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Analytic structure of the $n = 7$ scattering amplitude in $\mathcal{N} = 4$ SYM theory at multi-Regge kinematics: Conformal Regge Pole Contribution.

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Abstract: We investigate the analytic structure of the $2 \to 5$ scattering amplitude in the planar limit of $\mathcal{N} = 4$ SYM in multi-Regge kinematics in all physical regions. We demonstrate the close connection between Regge pole and Regge cut contributions: in a selected class of kinematic regions (Mandelstam regions) the usual factorizing Regge pole formula develops unphysical singularities which have to be absorbed and compensated by Regge cut contributions. This leads, in the corrections to the BDS formula, to conformal invariant 'renormalized' Regge pole expressions in the remainder function. We compute these renormalized Regge poles for the $2 \to 5$ scattering amplitude.

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1 Introduction

It is now well established that the Bern-Dixon-Smirnov (BDS) conjecture [1] for the MHV n-point scattering amplitude in the planar limit of the $\mathcal{N} = 4$ SYM theory is incomplete for $n \geq 6$. One of the first indications for this was found in [2, 3]. Corrections to the BDS-formula have been named 'remainder functions', $R^{(n)}$, and in recent years major efforts have been made for determining these remainder functions. For $n = 6$, the remainder function $R^{(6)}$ has been calculated for two, and three loops [4–15]. Beyond this loop expansion, it has turned out to be useful to consider a special kinematic limit, the multi-Regge limit. For the $n = 6$ point amplitude the comparison of the BDS conjecture with the leading logarithmic

approximation which extends over all orders of the coupling constant, has shown that the remainder function consists of a Regge cut contribution which vanishes in the euclidean region and in the physical region where all energies are positive, and it is nonzero only in special kinematic regions, named 'Mandelstam regions': these are physical regions where some of the energy variables are positive, others negative ('mixed regions': the precise definition will be given later on). These results have been generalized also beyond the leading logarithmic approximation, and there is no doubt that the multi-Regge limit plays a key role for the determination of the remainder functions.

In the comparison of the multi-Regge formula with the BDS conjecture in [2, 3, 16] it was crucial to make use of the analytic structure of the $2 \to 4$ scattering amplitude in the multi-Regge limit. It is well known that in non-abelian gauge theories the gauge bosons reggeize, and in the leading approximation the $2 \to n+1$ production amplitudes can be written in a simple factorizing form with the exchange of reggeized gluons in all t-channels. Beyond the leading approximation this factorizing form of the Regge-pole contribution remains valid in the region of all energies being positive, but the production vertices become complex-valued functions, in agreement with the results of Regge theory derived from dual models [17–19] or scalar theories [20]. But in [2, 16] in was also shown that the simple factorized form of the Regge pole contributions is valid only in the physical region with all energy variables being positive (and also in the euclidean region), but it takes a quite different form in all other regions, in particular in the Mandelstam regions mentioned before: in the expression for the Regge pole contribution a new term appears which contains an unphysical singularity and should be cancelled by other terms.

This representation of the Regge poles is equivalent to another representation, in which the scattering amplitude is written as a sum of k_n different terms, each of them belonging to a distinct set of non-vanishing simultaneous energy discontinuities: in this representation the agreement with the Steinmann relation is explicit. For the case of $n = 6$, there are five terms, i.e. $k_6 = 5$; for $n > 6$ the number increases rapidly: $k_7 = 14$, $k_8 = 42$ etc. As discussed in [2, 3], the perturbative analysis of Yang-Mills theories shows that some of these terms contain, in addition to the Regge poles, also Regge cut singularities. For the $2 \to 4$ case, this applies to two terms: in the notation of [2, 3], to W_3 and W_4. In the physical region where all energies are positive, the phase factors in front W_3 and W_4 are such that the Regge cut contributions in W_3 and W_4 cancel, whereas in the Mandelstam region they add up to a nonzero result. Both the discussions of the Regge cut contributions and of Regge poles have made it clear that a complete analysis of the analytic structure of scattering amplitudes must include the investigation of all physical regions.

The analysis of [16] for the $2 \to 4$ amplitude has shown that there is an important connection between the Regge poles and Regge cuts which has not been seen in earlier analysis of Regge pole models [19]. First, it was observed that the Regge cut appears in exactly the same kinematic regions in which the Regge pole expression contains the terms with the unphysical singularities. Furthermore, both this singular Regge pole piece and the Regge cut term have the same complex phase structure: this allows to absorb the singular Regge pole piece into the Regge cut contribution, leading to a 'renormalized' Regge pole which is free from unphysical divergences, and to a modified Regge cut definition. The

existence of Regge cuts therefore resolves the problem connected with appearance of the singular pieces of the Regge poles. Conversely, without Regge cuts the standard factorizing Regge pole expression appears to be problematic.

For the determination of the conformal invariant remainder function in $\mathcal{N} = 4$ SYM it is necessary to perform a careful analysis of the content of the BDS formula. In [2, 3, 16] it was shown that, in multi-Regge kinematics, the BDS formula does not agree with the analytic structure outlined above in two respects: (i) the Regge pole contribution is correctly described in the region of positive energies and in the euclidean region, but not in the Mandelstam region; (ii) in these Mandelstam regions the Regge cut contributions are contained only in the one loop approximation, but not to all orders. This implies that the conformal invariant remainder function must (i) correct the Regge pole contribution in all kinematic regions and (ii) provide the all-loop Regge cut contribution. In view of the described interdependence between Regge pole and Regge cut contributions, there must be a close connection between the solutions to both problems. It looks reasonable to start with the Regge pole part: here the main task is the subtraction of the singular pieces by Regge cut contributions. To be more concrete, one can attempt to use the known phase structure of the Regge pole terms in all kinematic regions to constrain the phases of the Regge cuts in such a way that they can absorb all singular terms of the Regge poles. In this subtraction, most powerful constraints follow from the conformal invariance of the remainder function: after absorbing the singular Regge pole pieces (which by themselves are not conformal invariant) into the Regge cut contributions, the remaining 'renormalized' Regge poles and the modified Regge cut terms must be conformal invariant.

In this paper we describe this subtraction procedure for the $2 \to 4$ and for $2 \to 5$ cases. For the former case, most the work has been done already in earlier publications: so we only briefly review and complete our previous studies and then generalize to the $2 \to 5$ case. In the first part (section 2) we analyze the general factorization formula of Regge pole contributions in all physical regions. Starting from the region of positive energies where factorization holds, we continue to other regions and derive the existence of terms with unphysical pole singularities which have to be compensated by Regge cut contributions. Particular attention will be given to the phase structure which is important in determining the phase structure of Regge cut contributions in $\mathcal{N} = 4$ SYM. We present explicit results for $2 \to 4$ and $2 \to 5$, but our analysis can also be generalized to the general case $2 \to n + 1$. In the second part (section 3) we present an analysis of the BDS formula in multi-Regge kinematics in all physical regions. This analysis is general and applies to the case $2 \to n + 1$. In the third part (section 5) we carry out the program described at the end of the previous paragraph. We first compute, for the case $2 \to 5$, phases of Regge cut contributions which allow to absorb the unphysical terms of the Regge poles calculated in the first part. We then define subtraction schemes for absorbing these pieces into the Regge cuts, leaving conformal invariant expressions for the Regge poles. In the final part of this section we combine these results with our findings of the BDS amplitude obtained in section 3, and we present predictions for the remainder function. It should be emphasized that, in this paper, we do not yet address the second part of the program, the construction of the conformal invariant Regge cut contributions. This will be left for a separate paper.

2 The Regge Pole framework

2.1 Factorizing Regge poles

We begin with the factorized form of the fully signatured $2 \to n+1$ production amplitude (Fig.1). The produced particles will be labelled by $a_1, ..., a_{n-1}$, and they can have positive or negative energies.

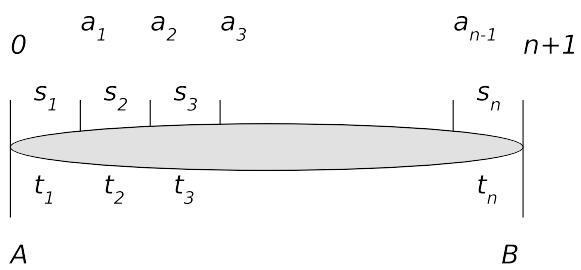

Figure 1: Notations for the $2 \to n-2$ amplitude

We want to describe all physical channels of these amplitudes in the multiregge kinematics $s \gg |s_1|, ..., |s_n| \gg -t_1, ..., -t_n$. We introduce, for each t-channel t_i, the signature label τ_i which takes the values $\tau_i = +1$ or $\tau_i = -1$. For $\tau_i = +1(-1)$ the scattering amplitude is even (odd) under twisting the t_i channel, i.e. under the crossing of the corresponding energy variables (for the simplest case, the $2 \to 2$ scattering, 'twisting the t-channel' is the same as $s \leftrightarrow u$ crossing). For our present discussion it is sufficient to consider signatured amplitudes as sums and differences of planar untwisted and twisted amplitudes. Denoting a twist by a simple cross, a signatured $2 \to 2$ scattering amplitude has the form

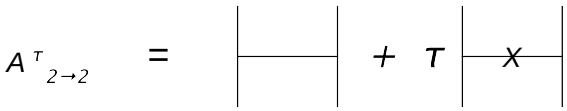

Figure 2: The signatured $2 \to 2$ amplitude

where the cross indicates the change of sign of the energies of the particles B and B'.

Generalizing this to arbitrary n, we write down the amplitude for the $2 \to n+1$ production amplitude in the following form

$$\frac{A^{\tau_i \tau_j ... \tau_n}_{2 \to n+1}}{\Gamma(t_1)\Gamma(t_n)} = |s_1|^{\omega_1} \xi_1 V^{\tau_1 \tau_2; a_1} |s_2|^{\omega_2} \xi_2 V^{\tau_2 \tau_3; a_2} |s_3|^{\omega_3} \xi_3 \times ... \\ \times |s_{n-1}|^{\omega_{n-1}} \xi_{n-1} V^{\tau_{n-1} \tau_n; a_{n-1}} |s_n|^{\omega_n} \xi_n, \tag{2.1}$$

where

$$\xi_i = e^{-i\pi\omega_i} - \tau_i \ ; \ \xi_{ij} = e^{-i\pi\omega_{ij}} + \tau_i \tau_j \ ; \ \xi_{ji} = e^{-i\pi\omega_{ji}} + \tau_i \tau_j \tag{2.2}$$

with

$$\omega_{ij} = \omega_i - \omega_j \tag{2.3}$$

denote the signature factors, and

$$V^{\tau_i\tau_j;a_j} \;=\; \frac{\xi_{ij}}{\xi_i}c_R^{ij;a_i} + \frac{\xi_{ji}}{\xi_j}c_L^{ij;a_i} \tag{2.4}$$

stands for the complex-valued production vertex.

As an example, for the case $2 \to 3$, the one particle production amplitude has a simple structure [16, 18]:

$$\frac{A_{2\to 3}^{\tau_1\tau_2}}{\Gamma(t_1)|s_1|^{\omega_1}|s_2|^{\omega_2}\Gamma(t_2)} \;=\; \xi_1 V^{\tau_1\tau_2;a_1}\xi_2 \;=\; \xi_{12}\xi_2 c_R^{12;a_1} + \xi_{21}\xi_1 c_L^{12;a_1} \;\equiv\; \tilde{V}^{\tau_1\tau_2;a_1}, \tag{2.5}$$

where $\Gamma(t)$ is the Regge pole residue and c_R and c_L the reggeon-reggeon-particle vertices. Similarly, the production of two particles has the form [16, 18]:

$$\frac{A_{2\to 4}^{\tau_1\tau_2\tau_3}}{\Gamma(t_1)|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\Gamma(t_3)} \;=\; \xi_1 V^{\tau_1\tau_2;a_1}\xi_2 V^{\tau_2\tau_3;a_2}\xi_3. \tag{2.6}$$

In order to arrive at a symmetric factorizing expression, we insert, for the t_2 channel, an additional signature factor and write

$$\frac{A_{2\to 4}^{\tau_1\tau_2\tau_3}}{\Gamma(t_1)|s_1|^{\omega_1}|s_2|^{\omega_2}|s_3|^{\omega_3}\Gamma(t_3)} \;=\; \tilde{V}^{\tau_1\tau_2;a_1}\frac{1}{\xi_2}\tilde{V}^{\tau_2\tau_3;a_2}, \tag{2.7}$$

where

$$\tilde{V}^{\tau_1\tau_2;a_1} = \xi_1 V^{\tau_1\tau_2;a_1}\xi_2. \tag{2.8}$$

Generalizing to the case $2 \to n+1$, we see that for each 'inner' t_i-channel, $t_2, ..., t_{n-1}$, we need an extra 'propagator' $1/\xi_i$. With this rule Eq.(2.1) can be written in the convenient form

$$\frac{A_{2\to n+1}^{\tau_i\tau_j\ldots\tau_n}}{\Gamma(t_1)|s_1|^{\omega_1}|s_2|^{\omega_2}\ldots|s_n|^{\omega_n}\Gamma(t_n)} \;=\; \tilde{V}^{\tau_1\tau_2;a_1}\frac{1}{\xi_2}\tilde{V}^{\tau_2\tau_3;a_2}\frac{1}{\xi_3}\ldots\frac{1}{\xi_{n-1}}\tilde{V}^{\tau_{n-1}\tau_n;a_{n-1}}. \tag{2.9}$$

It will be useful to write this formula as an expansion in monomials of signatures τ_i. In such an expansion, terms without any τ_i can be identified as the planar approximation in the kinematic region where all energies are positive. For the case of $n = 6$, terms proportional to $\tau_1\tau_3$ correspond to the planar amplitude where the particles a_1 and a_2 have become incoming: this is one of the Mandelstam regions where, according to the analysis in [2, 3], the Regge cut contribution will appear.

In order to obtain this representation we observe that the production vertex, Eq.(2.5), can be expanded as:

$$\begin{aligned}\tilde{V}^{\tau_1\tau_2;a} &= e^{-i\pi\omega_1}c_R^{12;a_1} + e^{-i\pi\omega_2}c_L^{12;a_1} - \tau_1 e^{-i\pi\omega_1}\left(e^{-i\pi\omega_1}c_R^{12;a_1} + e^{-i\pi\omega_2}c_L^{12;a_1}\right)\\ &-\tau_2 e^{-i\pi\omega_2}\left(e^{-i\pi\omega_1}c_R^{12;a_1} + e^{-i\pi\omega_2}c_L^{12;a_1}\right) + \tau_1\tau_2\left(e^{-i\pi\omega_2}c_R^{12;a_1} + e^{-i\pi\omega_1}c_L^{12;a_1}\right),\end{aligned} \tag{2.10}$$

and the propagator can be written in the form:

$$\frac{1}{\xi_2} \;=\; \frac{1}{e^{-i\pi\omega_2} - \tau_2} \;=\; \frac{e^{-i\pi\omega_2} + \tau_2}{-2i\sin(\pi\omega_2)e^{-i\pi\omega_2}}. \tag{2.11}$$

Note the appearance of the nonphysical poles $\sim 1/\sin(\pi\omega_2)$ which should be cancelled by the Regge cut contributions.

With these ingredients it is straightforward to find the expansion in monomials for the general $2 \to n+1$ amplitude.

2.2 Generating Function approach for the Regge Pole formula

To be definite, let us from now on concentrate on planar $\mathcal{N} = 4$ SYM. It will be convenient to define a generating function for the pole-term coefficients. Let us briefly introduce the idea behind it. We are interested in the analytical continuation of the planar scattering amplitude to arbitrary kinematic regions in multi-Regge kinematics. During such a continuation, various factors and phases may appear. As explained above, each particular kinematic region can be reached by a sequence of twists (crosses) of t-channels, and each such twist is denoted by a corresponding factor τ[1]. Thus, it is instructive to have a list of all possible phases and factors that appear due to continuation for each appropriate kinematic configuration. One may also think of a different point of view on the scattering amplitude. Instead of having one analytical function of kinematic invariants and then continuing to arbitrary physical and non-physical kinematic regions, one can introduce a *generating function*, $P_{2\to n}$, which is given as a sum of amplitudes in all physical regions. As a simple example, consider such a generating function of the $2 \to 3$ scattering process Fig.3:

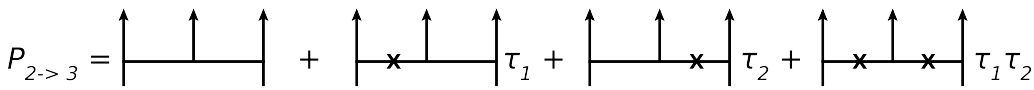

Figure 3: The generating function for the $2 \to 3$ scattering process written in terms of monomials of τ_1, τ_2

Turning now to the BDS formula, applied to the $2 \to 3$ amplitude [2], we have for the reggeon vertices in (2.4):

$$c_R^{ii+1;a} \;=\; |\Gamma_{i,i+1}|\frac{\sin(\pi\omega_i - \pi\omega_a)}{\sin(\pi\omega_i - \pi\omega_{i+1})} \quad ; \quad c_L^{ii+1;a} \;=\; |\Gamma_{i,i+1;a}|\frac{\sin(\pi\omega_{i+1} - \pi\omega_a)}{\sin(\pi\omega_{i+1} - \pi\omega_i)}. \tag{2.12}$$

Here i labels the t-channel (for the $2 \to 3$ case we have $i = 1$ only), a denotes the produced particle. Going to the physical region where all energies are positive, this allows to write the Reggeon-Reggeon-Gluon vertex $\Gamma_{i,j;a}$ (see Eq.(19-22) [16]) in the form:

$$\Gamma_{i,i+1;a}(\ln(\kappa_a - i\pi)) \;=\; |\Gamma_{i,i+1;a}|e^{i\pi\omega_a}. \tag{2.13}$$

Here the expansions in powers of $a = \frac{g^2 N_c}{8\pi^2}$ are given by:

$$\omega_i \;=\; -\frac{\gamma_K}{4}\ln\frac{|q_i|^2}{\lambda^2}\;,\;\; \gamma_K = 4a + O(a^2), \tag{2.14}$$

where γ_K is the cusp anomalous dimension and $\lambda^2 \equiv \mu^2 e^{1/\epsilon}$ for $D = 4 - 2\epsilon$ with $\epsilon \to 0^-$,

$$\omega_a \;=\; -\frac{\gamma_K}{8}\ln\frac{|q_i|^2|q_{i+1}|^2}{|k_{a_{i+1}}|^2\lambda^2} \tag{2.15}$$

[1]It should be clear that, from now on, τ is no longer related to signature but simply denotes kinematic regions

with $k_{a_{i+1}} = q_i - q_{i+1}$, and

$$\ln|\Gamma_{i,i+1}| = \frac{\gamma_K}{4}\left(-\frac{1}{4}\ln^2\frac{|q_i-q_{i+1}|^2}{\lambda^2} - \frac{1}{4}\ln^2\frac{|q_i|^2}{|q_{i+1}|^2} + \frac{1}{2}\ln\frac{|q_i|^2|q_{i+1}|^2}{\lambda^4}\ln\frac{|q_i-q_{i+1}|^2}{\mu^2} + \frac{5}{4}\zeta(2)\right). \tag{2.16}$$

Let us now return to the generating functions $P_{2\to n}$, to the sum of amplitudes in all kinematic regions. It is convenient to divide by factors which are common to all kinematic regions. Beginning with the case $2 \to 3$, Namely, using the explicit form Eq.(2.5) with Eq.(2.12) one arrives at

$$\begin{aligned} P_{2\to 3} &= \frac{A_{2\to 3}}{\Gamma(t_1)|s_1|^{\omega_1}||\Gamma_{1,2}||s_2|^{\omega_2}\Gamma(t_2)} \\ &= \tilde{V}_{red}^{\tau_1\tau_2;a} \\ &= \quad e^{-i\pi(\omega_1+\omega_2-\omega_a)} - e^{-i\pi(\omega_2-\omega_a)}\tau_1 - e^{-i\pi(\omega_1-\omega_a)}\tau_2 + e^{-i\pi\omega_a}\tau_1\tau_2. \end{aligned} \tag{2.17}$$

Here we have defined a reduced vertex by

$$\tilde{V}_{red}^{\tau_1\tau_2;a} = \frac{\tilde{V}^{\tau_1\tau_2;a}}{|\Gamma_{1,2}|} = e^{-i\pi(\omega_1+\omega_2-\omega_a)} - e^{-i\pi(\omega_2-\omega_a)}\tau_1 - e^{-i\pi(\omega_1-\omega_a)}\tau_2 + e^{-i\pi\omega_a}\tau_1\tau_2, \tag{2.18}$$

which consists of phases only.

