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New Results on the 3-Loop Heavy Flavor Corrections in Deep-Inelastic Scattering*

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We report on recent progress in the calculation of the 3-loop massive Wilson coefficients in deep-inelastic scattering at general values of N for neutral and charged current reactions in the asymptotic region $Q^2 \gg m^2$. Four new out of eight massive operator matrix elements and Wilson coefficients have been obtained recently. We also discuss recent results on Feynman graphs containing two-massive fermion lines and present complete results for the bubble topologies for all processes.

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1. Introduction

The Wilson coefficients for the heavy quark contributions are known to 2-loop order in semi-analytic form [1,2].¹ In the asymptotic region of large virtualities $Q^2 \gg m^2$, the Wilson coefficients were calculated analytically in [4–8]. This approximation holds in case of $F_2(x, Q^2)$ for scales of $Q^2/m^2 \gtrsim 10$ at the 1% level [4]. In 2009 a series of Mellin moments $N = 2 \dots 10(12, 14)$ was calculated for all massive operator matrix elements (OMEs) in [9] mapping these moments to massive tadpoles, which could be calculated using MATAD, [10]. In the asymptotic region also the 3-loop corrections for $F_L(x, Q^2)$ were computed [11], which are, however, only applicable at much higher scales.

Including the case of transversity [12] there are eight unpolarized massive Wilson coefficients at three loop order to be calculated. In 2010 the Wilson coefficients $L_{qg,Q}^{(3)}$ and $L_{qq,Q}^{(3),PS}$ were computed [13]. Here the exchanged gauge boson couples to a massless fermion line. Furthermore, all contributions due to the color factors $C_{F,A} T_F^2 N_F$ have been computed in [13, 14]. 3-loop ladder and V -topologies have been studied in detail in [15, 16]. In all these calculations after performing the Feynman parameter integrals, nested finite and infinite sums over hypergeometric terms, cf. [17], occur, which have to be solved by applying modern summation technologies.² These are encoded in the packages `Sigma` [19–27], `EvaluateMultiSums` [27–29], `SumProduction` [27], and `ρ -Sum` [30]. Algebraic and structural relations between sums of specific types, such as harmonic sums [31–33], multiple zeta values [34], harmonic polylogarithms [35], generalized harmonic sums and associated polylogarithms [36,37], cyclotomic harmonic sums and polylogarithms [38], as well as binomially weighted finite sums and the associated polylogarithms [39], are mutually applied in these calculations.³ The corresponding relations and algorithms are encoded in the package `HarmonicSums` [37,41].

During the last year we have calculated four more OMEs at three loop order, $A_{qq,Q}^{(3),NS}$, $A_{qq,Q}^{(3),NS,TR}$, $A_{gq,Q}^{(3)}$ and $A_{Qq}^{(3),PS}$ and the associated massive Wilson coefficients for $A_{qq,Q}^{(3),NS}$ and $A_{Qq}^{(3),PS}$ at large Q^2 . The corresponding topologies were first reduced to master integrals using integration-by-parts relations [42] using the package `Reduze2` [43,44]. The master integrals were finally calculated using different summation technologies being described above. Furthermore, we computed the bubble topologies for all OMEs containing one massless bubble. Progress has also been made in the calculation of the topologies with two massive lines of the same mass.

In this note we report on these series of results obtained recently. In Section 2 the yet missing results for the 2-bubble topologies, with one massless line beyond the results given in [13, 15], are presented. In Section 3 we discuss complete results obtained for four new massive OMEs and Wilson coefficients. Results on graphs with two massive quark lines of equal masses are discussed in Section 4. The asymptotic massive two-loop Wilson coefficients for charged current reactions are given in Section 5, and Section 6 contains the conclusions.

¹For a precise numerical implementation in Mellin space see [3].

²For a recent survey article of the summation packages see Ref. [18].

³For a recent review see [40].

2. Results for Bubble Graphs

All 2-bubble topologies have been calculated for all the massive operator matrix elements at general N . The contributions $\propto N_F T_F^2$ for $A_{Qq}^{(3)PS}$ and $A_{Qg}^{(3)}$ have been obtained before in [13] and for $A_{gg}^{(3)}$ in [14]. Likewise, the terms $\propto T_F^2$ were given in [45] for $A_{Qq}^{(3)PS}$.⁴ In the following we list the corresponding results for $a_{ij}^{k,b}$ in the pure-singlet-, gg - and Qg -cases. Here harmonic sums $S_{\bar{a}}(N) \equiv S_{\bar{a}}$ up to weight $w = 5$, including negative indices contribute. The calculation of the corresponding graphs has been carried out directly using (generalized) hypergeometric function techniques for the whole diagrams to convert them into sum-representations. The latter were solved using modern summation techniques as encoded in the packages `Sigma` [19–27], `EvaluateMultiSums` [27–29], `SumProduction` [27], and `ρ -Sum` [30].

For the pure-singlet OME all contributions are given. The constant part of the unrenormalized OME reads :

$$\begin{aligned}
a_{Qq}^{\text{PS,b}}(N) = & C_F^2 T_F \frac{1}{(N-1)N^2} \left\{ \frac{4P_{22}}{3(N-1)^3 N^3 (N+1)^5 (N+2)^4} - \frac{24(N^2+N+2)^2}{(N+1)^2 (N+2)} S_3 \right. \\
& + \left. \frac{2P_9}{(N-1)N(N+1)^3 (N+2)^2} [\zeta_2 + 4S_2] + \frac{28(N^2+N+2)^2 \zeta_3}{3(N+1)^2 (N+2)} \right\} \\
& + C_F T_F^2 \left\{ \frac{1}{(N-1)N^2} \left[\frac{32(N^2+N+2)^2}{27(N+1)^2 (N+2)} S_1^2 - \frac{160(N^2+N+2)^2}{9(N+1)^2 (N+2)} S_2 \right. \right. \\
& + \left. \left. \frac{64P_{18}}{81N^2 (N+1)^4 (N+2)^3 (N+3)(N+4)(N+5)} \right] S_1 \right. \\
& - \left. \frac{64P_{21}}{243(N-1)N^5 (N+1)^4 (N+2)^4 (N+3)(N+4)(N+5)} \right. \\
& + \left. \frac{32}{27(N-1)N^3 (N+1)^2 (N+2)^2 (N+3)(N+4)(N+5)} \left[P_{12} S_2 - \frac{P_{14} S_1^2}{N+1} \right] \right. \\
& - \frac{512(N^2+N+2)^2}{27(N-1)N^2 (N+1)^2 (N+2)} S_3 + \frac{128(N^2+N+2)^2}{3(N-1)N^2 (N+1)^2 (N+2)} S_{2,1} \\
& + \left[\frac{32(N^2+N+2)^2}{3(N-1)N^2 (N+1)^2 (N+2)} S_1 - \frac{32P_2}{9(N-1)N^3 (N+1)^2 (N+2)^2} \right] \zeta_2 \\
& - \left. \frac{1024(N^2+N+2)^2 \zeta_3}{9(N-1)N^2 (N+1)^2 (N+2)} \right\} + C_F T_F^2 n_f \left\{ - \frac{16(N^2+N+2)^2}{27(N-1)N^2 (N+1)^2 (N+2)} S_1^3 \right. \\
& + \frac{16P_5}{27(N-1)N^3 (N+1)^3 (N+2)^2} S_1^2 + \left[- \frac{208(N^2+N+2)^2}{9(N-1)N^2 (N+1)^2 (N+2)} S_2 \right. \\
& - \left. \frac{32P_{15}}{81(N-1)N^4 (N+1)^4 (N+2)^3} \right] S_1 + \frac{32P_{19}}{243(N-1)N^5 (N+1)^5 (N+2)^4} \\
& + \frac{208P_5}{27(N-1)N^3 (N+1)^3 (N+2)^2} S_2 - \frac{1760(N^2+N+2)^2}{27(N-1)N^2 (N+1)^2 (N+2)} S_3
\end{aligned}$$

⁴In (2.1) We corrected a typographical error in [45].

