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The Excited Hexagon Reloaded

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Abstract: This work revisits the computation of six-gluon scattering amplitudes in the high energy limit of strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. It is based on previous studies in which we showed that the amplitude simplifies in the Regge regime and outlined an efficient computational scheme. By exploiting a symmetry of the underlying equations we are now able to argue that a term we had seen in preliminary numerical studies must vanish identically. The derived formula for the Regge limit of the 6-gluon scattering amplitude at strong coupling differs from the one we had conjectured previously.

Keywords: AdS/CFT, gluon amplitudes, Regge limit, thermodynamic Bethe Ansatz

The computation of all-loop scattering amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory is an important problem in quantum field theory. This difficult task could become more tractable by considering appropriate kinematic limits. One such limit in which perturbative gauge theory computations are known to simplify is the multi-Regge or high energy limit. In [1] we observed that the prescription of Alday et al. [4, 5] to compute scattering amplitudes for $n = 6$ external gluons in strongly coupled $\mathcal{N} = 4$ SYM theory simplifies drastically at high energies. The analysis was extended to any number $n > 6$ gluons in [2]. Motivated by these simplifications in the high energy limit of scattering amplitudes both at weak and strong coupling we suggested that they might also occur for intermediate couplings and hence that the multi-Regge limit could be particularly well suited for an interpolation between the perturbative gauge theory and the string theory regimes. In order to constrain any such interpolation we developed a scheme to compute the high energy limit of scattering amplitudes in the strongly coupled gauge theory.

At strong coupling, the leading order of gluon scattering amplitudes may be obtained from the area of a minimal surface in AdS_5 that is approaching the boundary along a piecewise light-like loop [3]. As was shown in [4, 5], this geometric construction admits a beautiful mathematical reformulation in terms of the free energy of a certain 1-dimensional quantum integrable system. For six external gluons, the particle densities in the ground state of this 1-dimensional auxiliary system are determined by solving a system of coupled non-linear integral equations,

$$\log Y_a(\theta) = -m_a \cosh(\theta - i\phi) + C_a + \sum_{a'} \int_{\mathbb{R}+i\phi} d\theta' K_{aa'}(\theta - \theta') \log\left(1 + Y_{a'}(\theta')\right), \quad (1)$$

where $a, a' = 1, 2, 3$ enumerate different species of excitations. Their masses m_a and chemical potentials C_a are given by

$$m_1 = m = m_3 \;\;,\;\; m_2 = \sqrt{2}m \;\;,\;\; -C_1 = C = C_3 \;\;,\;\; C_2 = 0\,. \quad (2)$$

The kernel functions $K_{aa'}$ encode the interactions through their relations with the $2 \to 2$ scattering matrix,

$$(K_{ab}(\theta)) = \frac{1}{2\pi i}\frac{\partial}{\partial\theta}\begin{pmatrix} \log S_1 & \log S_2 & \log S_1 \\ \log S_2 & 2\log S_1 & \log S_2 \\ \log S_1 & \log S_2 & \log S_1 \end{pmatrix} \quad \text{with} \quad \begin{array}{l} S_1 = S_1(\theta) = i\frac{1-ie^\theta}{1+ie^\theta} \\ S_2 = S_2(\theta) = \frac{2i\sinh\theta - \sqrt{2}}{2i\sinh\theta + \sqrt{2}} \end{array}. \quad (3)$$

The integral equations (1) also contain a twist parameter ϕ that enters through the argument of the driving term and through the integration contour. Due to the decay behavior of the Y-functions for large $\mathrm{Re}(\theta)$, the contour can be shifted as long as no poles are crossed in the process. The Y-functions must satisfy Eqs.(1) as long as the argument θ satisfies $|\,\mathrm{Im}(\theta - i\phi)| < \pi/4$. If θ leaves this strip of width $\pi/2$ around the integration contour, the value of the Y function can be determined through appropriate shift equations.

