

Computing Three-Point Functions for Short Operators

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Abstract

We compute the three-point structure constants for short primary operators of $\mathcal{N} = 4$ super Yang–Mills theory to leading order in $1/\sqrt{\lambda}$ by mapping the problem to a flat-space string theory calculation. We check the validity of our procedure by comparing to known results for three chiral primaries. We then compute the three-point functions for any combination of chiral and non-chiral primaries, with the non-chiral primaries all dual to string states at the first massive level. Along the way we find many cancellations that leave us with simple expressions, suggesting that integrability is playing an important role.

Contents

1	Introduction	1
2	Review of Previous Results	4
3	Three-Point Functions	10
3.1	Three Chiral Primaries	11
3.2	Two Chiral Primaries	11
3.3	One Chiral Primary	12
3.4	Zero Chiral Primaries	13
4	Discussion	14
A	Bosonization Setup and OPE's	15
B	Primary Vertex Operators	18
C	Vertex function contractions	21
D	Mixed Correlators	31

1 Introduction

It is now possible, at least in principle, to compute the dimensions of single-trace local operators in planar $\mathcal{N} = 4$ super Yang–Mills for any value of the 't Hooft coupling λ . The ability to do this relies on the underlying integrability of the theory [1]. One of the most impressive results is the numerical computation of the dimension, Δ_K , of the Konishi operator, $\mathcal{O}_K = \text{Tr } \Phi^I \Phi_I$. Starting at weak coupling and interpolating to very large values of λ , one finds extremely precise results [2, 3]. What is clear from these studies is that for large values of λ , Δ_K asymptotes to

$$\Delta_K = 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \dots \quad \lambda \gg 1, \tag{1.1}$$

a very satisfying result since the leading behavior was predicted well beforehand based on a flat-space approximation of a string on $\text{AdS}_5 \times \text{S}^5$ [4]. The next two terms in this series can also be motivated by semi-classical string theory computations [5–8], or by analyzing the algebraic curve [9], or by a mini-superspace approach on the string world-sheet [10].

To fully solve planar $\mathcal{N} = 4$ SYM it is also necessary to know the three-point functions between local operators. Compared to the spectrum, the results here are rather limited. However, there are indications that integrability will play a crucial role in determining the structure constants. In particular, there have been important results at low orders of perturbation

theory that crucially rely on the integrable structure in the theory [11–13]. Impressively, in [12] it was shown that the weak-coupling three-point function for two semi-classical operators (with $\Delta \sim \sqrt{\lambda}$) and one short BPS operator, was consistent with the strong-coupling results derived from a semi-classical string calculation in [14].

Based on our experience with two-point functions, it will be much more difficult to compare weak and strong coupling results if the operators in the three-point functions are all short.¹ This is because finite-size effects are expected to become important at strong-coupling and so all of the machinery of the thermodynamic Bethe ansatz will likely be needed. Nonetheless, as with Δ_K , it will be very useful to know the strong-coupling target. In other words, analogous to the GKP result in [4], we wish to know the leading strong-coupling behavior for such three-point functions.

In this paper we continue the study of three-point functions for short operators at level one. The methods we describe take advantage of the fact that at large Yang–Mills coupling the string theory target space has small curvature and so one can approximate the vertex operators with flat-space vertex operators [15]. The curvature still plays a role as it determines how the states propagate in from the boundary. The accompanying Witten diagrams have an intersection point that should be integrated over, but for short operators with a large dimension the integral is dominant over a small region of the bulk² where one can ignore the curvature as well as the Ramond–Ramond flux. Within that region, one can then use flat space vertex operators to compute the couplings that determine the part of the three-point function not determined solely by the dimensions of the operators.

The challenge is to find the flat-space vertex operators and then compute the three-point function with those respective operators. In [15] it was described how to find such operators. In particular, it was argued that the relevant operators satisfy a twisted version of $Q_L = i Q_R$, where $Q_{L,R}$ are the left and right ten-dimensional supercharges in flat space. It was shown that this condition on the supercharges maps back to the AdS condition that $S = 0$, where S represents all superconformal charges. We impose this AdS condition since the three operators are assumed to be primaries. Consequently, the operators are necessarily combinations of NS-NS and R-R scalars.

Since the Konishi operator is primary but not chiral primary, its string dual is a massive state. In the large coupling limit, the dimensions of scalar operators with no R -charges have dimensions that are approximately $\Delta \approx 2\sqrt{n} \lambda^{1/4}$ corresponding to the energies of stringy modes when $\alpha' = 1/\sqrt{\lambda}$ [4]. There is now overwhelming evidence that the Konishi operator approaches one of these stringy modes with $n = 1$ [17, 2]. Hence, the three-point function for three Konishi operators at strong coupling can be well-approximated by determining the three-point amplitude for three massive string states in flat IIB string theory.

Unfortunately, the vertex operators are quite complicated, even at the lowest massive level, and to the best of our knowledge a three-point function involving three massive IIB vertex operators has never been computed. In this paper we will do just that for three scalar operators, although they are not scalars in the ten-dimensional theory. We will also compute three-point functions involving one, two and three chiral primaries, which are dual to massless string states, and whose dimensions equal the sum of their R -charges. In the

¹By “short” we mean local single-trace operators whose dual string states are short compared to the radii of curvature in $\text{AdS}_5 \times S^5$. We do not mean operators that are in short super-multiplets.

²The same thing leads to the focusing of wave-packets for AdS S -matrices [16].

case of three chiral primaries we will compare our result to the supergravity result in [18], showing that our procedure is consistent with their result. The correlator with one chiral primary whose R -charge J satisfies $J \ll \lambda^{1/4}$ is of significance because it comes with an exponential suppression factor that depends on J and not on the dimensions of the two non-chiral primaries.

The full three-point function of the operators also relies on overlap integrals on the S^5 . For chiral primaries, it is necessary to assume that the R -charges are large so that their corresponding dimensions are large. This will result in wave-functions that are strongly peaked on S^5 . However, for non-chiral primaries the dimensions are already large in the strong-coupling limit and there is otherwise no reason to assume large R -charges for these operators,³ provided that the total R -charge in the three-point function is conserved. In any case we will give expressions for three-point functions for general J values, but also give results for when the R -charges of the non-chiral primaries are zero, which happens for three Konishi operators.

The three-point function for three scalar operators in d space-time dimensions with scaling dimensions $1 \ll \Delta_i \ll \sqrt{\lambda}$ is given by [15, 19]

$$\langle \mathcal{O}_{\Delta_1}(x_1^\mu) \mathcal{O}_{\Delta_2}(x_2^\mu) \mathcal{O}_{\Delta_3}(x_3^\mu) \rangle = \frac{\mathcal{C}_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}, \quad (1.2)$$

where

$$\mathcal{C}_{123} \approx \frac{\pi^{\frac{2-d}{4}}}{4} \frac{(\Delta_1 \Delta_2 \Delta_3)^{d/4}}{(\alpha_1 \alpha_2 \alpha_3 \Sigma^{d+1})^{1/2}} \frac{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} \Sigma^\Sigma}{\Delta_1^{\Delta_1} \Delta_2^{\Delta_2} \Delta_3^{\Delta_3}} \mathcal{G}_{123}. \quad (1.3)$$

and

$$\alpha_1 = \frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1), \quad \alpha_2 = \frac{1}{2}(\Delta_3 + \Delta_1 - \Delta_2), \quad \alpha_3 = \frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3), \\ \Sigma = \frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3) = \alpha_1 + \alpha_2 + \alpha_3. \quad (1.4)$$

The coupling \mathcal{G}_{123} is given by

$$\mathcal{G}_{123} = \frac{8\pi}{g_c^2 \alpha'} \langle V_{k_1} V_{k_2} V_{k_3} \rangle \langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle, \quad (1.5)$$

which includes the contribution of the overlap integral $\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle$ on S^5 . The closed string coupling g_c and the inverse string tension are given under the AdS/CFT dictionary as $g_c = \pi^{3/2}/N$ and $\alpha' = 1/\sqrt{\lambda}$.

In this paper we will find for three level-one states with small R -charges $J_i \ll \lambda^{1/4}$ that the flat-space vertex operators satisfy

$$\langle V_{k_1} V_{k_2} V_{k_3} \rangle = g_c^3 \frac{3^8}{2^9} + \mathcal{O}(J_i^2 \lambda^{-1/2}) \quad (1.6)$$

Hence the result for the correlator of three short operators with small R -charges reads

$$\mathcal{C}_{123} \approx \frac{1}{N} \frac{3^{11/2} \pi^2}{32} \lambda^{1/4} \left(\frac{3}{4}\right)^{3\Delta/2} \langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle. \quad (1.7)$$

³We thank E. Witten for a discussion on this point.

For vanishing R -charges J_i we have that $\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle = \pi^{-3/2}$. Including the finite correction term in (1.1) [5], we can thus reexpress (1.7) for three Konishi operators as

$$\mathcal{C}_{123} \approx \frac{1}{N} 64 \pi^{1/2} \lambda^{1/4} \left(\frac{3}{4} \right)^{3\lambda^{1/4} + 5/2}. \quad (1.8)$$

In the intermediate steps the individual contributions to (1.6) involve larger prime factors, but after combining the different parts we end up with a much simpler expression. This suggests that there is an underlying symmetry playing an important role.

For the case of one chiral primary and two primaries at level one, we find that

$$\mathcal{C}_{123} \approx \frac{1}{N} \pi^2 \lambda^{1/4} 2^{-J_3} \langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle, \quad (1.9)$$

where $1 \ll J_3 \ll \lambda^{1/4}$ is the magnitude of the R -charge of the chiral primary, and $\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle$ depends on the distribution of R -charge on the level-one states. For $J_1 = 0$, $J_2 = J_3 = J$, the correlator becomes

$$\mathcal{C}_{123} \approx \frac{1}{N} \pi^{1/2} \lambda^{1/4} 2^{-J}. \quad (1.10)$$

Note that \mathcal{C}_{123} is exponentially suppressed if all $\Delta_i \gg 1$. However, if one of the operators is a chiral primary with $\Delta = J$ and J is much smaller than the other two dimensions, then the suppression factor is only by 2^{-J} .

The rest of this paper is organized as follows: In Section 2 we review some results of [15]. In Section 3, which is the main part of this article, we compute three-point amplitudes for three string states where zero, one, two or three are at the first massive level. We then use these to reach the main results in (1.7) and (1.9). The partial amplitudes involve combinations of three NS-NS scalars or one NS-NS scalar and two R-R scalars. The states are not Lorentz scalars in the full ten-dimensional flat space, but instead are twisted versions of these states. However, for $J = 0$ they remain scalars in the two reduced five-dimensional spaces which are the flattened versions of AdS_5 and S^5 .⁴ Section 4 contains our conclusions and discussion for various extensions of these results. Many technical details are contained in several appendices.

2 Review of Previous Results

In this section we collect some relevant results from [15].

We assume that the operators are short operators in the sense that their sizes are much smaller than the AdS_5 and S^5 radii. Hence at scales comparable to this radius the strings appear point-like. The three-point correlator is then closely related to the computation of a three-point correlator using Witten diagrams [20].

In the Witten diagram one integrates over all possible intersection points, but in the semiclassical approximation the Witten diagrams are dominated by peaks along the particles' classical trajectories in the $\text{AdS}_5 \times S^5$ background. The intersection point is calculable [19,15],

⁴They are no longer AdS scalars if $J \neq 0$ because the twisting happens in the four-dimensional subspace transverse to the AdS momentum, but the polarization can have components along both the AdS momentum and its transverse space.

and after including the contributions of the fluctuations one finds the coefficient in front of \mathcal{G}_{123} in (1.3). For large R -charges $J_i \gg 1$ we can also use a semiclassical approximation for the S^5 overlap integrals $\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle$ [21, 15]. In this approximation we assume that the R -charges are each highest weight with respect to some basis and satisfy $\vec{J}_1 + \vec{J}_2 + \vec{J}_3 = 0$. Then one finds

$$\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle \approx \frac{1}{2\pi^2} \frac{(J_1 J_2 J_3)^{3/2}}{(\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}^5)^{1/2}}, \quad (2.1)$$

where

$$\tilde{\Sigma} = \frac{J_1 + J_2 + J_3}{2}, \quad \tilde{\alpha}_1 = \frac{J_2 + J_3 - J_1}{2}, \quad \tilde{\alpha}_2 = \frac{J_3 + J_1 - J_2}{2}, \quad \tilde{\alpha}_3 = \frac{J_1 + J_2 - J_3}{2}. \quad (2.2)$$

For later reference, these can be written in terms of the standard spherical harmonic coefficients $C_{1\dots J}^J$ in [18] as

$$\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle \approx \frac{1}{2\pi^2} \frac{(J_1 J_2 J_3)^{3/2}}{(\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}^5)^{1/2}} \frac{J_1^{J_1} J_2^{J_2} J_3^{J_3}}{\tilde{\alpha}_1^{\tilde{\alpha}_1} \tilde{\alpha}_2^{\tilde{\alpha}_2} \tilde{\alpha}_3^{\tilde{\alpha}_3} \tilde{\Sigma}^{\tilde{\Sigma}}} \langle C^{J_1} C^{J_2} C^{J_3} \rangle, \quad (2.3)$$

where

$$\langle C^{J_1} C^{J_2} C^{J_3} \rangle \approx \frac{\tilde{\alpha}_1^{\tilde{\alpha}_1} \tilde{\alpha}_2^{\tilde{\alpha}_2} \tilde{\alpha}_3^{\tilde{\alpha}_3} \tilde{\Sigma}^{\tilde{\Sigma}}}{J_1^{J_1} J_2^{J_2} J_3^{J_3}}. \quad (2.4)$$

The classical trajectories come with constants of motion which are determined by the R -charges, the spins, the dimensions and by the positions of the operators on the boundary. The constants of the motion can be formulated in terms of the components of the conformal algebra, P_μ , K^μ , D and $M_{\mu\nu}$. For the operator $\mathcal{O}_i(x_i^\mu)$ in the three-point function the conserved charges include

$$\begin{aligned} \langle P_i^\mu \rangle &= -2i \sum_{i \neq j \neq k \neq i} \alpha_k \frac{x_{ij}^\mu}{x_{ij}^2} \\ \langle D_i \rangle &= i \left(\Delta_i - 2x_{i\mu} \sum_{i \neq j \neq k \neq i} \alpha_k \frac{x_{ij}^\mu}{x_{ij}^2} \right), \end{aligned} \quad (2.5)$$

where $x_{ij}^\mu = x_i^\mu - x_j^\mu$. The sums are over the other two operators in the three-point function. One can also construct the quadratic Casimir operator

$$-2\langle P_\mu \rangle \langle K^\mu \rangle + \langle D \rangle^2 - \frac{1}{2} \langle M_{\mu\nu} \rangle \langle M^{\mu\nu} \rangle = -\Delta^2. \quad (2.6)$$

At the point where the three trajectories intersect these constants of the motion are conserved. It is convenient to shift the operator positions such that the intersection point is at $x^\mu = 0$. In this case all $\langle M_{\mu\nu} \rangle = 0$ and the conserved charges $\langle K_i^\mu \rangle$ satisfy

$$\langle K_i^\mu \rangle = -\frac{\alpha_1 \alpha_2 \alpha_3 \Sigma}{F^2} x_{12}^2 x_{23}^2 x_{31}^2 \langle P_i^\mu \rangle, \quad (2.7)$$

where

$$F = \alpha_1 \alpha_2 x_{12}^2 + \alpha_2 \alpha_3 x_{23}^2 + \alpha_3 \alpha_1 x_{13}^2. \quad (2.8)$$

It is also convenient to write everything in a manifest $\text{SO}(2, d)$ formalism, with

$$M_{-1\mu} \equiv \frac{1}{\sqrt{2}}(\kappa P_\mu - \kappa^{-1} K_\mu) \quad M_{d\mu} \equiv \frac{1}{\sqrt{2}}(\kappa P_\mu + \kappa^{-1} K_\mu) \quad M_{-1d} \equiv -D, \quad (2.9)$$

where κ is arbitrary. In this convention the Casimir operator becomes $-\frac{1}{2}M_{rs}M^{rs} = -\Delta^2$, $r, s = -1 \dots d$. Setting $d = 4$, we can also combine this with the $\text{SO}(6)$ R -symmetry Casimir to write

$$-\frac{1}{2}M_{rs}M^{rs} + \frac{1}{2}R_{IJ}R^{IJ} = -\Delta^2 + J^2, \quad (2.10)$$

where R_{IJ} are the R -symmetry generators with $I, J = 5, \dots, 10$.