$$\begin{aligned}
& + \left[\frac{16P_5}{9(N-1)N^3(N+1)^3(N+2)^2} - \frac{16(N^2+N+2)^2}{3(N-1)N^2(N+1)^2(N+2)} S_1 \right] \zeta_2 \\
& + \frac{224(N^2+N+2)^2 \zeta_3}{9(N-1)N^2(N+1)^2(N+2)} \left. + C_A C_F T_F \left\{ \frac{(2N^3+5N^2-14N-24)}{72N^2(N+1)^2(N+2)} S_1^4 \right. \right. \\
& + \frac{P_6}{54(N-1)N^3(N+1)^3(N+2)^2} S_1^3 + \left[\frac{P_{13}}{54(N-1)N^4(N+1)^4(N+2)^3} \right. \\
& + \left. \frac{(74N^4+523N^3+733N^2+374N+440)}{12(N-1)N^2(N+1)^2(N+2)} S_2 \right] S_1^2 + \left[\frac{P_{20}}{81(N-1)N^5(N+1)^5(N+2)^4} \right. \\
& + \frac{P_7}{18(N-1)N^3(N+1)^3(N+2)^2} S_2 + \frac{(134N^4+1971N^3+3857N^2+2270N+936)}{9(N-1)N^2(N+1)^2(N+2)} S_3 \\
& - \left. \frac{2(4N^4+107N^3+255N^2+122N-32)}{3(N-1)N^2(N+1)^2(N+2)} S_{2,1} - \frac{64(N^3+6N^2+3N+2)}{3(N-1)N^2(N+1)^2} S_{-2,1} \right] S_1 \\
& + \frac{(82N^4+1211N^3+3061N^2+2982N+2008)}{24(N-1)N^2(N+1)^2(N+2)} S_2^2 \\
& + \frac{16(N^4+2N^3+7N^2+6N+4)}{3(N-1)N^2(N+1)^2(N+2)} S_{-2}^2 + \frac{P_{23}}{243(N-1)^5 N^5 (N+1)^6 (N+2)^5} \\
& + \left[\frac{8P_4}{3(N-1)N^3(N+1)^3(N+2)^2} + \frac{32(N^3+6N^2+3N+2)}{(N-1)N^2(N+1)^2} S_1 \right] S_{-3} \\
& + \frac{P_{17}}{54(N-1)^3 N^4 (N+1)^4 (N+2)^3} S_2 + \left[\frac{32(N^3+6N^2+3N+2)}{3(N-1)N^2(N+1)^2} S_1^2 \right. \\
& + \frac{32P_1}{3(N-1)N^3(N+1)^3(N+2)} S_1 - \frac{4P_{11}}{3(N-1)N^4(N+1)^4(N+2)^3} \\
& + \left. \frac{16(5N^4+58N^3+99N^2+46N+20)}{3(N-1)N^2(N+1)^2(N+2)} S_2 \right] S_{-2} + \frac{P_{10}}{27(N-1)^2 N^3 (N+1)^3 (N+2)^2} S_3 \\
& + \frac{(194N^4+4719N^3+10489N^2+6814N+2136)}{12(N-1)N^2(N+1)^2(N+2)} S_4 \\
& + \frac{32(3N^4+36N^3+61N^2+28N+12)}{3(N-1)N^2(N+1)^2(N+2)} S_{-4} - \frac{2P_3}{3(N-1)N^3(N+1)^3(N+2)^2} S_{2,1} \\
& - \frac{2(293N^3+813N^2+470N-80)}{3(N-1)N^2(N+1)^2(N+2)} S_{3,1} - \frac{32P_1}{3(N-1)N^3(N+1)^3(N+2)} S_{-2,1} \\
& - \frac{32(N^3+6N^2+3N+2)}{3(N-1)N^2(N+1)^2} [2S_{-2,2}+3S_{-3,1}] + \frac{64(N^3+6N^2+3N+2)}{3(N-1)N^2(N+1)^2} S_{-2,1,1} \\
& - \frac{2(2N^4-97N^3-267N^2-118N+88)}{3(N-1)N^2(N+1)^2(N+2)} S_{2,1,1} + \left[\frac{(2N^3+5N^2-14N-24)}{4N^2(N+1)^2(N+2)} S_1^2 \right. \\
& + \frac{P_6}{6(N-1)N^3(N+1)^3(N+2)^2} S_1 + \left. \frac{P_{16}}{18(N-1)^3 N^3 (N+1)^4 (N+2)^3} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left. \frac{(2N^4 + 95N^3 + 265N^2 + 222N + 88)}{4(N-1)N^2(N+1)^2(N+2)} S_2 + \frac{4(N^4 + 14N^3 + 23N^2 + 10N + 4)}{(N-1)N^2(N+1)^2(N+2)} S_{-2} \right] \zeta_2 \\
& + \left. \left[\frac{P_8}{9(N-1)^2 N^3 (N+1)^3 (N+2)^2} + \frac{(34N^3 + 109N^2 + 98N - 24)}{3N^2(N+1)^2(N+2)} S_1 \right] \zeta_3 \right\}. \quad (2.1)
\end{aligned}$$

The polynomials P_i read :

$$P_1 = N^6 + 11N^5 + 64N^4 + 87N^3 + 33N^2 + 16N + 4 \quad (2.2)$$

$$P_2 = 8N^6 + 29N^5 + 84N^4 + 193N^3 + 162N^2 + 124N + 24 \quad (2.3)$$

$$P_3 = 2N^7 + 65N^6 + 591N^5 + 1904N^4 + 2554N^3 + 1132N^2 - 120N - 48 \quad (2.4)$$

$$P_4 = 4N^7 + 67N^6 + 505N^5 + 1277N^4 + 1227N^3 + 476N^2 + 156N + 16 \quad (2.5)$$

$$P_5 = 8N^7 + 37N^6 + 68N^5 - 11N^4 - 86N^3 - 56N^2 - 104N - 48 \quad (2.6)$$

$$P_6 = 43N^7 + 221N^6 + 694N^5 + 722N^4 - 496N^3 - 424N^2 + 464N + 96 \quad (2.7)$$

$$P_7 = 631N^7 + 3965N^6 + 17170N^5 + 33194N^4 + 22160N^3 + 5912N^2 + 5456N + 864 \quad (2.8)$$

$$\begin{aligned}
P_8 = & -331N^8 - 1540N^7 - 3434N^6 - 1648N^5 + 4089N^4 + 1452N^3 - 2316N^2 \\
& + 560N + 1152 \quad (2.9)
\end{aligned}$$

$$P_9 = 2N^8 + 6N^7 - N^6 - 51N^5 - 59N^4 - 19N^3 - 26N^2 + 28N + 24 \quad (2.10)$$

$$\begin{aligned}
P_{10} = & 2365N^8 + 13228N^7 + 55085N^6 + 90910N^5 + 4596N^4 - 91944N^3 - 55632N^2 \\
& - 2960N - 96 \quad (2.11)
\end{aligned}$$

$$\begin{aligned}
P_{11} = & N^9 - 215N^8 - 2293N^7 - 7913N^6 - 12020N^5 - 8528N^4 - 3048N^3 - 848N^2 \\
& + 96N + 128 \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
P_{12} = & 40N^9 + 625N^8 + 3284N^7 + 5392N^6 - 7014N^5 - 33693N^4 - 47454N^3 \\
& - 46100N^2 - 26280N + 7200 \quad (2.13)
\end{aligned}$$

$$\begin{aligned}
P_{13} = & -359N^{10} - 2734N^9 - 9528N^8 - 14379N^7 - 11852N^6 - 28608N^5 - 46716N^4 \\
& - 8528N^3 + 22240N^2 + 7296N + 576 \quad (2.14)
\end{aligned}$$

$$\begin{aligned}
P_{14} = & 8N^{10} + 133N^9 + 1095N^8 + 5724N^7 + 18410N^6 + 34749N^5 + 40683N^4 \\
& + 37370N^3 + 22748N^2 - 3960N - 7200 \quad (2.15)
\end{aligned}$$

$$\begin{aligned}
P_{15} = & 25N^{10} + 176N^9 + 417N^8 + 30N^7 - 20N^6 + 1848N^5 + 2244N^4 + 1648N^3 \\
& + 3040N^2 + 2112N + 576 \quad (2.16)
\end{aligned}$$

$$\begin{aligned}
P_{16} = & -359N^{11} - 2025N^{10} - 4518N^9 + 2510N^8 + 21229N^7 + 14611N^6 - 14384N^5 \\
& - 16352N^4 - 6592N^3 + 152N^2 + 6784N + 4128 \quad (2.17)
\end{aligned}$$

$$\begin{aligned}
P_{17} = & -4739N^{12} - 27252N^{11} - 62919N^{10} + 29003N^9 + 277786N^8 + 167821N^7 \\
& - 215504N^6 - 163112N^5 - 31660N^4 - 19696N^3 + 67744N^2 + 46464N - 1728 \quad (2.18)
\end{aligned}$$

$$\begin{aligned}
P_{18} = & 52N^{13} + 746N^{12} + 4658N^{11} + 20431N^{10} + 79990N^9 + 251778N^8 + 553796N^7 \\
& + 837697N^6 + 886552N^5 + 599060N^4 + 155864N^3 - 82368N^2 - 76896N \\
& - 17280 \quad (2.19)
\end{aligned}$$