As the next example we calculate, from Eq.(2.7), the six-point generating function (cf.([16]):

$$\begin{aligned} P_{2\to 4} &= \frac{A_{2\to 4}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}|\Gamma_{2,3}||s_3|^{\omega_3}\Gamma(t_3)} \\ &= \tilde{V}_{red}^{\tau_1\tau_2;a}\frac{1}{\xi_2}\tilde{V}_{red}^{\tau_2\tau_3;b} \\ &= e^{-i\pi(\omega_1+\omega_2+\omega_3-\omega_a-\omega_b)} \\ &- e^{-i\pi(\omega_2+\omega_3-\omega_a-\omega_b)}\tau_1 - e^{-i\pi(\omega_1+\omega_3-\omega_a-\omega_b)}\tau_2 - e^{-i\pi(\omega_1+\omega_2-\omega_a-\omega_b)}\tau_3 \\ &+ e^{-i\pi(\omega_3+\omega_a-\omega_b)}\tau_1\tau_2 + e^{-i\pi(\omega_1-\omega_a+\omega_b)}\tau_2\tau_3 \\ &+ e^{-i\pi\omega_2}\left\{\cos(\pi\omega_{ab}) + i\left(\sin(\pi\omega_a+\pi\omega_b) - 2e^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right)\right\}\tau_1\tau_3 + \\ &- \left\{\cos(\pi\omega_{ab}) - i\left(\sin(\pi\omega_a+\pi\omega_b) - 2e^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right)\right\}\tau_1\tau_2\tau_3, \end{aligned} \tag{2.19}$$

where $\omega_{ab} = \omega_a - \omega_b$. The careful reader may notice that this expression has a mirror symmetry with respect to right and left ($a \leftrightarrow b$) exchange. This fact will be important in the future.

Concluding this part, on can write a general expression for the generating function for an arbitrary number of produced particles $2 \to n+1$:

$$\begin{aligned} P_{2\to n+1} &= \frac{A_{2\to n+1}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}...|\Gamma_{n-1,n}||s_n|^{\omega_n}\Gamma(t_n)} \\ &= \tilde{V}_{red}^{\tau_1\tau_2;a_1}\frac{1}{\xi_2}\tilde{V}_{red}^{\tau_2\tau_3;a_2}\frac{1}{\xi_3}...\frac{1}{\xi_{n-1}}\tilde{V}_{red}^{\tau_{n-1}\tau_n;a_{n-1}} \\ &= a_0 + a_1\tau_1 + a_2\tau_2 + a_{12}\tau_1\tau_2 + ... + a_{1..n}\tau_1...\tau_n. \end{aligned} \tag{2.20}$$

The r.h.s. can be written as a polynomial in the τ_i, and the coefficients consist of phases and trigonometric functions. In the Appendix A we list, for the cases $2 \to 3$, $2 \to 4$, and $2 \to 5$, all coefficients of the generating function.

2.3 Rules: a few particular cases

It will be useful to extract, from the particular cases given above, a few general rules. Let us begin with the case $n = 5$. As we have said before, the term without any τ belongs to the planar amplitude in the physical region with all positive energies. On the rhs of Eq.(2.17) we have:

$$e^{i\pi\omega_a} e^{-i\pi(\omega_1+\omega_2)}. \tag{2.21}$$

As expected, the amplitude has the simple factorized form, with phase factors for the produced particle, $e^{i\pi\omega_a}$, and for the exchange channels, $e^{-i\pi\omega_1}$ and $e^{-i\pi\omega_2}$. As to the remaining three terms for $n = 5$ we observe the following pattern: each t channel without a twist comes with a phase factor $e^{-i\pi\omega_i}$, each t-channel with a twist carries the factor -1:

- twisted propagator: $\to -1$
- untwisted propagator in channel t_i: $\to e^{-i\pi\omega_i}$

An illustration is given in the two figures below

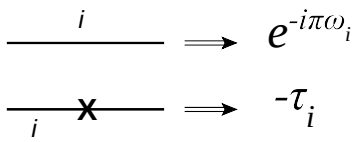

Figure 4: Two types of propagators in channel i.

Turning to $n = 6$, all but two terms are of the form which we have just described: phase factors for the propagators and for the production vertices. It is important to note that in all these terms the pole $\sim 1/\sin(\pi\omega_2)$ from the propagator of the t_2 channel cancels. New features appear for $\tau_1\tau_3$ and $\tau_1\tau_2\tau_3$, namely terms where the poles $\sim 1/\sin(\pi\omega_2)$ from the propagator (2.11) remain. The term proportional to $\tau_1\tau_3$ belongs to the planar amplitude continued into the physical regions where particles a and b are incoming:

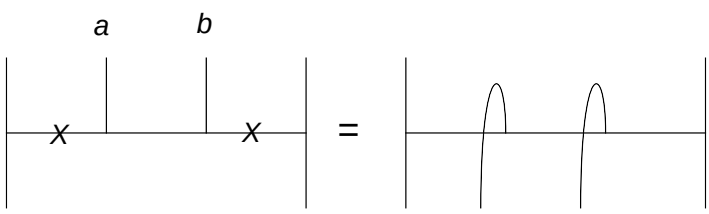

Figure 5: Illustration of the term $\tau_1\tau_3$

This kinematic region is the one in which the Regge cut appears [2, 3]. For this term we find from the rhs of Eq.(2.19)

$$= e^{-i\pi\omega_2}\Big[\cos(\pi(\omega_a-\omega_b)) + i\sin(\pi(\omega_a+\omega_b)) - 2i\frac{\cos(\pi\omega_2)\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\Big]. \tag{2.22}$$

which we rewrite as:

$$Eq.(2.22) = e^{-i\pi\omega_2}\left[e^{i\pi(\omega_a+\omega_b)} - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right]. \tag{2.23}$$

Here the first term is of the same form as discussed before, whereas the second term is new: it has an unphysical pole in $\sin(\pi\omega_2)$.

The important observation made in [16] is that the last two terms can be included in the Regge cut contribution, because they have the same phase structure as the Regge cut. This is the simplest example of the general feature that a Regge pole amplitude which, for positive energies, has the factorizing form, after analytic continuation, exhibits unphysical poles (in our case: $\sim 1/\sin(\pi\omega_2)$). From [2, 3] we know that, in Yang-Mills theories, the $2 \to 4$ amplitude contains a Regge cut contribution with the same phase $ie^{-i\pi\omega_2}$, which can absorb the singular piece in Eq.(2.22) of the Regge pole contribution.

An analogous discussion applies also to the term proportional to $\tau_1\tau_2\tau_3$. Note, however, that in this case the first term (see Appendix A) is of the form

$$-\,e^{-i\pi(\omega_a+\omega_b)}. \tag{2.24}$$

As expected, there are no phases from t-channel propagators, but for the production vertices we have $e^{-i\pi\omega_a}$ instead of $e^{i\pi\omega_a}$.

Moving on to $n = 7$, we again note the appearance of pole terms: the coefficient of $\tau_1\tau_3$ is illustrated below:

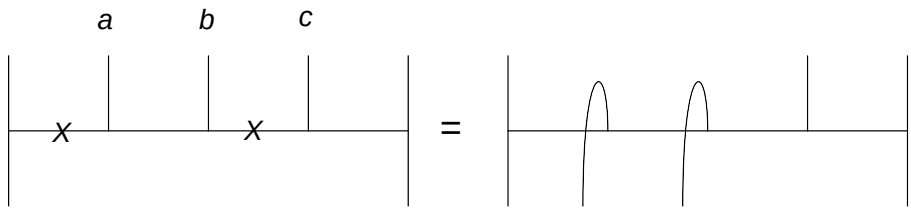

Figure 6: Illustration of the term $\tau_1\tau_3$

It has the form (Appendix A)

$$e^{-i\pi(\omega_2+\omega_4)}e^{i\pi\omega_c}\left[e^{i\pi(\omega_a+\omega_b)} - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right]. \tag{2.25}$$

It is easily obtained from the analogous term of the $2 \to 4$ amplitude by multiplication with $e^{i\pi\omega_c}$ (for the additional vertex of particle c) and by $e^{-i\pi\omega_4}$ (for the untwisted propagator of the t_4 channel). The pole term $\sim 1/\sin(\pi\omega_2)$ belongs to the t_2-channel, and later on we will show that it can be combined with the Regge cut contribution in the same t-channel.

An analogous discussion holds for the coefficient of $\tau_2\tau_4$. Next let us consider the coefficient of $\tau_1\tau_4$:

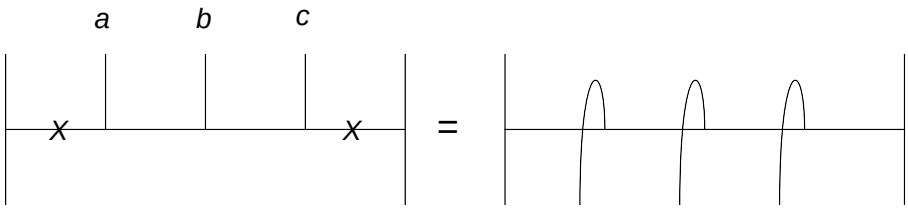

Figure 7: Illustration of the term $\tau_1\tau_4$

The corresponding term on the r.h.s. of Eq.(2.20) is (see Appendix A):

$$\left[e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_2+\omega_3)} - 2i\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right]. \tag{2.26}$$

Again, the first term is of the same form as the cases discussed above, whereas the double pole term belongs to the t_2 and t_3 channels and has to be combined with the Rege cut contribution extending over these two channels.

Finally, we look at the coefficient of $\tau_1\tau_2\tau_3\tau_4$:

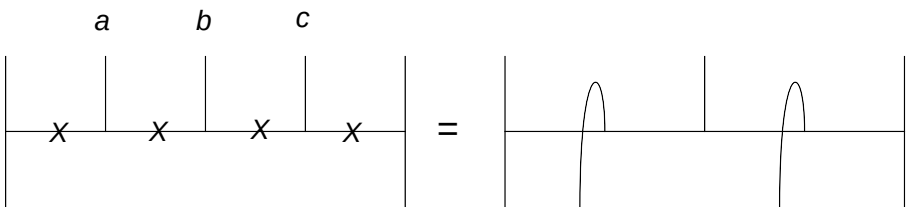

Figure 8: Illustration of the term $\tau_1\tau_2\tau_3\tau_4$

It has the form:

$$\left[e^{i\pi(\omega_b-\omega_a-\omega_c)} - 2i\frac{\sin(\pi\omega_2-\pi\omega_a)\sin(\pi\omega_b)\sin(\pi\omega_3-\pi\omega_c)}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right], \tag{2.27}$$

and there is again a double pole which has to absorbed by the Rege cut contribution extending over the t_2 and t_3-channels. The first term deviates from the previous cases: for the production vertex of particle b we have $e^{i\pi\omega_b}$, whereas particles a and c come the complex conjugate.

In Appendix A we present, for the cases $2 \to 3$, $2 \to 4$, and $2 \to 5$, a complete list of all coefficients of the generating function. In all cases we first find a term with a pure phase. For the 'generalized Mandelstam regions', there are, in addition, terms with simple, double, ... poles of the form $\sim 1/\left(\sin(\pi\omega_i)\sin(\pi\omega_j)...\sin(\pi\omega_k)\right)$. A closer inspection shows a one-to-one correspondence between these singular terms and Regge cut contributions: we will explicitly study the case $n = 7$ and show that these Regge cut pieces can be used to absorb all singular terms.

2.4 The general case: recurrence relations

In order to analyze the structure for the general case it is useful to make use of recurrence relations. To begin with, consider the generating function of the five point amplitude, $P_{2\to 3}^{\tau_1\tau_2}$ (Eq.(2.17)). Due to the factorization property Eq.(2.20), we can obtain the 6-point generating function by applying a *recurrence operator* $\tilde{K}$:

$$P_{2\to 4} = \tilde{V}_{red}^{\tau_1,\tau_2;a}\frac{1}{\xi_2}\tilde{V}_{red}^{\tau_2,\tau_3;b} = \tilde{V}_{red}^{\tau_1,\tau_2;a}\tilde{K}(\tau_2,\tau_3;b) \tag{2.28}$$

with

$$\tilde{K}(\tau_2,\tau_3;b) \;=\; \frac{1}{\xi_2}\tilde{V}_{red}^{\tau_2,\tau_3}. \tag{2.29}$$

Explicitly:

$$\tilde{K}(\tau_2,\tau_3;b) \;=\; e^{-i\pi(\omega_3-\omega_b)} - \frac{\sin(\pi\omega_2-\pi\omega_b)}{\sin(\pi\omega_2)}\tau_3 + \frac{\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\tau_2\tau_3. \tag{2.30}$$

Note that $\tilde{K}$ is not symmetric with respect to the monomial representation. In particular, it does not contain a term proportional to τ_2. Nevertheless, the resulting generating function, $P_{2\to n+1}$,

$$P_{2\to n+1} = \tilde{V}_{red}^{\tau_1,\tau_2;a_1}\;\tilde{K}(\tau_2,\tau_3;a_2)...\tilde{K}(\tau_{n-1}.\tau_n;a_{n-1}) \tag{2.31}$$

is symmetric.

In the Appendix B we present a more general discussion of the coefficients of different configuration of τ's. Here we only discuss one special case which corresponds to two crosses in the first (left) and in the last (right) channel Fig.9. As before, we consider the case $2\to n+1$ with n t-channels $(t_1,...,t_n)$ and $(\omega_1,...\omega_n)$, and $n-1$ produced particles labelled by $a_1,...,a_{n-1}$:

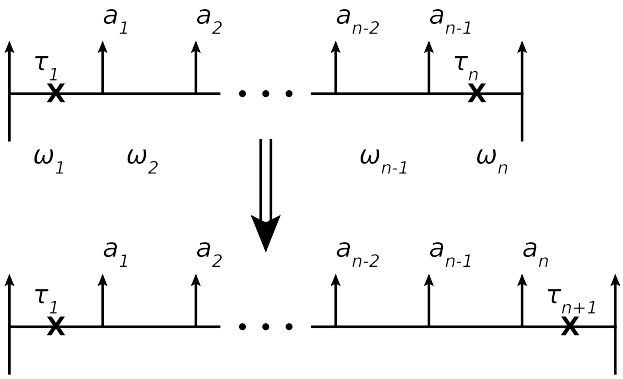

Figure 9: Initial configuration $\tau_1\tau_n$

and we want to prove, by induction, that the coefficient of $\tau_1\tau_n$ in $P_{2\to n+1}$ is given by

$$\Big\{e^{-i\pi(\omega_2+\omega_3+...\omega_{n-1})}e^{i\pi\left(\omega_{a_1}+\omega_{a_2}+...+\omega_{a_{n-1}}\right)}$$
$$-2i\frac{\sin(\pi\omega_{a_1})\sin(\pi\omega_{a_2})...\sin(\pi\omega_{a_{n-1}})}{\sin(\pi\omega_2)\sin(\pi\omega_3)...\sin(\pi\omega_{n-1})}\Big\}\,\tau_1\tau_n. \tag{2.32}$$

For this we also need to show that the coefficient proportional to τ_1 is

$$\left\{e^{-i\pi(\omega_2+\omega_3+\ldots\omega_n)}e^{i\pi\left(\omega_{a_1}+\omega_{a_2}+\ldots+\omega_{a_{n-1}}\right)}\right\}\tau_1. \tag{2.33}$$

To begin with the simplest case, $2\to 4$, we have for the coefficient $\tau_1\tau_3$ (Eq.(2.23) or Appendix A):

$$e^{-i\pi\omega_2}e^{i\pi(\omega_{a_1}+\omega_{a_2})}-2i\frac{\sin(\pi\omega_{a_1})\sin(\pi\omega_{a_2})}{\sin(\pi\omega_2)}, \tag{2.34}$$

whereas the coefficient of τ_1 is:

$$e^{-i\pi(\omega_2+\omega_3)}e^{i\pi\left(\omega_{a_1}+\omega_{a_2}\right)}. \tag{2.35}$$

Let us now prove, by induction, our assertion. In order to go from the case $2\to n+1$ to the case $2\to n+2$, we multiply $P_{2\to n+1}$ with the kernel $\tilde{K}(\tau_n,\tau_{n+1};a_n)$:

$$\tilde{K}(\tau_n,\tau_{n+1};a_n) \;=\; e^{-i\pi\omega_{n+1}}e^{i\pi\omega_{a_n}}-\frac{\sin(\pi\omega_n-\pi\omega_{a_n})}{\sin(\pi\omega_n)}\tau_{n+1}+\frac{\sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)}\tau_n\tau_{n+1}. \tag{2.36}$$

Within this product, the relevant terms are:

$$\begin{aligned} &P_{\to n+1}\cdot\tilde{K}(\tau_n,\tau_{n+1};a_n)= \\ &\Big[\{\ldots\}\tau_1+\{\ldots\}\boldsymbol{\tau_1\tau_n}\Big]\cdot\Big[1\{\ldots\}+\{\ldots\}\boldsymbol{\tau_{n+1}}+\{\ldots\}\tau_1\tau_{n+1}\Big], \end{aligned} \tag{2.37}$$

where, by assumption, in the first square bracket we use Eq.(2.32) and Eq.(2.33), and the second bracket is given in Eq.(2.36).