If Δ and J are large, then the $\text{SO}(2, 4) \times \text{SO}(6)$ algebra effectively reduces to a 10 dimensional Poincaré algebra. Assuming that all $\langle M_{\mu\nu} \rangle = 0$, and choosing κ to be,

$$\kappa = \frac{\sqrt{\alpha_1 \alpha_2 \alpha_3 \Sigma}}{F} |x_{12}| |x_{23}| |x_{31}|, \quad (2.11)$$

then the only non-zero components in (2.9) are $\langle M_{-1m} \rangle$, $m = 0, \dots, 4$ for all three operators. Assuming $\Delta^2 \gg 1$, and choosing a basis where the only non-zero R -symmetry components are $\langle R_{J,10} \rangle$, we can then identify the full 10-dimensional flat-space momentum as

$$k^M = (\langle M_{-1m} \rangle, \langle R_{J,10} \rangle), \quad (2.12)$$

which satisfies the on-shell condition

$$k \cdot k = -\Delta^2 + J^2 = -4n\sqrt{\lambda}. \quad (2.13)$$

If we fill out the algebra to the full superconformal $\text{PSU}(2, 2|4)$, then we also have fermionic generators. These include the supergenerators $Q_{\alpha a}$ and $\tilde{Q}_{\dot{\alpha} a}$, where α and $\dot{\alpha}$ are space-time spinor indices and a is an $\text{SO}(6)$ spinor index (raised or lowered depending on which spinor representation), and the superconformal generators S_α^a and $\tilde{S}_{\dot{\alpha} a}$. We can put these into a form that is manifestly $\text{SU}(2, 2) \simeq \text{SO}(2, 4)$ covariant by defining

$$\begin{aligned} Q_{\dot{a}a}^1 &\equiv (\kappa^{1/2} Q_{\alpha a}, \kappa^{-1/2} \tilde{S}_{\dot{\alpha} a}) \\ Q^{2, \dot{a}a} &\equiv (\kappa^{-1/2} \varepsilon^{\alpha\beta} S_\beta^a, \kappa^{1/2} \varepsilon^{\dot{\alpha}\dot{\beta}} \tilde{Q}_{\dot{\beta}}^a) \end{aligned} \quad (2.14)$$

where the lowered \dot{a} is an $\text{SO}(2, 4)$ spinor index and the raised \dot{a} is an index for the other spinor representation. The κ is the same that appears in (2.9). If we then define two sets of supercharges

$$Q_A^L = Q_{\dot{a}a}^1 + \gamma_{\dot{b}a}^{-1} \gamma_{ba}^6 Q^{2, \dot{b}b}, \quad Q_A^R = i(Q_{\dot{a}a}^1 - \gamma_{\dot{b}a}^{-1} \gamma_{ba}^6 Q^{2, \dot{b}b}), \quad (2.15)$$

then in the flat-space limit these approach the usual 10-dimensional super-Poincaré generators with

$$\{Q_A^{L,R}, Q_B^{L,R}\} = -2(\text{P}_+ \Gamma^M C)_{AB} P_M, \quad P_M = (M_{-1m}, R_{J,10}), \quad \{Q_A^L, Q_B^R\} = 0 \quad (2.16)$$

for $\Delta \gg 1$, $J \gg 1$, where P_+ is the positive-chirality projector.⁵

For a primary operator we have that $[S_\alpha^a, \mathcal{O}(0)] = [\tilde{S}_{\dot{\alpha}a}, \mathcal{O}(0)] = 0$. If we compare this to the new generators in (2.15) we see that this corresponds to

$$Q_A^L = \pm i Q_A^R \quad (2.17)$$

for the primary operators, where the sign depends on the spinor component. With the operator at $x_i^\mu = 0$ it is clear that the operator's charges have $\langle K^\mu \rangle = \langle M_{\mu\nu} \rangle = 0$. If we then assume that the intersection point is arbitrarily far away, or equivalently we are on a part of the trajectory arbitrarily close to the boundary, then for finite κ we have $M_{-1\mu} = M_{4\mu} = 0$, $M_{-14} = -i\Delta$, which puts the space-time directions transverse to the momentum in AdS₅. Going back to (2.15), we see that in the flat-space limit this imposes the condition on the operators

$$Q_{\alpha\tilde{a}}^L = i Q_{\alpha\tilde{a}}^R, \quad Q_{\dot{\alpha}\tilde{a}}^L = -i Q_{\dot{\alpha}\tilde{a}}^R, \quad (2.18)$$

where we use here the explicit four-dimensional space-time $(\alpha, \dot{\alpha})$ and orthogonal six-dimensional (\tilde{a}) spinor indices. Shifting the intersection point back to $x^\mu = 0$, the $M_{-1\mu}$ are no longer zero, but the primary operator condition is still defined by the directions transverse to the momentum in AdS₅. Hence, the four-dimensional spinor indices in (2.18) are replaced with the spinor indices for the four-dimensional space transverse to the particle momentum in AdS₅, while the six-dimensional spinor indices are replaced accordingly.

We then proceed by finding operators that satisfy the condition in (2.18). As was emphasized in [15], this leads to string states that are linear combinations of NS-NS and R-R modes. At the massless level this gives chiral primary operators while at the massive levels the states correspond only to primary operators. The relative sign in (2.18) complicates matters as the states cannot be scalars in the full 10-dimensional flat-space. At best they can be scalars in the two five-dimensional subspaces. Nonetheless, it is still convenient to start with twisted versions of our vertex operators that correspond to 10-dimensional scalars, and then untwist them afterwards. In particular, if we start with a vertex operator $V_{T,k}$ that satisfies

$$[Q_A^L, V_{T,k}] = [i Q_A^R, V_{T,k}], \quad (2.19)$$

then this is related to an operator V_k that satisfies (2.18) by

$$V_{T,k} = e^{\pi i(M_{0'1'}^R + M_{2'3'}^R)} V_k, \quad (2.20)$$

where the μ' components are transverse to k^M in the AdS₅ part and $M_{\mu'\nu'}^R$ is a generator of rotations for the right-movers.

For the level-zero vertex operators, those corresponding to the massless states, a general NS-NS vertex operator in the $(-1, -1)$ picture has the form [22]

$$W_{1,k}^{(-1,-1)}(z, \bar{z}) = g_c \varepsilon_{M\tilde{M}} \psi^M e^{-\phi} \tilde{\psi}^{\tilde{M}} e^{-\tilde{\phi}} e^{ik \cdot X}, \quad k^M \varepsilon_{M\tilde{M}} = k^{\tilde{M}} \varepsilon_{M\tilde{M}} = 0, \quad (2.21)$$

⁵For $J \sim \mathcal{O}(1)$, the correction terms in (2.16) are of the same order as the P_J components. However, as long as $\Delta \gg 1$ these are suppressed compared to the other momentum components and hence do not affect the conditions obeyed by the vertex operators.

where ϕ and $\tilde{\phi}$ are the left and right superconformal ghost fields. Eventually we will replace g_c and α' with their dictionary values, but for now leave them in the usual string-theory form. Since the total ghost number for the three-point function must be $(-2, -2)$, we will also need the vertex operator in the $(0, 0)$ picture, which is given by

$$W_{1,k}^{(0,0)}(z, \bar{z}) = -g_c \varepsilon_{M\tilde{M}} \frac{2}{\alpha'} (i\partial X^M + \frac{\alpha'}{2} k \cdot \psi \psi^M) (i\bar{\partial} X^{\tilde{M}} + \frac{\alpha'}{2} k \cdot \tilde{\psi} \tilde{\psi}^{\tilde{M}}) e^{ik \cdot X}. \quad (2.22)$$

A generic vertex operator in the R-R sector is [22]

$$W_{2,k}^{(-1/2,-1/2)}(z, \bar{z}) = g_c t_{AB} \tilde{\Theta}^A e^{-\frac{1}{2}\tilde{\phi}} \Theta^B e^{-\frac{1}{2}\phi} e^{ik \cdot X}, \quad t \not{k} = 0, \quad (2.23)$$

where Θ_A and $\tilde{\Theta}_B$ are the sixteen-component left and right twist fields. We only need these vertex operators in the $(-1/2, -1/2)$ picture.

The twisted vertex operator will be a linear combination of the NS-NS and the R-R vertex operators. Equation (2.19) can only be satisfied up to spurious terms and has one scalar solution,

$$W_T = W_{1,T} + \frac{1}{\sqrt{2}} W_{2,T}, \quad (2.24)$$

up to a normalization factor. Here, $W_{1,T}$ and $W_{2,T}$ are the NS-NS and R-R vertex operators with polarizations⁶

$$\varepsilon_T^{M\tilde{M}} = \eta^{M\tilde{M}} - \frac{k^M \bar{k}^{\tilde{M}} + k^{\tilde{M}} \bar{k}^M}{k \cdot \bar{k}}, \quad t_{T,AB} = \left(\frac{\alpha'}{2}\right)^{1/2} (C^\dagger \not{k})_{AB}, \quad (2.25)$$

where C is the charge conjugation matrix and \bar{k}^N is an arbitrary light-like vector satisfying $k \cdot \bar{k} \neq 0$. Untwisting the vertex operators the NS-NS polarization becomes

$$\varepsilon^{M\tilde{M}} = (-1)^{\sigma_k(\tilde{M})} \varepsilon_T^{M\tilde{M}} \quad (2.26)$$

where

$$\sigma_k(N) = \begin{cases} 1 & N = 0', \dots, 3' \\ 0 & N = 4', 5, \dots, 9. \end{cases} \quad (2.27)$$

Untwisting the R-R vertex operator, $t_{T,AB}$ gets replaced with

$$t_{AB} = \left(\frac{\alpha'}{2}\right)^{1/2} (C^\dagger i \Gamma^{0'} \Gamma^{1'} \Gamma^{2'} \Gamma^{3'} \not{k})_{AB}. \quad (2.28)$$

As before, the primed indices denote a frame in which the directions $0', \dots, 3'$ are transverse to k . The NS-NS vertex operator is for a graviton component in the full 10-dimensional space-time while the R-R vertex operator is that for a self-dual 4-form field. Further details on the massless vertices can be found in Appendix B.1.

⁶The coefficient of the R-R polarization t_{AB} depends on the chiralities of the spin fields $\tilde{\Theta}^A, \Theta^B$. Since the worldsheet theory is chiral, we will assume throughout the paper that all spin fields $\Theta, \tilde{\Theta}$ have positive chirality.

Moving now to level one there are two different types of NS-NS vertex operators that we can construct. The first has the form in the $(-1, -1)$ picture [23]

$$V_{1,k}^{(-1,-1)}(z, \bar{z}) = g_c \frac{2}{\alpha'} \varepsilon_{MN;\tilde{M}\tilde{N}} \psi^M i\partial X^N e^{-\phi} \tilde{\psi}^{\tilde{M}} i\bar{\partial} X^{\tilde{N}} e^{-\tilde{\phi}} e^{ik \cdot X}, \quad (2.29)$$

where the first two and last two indices of $\varepsilon_{MN;\tilde{M}\tilde{N}}$ are symmetric and traceless. The second vertex in the $(-1, -1)$ picture is given by

$$V_{2,k}^{(-1,-1)}(z, \bar{z}) = g_c \alpha_{MNL;\tilde{M}\tilde{N}\tilde{L}} \psi^M \psi^N \psi^L e^{-\phi} \tilde{\psi}^{\tilde{M}} \tilde{\psi}^{\tilde{N}} \tilde{\psi}^{\tilde{L}} e^{-\tilde{\phi}} e^{ik \cdot X}. \quad (2.30)$$

where the first three and last three indices of $\alpha_{MNL;\tilde{M}\tilde{N}\tilde{L}}$ are antisymmetric. For physical states the contraction of any index in $\varepsilon_{MN;\tilde{M}\tilde{N}}$ or $\alpha_{MNL;\tilde{M}\tilde{N}\tilde{L}}$ with k^M vanishes. There are 44×44 independent polarizations for V_1 and 84×84 for V_2 , but only one of each type is a ten-dimensional Lorentz scalar. These scalars correspond to the twisted vertex operators, $V_{1,T}$ and $V_{2,T}$. For $V_{1,T}$ the unnormalized polarization tensor is given by

$$\varepsilon_T^{MN;\tilde{M}\tilde{N}} = \frac{1}{2}(\hat{\eta}^{M\tilde{M}}\hat{\eta}^{N\tilde{N}} + \hat{\eta}^{M\tilde{N}}\hat{\eta}^{N\tilde{M}}) - \frac{1}{9}\hat{\eta}^{MN}\hat{\eta}^{\tilde{M}\tilde{N}}, \quad (2.31)$$

where $\hat{\eta}^{MN} \equiv \eta^{MN} - \frac{k^M k^N}{k^2}$. For $V_{2,T}$ the unnormalized polarization is

$$\alpha_T^{MNL;\tilde{M}\tilde{N}\tilde{L}} = \frac{1}{3!^2}(\hat{\eta}^{M\tilde{M}}\hat{\eta}^{N\tilde{N}}\hat{\eta}^{L\tilde{L}} \mp (5 \text{ permutations})). \quad (2.32)$$

We will also need the vertex operators in the $(0, 0)$ picture, and these are given by

$$V_{1,T,k}^{(0,0)} = -g_c \left(\frac{2}{\alpha'}\right)^2 \varepsilon_{T,MN;\tilde{M}\tilde{N}} (i\partial X^M (i\partial X^N + \frac{\alpha'}{2} k \cdot \psi \psi^N) + \frac{\alpha'}{2} \partial \psi^M \psi^N) \\ \times (i\bar{\partial} X^{\tilde{M}} (i\bar{\partial} X^{\tilde{N}} + \frac{\alpha'}{2} k \cdot \tilde{\psi} \tilde{\psi}^{\tilde{N}}) + \frac{\alpha'}{2} \bar{\partial} \tilde{\psi}^{\tilde{M}} \tilde{\psi}^{\tilde{N}}) e^{ik \cdot X}, \quad (2.33)$$

$$V_{2,T,k}^{(0,0)} = -g_c \frac{2}{\alpha'} \alpha_{T,MNL;\tilde{M}\tilde{N}\tilde{L}} (3 i\partial X^M + \frac{\alpha'}{2} k \cdot \psi \psi^M) \psi^N \psi^L \\ \times (3 i\bar{\partial} X^{\tilde{M}} + \frac{\alpha'}{2} k \cdot \tilde{\psi} \tilde{\psi}^{\tilde{M}}) \tilde{\psi}^{\tilde{N}} \tilde{\psi}^{\tilde{L}} e^{ik \cdot X}. \quad (2.34)$$

The vertex operators for level-one R-R string states in the $(-1/2, -1/2)$ picture were shown to have the form [23]

$$V_{3,k}^{(-1/2,-1/2)}(z, \bar{z}) = g_c (i\bar{\partial} X^M \tilde{\Theta} - \frac{1}{8} \frac{\alpha'}{2} \tilde{\psi}^M (\not{k} \tilde{\psi} \tilde{\Theta}))^A e^{-\tilde{\phi}/2} \\ \times t_{MA;NB} (i\partial X^N \Theta - \frac{1}{8} \frac{\alpha'}{2} \psi^N (\not{k} \psi \Theta))^B e^{-\phi/2} e^{ik \cdot X}, \quad (2.35)$$

with the twisted (scalar) states given by [15]⁷

$$t_{T,MA;NB} = \left(\frac{\alpha'}{2}\right)^{1/2} (C^\dagger \not{k} (\hat{\eta}_{MN} - \frac{1}{9} \hat{\Gamma}_M \hat{\Gamma}_N))_{AB}, \quad \hat{\Gamma}^M = \Gamma^M - \not{k} k^M / k^2. \quad (2.36)$$

If we now take a linear combination of $V_{1,T}$, $V_{2,T}$ and $V_{3,T}$ and solve for (2.19) up to spurious terms, we find that

$$V_T = V_{1,T} + V_{2,T} + \frac{1}{\sqrt{2}} V_{3,T} \quad (2.37)$$

does the trick up to an overall normalization constant. This vertex can then be untwisted in a fashion similar to the massless case. Further technical details including a careful derivation of the supersymmetry transformations and the normalization factors can be found in Appendix B.2.