$$P_{19} = 158N^{13} + 1663N^{12} + 7714N^{11} + 23003N^{10} + 56186N^9 + 89880N^8 + 59452N^7$$

$$-8896N^6 - 12856N^5 - 24944N^4 - 84608N^3 - 77952N^2 - 35712N - 6912 \quad (2.20)$$

$$\begin{aligned} P_{20} = & 1474N^{13} + 15137N^{12} + 67586N^{11} + 156550N^{10} + 284233N^9 + 530832N^8 \\ & + 412460N^7 - 695900N^6 - 1291340N^5 - 157480N^4 + 639968N^3 + 318720N^2 \\ & + 72000N + 3456 \end{aligned} \quad (2.21)$$

$$\begin{aligned} P_{21} = & 293N^{15} + 4670N^{14} + 32280N^{13} + 145948N^{12} + 559575N^{11} + 1871440N^{10} \\ & + 4877344N^9 + 9333994N^8 + 12958212N^7 + 12693884N^6 + 8472792N^5 \\ & + 4514336N^4 + 3109248N^3 + 2192832N^2 + 1026432N + 207360 \end{aligned} \quad (2.22)$$

$$\begin{aligned} P_{22} = & 4N^{16} + 30N^{15} + 101N^{14} + 301N^{13} + 561N^{12} - 1385N^{11} - 4474N^{10} + 324N^9 \\ & + 4667N^8 - 4115N^7 - 2529N^6 + 6629N^5 - 330N^4 - 3672N^3 - 1024N^2 \\ & + 976N + 480 \end{aligned} \quad (2.23)$$

$$\begin{aligned} P_{23} = & -8780N^{19} - 83054N^{18} - 302761N^{17} - 396603N^{16} + 104969N^{15} + 2043037N^{14} \\ & + 5908471N^{13} + 1725207N^{12} - 16095317N^{11} - 11836443N^{10} + 21978990N^9 \\ & + 16243568N^8 - 18166796N^7 - 7483912N^6 + 11581992N^5 + 1162152N^4 \\ & - 5841152N^3 - 833088N^2 + 1415808N + 563328. \end{aligned} \quad (2.24)$$

The new contributions to the finite part of the OME $A_{gg}^{(3)}$ read :

$$\begin{aligned} a_{gg}^b(N) = & T_F C_A^2 \left\{ -\frac{13}{36} S_1^5 + \frac{R_{14}}{864(N-1)N(N+1)(N+2)} S_1^4 + \left[\frac{R_{20}}{1944(N-1)^2 N^2 (N+1)^2 (N+2)^2} \right. \right. \\ & \left. \left. - \frac{235}{54} S_2 \right] S_1^3 + \left[\frac{R_{29}}{648(N-1)^3 N^3 (N+1)^3 (N+2)^3} + \frac{R_{15}}{432(N-1)N(N+1)(N+2)} S_2 \right. \right. \\ & \left. \left. - 5S_3 + \frac{4}{9} S_{2,1} + \frac{32}{9} S_{-2,1} \right] S_1^2 + \left[-\frac{49}{12} S_2^2 + \frac{R_{21}}{216(N-1)^2 N^2 (N+1)^2 (N+2)^2} S_2 \right. \right. \\ & \left. \left. + \frac{R_{30}}{972(N-1)^4 N^4 (N+1)^4 (N+2)^4} + \frac{R_{13}}{108(N-1)N(N+1)(N+2)} S_3 - \frac{179}{18} S_4 \right. \right. \\ & \left. \left. + \frac{R_6}{27(N-1)N(N+1)(N+2)} S_{2,1} - \frac{32}{9} S_{3,1} + \frac{64(5N+22)}{27(N+2)} S_{-2,1} + \frac{32}{9} S_{-2,2} - \frac{8}{9} S_{2,1,1} \right. \right. \\ & \left. \left. - \frac{128}{9} S_{-2,1,1} \right] S_1 + \frac{R_{16}}{864(N-1)N(N+1)(N+2)} S_2^2 \right. \\ & \left. + \frac{R_{32}}{46656(N-1)^5 N^5 (N+1)^5 (N+2)^5} + \frac{R_{22}}{972(N-1)^2 N^2 (N+1)^2 (N+2)^2} S_3^2 \right. \\ & \left. + \frac{R_{12}}{432(N-1)N(N+1)(N+2)} S_4 + 6S_5 + \frac{448}{81} S_{-3} - \frac{80}{27} S_{-4} + \frac{16}{9} S_{-5} \right. \\ & \left. + \frac{R_{17}}{81(N-1)N(N+1)^2 (N+2)^2} S_{2,1} + \left[-\frac{16}{27} S_1^3 - \frac{64(N+3)(41N+56)}{81(N+1)(N+2)} \right. \right. \\ & \left. \left. - \frac{16(5N+22)}{27(N+2)} S_1^2 + \left(\frac{16}{9} S_2 - \frac{128(7N^2+36N+38)}{81(N+1)(N+2)} \right) S_1 + \frac{16(5N+22)}{27(N+2)} S_2 - \frac{32}{27} S_3 \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{64}{9} S_{2,1} \left] S_{-2} + \frac{20}{9} S_{2,3} + \frac{64}{9} S_{2,-3} + \frac{R_3}{27(N-1)N(N+1)(N+2)} S_{3,1} - \frac{34}{9} S_{4,1} \right. \\
& + \left[\frac{R_{28}}{648(N-1)^3 N^3 (N+1)^3 (N+2)^3} - \frac{349}{27} S_3 - \frac{8}{9} S_{2,1} - \frac{32}{3} S_{-2,1} \right] S_2 \\
& + \frac{256(7N^2 + 36N + 38)}{81(N+1)(N+2)} S_{-2,1} + \frac{32(5N+22)}{27(N+2)} S_{-2,2} + \frac{R_1}{27(N-1)N(N+1)(N+2)} S_{2,1,1} \\
& - \frac{64}{9} S_{-2,3} - \frac{64}{9} S_{2,1,-2} - \frac{4}{9} S_{2,2,1} + \frac{52}{9} S_{3,1,1} - \frac{128(5N+22)}{27(N+2)} S_{-2,1,1} - \frac{64}{9} S_{-2,2,1} \\
& + \frac{64}{9} S_{2,1,1,1} + \frac{256}{9} S_{-2,1,1,1} + \left[-\frac{13}{6} S_1^3 + \frac{R_{11} S_1^2}{48(N-1)N(N+1)(N+2)} \right. \\
& + \left. \left(\frac{R_{18}}{72(N-1)N^2(N+1)^2(N+2)^2} - \frac{55}{6} S_2 \right) S_1 + \frac{R_{26}}{864(N-1)^3 N^3 (N+1)^3 (N+2)^3} \right. \\
& + \left. \left(-\frac{2(5N+18)}{3(N+2)} - \frac{4}{3} S_1 \right) S_{-2} + \frac{R_9}{48(N-1)N(N+1)(N+2)} S_2 + \frac{8}{3} S_3 + \frac{2}{3} S_{-3} \right. \\
& + \left. \frac{5}{3} S_{2,1} + \frac{8}{3} S_{-2,1} \right] \zeta_2 + \left[-\frac{49}{3} S_1^2 + \frac{7R_{10} S_1}{36(N-1)N(N+1)(N+2)} + 7S_2 + \frac{14}{3} S_{-2} \right. \\
& \left. - \frac{7R_{25}}{216(N-1)^2 N^2 (N+1)^2 (N+2)^2} \right] \zeta_3 \left. \right\}. \tag{2.25}
\end{aligned}$$