We immediately see that, on the rhs, the coefficient of τ_1 comes from the product of the first terms in each square bracket and equals:

$$e^{-i\pi(\omega_2+\ldots+\omega_{n+1})}e^{i\pi\left(\omega_{a_1}+\ldots+\omega_{a_n}\right)}. \tag{2.38}$$

This proves the second part of our assertion. Next, in order to calculate, the contribution proportional to $\tau_1\tau_{n+1}$, one should take into account two terms: the product of the term $\sim\tau_1\tau_n$ in the first bracket with the term $\tau_n\tau_{n+1}$ in the second bracket, and the product of the term τ_1 in the first bracket with the term τ_{n+1} in the second bracket. When combining these two contributions, the following identity is useful:

$$\frac{\sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)}+e^{-i\pi\omega_n}\frac{\sin(\pi\omega_n-\pi\omega_{a_n})}{\sin(\pi\omega_n)} \;=\; e^{-i\pi\omega_n}e^{i\pi\omega_{a_n}}. \tag{2.39}$$

One arrives at:

$$\begin{aligned} &\Big\{e^{-i\pi(\omega_2+\omega_3+\ldots\omega_n)}e^{i\pi\left(\omega_{a_1}+\omega_{a_2}+\ldots+\omega_{a_n}\right)} \\ &-2i\frac{\sin(\pi\omega_{a_1})\sin(\pi\omega_{a_2})\ldots\sin(\pi\omega_{a_n})}{\sin(\pi\omega_2)\sin(\pi\omega_3)\ldots\sin(\pi\omega_n)}\Big\}\tau_1\tau_{n+1}, \end{aligned} \tag{2.40}$$

which proves the first part our assertion.

Concluding this part, according to Eq.(2.31), each coefficient of the τ expansion in Eq.(2.20) can be calculated recursively, by multiplying the iterative kernel Eq.(2.36) with the initial expression $\tilde{V}_{red}^{\tau_1,\tau_2;a_1}$.[2]

[2] Although it is possible to calculate each coefficient in the expansion by using these recurrence relations, practically it is more efficient to use simple code with *Mathematica*, which generates these coefficients immediately. The simplest implementation might be iterative multiplication with the kernel $\tilde{K}$.

3 Generating function for the BDS amplitudes in the multi-Regge kinematics

3.1 Motivation

In order to determine the remainder function in each physical region for the pole and cut combinations, let us now find the phase structure of the BDS amplitude [1] in the different kinematic regions. Again, we find it convenient to define a generating function:

$$A_{BDS} \;=\; a_0 + a_1\tau_1 + a_2\tau_2 + ... + a_n\tau_n + a_{12}\tau_1\tau_2 + a_{13}\tau_1\tau_3 + ...a_{1..n}\tau_1\tau_2...\tau_n. \qquad (3.1)$$

In this expansion, each monomial of the twists $\tau_i...\tau_j$ defines a kinematic region, and the coefficient $a_{i...j}$ is the BDS prediction for this region. As before, each term in the expansion corresponds to a diagram of the type

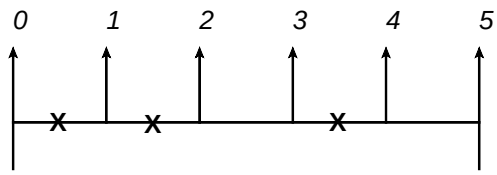

Figure 10: Example of a diagram with twists in the channels 1, 2, and 4. In Eq.(3.1) it corresponds to the term $\tau_1\tau_2\tau_4$.

The following discussion of the BDS formula will be similar to the previous study of the Regge pole model, but the results all be quite different.

The meaning of the "twist" or "crossed line" is the same as before. By twist we mean that the diagram is rotated around the direction of the exchanged momenta to the right of the cross ("**X**") sign. For example if one twists the diagram with respect to channel 1 (corresponding to ω_1), the result is

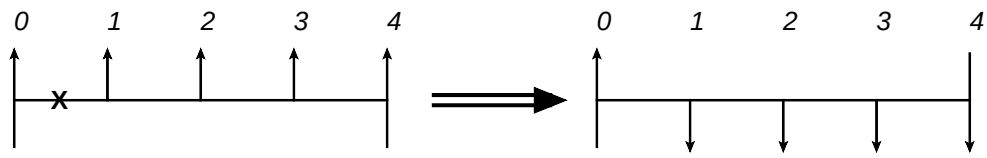

Figure 11: Example of diagram with a twist in channel 1, which in the expansion Eq.(3.1) corresponds to the term τ_1.

We can generalize the twisting of the diagram in order to reach other channels. For example, in Fig.12 we rotated twice. We move from left to right. The first twist brings the diagram similar to presented in Fig.11 and the second twist (cross in channel 3) rotates back the rest of the diagram to the right of the cross sign. It is important to stress that despite the fact that we rotate the diagram, it remains planar.

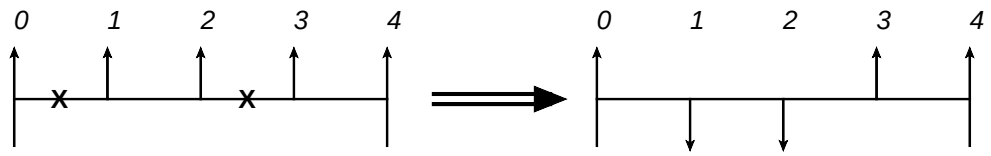

Figure 12: Example of double twisted diagram with twists in the channels 1 and 3, which in the expansion Eq.(3.1) corresponds to the term $\tau_1\tau_3$.

The diagram in Fig.12 corresponds to the following kinematic region:

$$s_1 < 0, s_2 > 0, s_3 < 0, s_4 > 0; s_{012} < 0, s_{123} < 0, s_{234} < 0; s_{0123} > 0, s_{1234} < 0; s > 0. \tag{3.2}$$

3.2 BDS predictions: examples

Let us begin with a brief review of the five point and the six point functions in the multiregge kinematics. As shown in [2], for the $2 \to 3$ amplitude in the region of positive energies (no τ factors) we have the simple exponential form:

$$\frac{M^{BDS}_{2\to 3}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}\Gamma(t_2)} \;=\; e^{-i\pi\omega_1}e^{i\pi\omega_a}e^{-i\pi\omega_2}. \tag{3.3}$$

Analogous expressions hold for the other regions. The exponents resemble those which we have discussed in the previous section. However, in contrast to our discussion of the Regge pole framework, for the BDS amplitudes we can formulate simple rules which also fix the signs of the exponents of the production vertices. Let us next consider the $2 \to 4$ case in the region belonging to the coefficient $\tau_1\tau_3$ (Mandelstam region). From [2, 16] we have

$$\frac{M^{BDS}_{2\to 4}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}||\Gamma_{2,3}||s_3|^{\omega_3}\Gamma(t_3)} \;=\; Ce^{-i\pi\omega_2}e^{i\pi(\omega_a+\omega_b)}. \tag{3.4}$$

Here C is the new phase factor, related to the one loop approximation of the Regge cut

$$C \;=\; e^{i\pi\left(\frac{\gamma_K}{4}\ln\frac{|q_1|^2|q_3|^2}{|k_a+k_b|^2\lambda^2}\right)} \tag{3.5}$$

with $k_a + k_b \;=\; q_1 - q_3$. The remaining parts of the phases are obtained from the rules of Section 2. It has been noticed in [16] that when combining this phase with the two vertex factors one arrives at a conformal invariant phase

$$Ce^{-i\pi(\omega_a+\omega_b)} = e^{i\delta} \tag{3.6}$$

with

$$\delta \;=\; \pi\frac{\gamma_K}{4}\ln\frac{|q_1||q_2||k_a||k_b|}{|k_a+k_b|^2|q_2|^2}. \tag{3.7}$$

It is important to recall the origin of the phase factor C: the BDS formula for the $2 \to 4$ amplitude contains three Li_2 functions (dilogarithms) which depend upon the three

independent anharmonic cross ratios. In the multi-Regge limit, one of these anharmonic cross ratios is a phase factor

$$\Phi = \frac{(-s_2)(-s)}{(-s_{012})(-s_{123})}, \tag{3.8}$$

with

$$\Phi - 1 = \frac{|k_a + k_b|^2}{s_2}, \tag{3.9}$$

whereas the remaining two ratios go to zero. The dilogarithm depending upon the phase Φ appears in the combination

$$R(\Phi) \;=\; -\frac{1}{4}\ln^2\Phi - \frac{1}{2}\ln\Phi\left(\ln\frac{(-t_1)(-t_3)}{(-s_2)\mu^2} - \frac{1}{\epsilon}\right) - \frac{1}{2}Li_2(1-\Phi). \tag{3.10}$$

It is easy to see that

$$R(\Phi = 1) = 0, \tag{3.11}$$

whereas for $\Phi = e^{\pm 2i\pi}$ the argument of the dilogarithm passes through a cut and

$$Li_2(1-\Phi) \to \mp 2\pi i \ln(1-\Phi) \tag{3.12}$$

with $\ln(1-\Phi)$ being real-valued. Concluding, one can see that the analytical continuation of the combination of the Li_2 function with the appropriate logarithms produces a logarithmic phase factor

$$R(|\Phi|e^{\mp 2\pi i}) \;=\; \pm i\pi\left(\ln\frac{|q_i|^2|q_j|^2}{|q_i - q_j|^2\lambda^2}\right), \tag{3.13}$$

which corresponds to the Mandelstam cut in the one loop approximation.. There is an overall factor $\gamma_K/4$ in front of the logarithm, which was omitted during the computation of R and should be restored in the final expression.

For the $2 \to 5$ amplitude there are three phases which have to be rotated. We first consider the kinematic region belonging to the coefficient of $\tau_1\tau_3$. Here we rotate only

$$\Phi_1 = \frac{(-s_{12})(-s_{0123})}{(-s_{012})(-s_{123})} \tag{3.14}$$

with

$$\Phi_1 - 1 = \frac{|k_a + k_b|^2}{s_{12}}, \tag{3.15}$$

whereas the two other phases are kept fixed. The BDS prediction is

$$\frac{M^{BDS}_{2\to 5}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}|\Gamma_{2,3}||s_3|^{\omega_3}|\Gamma_{3,4}||s_4|^{\omega_4}\Gamma(t_4)} \;=\; C_{13}e^{-i\pi(\omega_2+\omega_4)}e^{i\pi(\omega_a+\omega_b+\omega_c)} \tag{3.16}$$

with

$$C_{13} \;=\; e^{i\pi\left(\frac{\gamma_K}{4}\ln\frac{|q_1|^2|q_3|^2}{|k_a+k_b|^2\lambda^2}\right)} \quad ; \quad |k_a + k_b|^2 \;=\; |q_1 - q_3|^2. \tag{3.17}$$

We introduce the conformal invariant phase δ_{13}:

$$C_{13}e^{-i\pi(\omega_2+\omega_4)}e^{i\pi(\omega_a+\omega_b+\omega_c)} = e^{-i\pi(\omega_2+\omega_4)}e^{i\pi\omega_c}e^{i\delta_{13}}, \tag{3.18}$$

where

$$\delta_{13} = \pi \frac{\gamma_K}{4} \ln \frac{|q_1||q_3||k_a||k_b|}{|k_a+k_b|^2|q_2|^2}. \tag{3.19}$$

The coefficient of $\tau_2\tau_4$ (with the rotating phase Φ_2) is obtained from symmetry considerations. Next the region belonging to $\tau_1\tau_4$. The relevant phase which rotates to $e^{-2i\pi}$ is

$$\tilde{\Phi} = \frac{(-s_{123})(-s)}{(-s_{0123})(-s_{1234})} \quad ; \quad \tilde{\Phi} - 1 = \frac{|k_a+k_b+k_c|^2}{s_{123}}, \tag{3.20}$$

and the corresponding Li_2-function yields the phase factor

$$C_{14} \;=\; e^{i\pi\left(\frac{\gamma_K}{4}\ln\frac{|q_1|^2|q_4|^2}{|k_a+k_b+k_c|^2\lambda^2}\right)} \quad ; \quad |k_a+k_b+k_c|^2 \;=\; |q_1-q_4|^2. \tag{3.21}$$

The prediction of the BDS formula for this kinematic region is

$$\frac{M^{BDS}_{2\to 5}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}|\Gamma_{2,3}||s_3|^{\omega_3}|\Gamma_{3,4}||s_4|^{\omega_2}\Gamma(t_4)} \;=\; C_{14}e^{-i\pi(\omega_2+\omega_3)}e^{i\pi(\omega_a+\omega_b+\omega_c)}. \tag{3.22}$$

We write this as

$$C_{14}e^{-i\pi(\omega_2+\omega_3)}e^{i\pi(\omega_a+\omega_b+\omega_c)} = e^{-i\pi(\omega_2+\omega_3)}e^{i\pi\omega_b}e^{i\delta_{14}} \tag{3.23}$$

with the conformal invariant phase

$$\delta_{14} = \pi \frac{\gamma_K}{4} \ln \frac{|q_1||q_4||k_a||k_c|}{|k_a+k_b+k_c|^2|q_2||q_3|}. \tag{3.24}$$

One can spot that the contribution for a *single* Li_2-function belonging to a Mandelstam cut is given by the simple exponential expression (cf.(3.13))

$$C_{ij} \;=\; e^{i\pi\left(\frac{\gamma_K}{4}\ln\frac{|q_i|^2|q_j|^2}{|q_i-q_j|^2\lambda^2}\right)}. \tag{3.25}$$

The composite state of several single coefficients C_{ij} consists of a product of C's with appropriate signs of exponents, in accordance with the direction of the rotation of the analytical continuation.

Finally the coefficient of $\tau_1\tau_2\tau_2\tau_4$. Now we rotate $\tilde{\Phi}$ by $e^{-2i\pi}$ and Φ_1 and Φ_2 by $e^{+2i\pi}$. In terms of a single coefficient C_{ij}, the composite coefficient C_{1234} will be:

$$C_{1234} \;=\; C^{+}_{14}C^{-}_{13}C^{-}_{24}, \tag{3.26}$$

where C_{14} corresponds to the rotation of $\tilde{\Phi}$, Φ_1, and Φ_2 respectively. $\pm$ corresponds to the sign in front of $i\pi$ in the exponent. We obtain

$$\frac{M^{BDS}_{2\to 5}}{\Gamma(t_1)|s_1|^{\omega_1}|\Gamma_{1,2}||s_2|^{\omega_2}|\Gamma_{2,3}||s_3|^{\omega_3}|\Gamma_{3,4}||s_4|^{\omega_4}\Gamma(t_4)} \;=\; C_{1234}e^{-i\pi(\omega_a+\omega_b+\omega_c)} \tag{3.27}$$

with

$$C_{1234} = e^{-i\pi\left(\frac{\gamma_K}{4}\ln\frac{|q_2|^2|q_3|^2|k_a+k_b+k_c|^2}{|k_a+k_b|^2|k_b+k_c|^2\lambda^2}\right)} \tag{3.28}$$

and

$$C_{1234}e^{i\pi(\omega_a+\omega_c)} = e^{i\delta_{1234}} \quad \text{with} \quad \delta_{1234} = \pi\frac{\gamma_K}{4}\ln\frac{|q_1||q_4||k_a+k_b|^2|k_b+k_c|^2}{|k_a+k_b+k_c|^2|k_a||k_c||q_2||q_3|}. \tag{3.29}$$

In general, the definition of the phases $\delta_{ij\ldots}$ is not unique. It depends upon which vertex factors are combined with the phases resulting from the Li_2 functions. We will fix these phases at the end of section 5.3, after we have defined our renormalized Regge pole contributions.

3.3 Propagators, Vertices, and Li_2 functions

In order to generalize this discussion, we introduce 'Feynman rules' for the calculation of the terms in the generating function. From the previous discussion it follows that there are three building blocks: propagators, vertices, and phases resulting from the Li_2 functions. Beginning with the propagators, there are two types of propagators: one corresponds to untwisted t-channel lines, the other one to a twisted line (Fig.13). For each untwisted propagator one should put $e^{-i\pi\omega_i}$, and for the twisted propagator, one puts $-\tau_i$.

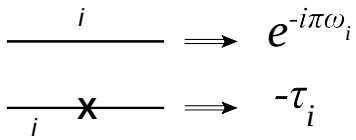

Figure 13: Two types of propagators in channel i.

The second ingredient is the production vertex for the particle a_i with the phase $\pi\omega_{a_i}$. We denote the produced momenta as k_{a_1}, k_{a_2}, k_{a_3}, ... There are four types of vertices. Three vertices are *simple* - with at most only one twisted propagator line (upper line in Fig 14), and the rule is $e^{i\pi\omega_{a_i}}$. For the "doubly-twisted" vertex (the lower line in the Fig 14) we have the conjugated rule $e^{-i\pi\omega_{a_k}}$:

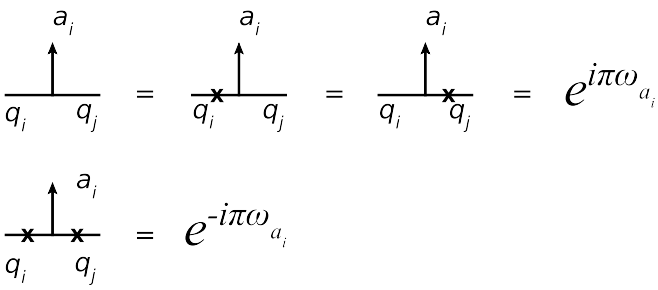

Figure 14: Four types of vertices for the production of a particle with momentm k_{a_i}.

For completeness we recapitulate the expressions for the different ω's presented here. The propagator in Fig.13 corresponds to the Regge trajectory, which is given by

$$\omega_i \;=\; -\frac{\gamma_K}{4}\ln\frac{|q_i|^2}{\lambda^2}, \tag{3.30}$$

while the vertex function ω_{a_i} corresponds to

$$\omega_{a_i} \;=\; -\frac{\gamma_K}{8}\ln\left(\frac{|q_i|^2|q_j|^2}{|q_i-q_j|^2\lambda^2}\right) \;\;;\;\; (j=i+1), \tag{3.31}$$

where $q_i - q_j \;=\; k_{a_i}$.