⁷Again we assume that all spin fields $\Theta, \tilde{\Theta}$ in (2.35) have positive chirality.

3 Three-Point Functions

In this section we compute the three-point functions for any combination of chiral primaries and level-one primaries, where we use the properly normalized and untwisted versions of the vertex operators (2.24,2.37).

The chiral primaries are operators of the form

$$\mathcal{O}_{\text{CP}} = C_{I_1 I_2 \dots I_J} \text{Tr}[\Phi^{I_1} \Phi^{I_2} \dots \Phi^{I_J}], \quad (3.1)$$

where $C_{I_1 I_2 \dots I_J}$ is symmetric and traceless. These are states in the $[0, J, 0]$ representation of $\text{SO}(6)$ and have the protected dimension $\Delta_{\text{CP}} = \Delta_0 = J$.

The other operators that make up the three-point functions are assumed to be primaries dual to string states at level one. These operators also transform in the $[0, J, 0]$ representation, but now the bare dimension is $\Delta_0 = J + 2$ [24]. Operators of this type include those with the form

$$\mathcal{O}_J = \text{Tr}[\Phi^I \Phi_I Z^J] + \dots \quad (3.2)$$

where Φ_I are the six scalars of $\mathcal{N} = 4$ SYM, $Z = (\Phi_5 + i\Phi_6)/\sqrt{2}$, and the ellipsis refers to different positions of the Φ_I in the trace, such that the corresponding magnon momenta are at level one [24, 25]. If $J > 0$ then levels higher than one are possible, but level one is the only possibility for the Konishi operator

$$\mathcal{O}_{\text{K}} = \mathcal{O}_{J=0} = \text{Tr}[\Phi^I \Phi_I]. \quad (3.3)$$

Our results are applicable if $\Delta \gg 1$. For the chiral primaries this means that $J \gg 1$, but for the level-one primaries small values of J are possible, as long as the R -charges in the full three-point function are conserved.

One should note that correlators with two or more twisted operators must vanish due to a supersymmetric Ward identity.⁸ The twisted vertices are annihilated by the sixteen supercharges $Q^A = Q_{\text{L}}^A - i Q_{\text{R}}^A$ and, since they have non-zero momenta, they can be written as the action of those supercharges on another vertex operator $V_{\text{T}} = \{Q^A, X\}$. It is then straightforward to see that

$$\langle V_{1,\text{T}} V_{2,\text{T}} V_{3,\text{T}} \rangle = \langle X [Q^A, V_{2,\text{T}}] V_{3,\text{T}} \rangle + \langle X V_{2,\text{T}} [Q^A, V_{3,\text{T}}] \rangle = 0. \quad (3.4)$$

If instead we consider the correlator $\langle V_{1,\text{T}} V_{2,\text{T}} V_3 \rangle$, the same argument follows except that we must choose the supersymmetries that also annihilate the untwisted vertex V_3 , which correspond to the eight supercharges $Q_{\alpha\tilde{a}}^- = Q_{\alpha\tilde{a}}^{\text{L}} - i Q_{\alpha\tilde{a}}^{\text{R}}$. In the case $\langle V_{1,\text{T}} V_2 V_3 \rangle$ of two untwisted vertices with momenta $k_{2,3}$, if one of them is closed under $Q_{\alpha\tilde{a}}^-$, $Q_{\dot{\alpha}}^{+,\tilde{a}} = Q_{\dot{\alpha}}^{\text{L},\tilde{a}} + i Q_{\dot{\alpha}}^{\text{R},\tilde{a}}$, then the other is closed under another set of Q^\pm , which is obtained from $Q_{\alpha\tilde{a}}^-$, $Q_{\dot{\alpha}}^{+,\tilde{a}}$ by a rotation that takes k_2 into k_3 . The rotation mixes α with $\dot{\alpha}$ components and lower with upper \tilde{a} components, and thus there is no common supercharge that annihilates both untwisted vertex operators. Hence, only correlators with at least two untwisted vertices can be non-zero.

⁸We thank E. Witten for a discussion on this point.

3.1 Three Chiral Primaries

In this subsection we compute the three-point function for three chiral primaries, showing that it is consistent with the result in [18].

As shown in Appendix B.1, the normalized vertex operator satisfying the primary operator condition is given by

$$W_k = -\frac{1}{4}(W_{1,k} + \frac{1}{\sqrt{2}}W_{2,k}). \quad (3.5)$$

We have included an extra sign that is not determined by the $Q_L = \pm i Q_R$ conditions nor by normalization, but is necessary to be consistent with [18]. We will continue to use this choice of sign in the later subsections.

The three-point amplitudes are of two general types, with either zero or two R-R vertex operators inside the amplitude. For NS-NS vertex operators only, two of the vertex operators should be given in the $(-1, -1)$ picture, while one should be in the $(0, 0)$ picture to have the correct ghost charge. In Appendix C.2 it is shown that

$$\langle W_{1,k_1}^{(-1,-1)} W_{1,k_2}^{(-1,-1)} W_{1,k_3}^{(0,0)} \rangle = \frac{4g_c^3 \alpha' \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}}{J_1^2 J_2^2 J_3^2} (J_1^4 + 6J_1^2 J_2^2 + J_2^4 + 6J_1^2 J_3^2 + 6J_2^2 J_3^2 + J_3^4) \quad (3.6)$$

For two R-R vertex operators and one NS-NS, we can choose the R-R vertex operators to be in the $(-1/2, -1/2)$ picture and the NS-NS in the $(-1, -1)$ picture. In Appendix C.2 we then show that

$$\langle W_{1,k_1}^{(-1,-1)} W_{2,k_2}^{(-1/2,-1/2)} W_{2,k_3}^{(-1/2,-1/2)} \rangle = \frac{32g_c^3 \alpha' \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}}{J_1^2 J_2 J_3} (3J_1^2 + J_2^2 + J_3^2) \quad (3.7)$$

Taking the combination in (3.5) and using (3.6) and (3.7), we find

$$\langle W_{k_1} W_{k_2} W_{k_3} \rangle = g_c^3 \alpha' \frac{\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}^5}{J_1^2 J_2^2 J_3^2}, \quad (3.8)$$

with $\tilde{\Sigma}$ and $\tilde{\alpha}_i$ defined in (2.2).

Using (1.3), (1.5), (2.3), (2.4) and the AdS/CFT dictionary value $g_c = \pi^{3/2}/N$, we obtain

$$\mathcal{C}_{123} \approx \frac{1}{N} \sqrt{J_1 J_2 J_3} \langle C^{J_1} C^{J_2} C^{J_3} \rangle, \quad (3.9)$$

reproducing the result in [18].

3.2 Two Chiral Primaries

We will next compute the correlator for two chiral primaries and one massive operator $\langle W_{k_1} W_{k_2} V_{k_3} \rangle$. As shown in Appendix B, the normalized vertex operators which satisfy the primary condition are (3.5) and

$$V_k = -\frac{1}{16}(V_{1,k} + V_{2,k} + \frac{1}{\sqrt{2}}V_{3,k}). \quad (3.10)$$

Here the overall minus sign is again not determined with the primary operator condition, but we make this choice to keep the sign of the structure constants obtained consistent with

the ones at weak coupling. Again, we have two types of three-point amplitudes. The first, with no R-R vertex operators, consists of $\langle W_{1,k_1} W_{1,k_2} V_{1,k_3} \rangle$ and $\langle W_{1,k_1} W_{1,k_2} V_{2,k_3} \rangle$, where two operators have ghost charges $(-1, -1)$ and one has $(0, 0)$. The second group, with two R-R vertices, is formed by $\langle W_{2,k_1} W_{2,k_2} V_{1,k_3} \rangle$, $\langle W_{2,k_1} W_{2,k_2} V_{2,k_3} \rangle$ and $\langle W_{1,k_1} W_{2,k_2} V_{3,k_3} \rangle$, with the R-R operators in the $(-1/2 - 1/2)$ frame and the NS-NS in the $(-1, -1)$.

All these correlators are computed in Appendix C.3. Using (3.5) and (3.10) we find

$$\langle W_{k_1} W_{k_2} V_{k_3} \rangle = g_c^3 \alpha'^2 \frac{\alpha_1^2 \alpha_2^2 \Sigma^4 \tilde{\alpha}_3^2 \tilde{\Sigma}^2}{\Delta_1^2 \Delta_2^2 \Delta_3^4}. \quad (3.11)$$

For the massless states (3.5), we have $\Delta = J$. When $J_1 = J_2 \equiv J$ and $J_3 = 0$, i.e. $\Delta \equiv \Delta_3 = 2/\sqrt{\alpha'}$, the correlator is given by

$$\langle W_{k_1} W_{k_2} V_{k_3} \rangle = \frac{g_c^3}{16} \alpha'^2 (J + \frac{1}{2} \Delta)^4. \quad (3.12)$$

Using (1.3) and (1.5), the structure constant becomes

$$\mathcal{C}_{123} \approx \frac{\sqrt{\pi}}{4N\sqrt{\lambda}} 2^{-\Delta} J^{2(1-J)} (J - \frac{1}{2} \Delta)^{J-\Delta/2-1/2} (J + \frac{1}{2} \Delta)^{J+\Delta/2+3/2}, \quad (3.13)$$

where we used that $\langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle = 1/\pi^{3/2}$ in this case.

We should say that the procedure we are using here requires that the operator dimensions satisfy a triangle rule, $\Delta_1 + \Delta_2 \geq \Delta_3$, which means that the result in (3.13) is only valid if $J \geq \Delta/2$. As we approach the extremal case $\Delta_1 + \Delta_2 = \Delta_3$ the saddle point approaches the boundary [19] and (1.3) develops a singularity [26]. The same thing can happen for three chiral primaries, but in that case the coupling also goes to zero and the overall structure constant stays finite [26]. That is not the case here as the coupling is non-zero at the extremal value. Furthermore, because of R -charge conservation it is not possible to go beyond the extremal case for three chiral primaries, but it is possible when one operator is non-chiral. This all suggests a breakdown of the particle approximation at the extremal point.⁹

3.3 One Chiral Primary

For the computation of the correlator $\langle V_{k_1} V_{k_2} W_{k_3} \rangle$ we use again the normalizations (3.5) and (3.10). The amplitudes with no R-R vertex operators are $\langle V_{1,k_1} V_{1,k_2} W_{1,k_3} \rangle$, $\langle V_{2,k_1} V_{2,k_2} W_{1,k_3} \rangle$ and $\langle V_{2,k_1} V_{1,k_2} W_{1,k_3} \rangle$, where two operators have ghost charges $(-1, -1)$ and one has $(0, 0)$. The second group, with two R-R vertices in the $(-1/2, -1/2)$ picture and the NS-NS vertex in the $(-1, -1)$ picture, is formed by $\langle V_{1,k_1} V_{3,k_2} W_{2,k_3} \rangle$, $\langle V_{2,k_1} V_{3,k_2} W_{2,k_3} \rangle$ and $\langle V_{3,k_1} V_{3,k_2} W_{1,k_3} \rangle$. Using the formulas for these correlators from Appendix C.4, we get in this case a more complicated result

$$\langle V_{k_1} V_{k_2} W_{k_3} \rangle = \frac{g_c^3 \Sigma^4}{\Delta_1^4 \Delta_2^4 \Delta_3^2} \left(-\frac{1}{2} \Omega_1^2 \alpha_3^4 + \alpha_3^2 (3\Omega_1^2 + 2\Omega_1 \alpha_3 + \alpha_3^2) \Omega_4 - \frac{1}{2} \Omega_1 (\Omega_1 - 4\alpha_3) \Omega_4^2 + \Omega_4^3 \right), \quad (3.14)$$

⁹One could analytically continue the integral, but in general one would find complex structure constants which would violate unitarity. It would be interesting to better understand this point.

where

$$\Omega_1 = \alpha_1 + \alpha_2, \quad \Omega_4 = \alpha' \alpha_1 \alpha_2 \alpha_3 \Sigma. \quad (3.15)$$

The result simplifies if we assume all $J_i \ll \lambda^{1/4}$, in which case we find

$$\langle V_{k_1} V_{k_2} W_{k_3} \rangle \approx \frac{g_c^3}{2} (1 + 2\sqrt{\alpha'} J_3 + \dots). \quad (3.16)$$

Using (1.3), (1.5) we obtain in this limit that the structure constant is

$$\mathcal{C}_{123} \approx \frac{\pi^2}{N} 2^{-J_3} (\lambda^{1/4} + \frac{3}{2} J_3 + \mathcal{O}(\lambda^0)) \langle \psi_{J_1} \psi_{J_2} \psi_{J_3} \rangle, \quad (3.17)$$

Choosing $J_1 = 0$ and $J_2 = J_3 = J$, it becomes

$$\mathcal{C}_{123} \approx \frac{1}{N} \pi^{1/2} (\lambda^{1/4} + \frac{3}{2} J) 2^{-J}. \quad (3.18)$$