The polynomials R_i are

$$R_1 = -353N^4 - 832N^3 + 443N^2 + 934N + 96 \tag{2.26}$$

$$R_3 = 7N^4 - 280N^3 - 277N^2 - 26N - 288 \tag{2.27}$$

$$R_6 = 121N^4 + 368N^3 - 211N^2 - 470N - 96 \tag{2.28}$$

$$R_9 = 343N^4 + 1036N^3 + 2033N^2 + 2908N + 3316 \tag{2.29}$$

$$R_{10} = 351N^4 + 824N^3 + 385N^2 + 456N + 1068 \tag{2.30}$$

$$R_{11} = 815N^4 + 1932N^3 + 537N^2 + 60N + 1684 \tag{2.31}$$

$$R_{12} = 1957N^4 + 14588N^3 + 38867N^2 + 50236N + 55020 \tag{2.32}$$

$$R_{13} = 3259N^4 + 9004N^3 + 4069N^2 + 980N + 8596 \tag{2.33}$$

$$R_{14} = 3311N^4 + 8148N^3 + 4041N^2 + 3204N + 9508 \tag{2.34}$$

$$R_{15} = 6869N^4 + 16876N^3 + 18403N^2 + 22892N + 33420 \tag{2.35}$$

$$R_{16} = 12653N^4 + 32092N^3 + 4987N^2 - 8404N + 20268 \tag{2.36}$$

$$R_{17} = 1270N^6 + 8222N^5 + 16333N^4 + 7130N^3 - 7481N^2 - 2050N + 2928 \tag{2.37}$$

$$\begin{aligned}
R_{18} = & -1571N^7 - 9661N^6 - 26791N^5 - 49153N^4 - 67528N^3 - 55096N^2 - 11384N \\
& + 8064 \tag{2.38}
\end{aligned}$$

$$\begin{aligned}
R_{20} = & -40553N^8 - 185774N^7 - 259150N^6 - 122366N^5 - 173461N^4 - 129392N^3 \\
& + 366064N^2 + 293208N - 59616 \tag{2.39}
\end{aligned}$$

$$R_{21} = -17939N^8 - 83762N^7 - 104386N^6 + 5070N^5 - 24959N^4 - 144464N^3$$

$$+25472N^2 + 106408N - 3360 \quad (2.40)$$

$$R_{22} = -3185N^8 - 62618N^7 - 314842N^6 - 626834N^5 - 451705N^4 + 424288N^3 \\ + 1063528N^2 + 416328N - 133920 \quad (2.41)$$

$$R_{25} = 763N^8 + 4224N^7 + 10030N^6 + 18476N^5 + 22927N^4 + 8348N^3 - 9888N^2 \\ - 8944N + 5904 \quad (2.42)$$

$$R_{26} = 2977N^{12} + 27302N^{11} + 121749N^{10} + 400754N^9 + 654511N^8 - 19518N^7 - 1184809N^6 \\ - 1028282N^5 - 42444N^4 - 316832N^3 - 1364944N^2 - 481248N + 120384 \quad (2.43)$$

$$R_{28} = 23572N^{12} + 191008N^{11} + 720025N^{10} + 1584161N^9 + 1802946N^8 - 206032N^7 \\ - 3384669N^6 - 3005177N^5 + 1544742N^4 + 2995680N^3 + 265168N^2 \\ - 238944N + 195840 \quad (2.44)$$

$$R_{29} = 96724N^{12} + 618064N^{11} + 1208881N^{10} + 170137N^9 - 1519638N^8 - 213008N^7 \\ + 897211N^6 - 1994577N^5 - 2283122N^4 + 584016N^3 + 268880N^2 - 319584N \\ - 2304 \quad (2.45)$$

$$R_{30} = -290424N^{16} - 2538872N^{15} - 8370109N^{14} - 11564871N^{13} - 1339582N^{12} \\ + 15569491N^{11} + 25755378N^{10} + 32901569N^9 + 17718754N^8 - 29577639N^7 \\ - 33515273N^6 + 12365378N^5 + 17499480N^4 + 73824N^3 - 625344N^2 + 266688N \\ - 736128 \quad (2.46)$$

$$R_{32} = 3752873N^{20} + 44889498N^{19} + 231324635N^{18} + 699986798N^{17} + 1323895202N^{16} \\ + 1047978356N^{15} - 2040696426N^{14} - 6981260596N^{13} - 6762635091N^{12} \\ + 2129282098N^{11} + 9092892879N^{10} + 5284986214N^9 - 791167784N^8 - 3464256800N^7 \\ - 6299611472N^6 - 5601882048N^5 - 1407467456N^4 + 248136192N^3 - 362165760N^2 \\ - 161782272N + 60134400. \quad (2.47)$$

Likewise, the new contributions to a_{Qg} beyond the results given in [13], are given by :

$$a_{Qg}^b(N) = T_F C_A^2 \left\{ \frac{(8-N)}{12N(N+1)(N+2)} S_1^5 + \frac{Q_9}{108(N-1)N(N+1)^2(N+2)^2} S_1^4 \right. \\ + \left[\frac{Q_{20}}{54(N-1)N^2(N+1)^3(N+2)^3} + \frac{(88-39N)}{18N(N+1)(N+2)} S_2 \right] S_1^3 \\ + \left[\frac{Q_{28}}{162(N-1)N^3(N+1)^4(N+2)^4} + \frac{Q_{11}}{18(N-1)N(N+1)^2(N+2)^2} S_2 \right. \\ + \left. \frac{(8-5N)}{3N(N+1)(N+2)} S_3 - \frac{2(N-72)}{3N(N+1)(N+2)} S_{2,1} + \frac{32}{(N+1)(N+2)} S_{-2,1} \right] S_1^2 \\ + \left[\frac{(184-119N)}{12N(N+1)(N+2)} S_2^2 + \frac{Q_{19}}{54(N-1)N^2(N+1)^3(N+2)^3} S_2 \right. \\ + \left. \frac{Q_{32}}{81(N-1)N^4(N+1)^5(N+2)^5} + \frac{2Q_5}{27(N-1)N(N+1)^2(N+2)^2} S_3 \right]$$

$$\begin{aligned}
& + \frac{(53N-8)}{6N(N+1)(N+2)} S_4 + \frac{4(95N^3-787N^2-2504N-1310)}{9N(N+1)^2(N+2)^2} S_{2,1} \\
& + \frac{4(N+48)}{3N(N+1)(N+2)} S_{3,1} - \frac{32(3N^2+N-6)}{(N+1)^2(N+2)^2} S_{-2,1} + \frac{160}{3(N+1)(N+2)} S_{-2,2} \\
& + \left. \frac{64}{(N+1)(N+2)} S_{-3,1} - \frac{4(5N+48)}{3N(N+1)(N+2)} S_{2,1,1} - \frac{256}{3(N+1)(N+2)} S_{-2,1,1} \right] S_1 \\
& + \frac{Q_{10}}{36(N-1)N(N+1)^2(N+2)^2} S_2^2 - \frac{16(3N^2-23N-20)}{3(N-1)N(N+1)^2(N+2)} S_{-2}^2 \\
& + \frac{Q_{34}}{243(N-1)^4 N^5 (N+1)^5 (N+2)^6} + \left[\frac{8Q_7}{3(N-1)N(N+1)^2(N+2)^2} \right. \\
& - \left. \frac{176}{3(N+1)(N+2)} S_1 \right] S_{-4} + \left[-\frac{16}{(N+1)(N+2)} S_1^2 + \frac{16(3N^2+N-6)}{(N+1)^2(N+2)^2} S_1 \right. \\
& - \left. \frac{8Q_{17}}{3(N-1)N(N+1)^3(N+2)^3} - \frac{32}{(N+1)(N+2)} S_2 \right] S_{-3} \\
& + \frac{Q_{23}}{27(N-1)N^2(N+1)^3(N+2)^3} S_3 + \frac{Q_1}{18(N-1)N(N+1)^2(N+2)^2} S_4 \\
& + \frac{16(11N-16)}{3N(N+1)(N+2)} S_5 - \frac{296}{3(N+1)(N+2)} S_{-5} - \frac{2Q_{15}}{27N(N+1)^3(N+2)^3} S_{2,1} \\
& + \left[-\frac{32}{9(N+1)(N+2)} S_1^3 + \frac{16(3N^2+N-6)}{3(N+1)^2(N+2)^2} S_1^2 + \left[-\frac{16Q_3}{3(N+1)^3(N+2)^3} \right. \right. \\
& - \left. \left. \frac{32}{(N+1)(N+2)} S_2 \right] S_1 + \frac{8Q_{24}}{3(N-1)N(N+1)^4(N+2)^4} + \frac{16Q_4}{(N-1)N(N+1)^2(N+2)^2} S_2 \right. \\
& - \left. \frac{448}{9(N+1)(N+2)} S_3 + \frac{224}{3(N+1)(N+2)} S_{2,1} \right] S_{-2} + \frac{4(N-8)}{3N(N+1)(N+2)} S_{2,3} \\
& + \frac{176}{3(N+1)(N+2)} S_{2,-3} + \frac{4(87N^3-157N^2-844N-498)}{3N(N+1)^2(N+2)^2} S_{3,1} - \frac{56}{3(N+1)(N+2)} S_{4,1} \\
& + \frac{16Q_3}{(N+1)^3(N+2)^3} S_{-2,1} + \left[\frac{Q_{31}}{162(N-1)^2 N^3 (N+1)^4 (N+2)^4} + \frac{(-39N-584)}{9N(N+1)(N+2)} S_3 \right. \\
& + \left. \frac{2(9N-8)}{3N(N+1)(N+2)} S_{2,1} + \frac{64}{3(N+1)(N+2)} S_{-2,1} \right] S_2 - \frac{80(3N^2+N-6)}{3(N+1)^2(N+2)^2} S_{-2,2} \\
& + \frac{48}{(N+1)(N+2)} S_{-2,3} - \frac{32(3N^2+N-6)}{(N+1)^2(N+2)^2} S_{-3,1} + \frac{400}{3(N+1)(N+2)} S_{-4,1} \\
& - \frac{8(10N^3-344N^2-991N-523)}{9N(N+1)^2(N+2)^2} S_{2,1,1} - \frac{224}{3(N+1)(N+2)} S_{2,1,-2} \\
& - \frac{4(17N-72)}{3N(N+1)(N+2)} S_{2,2,1} - \frac{4(23N+8)}{3N(N+1)(N+2)} S_{3,1,1} + \frac{128(3N^2+N-6)}{3(N+1)^2(N+2)^2} S_{-2,1,1} \\
& - \frac{64}{(N+1)(N+2)} S_{-2,2,1} - \frac{80}{(N+1)(N+2)} S_{-3,1,1} + \frac{8(3N+4)}{3N(N+1)(N+2)} S_{2,1,1,1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{96}{(N+1)(N+2)} S_{-2,1,1,1} + \left[\frac{(8-N)}{2N(N+1)(N+2)} S_1^3 + \frac{Q_8}{6(N-1)N(N+1)^2(N+2)^2} S_1^2 \right. \\
& + \left. \left[\frac{Q_{21}}{6(N-1)N^2(N+1)^3(N+2)^3} + \frac{(-3N-8)}{2N(N+1)(N+2)} S_2 \right] S_1 \right. \\
& + \frac{Q_{29}}{18(N-1)^2 N^3 (N+1)^3 (N+2)^4} + \left[\frac{4Q_4}{(N-1)N(N+1)^2(N+2)^2} \right. \\
& - \left. \frac{8}{(N+1)(N+2)} S_1 \right] S_{-2} + \frac{Q_2}{6(N-1)N(N+1)^2(N+2)^2} S_2 + \frac{2(3N-8)}{N(N+1)(N+2)} S_3 \\
& - \left. \frac{12}{(N+1)(N+2)} S_{-3} + \frac{16}{N(N+1)(N+2)} S_{2,1} + \frac{24}{(N+1)(N+2)} S_{-2,1} \right] \zeta_2 \\
& + \left[\frac{7(N-8)}{3N(N+1)(N+2)} S_1^2 - \frac{14Q_6}{3(N-1)N(N+1)^2(N+2)^2} S_1 \right. \\
& \left. + \frac{7Q_{22}}{9(N-1)N^2(N+1)^2(N+2)^3} + \frac{7(7N+8)}{3N(N+1)(N+2)} S_2 + \frac{56}{3(N+1)(N+2)} S_{-2} \right] \zeta_3 \left. \right\}, \tag{2.48}
\end{aligned}$$

