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HYPERDIRE
HYPERgeometric functions DIfferential REduction: MATHEMATICA based packages for differential reduction of generalized hypergeometric functions: F_D and F_S Horn-type hypergeometric functions of three variables.

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Abstract

HYPERDIRE is a project devoted to the creation of a set of `Mathematica` based programs for the differential reduction of hypergeometric functions. The current version includes two parts: the first one, **FdFunction**, for manipulations with Appell hypergeometric functions F_D of r variables; and the second one, **FsFunction**, for manipulations with Lauricella-Saran hypergeometric functions F_S of three variables. Both functions are related with one-loop Feynman diagrams.

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Keywords: Hypergeometric functions; Differential reduction; Feynman diagrams

PROGRAM SUMMARY

Title of program: HYPERDIRE

Version: 1.0.0 *Release*: 1.0.0 *Catalogue number*:

Program obtained from `https://sites.google.com/site/loopcalculations/home`:

E-mail: bvv@jinr.ru

Computers: all computers running `Mathematica`

Operating systems: operating systems running `Mathematica`

Programming language: `Mathematica`

Keywords: multivariable Lauricella functions, Horn functions, Feynman integrals.

Nature of the problem: Reduction of hypergeometric functions F_D and F_S to set of basis functions.

Method of solution: Differential reduction

Restriction on the complexity of the problem: none

Typical running time: Depending on the complexity of problem.

LONG WRITE-UP

1 Introduction

The study of solutions of linear partial differential equations (PDEs) of a few variables in terms of multiple series, i.e., a multivariable generalization of Gauss hypergeometric function [1], was started a long time ago [2]. Following the Horn definition [1], a multiple series is called Horn-type hypergeometric function [4], if around some point $\vec{z}=\vec{z_0}$, there are series representations

$$H(\vec{z})=\sum_{\vec{m}} C(\vec{m})\vec{z}^{\,\vec{m}},$$

where $\vec{m}$ is a set of integers and the ratio of two coefficients can be represented as a ratio of two polynomials:

$$\frac{C(\vec{m}+\vec{e}_j)}{C(\vec{m})}=\frac{P_j(\vec{m})}{Q_j(\vec{m})}\ , \tag{1}$$

where $\vec{e}_j=(0,\cdots,0,1,0,\cdots,0)$, is the j^{th} unit vector. The coefficients $C(\vec{m})$ of such a series are expressible as product/ratio of Gamma-functions (up to some factors irrelevant for our consideration) [5]:

$$C(\vec{m})=\frac{\prod\limits_{j=1}^{p}\Gamma\left(\sum_{a=1}^{r}\mu_{ja}m_a+\gamma_j\right)}{\prod\limits_{k=1}^{q}\Gamma\left(\sum_{b=1}^{r}\nu_{kb}m_b+\sigma_k\right)}\ , \tag{2}$$

where $\mu_{ja},\nu_{kb},\sigma_j,\gamma_j\in\mathbb{Z}$ and m_a are elements of $\vec{m}$.

The Horn-type hypergeometric function, Eq. (1), satisfies the following system of differential equations:

$$0=D_j(\vec{z})H(\vec{z})=\left[Q_j\left(\sum_{k=1}^{r}z_k\frac{\partial}{\partial z_k}\right)\frac{1}{z_j}-P_j\left(\sum_{k=1}^{r}z_k\frac{\partial}{\partial z_k}\right)\right]H(\vec{z})\ , \tag{3}$$

where $j=1,\ldots,r$. The degree of polynomials P_i and Q_i is p_i and q_i, respectively. The largest of these numbers, $r=\max\{p_i,q_j\}$, is called the order of the hypergeometric series.

Any Horn-type hypergeometric function is a function of two kind of variables, *continuous* variables, $z_1,z_2,\cdots,z_r$ and *discrete* variables: $\{J_a\}:=\{\gamma_k,\sigma_r\}$, where the latter can change by integer numbers and are often referred to as the *parameters* of the hypergeometric function. For any Horn-hypergeometric function, there are linear differential operators changing the value of the discrete variables by one unit:

$$R_K(\vec{z})\frac{\partial^K}{\partial\vec{z}}H(\vec{J};\vec{z})=H(\vec{J}\pm e_K;\vec{z})\ , \tag{4}$$

[1] The modern approach to hypergeometric functions has been presented in [3].

where $R_K(\vec{z})$ are polynomial (rational) functions. In Refs. [6,7] it was shown that there is algorithmic solution for the construction of inverse linear differential operators:

$$B_L(\vec{z})\frac{\partial^L}{\partial\vec{z}}H(\vec{J};\vec{z}) = H(\vec{J}\mp e_L;\vec{z}) \ , \tag{5}$$

or, expressed in another form,

$$B_L(\vec{z})\frac{\partial^L}{\partial\vec{z}}\left(R_K(\vec{z})\frac{\partial^K}{\partial\vec{z}}\right)H(\vec{J};\vec{z}) = H(\vec{J}\pm e_K\mp e_L;\vec{z}) \ . \tag{6}$$

Applying the direct or inverse differential operators to the hypergeometric function, the value of parameters can be changed by an arbitrary integer numbers:

$$S(\vec{z})H(\vec{J}+\vec{m};\vec{z}) = \sum_{j=0}^{r} S_j(\vec{z})\frac{\partial^j}{\partial\vec{z}}H(\vec{J};\vec{z}) \ , \tag{7}$$

where $\vec{m}$ is a set of integers, S and S_j are polynomials and r is the holonomic rank (the number of linearly independent solutions) of the system of differential equations, Eq. (3). Additionally, the construction of inverse differential operators defined by Eq. (5) (or by Eq. (6)) allows to

(i) find a set of exceptional parameters for any hypergeometric function, and this set coincides with the condition of reducibility of the monodromy group of the corresponding hypergeometric functions (see discussion in [8]);

(ii) convert the system of linear PDEs, Eq. (3), into Pfaff form for any hypergeometric functions, including functions with Puiseux monomials as one of the solution, see details in [9].

The interest of physicists in hypergeometric functions is related with

(i) the necessity of an analytical evaluation of multiple series generated by multiple residues of Mellin-Barnes integrals [10];

(ii) the restricted set of values of parameters of hypergeometric functions or multiple series, where the algorithms [11,12,14,16] are applicable;

(iii) the complicated analytical structure of one-loop massive Feynman diagrams, where, nevertheless, a simple hypergeometric representation exist [17,18,20].

It was pointed out in [21] that the differential reduction algorithm, defined as a full system of differential operators, Eqs. (3), (4), (5), can be applied to the construction of analytical coefficients of the so-called ε-expansions of hypergeometric functions about any rational values of parameters via the direct solution of the linear systems of differential equations.

This is the motivation for creating a package for the manipulation of the parameters of Horn-type hypergeometric functions of several variables.

In the previous publications the algebraic reduction of ${}_2F_1$ functions has been considered [22], the program **pfq**, for the manipulation of hypergeometric functions, ${}_{p+1}F_p$ ($p \geq 1$) [8], the program **AppellF1F4**, for the manipulation of Appell hypergeometric functions, F_1, F_2, F_3 and F_4 [23], the program **Horn**, for the manipulation of Horn-hypergeometric functions of two variables (30 hypergeometric functions in addition to four Appell functions) [9].

The aim of this paper is to present a further extension of the `Mathematica` [24] based package **HYPERDIRE** for the differential reduction of the Horn-type hypergeometric function with arbitrary values of parameters to a set of basis functions. The current version consists of two parts: one, **FdFunction**, for the manipulation of Lauricella hypergeometric functions, F_D, of r variables, and the second one, **FsFunction**, for the manipulation with Lauricella-Saran hypergeometric functions F_S with three variables.

2 The structure of hypergeometric functions related with one-loop off-shell Feynman diagrams

A generic scalar one-loop N-point function is defined by the following integral in d space-time dimensions

$$I^{(d)}_{N;a_1,\cdots,a_N} = \int \frac{d^d l}{(2\pi)^d} \frac{1}{((l-p_1)^2 - m_{12}^2)^{a_1} \ldots ((l-(p_1+\ldots p_{N-1}))^2 - m^2_{N-1,N})^{a_{N-1}} (l^2 - m^2_{N,1})^{a_N}} , \quad (8)$$

where l is the loop momentum to be integrated, p_i are the external momenta and $m^2_{i,j}$ the masses of the internal propagators, $i, j = 1, \ldots, N$. Energy-momentum conservation enforces $\sum_i p_i = 0$.