There is one observation concerning the solutions of Eqs.(1) that we will play a crucial role later on: The Y-functions enjoy the shifted symmetry

$$Y_a(\theta + i\phi) = Y_a(-\theta + i\phi) \;\; . \quad (4)$$

Let us explain this in a bit more detail. Keeping in mind the way we solve the Y-system numerically, we can prove the symmetry iteratively, starting with the driving term $Y_a^{(0)}(\theta) = -m_a \cosh(\theta - i\phi) + C_a$ itself. It is obvious that $Y_a^{(0)}$ obey Eq.(4). Assume now that the n-th iteration satisfies $Y_a^{(n)}(\theta + i\phi) = Y_a^{(n)}(-\theta + i\phi)$. We want to show that the same is true for the $(n+1)$-th iteration,

$$\log Y_a^{(n+1)}(\tilde{\theta}) = -m_a \cosh\left(\tilde{\theta} - i\phi\right) + C_a + \sum_{a'} \int_{\mathbb{R}+i\phi} d\theta' K_{aa'}\left(\tilde{\theta} - \theta'\right) \log\left(1 + Y_{a'}^{(n)}(\theta')\right).$$

Setting $\tilde{\theta} = \theta + i\phi$, the integral contribution reads

$$\int_{\mathbb{R}+i\phi} d\theta' K\left(\theta + i\phi - \theta'\right) \log\left(1 + Y^{(n)}(\theta')\right) = \int_{\mathbb{R}} dx K\left(\theta + x\right) \log\left(1 + Y^{(n)}(-x + i\phi)\right) =$$

$$= \int_{\mathbb{R}} dx K\left(-\theta - x\right) \log\left(1 + Y^{(n)}(x + i\phi)\right) = \int_{\mathbb{R}+i\phi} d\theta' K\left(-\theta + i\phi - \theta'\right) \log\left(1 + Y^{(n)}(\theta')\right),$$

where we suppressed all indices. In passing from the first to the second line we have used the symmetry $K(x) = K(-x)$ which is special to the kernels appearing in the six-point case and we inserted our assumption on the symmetry for the n-th approximation. Since the final result is the integral contribution for $\tilde{\theta} = -\theta + i\phi$, we have demonstrated that the $(n+1)$-th iteration of all three Y-functions has the claimed symmetry.

After these general comments we shall now focus on the Regge regime. As we have shown in previous work, the high energy limit in the gauge theory corresponds to sending the mass parameter m to infinity while keeping C and $\log w := m \sin\phi$ fixed. For large mass m, one can neglect the integral contributions in Eqs.(1) unless a solution of $Y_a(\theta_*) = -1$ comes close to the integration contour. Note that the position θ_* depends on the choice of parameters m, C, ϕ. Since the system parameters are related to dual conformal invariant cross ratios u_a, the positions θ_* depend on the kinematics of the process under consideration. If the Regge limit is performed in the Euclidean region, i.e. $u_3 \to 1$ while $u_1, u_2 \to 0$ with $u_1, u_2 > 0$, none of the solutions of $Y_a(\theta_*) = -1$ comes close to the real axis so that, in the large m limit, the Y-functions are well approximated by the driving terms. In order to approach the Regge limit in the so-called mixed regime where $u_1, u_2 < 0$, one has to continue the cross ratios from the physical regime, which translates into an analytic continuation for the Y-system parameters. During this continuation of the system parameters, it turns out that a few of the solutions θ_* get close to or cross the integration contour. Whenever this happens, the integral terms can no longer be dropped, i.e. they contribute to the Y-functions at the end of the continuation. However, the new terms are easy to evaluate. In fact, each solution of $Y_a(\theta_*)$ that crosses the contour during the continuation contributes terms of the form $\pm \log S_b(\theta - \theta_*)$ to the logarithm of the Y-functions, with the overall sign depending on the direction in which the contour is crossed. While these additional terms modify the Y-functions, they preserve the symmetry (4) as we have shown above.

In order to reach the mixed region in [1] we continued the cross ratios u_a along the curve

$$u_3(\varphi) = e^{-2i\varphi} u_3, \tag{5}$$

while keeping both u_1 and u_2 fixed. Most of the numerical evaluation was performed for configurations with $u_1 = u_2$. These are associated with the twist parameter $\phi = 0$. The dependence of the system parameters $m = m(\varphi)$ and $C = C(\varphi)$ on the parameter φ of the continuation is shown in figures of our original publication [1]. For these paths, a pair of solutions to $Y_{2\pm1}(\theta_*) = -1$ crosses the integration contour while a second pair of solutions to $Y_2(\theta_*) = -1$ is seen to collide at the end of the continuation process. This general pattern does not change when $u_1 \neq u_2$ at least within the finite region $10^{-2} \leq u_1/u_2 \leq 10^2$ that we scanned numerically. It should be noted that for $\phi \neq 0$, the mass parameter always stays large during the continuation, while ϕ always stays small. C is a finite quantity at the beginning and the end of the continuation, but reaches values of $\mathcal{O}(m)$ during the continuation.