with the polynomials Q_i given by

$$Q_1 = -3327N^4 - 5641N^3 - 5102N^2 - 13268N - 7582 \tag{2.49}$$

$$Q_2 = -51N^4 - 361N^3 - 434N^2 + 196N + 290 \tag{2.50}$$

$$Q_3 = 3N^4 - 6N^3 - 21N^2 + 24N + 52 \tag{2.51}$$

$$Q_4 = 3N^4 + N^3 - 24N^2 - 60N - 40 \tag{2.52}$$

$$Q_5 = 5N^4 - 427N^3 - 4524N^2 - 9344N - 5510 \tag{2.53}$$

$$Q_6 = 6N^4 - 19N^3 - 73N^2 - 28N - 6 \tag{2.54}$$

$$Q_7 = 33N^4 + 2N^3 - 213N^2 - 462N - 320 \tag{2.55}$$

$$Q_8 = 47N^4 - 313N^3 - 708N^2 + 244N + 370 \tag{2.56}$$

$$Q_9 = 221N^4 - 1849N^3 - 3642N^2 + 2212N + 2698 \tag{2.57}$$

$$Q_{10} = 269N^4 - 7345N^3 - 13506N^2 + 5188N + 6394 \tag{2.58}$$

$$Q_{11} = 357N^4 - 1381N^3 - 4142N^2 - 356N + 842 \tag{2.59}$$

$$Q_{15} = 1504N^5 - 8063N^4 - 60746N^3 - 111983N^2 - 79376N - 21632 \tag{2.60}$$

$$Q_{17} = 9N^6 - N^5 - 19N^4 + 87N^3 + 488N^2 + 940N + 608 \tag{2.61}$$

$$\begin{aligned}
Q_{19} = & -1430N^7 + 29061N^6 + 168141N^5 + 311889N^4 + 262827N^3 + 154488N^2 \\
& + 113360N + 62400 \tag{2.62}
\end{aligned}$$

$$\begin{aligned}
Q_{20} = & -778N^7 + 8151N^6 + 39567N^5 + 40819N^4 - 14631N^3 - 34136N^2 - 12368N \\
& + 1600 \tag{2.63}
\end{aligned}$$

$$Q_{21} = -122N^7 + 1331N^6 + 7459N^5 + 10911N^4 + 4621N^3 + 648N^2 + 1776N + 1600 \tag{2.64}$$

$$Q_{22} = 20N^7 + 18N^6 - 333N^5 - 1450N^4 - 3273N^3 - 4570N^2 - 3692N - 1480 \tag{2.65}$$

$$Q_{23} = -1080N^8 + 1898N^7 + 22443N^6 + 92307N^5 + 267403N^4 + 421473N^3 + 361900N^2 + 222376N + 81520 \quad (2.66)$$

$$Q_{24} = 3N^8 + 21N^7 + 174N^6 + 700N^5 + 1506N^4 + 1216N^3 - 1676N^2 - 4024N - 2144 \quad (2.67)$$

$$Q_{28} = 2080N^{10} - 155867N^9 - 992144N^8 - 2266725N^7 - 2100345N^6 + 59166N^5 + 1939307N^4 + 1900784N^3 + 806384N^2 + 252096N + 57600 \quad (2.68)$$

$$Q_{29} = 184N^{11} + 176N^{10} - 4108N^9 - 16762N^8 - 25657N^7 - 15063N^6 - 883N^5 + 14485N^4 + 37996N^3 + 13360N^2 - 32048N - 17040 \quad (2.69)$$

$$Q_{31} = 6624N^{12} + 5292N^{11} - 210971N^{10} - 1104257N^9 - 2480453N^8 - 2265264N^7 + 636087N^6 + 2871225N^5 + 2408885N^4 + 828944N^3 - 1329328N^2 - 1961664N - 671040 \quad (2.70)$$

$$Q_{32} = 15604N^{13} + 361847N^{12} + 2453891N^{11} + 8204366N^{10} + 15666936N^9 + 16766294N^8 + 5755934N^7 - 9519761N^6 - 13953239N^5 - 6072896N^4 + 1787904N^3 + 3241984N^2 + 1353984N + 230400 \quad (2.71)$$

$$Q_{34} = 6208N^{19} - 86928N^{18} - 1344972N^{17} - 6002889N^{16} - 9808011N^{15} + 4340125N^{14} + 32811393N^{13} + 24313093N^{12} - 30513058N^{11} - 46961276N^{10} + 3785621N^9 + 36663986N^8 + 1686347N^7 - 40115539N^6 - 14945624N^5 + 25303412N^4 + 16493728N^3 - 1302672N^2 - 6643584N - 2376000. \quad (2.72)$$

These quantities will be used in the later calculation of the full massive OMEs.

3. Complete Wilson Coefficients

After the first two massive OMEs, $L_{qg,Q}^{(3)}$ and $L_{qq,Q}^{(3),PS}$, at 3-loop order were calculated in [13], during the last months we computed four other OMEs and associated massive Wilson coefficients in the asymptotic region $Q^2 \gg m^2$. These are the non-singlet OME $A_{qq,Q}^{(3),NS}$, that of transversity $A_{qq,Q}^{(3),NS,TR}$, $A_{gq,Q}^{(3)}$, and very recently also the pure singlet $A_{gq,Q}^{(3),PS}$. These matrix elements contain topology up to Benz-graphs with respective local operator insertions. We used `Reduze2` [43, 44] to reduce the diagrams to master integrals applying the integration-by-parts relations for Feynman diagrams containing local operator insertions. In the first three cases the master integrals could be calculated using hypergeometric function techniques and Mellin-Barnes [46, 47] representations to map the integrals into nested finite and infinite sums, which were then solved using the summation technologies of [19–29, 37, 41]. For $A_{qq,Q}^{(3),NS}$, $A_{qq,Q}^{(3),NS,TR}$ and $A_{gq,Q}^{(3)}$ the results can be represented using harmonic sums only.