2.1 Massive case

In accordance with algorithm described in [25], one-loop N-point diagrams with all powers of propagators equal to unity, i.e., all $a_i = 1$ in Eq. (8), satisfy to the following difference equation

$$I_N^{(d)} = b_N(d) + \sum_{k=1}^{N} \left(\frac{\partial_k \Delta_N}{2\Delta_N} \right) \sum_{r=0}^{\infty} \left(\frac{d-N+1}{2} \right)_r \left(\frac{G_{N-1}}{\Delta_N} \right)^r \mathbf{k}^- I_N^{(d+2r)} , \quad (9)$$

where $I_N^{(d)} \equiv I^{(d)}_{N;a_1,\cdots,a_N}$, $(a)_k$ is a Pochhammer symbol, $(a)_k = \Gamma(a+k)/\Gamma(a)$. d is dimension of space-time, $b_N(d)$ is a some function of space-time dimension, and we are working in Euclidean space-time, which is the source of the sign "+" instead of "-" as it was defined in [25], G_N is a Gram determinant, Δ_N is a Cayley determinant for the N-point diagram and $\partial_k \Delta_N = \frac{\partial}{\partial m_k^2} \Delta_N$. For details we refer to [17,25]. Eq. (9) can be solved iteratively [17,25] and the result for the one-loop N-point diagram in an arbitrary dimension d can be written as linear combinations of the following hypergeometric functions:

$$
\begin{aligned}
I^{(d)}_{N\geq 2} \sim & \prod_{j=2}^{N} \left(\frac{\partial_{k_j}\Delta_j}{2\Delta_j} \right) \times \sum_{r_1,r_2,r_3,\cdots,r_{N-1}=0}^{\infty} \left(-m^2_{N-1,N} \frac{G_{N-1}}{\Delta_N} \right)^{r_{N-1}} \cdots \left(-m^2_{12} \frac{G_1}{\Delta_2} \right)^{r_1} \\
& \times \frac{\Gamma\left(\frac{d-N+1}{2}+r_{N-1}\right)}{\Gamma\left(\frac{d-N+1}{2}\right)} \cdots \frac{\Gamma\left(\frac{d-4}{2}+r_4\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-4}{2}+r_5\cdots+r_{N-1}\right)} \frac{\Gamma\left(\frac{d-3}{2}+r_3+r_4\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-3}{2}+r_4\cdots+r_{N-1}\right)} \\
& \times \frac{\Gamma\left(\frac{d-2}{2}+r_2+r_3\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-2}{2}+r_3\cdots+r_{N-1}\right)} \frac{\Gamma\left(\frac{d-1}{2}+r_1+r_2\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-1}{2}+r_2\cdots+r_{N-1}\right)} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}+r_1+r_2\cdots+r_{N-1}\right)} \\
& + \sum_{j=3}^{N} b_j(d) c_j(d) \,, \qquad (10)
\end{aligned}
$$

where $m^2_{i,j}$ are some masses, cf., Eq. (8). In accordance with **Proposition 1** of [26] [2], the system of differential equations for the last terms in Eq. (10) has the same order as another terms. Dropping all irrelevant factors, the hypergeometric function related with one-loop N-point off-shell massive Feynman diagram is (it has a simpler form in contrast to the results of [27]):

$$
\begin{aligned}
H^{(d)}_{N\geq 2} = & \sum_{r_1,r_2,r_3,\cdots,r_{N-1}=0}^{\infty} \frac{\Gamma\left(\frac{d-1}{2}+r_1+r_2\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d}{2}+r_1+r_2\cdots+r_{N-1}\right)} \\
& \times \frac{\Gamma\left(\frac{d-2}{2}+r_2+r_3\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-1}{2}+r_2\cdots+r_{N-1}\right)} \frac{\Gamma\left(\frac{d-3}{2}+r_3+r_4\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-2}{2}+r_3\cdots+r_{N-1}\right)} \\
& \times \frac{\Gamma\left(\frac{d-4}{2}+r_4\cdots+r_{N-1}\right)}{\Gamma\left(\frac{d-3}{2}+r_4\cdots+r_{N-1}\right)} \cdots \frac{\Gamma\left(\frac{d-N+1}{2}+r_{N-1}\right)}{\Gamma\left(\frac{d-N+2}{2}+r_{N-1}\right)} z_1^{r_1} z_2^{r_2} \cdots z_{N-1}^{r_{N-1}} \,. \qquad (11)
\end{aligned}
$$

For the lowest values of $N = 2, 3, 4, 5$, Eq. (11) has the following form:

$$
H^{(d)}_2 = \sum_{r_1} \frac{\left(\frac{d-1}{2}\right)_{r_1}}{\left(\frac{d}{2}\right)_{r_1}} z_1^{r_1} \,, \qquad (12)
$$

$$
H^{(d)}_3 = \sum_{r_1,r_2} \frac{\left(\frac{d-1}{2}\right)_{r_1+r_2} \left(\frac{d-2}{2}\right)_{r_2}}{\left(\frac{d}{2}\right)_{r_1+r_2} \left(\frac{d-1}{2}\right)_{r_2}} z_1^{r_1} z_2^{r_2} \,, \qquad (13)
$$

$$
H^{(d)}_4 = \sum_{r_1,r_2,r_3} \frac{\left(\frac{d-1}{2}\right)_{r_1+r_2+r_3} \left(\frac{d-2}{2}\right)_{r_2+r_3} \left(\frac{d-3}{2}\right)_{r_3}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3} \left(\frac{d-1}{2}\right)_{r_2+r_3} \left(\frac{d-2}{2}\right)_{r_3}} z_1^{r_1} z_2^{r_2} z_3^{r_3} \,, \qquad (14)
$$

$$
H^{(d)}_5 = \sum_{r_1,r_2,r_3,r_4} \frac{\left(\frac{d-1}{2}\right)_{r_1+r_2+r_3+r_4} \left(\frac{d-2}{2}\right)_{r_2+r_3+r_4} \left(\frac{d-3}{2}\right)_{r_3+r_4} \left(\frac{d-4}{2}\right)_{r_4}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4} \left(\frac{d-1}{2}\right)_{r_2+r_3+r_4} \left(\frac{d-2}{2}\right)_{r_3+r_4} \left(\frac{d-3}{2}\right)_{r_4}} z_1^{r_1} z_2^{r_2} z_3^{r_3} z_4^{r_4} \,. \qquad (15)
$$

[2] For completeness, we recall it here: A multiple Mellin-Barnes integrals can be presented as a linear combination of Horn-type hypergeometric functions about some point. Therefore, the holonomic rank of the corresponding system of linear differential equations related with the Mellin-Barnes integral is equal to the holonomic rank of any hypergeometric function in the corresponding hypergeometric representation.

In accordance with Eq. (3), the order of differential equations of hypergeometric functions Eq. (11), increase with number of external legs:

$$\frac{P_N^j}{Q_N^j} = j \; ,$$

where the index j is the same as the summation index r_j and $j+1$ is equal to the number of external legs of the Feynman diagrams, cf., Eq. (8). To reduce the order of differential equations of the hypergeometric function Eq. (11), we apply recursively the following transformation:

$$\begin{aligned}\sum_{r=0}^{\infty} \frac{\Gamma(A+r)}{\Gamma(B+r)} z^r &= \frac{\Gamma(A)}{\Gamma(B)} {}_2F_1\left(\left. \begin{array}{c} 1, A \\ B \end{array} \right| z \right) = \frac{1}{1-z} \frac{\Gamma(A)}{\Gamma(B)} {}_2F_1\left(\left. \begin{array}{c} 1, B-A \\ B \end{array} \right| \frac{z}{z-1} \right) \\ &= \frac{1}{1-z} \frac{\Gamma(A)}{\Gamma(B-A)} \sum_{r=0}^{\infty} \left(\frac{z}{z-1} \right)^r \frac{\Gamma(B-A+r)}{\Gamma(B+r)} \; . \end{aligned} \tag{16}$$

Let us introduce new variables:

$$x_i = -\frac{z_i}{1-z_i} \; , \quad z_i = -\frac{x_i}{1-x_i} \; , \quad 1 - z_i = \frac{1}{1-x_i} \; . \tag{17}$$