Once we accept this general pattern of the migration of solutions, we can exploit the symmetry (4) to obtain precise analytical expressions for the Y-functions Y'_a in the mixed region. After we have made the mass parameter large to reach the Regge regime, we can again neglect the integral contributions at the end of the continuation and the equation governing Y'_3 reads

$$\log Y'_3(\theta) = -m' \cosh\left(\theta - i\phi'\right) + C' + \log\left(\frac{S_1(\theta - \theta_-)}{S_1(\theta - \theta_+)}\right), \tag{6}$$

where primes indicate quantities at the end of the continuation and θ_+ is the position of the crossing solution of $Y_3(\theta) = -1$ that has $\mathrm{Im}(\theta_+) > 0$ (and analogously for θ_-). By definition, we have that $Y'_3(\theta_+) = -1$ and therefore

$$i\pi = \log Y'_3(\theta_+) = -m' \cosh(\theta_+ - i\phi') + C' + \log\left(\frac{S_1(\theta_+ - \theta_-)}{S_1(0)}\right). \tag{7}$$

Since the left-hand side is finite, so is the right-hand side. As we send m' to infinity, this is only possible if $\theta_+ - \theta_-$ approaches a pole of the S-matrix, since C' is a finite quantity. From the definition of $S_1(x)$, Eq.(3), we obtain

$$\theta_+ - \theta_- = i\frac{\pi}{2}. \tag{8}$$

Taking into account the symmetry (4) of the Y-functions we can conclude that

$$\theta_- + \theta_+ = 2i\phi'. \tag{9}$$

Together, Eqs.(8) and (9) imply

$$\theta_\pm = \pm i\frac{\pi}{4} + i\phi' \ . \tag{10}$$

One may object that the solutions $\theta_\pm$ could leave the fundamental strip $|\mathrm{Im}(\theta - i\phi)| < \pi/4$ and that we have to pick up contributions of the form $\log\left(1 + Y_2\left(\theta \pm i\frac{\pi}{4}\right)\right)$, which could also compensate the diverging term in Eq.(7). This, however, is not possible. If we assume, for example, that $\mathrm{Im}(\theta - i\phi) > \frac{\pi}{4}$, we find that

$$\mathrm{Im}\left(\theta_- - i\phi\right) = \mathrm{Im}\left(-\theta_+ + i\phi\right) = -\mathrm{Im}\left(\theta_+ - i\phi\right) < -\frac{\pi}{4}, \tag{11}$$

and therefore there are no such contributions for θ_-. The analogous statement is true if $\text{Im}(\theta_- - i\phi) < -\frac{\pi}{4}$. We therefore see that we always have at least one position of the solutions $\theta_\pm$ for which Eq.(6) holds, which is enough to use our argument.

We now turn to $Y_2'(\theta)$, which is governed by the equation

$$\log Y_2'(\theta) = -\sqrt{2}m' \cosh\left(\theta - i\phi'\right) + \log\left(\frac{S_2(\theta-\theta_-)}{S_2(\theta-\theta_+)}\right). \tag{12}$$

To find the positions of the solutions to $Y_2'(\theta_*) = -1$ at the end of the continuation we can repeat the above argument to conclude that they must lie on poles of the function

$$\frac{S_2\left(\theta + i\frac{\pi}{4} - i\phi'\right)}{S_2\left(\theta - i\frac{\pi}{4} - i\phi'\right)} = \coth\left(\frac{1}{2}(\theta - i\phi')\right)^2. \tag{13}$$

We therefore find that both solutions have to approach $\theta = i\phi'$. As this coincides with the imaginary part of the integration contour, these two solutions do not give rise to new contributions. This conclusion deviates from [1] where it seemed, for our numerical investigations performed for w too close to 1, as if the two solutions would meet at $\theta_* = 0$. Now we understand that such a behavior would be incompatible with the symmetry (4). New numerical results with larger ϕ confirm the above analytical derivation.