The final ingredient is the phase resulting from the Li_2 functions. It depends on the kinematic regions, and it is convenient to find graphical rules for deriving these contribution. The idea of twisting the diagram is equivalent to changing the kinematic regions of energy variables $s_{ij..k}$. Consider the diagram in Fig.15:

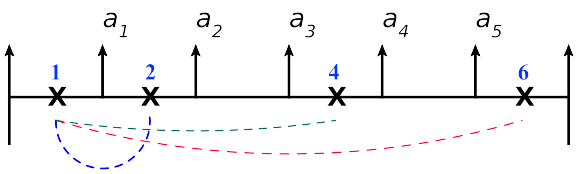

Figure 15: Rules for obtaining the Li_2 functions for a particular kinematic region (see text).

We connect crosses by lines. Each connecting line - except for those which embrace a single production vertex - corresponds to a phase (anharmonic cross ratio) which has been rotated: $\Phi \to e^{\pm 2\pi i}$, and for each rotated phase the corresponding Li_2 function has to be analytically continued and produces a non vanishing phase. The sign in the exponent can be determined by counting the number of crosses embraced by the line: if the number is even, we have $\Phi \to e^{-2\pi i}$, otherwise $\Phi \to e^{+2\pi i}$. A simple example has already been given above, the case $2 \to 4$:

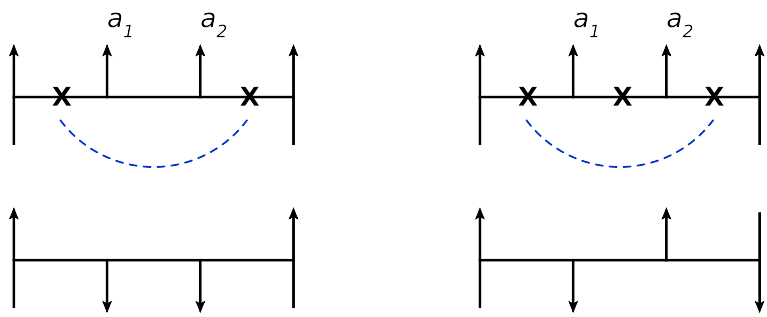

Figure 16: Example for the relation between connecting lines and kinematic regions

For the coefficient $\tau_1\tau_3$ (left part of Fig.16) there is only one such line which corresponds to the phase $\Phi = \frac{(-s)(-s_2)}{(-s_{012})(-s_{1223})}$ (Eq.(3.8)), and there is no cross ('zero cross') inside the line. This phase is rotated by $\Phi \to e^{-2\pi i}$. The analytic continuation of the Li_2-function

leads to the expression Eq.(3.13) which we denote by the 'potential' V_{13}:

$$e^{i\pi V_{13}} = e^{i\pi \frac{\gamma_K}{4} \ln \frac{|q_1|^2 |q_3|^2}{|q_1 - q_3|^2 \lambda^2}}. \tag{3.32}$$

If we apply the same discussion to the coefficient of $\tau_1\tau_2\tau_3$ (right part of Fig.16), we have one cross inside the line, the phase is rotated by $\Phi \to e^{+2\pi i}$, and the analytic continuation of the Li_2-function gives

$$e^{-i\pi V_{13}} = e^{-i\pi \frac{\gamma_K}{4} \ln \frac{|q_1|^2 |q_3|^2}{|q_1 - q_3|^2 \lambda^2}} \tag{3.33}$$

We generalize the notion of a 'potential' for the interaction between two crosses in the t_i-channel and the t_j-channel:

$$V_{ij} \;=\; \frac{\gamma_K}{4} \ln \frac{|q_i|^2 |q_j|^2}{|q_i - q_j|^2 \lambda^2}. \tag{3.34}$$

Returning to the production vertices ω_{a_i}, it is convenient to extend the notion of the 'potential' also to neighboring lines which encircle not more than one production vertex:

$$V_{ii+1} = -2\omega_{a_i}. \tag{3.35}$$

With this definition we modify our rules for the production vertex: instead of writing $e^{\pm i\pi\omega_{a_i}}$ (depending on whether we have crosses on both sides of the produced particle a_i) we adopt the following rule: for each vertex we write the unique factor $e^{+i\pi\omega_a}$, and for production vertices with crosses on both sides we include the additional factor

$$e^{i\pi V_{ii+1}}. \tag{3.36}$$

This allows to include into our rules, in Fig.15, also the short line around the vertex a_1: now each line which connects crosses in the t_i and the t_j channel obtains a factor

$$e^{\pm i\pi V_{ij}}. \tag{3.37}$$

If the channels i and j are adjacent (i.e. $j = i + 1$ and they enclose a production vertex) the sign is always positive. Otherwise the counting rules of crosses inside the lines apply:

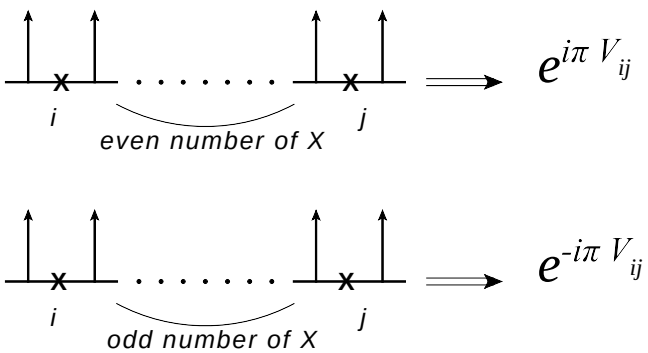

Figure 17: Sign of the phase depending on the number of crosses "**X**" between two twists i and j.

Concluding everything, we formulate 'Feynman'-like rules for the calculation of the coefficients in the monomial expansion of the amplitude A_{BDS} (Eq.(3.1)). Each coefficient $a_{i..j}$ will be written in the form

$$a_{i..j} = \pm|a_{i..j}|e^{i\varphi_{i...j}}e^{i\delta_{i...j}}, \tag{3.38}$$

and for the overall sign and for the sum of the phases in the exponent we have the following rules:

- for each t-channel we write a propagator (twisted and untwisted) according to the rules:
  - twisted propagator: $\to -1$
  - untwisted propagator in channel t_i: $\to e^{-i\pi\omega_i}$
- write the product of phase factors of vertices for all produced particles: $e^{i\pi\left(\omega_{a_1}+\omega_{a_2}+\dots\right)}$
- write all pairwise interactions $e^{\pm i\pi V_{ij}}$, $i \neq j$ with the sign $(-1)^n$ in the exponent. Here n is the number of crosses encircled by the pair (ij).

These rules uniquely define the *sum of all phases*. For our purposes, however, we go one step further and divide this sum into two terms, $i(\varphi_{i...j} + \delta_{i...j})$. Examples have been given in section 3.2 for the case $2 \to 4$. The first part, $i\varphi_{i...j}$, contains all the propagators, and it may contain some of the production vertices. The second part has to be conformal invariant. From these requirements alone, we do not find a unique separation into the two terms, $i(\varphi_{i...j} + \delta_{i...j})$. We will come back to this quetion in our final section 4.4. As an example of applying these rules, we return to the diagram in Fig.15:

- propagators: $(-)(-)e^{-i\pi\omega_3}(-)e^{-i\pi\omega_5}(-)$
- vertices: $e^{i\pi\left(\omega_{a_1}+\omega_{a_2}+\omega_{a_3}+\omega_{a_4}+\omega_{a_5}\right)}$
- potentials: $e^{i\pi(V_{12}-V_{14}+V_{16}+V_{24}-V_{26}+V_{46})}$

The final expression for Fig.15 becomes:

$$e^{i\pi\left(\omega_{a_1}+\omega_{a_2}+\omega_{a_3}+\omega_{a_4}+\omega_{a_5}\right)}e^{-i\pi(\omega_3+\omega_5)}e^{i\pi(V_{12}-V_{14}+V_{16}+V_{24}-V_{26}+V_{46})}\tau_1\tau_2\tau_4\tau_6. \tag{3.39}$$

The logarithmic form of the potential Eq.(3.34), together with the exponential form of the coefficient of the monomial in Eq.(3.39), allow an interesting analogy. Namely, we can interpret V_{ij} as a two dimensional Coulomb potential of the interaction of two point charges i and j, derived from the Polyakov string action. In more detail, we consider the product of k vertex operators, i.e. correlators of the form

$$\langle 0|e^{i\pi\sum_{r=1}^{k} c_r[\phi(\vec{\rho}_r)-\phi(\vec{\rho}_0)]}|0\rangle, \tag{3.40}$$

where the averaging is done with the free action

$$e^{i\frac{1}{2}\int d^2\vec{\rho}[\partial_\sigma\phi(\vec{\rho})]^2} \tag{3.41}$$

and $c_r = (-1)^r$ is the charge. It is convenient to introduce the following currents:

$$\pi \sum_{r=1}^{k} c_r \left[\phi(\vec{\rho}_r) - \phi(\vec{\rho}_0)\right] = \pi \int d^2\vec{\rho}\, \phi(\vec{\rho}_r) \sum_{r=1}^{k} c_r \left(\delta^{(2)}(\vec{\rho} - \vec{\rho}_r) - \delta^{(2)}(\vec{\rho} - \vec{\rho}_0)\right) = \\ = \int d^2\vec{\rho}\, \phi(\vec{\rho}) J(\vec{\rho}), \tag{3.42}$$

with

$$J(\vec{\rho}) = \pi \sum_{r=1}^{k} c_r \left(\delta^{(2)}(\vec{\rho} - \vec{\rho}_r) - \delta^{(2)}(\vec{\rho} - \vec{\rho}_0)\right). \tag{3.43}$$

One can calculate the gaussian integral of the neutral system

$$Z[J] = \int e^{i \int d^2\vec{\rho} \left[\frac{1}{2}(\partial_\sigma \phi(\vec{\rho})^2 + \phi(\vec{\rho}) J(\vec{\rho})\right]} \mathcal{D}\phi \tag{3.44}$$

by using the inverse of the two dimensional Laplacian:

$$\partial_\sigma^2 \tilde{\phi}(\vec{\rho}) = J(\vec{\rho}) \rightarrow \tilde{\phi}(\vec{\rho}) = \frac{1}{4\pi} \int d^2\vec{\rho}' J(\rho') \log\left(|\vec{\rho} - \vec{\rho}'|^2\right) \tag{3.45}$$

and by shifting the field variables: $\phi = \phi' + \tilde{\phi}$. One obtains:

$$\int d^2\vec{\rho} \left[\frac{1}{2}\left(\partial_\sigma \phi(\vec{\rho})\right)^2 + \phi(\vec{\rho}) J(\vec{\rho})\right] = \\ = \frac{1}{8\pi} \int\int d^2\vec{\rho} d^2\vec{\rho}' J(\vec{\rho}) \log\left(|\vec{\rho} - \vec{\rho}'|^2\right) J(\vec{\rho}') + \frac{1}{2} \int d^2\vec{\rho} \left(\partial_\sigma \phi'\right)^2. \tag{3.46}$$

From this expression one derives, for the correlator (3.40), an exponential of the form:

$$V_{ij} = \frac{\pi}{8} \sum_{r,r'=1}^{k} c_r\, c_{r'} \left[\log|\vec{\rho_r} - \vec{\rho_r'}|^2 - \log|\vec{\rho_r} - \vec{\rho_0}|^2 - \log|\vec{\rho_{r'}} - \vec{\rho_0}|^2 + \log|\vec{\rho_{00}}|^2\right]. \tag{3.47}$$

In the first term one recognizes the logarithmic part of the 'potential' V_{ij} between two crosses defined in (3.34). In particular, we notice the universal short range interaction between two adjacent crosses:

$$V_{i,i+1} \sim \log|\vec{\rho_i} - \vec{\rho}_{r_{i+1}}|^2. \tag{3.48}$$

Finally, returning to the generating function introduced at the beginning of this section:

$$A_{BDS} = a_0 + a_1\tau_1 + a_2\tau_2 + ... + a_n\tau_n + a_{12}\tau_1\tau_2 + a_{13}\tau_1\tau_3 + ...a_{1...n}\tau_1\tau_2...\tau_n,$$

we can interpret this expression also as a partition function, where each terms represents one of the correlators described above. For the rest of this paper, we will not pursue this analogy any further.

4 Subtractions from Regge pole contributions

In the previous sections we have seen that the Regge pole formula, based upon factorization and the analytic decomposition into 5 terms (for the case $2 \to 4$) or 14 terms (for the case $2 \to 5$), exhibits, when continued into different kinematic regions with positive and negative energies, terms with unphysical singularities. At the end of section 3 we have indicated that Regge cut terms are needed in order to compensate these unwanted singularities. The subsequent analysis of the BDS predictions, on the other hand, has shown that the BDS formula is not in agreement with the Regge pole structure, because it contains contributions from the Li_2 functions. As a consequence, depending on the kinematic region, it contains phases which, in the $2 \to 4$ case [2], have been understood as a signal of the beginning of Regge cut contributions. In this final section we concentrate on the case $n = 7$, and we show that Regge cut contributions can be determined which satisfy the two conditions:

1. the terms with Regge cuts have the correct phase structure for absorbing the unwanted pole terms,

2. after absorbing the unphysical pole pieces of the Regge poles into the Regge cut terms, we are left with conformal invariant Regge pole contributions.

To be definite, our construction proceeds as follows. Initially we have the Regge pole terms which, as we have stated, factorize in the kinematic region of positive energies but, when analytically continued, lead to unphysical singularities. They have to be absorbed into Regge cut contributions. Schematically we therefore write:

$$A = A_{pole} + A_{cut}, \tag{4.1}$$

where the pole contributions are listed in Appendix A, and the phase structure of the cut contributions have to be discussed in the following. Their contributions to the scattering amplitude depend upon the kinematic region: they vanish for positive energies (and in the euclidean region), and they are nonzero in exactly those kinematic regions where the Regge poles exhibit the unphysical singularities. After having fixed the subtractions we will arrive at modified expressions:

$$A = A'_{pole} + A'_{cut}, \tag{4.2}$$

where the primes indicate that, in each physical region with Regge cuts and singular Regge pole pieces, the unphysical singular pieces have been absorbed by the Regge cuts. In this new representation the amplitude, for each region $\tau_i ... \tau_j$, will be written in the form:

$$A = A_{BDS} R, \tag{4.3}$$

where A_{BDS} contains the phase factors $\varphi_{i...j}$ and $\delta_{i...j}$ calculated in the previous section 3.3, and the conformal invariant remainder function R is of the form:

$$Re^{i\delta} = \text{conformal pole} + \text{conformal Regge cut}. \tag{4.4}$$

For illustration we return, once more, very briefly to the $2 \to 4$ case [16]. As shown in [2, 3], the Regge cut piece has the phase $ie^{-i\pi\omega_2}$. To see this we remind that, in the decompositon of the $2 \to 4$, amplitude, the Regge cut appears in two of the five terms. Their phase structure follows from the energy factors which, in the notation of [2, 3], is

$$\begin{aligned} W_3 &\sim (-s_2)^{\omega_{21}}(-s_{012})^{\omega_{13}}(-s)^{\omega_3}V_{cut} \\ W_4 &\sim -(-s_2)^{\omega_{23}}(-s_{123})^{\omega_{31}}(-s)^{\omega_1}V_{cut}. \end{aligned} \tag{4.5}$$

The coefficient V_{cut} is the same in both terms, and there is relative minus sign between the two partial waves. From this structure one derives easily that the sum of these two contributions vanishes in the physical region where all energies are positive (a phase factor $e^{-i\pi}$ form each energy), in the euclidean region (all energies negative, i.e. all phases reduce to unity), and also in the region where only one energy is negative. In contrast to this, in the region $s, s_2 > 0$, $s_{012}, s_{123} < 0$, the sum is proportional to $ie^{-i\pi\omega_2}$. On the other hand, the Regge pole, when continued into this kinematic region, takes the form Eq.(2.22), i.e. we have one term proportional to $e^{-i\pi\omega_2}$, and two terms proportional to $ie^{-i\pi\omega_2}$. The latter ones have the same phase structure as the Regge cut contribution, and thus they can be combined with the Regge cut: we can remove them by a special contribution ('subtraction') inside the Regge cut. What is then left is the first term of the Regge pole contribution

$$e^{-i\pi\omega_2}\cos\pi(\omega_a - \omega_b). \tag{4.6}$$

Here the argument of the cos is conformal invariant. Therefore, this expression defines, for this kinematic region, a 'conformal' Regge pole contribution. The amplitude can be written as

$$A = A_{BDS}R, \tag{4.7}$$

where A_{BDS} contains the phase factor $e^{-i\pi\omega_2}$, and

$$Re^{i\delta} = \cos\pi(\omega_a - \omega_b) + i\ \text{ReggeCut}. \tag{4.8}$$

The new Regge cut contribution is expected to be conformal invariant.