The recursive application of the linear-fractional transformation, Eq. (16), to Eq. (11) gives rise to the following hypergeometric function:

$$\begin{aligned}&\prod_{k=1}^{N-1} (1-z_k) H_{N\geq 2}^{(d)} = \\ &= \sum_{r_1,r_2,r_3,r_4,\cdots,r_{N-1}=0}^{\infty} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}+r_1+\cdots+r_{N-1}\right)} \prod_{i=1}^{N-1} x_i^{r_i} \left[\frac{\Gamma\left(\frac{i}{2}+r_1+\cdots+r_i\right)}{\Gamma\left(\frac{i}{2}+r_1+\cdots+r_{i-1}\right)} \right] . \end{aligned} \tag{18}$$

For completeness, we present explicitly the hypergeometric terms defined by Eq. (18) for the first few values of $N = 2, 3, 4, 5$: [3]

$$H_2^{(d)} = \sum_{r_1} \frac{\left(\frac{1}{2}\right)_{r_1}}{\left(\frac{d}{2}\right)_{r_1}} x_1^{r_1} \; , \tag{19}$$

$$H_3^{(d)} = \sum_{r_1,r_2} \frac{\left(\frac{1}{2}\right)_{r_1} \left(\frac{d-2}{2}\right)_{r_2}}{\left(\frac{d}{2}\right)_{r_1+r_2}} x_1^{r_1} z_2^{r_2} \; , \tag{20}$$

$$= \sum_{r_1,r_2} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2}}{(1)_{r_1} \left(\frac{d}{2}\right)_{r_1+r_2}} x_1^{r_1} x_2^{r_2} \; , \tag{21}$$

[3] For our discussion we drop all irrelevant factors, like $(1-z_i)^{\pm 1}$ and assume, wherever it does not cause any problems, that $\Gamma\left(\frac{d}{2} \pm k + \vec{r}\right) \equiv \left(\frac{d}{2} \pm k\right)_{\vec{r}}$ with k being integer.

$$H_4^{(d)} = \sum_{r_1,r_2,r_3} \frac{\left(\frac{1}{2}\right)_{r_1} \left(\frac{d-2}{2}\right)_{r_2+r_3} \left(\frac{d-3}{2}\right)_{r_3}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3} \left(\frac{d-2}{2}\right)_{r_3}} x_1^{r_1} z_2^{r_2} z_3^{r_3} \tag{22}$$

$$= \sum_{r_1,r_2,r_3} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{d-3}{2}\right)_{r_3}}{(1)_{r_1} \left(\frac{d}{2}\right)_{r_1+r_2+r_3}} x_1^{r_1} x_2^{r_2} z_3^{r_3} \tag{23}$$

$$\equiv \sum_{r_1,r_2,r_3} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{3}{2}\right)_{r_1+r_2+r_3}}{(1)_{r_1} \left(\frac{3}{2}\right)_{r_1+r_2} \left(\frac{d}{2}\right)_{r_1+r_2+r_3}} x_1^{r_1} x_2^{r_2} x_3^{r_3} \,, \tag{24}$$

where the x_i are defined in Eq. (17). As follows from Eq. (20) and Eq. (21), the vertex diagrams are described by the Appell hypergeometric functions F_3 or F_1 [28]. For the pentagon ($N = 5$), the hypergeometric function has the following form:

$$H_5^{(d)} = \sum_{r_1,r_2,r_3,r_4}^{\infty} \frac{\left(\frac{1}{2}\right)_{r_1} \left(\frac{d-2}{2}\right)_{r_2+r_3+r_4}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_4}} \frac{\left(\frac{d-3}{2}\right)_{r_3+r_4}}{\left(\frac{d-2}{2}\right)_{r_3+r_4}} \frac{\left(\frac{d-4}{2}\right)_{r_4}}{\left(\frac{d-3}{2}\right)_{r_4}} x_1^{r_1} z_2^{r_2} z_3^{r_3} z_4^{r_4} \,, \tag{25}$$

$$= \sum_{r_1,r_2,r_3,r_4=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2}}{(1)_{r_1}} \frac{\left(\frac{d-3}{2}\right)_{r_3+r_4}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4}} \frac{\left(\frac{d-4}{2}\right)_{r_4}}{\left(\frac{d-3}{2}\right)_{r_4}} x_1^{r_1} x_2^{r_2} z_3^{r_3} z_4^{r_4} \,, \tag{26}$$

$$= \sum_{r_1,r_2,r_3,r_4=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2}}{(1)_{r_1}} \frac{\left(\frac{3}{2}\right)_{r_1+r_2+r_3}}{\left(\frac{3}{2}\right)_{r_1+r_2}} \frac{\left(\frac{d-4}{2}\right)_{r_4}}{\left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4}} x_1^{r_1} x_2^{r_2} x_3^{r_3} z_4^{r_4} \,, \tag{27}$$

$$= \sum_{r_1,r_2,r_3,r_4} \frac{\left(\frac{1}{2}\right)_{r_1} (1)_{r_1+r_2} \left(\frac{3}{2}\right)_{r_1+r_2+r_3} \left(\frac{4}{2}\right)_{r_1+r_2+r_3+r_4}}{(1)_{r_1} \left(\frac{3}{2}\right)_{r_1+r_2} \left(\frac{4}{2}\right)_{r_1+r_2+r_3} \left(\frac{d}{2}\right)_{r_1+r_2+r_3+r_4}} x_1^{r_1} x_2^{r_2} x_3^{r_3} x_4^{r_4} \,. \tag{28}$$

After the linear-fractional transformation, Eq. (16), the order of the differential equation for the hypergeometric function related to the box diagram is reduced from three to two, see Eqs. (22) and (23), [17]. The pentagon, Eq. (26), corresponds to a hypergeometric function satisfying a differential equation of order two [4]. As follows from Eqs. (11) and (18), the massive hexagon is expressible in terms of hypergeometric functions of five variables satisfying a differential equations of order three. However, since the difference between parameters of hypergeometric functions, Eqs. (11) or (18), are integer or half-integer, these functions possess extended symmetries with respect to non-linear transformations of their arguments [29] (multivariable generalizations of quadratic transformations related to Gauss hypergeometric functions) [5]. It is still open question, whether or not is it possible, to reduce the order of differential equations with the help of non-linear transformations.

[4] At present, a full classification of Horn-type hypergeometric functions of four variables does not exist [4].

[5] All hypergeometric functions, defined by Eqs. (12)-(28), belong to the class of multiple Gauss hypergeometric functions [4]: the series representation can be written as infinite sum(s) with respect to the index of summation over the parameters of ${}_2F_1$ hypergeometric functions: in Eqs. (21), (24), (28) the Gauss hypergeometric functions enter via last index of summation; in Eq. (22) via summation over r_1; in Eq. (23) via summation over r_2 or r_3; in Eq. (25) via summation over r_1 or r_2; in Eq. (26) via summation over r_2 or r_3; in Eq. (27) via summation over r_3. Exploring the transformation properties of Gauss hypergeometric functions, see Eq. (16) for an example, the transformation of hypergeometric functions, Eqs. (12)-(28), can be performed.

2.2 Off-shell massless case

Let us consider an off-shell massless one-loop N-point diagram, for where, for some i, we have $\{p_i^2\} \neq 0$, cf., Eq. (8). In these kinematics, the three-point diagram (vertex) is not algebraically reducible to a simpler diagram. The $I_2^{(d)}$ integral can be written (up to some irrelevant normalization) as

$$I_2^{(d)} = \frac{1}{\Gamma\left(\frac{d-1}{2}\right)} \,, \tag{29}$$

and the iterative solution of Eq. (9) is

$$\begin{aligned} H_{N\geq 3}^{(d)}\Big|_{\{p_i^2\}\neq 0} &= \sum_{r_1,r_2,r_3,\cdots,r_{N-2}=0}^{\infty} \frac{\Gamma\left(\frac{d-2}{2}+r_1+r_2\cdots+r_{N-2}\right)}{\Gamma\left(\frac{d-1}{2}+r_1+r_2\cdots+r_{N-2}\right)} \\ &\times \frac{\Gamma\left(\frac{d-3}{2}+r_2+r_3\cdots+r_{N-2}\right)}{\Gamma\left(\frac{d-2}{2}+r_2+r_3\cdots+r_{N-2}\right)} \frac{\Gamma\left(\frac{d-4}{2}+r_3+r_4\cdots+r_{N-2}\right)}{\Gamma\left(\frac{d-3}{2}+r_3+r_4\cdots+r_{N-2}\right)} \\ &\times \frac{\Gamma\left(\frac{d-5}{2}+r_4\cdots+r_{N-2}\right)}{\Gamma\left(\frac{d-4}{2}+r_4\cdots+r_{N-2}\right)} \cdots \frac{\Gamma\left(\frac{d-N+1}{2}+r_{N-2}\right)}{\Gamma\left(\frac{d-N+2}{2}+r_{N-2}\right)} z_1^{r_1} z_2^{r_2} \cdots z_{N-2}^{r_{N-2}} \,. \end{aligned} \tag{30}$$

From Eqs. (11) and (30) we see, that the structure of hypergeometric functions related with off-shell massive and off-shell massless integrals is related as follows [20]:

$$H_{N+1}^{(d)}\Big|_{\{p_i^2\}\neq 0} \sim H_N^{(d-1)} \,, \tag{31}$$

where d is the dimension of space-time and N denotes the number of external legs. The symbol $\sim$ in Eq. (31) indicates that this relation is valid for hypergeometric functions related with the corresponding Feynman diagram. Eq. (31) is also valid for hypergeometric functions, Eq. (18), after application of the linear-fractional transformation, Eq. (16).