3.4 Zero Chiral Primaries

Finally, we compute the three-point function of three Konishi-like operators $\langle V_{k_1} V_{k_2} V_{k_3} \rangle$, using the vertex (3.10). The amplitudes with no R-R vertex operators are $\langle V_{1,k_1} V_{1,k_2} V_{1,k_3} \rangle$, $\langle V_{2,k_1} V_{2,k_2} V_{2,k_3} \rangle$, $\langle V_{2,k_1} V_{2,k_2} V_{1,k_3} \rangle$ and $\langle V_{1,k_1} V_{1,k_2} V_{2,k_3} \rangle$, where two operators have ghost charges $(-1, -1)$ and one has ghost charge $(0, 0)$. The correlators with two R-R vertices in the $(-1/2, -1/2)$ picture and one NS-NS vertex in the $(-1, -1)$ picture are $\langle V_{1,k_1} V_{3,k_2} V_{3,k_3} \rangle$ and $\langle V_{2,k_1} V_{3,k_2} V_{3,k_3} \rangle$. Once again, the details of the calculation of these amplitudes are given in Appendix C.5. Putting all cases together we get

$$\begin{aligned} \langle V_{k_1} V_{k_2} V_{k_3} \rangle &= \frac{g_c^3 \Sigma^4}{\Delta_1^4 \Delta_2^4 \Delta_3^4} \left(\frac{1}{2} \Sigma_2^4 + \frac{9}{2} \Sigma_4^2 + (2\Sigma_2^3 - 3\Sigma_2^2 \Sigma_2^2 + 6\Sigma_2^2 \Sigma_4 + 3\Sigma_2 \Sigma_4) \alpha' \Sigma_4 \right. \\ &\quad \left. + \frac{1}{2} (3\Sigma^4 + 7\Sigma_2^2 - 8\Sigma^2 \Sigma_2 + 6\Sigma_4) (\alpha' \Sigma_4)^2 - (\Sigma^2 - 3\Sigma_2) (\alpha' \Sigma_4)^3 + (\alpha' \Sigma_4)^4 \right). \end{aligned} \quad (3.19)$$

where

$$\Sigma_2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3, \quad \Sigma_4 = \alpha_1 \alpha_2 \alpha_3 \Sigma, \quad (3.20)$$

If we now consider the case of all $J_i = 0$ (i.e. $\Delta_i = 2/\sqrt{\alpha'}$), we obtain the three-point function (C.68) for three Konishi operators

$$\langle V_{k_1} V_{k_2} V_{k_3} \rangle \approx g_c^3 \frac{3^8}{29}. \quad (3.21)$$

Plugging this result in (1.3), we find the structure constant in (1.8),

$$\mathcal{C}_{123} \approx \frac{1}{N} 64 \pi^{1/2} \lambda^{1/4} \left(\frac{3}{4} \right)^{3\lambda^{1/4} + 5/2}. \quad (3.22)$$

4 Discussion

In this work we computed the three-point functions for certain short scalar operators in $\mathcal{N} = 4$ SYM at strong coupling. The set of operators we considered are the chiral primaries as well as primaries which are dual to string states at the first massive level, which includes the Konishi operator. Following the analysis in [27, 15, 19], the action is dominated by a saddle point composed of three geodesic trajectories propagating from the AdS boundary to a common intersection point, whose location is determined by the conservation of the canonical momentum. For short operators (whose dimensions scale as $\Delta \sim \sqrt[4]{\lambda}$), the interaction region is small compared to the AdS radius, and hence the coupling at the intersection point is well approximated by a flat-space correlator of appropriate type IIB string vertex operators. For primary operators, the vertex operators have to satisfy a twisted version of the $Q_L = i Q_R$ equation. This constraint uniquely singles out the appropriate string states at the massless and first massive level.¹⁰ The corresponding vertex operators are certain twisted linear combinations of ten-dimensional flat-space NS-NS and R-R scalar modes. In addition, there is a contribution from the wave-function overlap on the S^5 part of the space.

Identifying the correct string vertex operators corresponding to given local operators in the boundary CFT is a long-standing open problem in the AdS/CFT duality. The procedure applied here could provide insights as to what the full $\text{AdS}_5 \times S^5$ vertex operators might look like. It should also be mentioned that the general method used here and in [15] is not just restricted to $\mathcal{N} = 4$ SYM with its IIB dual in $\text{AdS}_5 \times S^5$. It should in principle apply to all AdS/CFT theories with a string theory as a gravitational dual, up to the construction of the appropriate vertex operators. Here we have relied on $\mathcal{N} = 4$ supersymmetry to identify the vertex operators.

Our results for the structure constants for any combination of scalar massless and level-one states are given in (1.7), (3.9), (3.13), and (3.17) above. Carrying out the contractions of the primary vertices requires substantial effort due to the combinatorial complexity. Moreover, many intermediate results (partial correlators) contain unappealing prime factors. In comparison, the complete three-point correlators are astonishingly simple. Of course this is not completely unexpected, as planar $\mathcal{N} = 4$ SYM is believed to be integrable.¹¹ The simplicity of our results suggests that integrability will play an equally important role in the computation of three-point functions as it did for the spectral problem. But even disregarding integrability, our results demonstrate that the holographic computation of massive correlators, albeit technically involved, are in fact feasible.

We find that the structure constants for massive states typically are exponentially small in the scaling dimension $\Delta \sim \sqrt[4]{\lambda}$. However, the suppression is still moderate in interesting regimes. For example, the structure constant (1.8) for three Konishi states is $\mathcal{C}_{123} \approx 2.4/N$ for $\lambda = 10^3$, a regime in which the Konishi dimension clearly shows GKP behavior [2] and hence the semiclassical approximation should be applicable. Also, in the case of one chiral primary and two massive primaries, the suppression is only exponential in the R-charge J of the chiral primary operator.

There are several directions for future work: It would be very interesting to compute sub-

¹⁰There is only one long multiplet at the first massive level, and hence a unique solution to $Q_L = i Q_R$.

¹¹See [1] for an introductory review.

leading corrections to our results in the α' expansion. Subleading terms can be obtained by inserting the expansions for the dimensions $\Delta_{1,2,3}$ of the respective operators, and by calculating loop corrections to the worldsheet amplitude of the appropriate untwisted vertex operators. Analytic corrections to our results will be an expansion in powers of α' . There is a possibility of non-analytic corrections that come with half-integer powers, as happens for the operator dimensions [7]. But we suspect that such terms only enter the correlators indirectly, through the corrections to the operator dimensions. The three-point vertices are functions of the momenta, whose corrections will come in half-integer powers of α' through their dependence on the scaling dimensions. It is also likely that (1.3) itself will get modified as we move away from the point-like limit, as indicated by the singularities that appear as we approach the extremal limit.

Another exciting venture would be to study four-point functions of short operators, aiming at a holographic operator product expansion. For related references, see e.g. [28, 29]. It can be hoped that the methods used here will also be useful for higher-point correlators. Likewise, information from the four-point correlation functions of protected operators can be used to check the validity of our results. For example, in [29] there is a prediction for the structure constant between two operators dual to the dilaton and a third operator dual to a higher-spin state on the graviton Regge trajectory. It would be interesting to find their result for the spin 4 operator, which is at the first massive level, using the methods here.

Finally, a bigger goal would be to employ integrability in the computation of short-string correlators. Based on the experience that integrability techniques usually start with a large-volume limit, this would probably require the adaption of a full thermodynamic Bethe ansatz [30] or quantum spectral curve [31] machinery.

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A Bosonization Setup and OPE's

We follow the conventions of Polchinski [32]. In all products of operators evaluated at the same worldsheet point, normal ordering is implicitly assumed.

Bosonization. For OPE's and correlators involving R-R vertices, we use bosonization as described in [22, 33]. Namely, the ten real fermions ψ^M are written as five complex bosons ϕ^j . First convert the ψ^M to a Cartan–Weyl basis,

$$\psi^{\pm e_0} = \frac{1}{\sqrt{2}}(\pm\psi^0 + \psi^1), \quad \psi^{\pm e_j} = \frac{1}{\sqrt{2}}(\psi^{2j} \pm i\psi^{2j+1}), \quad j = 1, \dots, 4. \quad (\text{A.1})$$

Here, e_j denotes a unit vector in direction j on the Euclidean weight lattice of $\mathfrak{o}(1, 9)$, where $\pm e_i \pm e_j$ form the non-zero roots. Now write the $\psi^{\pm e_j}$ as

$$\psi^{\pm e_j}(z) = e^{\pm e_j \cdot \phi(z)} c_{\pm e_j}, \quad c_{\pm e_j} = (-1)^{N_1 + \dots + N_{j-1}} = e^{i\pi(\pm e_j \cdot M \cdot N)}, \quad (\text{A.2})$$

where $\phi = (\phi^1, \dots, \phi^5)$ is the vector of bosonic fields, and $c_{\pm e_j}$ is a ‘‘cocycle factor’’ that ensures the right commutation relations: $N = (N_1, \dots, N_5)$ is the vector of fermion number operators N_j for the fermions $\psi^{\pm e_j}$, and M is a lower-triangular sign matrix, given in Appendix E of [33]. More generally, any operator of conformal weight λ can be written as $e^{\lambda \cdot \phi} c_\lambda$. In particular, the twist fields Θ^A become

$$\Theta^A = e^{A \cdot \phi} c_A, \quad c_A = e^{i\pi A \cdot M \cdot N}, \quad (\text{A.3})$$

where $A = (\pm, \dots, \pm)/2$ is a five-component spinor weight. Now the OPE for two bosonized operators of weight λ, λ' takes the form

$$\begin{aligned} e^{\lambda \cdot \phi(z_1)} c_\lambda e^{\lambda' \cdot \phi(z_2)} c_{\lambda'} &= z_{12}^{\lambda \cdot \lambda'} e^{i\pi \lambda \cdot M \cdot \lambda'} : e^{\lambda \cdot \phi(z_1) + \lambda' \cdot \phi(z_2)} : c_{\lambda + \lambda'} \\ &= z_{12}^{\lambda \cdot \lambda'} e^{i\pi \lambda \cdot M \cdot \lambda'} : e^{(\lambda + \lambda') \cdot \phi(z_2)} \left(1 + z_{12} \lambda \cdot \partial \phi(z_2) + \mathcal{O}(z_{12})^2 \right) : c_{\lambda + \lambda'} \end{aligned} \quad (\text{A.4})$$

Including the bosonic ghost field $\phi_g \equiv \phi^6$, the weight lattice becomes Lorentzian with a ‘‘timelike’’ sixth dimension, and with the sign matrix M now given by (E.7) in [33]. The OPE formula (A.4) continues to hold. For example,

$$\psi^{\pm e_j} e^{-\phi_g} = e^{\lambda \cdot \phi} c_\lambda, \quad \lambda = \pm e_j + e_6, \quad c_\lambda = e^{i\pi \lambda \cdot M \cdot N}. \quad (\text{A.5})$$

Using the OPE (A.4), the correlator of n fields with (six-dimensional) weights $\lambda_1, \dots, \lambda_n$ becomes

$$\langle e^{\lambda_1 \cdot \phi(z_1)} c_{\lambda_1} \dots e^{\lambda_n \cdot \phi(z_n)} c_{\lambda_n} \rangle = \delta(\sum_j \lambda_j - 2e_6) \prod_{1 \leq j < k \leq n} z_{jk}^{\lambda_j \cdot \lambda_k} e^{i\pi \lambda_j \cdot M \cdot \lambda_k}, \quad (\text{A.6})$$

where the scalar product in the exponents is Lorentzian with signature $(1, \dots, 1, -1)$.

Gamma Matrices. In the Cartan–Weyl basis, we use the gamma matrices

$$(\Gamma^{\pm e_k})^A_B = \sqrt{2} e^{i\pi(\pm e_k) \cdot M \cdot A} \delta(\pm e_k + A, B), \quad k = 1, \dots, 5, \quad A = (\pm, \dots, \pm)/2. \quad (\text{A.7})$$

They are related to the gamma matrices Γ^M in the covariant basis in the same way $\psi^{\pm e_j}$ are related to ψ^M (A.1). For the charge conjugation matrix we use

$$\begin{aligned} C^{AB} &= e^{i\pi 3/4} \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \\ &= \delta(A + B) \begin{cases} e^{-i\pi A_{-1/2} \cdot M \cdot A_{-1/2}} & \text{for } A \text{ with positive chirality} \\ -e^{-i\pi A_{+1/2} \cdot M \cdot A_{+1/2}} & \text{for } A \text{ with negative chirality} \end{cases} \end{aligned} \quad (\text{A.8})$$

Here, σ^j are the usual Pauli matrices, and $A_{\pm 1/2} = (A, \pm \frac{1}{2})$ is the spinor weight A extended by the ghost weight $\pm 1/2$.

OPE's. The non-vanishing OPE's among the elementary fields are

$$X_i^M X_j^N = -\frac{\alpha'}{2} \eta^{MN} \log |z_{ij}|^2 + :X_i^M X_j^N:, \quad \psi_i^M \psi_j^M = \frac{\eta^{MN}}{z_{ij}} + :\psi_i^M \psi_j^N:, \quad (\text{A.9})$$

where subscript i denotes dependence on worldsheet coordinates (z_i, \bar{z}_i) , and $z_{ij} \equiv z_i - z_j$. In three-point correlators, one often needs to use the OPE of a coordinate derivative with two plane-wave factors, which takes the convenient form

$$i\partial X_1^M e^{ik_2 \cdot X_2} e^{ik_3 \cdot X_3} = \frac{\alpha'}{2} \frac{k_2 z_{23}}{z_{12} z_{13}} e^{ik_2 \cdot X_2} e^{ik_3 \cdot X_3} + \dots \quad \text{for } k_1 + k_2 + k_3 = 0. \quad (\text{A.10})$$

For the correlator of plane-wave factors, we use

$$\left\langle \prod_{j=1}^n :e^{ik_j \cdot X_j}: \right\rangle = \delta(\sum_j k_j) \prod_{1 \leq j < k \leq n} |z_{jk}|^{\alpha' k_j \cdot k_k}. \quad (\text{A.11})$$

where for simplicity of notation we write $\delta(\sum_j k_j) = (2\pi)^{10} \delta^{10}(\sum_j k_j)$. The OPE's for bosonized fields are given in (A.4). Further relevant OPE's for bosonized fields ϕ^j are ($j = 1, \dots, 6$)

$$\phi^i(z_1) \phi^j(z_2) = \gamma^{ij} \log |z_{12}|^2 + :\phi^i(z_1) \phi^j(z_2):, \quad \gamma = \text{diag}(1, 1, 1, 1, 1, -1), \quad (\text{A.12})$$

$$\partial \phi^j(z_1) e^{\lambda \cdot \phi(z_2)} = \frac{\lambda \cdot e_j}{z_{12}} e^{\lambda \cdot \phi(z_2)} + :\partial \phi^j(z_1) e^{\lambda \cdot \phi(z_2)}: \quad (\text{A.13})$$

When computing three-point functions, the worldsheet integrations will be replaced by the corresponding c, \tilde{c} ghosts. Three-point correlators are then accompanied by the following expectation value

$$\langle c_1 c_2 c_3 \tilde{c}_1 \tilde{c}_2 \tilde{c}_3 \rangle = |z_{12}|^2 |z_{13}|^2 |z_{23}|^2 \quad (\text{A.14})$$

For simplicity of notation, in the following sections we will omit these ghosts from the correlators and their presence will be implicit, making the correlators have no dependence on the worldsheet coordinates.