4.1 Analytic structure of the $2 \to 5$ scattering amplitude

In the following we will extend this analysis to the $2 \to 5$ case. We now have three different Regge cut contributions. They are illustrated in the following figure:

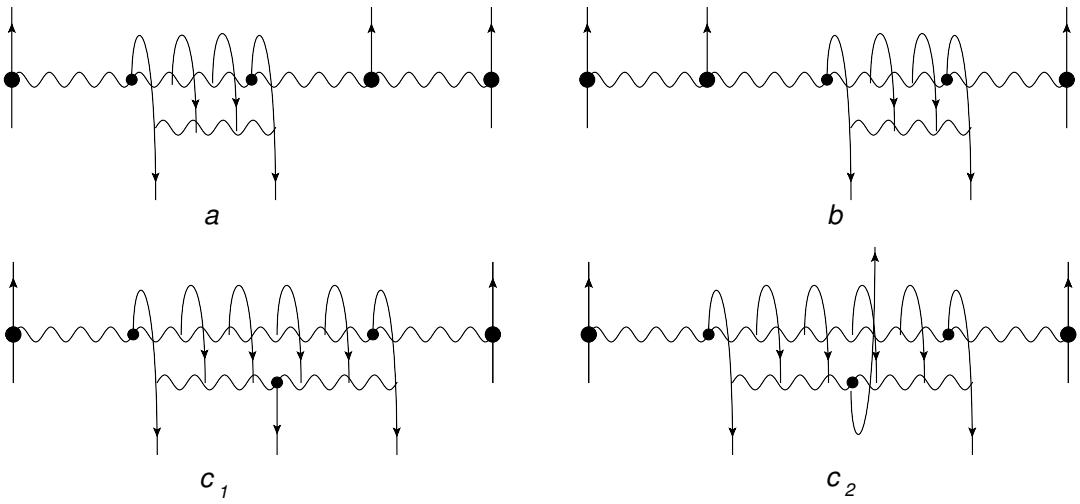

Figure 18: Regge cut contributions for the $2 \to 5$ scattering amplitude

In addition to the t-channels where the Regge cuts appear, we have also indicated a few kinematic regions in which these Regge cut contribute. In the generating functional, these kinematic regions correspond to the coefficients of $\tau_1\tau_3$, $\tau_2\tau_4$, $\tau_1\tau_4$, and $\tau_1\tau_2\tau_3\tau_4$. The analytic representation of the $2 \to 5$ amplitude contains 14 different terms. They are illustrated below:

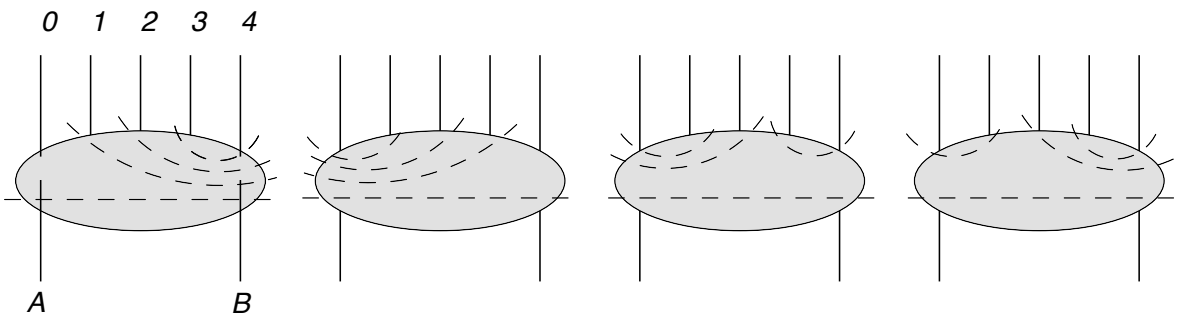

Figure 19: Terms without Regge cuts

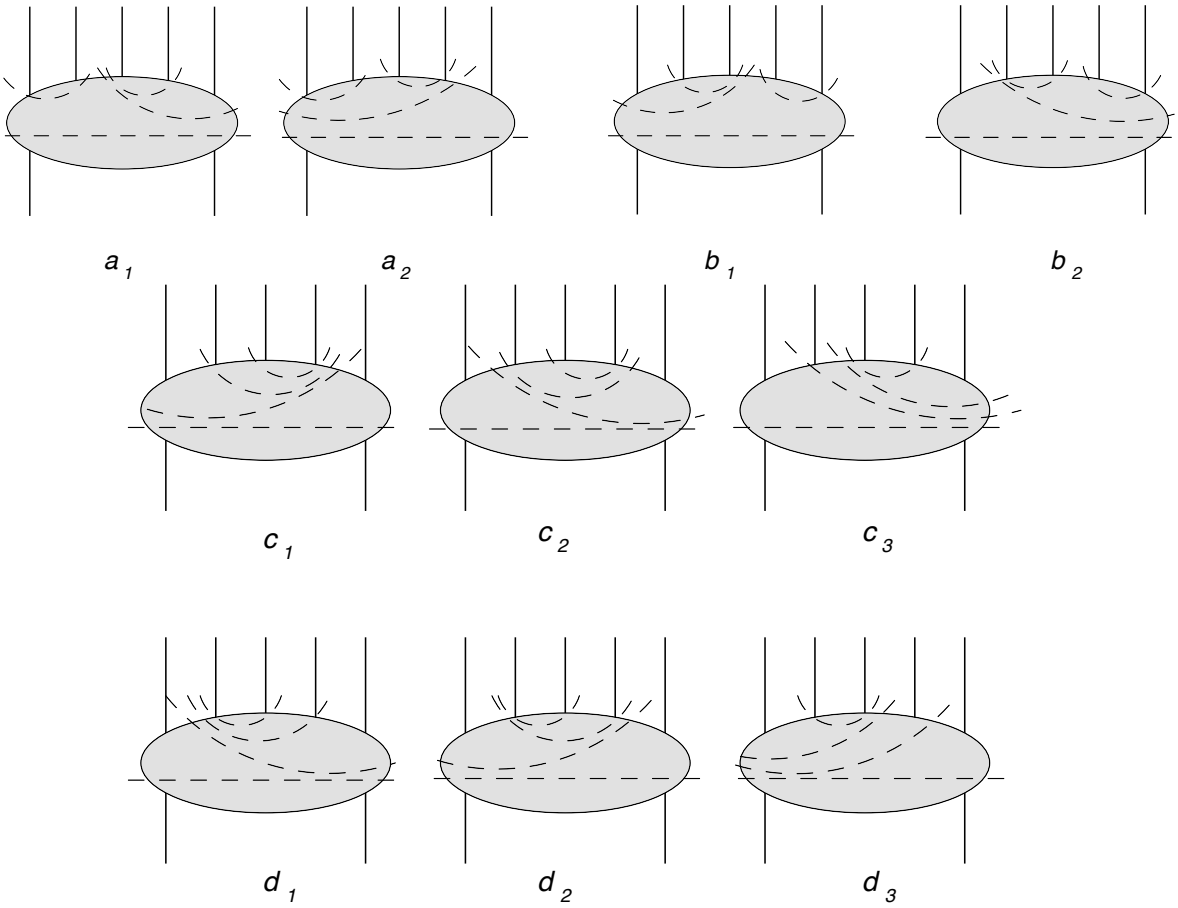

Figure 20: Terms which contain Regge cut contributions: two doublets (a) and (b), and two triplets (c) and (d)

Here each term is written as a multiple Sommerfeld Watson integral over ω variables, and the integrand comes as a product of energy factors which contain all the phases and a real-valued partial wave. For simplicity, we will disregard the ω-integration in the rest of our paper.. The analytic structure of these terms is in agreement with the Steinmann relations, i.e. each of these 14 terms has a maximal set of energy discontinuities in non-overlapping channels (denoted by dashed lines).

Only ten of these 14 terms contain Regge cut contributions: they can be arranged as two doublets a, b and two triplets c, d. The 'short' Regge cut in the t_3 channel (Fig.18b) is contained in the first doublet a_1 and a_2 and in the first triplet, c_1, c_2, and c_3. Similarly, the 'short' Regge cut in the t_2 channel (Fig.18a) is contained in the second doublet, b_1 and b_2, and in the second triplet, d_1, d_2, and d_3. Finally, the 'long' cut in Fig.18c_1 and Fig.18c_1 appears in the first two terms of both triplets. In each term, these Regge cut contributions are additive. As an example, the first two terms of the triplets are sums of two terms, each of a 'short' cut and of the 'long' cut.

Next we are interested in the phase structure of these terms: it follows from the energy

factors which we list in the following. For the doublets we have:

$$\begin{aligned} a_1 &= (-s_1)^{\omega_{12}} (-s_3)^{\omega_{34}} (-s_{234})^{\omega_{42}} (-s)^{\omega_2} \\ a_2 &= (-s_1)^{\omega_{12}} (-s_3)^{\omega_{32}} (-s_{0123})^{\omega_{24}} (-s)^{\omega_4} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} b_1 &= (-s_2)^{\omega_{21}} (-s_{012})^{\omega_{13}} (-s_4)^{\omega_{43}} (-s)^{\omega_3} \\ b_2 &= (-s_2)^{\omega_{23}} (-s_4)^{\omega_{43}} (-s_{1234})^{\omega_{31}} (-s)^{\omega_1} . \end{aligned} \tag{4.10}$$

Similarly for the two triplets:

$$\begin{aligned} c_1 &= (-s_3)^{\omega_{32}} (-s_{123})^{\omega_{21}} (-s_{0123})^{\omega_{14}} (-s)^{\omega_4} \\ c_2 &= (-s_3)^{\omega_{32}} (-s_{123})^{\omega_{24}} (-s_{1234})^{\omega_{41}} (-s)^{\omega_1} \\ c_3 &= (-s_3)^{\omega_{34}} (-s_{234})^{\omega_{42}} (-s_{1234})^{\omega_{21}} (-s)^{\omega_1} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} d_1 &= (-s_2)^{\omega_{23}} (-s_{123})^{\omega_{34}} (-s_{1234})^{\omega_{41}} (-s)^{\omega_1} \\ d_2 &= (-s_2)^{\omega_{23}} (-s_{123})^{\omega_{31}} (-s_{0123})^{\omega_{14}} (-s)^{\omega_4} \\ d_3 &= (-s_2)^{\omega_{21}} (-s_{012})^{\omega_{13}} (-s_{0123})^{\omega_{34}} (-s)^{\omega_4} . \end{aligned} \tag{4.12}$$

It should be noted that in these expressions, for simplicity, we have disregarded κ factors as well as energy scales. As an example, the complete form of d_1 from Eq.(4.12) which includes these coefficients has the form [2, 3]:

$$d_1 = \left(\frac{-s_2}{\mu^2}\right)^{\omega_{23}} \left(\frac{-s_{123}\kappa_{23}}{\mu^4}\right)^{\omega_{34}} \left(\frac{-s_{1234}\kappa_{23}\kappa_{34}}{\mu^6}\right)^{\omega_{41}} \left(\frac{-s\kappa_{12}\kappa_{23}\kappa_{34}}{\mu^8}\right)^{\omega_1} ,$$

where

$$\kappa_{ii+1} = \frac{s_i s_{i+1}}{s_{i-1ii+1}} = |q_i - q_{i+1}|^2 \quad , \quad \text{and the usual convention:} \quad s_i \equiv s_{i-1i}. \tag{4.13}$$

As a result of these κ-factors, all energy factors d_1 etc. can be written in the common form

$$d_1 = \text{phase factor} \times \left(\frac{|s_1|}{\mu^2}\right)^{\omega_1} \left(\frac{|s_2|}{\mu^2}\right)^{\omega_2} \left(\frac{|s_3|}{\mu^2}\right)^{\omega_3} \left(\frac{|s_4|}{\mu^2}\right)^{\omega_4} \tag{4.14}$$

In the following, our interest will first be devoted to the phase factors derived from Eqs.(4.9) - (4.12): they depend upon the kinematic regions. In the next step, we will determine the coefficients that accompany the phase factors; they are real-valued and do not depend upon the kinematic region we are considering.

4.2 Determination of the coefficients of the partial waves

As we have said before,the kinematic regions in which the Regge pole expressions (listed in Appendix A) contain poles of the type $1/\sin(\pi\omega_i)$ are the same regions for which we also have Regge cut contributions [3]. For each such region we write schematically:

$$f = f_{pole} + f_{cut}. \tag{4.15}$$

In this notation, f denotes the sum of all those terms which contribute to this region, and it contains both the energy (and phase) factors and their real-valued coefficient, the partial waves. As a consequence, the form of the f will be different in different kinematic regions. In general, the Regge cut piece will a sum of several terms: for example, the coefficient of $\tau_1\tau_2\tau_3\tau_4$ contains the two 'short' cuts and the 'long' cut:

$$f_{cut} = f_{\omega_2} + f_{\omega_3} + f_{\omega_2\omega_3}. \tag{4.16}$$

In this paper we will not address the full structure of these Regge cut terms. Instead, we will concentrate on the overall phases, f_{cut}^{phase}, and only those pieces of the Regge cuts which absorb the 'unwanted' pieces of the Regge pole contributions, i.e. those terms which have the unphysical poles of the form $1/\sin(\pi\omega_i)$: δf_{cut}, namely:

$$f_{\omega_i} \;=\; N_{\omega_i} f_{\omega_i}^{phase} \delta f_{\omega_i}\,. \tag{4.17}$$

We therefore have to keep in mind that the f_{cut} which we discuss in the following contain only the subtraction terms but not the full Regge cut terms. We will name this procedure 'subtraction', in analogy to the removal of ultraviolet divergences in the renormalization of quantum field theory.

In more detail, for the two doublets and for the two triplets, we will find a set of coefficients which should satisfy the following requirements:
(i) the Regge cuts contribute only in specific kinematic regions where the so-called Mandelstam conditions are fulfilled. In particular, Regge cuts do not contribute to the euclidean region or to the physical region where all energies are positive.
(ii) Phases of the Regge cut contributions have to match the 'unwanted' pieces of the Regge pole contributions, i.e. those terms which have the unphysical poles of the form $1/\sin(\pi\omega_i)$.
(iii) After having absorbed these 'unwanted' pole terms into the Regge cut terms, the remaining Regge pole contributions have to be conformal invariant.

Let us begin with the 'short' Regge cut in the t_3 which appears in the terms labelled by a_1, a_2, c_1, c_2, and c_3. We are searching for real-valued coefficients of theses terms which, for the sum of all five terms, lead to correct phases in all kinematic regions. First we notice that, in the region of all energies being positive, all c_i have the common phase $e^{-i\pi\omega_3}$, and all a_i the common phase $e^{-i\pi(\omega_1-\omega_2+\omega_3)}$. The absence of the Regge cut in this region implies that the sum of the terms a_1, a_2 and the sum of the terms c_1, c_2, c_3 must be zero separately. This alone does not fix the coefficients of the c_i. We make the ansatz (which

[3] Conditions for the existence of the Regge cuts have been formulated in the appendix of [21]

will be justified in a moment) and chose, for the coefficients of c_1, c_2, and c_3, the relative weights $\frac{1}{2}$, $\frac{1}{2}$, and -1, resp. Similarly, for the coefficients of a_1 and a_2 the relative weights are $+1$ and -1, resp. In order to determine the common factors of the c_i, we go to the region $\tau_2\tau_4$: here the terms c_1 and c_2 have the common phase $e^{-i\pi\omega_3}e^{-i\pi(\omega_4-\omega_2)}$, whereas c_3 has the phase $e^{-i\pi\omega_3}e^{-i\pi(\omega_2-\omega_4)}$. Taking into account the relative weights given above, the sum of the terms c_i gives the phase $e^{-i\pi\omega_3}2i\sin\pi(\omega_2-\omega_4)$. In the same way, the sum of a_1 and a_2 lead to the factor $e^{-i\pi(\omega_3+\omega_1-\omega_2}2i\sin\pi(\omega_4-\omega_2)$. Combining the sum of the c_i-terms with the sum of the a_i terms we still have the freedom to chose coefficients: with the choice $\sin(\pi\omega_{2a})$ and $\sin(\pi\omega_{1a})$ we have, again for the region $\tau_2\tau_4$, the result:

$$\begin{aligned}\sin(\pi\omega_{2a})\left\{\frac{1}{2}c_1+\frac{1}{2}c_2-c_3\right\}+\sin(\pi\omega_{1a})\left\{a_1-a_2\right\}\\ =2i\sin(\pi\omega_{12})\sin(\pi\omega_{34})e^{-i\pi\omega_1}e^{i\pi\omega_a}e^{-i\pi\omega_3}.\end{aligned} \tag{4.18}$$

The phases are in agreement with what one expects from Regge factorization: the Regge cut in the t_3 channel has the same phase in the $2\to 4$ amplitude: $ie^{-i\pi\omega_3}$, and the phase of the t_1 channel together with the production vertex of particle a factorizes: $e^{-i\pi\omega_1}e^{i\pi\omega_a}$.

However, this is not yet the final answer for the cut in the ω_3 channel. Namely, when going to the region $\tau_1\tau_4$, we find the phases:

$$a_1-a_2=e^{-i\pi\omega_3}2i\sin(\pi\omega_{24}) \tag{4.19}$$

$$\frac{1}{2}(c_1+c_2)-c_3=e^{-i\pi\omega_3}i\sin(\pi\omega_{14}). \tag{4.20}$$

Together with the prefactors $\sin(\pi\omega_{1a})$, $\sin(\pi\omega_{2a})$, these terms cannot be combined to arrive at the the expected phase $e^{-i\pi(\omega_2+\omega_3)}$. As a solution, we chose to completely cancel this contribution by adding a term proportional to c_1-c_2. In the region $\tau_1\tau_4$ we have:

$$c_1-c_2=e^{-i\pi\omega_3}2i\sin(\pi\omega_{14}), \tag{4.21}$$

and with the following coefficients we arrive at our final answer for the 'short' cut in the ω_3-channel:

$$\begin{aligned}N_{\omega_3}f_{\omega_3}^{phase} &= \sin(\pi\omega_{2a})\left\{\frac{1}{2}c_1+\frac{1}{2}c_2-c_3\right\}+\sin(\pi\omega_{1a})\left\{a_1-a_2\right\}-\\ &-\frac{1}{\sin(\pi\omega_{14})}\left(\frac{1}{2}\sin(\pi\omega_{14})\sin(\pi\omega_{2a})+\sin(\pi\omega_{42})\sin(\pi\omega_{1a})\right)\left\{c_1-c_2\right\}.\end{aligned} \tag{4.22}$$

We make sure that, by analytically continuing this function f_{ω_3} into different kinematic regions, we find correct answers. In detail, the results are the following: non-zero values appear only in the four kinematic regions $\tau_2\tau_4$, $\tau_1\tau_2\tau_3\tau_4$, $\tau_1\tau_2\tau_4$ and $\tau_2\tau_3\tau_4$:

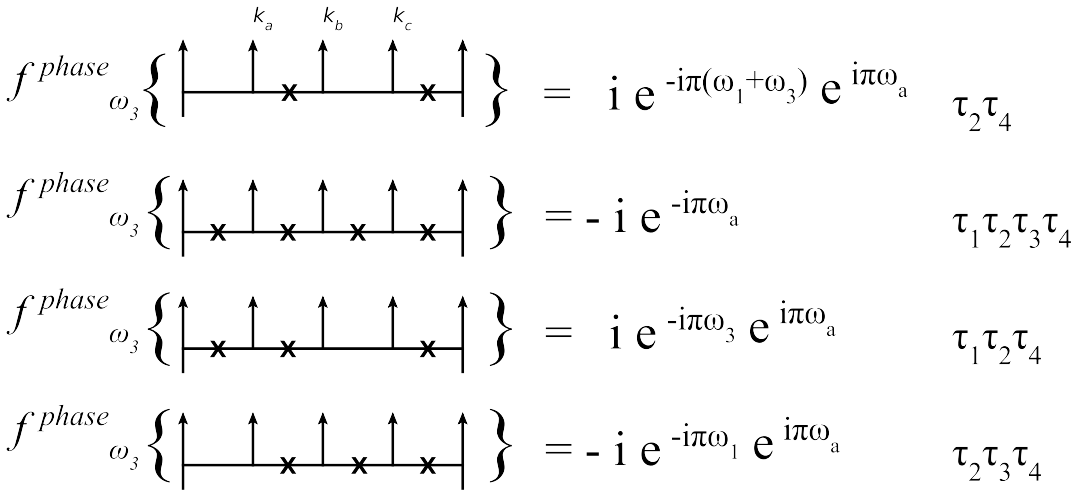

Figure 21: Analytical continuation of f_{ω_3}.