3 Differential reduction of Horn-type hypergeometric functions of three variables

3.1 System of differential equations

Let us consider the system of linear differential operators of second order L_j for the hypergeometric functions $\omega(\vec{z})$:

$$L_i\omega(\vec{z}) : \quad \theta_i^2\omega(\vec{z}) = \left[\sum_{j;j\neq i} P_{ij}\theta_j\theta_i + \sum_m R_{im}\theta_m + S_i\right]\omega(\vec{z}) \,, \quad i = 1, \cdots, 3, \tag{32}$$

where $\vec{z} = (z_1, z_2, z_3)$ with z_1, z_2, z_3 being variables, $\{P_{i,j}, R_{i,ab}, S_j\}$ are rational functions, $\theta_j = z_j\partial_{z_j}$ for $j = 1, 2, 3$, and $\theta_{i_1\cdots i_k} = \theta_{i_i} \cdots \theta_{i_k}$. Taking the derivative, $\theta_k L_i\omega(\vec{z})$, we finally obtain from Eq. (32):

$$\theta_k L_i \omega(\vec{z}) : \quad \left[(1 - P_{ik}P_{ki})\, \theta_k \theta_i^2 - \sum_{J \neq k \neq i = 1}^{3} (P_{iJ} + P_{ik}P_{kJ})\, \theta_i \theta_k \theta_J \right] \omega(\vec{z})$$

$$= \Bigg\{ [P_{ik}R_{ki}P_{ik} + R_{ik}P_{ki} + R_{ii} + P_{ik}R_{kk} + (\theta_k P_{ik})]\, \theta_{ik}$$

$$+ \sum_{J \neq k \neq i = 1}^{3} [P_{ik}R_{ki}P_{iJ} + P_{ik}R_{kJ} + (\theta_k P_{iJ})]\, \theta_i \theta_J + \sum_{J \neq i \neq k = 1}^{3} [R_{ik}P_{kJ} + R_{iJ}]\, \theta_k \theta_J$$

$$+ \sum_{m=1}^{3} [P_{ik}R_{ki}R_{im} + R_{ik}R_{km} + (\theta_k R_{im})]\, \theta_m + P_{ik}S_k\theta_i + S_i\theta_k$$

$$+ P_{ik}R_{ki}S_i + R_{ik}S_k + (\theta_k S_i) \Bigg\} \omega(\vec{z}) \,. \tag{33}$$

For a function of three variables the sum $\sum_{J \neq k \neq i}$ can be replaced by the index j, where $j \neq i \neq k$. The conditions of complete integrability are defined via the relations:

$$\theta_i \left[\theta_j L_k\right] \omega(\vec{z}) = \theta_j \left[\theta_i L_k\right] \omega(\vec{z}) \;, \quad i,j = 1 \cdots 3. \tag{34}$$

The number of independent solutions of the system of differential equations, Eq. (32), of three variables is defined by coefficients in l.h.s. of Eq. (33) and the validity of Eq. (34). When the coefficients

$$(1 - P_{ik}P_{ki}) \;, \quad \{i,k\} = 1,2,3 \;, \tag{35}$$

and

$$(P_{iJ} + P_{ik}P_{kJ}) \;, \quad J \neq i,k \;, \quad \{J,i,k\} = 1,2,3 \;, \tag{36}$$

are not equal to zero for all i, j, k, Eqs. (32) and (33) can be reduced to the Pfaff system of eight independent differential equations:

$$d\vec{f} = R\vec{f} \;, \tag{37}$$

where $\vec{f} = (\omega(\vec{z}), \theta_1\omega(\vec{z}), \theta_2\omega(\vec{z}), \theta_3\omega(\vec{z}), \theta_{12}\omega(\vec{z}), \theta_{13}\omega(\vec{z}), \theta_{23}\omega(\vec{z}), \theta_{123}\omega(\vec{z}))$. When some of the coefficients in Eq. (35) are zero, the coefficients in front of the terms $\theta_{123}\omega(\vec{z})$, defined by Eq. (36), start to play a role. For non-zero values of Eq. (36), the terms $\theta_{123}\omega(\vec{z})$ can be excluded, and the rank of differential system is reduced to seven independent functions. When for some i and k both coefficients, defined by Eq. (35) and Eq. (36) are zero, a further simplification can be performed, so that the rank of system is reduced to six or to an even smaller number.

The locus of singularities L_{ij} of the linear system of differential equations of second order of three variables defined by Eq. (32) follows from singularities of higher rank differential operators in the l.h.s. of Eq. (32) and Eq. (33):

$$L_{ij} = \cup_{i=1}^{3}\{z_i\} \cup_{i,k=1}^{3} \{P_{ik}^{-1}\} \cup_{i,k=1}^{3} \{(1 - P_{ik}P_{ki})^{-1}\} \cup_{i,j,k=1}^{3} \{(P_{ij} - P_{ik}P_{kj})^{-1}\} \;. \tag{38}$$

3.2 Lauricella hypergeometric function F_D

3.2.1 General consideration

Let us consider the $F_D^{(r)}$ functions of r variables, defined around $x_i = 0$ as

$$F_D^{(r)}(a; b_1, \cdots, b_k; c; z_1, \cdots, z_r) = \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{(a)_{|\vec{m}|}}{(c)_{\vec{|m|}}} \prod_{j=1}^{r} (b_j)_{m_j} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_k^{m_r}}{m_r!} \,, \tag{39}$$

For $r = 1$ this functions coincides with the Gauss hypergeometric function, for $r = 2$, it coincides with Appell function F_1 [30]. As follows from the definition, Eq. (39), this function is symmetric with respect to the transformation

$$b_i \Leftrightarrow b_j \,, \quad z_i \Leftrightarrow z_j \,.$$

Generally, F_D functions and their properties have been analyzed in detail in many references [6, 31]. The differential operators for F_D hypergeometric function, Eq. (3), are given by

$$\begin{aligned} D_i F_D^{(r)} : \quad & \partial_i \left(c - 1 + \sum_{j=1}^{r} \theta_j \right) F_D^{(r)} = \left(a + \sum_{j=1}^{r} \theta_j \right) (b_i + \theta_i) \, F_D^{(r)} \,, \\ & i = 1, \cdots, r. \end{aligned} \tag{40}$$

where

$$F_D \equiv F_D^{(r)}(a; b_1, \cdots, b_r; c; z_1, \cdots, z_r) \,. \tag{41}$$

They can be written in canonical form, cf. Eq. (32):

$$\begin{aligned} L_i F_D : \quad & \theta_i^2 F_D = \left[-\theta_i \sum_{j;j\neq i} \theta_j + \frac{(a+b_i)z_i - (c-1)}{1-z_i} \theta_i + \frac{b_i z_i}{1-z_i} \sum_{j;j\neq i} \theta_j + \frac{a b_i z_i}{1-z_i} \right] F_D \,, \\ & i = 1, \cdots, r. \end{aligned} \tag{42}$$

From these equations we have:

$$\begin{aligned} & P_{ij} = P_{ji} = -1 \,, \quad S_i = \frac{a b_i z_i}{1 - z_i} \equiv a P_i \,, \\ & R_{ii} = \frac{(a + b_i) z_i - (c-1)}{1 - z_i} \equiv R_i \,, \quad R_{im} = \frac{b_i z_i}{1 - z_i} \equiv P_i \,, \quad m \neq i \,. \end{aligned} \tag{43}$$