As an example we show the constant part of the unrenormalized OME for transversity $a_{qq}^{NS,TR(3)}$ for even and odd values of N :

$$a_{qq}^{NS,TR(3)}(N) = C_F^2 T_F \left\{ \frac{128}{27} S_2 S_1^3 + \left[\frac{64}{3} S_3 - \frac{128}{9} S_{2,1} - \frac{256}{9} S_{-2,1} - \frac{16}{9N} - \frac{32(-1)^N}{9N(N+1)} \right] S_1^2 + \left[-\frac{64}{9} S_2^2 + \frac{7168 S_2}{81} + \frac{32(-1)^N(13N+7)}{27N(N+1)^2} - \frac{2560 S_3}{27} + \frac{704 S_4}{9} - \frac{320}{9} S_{3,1} \right] \right.$$

$$\begin{aligned}
& -\frac{2560}{27}S_{-2,1} - \frac{256}{9}S_{-2,2} + \frac{64}{3}S_{2,1,1} + \frac{1024}{9}S_{-2,1,1} \\
& + \frac{8(769N^4 + 1547N^3 + 787N^2 - 15N - 12)}{27N^2(N+1)^2} \left] S_1 - \frac{496}{27}S_2^2 \right. \\
& - \frac{16(-1)^N(133N^4 + 188N^3 + 46N^2 - 45N - 18)}{81N^3(N+1)^3} \\
& - \frac{2(6327N^6 + 18981N^5 + 18457N^4 + 5687N^3 - 260N^2 + 144N + 144)}{81N^3(N+1)^3} \\
& + \left[16 - \frac{64}{3}S_1 \right] B_4 + \left[\frac{256}{9}S_1 - \frac{1280}{27} \right] S_{-4} + \left[96S_1 - 72 \right] \zeta_4 + \left[\frac{128}{9}S_1^2 - \frac{1280}{27}S_1 \right. \\
& + \frac{128}{9}S_2 + \frac{7168}{81} \left. \right] S_{-3} + \frac{10408}{81}S_3 - \frac{2992}{27}S_4 + \frac{512}{9}S_5 + \frac{256}{9}S_{-5} + \left[\frac{256}{27}S_1^3 \right. \\
& + \frac{14336}{81}S_1 - \frac{1280}{27}S_2 + \frac{512}{27}S_3 - \frac{512}{9}S_{2,1} - \frac{64}{9N(N+1)} \left. \right] S_{-2} + \frac{112}{9}S_{2,1} + \frac{256}{9}S_{2,3} \\
& - \frac{512}{9}S_{2,-3} + \frac{1424}{27}S_{3,1} - \frac{512}{9}S_{4,1} - \frac{14336}{81}S_{-2,1} + \left[-\frac{16(169N^2 + 169N + 6)}{27N(N+1)} \right. \\
& + \frac{256S_3}{27} + \frac{256}{3}S_{-2,1} - \frac{32(-1)^N}{9N(N+1)} \left. \right] S_2 - \frac{1280}{27}S_{-2,2} + \frac{512}{9}S_{-2,3} - 16S_{2,1,1} + \frac{512}{9}S_{2,1,-2} \\
& + \frac{256}{9}S_{3,1,1} + \frac{5120}{27}S_{-2,1,1} + \frac{512}{9}S_{-2,2,1} - \frac{2048}{9}S_{-2,1,1,1} \\
& + \left[-\frac{2(45N^2 + 45N - 4)}{3N(N+1)} + \frac{64}{3}S_{-2}S_1 - 8S_2 + \left[\frac{32}{3}S_2 + 40 \right] S_1 + \frac{32}{3}S_3 + \frac{32}{3}S_{-3} \right. \\
& \left. - \frac{64}{3}S_{-2,1} - \frac{8(-1)^N}{3N(N+1)} \right] \zeta_2 + \left[-\frac{1208}{9}S_1 - \frac{64}{3}S_2 + \frac{350}{3} \right] \zeta_3 \left. \right\} \\
& + C_F T_F^2 \left\{ \frac{8(157N^4 + 314N^3 + 277N^2 - 24N - 72)}{243N^2(N+1)^2} - \frac{19424}{729}S_1 + \frac{1856}{81}S_2 - \frac{640}{81}S_3 \right. \\
& + \frac{128}{27}S_4 + N_F \left[\frac{32(308N^4 + 616N^3 + 323N^2 - 3N - 9)}{243N^2(N+1)^2} - \frac{55552}{729}S_1 + \frac{640}{27}S_2 \right. \\
& \left. - \frac{320}{81}S_3 + \frac{64}{27}S_4 \right] + \left[-\frac{320}{27}S_1 + \frac{64}{9}S_2 + N_F \left[-\frac{160}{27}S_1 + \frac{32}{9}S_2 + \frac{16}{9} \right] + \frac{32}{9} \right] \zeta_2 \\
& + \left[-\frac{1024}{27}S_1 + N_F \left[\frac{448}{27}S_1 - \frac{112}{9} \right] + \frac{256}{9} \right] \zeta_3 \left. \right\} \\
& + C_A C_F T_F \left\{ -\frac{64}{27}S_2 S_1^3 + \left[\frac{4(3N+2)}{9N(N+1)} - \frac{80}{9}S_3 + \frac{128}{9}S_{2,1} + \frac{128}{9}S_{-2,1} + \frac{16(-1)^N}{9N(N+1)} \right] S_1^2 \right. \\
& + \left[\frac{112}{9}S_2^2 - \frac{16(N-2)(2N+3)}{9(N+1)(N+2)}S_2 - \frac{16(-1)^N(13N+7)}{27N(N+1)^2} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{4(6197N^3 + 18591N^2 + 15850N + 4320)}{729N(N+1)(N+2)} + \frac{320}{9}S_3 - \frac{208}{9}S_4 - 8S_{2,1} + \frac{64}{3}S_{3,1} \\
& + \frac{1280}{27}S_{-2,1} + \frac{128}{9}S_{-2,2} - 32S_{2,1,1} - \frac{512}{9}S_{-2,1,1} \left] S_1 - \frac{20}{3}S_2^2 \right. \\
& + \frac{8(-1)^N(133N^4 + 188N^3 + 46N^2 - 45N - 18)}{81N^3(N+1)^3} \\
& + \frac{-1013N^6 - 3039N^5 - 5751N^4 - 2981N^3 + 1752N^2 + 1872N + 432}{243N^3(N+1)^3} \\
& + \left[72 - 96S_1 \right] \zeta_4 + \left[\frac{640}{27} - \frac{128}{9}S_1 \right] S_{-4} + \left[\frac{32}{3}S_1 - 8 \right] B_4 + \left[-\frac{64}{9}S_1^2 + \frac{640}{27}S_1 \right. \\
& - \frac{64}{9}S_2 - \frac{3584}{81} \left. \right] S_{-3} - \frac{8(27N^3 + 560N^2 + 1365N + 778)}{81(N+1)(N+2)} S_3 + \frac{1244}{27}S_4 - \frac{224}{9}S_5 \\
& - \frac{128}{9}S_{-5} - \frac{32(3N^3 + 7N^2 + 7N + 6)}{9(N+1)(N+2)} S_{2,1} + \left[-\frac{128}{27}S_1^3 - \frac{7168}{81}S_1 \right. \\
& + \frac{640}{27}S_2 - \frac{256}{27}S_3 + \frac{256}{9}S_{2,1} + \frac{32}{9N(N+1)} \left. \right] S_{-2} - \frac{128}{3}S_{2,3} + \frac{256}{9}S_{2,-3} - \frac{1352}{27}S_{3,1} \\
& + \frac{256}{9}S_{4,1} + \left[-\frac{4(364N^3 + 1227N^2 + 872N + 36)}{81N(N+1)(N+2)} + \frac{496}{27}S_3 - \frac{64}{3}S_{2,1} \right. \\
& - \frac{128}{3}S_{-2,1} + \frac{16(-1)^N}{9N(N+1)} \left. \right] S_2 + \frac{7168}{81}S_{-2,1} + \frac{640}{27}S_{-2,2} - \frac{256}{9}S_{-2,3} + 24S_{2,1,1} \\
& - \frac{256}{9}S_{2,1,-2} + \frac{64}{3}S_{2,2,1} - \frac{256}{9}S_{3,1,1} - \frac{2560}{27}S_{-2,1,1} - \frac{256}{9}S_{-2,2,1} + \frac{224}{9}S_{2,1,1,1} \\
& + \frac{1024}{9}S_{-2,1,1,1} + \left[\frac{2(35N^2 + 35N - 6)}{9N(N+1)} - \frac{32}{3}S_{-2}S_1 - \frac{16}{27}S_1 - \frac{88}{9}S_2 - \frac{16}{3}S_3 \right. \\
& - \frac{16}{3}S_{-3} + \frac{32}{3}S_{-2,1} + \frac{4(-1)^N}{3N(N+1)} \left. \right] \zeta_2 + \left[-16S_1^2 + \frac{2548S_1}{27} \right. \\
& \left. + \frac{2(108N^3 - 239N^2 - 1137N - 646)}{9(N+1)(N+2)} + 16S_2 \right] \zeta_3 \left. \right\} ,
\end{aligned}$$

with B_4

$$B_4 = -4\zeta_2 \ln^2(2) + \frac{2}{3} \ln^4(2) - \frac{13}{2} \zeta_4 + 16\text{Li}_4\left(\frac{1}{2}\right) . \quad (3.1)$$

It is represented by harmonic sums up to $w = 5$. The logarithmic contributions and other pieces of the constant term stemming from lower order quantities are given in [48]. The renormalized expression both in N and x -space is presented in [49]. The analytic continuation to complex values of N is obtained using the relations being given in Refs. [33, 50].