From these results we can determine the value of the cross ratios when we reach the end-point of the continuation. Inserting the expression (12) for the Y-function Y_2 at the end of the continuation we find to leading order in ε'

$$u_1 = \frac{Y_2'(-i\frac{\pi}{4})}{1+Y_2'(-i\frac{\pi}{4})} = \gamma\varepsilon' w' + \mathcal{O}((\varepsilon')^2) \quad , \quad u_2 = \frac{Y_2'(i\frac{\pi}{4})}{1+Y_2'(i\frac{\pi}{4})} = \gamma\frac{\varepsilon'}{w'} + \mathcal{O}((\varepsilon')^2),$$

with $\gamma = -(3+2\sqrt{2})$ and with parameters $w' = \exp(m'\sin\phi')$ and $\varepsilon' = \exp(-m'\cos\phi')$. Before the continuation, the same cross ratios take the form $u_1 \sim w\varepsilon$ and $u_2 \sim w^{-1}\varepsilon$ so that we conclude

$$\varepsilon' \;=\; \gamma^{-1}\varepsilon + \mathcal{O}\left(\varepsilon^2\right), \quad w' \;=\; w + \mathcal{O}\left(\varepsilon^2\right) \quad , \tag{14}$$

see [1] for more details. With this preparation we can evaluate the amplitude. The most important contribution comes from the free energy

$$A'^{(6)}_{\text{free}} = \int\limits_{\mathbb{R}+i\phi'} \frac{d\theta}{2\pi} m' \cosh(\theta - i\phi') \log\left[(1+Y_1'(\theta))(1+Y_3'(\theta))(1+Y_2'(\theta))^{\sqrt{2}}\right]$$
$$+m' i \sinh(\theta_+ - i\phi') - m' i \sinh(\theta_- - i\phi').$$

In comparison with the corresponding formula (5.19) in our original publication [1], we have corrected a sign mistake and we dropped the last term that seemed to arise from the solutions of $Y_2(\theta_*) = -1$. Its absence is a direct consequence of the symmetry (4). After these corrections, Eq.(5.20) of [1] takes the form

$$\begin{aligned} A'^{(6)}_{\text{free}} &= -\sqrt{2}m' + \mathcal{O}\left(\varepsilon'\right) \;=\; \sqrt{2}\log\varepsilon' + \mathcal{O}\left(\varepsilon'\right) \\ &= \sqrt{2}\log\varepsilon - \sqrt{2}\log\gamma + A^{(6)}_{\text{free}} + \mathcal{O}\left(\varepsilon\right). \end{aligned}$$

Following the steps that were performed in [1] we finally arrive at an expression for the 6-gluon remainder function in the Regge limit of strongly coupled $\mathcal{N} = 4$ SYM theory

$$\mathrm{e}^{\frac{\sqrt{\lambda}}{2\pi}R'+i\delta} \;\sim\; \left((1-u_3)\sqrt{\tilde{u}_1\tilde{u}_2}\right)^{\frac{\sqrt{\lambda}}{2\pi}e_2} \tag{15}$$

with

$$e_2 \;=\; \left(-\sqrt{2}+\tfrac{1}{2}\log(3+2\sqrt{2})\right) \sim -0.533$$

and where we have factored out a phase factor $e^{i\delta}$ that cancels the same term with opposite sign in the BDS-part of the amplitude. Note that the correct value of e_2 deviates from the one spelled out in [1] by an internal sign. After correcting this mistake, the number e_2 that appears in the final expression of the amplitude possesses the same sign as the BFKL eigenvalue $E_2 = -\frac{\lambda}{2}(2\log 2 - 1)$ at weak coupling.

The result Eq.(15) we have obtained in this note deviates from the expression we had proposed in the original publication in a more fundamental way: It does not contain the factor depending on the ratio u_1/u_2 that we had proposed previously. This has far reaching implications. Recall that at weak coupling the correction to the BDS Ansatz for the amplitude can be written as a sum/integral over the quantum numbers n and ν of the 2-dimensional conformal group. The integral over ν is dominated by a saddle point at $i\nu = 0$. Assuming that a similar representation of the amplitude also holds at strong coupling, the term depending on u_1/u_2 that appeared in [1] suggested that the saddle point moves to infinity at strong coupling. The absence of this term, however, means that the saddle point appears at $i\nu = 0$, just as it does at weak coupling. This is consistent with the recent analysis by Simon Caron-Huot in [6].

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