Upon substitution of these values for P_{ij}, R_{ab}, S_i into Eq. (33), we obtain

$$\left([P_k - P_i + R_i - R_k] \, \theta_i \theta_k - [R_{ki} R_{im} - R_{ik} R_{km}] \sum_{m=1}^{r} \theta_m - S_k \theta_i + S_i \theta_k - P_k S_i + P_i S_k \right) F_D = 0 \,. \tag{44}$$

Eq. (44) can be simplified with the help of Eq. (43) by taking into account that the sum of the last two terms in Eq. (44), $P_iS_k - P_kS_i$, is equal to zero, and by splitting the sum over m into $i, k, j \neq i \neq k$. In this way, we get

$$\left(R_k - P_k - R_i + P_i \right) \theta_i \theta_k F_D = \left(P_k \left[P_i - R_i - a \right] \theta_i - P_i \left[P_k - R_k - a \right] \theta_k \right) F_D \ , \quad (45)$$

where

$$R_i - P_i \equiv R_{ii} - R_{ik} = \frac{a z_i - (c-1)}{1 - z_i} \ . \quad (46)$$

Eq. (45) can be rewritten in a more familiar form, see [31]:

$$\left[(z_i - z_j) \theta_i \theta_j - b_j z_j \theta_i + b_i z_i \theta_j \right] F_D = 0 \ . \quad (47)$$

After factorization of z_i, z_j, Eq. (47) can be expressed as follows,

$$\left[(z_i - z_j) \partial_{ij} + b_i \partial_j - b_j \partial_i \right] F_D = 0. \quad (48)$$

In this way, all second derivatives of an F_D function are expressible in terms of the corresponding first derivatives and function, see Eqs. (42) and (47). As consequence, there are only $r+1$ linearly independent solutions of linear differential equations, Eq. (40). The locus of the singularities L_{ij} of an F_D function is defined from the singularities of the differential equations, Eqs. (42) and (47):

$$L_{ij} = \cup_{i=1}^{r} \{ z_i = 0 \} \cup_{1=i<j=r} \{ z_i - z_j = 0 \} \cup_{i=1}^{r} \{ z_i = 1 \} \ .$$

The Pfaff system for an F_D hypergeometric function has the following form:

$$d\omega(\vec{z}) = \left(\sum_{i<j} A_{ij} d \log(z_i - z_j) \right) \omega(\vec{z}) \ ,$$

where $\omega(\vec{z}) = \{ F_D, \theta_j F_D \}$ and the matrices A_{ij} have been constructed explicitly in [31].

3.2.2 Differential reduction of F_D

The direct differential operators are the following:

$$\begin{aligned}
a F_D^{(r)}(a+1; b_1, \cdots, b_r; c; z_1, \cdots, z_r) &= \left(a + \prod_{i=1}^{r} \theta_i \right) F_D \ , \\
b_i F_D^{(r)}(a; b_1, \cdots, b_i + 1, \cdots, b_r; c; z_1, \cdots, z_r) &= (b_i + \theta_i) F_D \ , \\
(c-1) F_D^{(r)}(a; b_1, \cdots, b_r; c-1; z_1, \cdots, z_r) &= \left(c - 1 + \prod_{i=1}^{r} \theta_i \right) F_D \ ,
\end{aligned} \quad (49)$$

and F_D is defined by Eq. (41). The inverse differential operators have been constructed in [6]:

$$
\begin{aligned}
& (c-a)F_D^{(r)}(a-1;b_1,\cdots,b_r;c;z_1,\cdots,z_r) = \\
& \qquad \left[\sum_{j=1}^{r}(1-z_j)\theta_j - \sum_{j=1}^{r} b_j z_j + c - a\right] F_D \; , \qquad (50) \\
& (c-\sum_{j=1}^{r} b_j)F_D^{(r)}(a;b_1,\cdots,b_i-1,\cdots,b_r;c;z_1,\cdots,z_r) = \\
& \qquad \left[z_i\sum_{j=1}^{r}(1-z_j)\partial_j - a z_i + c - \sum_{j=1}^{r} b_j\right] F_D \; , \qquad (51) \\
& (c-a)(c-\sum_{j=1}^{r} b_j)F_D^{(r)}(a;b_1,\cdots,b_r;c+1;z_1,\cdots,z_r) = \\
& \qquad c\left[\sum_{j=1}^{r}(1-z_j)\partial_j + c - a - \sum_{j=1}^{r} b_j\right] F_D \; , \qquad (52)
\end{aligned}
$$

where F_D is defined by Eq. (41). In this case, the results of the differential reduction, Eq. (7), have the following form

$$
S(\vec{z})F_D((a;\vec{b};c)+\vec{m};\vec{z}) = S_0(\vec{z})F_D((a;\vec{b};c)s;\vec{z}) + \sum_{i=1}^{r} S_i(\vec{z})\frac{\partial}{\partial z_i}F_D^{(r)}(a;\vec{b};c;\vec{z}) \; , \qquad (53)
$$

where $\vec{m}$ is a set of integers and S, S_j are polynomials.

3.3 Hypergeometric function F_S

3.3.1 General consideration

The Lauricella-Saran hypergeometric function of three variables F_S [32] (F_7 in notations of [2]) is defined around the point $z_1 = z_2 = z_3 = 0$ as follows

$$
F_S(a_1;a_2;b_1,b_2,b_3;c;z_1,z_2,z_3) = \sum_{m_1,m_2,m_3=0}^{\infty} \frac{(a_1)_{m_1}(a_2)_{m_2+m_3}}{(c)_{m_1+m_2+m_3}} \prod_{j=1}^{3}(b_j)_{m_j} \frac{z_1^{m_1} z_2^{m_2} z_3^{m_3}}{m_1! m_2! m_3!} \, . \qquad (54)
$$

It is one of the 14 functions of three variables of order two [6], introduced by Lauricella [2]. In this case, the differential operators, Eq. (3), are

$$
D_1 F_S : \qquad \partial_1\left(c-1+\sum_{j=1}^{3}\theta_j\right) F_S = (a_1+\theta_1)(b_1+\theta_1) F_S \; , \qquad (55)
$$

[6] The complete set of programs for the differential reduction for other functions from the Lauricella-Srivastava list [4] will be presented in separate publication.

$$D_iF_S : \qquad \partial_i \left(c-1+\sum_{j=1}^{3}\theta_j \right) F_S = (a_2+\theta_2+\theta_3)\,(b_i+\theta_i)\,F_S \ , \quad i=2,3 \ , \tag{56}$$

where

$$F_S = F_S(a_1,a_2;b_1,b_2,b_3;c;z_1,z_2,z_3) \ . \tag{57}$$

The canonical form of these differential equations are the following:

$$L_1F_S : \qquad \theta_1^2 F_S = \left[-\frac{1}{1-z_1}\theta_1\,(\theta_2+\theta_3) + \frac{(a_1+b_1)z_1-(c-1)}{1-z_1}\theta_1 + \frac{a_1b_1z_1}{1-z_1} \right] F_S \ , \tag{58}$$

$$L_2F_S : \qquad \theta_2^2 F_S = \left[-\theta_2\theta_3 - \frac{1}{1-z_2}\theta_2\theta_1 + \frac{(a_2+b_2)z_2-(c-1)}{1-z_2}\theta_2 + \frac{b_2z_2}{1-z_2}\theta_3 + \frac{a_2b_2z_2}{1-z_2} \right] F_S \ , \tag{59}$$

$$L_3F_S : \qquad \theta_3^2 F_S = \left[-\theta_3\theta_2 - \frac{1}{1-z_3}\theta_3\theta_1 + \frac{(a_2+b_3)z_3-(c-1)}{1-z_3}\theta_3 + \frac{b_3z_3}{1-z_3}\theta_2 + \frac{a_2b_3z_3}{1-z_3} \right] F_S \ . \tag{60}$$

These equations define the values of functions P_{ij}, R_{ab}, S_i entering in Eq. (32):

$$\begin{aligned}
& R_{12} = R_{13} = R_{21} = R_{31} = 0 \ , \quad P_{23} = P_{32} = -1 \ , \\
& P_{12} = P_{13} = -\frac{1}{1-z_1} \ , \quad P_{21} = -\frac{1}{1-z_2} \ , \quad P_{31} = -\frac{1}{1-z_3} \ , \\
& R_{11} = \frac{(a_1+b_1)z_1-(c-1)}{1-z_1} \ , \quad R_{ii} = \frac{(a_2+b_i)z_i-(c-1)}{1-z_i} \ , i=2,3 \ , \\
& R_{23} = \frac{b_2z_2}{1-z_2} \ , \quad R_{32} = \frac{b_3z_3}{1-z_3} \ , \quad S_1 = \frac{a_1b_1z_1}{1-z_1} \ , \quad S_i = a_2\frac{b_iz_i}{1-z_i} \ , i=2,3 \ .
\end{aligned} \tag{61}$$

With the substitution of these values of P_{ij} into Eq. (33) and, since $1-P_{23}P_{32}=0$, we can express the third mixing derivatives of hypergeometric function, $\theta_{123}\omega(\vec{z})$, via second derivatives of hypergeometric function.