In all other kinematic regions $f^{phase}_{\omega_3}$ vanishes. The common factor N_{ω_3} is found to be:

$$N_{\omega_3} = 2\sin(\pi\omega_{24})\sin(\pi\omega_{21})\left(\frac{|s_1|}{\mu^2}\right)^{\omega_1}\left(\frac{|s_2|}{\mu^2}\right)^{\omega_2}\left(\frac{|s_3|}{\mu^2}\right)^{\omega_3}\left(\frac{|s_4|}{\mu^2}\right)^{\omega_4}. \tag{4.23}$$

A comment is in place about the second line in (4.22) which is proportional to $c_1 - c_2$. As we will show in a few moments, the combination $c_1 - c_2$ belongs to the 'long' cut in the ω_2 and ω_3 channel. The fact that this combination also participates in our calculations of the 'short' cut hints at the fact that the 'long' cut contribution may contain terms which have the ω-plane singularity structure of the 'short' cuts, i.e. there is a mixing between the different Regge cuts. We will come back to this question in a forthcoming paper.

An analogous discussion applies to the 'short' cut in the f_{ω_2} channel:

$$\begin{aligned} N_{\omega_2} f^{phase}_{\omega_2} &= \sin(\pi\omega_{3c})\left\{\frac{1}{2}d_1 + \frac{1}{2}d_2 - d_3\right\} + \sin(\pi\omega_{4c})\left\{b_1 - b_2\right\} - \\ &- \frac{1}{\sin(\pi\omega_{41})}\left(\frac{1}{2}\sin(\pi\omega_{41})\sin(\pi\omega_{3c}) + \sin(\pi\omega_{13})\sin(\pi\omega_{4c})\right)\left\{d_1 - d_2\right\}. \end{aligned} \tag{4.24}$$

We continue the function $f^{phase}_{\omega_2}$ to those four different kinematic regions where it is non-zero:

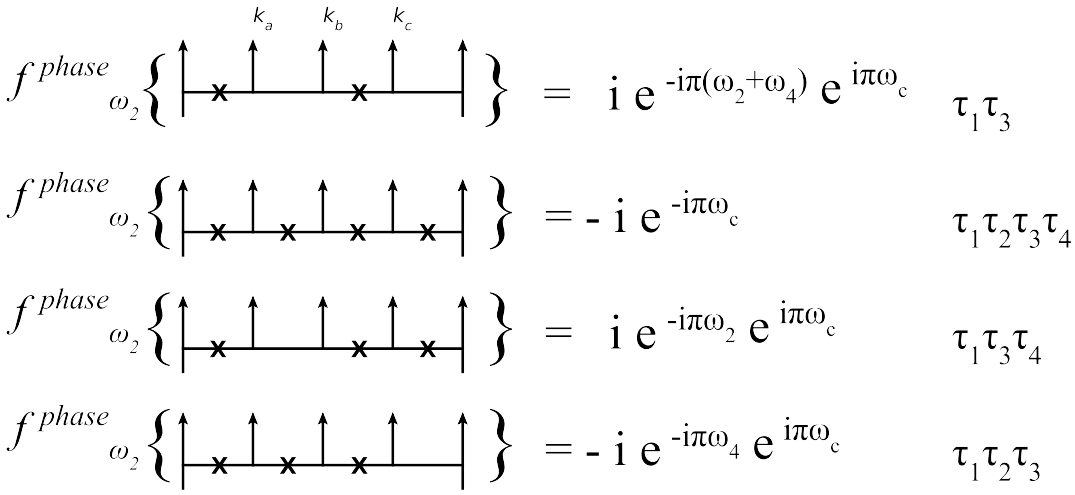

Figure 22: Analytical continuation of f_{ω_2}.

with the common factor

$$N_{\omega_2} = 2\sin(\pi\omega_{31})\sin(\pi\omega_{34})\left(\frac{|s_1|}{\mu^2}\right)^{\omega_1}\left(\frac{|s_2|}{\mu^2}\right)^{\omega_2}\left(\frac{|s_3|}{\mu^2}\right)^{\omega_3}\left(\frac{|s_4|}{\mu^2}\right)^{\omega_4}. \tag{4.25}$$

In all other kinematic regions we have $f^{phase}_{\omega_2} = 0$.

Next we turn to the 'long' Regge cut in the ω_2 and ω_3 channels simultaneously. This cut is contained in the first two terms of the triplets - c_1, c_2, d_1, d_2 of Fig.20 with the corresponding phases c_1, c_2, d_1, and d_2 in Eq.(4.11) and Eq.(4.12). Repeating our line of arguments, we first consider the region where all energies are positive: since all c_i are proportional to $e^{-i\pi\omega_3}$, all d_i proportional to $e^{-i\pi\omega_2}$, the coefficient of c_1 has to be opposite equal to that c_2, and similarly for d_1 and d_2. Turning to the region $\tau_1\tau_2\tau_3\tau_4$, the phases of $c_1 - c_2$ are

$$c_1 - c_2 = 2ie^{-i\pi\omega_2}\sin(\pi\omega_{14}). \tag{4.26}$$

We take the following linear combination:

$$\sin(\pi\omega_{3x})\{c_1 - c_2\} + \sin(\pi\omega_{2x})\{d_1 - d_2\} = 2ie^{-i\pi\omega_x}\sin(\pi\omega_{14})\sin(\pi\omega_{32}) \tag{4.27}$$

with $x = a, b, c$. Obviously, x_b would be a symmetric choice; however the singular term in the Regge pole contribution (Appendix A) has no phase $e^{-i\pi\omega_b}$, and therefore this ansatz for the Regge cut cannot be used to subtract for the subtraction. Instead, we take the linear combination of two contributions:

$$N_{\omega_2\omega_3} f^{a;phase}_{(\omega_2\omega_3)} = \sin(\pi\omega_{3a})\{c_1 - c_2\} + \sin(\pi\omega_{2a})\{d_1 - d_2\} \tag{4.28}$$

and

$$N_{\omega_2\omega_3} f^{c;phase}_{(\omega_2\omega_3)} = \sin(\pi\omega_{3c})\{c_1 - c_2\} + \sin(\pi\omega_{2c})\{d_1 - d_2\}, \tag{4.29}$$

and in the combination $Af^{a;phase}_{(\omega_2\omega_3)} + Cf^{c;phase}_{(\omega_2\omega_3)}$ we will determine real valued coefficients $A = \delta f^a_{\omega_2\omega_3}$ and $C = \delta f^c_{\omega_2\omega_3}$ which subtract the singular part of the Regge pole contribution.

Let us first study the other kinematic regions. The functions $f^{a;phase}_{\omega_2\omega_3}$ and $f^{c;phase}_{\omega_2\omega_3}$ have non-zero values in four particular kinematic regions:

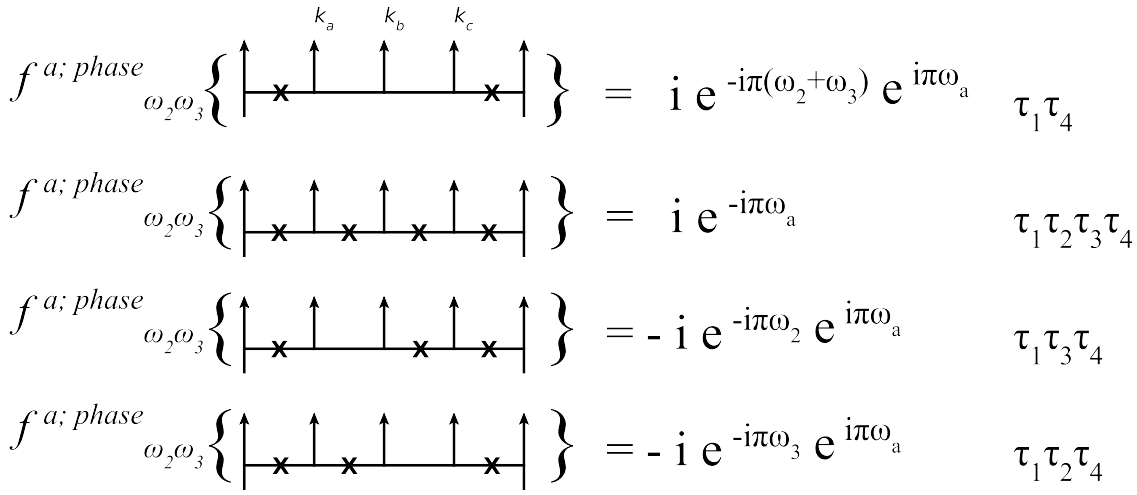

Figure 23: Analytical continuation of $f^{a;phase}_{\omega_2\omega_3}$.

and

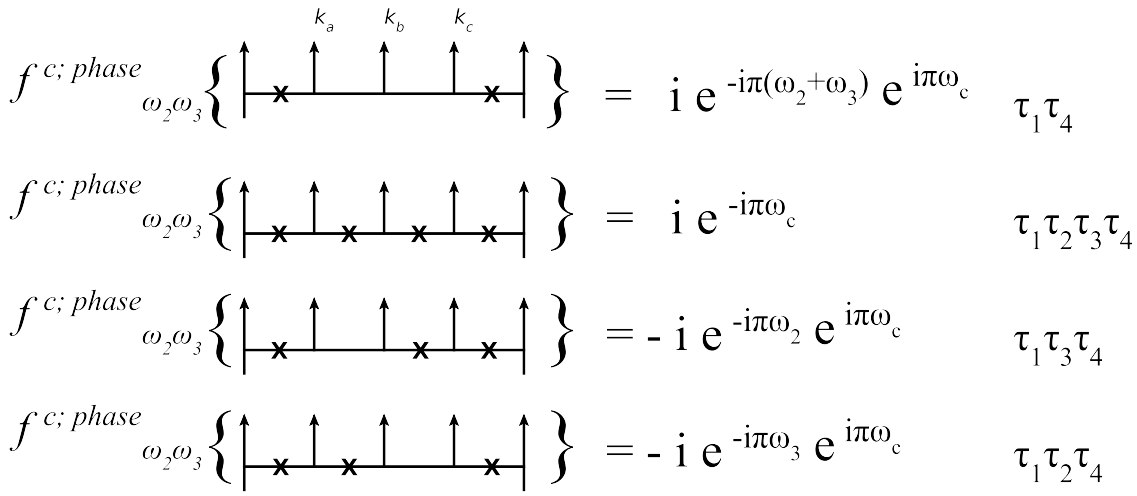

Figure 24: Analytical continuation of $f^{c;phase}_{\omega_2\omega_3}$.

The common factor is the same for $f^{a;phase}_{\omega_2\omega_3}$ and for $f^{c;phase}_{\omega_2\omega_3}$:

$$N_{\omega_2\omega_3} = 2\sin(\pi\omega_{14})\sin(\pi\omega_{32})\left(\frac{|s_1|}{\mu^2}\right)^{\omega_1}\left(\frac{|s_2|}{\mu^2}\right)^{\omega_2}\left(\frac{|s_3|}{\mu^2}\right)^{\omega_3}\left(\frac{|s_4|}{\mu^2}\right)^{\omega_4}. \tag{4.30}$$

For all other possible configuration of analytical continuation, the result is zero. Thus, the 'long' cut contributes only to these four particular kinematic regions. We combine these two terms

$$\Delta f_{\omega_2\omega_3} \;=\; \delta f^{a}_{\omega_2\omega_3} f^{a;phase}_{\omega_2\omega_3} + \delta f^{c}_{\omega_2\omega_3} f^{c;phase}_{\omega_2\omega_3} \tag{4.31}$$

with real coefficients $\delta f^a_{\omega_2\omega_3}$ and $\delta f^c_{\omega_2\omega_3}$, and we find for the different regions:[4]

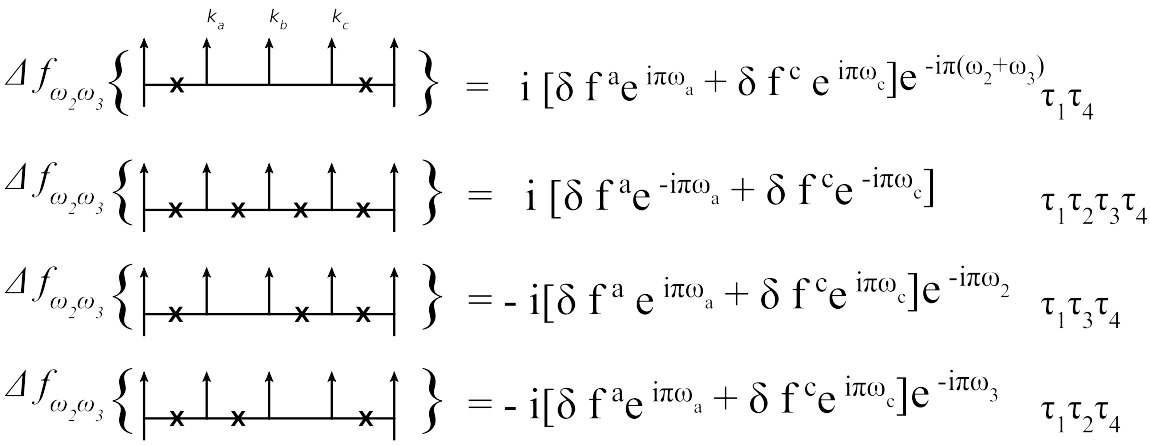

Figure 25: Analytical continuation of $\Delta f_{\omega_2\omega_3}$.

It is remarkable that the square bracket is the same in all four cases, up to complex conjugation of the phases. Below we will determine the coefficients $\delta f^a_{\omega_2\omega_3}$ and $\delta f^c_{\omega_2\omega_3}$. Summarizing this subsection, we have determined coefficients of the partial waves $a_1, ...d_3$ which, for all those kinematic region which contains Regge cuts, can be combined to give a 'good' phase structure. Returning to Eq.(4.17), we have determined the normalization factors N and the phases f^{phase}_ω. In the following we still have to calculate the coefficients δf_ω, and we have to show that our ansatz matches the phases of the singular pieces of the Regge pole terms (studied in section 2) and thus allows to absorb these singularities by the Regge cuts.

4.3 Redefinitions of Regge pole terms: subtractions

Let us now turn to the subtraction procedure. Figs.21, 22, 25 show the kinematic regions in which the different Regge cuts, f_{ω_3}, f_{ω_2}, and $f_{\omega_2\omega_3}$, contribute. There are two regions ($\tau_2\tau_4$ and $\tau_2\tau_3\tau_4$) in which only f_{ω_3} contributes, two regions ($\tau_1\tau_3$ and $\tau_1\tau_2\tau_3$) where only f_{ω_2} is nonzero, and one region ($\tau_1\tau_4$) where only the 'long' cut appears. In the remaining three regions we have combinations of several Regge cuts. In particular, the region $\tau_1\tau_2\tau_3\tau_4$ sees all cut contributions. We begin with the 'short' cut f_{ω_3}: from the region $\tau_2\tau_4$ we determine the subtraction δf_{ω_3} which then fixes the subtractions in all regions listed in Fig.21. Similarly, δf_{ω_2} is obtained from the region $\tau_1\tau_3$ and will be used in all regions listed in Fig.22. Finally, in the region $\tau_1\tau_2\tau_3\tau_4$ we can fix the remaining subtraction, $\delta f_{\omega_2\omega_3}$.

We begin with the region $\tau_2\tau_4$ where only the 'short' cut in the t_3 channel contributes. From Appendix A we read off the Regge pole contribution in the region $\tau_2\tau_4$:

$$f^{\tau_2\tau_4}_{pole} = e^{-i\pi(\omega_1+\omega_3)} e^{i\pi\omega_a} \left(\cos(\pi\omega_{bc}) + \left[e^{i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) - 2i e^{i\pi\omega_3} \frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)} \right] \right). \tag{4.32}$$

[4]We omitted the subscript '$\omega_2\omega_3$' of the $\delta f^{a,c}$ in the figure for the sake of simplicity.

The square bracket expression on the rhs can also be written as (Eq.(2.22) and Eq.(2.23))

$$[...] = +i\sin(\pi(\omega_b+\omega_c)) - 2i\frac{\cos(\pi\omega_3)\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}, \tag{4.33}$$

which shows that it is purely imaginary and can also be written as

$$[...] = -\left[e^{-i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]. \tag{4.34}$$

Thus the phase structure of the second part of Eq.(4.32) is the same as that of the cut contribution $f^{phase}_{\omega_3}$ in the first line of Fig.21, and we define the subtraction term as follows:

$$\begin{aligned}\delta f_{\omega_3} &= i\left[e^{i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) - 2ie^{i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]\\ &= -i\left[e^{-i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]\\ &= -\left[\sin(\pi(\omega_b+\omega_c)) - 2\frac{\cos(\pi\omega_3)\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right].\end{aligned} \tag{4.35}$$

Having fixed the subtraction δf_{ω_3} in the $\tau_2\tau_4$ region, we know the subtraction for all kinematic regions in which the ω_3-cut appears (these regions are listed in Fig.21). In our generating function we therefore have the following contributions[5]

$$\begin{aligned}&-\left[\tau_2\tau_4 e^{-i\pi(\omega_1+\omega_3)}e^{i\pi\omega_a} - \tau_1\tau_2\tau_4 e^{-i\pi\omega_3}e^{i\pi\omega_a}\right]\left[e^{i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) - 2ie^{i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]\\ &+\left[-\tau_2\tau_3\tau_4 e^{-i\pi\omega_1}e^{i\pi\omega_a} + \tau_1\tau_2\tau_3\tau_4 e^{-i\pi\omega_a}\right]\left[e^{-i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right].\end{aligned} \tag{4.36}$$

For the regions $\tau_2\tau_4$ and $\tau_2\tau_3\tau_4$ this are the only subtractions, and by subtracting the corresponding parts of Eq.(4.36) from their Regge pole terms (Appendix A), all unwanted singular terms must cancel. Indeed, for the region $\tau_2\tau_4$ we find

$$\begin{aligned}f^{\tau_2\tau_4}_{ren;pole} &= f^{\tau_2\tau_4}_{pole} + ie^{-i\pi(\omega_1+\omega_3)}e^{i\pi\omega_a}\delta f_{\omega_3}\\ &= e^{-i\pi(\omega_1+\omega_3)}e^{i\pi\omega_a}\cos(\pi\omega_{bc}),\end{aligned} \tag{4.37}$$

which consists of a phase factor and a conformal invariant expression: the latter will be called 'conformal Regge pole'. Similarly the region $\tau_2\tau_3\tau_4$: together with the Regge pole contribution from Appendix A which we write as

$$f^{\tau_2\tau_3\tau_4}_{pole} = -e^{-i\pi\omega_1}e^{i\pi\omega_a}\left(\cos(\pi\omega_{bc}) + \left[e^{-i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]\right) \tag{4.38}$$

[5]Note that here we follow our convention that terms promotional to an odd number of factors τ receive an additional minus sign.

we obtain

$$
\begin{aligned}
f^{\tau_2\tau_3\tau_4}_{ren;pole} &= f^{\tau_2\tau_3\tau_4}_{pole} - ie^{-i\pi\omega_1}e^{i\pi\omega_a}\delta f_{\omega_3} \\
&= -e^{-i\pi\omega_1}e^{i\pi\omega_a}\cos(\pi\omega_{bc}).
\end{aligned}
\tag{4.39}
$$

This defines our renormalized pole contribution in the region $\tau_2\tau_3\tau_4$. The other two regions, $\tau_1\tau_2\tau_4$ and $\tau_1\tau_2\tau_3\tau_4$ receive contributions also from the 'long' cut: they will be discussed further below.