Point-Splitting. We define products of fields evaluated at the same worldsheet coordinate by a point-splitting regularization, see e.g. [23]. Specifically,

$$\begin{aligned} \psi^{+e_i} \psi^{+e_i}(z) &= \psi^{-e_i} \psi^{-e_i}(z) = 0, \\ \psi^{+e_i} \psi^{-e_i}(z) &= -\psi^{-e_i} \psi^{+e_i}(z) = \partial \phi^i(z). \end{aligned} \quad (\text{A.15})$$

Moreover, for $i \neq j$ (the two signs \pm are independent),

$$\begin{aligned} \psi^{\pm e_i} \psi^{\pm e_j} \Theta^A(z) &= e^{i\pi(\pm e_i) \cdot M \cdot (\pm e_j)} e^{i\pi(\pm e_i \pm e_j) \cdot M \cdot A}, \\ &\cdot \begin{cases} (\pm e_i \pm e_j) \cdot \partial \phi \Theta^{A \pm e_i \pm e_j}(z) & \text{for } (\pm e_i \pm e_j) \cdot A = -1, \\ \Theta^{A \pm e_i \pm e_j}(z) & \text{for } (\pm e_i \pm e_j) \cdot A = 0, \\ 0 & \text{for } (\pm e_i \pm e_j) \cdot A = 1. \end{cases} \end{aligned} \quad (\text{A.16})$$

Here, the boldface Θ^A is defined as in (A.3), but also for non-spinorial weight A . Applying the point-splitting prescription to the term $\psi^{\pm e_k}(\psi\Theta)^A$ appearing in the vertex (2.35) yields

$$\psi^{\pm e_k}(\psi\Theta)^A = \delta(A_k \mp \frac{1}{2})4\sqrt{2}e^{\pm i\pi e_k \cdot M \cdot A} \Theta^{A \pm e_k} + (2A \pm 6e_k) \cdot \partial\phi (\Gamma^{\pm e_k} \Theta)^A, \quad (\text{A.17})$$

This also implies

$$(\psi\psi\Theta)^A = 36(A \cdot \partial\phi)\Theta^A. \quad (\text{A.18})$$

When computing correlation functions, terms like (A.17) will always appear multiplied by the polarization tensor $t_{\tilde{M}\tilde{A},MA}$ which obeys $t_{\tilde{M}\tilde{A},MA}(\Gamma^M)^A_C = 0$, so we are free to add terms of the form $B \cdot \partial\phi (\Gamma^{\pm e_k})^A_B \Theta^B$ to (A.17). We will then use the following formula

$$\psi^{\pm e_k}(\psi\Theta)^A \simeq \delta(A_k \mp \frac{1}{2})4\sqrt{2}e^{\pm i\pi e_k \cdot M \cdot A} \Theta^{A \pm e_k} - \frac{8}{5}(A \mp \frac{3}{2}e_k) \cdot \partial\phi (\Gamma^{\pm e_k} \Theta)^A, \quad (\text{A.19})$$

when computing three-point functions. This will ensure a manifestly correct dependence on the worldsheet coordinates, since $(A \mp \frac{3}{2}e_k) \cdot (A \pm e_k) = 0$.

B Primary Vertex Operators

To find the massless and massive primary operators, one needs to compute the action of the supersymmetries on the flat-space vertices. We will act with the left supercharge in the (+1/2) picture on the NS-NS vertices, and compare to the action of the right supercharge in the (-1/2) picture on the R-R vertices. The supercharges are given by [23]

$$Q_R^{A(-1/2)} = \oint \frac{d\tilde{z}}{2\pi i} \tilde{\Theta}^A e^{-\tilde{\phi}_6/2} \quad (\text{B.1})$$

and, using the definition of Q_{BRST} from [32], we can change to the frame with ghost charge 1/2 in the following way¹²

$$Q_L^{A(+1/2)} = \left[i Q_{\text{BRST}}, \xi Q_L^{A(-1/2)} \right] = \frac{1}{\sqrt{2}} \left(\frac{2}{\alpha'} \right)^{1/2} \oint \frac{dz}{2\pi i} i \partial X^M (\Gamma_M \Theta)^A e^{+\phi_6/2}. \quad (\text{B.2})$$

B.1 Massless Vertex Operators

The supercharges act on the massless vertices in the following way

$$[Q_L^{A(+1/2)}, W_{1,T}^{(-1,-1)}] = i \frac{g_c}{2} \left(\frac{\alpha'}{2} \right)^{1/2} \varepsilon_{T,MN} (\not{k} \Gamma^M)^A_B \Theta^B e^{-\phi_6/2} \tilde{\psi}^N e^{-\tilde{\phi}_6} e^{ik \cdot X} \quad (\text{B.3})$$

and

$$[i Q_R^{A(-1/2)}, W_{2,T}^{(-1/2,-1/2)}] = -i \frac{g_c}{\sqrt{2}} \left(\frac{\alpha'}{2} \right)^{1/2} (\Gamma_M \not{k})^A_B \Theta^B e^{-\phi_6/2} \tilde{\psi}^M e^{-\tilde{\phi}_6} e^{ik \cdot X}. \quad (\text{B.4})$$

Since $\not{k}\not{k} = 0$ and

$$k_M \tilde{\psi}^M e^{-\tilde{\phi}_6} e^{ik \cdot X} \quad (\text{B.5})$$

¹²We will need this operator for A with positive chirality. For A with negative chirality, (B.2) would have the opposite sign. Note also that the i in front of Q_{BRST} is different from the convention in [22], but necessary to get the overall sign of the vertex operators in the (0,0) frame consistent with [32].

is a spurious term, then we can write

$$[i Q_R^{A(-1/2)}, W_{2,T}^{(-1/2,-1/2)}] = i \frac{g_c}{\sqrt{2}} \left(\frac{\alpha'}{2}\right)^{1/2} \varepsilon_{T,MN} (\not{k} \Gamma^M)^A_B \Theta^B e^{-\phi_6/2} \tilde{\psi}^N e^{-\tilde{\phi}_6} e^{ik \cdot X}. \quad (\text{B.6})$$

We then get that the massless vertex operator must be of the form $W_T \sim W_{1,T} + \frac{1}{\sqrt{2}} W_{2,T}$. Its normalization is

$$\frac{1}{g_c^2} \langle (W_{1,T}^k + \frac{1}{\sqrt{2}} W_{2,T}^k)^\dagger (W_{1,T}^k + \frac{1}{\sqrt{2}} W_{2,T}^k) \rangle = 8 + 8 = 16. \quad (\text{B.7})$$

Note that the vertex operator $(W_{1,T}^k + \frac{1}{\sqrt{2}} W_{2,T}^k)^\dagger$ satisfies the opposite condition $Q_L^A = -i Q_R^A$. The normalized massless primary state is then given by

$$W_T = \frac{1}{4} \left(W_{1,T} + \frac{W_{2,T}}{\sqrt{2}} \right). \quad (\text{B.8})$$

B.2 Massive Vertex Operators

The calculation for the massive case is quite lengthy and tedious so we will present only guidelines of the computation. Starting with $V_{1,T}$ and using that $\varepsilon_{T,MN,\tilde{M}\tilde{N}}$ is symmetric, traceless and orthogonal to k , one can obtain

$$[Q_L^{A(+1/2)}, V_{1,T}^{(-1,-1)}] = i \frac{g_c}{2} \left(\frac{2}{\alpha'}\right)^{1/2} (\not{k} \hat{\Gamma}^M)^A_B \left(i \partial X_N \Theta - \frac{1}{8} \frac{\alpha'}{2} \psi_N (\not{k} \psi \Theta) \right)^B e^{-\phi_6/2} \hat{\eta}^{NP} \times \left(\frac{1}{2} \tilde{\psi}_{\{M} i \tilde{\partial} X_{P\}} - \frac{1}{9} \hat{\eta}_{MP} (\tilde{\psi} \cdot i \tilde{\partial} X) \right) e^{-\tilde{\phi}_6} e^{ik \cdot X}. \quad (\text{B.9})$$

Since we can add total derivatives to the vertices, one can use the following equation to eliminate terms with $\partial \phi_6$

$$\frac{1}{2} \partial \phi_6 \Theta^B e^{-\phi_6/2} e^{ik \cdot X} = ik \cdot \partial X \Theta^B e^{-\phi_6/2} e^{ik \cdot X} + \frac{1}{36} (\psi \psi \Theta)^B e^{-\phi_6/2} e^{ik \cdot X}. \quad (\text{B.10})$$

Using that $\alpha_{T,MNP,\tilde{M}\tilde{N}\tilde{P}}$ is antisymmetric and noting that

$$\left(\frac{1}{48} (\Gamma_L + \frac{6}{k^2} \not{k} k_L)^{AB} \psi^L (\psi \Theta)^B + (k_L - \frac{1}{4} \not{k} \Gamma_L)^{AB} i \partial X^L \Theta^B \right) e^{-\phi_6/2} e^{ik \cdot X} \quad (\text{B.11})$$

is a spurious term, then we obtain that the action of Q_L on $V_{2,T}$ is

$$[Q_L^{A(+1/2)}, V_{2,T}^{(-1,-1)}] = i \frac{g_c}{4} \left(\frac{2}{\alpha'}\right)^{1/2} \left(\hat{\Gamma}_N \hat{\Gamma}_P (\hat{\eta}_{MR} - \frac{1}{9} \hat{\Gamma}_M \hat{\Gamma}_R) \right)^A_B \times \left(i \partial X^R \Theta - \frac{1}{8} \left(\frac{\alpha'}{2}\right) \psi^R (\not{k} \psi \Theta) \right)^B e^{-\phi_6/2} \tilde{\psi}^M \tilde{\psi}^N \tilde{\psi}^P e^{-\tilde{\phi}_6} e^{ik \cdot X}. \quad (\text{B.12})$$

Finally we need only to compute the action of $i Q_R$ on $V_{3,T}$. Using that $t_{T,\tilde{M}A,MB}$ is orthogonal to k and $(\Gamma^M C t_{T,M,N}) = 0$, eliminating terms with $\partial \phi_6$ using

$$\tilde{\partial} \tilde{\phi}_6 \tilde{\psi}^M e^{-\tilde{\phi}_6} e^{ik \cdot X} = ik \cdot \tilde{\partial} X \tilde{\psi}^M e^{-\tilde{\phi}_6} e^{ik \cdot X} + \tilde{\partial} \tilde{\psi}^M e^{-\tilde{\phi}_6} e^{ik \cdot X}, \quad (\text{B.13})$$

and noting that the following are spurious terms [23]

$$\left(\frac{2}{\alpha'} \frac{1}{2} i \tilde{\partial} X^{[M} \tilde{\psi}^{P]} - \frac{1}{2} \tilde{\psi}^M \tilde{\psi}^R \tilde{\psi}^P k_R \right) (\hat{\Gamma}_P C t_{T,MA;NB}) e^{-\tilde{\phi}_6} e^{ik \cdot X}, \quad (\text{B.14})$$

$$\left(\bar{\partial}\tilde{\psi}^M + \frac{1}{2}i\bar{\partial}X^{\{M}\tilde{\psi}^{P\}}k_P\right)(\not{k}Ct_{T,MA;NB})e^{-\tilde{\phi}_6}e^{ik\cdot X}, \quad (\text{B.15})$$

we obtain that

$$\begin{aligned} [iQ_R^{A(-1/2)}, V_{3,T}^{(-1/2,-1/2)}] = \\ \left(\frac{i}{2}\bar{\partial}X^{\{M}\tilde{\psi}^{P\}}\hat{\Gamma}_P + \frac{1}{4}\left(\frac{\alpha'}{2}\right)\tilde{\psi}^M\tilde{\psi}^P\tilde{\psi}^R\hat{\Gamma}_P\hat{\Gamma}_R\not{k}\right)\not{k}(\hat{\eta}_{MN} - \frac{1}{9}\hat{\Gamma}_M\hat{\Gamma}_N)^A_B e^{-\tilde{\phi}_6} \\ \times \left(-i\frac{g_c}{\sqrt{2}}\right)\left(\frac{2}{\alpha'}\right)^{1/2}\left(i\partial X^N\Theta - \frac{1}{8}\left(\frac{\alpha'}{2}\right)\psi^N(\not{k}\psi\Theta)\right)^B e^{-\phi_6/2}e^{ik\cdot X}. \end{aligned} \quad (\text{B.16})$$

We can now easily see from equations (B.9,B.12,B.16) that the primary constraint is satisfied for a vertex operator of the form $V_T \sim V_{1,T} + V_{2,T} + \frac{1}{\sqrt{2}}V_{3,T}$. To normalize it we only need to compute the two-point function with its adjoint, which gives

$$\frac{1}{g_c^2}\left\langle\left(V_{1,T}^k + V_{2,T}^k + \frac{1}{\sqrt{2}}V_{3,T}^k\right)^\dagger\left(V_{1,T}^k + V_{2,T}^k + \frac{1}{\sqrt{2}}V_{3,T}^k\right)\right\rangle = 44 + 84 + 128 = 256. \quad (\text{B.17})$$

So we have finally obtained that the primary vertex for the first massive level is

$$V_T = \frac{1}{16}\left(V_{1,T} + V_{2,T} + \frac{V_{3,T}}{\sqrt{2}}\right). \quad (\text{B.18})$$

B.3 Untwisting

The primary constraint (2.19) was for the twisted vertex operators W_T and V_T , but we must compute the three-point function with the untwisted versions of these operators, W and V . For the massless vertex the modifications are

$$\varepsilon_{M,\tilde{M}}^{k_i} = (-1)^{\sigma_{k_i}(\tilde{M})}\varepsilon_{T,M,\tilde{M}}^{k_i}, \quad (\text{B.19})$$

$$t_{A,B}^{k_i} = \left(\frac{\alpha'}{2}\right)^{1/2}\left(\mathcal{C}^\dagger i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}\not{k}\right)_{AB}. \quad (\text{B.20})$$

For the massive case we have the following modification of the polarization tensors

$$\varepsilon_{MN,\tilde{M}\tilde{N}}^{k_i} = (-1)^{\sigma_{k_i}(\tilde{M})+\sigma_{k_i}(\tilde{N})}\varepsilon_{T,MN,\tilde{M}\tilde{N}}^{k_i}, \quad (\text{B.21})$$

$$\alpha_{MNP,\tilde{M}\tilde{N}\tilde{P}}^{k_i} = (-1)^{\sigma_{k_i}(\tilde{M})+\sigma_{k_i}(\tilde{N})+\sigma_{k_i}(\tilde{P})}\alpha_{T,MNP,\tilde{M}\tilde{N}\tilde{P}}^{k_i}, \quad (\text{B.22})$$

$$t_{\tilde{M}A,\tilde{M}B}^{k_i} = (-1)^{\sigma_{k_i}(\tilde{M})}\left(\frac{\alpha'}{2}\right)^{1/2}\left(\mathcal{C}^\dagger i\Gamma^{0'}\Gamma^{1'}\Gamma^{2'}\Gamma^{3'}\not{k}\left(\hat{\eta}^{M\tilde{M}} - \frac{1}{9}\hat{\Gamma}^{\tilde{M}}\hat{\Gamma}^M\right)\right)_{AB}. \quad (\text{B.23})$$

Here we defined the twist factor

$$\sigma_{k_i}(M) = \begin{cases} 1 & \text{if } M = 0', \dots, 3', \\ 0 & \text{if } M = 4', 5', \dots, 9, \end{cases} \quad (\text{B.24})$$

and $0', \dots, 3'$ represent the AdS directions perpendicular to k_i . It will be useful computationally to consider the relation

$$(-1)^{\sigma_{k_i}(M)}\eta^{MN} = (-1)^{\sigma(M)}\eta^{MN} - 2\frac{k_{i,A}^M k_{i,A}^N}{\Delta_i^2}, \quad (\text{B.25})$$

where we introduce

$$\sigma(M) = \begin{cases} 1 & \text{if } M = 0, \dots, 4, \\ 0 & \text{if } M = 5, \dots, 9, \end{cases} \quad (\text{B.26})$$

and where we denote by $k_{i,A} = (k_i^0, \dots, k_i^4, 0, \dots, 0)$ the projection of k_i onto the AdS_5 part, and $-\Delta_i^2 = k_{i,A}^2$. We then have the following useful relations

$$i\Gamma^{0'} \Gamma^{1'} \Gamma^{2'} \Gamma^{3'} = \frac{1}{\Delta_i} \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \not{k}_{i,A}, \quad (\text{B.27})$$

$$\Gamma^4 \Gamma^3 \Gamma^2 \Gamma^1 \Gamma^0 (\Gamma^{M_1} \dots \Gamma^{M_n}) \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 = (-1)^{1+n+\sigma(M_1)+\dots+\sigma(M_n)} \Gamma^{M_1} \dots \Gamma^{M_n}. \quad (\text{B.28})$$

Note that for the untwisted case, the same sixteen charges $Q_{L,\alpha\bar{a}} - i Q_{R,\alpha\bar{a}}$ and $Q_{L,\dot{\alpha}}^{\bar{a}} + i Q_{R,\dot{\alpha}}^{\bar{a}}$ annihilate both the vertex operator and its adjoint.