The series representation of the hypergeometric function F_S can be rewritten in the following form:

$$\begin{aligned}
& F_S(a_1,a_2,a_2;b_1,b_2,b_3;c,c,c;z_1,z_2,z_3) \\
& = \sum_{m_1=0}^{\infty} \frac{(a_1)_{m_1}(b_1)_{m_1}}{(c)_{m_1}} \frac{z_1^{m_1}}{m_1!} F_1(a_2;b_2,b_3;c+m_1;z_2,z_3) \ ,
\end{aligned} \tag{62}$$

where $F_1(a;b_1,b_2;c;z_1,z_2)$ is the Appell function of two variables: $F_1 \equiv F_D^{(2)}$. From this represenation it is easy to get the following relation:

$$\left[(z_2-z_3)\theta_{23}\right] F_S = (b_3z_3\theta_2 - b_2z_2\theta_3)\,F_S \ . \tag{63}$$

Eq. (33) allows us to express all higher derivatives of hypergeometric functions F_S in terms of second derivatives only. In particular,

$$\theta_2 L_1 : \left(1 - \frac{1}{(1-z_1)(1-z_2)}\right)\theta_{112}F_S$$
$$= \left\{(P_{12}R_{22}+R_{11})\theta_{12}+P_{12}R_{23}\theta_{13}+P_{12}S_2\theta_1+S_1\theta_2\right\}F_S$$
$$= \left\{\left(\frac{(a_1+b_1)z_1-(c-1)}{1-z_1} - \frac{(a_2+b_2)z_2-(c-1)}{(1-z_1)(1-z_2)}\right)\theta_{12}\right.$$
$$\left. - \frac{b_2 z_2}{(1-z_1)(1-z_2)}\theta_{13} - \frac{a_2 b_2 z_2}{(1-z_1)(1-z_2)}\theta_1 + \frac{a_1 b_1 z_1}{1-z_1}\theta_2\right\}F_S \ , \tag{64}$$

$$\theta_3 L_1 : \left(1 - \frac{1}{(1-z_1)(1-z_3)}\right)\theta_{113}F_S$$
$$= \left\{(P_{13}R_{33}+R_{11})\theta_{13}+P_{13}R_{32}\theta_{12}+P_{13}S_3\theta_1+S_1\theta_3\right\}F_S$$
$$= \left\{\left(\frac{(a_1+b_1)z_1-(c-1)}{1-z_1} - \frac{(a_2+b_3)z_3-(c-1)}{(1-z_1)(1-z_3)}\right)\theta_{13}\right.$$
$$\left. - \frac{b_3 z_3}{(1-z_1)(1-z_3)}\theta_{12} - \frac{a_2 b_3 z_3}{(1-z_1)(1-z_3)}\theta_1 + \frac{a_1 b_1 z_1}{1-z_1}\theta_3\right\}F_S \ , \tag{65}$$

$$\theta_3 L_2 = -\theta_2 L_3 :$$
$$\frac{(z_3-z_2)}{(1-z_2)(1-z_3)}\theta_{123}F_S = \left\{P_{21}R_{32}\theta_{12} - P_{31}R_{23}\theta_{13}\right.$$
$$-\left(R_{22}-R_{23}+R_{32}-R_{33}\right)\frac{1}{z_2-z_3}\left(b_3 z_3\theta_2 - b_2 z_2\theta_3\right)$$
$$\left. -R_{23}\left(a_2+R_{33}-R_{32}\right)\theta_3 + R_{32}\left(a_2+R_{22}-R_{23}\right)\theta_2\right\}F_S$$
$$= \left\{\frac{b_2 z_2}{(1-z_2)(1-z_3)}\theta_{13} - \frac{b_3 z_3}{(1-z_2)(1-z_3)}\theta_{12}\right\}F_S \ , \tag{66}$$

where Eq. (66) follows from Eq. (63). The remaining two differential equations we can write in the following form:

$$\theta_1 L_2 : \left(1 - \frac{1}{(1-z_1)(1-z_2)}\right)\theta_{122} F_S = -\left(1 - \frac{1}{(1-z_1)(1-z_2)}\right)\theta_{123} F_S$$
$$+\Biggl\{ (P_{21}R_{11} + R_{22})\,\theta_{12} + R_{23}\theta_{13} + P_2 S_1 \theta_2 + S_2\theta_1 \Biggr\} F_S$$
$$= -\left(1 - \frac{1}{(1-z_1)(1-z_2)}\right)\theta_{123} F_S$$
$$+\Biggl\{ \left(\frac{(a_2+b_2)z_2 - (c-1)}{1-z_2} - \frac{(a_1+b_1)z_1 - (c-1)}{(1-z_1)(1-z_2)} \right)\theta_{12}$$
$$+\frac{b_2 z_2}{(1-z_2)}\theta_{13} - \frac{a_1 b_1 z_1}{(1-z_1)(1-z_2)}\theta_2 + \frac{a_2 b_2 z_2}{1-z_2}\theta_1 \Biggr\} F_S \ , \tag{67}$$

$$\theta_1 L_3 : \left(1 - \frac{1}{(1-z_1)(1-z_3)}\right)\theta_{133} F_S = -\left(1 - \frac{1}{(1-z_1)(1-z_3)}\right)\theta_{123} F_S$$
$$+\left\{ (P_{31}R_{11} + R_{33})\,\theta_{13} + R_{32}\theta_{12} + P_{31} S_1 \theta_3 + S_3\theta_1 \right\} F_S$$
$$= -\left(1 - \frac{1}{(1-z_1)(1-z_2)}\right)\theta_{123} F_S$$
$$+\Biggl\{ \left(\frac{(a_2+b_3)z_3 - (c-1)}{1-z_3} - \frac{(a_1+b_1)z_1 - (c-1)}{(1-z_1)(1-z_3)} \right)\theta_{13}$$
$$+\frac{b_3 z_3}{(1-z_3)}\theta_{12} - \frac{a_1 b_1 z_1}{(1-z_1)(1-z_3)}\theta_3 + \frac{a_2 b_3 z_3}{1-z_3}\theta_1 \Biggr\} F_S \ , \tag{68}$$

where the mixed derivative $\theta_{123} F_S$ is defined by Eq. (66). In this way, we have proven:
Theorem 1:
The Lauricella-Saran hypergeometric function F_S of three variables, Eq. (54), has six linearly independent solutions around the points $z_1 = z_2 = z_3 = 0$.