A similar discussion applies to the symmetric region $\tau_1\tau_3$ which is used to calculate the subtraction contained in f_{ω_2}:

$$
\begin{aligned}
\delta f_{\omega_2} &= i\left[e^{i\pi(\omega_a+\omega_b)} - \cos(\pi\omega_{ab}) - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] \\
&= -i\left[e^{-i\pi(\omega_a+\omega_b)} - \cos(\pi\omega_{ab}) + 2ie^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] \\
&= -\left[\sin(\pi(\omega_a+\omega_b)) - 2\frac{\cos(\pi\omega_2)\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right].
\end{aligned}
\tag{4.40}
$$

From Fig.22 it follows that the same subtraction contributes also to the regions $\tau_1\tau_2\tau_3$, $\tau_1\tau_3\tau_4$, and $\tau_1\tau_2\tau_3\tau_4$. The analogue of Eq.(4.36) reads:

$$
\begin{aligned}
&-\left[\tau_1\tau_3 e^{-i\pi(\omega_2+\omega_4)}e^{i\pi\omega_c} - \tau_1\tau_3\tau_4 e^{-i\pi\omega_2}e^{i\pi\omega_c}\right]\left[e^{i\pi(\omega_a+\omega_b)} - \cos(\pi\omega_{ab}) - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] \\
&+\left[-\tau_1\tau_2\tau_3 e^{-i\pi\omega_1}e^{i\pi\omega_c} + \tau_1\tau_2\tau_3\tau_4 e^{-i\pi\omega_c}\right]\left[e^{-i\pi(\omega_a+\omega_b)} - \cos(\pi\omega_{ab}) + 2ie^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right],
\end{aligned}
\tag{4.41}
$$

and the renormalized Regge poles in the regions $\tau_1\tau_3$ and $\tau_1\tau_2\tau_3$ have the form:

$$
\begin{aligned}
f^{\tau_1\tau_3}_{ren;pole} &= f^{\tau_1\tau_3}_{pole} + ie^{-i\pi(\omega_2+\omega_4)}e^{i\pi\omega_c}\delta f_{\omega_2} \\
&= e^{-i\pi(\omega_2+\omega_4)}e^{i\pi\omega_c}\cos(\pi\omega_{ab}),
\end{aligned}
\tag{4.42}
$$

and

$$
\begin{aligned}
f^{\tau_1\tau_2\tau_3}_{ren;pole} &= f^{\tau_1\tau_2\tau_3}_{pole} - ie^{-i\pi\omega_4}e^{i\pi\omega_c}\delta f_{\omega_2} \\
&= -e^{-i\pi\omega_4}e^{i\pi\omega_c}\cos(\pi\omega_{ab}).
\end{aligned}
\tag{4.43}
$$

Finally we turn to the contributions of the 'long' cut which contributes to the regions listed in Fig.25. We start with the region $\tau_1\tau_2\tau_3\tau_4$: in this region all three cuts contribute. The subtractions contained in the two 'short' cuts have already been determined before, and we can use these results for fixing the subtraction due to the 'long' cut. We again begin with the Regge pole expression (from the Appendix A):

$$
f^{\tau_1\tau_2\tau_3\tau_4}_{pole} = e^{i\pi(-\omega_a+\omega_b-\omega_c)} - 2i\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)}.
\tag{4.44}
$$

The subtractions from the 'short' cuts, δf_{ω_3} and δf_{ω_2}, have been defined above: δf_{ω_3} in Eq.(4.35) and Eq.(4.36), and δf_{ω_2} in Eq.(4.40) and Eq.(4.41). Before the subtraction due to the 'long' cut we have:

$$f_{pole}^{\tau_1\tau_2\tau_3\tau_4} - ie^{-i\pi\omega_c}\delta f_{\omega_2} - ie^{-i\pi\omega_a}\delta f_{\omega_3} \tag{4.45}$$

which contains a double pole term $\sim 1/\left(\sin(\pi\omega_2)\sin(\pi\omega_3)\right)$ (from $f_{pole}^{\tau_1\tau_2\tau_3\tau_4}$ in Eq.(4.44)) and single poles $\sim 1/\sin(\pi\omega_i)$ (i=1,2) (from δf_{ω_3} and δf_{ω_2}). We now use the freedom of having another subtraction connected with the 'long' cut, $f_{\omega_2\omega_3}^{a,c}$: we chose these remaining subtractions $\delta f_{\omega_2\omega_3}^{a,c}$ in such a way that they remove all double poles $\sim 1/\left(\sin(\pi\omega_2)\sin(\pi\omega_3)\right)$, all single poles $\sim 1/\sin(\pi\omega_i)$ (i=1,2), and makes the resulting expression conformally invariant. This leads to:

$$\begin{aligned}\Delta f_{\omega_2\omega_3} = \Bigg\{ & -2\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)} - \\ & -e^{-i\pi\omega_a} i\left[e^{-i\pi(\omega_b+\omega_c)} - \cos(\pi\omega_{bc}) + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right] - \\ & -e^{-i\pi\omega_c} i\left[e^{-i\pi(\omega_a+\omega_b)} - \cos(\pi\omega_{ab}) + 2ie^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right]\Bigg\}.\end{aligned} \tag{4.46}$$

Here we remind that, according to Eq.(4.33), the square brackets in the second and third rows are purely imaginary. The first term can also be written in the form

$$\begin{aligned} & -2\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)} = \\ & \left(e^{-i\pi\omega_a}\frac{\sin(\pi\omega_c)}{\sin(\pi\omega_{ac})} + e^{-i\pi\omega_c}\frac{\sin(\pi\omega_a)}{\sin(\pi\omega_{ca})}\right) 2\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)}.\end{aligned} \tag{4.47}$$

Inserting this into Eq.(4.46) one sees that, in fact, $\Delta f_{\omega_2\omega_3}$ can be written as:

$$\Delta f_{\omega_2\omega_3} = \delta f_{\omega_2\omega_3}^{a} e^{-i\pi\omega_a} + \delta f_{\omega_2\omega_3}^{c} e^{-i\pi\omega_c} \tag{4.48}$$

with real coefficients $\delta f_{\omega_2\omega_3}^{a}$ and $\delta f_{\omega_2\omega_3}^{c}$.

Having fixed the subtractions due to the 'long' cut, $\delta f_{\omega_2\omega_3}^{a}$ and $\delta f_{\omega_2\omega_3}^{c}$, we must show that in all four kinematic regions in which the 'long' cut is nonzero (Fig.25), the unphysical singularities of the Regge pole contributions cancel. We collect these subtractions by

writing it as part of the partition function:

$$\left\{e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_2+\omega_3)} - 2i\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right\}\tau_1\tau_4 \tag{4.49}$$

$$\begin{aligned}
&\left\{e^{i\pi(-\omega_a+\omega_b-\omega_c)} - 2i\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)} - \right.\\
&-e^{-i\pi\omega_a}\left[e^{-i\pi(\omega_b+\omega_c)} - \cos(\omega_{bc}) + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]\\
&\left.-e^{-i\pi\omega_c}\left[e^{-i\pi(\omega_a+\omega_b)} - \cos(\omega_{ab}) + 2ie^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right]\right\}\tau_1\tau_2\tau_3\tau_4\\
-&\left\{e^{i\pi(-\omega_a+\omega_b+\omega_c)}e^{-i\pi\omega_3} - 2i\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_2)\sin(\pi\omega_3)} - \right.\\
&\left.- e^{-i\pi\omega_3}e^{i\pi\omega_a}\left[e^{i\pi(\omega_b+\omega_c)} - \cos(\omega_{bc}) - 2ie^{i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right]\right\}\tau_1\tau_2\tau_4\\
-&\left\{e^{i\pi(\omega_a+\omega_b-\omega_c)}e^{-i\pi\omega_2} - 2i\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)} - \right.\\
&\left.- e^{-i\pi\omega_2}e^{i\pi\omega_c}\left[e^{i\pi(\omega_a+\omega_b)} - \cos(\omega_{ab}) - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right]\right\}\tau_1\tau_3\tau_4.
\end{aligned} \tag{4.50}$$

It is now a matter of straightforward algebra to calculate the conformal Regge poles for the four different kinematic regions. For the region $\tau_1\tau_2\tau_3\tau_4$ we return to Eq.(4.45) and find:

$$\begin{aligned}
f^{\tau_1\tau_2\tau_3\tau_4}_{ren;pole} &= f^{\tau_1\tau_2\tau_3\tau_4}_{pole} - ie^{-i\pi\omega_c}\delta f_{\omega_2} - ie^{-i\pi\omega_a}\delta f_{\omega_3} + i\Delta f_{\omega_2\omega_3}\\
&= e^{i\pi(-\omega_a+\omega_b-\omega_c)}.
\end{aligned} \tag{4.51}$$

Here the 'conformal Regge pole' equals unity. In the same way we find for the other regions:

$$\begin{aligned}
f^{\tau_1\tau_4}_{ren;pole} &= f^{\tau_1\tau_4}_{pole} + ie^{i\pi(\omega_2+\omega_3)}\Delta f_{\omega_2\omega_3}\\
&= e^{-i\pi(\omega_2+\omega_3)}e^{i\pi\omega_b}\cos(\pi\omega_{ac}),
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
f^{\tau_1\tau_2\tau_4}_{ren;pole} &= f^{\tau_1\tau_2\tau_4}_{pole} + ie^{-i\pi\omega_3}\delta f_{\omega_3} - ie^{-i\pi\omega_3}\Delta f_{\omega_2\omega_3}\\
&= -e^{-i\pi\omega_3}e^{i\pi\omega_c}\cos(\pi\omega_{ab}),
\end{aligned} \tag{4.53}$$

and

$$\begin{aligned}
f^{\tau_1\tau_3\tau_4}_{ren;pole} &= f^{\tau_1\tau_3\tau_4}_{pole} - ie^{-i\pi\omega_2}\delta f_{\omega_2} - ie^{-i\pi\omega_2}\Delta f_{\omega_2\omega_3}\\
&= -e^{-i\pi\omega_2}e^{i\pi\omega_a}\cos(\pi\omega_{bc}).
\end{aligned} \tag{4.54}$$

4.4 Predictions for the remainder function of the $2 \to 5$ amplitude

Let us summarize our results for those eight kinematic regions for which the Regge pole terms need to be renormalized. This are also the regions which contain Regge cuts. We again use our notation of a generating function and write for the scattering amplitude A:

$$A = A_0 + A_1\tau_1 + ... + A_{12}\tau_1\tau_2 + ... + A_{1234}\tau_1\tau_2\tau_3\tau_4. \tag{4.55}$$

Here each term proportional to $\tau_i ... \tau_j$ is written as a product of the BDS prediction and a remainder function:

$$A_{i..j} = A_{BDS;i..j} R_{\tau_i ... \tau_j}, \tag{4.56}$$

and in section 3 it has been shown that the BDS amplitude $A_{BDS;i..j}$ can be written as product of a real part, a kinematic phase factor, and a second phase factor $e^{i\delta_{i...j}}$, where the conformal invariant $\delta_{i...j}$ result from the Li_2 functions and represent the one-loop approximations to Regge cut contributions:

$$A_{BDS;i..j} = \pm |A_{BDS;i..j}| e^{i\varphi_{i...j}} e^{i\delta_{i...j}}. \tag{4.57}$$

In the following Fig.26 we list the phase factors $e^{i\varphi_{i...j}}$:

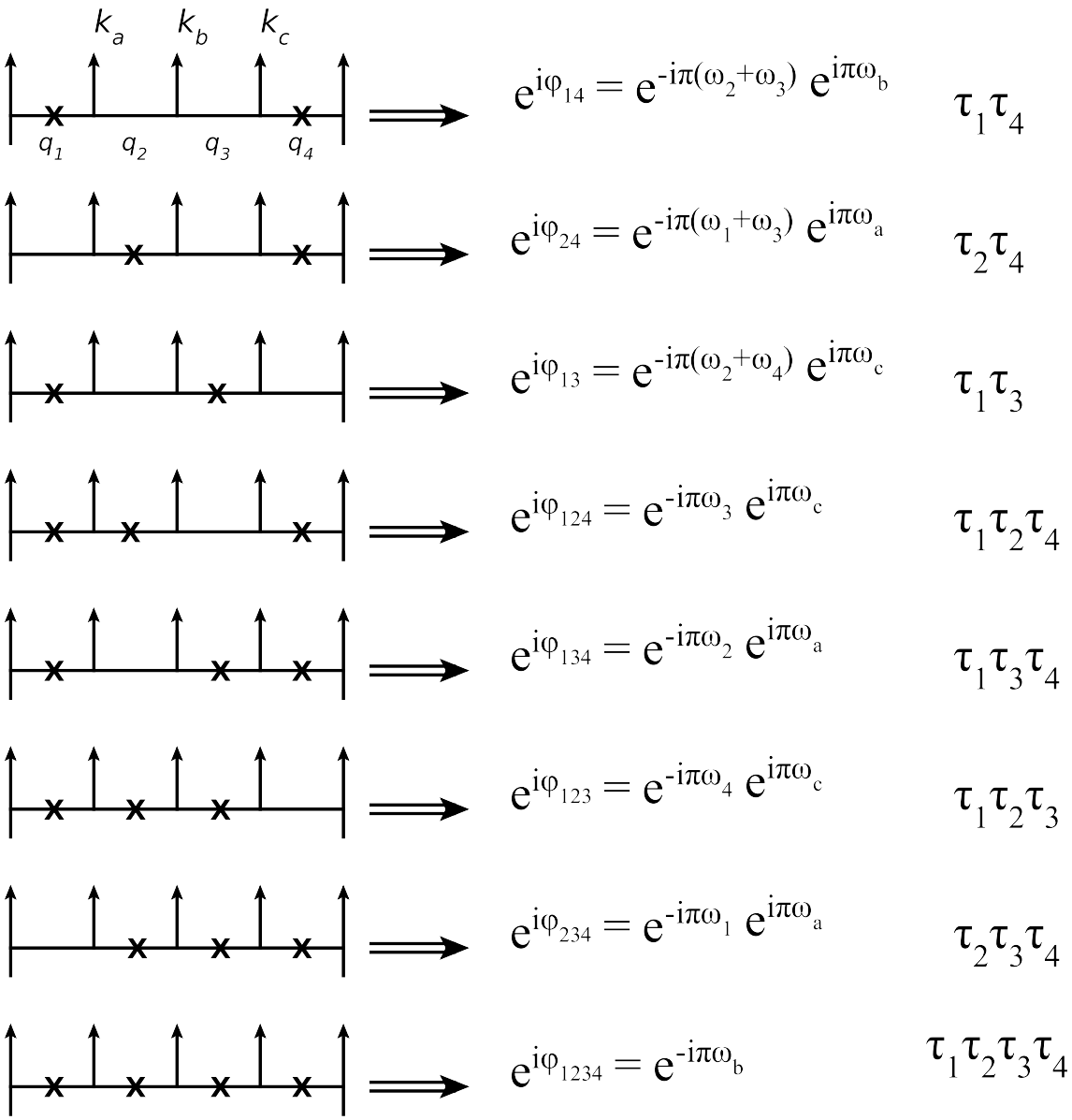

Figure 26: Phase factors $\varphi_{i...j}$ of the $2 \to 5$ amplitude.