C Vertex function contractions

C.1 Fermion VEVs

When computing correlators involving R-R vertices, we will need vacuum expectation values for a number of fermion field combinations. Here we precompute these vacuum expectation values, using bosonization as outlined in Section A.¹³ In the following, it is assumed that factors $\psi^M (\psi\Theta)^A$ get contracted with massive polarization tensors $t_{k,\tilde{M}\tilde{A},MA}$ (2.36), which obey

$$t_{k,\tilde{M}\tilde{A},MA} (\Gamma^M)^A_B = 0, \quad (\Gamma^{\tilde{M}} C)^{\tilde{A}\tilde{B}} t_{k,\tilde{M}\tilde{B},MB} = 0. \quad (\text{C.1})$$

These relations hold for the twisted as well as for the untwisted tensors. In computing the below correlators, we have added terms that vanish by these relations in order to obtain a uniform dependence on the worldsheet coordinates. It is also assumed that the term $\psi^M \psi^N \psi^P$ is always contracted with the polarization tensor (2.32), such that all terms symmetric in M, N, P are discarded. In the relations below, the “ \simeq ” signs mean that the respective relations only hold under these assumptions. The relevant VEVs are¹⁴

$$\left\langle \psi_1^M e^{-\phi_1} \Theta_2^A e^{-\phi_2/2} \Theta_3^B e^{-\phi_3/2} \right\rangle = \frac{1}{\sqrt{2} z_{12} z_{13} z_{23}} (\Gamma^M C)^{AB}, \quad (\text{C.2})$$

$$\left\langle \psi_1^M \psi_1^N \psi_1^P e^{-\phi_1} \Theta_2^A e^{-\phi_2/2} \Theta_3^B e^{-\phi_3/2} \right\rangle = \frac{-1}{2\sqrt{2} z_{12}^2 z_{13}^2} (\Gamma^{MNP} C)^{AB}, \quad (\text{C.3})$$

$$\left\langle \psi_1^M e^{-\phi_1} \psi_2^N (\psi_2 \Theta_2)^A e^{-\phi_2/2} \Theta_3^B e^{-\phi_3/2} \right\rangle \simeq \frac{2\sqrt{2}}{5 z_{12}^2 z_{23}^2} (\Gamma^{MN} C + 8\eta^{MN} C)^{AB}, \quad (\text{C.4})$$

$$\left\langle \psi_1^M \psi_1^N \psi_1^P e^{-\phi_1} \psi_2^Q (\psi_2 \Theta_2)^A e^{-\phi_2/2} \Theta_3^B e^{-\phi_3/2} \right\rangle \simeq \frac{-\sqrt{2}}{5 z_{12}^3 z_{13} z_{23}} (3\Gamma^{MNPQ} C + 12\eta^{MQ} \Gamma^{NP} C)^{AB}, \quad (\text{C.5})$$

¹³We thank Oliver Schlotterer for mentioning that VEVs involving terms $\psi^M (\psi\Theta)^A$ can alternatively be computed using the “excited spin fields” described in [34].

¹⁴The subscripts 1, 2, 3 on the fields indicate dependence on the worldsheet coordinate $z_{1,2,3}$. We assume that all spin fields Θ^A have positive chirality, and we use the shorthand notation $\Gamma^{MNP\dots}$ for $\Gamma^M \Gamma^N \Gamma^P \dots$.

$$\begin{aligned} \left\langle \psi_1^M e^{-\phi_1} \psi_2^N (\psi_2 \Theta_2)^A e^{-\phi_2/2} \psi_3^P (\psi_3 \Theta_3)^B e^{-\phi_3/2} \right\rangle &\simeq \frac{16\sqrt{2}}{25z_{12}z_{13}z_{23}^3} \\ &\cdot \left(3\Gamma^{MNP}C - \eta^{MN}\Gamma^PC + 5\eta^{MP}\Gamma^NC - 25\eta^{NP}\Gamma^MC \right)^{AB}, \end{aligned} \quad (\text{C.6})$$

$$\begin{aligned} \left\langle \psi_1^M \psi_1^N \psi_1^P e^{-\phi_1} \psi_2^Q (\psi_2 \Theta_2)^A e^{-\phi_3/2} \psi_3^R (\psi_3 \Theta_3)^B e^{-\phi_3/2} \right\rangle &\simeq \frac{-64\sqrt{2}}{5z_{12}^2z_{13}^2z_{23}^2} \\ &\cdot \left(\frac{1}{40}\Gamma^{MNPQR}C - \frac{3}{8}\eta^{MR}\Gamma^{NPQ}C - \frac{21}{40}\eta^{MQ}\Gamma^{NPR}C - \frac{5}{8}\eta^{QR}\Gamma^{MNP}C - 6\eta^{MQ}\eta^{NR}\Gamma^PC \right)^{AB}. \end{aligned} \quad (\text{C.7})$$

C.2 Contractions with Three Chiral Primaries

In a three-point function of three massless vertices $W_{k_{1,2,3}}$, we have $k_1^2 = k_2^2 = k_3^2 = 0$ and $k_i \cdot k_j = 0$: All scalar products among the momenta vanish.

$\langle \mathbf{W}_1 \mathbf{W}_1 \mathbf{W}_1 \rangle$. The correlator of three scalar level-zero NS-NS vertices W_1 (2.21, 2.22) reads¹⁵

$$\begin{aligned} \left\langle W_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \right\rangle &= -g_c^3 \delta(\sum_j k_j) \cdot \\ &\cdot \varepsilon_{k_1, M\tilde{M}} \varepsilon_{k_2, N\tilde{N}} \varepsilon_{k_3, P\tilde{P}} \frac{\alpha'}{2} (\eta^{MN} k_1^P - k_1^N \eta^{MP} + k_2^M \eta^{NP}) (\eta^{\tilde{M}\tilde{N}} k_1^{\tilde{P}} - k_1^{\tilde{N}} \eta^{\tilde{M}\tilde{P}} + k_2^{\tilde{M}} \eta^{\tilde{N}\tilde{P}}). \end{aligned} \quad (\text{C.8})$$

Using the twisted polarization tensors ε_T , the result vanishes because all scalar products $k_i \cdot k_j$ among the three momenta vanish. But with untwisted polarization tensors, the result becomes

$$\begin{aligned} \left\langle W_{1,k_1}(z_1, \bar{z}_1) W_{1,k_2}(z_2, \bar{z}_2) W_{1,k_3}(z_3, \bar{z}_3) \right\rangle \\ = g_c^3 \delta(\sum_j k_j) \frac{\alpha'}{2} \frac{-8(J_1^4 + 6J_1^2 J_2^2 + J_2^4 + 6J_1^2 J_3^2 + 6J_2^2 J_3^2 + J_3^4) \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}}{J_1^2 J_2^2 J_3^2}, \end{aligned} \quad (\text{C.9})$$

where $\tilde{\alpha}_j$, $\tilde{\Sigma}$ are defined in (2.2), and where $J_j = \sqrt{(k_j - k_{j,A})^2}$ is the R-charge of the operator W_{1,k_j} .¹⁶

$\langle \mathbf{W}_1 \mathbf{W}_2 \mathbf{W}_2 \rangle$. Using the VEV (C.2), the correlator of one flat-space scalar level-zero NS-NS vertex W_1 (2.21) with two level-zero R-R vertices W_2 (2.23) reads

$$\begin{aligned} \left\langle W_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) W_{2,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \right\rangle \\ = -g_c^3 \delta(\sum_j k_j) \varepsilon_{k_1, M\tilde{M}} t_{k_2, AB} t_{k_3, CD} \frac{1}{2} (\Gamma^M C)^{BD} (\Gamma^N C)^{AC}. \end{aligned} \quad (\text{C.10})$$

Using the twisted polarization tensors ε_T , t_T , the result is zero again, while with untwisted polarizations it becomes¹⁷

$$\left\langle W_{1,k_1}(z_1, \bar{z}_1) W_{2,k_2}(z_2, \bar{z}_2) W_{2,k_3}(z_3, \bar{z}_3) \right\rangle$$

¹⁵Here and everywhere below, we do not write the fermionic ghosts c , \tilde{c} explicitly, but we include their effect in the correlators.

¹⁶As before, we denote by $k_{j,A} = (k_j^0, \dots, k_j^4, 0, \dots, 0)$ the projection of k_j onto the AdS₅ part.

¹⁷Because the theory is chiral, we use $\text{Tr}(\mathbf{1}) = 16$.

$$\begin{aligned}
&= -g_c^3 \delta(\sum_j k_j) \varepsilon_{\mathbb{T}, k_1, M\tilde{M}} \frac{\alpha'}{2} \frac{1}{\Delta_2 \Delta_3} \frac{1}{2} \text{Tr}[\Gamma^M \not{k}_3 \not{k}_{3,A} \Gamma^{43210} \Gamma^{\tilde{M}} \Gamma^{01234} \not{k}_{2,A} \not{k}_2] \\
&= -g_c^3 \delta(\sum_j k_j) \varepsilon_{\mathbb{T}, k_1, M\tilde{M}} \frac{\alpha'}{2} \frac{1}{\Delta_2 \Delta_3} \frac{1}{2} (-1)^{\sigma(\tilde{M})} \text{Tr}[\Gamma^M \not{k}_3 \not{k}_{3,A} \Gamma^{\tilde{M}} \not{k}_{2,A} \not{k}_2] \\
&= g_c^3 \delta(\sum_j k_j) \frac{\alpha' - 64(3J_1^2 + J_2^2 + J_3^2) \tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}}{2 J_1^2 J_2 J_3}. \tag{C.11}
\end{aligned}$$

$\langle \mathbf{WWW} \rangle$. We are now in a position to assemble the correlator of three massless chiral primaries. When all three operators are twisted the result vanishes trivially since all scalar products among the momenta vanish. As a check, we can also compute the correlator with two twisted operators and again one obtains that it vanishes, which is due to a supersymmetric Ward identity. Finally the correlator for three primaries (3.5) yields (3.8)

$$\langle W_{k_1}(z_1, \bar{z}_1) W_{k_2}(z_2, \bar{z}_2) W_{k_3}(z_3, \bar{z}_3) \rangle = g_c^3 \delta(\sum_j k_j) \frac{\alpha' 2\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\alpha}_3 \tilde{\Sigma}^5}{2 J_1^2 J_2^2 J_3^2}. \tag{C.12}$$

C.3 Contractions with Two Chiral Primaries

With two massless vertices, $W_{k_{1,2}}$ and the level-one vertex V_{k_3} we have $k_1^2 = k_2^2 = 0$, $k_3^2 = -4/\alpha'$, $k_{1,2} \cdot k_3 = 2/\alpha'$ and $k_1 \cdot k_2 = -2/\alpha'$.

$\langle \mathbf{W}_1 \mathbf{W}_1 \mathbf{V}_1 \rangle$. Both the left and right moving part of the correlator $\langle W_{1,k_1}^{(-1,-1)} W_{1,k_2}^{(-1,-1)} V_{1,k_3}^{(0,0)} \rangle$ are split into three parts corresponding to the terms in $V_{1,k_3}^{(0,0)}$. Using the symmetries of the polarization tensors, combining the three parts and multiplying contractions of the left-moving with the right-moving part, we obtain¹⁸

$$\begin{aligned}
&\langle W_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\
&= -g_c^3 \delta(\sum_j k_j) \varepsilon_{M,\tilde{M}}^{k_1} \varepsilon_{N,\tilde{N}}^{k_2} \varepsilon_{RS,\tilde{R}\tilde{S}}^{k_3} X^{MNR\tilde{S}} X^{\tilde{M}\tilde{N}\tilde{R}\tilde{S}}, \tag{C.13}
\end{aligned}$$

where

$$X^{MNR\tilde{S}} = \frac{\alpha'}{2} \left(\eta^{MN} k_1^R k_1^{\tilde{S}} + \frac{2}{\alpha'} \eta^{MS} \eta^{NR} + (\eta^{MR} k_3^N - \eta^{NR} k_3^M) k_1^{\tilde{S}} \right). \tag{C.14}$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle W_{1,\mathbb{T},k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{1,\mathbb{T},k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{1,\mathbb{T},k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = -2^2 \times 3^2 g_c^3 \delta(\sum_j k_j). \tag{C.15}$$

When contracting with the untwisted polarization tensors, the result becomes a lengthy rational function of $\Delta_{1,2,3}$ that we do not quote here.

$\langle \mathbf{W}_1 \mathbf{W}_1 \mathbf{V}_2 \rangle$. For the correlator $\langle W_{1,k_1}^{(-1,-1)} W_{1,k_2}^{(-1,-1)} V_{2,k_3}^{(0,0)} \rangle$, we have to consider only the first term of $V_{2,k_3}^{(0,0)}$ since for the second there are ψ 's which cannot be contracted. Assuming antisymmetry of the polarization tensor α , the result is

¹⁸We do not denote the fermionic ghosts c, \tilde{c} explicitly, but include their effect in the result.

$$\begin{aligned} & \langle W_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{2,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\ & = -g_c^3 \delta(\sum_j k_j) \varepsilon_{M,\tilde{M}}^{k_1} \varepsilon_{N,\tilde{N}}^{k_2} \alpha_{RST,\tilde{R}\tilde{S}\tilde{T}}^{k_3} \frac{\alpha'}{2} 36 \eta^{MT} \eta^{NS} k_1^R \eta^{\tilde{M}\tilde{T}} \eta^{\tilde{N}\tilde{S}} k_1^{\tilde{R}}. \end{aligned} \quad (\text{C.16})$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle W_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{1,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{2,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = -2^2 \times 7 g_c^3 \delta(\sum_j k_j). \quad (\text{C.17})$$

When contracting with the untwisted polarization tensors, the result again becomes a complicated rational function of $\Delta_{1,2,3}$ that we do not quote here.

$\langle \mathbf{V}_1 \mathbf{W}_2 \mathbf{W}_2 \rangle$. Using the VEV (C.2), we can compute the correlator

$$\begin{aligned} & \langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) W_{2,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ & = -g_c^3 \delta(\sum_j k_j) \varepsilon_{MN,\tilde{M}\tilde{N}}^{k_1} \frac{1}{2} \text{Tr} \left[t^{k_2} X^{MN} (t^{k_3})^\top (X^{\tilde{M}\tilde{N}})^\top \right], \end{aligned} \quad (\text{C.18})$$

where superscript \top denotes transposition in spinor indices, and

$$X^{MN} = \sqrt{\frac{\alpha'}{2}} k_2^N \Gamma^M C \quad (\text{C.19})$$

are the left/right-moving field contractions. Expanding the twisted polarization tensors and performing the index contractions and matrix algebra gives

$$\langle V_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) W_{2,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = -2^4 g_c^3 \delta(\sum_j k_j). \quad (\text{C.20})$$

Contracting with the untwisted polarization tensors gives an expression that we will refrain from quoting here.