Very recently also the massive OME of the pure singlet case has been calculated [51]. Here the master integrals are somewhat more demanding than for A_{gq} and $A_{qq}^{\text{NS,TR}}$ and they

were partly solved using differential and difference equations applying the packages `Sigma`, `EvaluateMultiSums`, `SumProduction` and `HarmonicSums`. Here the structure of the Wilson coefficient contains a series of generalized harmonic sums as a fully inclusive quantity in QCD. They are of the type

$$S_{2,1}(2, 1; N), \quad S_{1,1,2}\left(2, \frac{1}{2}, 1; N\right), \quad S_3(2; N), \quad \text{etc.} \quad (3.2)$$

These sums may individually diverge as $N \rightarrow \infty$. However, the asymptotic expansion of the complete expression is well behaved. In the representation in x -space, generalized harmonic polylogarithms emerge. For QCD corrections being related to deep-inelastic scattering quantities of this kind are observed for the first time.

4. Graphs with two massive quark lines of equal masses

Starting from 3-loop order graphs with two distinct internal massive lines occur in the calculation of the massive operator matrix elements. The corresponding contributions to the operator matrix elements A_{gg} and A_{gq} are characterized by the color factors $T_F^2 C_A(C_F)$ without additional factors N_f . The challenge in computing these diagrams derives from the fact that identifying hypergeometric series directly, as used in earlier 3-loop calculations [13–15], leads to divergent sums. In fact the degree of divergence even grows linearly with N due to factors

$$B(N+i+\alpha, -i+\beta) = \frac{\Gamma(N+i+\alpha)\Gamma(-i+\beta)}{\Gamma(N+\alpha+\beta)}, \quad (4.1)$$

where N is the Mellin variable, i the summation index of an infinite sum, and α, β are independent of N and i . To avoid this source of divergence, a Mellin-Barnes representation is introduced and the Beta-function of the above type is kept in the form of a Feynman parameter integral. As a result, the divergent pattern can be removed by observing that the contour of the Mellin-Barnes integral either must be closed to the right *or* to the left, depending on the value of the remaining Feynman parameter. Due to this distinction the remaining integral does not represent a Beta-function anymore, but will be performed in the space of certain iterated integrals at a later stage. The sum of residues is simplified using symbolic summation technologies [19–29].

In order to perform the last integral in x , say, a generating function for the Mellin moments is introduced with

$$\sum_{N=0}^{\infty} (\kappa R(x))^N = \frac{1}{1 - \kappa R(x)}, \quad (4.2)$$

where $R(x) \in \{1/(1+x^2), x^2/(1+x^2)\}$. This introduces cyclotomic letters weighted by the tracing parameter κ . The resulting expression involves cyclotomic harmonic polylogarithms (HPLs) which depend on the variable κ . In order to extract the Mellin-space expression the N th Taylor coefficient has to be calculated, which is possible using the packages `HarmonicSums` [37, 38, 41] and `Sigma` [19–27]. The resulting multi-sum expressions are simplified using the package `EvaluateMultiSums` [27–29] and expressed in a basis of indefinite (nested) sums.

As a proof of principle we calculated all scalar graphs which correspond to the T_F^2 -contributions to A_{gg} . Also the calculation of the full T_F^2 contributions will be finished soon. One of the QCD-graphs is shown in Figure 1.

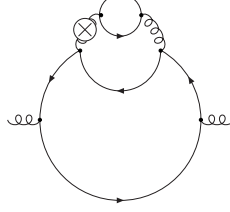


Figure 1: A digram with two massive quark cycles contributing to A_{gg} .

The N -space result has the following form:

$$\begin{aligned}
 I_{560} = & \frac{2\bar{P}_4}{3N(N+1)^2(N+2)(2N-5)(2N-3)(2N-1)} \frac{1}{4^N} \binom{2N}{N} \left[\sum_{j=1}^N \frac{4^j S_1(j)}{\binom{2j}{j} j^2} - \sum_{j=1}^N \frac{4^j}{\binom{2j}{j} j^3} - 7\zeta_3 \right] \\
 & + \frac{N^2 + N + 2}{27(N-1)N^2(N+1)} \left[-144S_{2,1} - 36\zeta_2 S_1 - 4S_1^3 + 36S_2 S_1 + 88S_3 \right. \\
 & \left. + 312\zeta_3 \right] - \frac{4\bar{P}_2}{6075(N-2)^2(N-1)^4 N^5 (N+1)^4 (N+2)(2N-5)(2N-3)(2N-1)} \\
 & - \frac{64(N^2 + N + 2)}{9\epsilon^3(N-1)N^2(N+1)} + \frac{1}{\epsilon^2} \left\{ -\frac{32\bar{P}_6}{27(N-1)^2 N^3 (N+1)^2 (N+2)} - \frac{32(N^2 + N + 2)}{9(N-1)N^2(N+1)} S_1 \right\} \\
 & + \frac{1}{\epsilon} \left\{ -\frac{8\bar{P}_7}{405(N-2)(N-1)^3 N^4 (N+1)^3 (N+2)} - \frac{16\bar{P}_6}{27(N-1)^2 N^3 (N+1)^2 (N+2)} S_1 \right. \\
 & \left. - \frac{8(N^2 + N + 2)(S_1^2 - 3S_2 + 3\zeta_2)}{9(N-1)N^2(N+1)} - \frac{8(55N^3 + 235N^2 - 52N + 20)}{15(N-2)(N-1)N(N+1)^2(N+2)} \right\} \\
 & - \frac{4\bar{P}_5 S_1}{81(N-1)^3 N^4 (N+1)^3 (N+2)(2N-5)(2N-3)(2N-1)} \\
 & - \frac{4\bar{P}_6 (S_1^2 - 3S_2 + 3\zeta_2)}{27(N-1)^2 N^3 (N+1)^2 (N+2)} - \frac{4\bar{P}_1}{225(N-2)^2 (N-1)^2 N^2 (N+1)^3 (N+2)}. \quad (4.3)
 \end{aligned}$$

Here the functions \bar{P}_i are polynomials in N up to degree $d = 17$ and we used the shorthand notation $S_{\bar{a}}(N) \equiv S_{\bar{a}}$ for the harmonic sums. Besides the well-known harmonic sums the above diagram depends on the new structure

$$\frac{1}{4^N} \binom{2N}{N} \left[\sum_{j=1}^N \frac{4^j S_1(j)}{\binom{2j}{j} j^2} - \sum_{j=1}^N \frac{4^j}{\binom{2j}{j} j^3} - 7\zeta_3 \right], \quad (4.4)$$

which involves binomially weighted harmonic sums within finite sums.⁵ These objects cannot be represented in terms of (generalized) harmonic sums or (generalized) cyclotomic sums. In all scalar diagrams, and all considered QCD diagrams contributing to the color factors $T_F^2 C_A(C_F)$ these sums occur in the same combination, which is hence a property of the corresponding Feynman diagrams.

⁵Infinite binomial and inverse binomial sums have been considered in Refs. [52–55].

In some terms denominators occur, which introduce poles at points $N = \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}$. However, the rightmost singularity expected for these diagrams is $N = 1$. Interestingly, these poles can be shown to be removable by expanding (4.3) in a Laurent series around these points.