The locus of singularities L_{ij} of the hypergeometric function F_S follows from the singularities of the differential operators, Eqs. (58)-(60), (66)-(68):

$$L_{ij} = \cup_{i=1}^{3} \{z_i = 0\} \cup \{z_2 - z_3 = 0\} \cup_{i=1}^{3} \{z_i = 1\} \ .$$

3.3.2 Differential reduction of F_S

The direct differential operators are the following:

$$\begin{aligned}
a_1 F_S(a_1+1, a_2; \vec{b}; c; \vec{x}) &= (a_1+\theta_1)\, F_S \ , \\
a_2 F_S(a_1, a_2+1; \vec{b}; c; \vec{x}) &= (a_2+\theta_2+\theta_3)\, F_S \ , \\
b_i F_S(a_1, a_2; \cdots, b_i+1, \cdots; c; \vec{x}) &= (b_i+\theta_i)\, F_S \ , \\
(c-1) F_S(\vec{a}; \vec{b}; c-1; \vec{x}) &= \left(c-1+\prod_{j=1}^{3}\theta_j \right) F_S \ .
\end{aligned} \tag{69}$$

and F_S is defined by Eq. (57). The corresponding inverse differential operators we define for parameters a_1, a_2, b_1, b_2, b_3 as follows:

$$
\begin{aligned}
& F_S(\mathrm{Index}_{\{a_1,a_2,b_1,b_2,b_3\}}; \vec{z}) = \\
& \left[A_{\mathrm{Index},Fs} + B_{\mathrm{Index},Fs}\theta_1 + C_{\mathrm{Index},Fs}\theta_2 + D_{\mathrm{Index},Fs}\theta_3 + E_{\mathrm{Index},Fs}\theta_{12} + F_{\mathrm{Index},Fs}\theta_{13} \right] \\
& \times F_S(\mathrm{Index}_{\{a_1,a_2,b_1,b_2,b_3\}+1}; \vec{z}) \;, \qquad (70)
\end{aligned}
$$

and for parameter c:

$$
\begin{aligned}
& F_S(a_1, a_2; b_1, b_2, b_3; c; \vec{z}) = \\
& \left[A_{c,Fs} + B_{c,Fs}\theta_1 + C_{c,Fs}\theta_2 + D_{c,Fs}\theta_3 + E_{c,Fs}\theta_{12} + F_{c,Fs}\theta_{13} \right] F_S(a_1, a_2; b_1, b_2, b_3; c-1; \vec{z}) \;. \qquad (71)
\end{aligned}
$$

The full list of inverse differential operators are the following:

$$
\begin{aligned}
A_{a_1,Fs} &= \frac{a_1^2 + a_1(b_1 z_1 + D_1 + D_3 - 2b_1) + a_2(b_1 z_1 + D_2 - a_1) + (b_1 z_1 - c + 1)(D_2 - a_1)}{D_0 D_2} \;, \\
B_{a_1,Fs} &= \frac{(z_1 - 1)(a_2 + D_2)}{D_0 D_2} \;, \quad C_{a_1,Fs} = \frac{b_1 z_1 (z_2 - 1)}{z_2 D_0 D_2} \;, \quad D_{a_1,Fs} = \frac{b_1 z_1 (z_3 - 1)}{z_3 D_0 D_2} \;, \\
E_{a_1,Fs} &= -\frac{z_1 + z_2 - z_1 z_2}{z_2 D_0 D_2} \;, \quad F_{a_1,Fs} = -\frac{z_1 + z_3 - z_1 z_3}{z_3 D_0 D_2} \;, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
A_{a_2,Fs} &= \frac{(b_2 z_2 + b_3 z_3 + D_1)(a_1 + D_1) - b_1 D_1}{D_0 D_1} \;, \quad B_{a_2,Fs} = \frac{(z_1 - 1)(b_2 z_2 + b_3 z_3)}{z_1 D_0 D_1} \;, \\
C_{a_2,Fs} &= \frac{(z_2 - 1)(b_1 + D_0)}{D_0 D_1} \;, \quad D_{a_2,Fs} = \frac{(z_3 - 1)(b_1 + D_0)}{D_0 D_1} \;, \\
E_{a_2,Fs} &= -\frac{z_1 + z_2 - z_1 z_3}{z_1 D_0 D_1} \;, \quad F_{a_2,Fs} = -\frac{z_1 + z_3 - z_1 z_3}{z_1 D_0 D_1} \;, \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,Fs} &= -\frac{(c-1)}{D_0 D_1 D_2 D_3} \Big[a_1(a_2 + D_3)(D_1 + D_3) + D_1(D_2 + a_2 - a_1) D_3 \Big] \;, \qquad (74) \\
B_{c,Fs} &= -\frac{(c-1)(z_1 - 1)\left(a_2(D_1 + D_2) + D_2 D_3\right)}{z_1 D_0 D_1 D_2 D_3} \;, \\
C_{c,Fs} &= \frac{(c-1)(1 - z_2)\left(a_1(D_1 + D_2) + D_1 D_3\right)}{z_2 D_0 D_1 D_2 D_3} \;, \\
D_{c,Fs} &= \frac{(c-1)(1 - z_3)\left(a_1(D_1 + D_2) + D_1 D_3\right)}{z_3 D_0 D_1 D_2 D_3} \;, \\
E_{c,Fs} &= \frac{(c-1)(z_1 + z_2 - z_1 z_2)(D_1 + D_2)}{z_1 z_2 D_0 D_1 D_2 D_3} \;, \quad F_{c,Fs} = \frac{(c-1)(z_1 + z_3 - z_1 z_3)(D_1 + D_2)}{z_1 z_3 D_0 D_1 D_2 D_3} \;,
\end{aligned}
$$

$$
\begin{aligned}
A_{b_1,Fs} &= \frac{a_2(a_1z_1+D_3)+(a_1z_1+D_1-a_2)D_3}{D_1D_3} \;, \quad B_{b_1,Fs} = \frac{(z_1-1)(a_2+D_3)}{D_1D_3} \;, \\
C_{b_1,Fs} &= \frac{a_1z_1(z_2-1)}{z_2D_1D_3} \;, \quad D_{b_1,Fs} = \frac{a_1z_1(z_3-1)}{z_3D_1D_3} \;, \\
E_{b_1,Fs} &= -\frac{z_1+z_2-z_1z_2}{z_2D_1D_3} \;, \quad F_{b_1,Fs} = -\frac{z_1+z_3-z_1z_3}{z_3D_1D_3} \;,
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
A_{b_2,Fs} &= \frac{a_1(a_2z_2+D_3)+D_3(a_2z_2+D_3-b_1)}{D_2D_3} \;, \quad B_{b_2,Fs} = \frac{a_2(z_1-1)z_2}{z_1D_2D_3} \;, \\
C_{b_2,Fs} &= \frac{(z_2-1)(a_1+D_3)}{D_2D_3} \;, \quad D_{b_2,Fs} = \frac{z_2(z_3-1)(a_1+D_3)}{z_3D_2D_3} \;, \\
E_{b_2,Fs} &= -\frac{z_1+z_2-z_1z_2}{z_1D_2D_3} \;, \quad F_{b_2,Fs} = -\frac{z_2(z_1+z_3-z_1z_3)}{z_1z_3D_2D_3} \;.
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
A_{b_3,Fs} &= \frac{a_1(a_2z_3+D_3)+D_3(a_2z_3+D_3-b_1)}{D_2D_3} \;, \quad B_{b_3,Fs} = \frac{a_2(z_1-1)z_3}{z_1D_2D_3} \;, \\
C_{b_3,Fs} &= \frac{(z_2-1)z_3(a_1+D_3)}{z_2D_2D_3} \;, \quad D_{b_3,Fs} = \frac{(z_3-1)(a_1+D_3)}{D_2D_3} \;, \\
E_{b_3,Fs} &= -\frac{z_3(z_1+z_2-z_1z_2)}{z_1z_2D_2D_3} \;, \quad F_{b_3,Fs} = -\frac{z_1+z_3-z_1z_3}{z_1D_2D_3} \;,
\end{aligned} \tag{77}
$$

where

$$D_0 = a_1+a_2-(c-1) \;, \tag{78}$$
$$D_1 = a_2+b_1-(c-1) \;, \tag{79}$$
$$D_2 = a_1+b_2+b_3-(c-1) \;, \tag{80}$$
$$D_3 = b_1+b_2+b_3-(c-1) \;, \tag{81}$$

and

$$D_1+D_2 = D_0+D_3 \;. \tag{82}$$

The results of the differential reduction, Eq. (7), have the following form in this case:

$$
\begin{aligned}
& S(\vec{z})F_S((\vec{a};\vec{b};c)+\vec{m};\vec{z}) = \\
& \quad S_0(\vec{z})F_S(\vec{a};\vec{b};c;\vec{z}) + \sum_{i=1}^{3} S_i(\vec{z})\frac{\partial}{\partial z_i}F_S(\vec{a};\vec{b};c;\vec{z}) + \sum_{j=2}^{3} S_{1j}(\vec{z})\frac{\partial^2}{\partial z_1\partial z_j}F_S(\vec{a};\vec{b};c;\vec{z}) \;,
\end{aligned} \tag{83}
$$

where $\vec{m}$ is a set of integers, S, S_j and S_{ij} are polynomials.