$\langle \mathbf{V}_2 \mathbf{W}_2 \mathbf{W}_2 \rangle$. Using the VEV (C.3), the correlator is given by

$$\begin{aligned} & \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) W_{2,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ & = -g_c^3 \delta(\sum_j k_j) \alpha_{MNP,\tilde{M}\tilde{N}\tilde{P}}^{k_1} \frac{1}{2} \text{Tr} \left[t^{k_2} X^{MNP} (t^{k_3})^\top (X^{\tilde{M}\tilde{N}\tilde{P}})^\top \right], \end{aligned} \quad (\text{C.21})$$

where

$$X^{MNP} = \frac{1}{2} \Gamma^M \Gamma^N \Gamma^P C \quad (\text{C.22})$$

are the left/right-moving field contractions. Expanding the twisted polarization tensors and performing the index contractions and matrix algebra gives

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) W_{2,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = -2^4 \times 7 g_c^3 \delta(\sum_j k_j). \quad (\text{C.23})$$

Once again, contracting with the untwisted polarization tensors gives a complicated expression that we will not write here.

$\langle \mathbf{W}_1 \mathbf{W}_2 \mathbf{V}_3 \rangle$. Using the VEVs (C.2,C.4), we compute the last correlator

$$\begin{aligned} \langle W_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ = -g_c^3 \delta(\sum_j k_j) \varepsilon_{M,\tilde{M}}^{k_1} \frac{1}{2} \text{Tr} \left[t^{k_2} X^{MN} (t_{\tilde{N},N}^{k_3})^\top (X^{\tilde{M}\tilde{N}})^\top \right], \end{aligned} \quad (\text{C.24})$$

where

$$X^{MN} = \sqrt{\frac{\alpha'}{2}} (k_1^N \Gamma^M C + \eta^{MN} \not{k}_3 C) \quad (\text{C.25})$$

are the left/right-moving field contractions, which have been simplified using (C.1) as well as the fact that all contractions of k_j with ε_{k_j} , t_{k_j} vanish. Expanding the twisted polarization tensors and performing the index contractions and matrix algebra gives

$$\langle W_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = 2^7 g_c^3 \delta(\sum_j k_j). \quad (\text{C.26})$$

Contracting with the untwisted polarization tensors gives a lengthy expression that we will refrain from writing here.

$\langle \mathbf{W} \mathbf{W} \mathbf{V} \rangle$. We can now assemble the different parts of the correlator with two massless vertices W (3.5) and one massive vertex V (3.10). As a check, we first compute the correlators with two and three twisted vertex operators, and we obtain that all of them vanish, which is explained by a supersymmetric Ward identity. For three untwisted vertices, meaning they are all primaries, all the complicated results of the different parts combine into the simple expression (3.11)

$$\langle W_{k_1}(z_1, \bar{z}_1) W_{k_2}(z_2, \bar{z}_2) V_{k_3}(z_3, \bar{z}_3) \rangle = g_c^3 \delta(\sum_j k_j) \alpha'^2 \frac{\alpha_1^2 \alpha_2^2 \Sigma^4 \tilde{\alpha}_3^2 \tilde{\Sigma}^2}{\Delta_1^2 \Delta_2^2 \Delta_3^4}. \quad (\text{C.27})$$

C.4 Contractions with One Chiral Primary

With two level-one vertices, $V_{k_{1,2}}$, and the massless vertex W_{k_3} we have $k_1^2 = k_2^2 = -4/\alpha'$, $k_3^2 = 0$, $k_{1,2} \cdot k_3 = 0$ and $k_1 \cdot k_2 = 4/\alpha'$.

$\langle \mathbf{V}_1 \mathbf{V}_1 \mathbf{W}_1 \rangle$. The left and right-moving parts of the correlator $\langle V_{1,k_1}^{(-1,-1)} V_{1,k_2}^{(-1,-1)} W_{1,k_3}^{(0,0)} \rangle$ are split into two parts corresponding to the terms in $W_{1,k_3}^{(0,0)}$. Using the symmetries of the polarization tensors, combining those parts and multiplying left and right-moving contractions we obtain ¹⁹

$$\begin{aligned} \langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\ = -g_c^3 \delta(\sum_j k_j) \varepsilon_{MN,\tilde{M}\tilde{N}}^{k_1} \varepsilon_{PQ,\tilde{P}\tilde{Q}}^{k_2} \varepsilon_{R,\tilde{R}}^{k_3} X^{MNPQR} X^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}\tilde{R}}, \end{aligned} \quad (\text{C.28})$$

where

¹⁹Once again, we do not denote the fermionic ghosts c , \tilde{c} explicitly, but include their effect in the result.

$$X^{MNPQR} = \sqrt{\frac{\alpha'}{2}} \eta^{MP} (\eta^{NQ} k_1^R + 2\eta^{QR} k_2^N + 2\eta^{NR} k_3^Q + \frac{\alpha'}{2} k_2^N k_3^Q k_1^R) + \left(\frac{\alpha'}{2}\right)^{3/2} (\eta^{MR} k_3^P + \eta^{PR} k_2^M) k_2^N k_3^Q. \quad (\text{C.29})$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle V_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = 2^3 \times 11 g_c^3 \delta(\sum_j k_j). \quad (\text{C.30})$$

When contracting with the untwisted polarization tensors, the result becomes a lengthy rational function of $\Delta_{1,2,3}$ that we do not quote here.

$\langle \mathbf{V}_2 \mathbf{V}_2 \mathbf{W}_1 \rangle$. As before, there are two parts in the correlator $\langle V_{2,k_1}^{(-1,-1)} V_{2,k_2}^{(-1,-1)} W_{1,k_3}^{(0,0)} \rangle$. Assuming antisymmetry of the polarization tensor α , they combine into

$$\begin{aligned} \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{2,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle &= -g_c^3 \delta(\sum_j k_j) \alpha_{MNP, \tilde{M}\tilde{N}\tilde{P}}^{k_1} \alpha_{QRS, \tilde{Q}\tilde{R}\tilde{S}}^{k_2} \varepsilon_{T, \tilde{T}}^{k_3} \\ &\cdot \frac{\alpha'}{2} 36 \eta^{NS} \eta^{PR} \left[k_1^T \eta^{MQ} + 3(\eta^{MT} k_3^Q - \eta^{QT} k_3^M) \right] \eta^{\tilde{N}\tilde{S}} \eta^{\tilde{P}\tilde{R}} \left[k_1^{\tilde{T}} \eta^{\tilde{M}\tilde{Q}} + 3(\eta^{\tilde{M}\tilde{T}} k_3^{\tilde{Q}} - \eta^{\tilde{Q}\tilde{T}} k_3^{\tilde{M}}) \right]. \end{aligned} \quad (\text{C.31})$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{2,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = 2^3 \times 3 \times 7 g_c^3 \delta(\sum_j k_j). \quad (\text{C.32})$$

Contracting with the untwisted polarization tensors, the result becomes a complicated function of $\Delta_{1,2,3}$ that we will not quote here.

$\langle \mathbf{V}_2 \mathbf{V}_1 \mathbf{W}_1 \rangle$. In the case of $\langle V_{2,k_1}^{(-1,-1)} V_{1,k_2}^{(-1,-1)} W_{1,k_3}^{(0,0)} \rangle$, we have to consider only the second term of $W_{1,k_3}^{(0,0)}$ since for the first there are ψ 's which cannot be contracted. Then, assuming antisymmetry of the polarization tensor α , the correlator is given by

$$\begin{aligned} \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\ = -g_c^3 \delta(\sum_j k_j) \alpha_{MNP, \tilde{M}\tilde{N}\tilde{P}}^{k_1} \varepsilon_{RS, \tilde{R}\tilde{S}}^{k_2} \varepsilon_{T, \tilde{T}}^{k_3} \left(\frac{\alpha'}{2}\right)^2 36 k_3^S k_3^N \eta^{MT} \eta^{PR} k_3^{\tilde{S}} k_3^{\tilde{N}} \eta^{\tilde{M}\tilde{T}} \eta^{\tilde{P}\tilde{R}}. \end{aligned} \quad (\text{C.33})$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) W_{1,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = 0. \quad (\text{C.34})$$

As before, we do not quote the result of contracting with the untwisted polarization tensors.

$\langle \mathbf{V}_1 \mathbf{W}_2 \mathbf{V}_3 \rangle$. Using the VEVs (C.2,C.4), we can compute the correlator

$$\begin{aligned} \langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ = -g_c^3 \delta(\sum_j k_j) \varepsilon_{MN, \tilde{M}\tilde{N}}^{k_1} \frac{1}{2} \text{Tr} \left[t^{k_2} X^{MNP} (t_{\tilde{P}, P}^{k_3})^\top (X^{\tilde{M}\tilde{N}\tilde{P}})^\top \right], \end{aligned} \quad (\text{C.35})$$

where

$$X^{MNP} = (\eta^{NP} + \frac{\alpha'}{2} k_2^N k_1^P) \Gamma^M C + \frac{\alpha'}{2} k_2^N \eta^{MP} \not{k}_3 C \quad (\text{C.36})$$

are the left/right-moving field contractions, which have been simplified using (C.1) as well as the symmetry of ε_{k_1} and the fact that all contractions of k_j with $\varepsilon_{k_j}, t_{k_j}$ vanish. Expanding the twisted polarization tensors and performing the index contractions and matrix algebra gives

$$\langle V_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = 0. \quad (\text{C.37})$$

Once again, contracting with the untwisted polarization tensors gives an expression that we will refrain from quoting here.

$\langle \mathbf{V}_2 \mathbf{W}_2 \mathbf{V}_3 \rangle$. Using the VEVs (C.3,C.5), the correlator is given by

$$\begin{aligned} & \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ &= -g_c^3 \delta(\sum_j k_j) \alpha_{MNP, \tilde{M}\tilde{N}\tilde{P}}^{k_1} \frac{1}{2} \text{Tr} \left[t^{k_2} X^{MNPR} (t_{\tilde{R},R}^{k_3})^\top (X^{\tilde{M}\tilde{N}\tilde{P}\tilde{R}})^\top \right], \end{aligned} \quad (\text{C.38})$$

where

$$X^{MNPR} = \sqrt{\frac{\alpha'}{2}} \frac{1}{2} k_1^R \Gamma^M \Gamma^N \Gamma^P C + \sqrt{\frac{\alpha'}{2}} \frac{3}{2} \eta^{MR} \Gamma^N \Gamma^P \not{k}_3 C \quad (\text{C.39})$$

are the left/right-moving field contractions, which have been simplified using (C.1) as well as the symmetry of α_{k_1} and the fact that all contractions of k_j with α_{k_j}, t_{k_j} vanish. Expanding the twisted polarization tensors and performing the index contractions and matrix algebra gives

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) W_{2,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = 0. \quad (\text{C.40})$$

Contracting with the untwisted polarization tensors gives a complicated function of $\Delta_{1,2,3}$ that we will refrain from quoting here.

$\langle \mathbf{W}_1 \mathbf{V}_3 \mathbf{V}_3 \rangle$. Using the VEVs (C.2,C.4,C.6), the last correlator is

$$\begin{aligned} & \langle W_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ &= -g_c^3 \delta(\sum_j k_j) \varepsilon_{M,\tilde{M}}^{k_1} \frac{1}{2} \text{Tr} \left[t_{\tilde{N},N}^{k_2} X^{MNP} (t_{\tilde{P},P}^{k_3})^\top (X^{\tilde{M}\tilde{N}\tilde{P}})^\top \right], \end{aligned} \quad (\text{C.41})$$

where

$$X^{MNP} = (\eta^{NP} + \frac{\alpha'}{2} k_3^N k_1^P) \Gamma^M C + \frac{\alpha'}{2} k_3^N \eta^{MP} \not{k}_3 C - \frac{\alpha'}{2} k_1^P \eta^{MN} \not{k}_2 C + \frac{\alpha'}{2} \frac{1}{2} \eta^{NP} \not{k}_2 \Gamma^M \not{k}_3 C \quad (\text{C.42})$$

are the left/right-moving field contractions, which have been simplified using (C.1) as well as the fact that all contractions of k_j with $\varepsilon_{k_j}, t_{k_j}$ vanish. Expanding the twisted polarization tensors and performing the index contractions and matrix algebra gives

$$\langle W_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = -2^9 g_c^3 \delta(\sum_j k_j). \quad (\text{C.43})$$

Again, we do not write here the result of contracting with the untwisted polarization tensors.

$\langle \mathbf{V}\mathbf{V}\mathbf{W} \rangle$. We now can assemble all the different parts of the correlator of a massless vertex W (3.5) and two massive vertices V (3.10). When two or three of them are twisted, we find once again that the results vanish, which matches the supersymmetric Ward identity. For three primaries, the correlator becomes (3.14)

$$\begin{aligned} \langle V_{k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) W_{k_3}(z_3, \bar{z}_3) \rangle &= g_c^3 \delta(\sum_j k_j) \frac{\Sigma^4}{\Delta_1^4 \Delta_2^4 \Delta_3^2} \\ &\cdot \left(-\frac{1}{2} \Omega_1^2 \alpha_3^4 + \alpha_3^2 (3\Omega_1^2 + 2\Omega_1 \alpha_3 + \alpha_3^2) \Omega_4 - \frac{1}{2} \Omega_1 (\Omega_1 - 4\alpha_3) \Omega_4^2 + \Omega_4^3 \right). \end{aligned} \quad (\text{C.44})$$

where

$$\Omega_1 = \alpha_1 + \alpha_2, \quad \Omega_4 = \alpha' \alpha_1 \alpha_2 \alpha_3 \Sigma. \quad (\text{C.45})$$

C.5 Contractions with Zero Chiral Primaries

With three level-one vertices, we have $k_j^2 = -4/\alpha'$, and $k_i \cdot k_j = 2/\alpha'$ for $i \neq j$.

$\langle \mathbf{V}_1 \mathbf{V}_1 \mathbf{V}_1 \rangle$. The correlator of three vertices V_1 (2.29, 2.33) evaluates to²⁰

$$\begin{aligned} \langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\ = -g_c^3 \delta(\sum_j k_j) \varepsilon_{MN, \tilde{M}\tilde{N}}^{k_1} \varepsilon_{PQ, \tilde{P}\tilde{Q}}^{k_2} \varepsilon_{RS, \tilde{R}\tilde{S}}^{k_3} X^{MNPQRS} X^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}\tilde{R}\tilde{S}}, \end{aligned} \quad (\text{C.46})$$

where, using the antisymmetry of the polarization tensor $\varepsilon_{MN, \tilde{M}\tilde{N}}$ as well as $k_j \cdot \varepsilon^{k_j} = 0$,

$$\begin{aligned} X^{MNPQRS} &= \frac{1}{2} \eta^{MP} \left(\eta^{NR} (\eta^{QS} - \frac{\alpha'}{2} 3k_1^Q k_1^S) - \frac{\alpha'}{2} (\eta^{NQ} - \frac{\alpha'}{2} k_3^N k_3^Q) k_1^R k_2^S \right) \\ &+ (5 \text{ permutations of } \{(M, N, k_1), (P, Q, k_2), (R, S, k_3)\}). \end{aligned} \quad (\text{C.47})$$

Contracting with the twisted polarization tensors ε_T (2.31) gives

$$\langle V_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{1,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = -2^4 \times 3 \times 53 g_c^3 \delta(\sum_j k_j). \quad (\text{C.48})$$

When contracting with the untwisted polarization tensors, the result becomes a lengthy rational function of $\Delta_1, \Delta_2, \Delta_3$ (or equivalently J_1, J_2, J_3) that we do not quote here. For $J_i \ll \sqrt{1/\alpha'}$ (i.e. $\Delta_i \approx \sqrt{4/\alpha'}$), it becomes exactly (C.48), plus $\mathcal{O}(J_i^2 \alpha')$ terms. This can be understood by noting that all terms in the contraction contain an even number of untwisted index pairs.