5. Massive quark production in charged current DIS at 2-loop order

The $O(\alpha_s)$ corrections to heavy flavor production in charged current deep-inelastic scattering have been calculated in [56–58]. Here the $O(\alpha_s^2)$ corrections are presented in the asymptotic region $Q^2 \gg m^2$ [59], comparing to an earlier calculation in Ref. [60]. This process is particularly important because of its sensitivity to the sea quark densities $\bar{s}(x, Q^2)$, $\bar{d}(x, Q^2)$ and $\bar{u}(x, Q^2)$. Furthermore the asymptotic representation is fully justified since the corresponding data are measured mostly at high virtualities $Q^2 \gtrsim 100 \text{ GeV}^2$.⁶

The charged current cross sections for deep inelastic lepton-nucleon scattering is commonly parameterized in three structure functions F_1, F_2, F_3 :

$$\frac{d\sigma^{v(\bar{v})}}{dxdy} = \frac{G_{FS}^2}{4\pi} \left\{ (1 + (1-y)^2)F_2^{W^\pm} - y^2 F_L^{W^\pm} \pm (1 - (1-y)^2)x F_3^{W^\pm} \right\}, \quad (5.1)$$

$$\frac{d\sigma^{e^-(e^+)}}{dxdy} = \frac{G_{FS}^2}{4\pi} \left\{ (1 + (1-y)^2)F_2^{W^\mp} - y^2 F_L^{W^\mp} \pm (1 - (1-y)^2)x F_3^{W^\mp} \right\}. \quad (5.2)$$

The expressions of the heavy flavor Wilson coefficients in the asymptotic region are constructed in terms of light flavor Wilson coefficients and massive operator matrix elements (OMEs). This is achieved by exploiting the process independence of the PDFs and OMEs and constructing the 4-flavor expressions in the variable flavor number scheme in [5, 9]. By matching them back onto the 3-flavor scheme one finds the factorization formulae in the asymptotic region :

$$\begin{aligned} L_{i,q}^{W^+\pm W^-, \text{NS}, (2)} &= \delta_{i,2} A_{qq,Q}^{\text{NS}, (2)} + C_{i,q}^{W^+\pm W^-, \text{NS}, (2)}(n_f + 1) - C_{i,q}^{W^+\pm W^-, \text{NS}, (2)}(n_f), \\ H_{i,q}^{W^+\pm W^-, \text{NS}, (2)} &= \delta_{i,2} A_{qq,Q}^{\text{NS}, (2)} + C_{i,q}^{W^+\pm W^-, \text{NS}, (2)}(n_f + 1), \\ L_{i,q}^{W, \text{PS}, (2)} &= C_{i,q}^{W, \text{PS}, (2)}(n_f + 1) - C_{i,q}^{W, \text{PS}, (2)}(n_f) = 0, \\ H_{i,q}^{W, \text{PS}, (2)} &= \frac{1}{2} \delta_{i,2} A_{Qq}^{\text{PS}, (2)} + C_{i,q}^{W, \text{PS}, (2)}(n_f + 1), \\ L_{i,g}^{W, (2)} &= A_{gg,Q}^{(1)} C_{i,g}^{W, (1)}(n_f + 1) + C_{i,g}^{W, (2)}(n_f + 1) - C_{i,g}^{W, (2)}(n_f), \\ H_{i,g}^{W, (2)} &= A_{gg,Q}^{(1)} C_{i,g}^{W, (1)}(n_f + 1) + C_{i,g}^{W, (2)}(n_f + 1) \\ &\quad + \frac{1}{2} \left(\delta_{i,2} A_{Qg}^{(2)} + A_{Qg}^{(1)} C_{i,q}^{W^+ + W^-, \text{NS}, (1)}(n_f + 1) \right), \\ L_{3,q}^{W^+\pm W^-, \text{NS}, (2)} &= A_{qq,Q}^{\text{NS}, (2)} + C_{3,q}^{W^+\pm W^-, \text{NS}, (2)}(n_f + 1) - C_{3,q}^{W^+\pm W^-, \text{NS}, (2)}(n_f), \\ H_{3,q}^{W^+\pm W^-, \text{NS}, (2)} &= A_{qq,Q}^{\text{NS}, (2)} + C_{3,q}^{W^+\pm W^-, \text{NS}, (2)}(n_f + 1), \\ H_{3,q}^{W, \text{PS}, (2)} &= -\frac{1}{2} A_{Qq}^{\text{PS}, (2)}, \\ H_{3,g}^{W, (2)} &= \frac{1}{2} \left(-A_{Qg}^{(2)} - A_{Qg}^{(1)} C_{3,q}^{W^+ + W^-, \text{NS}, (1)}(n_f + 1) \right). \end{aligned} \quad (5.3)$$

⁶For a discussion of the accuracy of the asymptotic representation see [4].

Here $A_{ij}^{(k)}$, $k = 1, 2$ denote the massive OMEs [2, 4, 6–8] and $C_{l,m}$ the massless Wilson coefficients [61–65] up to 2-loop order. Note that the $O(\alpha_s^2)$ corrections contain, besides a single heavy quark excitation also contributions due to heavy quark pair-production.

The Wilson coefficients (5.3) correct and complete an earlier derivation of the heavy flavor Wilson coefficients [60]; for details see Ref. [59]. The difference lies in factors (-1) in the expressions for $H_{3,q}^{W,PS,(2)}$ and $H_{3,g}^{W,(2)}$. The correctness of the present result was checked by an explicit calculation of the leading logarithmic parts using the same idea as in [66], referring to the ladder graph contributions in physical gauge. Furthermore the construction of $H_{3,g}^{W,(1)}$ in the same way delivers a minus sign that can be reproduced by the asymptotic expansion of the exact 1-loop result [58]. We also calculated the terms to $O(\alpha_s^2)$ having been left out in Ref. [60].

The representation of the Wilson coefficients (5.3) both in Mellin- N and x -space have been derived [59] using the package `HarmonicSums` [37, 38, 41]. For the use in phenomenological applications we implemented both these expressions into FORTRAN-programs, which are available on request.

In Figures 2 numerical illustrations of the charm quark corrections to $F_2(x, Q^2)$ and $x F_3(x, Q^2)$ are given in leading (LO), next-to-leading (NLO) and next-to-next-to-leading order (NNLO) at different scales of Q^2 . The difference between LO and NLO turns out to be large since the NLO corrections are dominated by the gluon- W fusion process, which contributes for the first time, and which reflects the size of the gluon distribution. They get much smaller comparing NLO and NNLO, where also the factorization scale uncertainty is expected to stabilize.

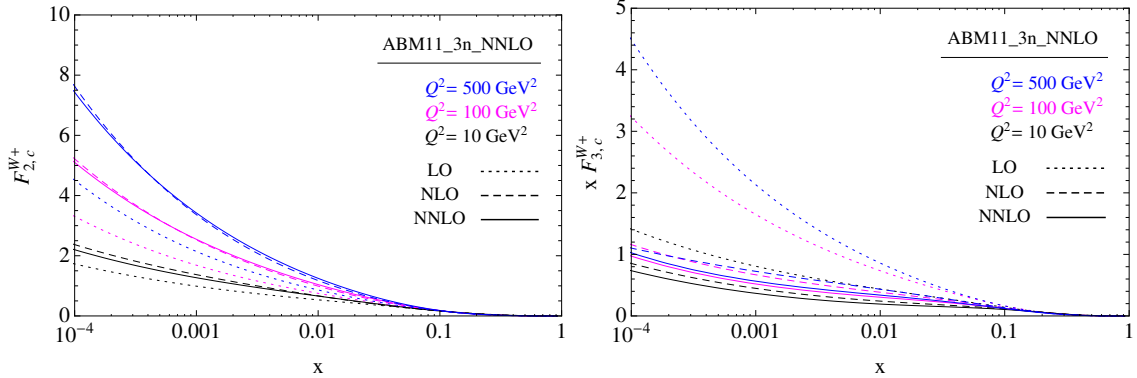


Figure 2: Charm contributions to the structure functions F_2 and $x F_3$ of deep-inelastic scattering via W^+ -exchange at LO, NLO, NNLO.

Here the ABM11 PDF set [67] at NNLO in the 3-flavor scheme was used.

6. Conclusions

We reported on recent progress in the calculation of massive 3-loop operator matrix elements and Wilson Coefficients for deep-inelastic scattering for general values of the Mellin variable N . Four years after a larger amount of Mellin moments for these quantities had been computed, six out of eight OMEs and corresponding Wilson coefficients in the region $Q^2 \gg m^2$ have been calculated analytically. In parallel, quite a series of theoretical, mathematical and computer-algebraic

technologies had to be newly developed and put significantly forward to make the present results possible. Here we would like to note in particular the automated use of IBP-identities for massive 3-loop diagrams also containing local operator insertions in `Reduze2` and modern summation technologies built in several advanced summation packages by the Linz-Group along with the development of the present project. These and related technologies are assumed to have a significant potential to be used in many other calculations in quantum field theory in the future. We also obtained a better insight into the calculation of the more difficult topologies, like those of V -graphs and the treatment of graphs with two massive fermion lines, being necessary to perform the forthcoming calculations. Furthermore, we obtained the asymptotic $O(\alpha_s^2)$ heavy-flavor corrections for deep-inelastic charged current scattering. The remaining part of the present project is still putting a series of very interesting challenges to be mastered.

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