Next, we collect the phases $\delta_{i\ldots j}$:

$$
\begin{aligned}
\delta_{14} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_a||k_c||q_1||q_4|}{|k_a+k_b+k_c|^2|q_2||q_3|}\right) \\
\delta_{24} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_b||k_c||q_2||q_4|}{|k_b+k_c|^2|q_3|^2}\right) \\
\delta_{13} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_a||k_b||q_1||q_3|}{|k_a+k_b|^2|q_2|^2}\right) \\
\delta_{124} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_a+k_b+k_c|^2|k_a||q_2|^2|q_3|}{|k_b+k_c|^2|k_b||q_1|^3}\right) \\
\delta_{134} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_a+k_b+k_c|^2|k_c||q_2||q_3|^2}{|k_a+k_b|^2|k_b||q_4|^3}\right) \\
\delta_{123} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_a||k_b||q_1||q_3|}{|k_a+k_b|^2|q_2|^2}\right) \\
\delta_{234} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_b||k_c||q_2||q_4|}{|k_b+k_c|^2|q_3|^2}\right) \\
\delta_{1234} &= \pi\frac{\gamma_K}{4}\ln\left(\frac{|k_a+k_b|^2|k_b+k_c|^2|q_1||q_4|}{|k_a+k_b+k_c|^2|k_a||k_c||q_2||q_3|}\right).
\end{aligned}
\tag{4.58}
$$

Finally, we collect the conformal invariant Regge pole and cut terms which have been calculated in the previous subsection and represent the main results of section 4. They define our predictions for the remainder function R, more precisely for the products $R_{\tau_i\ldots\tau_j}e^{i\delta_{ij}}$:

$$
\begin{aligned}
\tau_1\tau_4: &\quad \cos(\pi\omega_{ac}) + i\,\left(e^{i\pi\omega_{ba}}f^a_{\omega_2\omega_3} + e^{i\pi\omega_{bc}}f^c_{\omega_2\omega_3}\right) \\
\tau_2\tau_4: &\quad \cos(\pi\omega_{bc}) + i\,f_{\omega_3} \\
\tau_1\tau_3: &\quad \cos(\pi\omega_{ac}) + i\,f_{\omega_2} \\
\tau_1\tau_2\tau_4: &\quad -\cos(\pi\omega_{ab}) - i\,e^{i\pi\omega_{ac}}f_{\omega_3} - i\,e^{i\pi\omega_{ac}}f^a_{\omega_2\omega_3} - i\,f^c_{\omega_2\omega_3} \\
\tau_1\tau_3\tau_4: &\quad -\cos(\pi\omega_{bc}) - i\,e^{i\pi\omega_{ca}}f_{\omega_2} - i\,f^a_{\omega_2\omega_3} - i\,e^{i\pi\omega_{ca}}f^c_{\omega_2\omega_3} \\
\tau_1\tau_2\tau_3: &\quad -\cos(\pi\omega_{ab}) - i\,f_{\omega_2} \\
\tau_2\tau_3\tau_4: &\quad -\cos(\pi\omega_{bc}) - i\,f_{\omega_3} \\
\tau_1\tau_2\tau_3\tau_4: &\quad e^{i\pi\omega_{ba}}e^{i\pi\omega_{bc}} - i\,e^{i\pi\omega_{ba}}f_{\omega_3} - i\,e^{i\pi\omega_{bc}}f_{\omega_2} + i\,e^{i\pi\omega_{ba}}f^a_{\omega_2\omega_3} + i\,e^{i\pi\omega_{bc}}f^c_{\omega_2\omega_3}
\end{aligned}
\tag{4.59}
$$

The conformal invariant Regge cut terms f_{ω_2}, f_{ω_3}, $f^{a,c}_{\omega_2\omega_3}$ contain, in addition to the subtraction terms δf_{ω_2}, δf_{ω_3}, $\delta f^{a,c}_{\omega_2\omega_3}$, resp., which we have discussed in subsection 4.3, the terms with Regge cut singularities. In this paper, we have not addressed yet the general structure of these amplitudes. This will be the subject of a forthcoming paper.

5 Conclusions

In this paper we have addressed different aspects of scattering amplitudes in the multi-Regge region: starting from Regge pole models which factorize in the kinematic region of positive energies we have seen that, after analytic continuation to other kinematic regions,

terms with unphysical poles appear which need to be compensated by other terms. Specializing to the planar approximation of the conformal $\mathcal{N} = 4$ SYM theory, we have studied the cases $2 \to 4$ and $2 \to 5$, and we have shown that it is possible to compute, in agreement with the analytic structure dictated by the Steinmann relations, coefficients of Regge cut contributions which match the singular Regge pole pieces and thus can be used to absorb the singularities. We have outlined a 'renormalization scheme' which consistently removes the singularities and leads to conformal invariant.

Since most of this has been motivated by the goal of determining the remainder function $R^{(n)}$ in $\mathcal{N} = 4$ SYM theory, we have systematically studied the predictions of the BDS formula in multi-regge kinematics for the different kinematic regions, and compared them with our results for Regge pole models and Regge cuts. This has lead us to the definition of a remainder function which contains, apart from the Regge cut contribution, a conformal invariant Regge pole term. In this paper, we have not addressed the detailed structure of the Regge cut terms; this will be the content of a separate paper.

In a future study we will extend our study to the case $2 \to 6$ which is expected to contain a new form of the Regge cut consisting of three reggeized gluons.

6 Acknowledgments

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A Explicit results of the τ expansion for the $n = 5, 6, 7$ point amplitudes

In this part we summarize the explicit coefficients of the τ expansions, $P_{2\to n}$, for the cases $2 \to 3$, $2 \to 4$, $2 \to 5$, and $2 \to 6$. We start with the simplest case of $n = 5$ amplitude and list all terms:

$$
\begin{aligned}
P_{2\to 3} \;=\; & e^{i\pi\omega_a} e^{-i\pi(\omega_1+\omega_2)} && \text{(free term)} \\
& -e^{i\pi\omega_a} e^{-i\pi\omega_2} && \tau_1 \\
& -e^{i\pi\omega_a} e^{-i\pi\omega_1} && \tau_2 \\
& e^{-i\pi\omega_a} && \tau_1\tau_2
\end{aligned}
$$

Table 1: All terms of the production amplitude $P_{2\to 3}$.

Next we summarize the $n = 6$ amplitude:

$$
\begin{aligned}
P_{2\to 4} \;=\; & e^{i\pi(\omega_a+\omega_b)} e^{-i\pi(\omega_1+\omega_2+\omega_3)} && \text{(free term)} \\
& -e^{i\pi(\omega_a+\omega_b)} e^{-i\pi(\omega_2+\omega_3)} && \tau_1 \\
& -e^{i\pi(\omega_a+\omega_b)} e^{-i\pi(\omega_1+\omega_3)} && \tau_2 \\
& -e^{i\pi(\omega_a+\omega_b)} e^{-i\pi(\omega_1+\omega_2)} && \tau_3 \\
& e^{i\pi(\omega_b-\omega_a)} e^{-i\pi\omega_3} && \tau_1\tau_2 \\
& e^{i\pi(\omega_a-\omega_b)} e^{-i\pi\omega_1} && \tau_2\tau_3 \\
& e^{-i\pi\omega_2}\left[e^{i\pi(\omega_a+\omega_b)} - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] && \tau_1\tau_3 \\
& -\left[e^{-i\pi(\omega_a+\omega_b)} + 2ie^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] && \tau_1\tau_2\tau_3
\end{aligned}
$$

Table 2: All terms of the production amplitude $P_{2\to 4}$.

Finally, the coefficients of $n = 7$ amplitude:

$$
\begin{aligned}
P_{2\to 5} \;=\; & e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_1+\omega_2+\omega_3+\omega_4)} && \text{(free term)}\\
& -e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_2+\omega_3+\omega_4)} && \tau_1\\
& -e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_1+\omega_3+\omega_4)} && \tau_2\\
& -e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_1+\omega_2+\omega_4)} && \tau_3\\
& -e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_1+\omega_2+\omega_3)} && \tau_4\\
& e^{-i\pi(\omega_a-\omega_b-\omega_c)}e^{-i\pi(\omega_3+\omega_4)} && \tau_1\tau_2\\
& e^{-i\pi(\omega_2+\omega_4)}e^{i\pi\omega_c}\left[e^{i\pi(\omega_a+\omega_b)} - 2ie^{i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] && \tau_1\tau_3\\
& \left[e^{i\pi(\omega_a+\omega_b+\omega_c)}e^{-i\pi(\omega_2+\omega_3)} - 2i\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right] && \tau_1\tau_4\\
& e^{i\pi(\omega_a-\omega_b+\omega_c)}e^{-i\pi(\omega_1+\omega_4)} && \tau_2\tau_3\\
& e^{-i\pi(\omega_1+\omega_3)}e^{i\pi\omega_a}\left[e^{i\pi(\omega_b+\omega_c)} - 2ie^{i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right] && \tau_2\tau_4\\
& e^{i\pi(\omega_a+\omega_b-\omega_c)}e^{-i\pi(\omega_1+\omega_2)} && \tau_3\tau_4\\
& -e^{-i\pi\omega_4}e^{i\pi\omega_c}\left[e^{-i\pi(\omega_a+\omega_b)} + 2ie^{-i\pi\omega_2}\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)}{\sin(\pi\omega_2)}\right] && \tau_1\tau_2\tau_3\\
& -\left[e^{i\pi(-\omega_a+\omega_b+\omega_c)}e^{-i\pi\omega_3} - 2i\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right] && \tau_1\tau_2\tau_4\\
& -\left[e^{i\pi(\omega_a+\omega_b-\omega_c)}e^{-i\pi\omega_2} - 2i\frac{\sin(\pi\omega_a)\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right] && \tau_1\tau_3\tau_4\\
& -e^{-i\pi\omega_1}e^{i\pi\omega_a}\left[e^{-i\pi(\omega_b+\omega_c)} + 2ie^{-i\pi\omega_3}\frac{\sin(\pi\omega_b)\sin(\pi\omega_c)}{\sin(\pi\omega_3)}\right] && \tau_2\tau_3\tau_4\\
& \left[e^{i\pi(-\omega_a+\omega_b-\omega_c)} - 2i\frac{\sin(\pi\omega_{2a})\sin(\pi\omega_b)\sin(\pi\omega_{3c})}{\sin(\pi\omega_2)\sin(\pi\omega_3)}\right] && \tau_1\tau_2\tau_3\tau_4
\end{aligned}
$$

Table 3: All terms of the production amplitude $P_{2\to 5}$.

B Recurrence relations for the coefficients of the expansion in the Regge framework

Consider a configuration of k crosses ("twists") on the left side and one cross on the right side n (Fig.27)

$$
(-1)^k \tau_{i_1}\tau_{i_2}...\tau_{i_k}\left\{A^n_{i_1 i_2 ... i_k}\right\} \quad \text{and} \quad (-1)^{k+1}\tau_{i_1}\tau_{i_2}...\tau_{i_k}\tau_n\left\{B^n_{i_1 i_2 ... i_k}\right\} \quad ; \quad (n > i_k + 1) \tag{B.1}
$$

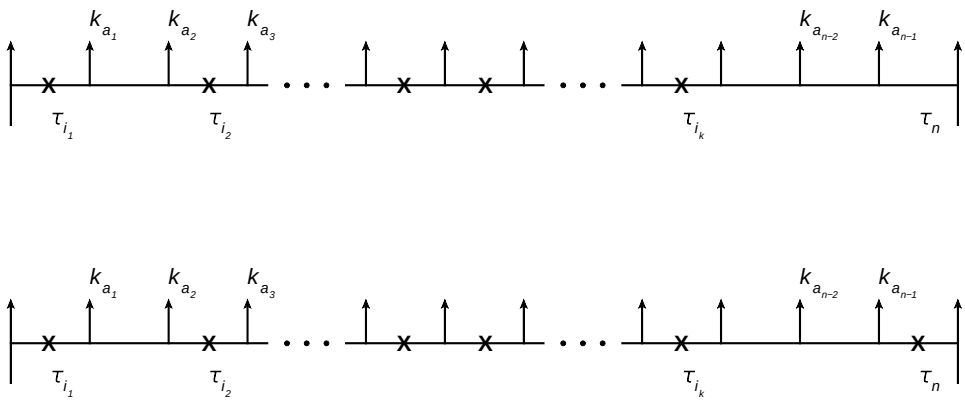

Figure 27: Diagrams which correspond to $A^n_{i_1 i_2 \ldots i_k}$ (up) and $B^n_{i_1 i_2 \ldots i_k}$ (bottom) configuration.

the recurrence relation reads as

$$B^{n+1}_{i_1 \ldots i_k} = B^n_{i_1 \ldots i_k} \frac{sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)} + A^n_{i_1 \ldots i_k} \frac{sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} \tag{B.2}$$

This (Eq.(B.2)) can be rewritten, using Eq.(2.39):

$$B^n_{i_1 \ldots i_k} = b^n_{i_1 \ldots i_k} + a^n_{i_1 \ldots i_k} \tag{B.3}$$

with

$$A^n_{i_1 \ldots i_k} = e^{-i\pi\omega_n} a^n_{i_1 \ldots i_k} \tag{B.4}$$

as

$$b^{n+1}_{i_1 \ldots i_k} = b^n_{i_1 \ldots i_k} \frac{sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)} \quad \text{and} \quad a^{n+1}_{i_1 \ldots i_k} = a^n_{i_1 \ldots i_k} e^{-i\pi\omega_n} e^{i\pi\omega_{a_n}} \tag{B.5}$$

with initial conditions

$$A^{i_k+1}_{i_1 \ldots i_k} = e^{-i\pi\omega_{i_k+1}} e^{i\pi\omega_{a_{i_k}}} A^{i_k}_{i_1 \ldots i_k} \quad \text{and} \quad B^{i_k+1}_{i_1 \ldots i_k} = b^{i_k+1}_{i_1 \ldots i_k} + e^{i\pi\omega_{i_k+1}} A^{i_k+1}_{i_1 \ldots i_k}. \tag{B.6}$$

Let us generalize for the case $i_k < n-2$, $n+1$ produced particles

$$(-1)^k \tau_{i_1} \tau_{i_2} \ldots \tau_{i_k} \tau_{n-1} \tau_n \tilde{B}^n_{i_1 i_2 \ldots i_k}$$

From the recurrence relation we have:

$$\tilde{B}^{n+1}_{i_1 i_2 \ldots i_k} = \frac{sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)} A^n_{i_1 \ldots i_k} + \frac{sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} B^n_{i_1 \ldots i_k} \tag{B.7}$$

with

$$\begin{aligned} B^n_{i_1 i_2 \ldots i_k} &= b^n_{i_1 i_2 \ldots i_k} + a^n_{i_1 i_2 \ldots i_k} \\ A^n_{i_1 i_2 \ldots i_k} &= e^{-i\pi\omega_n} a^n_{i_1 i_2 \ldots i_k} \\ a^n_{i_1 i_2 \ldots i_k} &= e^{-i\pi\omega_n} e^{i\pi\omega_{a_n}} a^n_{i_1 i_2 \ldots i_k}. \end{aligned} \tag{B.8}$$

We obtain using the ansatz:

$$\tilde{B}^n_{i_1 i_2 \ldots i_k} \;=\; \tilde{b}^{n+1}_{i_1 i_2 \ldots i_k} + \tilde{a}^n_{i_1 i_2 \ldots i_k}, \tag{B.9}$$

with

$$\tilde{a}^n_{i_1 i_2 \ldots i_k} \;=\; e^{i\pi\omega_n} a^n_{i_1 i_2 \ldots i_k}. \tag{B.10}$$

The result is:

$$\begin{aligned} \tilde{b}^n_{i_1 i_2 \ldots i_k} &= \frac{\sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n} b^n_{i_1 i_2 \ldots i_k} \\ \tilde{a}^{n+1}_{i_1 i_2 \ldots i_k} &= e^{-i\pi\omega_{a_n}}\, a^n_{i_1 i_2 \ldots i_k} \;=\; e^{-i\pi\omega_n} e^{-i\pi\omega_{a_n}} \tilde{a}^n_{i_1 i_2 \ldots i_k}. \end{aligned} \tag{B.11}$$

Now we consider the most general case Fig.28:

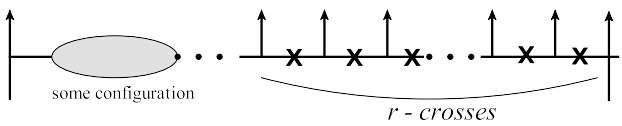

Figure 28: Configuration with r-crosses on the most right hand side and some arbitrary configuration on the left (grey blob).

$$(-1)^{k+r} \tau_{i_1} \tau_{i_2} \ldots \tau_{i_k} \tau_{n-r} \tau_{n-r+1} \ldots \tau_n B^n_r \quad \text{for} \;\; i_k < n-r-1, \tag{B.12}$$

Then the recurrence relation becomes the *three-term relation* for B^n_r:

$$B^{n+1}_{r+1} \;=\; e^{-i\pi(\omega_n - \omega_{a_{n-1}})} \frac{\sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)} B^{n-1}_{r-1} + \frac{\sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} B^n_r, \tag{B.13}$$

with the initial conditions:

$$B^n_1 \;=\; B^n_{i_1 i_2 \ldots i_k} \quad \text{and} \quad B^n_2 \;=\; \tilde{B}^n_{i_1 i_2 \ldots i_k}. \tag{B.14}$$

The recurrence relation for $\tilde{B}^n_{i_1 i_2 \ldots i_k}$ and $B^n_{i_1 i_2 \ldots i_k}$ are given in above (really, $\tilde{B}^n_{i_1 i_2 \ldots i_k}$) is expressed in terms of $B^n_{i_1 i_2 \ldots i_k}$ and $B^n_{i_1 i_2 \ldots i_k}$ satisfies *two-term* recurrence relation and is expressed again in terms of B^n_r). Therefore, we can construct everything in terms of very simple recurrence relations. Consider a case $(-1)^{k+r} \tau_{i_1} \tau_{i_2} \ldots \tau_{i_k} .. \tau_{n-r} \tau_{n-r+1} \ldots \tau_n B^n_r$ we obtain the recurrence relation:

$$B^{n+1}_{r+1} \;=\; \frac{\sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)} e^{-i\pi\omega_n} e^{-i\pi\omega_{a_{n-1}}} B^{n-1}_{r-1} + \frac{\sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} B^n_r. \tag{B.15}$$

In particular,

$$B^{n+2-r}_2 \;=\; \frac{\sin(\pi\omega_{a_{n+1-r}})}{\sin(\pi\omega_{n+1-r})} e^{-i\pi\omega_{n+1-r}} e^{-i\pi\omega_{a_{n-r}}} B^{n-r}_0 + \frac{\sin(\pi\omega_{n+1-r} - \pi\omega_{a_{n+1-r}})}{\sin(\pi\omega_{n+1-r})} B^{n+1-r}_1 \tag{B.16}$$

where we can write:

$$B_2 \;=\; b_2 + a_2; \quad B_1^n \;=\; b_1^n + a_1^n; \quad a_1 \;=\; e^{i\pi\omega_{a_{n-1}}} B_0^{n-1}; \quad b_2^{n+1} \;=\; \frac{\sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} b_1^n \tag{B.17}$$

and

$$b_1^{r+1} \;=\; b_1^2 \frac{\sin(\pi\omega_{a_r})}{\sin(\pi\omega_r)}; \quad a_1^{r+1} \;=\; a_1^r e^{-i\pi\omega_r} e^{i\pi\omega_{a_r}}. \tag{B.18}$$

We have:

$$B_1^{n+1} \;=\; \frac{\sin(\pi\omega_{a_n})}{\sin(\pi\omega_n)} B_1^n + \frac{\sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} A_1^n; \quad \text{with} \quad A_1^n \;=\; e^{-i\pi\omega_n} a_1^n. \tag{B.19}$$

Indeed,

$$a_1^{n+1} \;=\; \left[\frac{\sin(\pi\omega_{a_n}) + e^{-i\pi\omega_n} \sin(\pi\omega_n - \pi\omega_{a_n})}{\sin(\pi\omega_n)} \right] a_1^n \;=\; e^{-i\pi\omega_n} e^{i\pi\omega_{a_n}} a_1^n. \tag{B.20}$$

Concluding this part, one can see a clear recurrence relation for arbitrary number of crosses and "holes" (untwisted propagators). Thus, more complicated configurations might be reduced to the more simple ones using the recurrence relations formulated in above.

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