$\langle \mathbf{V}_1 \mathbf{V}_1 \mathbf{V}_2 \rangle$. In the correlator $\langle V_{1,k_1}^{(-1,-1)} V_{1,k_2}^{(-1,-1)} V_{2,k_3}^{(0,0)} \rangle$, only the first term in $V_{2,k_3}^{(0,0)}$ (2.34) contributes, since the second term yields an odd number of ψ 's, which cannot be fully contracted. Assuming antisymmetry of the polarization tensor α , the result is

$$\langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{2,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle$$

²⁰As before, we do not denote the fermionic ghosts c, \tilde{c} explicitly, but include their effect in the result.

$$= -g_c^3 \delta(\sum_j k_j) \varepsilon_{MN, \tilde{M}\tilde{N}}^{k_1} \varepsilon_{PQ, \tilde{P}\tilde{Q}}^{k_2} \alpha_{RST, \tilde{R}\tilde{S}\tilde{T}}^{k_3} X^{MNPQRST} X^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}\tilde{R}\tilde{S}\tilde{T}}, \quad (\text{C.49})$$

where

$$X^{MNPQRST} = \sqrt{\frac{\alpha'}{2}} 6\eta^{MR} \eta^{PS} (\eta^{NT} k_1^Q - \eta^{NQ} k_1^T - \eta^{QT} k_2^N + \frac{\alpha'}{2} k_2^N k_1^Q k_1^T). \quad (\text{C.50})$$

Contracting with the twisted polarization tensors ε_T (2.31) and α_T (2.32) gives

$$\langle V_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{1,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{2,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = 2 \times 3 \times 7^2 g_c^3 \delta(\sum_j k_j). \quad (\text{C.51})$$

The result for untwisted vertices with general J_i is again lengthy and we refrain from quoting it. For $J_i \ll \sqrt{1/\alpha'}$, it becomes exactly the negative of (C.51), plus $\mathcal{O}(J_i^2 \alpha')$ terms, which follows from the fact that all terms in the contraction contain an odd number of untwisted index pairs.

$\langle \mathbf{V}_2 \mathbf{V}_2 \mathbf{V}_1 \rangle$. Performing the field contractions, the correlator $\langle V_{2,k_1} V_{2,k_2} V_{1,k_3} \rangle$ becomes

$$\begin{aligned} & \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{2,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{1,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\ &= -g_c^3 \delta(\sum_j k_j) \alpha_{MNP, \tilde{M}\tilde{N}\tilde{P}}^{k_1} \alpha_{QRS, \tilde{Q}\tilde{R}\tilde{S}}^{k_2} \varepsilon_{TU, \tilde{T}\tilde{U}}^{k_3} X^{MNPQRSTU} X^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}\tilde{R}\tilde{S}\tilde{T}\tilde{U}}, \end{aligned} \quad (\text{C.52})$$

where

$$X^{MNPQRSTU} = 6\eta^{NR} \eta^{PS} \left(3\eta^{MU} (\eta^{QT} - \frac{\alpha'}{2} k_1^Q k_1^T) + \frac{\alpha'}{2} (\eta^{MQ} k_1^U + 3\eta^{QU} k_2^M) k_1^T \right). \quad (\text{C.53})$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{2,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{1,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = -2^2 \times 3 \times 7 \times 19 g_c^3 \delta(\sum_j k_j). \quad (\text{C.54})$$

We refrain from writing out the lengthy result for untwisted vertices. Again, for $J_i \ll \sqrt{1/\alpha'}$, it becomes exactly (C.54), plus $\mathcal{O}(J_i^2 \alpha')$ terms, because each term contains an even number of untwisted index contractions.

$\langle \mathbf{V}_2 \mathbf{V}_2 \mathbf{V}_2 \rangle$. Performing the field contractions, the correlator $\langle V_{2,k_1} V_{2,k_2} V_{2,k_3} \rangle$ becomes (using the antisymmetry of the polarization tensors α)

$$\begin{aligned} & \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{2,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{2,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle \\ &= -g_c^3 \delta(\sum_j k_j) \alpha_{MNP, \tilde{M}\tilde{N}\tilde{P}}^{k_1} \alpha_{QRS, \tilde{Q}\tilde{R}\tilde{S}}^{k_2} \alpha_{TUV, \tilde{T}\tilde{U}\tilde{V}}^{k_3} X^{MNPQRSTUV} X^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}\tilde{R}\tilde{S}\tilde{T}\tilde{U}\tilde{V}}, \end{aligned} \quad (\text{C.55})$$

where

$$X^{MNPQRSTUV} = 2^2 3^3 \eta^{MQ} \eta^{SV} \sqrt{\frac{\alpha'}{2}} \left(\eta^{NR} \eta^{PU} k_1^T + \eta^{NT} (\eta^{RU} k_2^P - \eta^{PU} k_1^R) \right). \quad (\text{C.56})$$

Expanding the twisted polarization tensors and performing the index contractions gives

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{2,T,k_2}^{(-1,-1)}(z_2, \bar{z}_2) V_{2,T,k_3}^{(0,0)}(z_3, \bar{z}_3) \rangle = 2 \times 3^3 \times 7 \times 13 g_c^3 \delta(\sum_j k_j). \quad (\text{C.57})$$

The result for untwisted vertices again reduces to the negative of (C.57) for $J_i \ll \sqrt{1/\alpha'}$, plus $\mathcal{O}(J_i^2 \alpha')$ terms.

$\langle \mathbf{V}_1 \mathbf{V}_3 \mathbf{V}_3 \rangle$. Using the VEVs (C.2,C.4,C.6), we can compute the correlator

$$\begin{aligned} & \langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ & = -g_c^3 \delta(\sum_j k_j) \varepsilon_{MN, \tilde{M}\tilde{N}}^{k_1} t_{\tilde{P}\tilde{A}, PA}^{k_2} t_{\tilde{Q}\tilde{B}, QB}^{k_3} X_L^{MNPQ, AB} X_R^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}, \tilde{A}\tilde{B}}, \end{aligned} \quad (\text{C.58})$$

where

$$\begin{aligned} X_{L,R}^{MNPQ, AB} & = \frac{\sqrt{\alpha'}}{2} \left[(-\eta^{NP} k_1^Q + \eta^{NQ} k_1^P - \eta^{PQ} k_2^N + \frac{\alpha'}{2} k_1^Q k_1^P k_2^N) \Gamma^M C \right. \\ & \quad \left. + (\eta^{NQ} + \frac{\alpha'}{2} k_1^Q k_2^N) \eta^{MP} \not{k}_2 C + (-\eta^{NP} + \frac{\alpha'}{2} k_1^P k_2^N) \eta^{MQ} \not{k}_3 C - \frac{\alpha'}{2} \frac{1}{2} k_2^N \eta^{PQ} \not{k}_2 \Gamma^M \not{k}_3 C \right]^{AB} \end{aligned} \quad (\text{C.59})$$

are the left/right-moving field contractions, which have been simplified using (C.1) as well as the symmetry of ε_{k_1} and the fact that all contractions of k_j with ε_{k_j} , t_{k_j} vanish. Performing the matrix algebra and all the contractions, and using the twisted polarization tensors t_T , ε_T , the correlator evaluates to

$$\langle V_{1,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = 2^8 \times 5^2 g_c^3 \delta(\sum_j k_j). \quad (\text{C.60})$$

For untwisted vertices, the correlator is again a lengthy rational expression in J_1 , J_2 , J_3 . For $J_i \ll \sqrt{1/\alpha'}$, it reduces to

$$\langle V_{1,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = -2^5 \times 5 \times 59 g_c^3 \delta(\sum_j k_j) + \mathcal{O}(J_i^2 \alpha'). \quad (\text{C.61})$$

$\langle \mathbf{V}_2 \mathbf{V}_3 \mathbf{V}_3 \rangle$. For the correlator $\langle V_{2,k_1}^{(-1,-1)} V_{3,k_2}^{(-1/2,-1/2)} V_{3,k_3}^{(-1/2,-1/2)} \rangle$, the required fermionic correlators are (C.3,C.5,C.7). Assembling all terms, and taking into account the asymmetry of the polarization tensor α as well as $k \cdot \alpha_k = k \cdot t_k = 0$, the wanted correlator becomes

$$\begin{aligned} & \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ & = -g_c^3 \delta(\sum_j k_j) \alpha_{MNP, \tilde{M}\tilde{N}\tilde{P}}^{k_1} t_{\tilde{Q}\tilde{A}, QA}^{k_2} t_{\tilde{R}\tilde{B}, RB}^{k_3} X_L^{MNPQR, AB} X_R^{\tilde{M}\tilde{N}\tilde{P}\tilde{Q}\tilde{R}, \tilde{A}\tilde{B}}, \end{aligned} \quad (\text{C.62})$$

where

$$\begin{aligned} X_{L,R}^{MNPQR, AB} & = \sqrt{2} \left[-\frac{1}{4} (\eta^{QR} - \frac{\alpha'}{2} k_1^Q k_1^R) \Gamma^{MNP} C + \frac{3}{4} \frac{\alpha'}{2} k_1^Q \eta^{MR} \Gamma^{NP} \not{k}_3 C \right. \\ & \quad \left. + \frac{3}{4} \frac{\alpha'}{2} k_1^R \eta^{MQ} \not{k}_2 \Gamma^{NP} C + \frac{1}{8} \frac{\alpha'}{2} \not{k}_2 \left(-\eta^{QR} \Gamma^{MNP} + 12 \eta^{MR} \eta^{NQ} \Gamma^P \right) \not{k}_3 C \right]^{AB} \end{aligned} \quad (\text{C.63})$$

are the left/right-moving field contractions. Expanding the polarization tensors, the full correlator can be computed in the same way as for $\langle \mathbf{V}_1 \mathbf{V}_3 \mathbf{V}_3 \rangle$ above. With twisted polarization tensors, the result is

$$\langle V_{2,T,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,T,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,T,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle = -2^8 \times 3 \times 7 g_c^3 \delta(\sum_j k_j). \quad (\text{C.64})$$

For untwisted vertices the result is a lengthy expression again; for $J_i \ll \sqrt{1/\alpha'}$, it becomes

$$\begin{aligned} & \langle V_{2,k_1}^{(-1,-1)}(z_1, \bar{z}_1) V_{3,k_2}^{(-1/2,-1/2)}(z_2, \bar{z}_2) V_{3,k_3}^{(-1/2,-1/2)}(z_3, \bar{z}_3) \rangle \\ & = -2^5 \times 3 \times 5^2 \times 7 g_c^3 \delta(\sum_j k_j) + \mathcal{O}(J_i^2 \alpha'). \end{aligned} \quad (\text{C.65})$$

$\langle \mathbf{V}\mathbf{V}\mathbf{V} \rangle$. We now can assemble the correlator of three massive scalars. If we compute the correlators with two or three twisted operators we obtain that all of them vanish, which is accounted for by a supersymmetric Ward identity. For three primaries, the correlator becomes (3.19)

$$\begin{aligned} & \langle V_{k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) V_{k_3}(z_3, \bar{z}_3) \rangle \\ &= g_c^3 \delta(\sum_j k_j) \frac{\Sigma^4}{\Delta_1^4 \Delta_2^4 \Delta_3^4} \left(\frac{1}{2} \Sigma_2^4 + \frac{9}{2} \Sigma_4^2 + (2\Sigma_2^3 - 3\Sigma^2 \Sigma_2^2 + 6\Sigma^2 \Sigma_4 + 3\Sigma_2 \Sigma_4) \alpha' \Sigma_4 \right. \\ & \quad \left. + \frac{1}{2} (3\Sigma^4 + 7\Sigma_2^2 - 8\Sigma^2 \Sigma_2 + 6\Sigma_4) (\alpha' \Sigma_4)^2 - (\Sigma^2 - 3\Sigma_2) (\alpha' \Sigma_4)^3 + (\alpha' \Sigma_4)^4 \right), \end{aligned} \quad (\text{C.66})$$

where

$$\Sigma_2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3, \quad \Sigma_4 = \alpha_1 \alpha_2 \alpha_3 \Sigma, \quad (\text{C.67})$$

and Σ, α_j are defined in (1.4). For $J_i \ll \sqrt{1/\alpha'}$, the correlator reduces to (1.6)

$$\langle V_{k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) V_{k_3}(z_3, \bar{z}_3) \rangle = \frac{3^8}{2^9} g_c^3 \delta(\sum_j k_j) + \mathcal{O}(J_i^2 \alpha'). \quad (\text{C.68})$$

D Mixed Correlators

In the previous section, we checked that all correlators with two or three twisted operators vanish and we also computed the correlators with three untwisted vertices, which correspond to three-point functions of primary operators.

In this section we state the results of correlators with two untwisted vertex operators, which we expect to represent correlation functions of both primary and descendant operators. Starting with the three vertices at the massless level, we have

$$\langle W_{T,k_1}(z_1, \bar{z}_1) W_{k_2}(z_2, \bar{z}_2) W_{k_3}(z_3, \bar{z}_3) \rangle = 0. \quad (\text{D.1})$$

For one massive vertex and two chiral vertices, we also have cancellations for both correlators

$$\langle W_{k_1}(z_1, \bar{z}_1) W_{k_2}(z_2, \bar{z}_2) V_{T,k_3}(z_3, \bar{z}_3) \rangle = 0, \quad (\text{D.2})$$

$$\langle W_{T,k_1}(z_1, \bar{z}_1) W_{k_2}(z_2, \bar{z}_2) V_{k_3}(z_3, \bar{z}_3) \rangle = 0. \quad (\text{D.3})$$

In the case of two massive operators and a massless vertex we get

$$\langle V_{T,k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) W_{k_3}(z_3, \bar{z}_3) \rangle = 0, \quad (\text{D.4})$$

$$\langle V_{k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) W_{T,k_3}(z_3, \bar{z}_3) \rangle = -g_c^3 \delta(\sum_j k_j) \frac{\alpha_3^4 \Sigma^4}{2 \Delta_1^4 \Delta_2^4}. \quad (\text{D.5})$$

Finally, for the case of three massive vertices we have

$$\langle V_{T,k_1}(z_1, \bar{z}_1) V_{k_2}(z_2, \bar{z}_2) V_{k_3}(z_3, \bar{z}_3) \rangle = g_c^3 \delta(\sum_j k_j) \frac{\alpha_1^4 \Sigma^4}{2 \Delta_2^4 \Delta_3^4}. \quad (\text{D.6})$$

In Section 3 we explained that there cannot be a supercharge that annihilates two untwisted vertex operators, so it might at first seem surprising that four of these correlators actually

vanish. Notice though that all of those include a massless vertex operator W , which corresponds to a chiral operator. Chiral primaries are annihilated not only by the superconformal charges that in our setup correspond to Q^\pm , but also by half of the supergenerators $Q_{\alpha\alpha}$ and $\tilde{Q}_\alpha^{\dot{\alpha}}$. Massless fields are then annihilated by an extra eight supercharges, which explains why correlators with at least one twisted vertex and at least one massless operator are indeed vanishing.

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