Liouville field theory should be a central ingredient in the construction of a new 3-dimensional topological quantum field theory, just as Faddeev's quantum dilogarithm [22, 23], i.e. the building block of the fusing matrix on Liouville field theory, is used to construct SL(2) Chern-Simons or quantum Teichmueller theory, see e.g. [24, 25, 26, 27, 28, 29, 30]. We will explore these aspects of our work in a future publication.

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A Double sine functions

The double sine function $S_b(x)$ is given in terms of Barnes' double Gamma function through

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q - x)} \tag{A.1}$$

and has poles in positions x such that

$$S_b(x)^{-1} = 0 \qquad \Longleftrightarrow \qquad x = -nb - mb^{-1}\,, \qquad n, m \in \mathbb{Z}_{\geq 0}\,. \tag{A.2}$$

It satisfies the shift relations

$$S_b(x + b^{\pm 1}) = 2\sin(\pi b^{\pm 1} x)\, S_b(x) \tag{A.3}$$

which imply that one can evaluate

$$\begin{aligned} S_b(-kb) &= \prod_{j=1}^{k} \left(2\sin(-\pi j b^2)\right)^{-1}\, S_b(0) = \left(-2\sin\left(\pi b^2\right)\right)^{-k}\, \frac{S_b(0)}{[k]!} \\ S_b(-kb - Q) &= \left(2\sin\left(\pi b^2\right)\right)^{-k-1} \left(2\sin\left(-\pi b^{-2}\right)\right)^{-1}\, \frac{S_b(0)}{[k+1]!} \end{aligned} \tag{A.4}$$

for $k \in \mathbb{N}$. We have also used the q-number $[x] = \frac{\sin(\pi b^2 x)}{\sin \pi b^2}$ that was defined in the main text already.

The supersymmetric double sine functions are constructed from Barnes' double Gamma functions as

$$\begin{aligned} S_1(x) &= S_{NS}(x) = \frac{\Gamma_b\left(\frac{x}{2}\right)\, \Gamma_b\left(\frac{x+Q}{2}\right)}{\Gamma_b\left(\frac{Q-x}{2}\right)\Gamma_b\left(\frac{2Q-x}{2}\right)} \\ S_0(x) &= S_R(x) = \frac{\Gamma_b\left(\frac{x+b}{2}\right)\, \Gamma_b\left(\frac{x+b^{-1}}{2}\right)}{\Gamma_b\left(\frac{Q-x+b}{2}\right)\Gamma_b\left(\frac{Q-x+b^{-1}}{2}\right)} \end{aligned} \tag{A.5}$$

$S_1(x)$ has poles at $x = kb + l/b$ for $k + l \in 2\mathbb{N}$, while $S_0(x)$ has poles at $x = kb + l/b$ for $k + l \in 2\mathbb{N} + 1$. They obey the shift relations:

$$S_1(x + b^{\pm 1}) = 2\cos(\frac{\pi b^{\pm 1} x}{2}) S_0(x), \qquad S_0(x + b^{\pm 1}) = 2\sin(\frac{\pi b^{\pm 1} x}{2}) S_1(x). \tag{A.6}$$

For x integer such that $x \in 2\mathbb{N} + (1 - \nu)$ the double sine functions can be written as:

$$\begin{aligned} S_\nu(-xb) &= \frac{S_1(0)}{\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x}\, [x]_+!} \\ S_\nu(-xb - Q) &= \frac{(-1)^{-\frac{x+1}{2} - \frac{1}{2}\delta_{\nu,1}}\, S_1(0)}{2\cos\left(\frac{\pi}{2b^2}\right)\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x+1}\, [x+1]_+!} = \frac{(-1)^{-\frac{x(x-1)}{2} + 1}\, S_1(0)}{2\cos\left(\frac{\pi}{2b^2}\right)\left(2\cos\left(\frac{\pi b^2}{2}\right)\right)^{x+1}\, [x+1]_+!} \end{aligned} \tag{A.7}$$

where

$$[n]_+! = \begin{cases} \prod_{j=1\, mod\, 2}^{n-1} \cos(j\frac{\pi b^2}{2}) \prod_{j=2\, mod\, 2}^{n} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N} \\ \prod_{j=1\, mod\, 2}^{n} \cos(j\frac{\pi b^2}{2}) \prod_{j=2\, mod\, 2}^{n-1} \sin(-j\frac{\pi b^2}{2}) \left(\cos(\frac{\pi b^2}{2})\right)^{-n}, & \text{for } n \in 2\mathbb{N}+1 \end{cases}$$

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