$F_D^{(r)}$	$\{a, b_j, c-a, c-\sum_{j=1}^r b_j\} \in \mathbb{Z}$
F_S	$\{a_1, a_2, b_j, c-a_1-a_2, c-b_1-b_2-b_3, a_1+b_2+b_3-c, a_2+b_1-c\} \in \mathbb{Z}$

Table 1: Exceptional set of parameters for the hypergeometric functions $F_D^{(r)}$ and F_S.

3.4 Exceptional values of parameters: F_D and F_S

It was pointed out in [8], that the subset of parameters for which the results of the differential reduction, Eqs. (53) and (83), have simpler forms, can be defined from the conditions

(i) that the hypergeometric function entering the l.h.s. of Eqs. (50)–(52), (72)–(77), is expressible in terms of simpler hypergeometric functions ($F_D^{(r-1)}$ for $F_D^{(r)}$ and ${}_2F_1, F_1$ or F_3 for F_S hypergeometric function);

(ii) that some of the coefficients entering the inverse differential relations are equal to zero (infinity).

For the hypergeometric functions F_D and F_S, the exceptional sets of parameters are listed in Table 1.

4 Mathematica based program for the differential reduction of F_D and F_S hypergeometric functions

In this section, we will present the `Mathematica` based programs **FdFunction** and **FsFunction** for the differential reduction of Horn-type hypergeometric functions F_D of r variables and F_S of three variables [7]. In particular, in application to Lauricella functions F_D, the reduction algorithm, Eq. (7), has the following form:

$$R(x,y)F_D^{(r)}(a+m_a;\vec{b}+\vec{m}_b;c+m_c;\vec{z}) = [P_0(\vec{z})+P_1(\vec{z})\theta_{z_1}+\ldots+P_r(\vec{z})\theta_{z_r}]\, F_D^{(r)}(a;\vec{b};c;\vec{z})\ , \quad (84)$$

where $\vec{m}_b$, m_a, m_c are sets of integers and $\vec{b}$, a, c denote the set of parameters. R, P_i are some polynomial and $\theta_{z_i} = z_i\partial_{z_i}$. The differential reduction algorithm in application to the Lauricella-Saran function F_S is:

$$R(\vec{z})F_S(\vec{a}+\vec{m}_a;\vec{b}+\vec{m}_b;c+m_c;\vec{z}) = \\ [P_0(\vec{z}) + P_1(\vec{z})\theta_{z_1} + P_2(\vec{z})\theta_{z_2} + P_3(\vec{z})\theta_{z_3} + P_{12}(\vec{z})\theta_{z_1}\theta_{z_2} + P_{13}(\vec{z})\theta_{z_1}\theta_{z_3}]\, F_S(\vec{a};\vec{b};c;\vec{z})\ , \quad (85)$$

where, again, $\vec{m}_a$, $\vec{m}_b$, m_c denote sets of integers, $\vec{a}$, $\vec{b}$, c sets of parameters, and $R, \{P_j\}, \{P_{ij}\}$ some polynomials.

The program is freely available from [34] subject to the license conditions specified. The current version, 1.0, deals with non-exceptional values of a parameters only.

[7]The programs have been tested for `Mathematica` version 8.0.

4.1 Package FdFunction

The package can be loaded in the standard way:

$$<< \text{"FdFunction.m"}$$

and it includes the following basic routines:

$$\mathbf{FdIndexChange}[\text{changingVector}, \text{parameterVector}], \tag{86}$$

and

$$\mathbf{FdDiffSeries}[\ldots]\ , \mathbf{FdSeries}[\ldots] \tag{87}$$

The list "changingVector" in Eq. (86) provides the set of integers by which the values of parameters of Lauricella function F_D are to be changed, i.e., the vector $m_a, \{\vec{m}_b\}, m_c$ in Eqs. (84). The set of initial parameters of F_D function are defined in the list "parameterVector" corresponding to the vector $a + m_a; \vec{b} + \vec{m}_b, c + m_c$ and arguments $\vec{z}$ in the l.h.s. of Eqs. (84).

The structure of the output of **FdIndexChange**[] is the following:

$$\{\{A_1, A_2, \ldots, A_{r+1}\}, \{\text{parameterVectorNew}\}\}, \tag{88}$$

where

(i) "parameterVectorNew" is the set of new parameters of $F_D^{(r)}$ hypergeometric function;

(ii) $A_1, A_2, \ldots, A_{r+1}$ are the rational functions corresponding to the ratios of $P_0/R, P_1/R, P_2/R \ldots$ of functions entering in Eq. (84).

The functions **FdDiffSeries**[] and **FdSeries**[] are designed for the numerical evaluation of F_D hypergeometric functions. They return the Taylor series of F_D in its derivatives, respectively:

$$\mathbf{FdDiffSeries}[\text{numberOfvariable}, \text{vectorInit}, \text{numbSer}]\ , \tag{89}$$

$$\mathbf{FdSeries}[\text{vectorInit}, \text{numbSer}]\ , \tag{90}$$

where

(i) "numberOfvariable" is the list of variable numbers for differentiation;

(ii) "vectorInit" is the set of Fd parameters;

(iii) "numbSer" is the number of terms in Taylor expansion.

Let us present a number of examples for the usage[8]

[8] All functions in the package **HYPERDIRE** generate output without additional simplification. This is done for the maximum efficiency of the algorithm. To bring the output into a simpler form, we recommend to use in addition the command **Simplify**. All examples considered here have been treated with the command **Simplify[. . .]**. subsequent to the call of **HYPERDIRE**.

Example 1. Differential reduction of the hypergeometric function $F_D^{(2)}$ of two variables [9].

FdIndexChange[$\{-1,\{1,0\},1\}$, $\{a,\{b_1,b_2\},c,\{z_1,z_2\}\}$]

$$\Bigg\{\Bigg\{\frac{c(-1+z_2)-b_2z_2+z_1(-1+a+b_1(-1+z_2)+z_2-az_2+b_2z_2)}{c(-1+z_2)},$$
$$\frac{1-z_2-b_2z_2+z_1(-1+b_1(-1+z_2)+z_2+b_2z_2)+a(-1+z_1+z_2-z_1z_2)}{(-1+a)c(-1+z_2)},$$
$$\frac{1-a(1+z_1(-2+z_2))-b_2z_2+cz_2+z_1(-2-c+b_1(-1+z_2)+z_2+b_2z_2)}{(-1+a)c(-1+z_2)},$$
$$\{-1+a,\{1+b_1,b_2\},1+c,\{z_1,z_2\}\Bigg\}\Bigg\} \tag{91}$$

In an explicit form:

$$F_D^{(2)}(a;b_1,b_2,c;z_1,z_2) =$$
$$\Bigg[\frac{c(-1+z_2)-b_2z_2+z_1(-1+a+b_1(-1+z_2)+z_2-az_2+b_2z_2)}{c(-1+z_2)}$$
$$+\frac{1-z_2-b_2z_2+z_1(-1+b_1(-1+z_2)+z_2+b_2z_2)+a(-1+z_1+z_2-z_1z_2)}{(-1+a)c(-1+z_2)}\theta_1$$
$$+\frac{1-a(1+z_1(-2+z_2))-b_2z_2+cz_2+z_1(-2-c+b_1(-1+z_2)+z_2+b_2z_2)}{(-1+a)c(-1+z_2)}\theta_2\Bigg]$$
$$\times F_D^{(2)}(a-1;b_1+1,b_2,c+1;z_1,z_2). \tag{92}$$

Example 2. Reduction of the hypergeometric function $F_D^{(3)}$ of three variables.

FdIndexChange[$\{-1,\{1,-1,0\},0\}$, $\{a,\{b_1,b_2,b_3\},c,\{z_1,z_2,z_3\}\}$]

$$\{\{\tfrac{z_1-1}{z_2-1}, \tfrac{z_1-1}{(a-1)(z_2-1)}, \tfrac{z_1(a-c+(b_2-1)z_2)-(a-c+b_2-1)z_2}{(a-1)(b_2-1)(z_2-1)z_2}, \tfrac{z_1-1}{(a-1)(z_2-1)}\},$$
$$\{a-1,\{b_1+1,b_2-1,b_3\},c,\{z_1,z_2,z_3\}\}\}$$

[9] When $r=2$, the Lauricella function F_D coincides with the Appell function F_1 and the package **AppellF1F4** [8] can be used for the differential reduction.

This has the explicit form:

$$F_D^{(3)}(a;b_1,b_2,b_3;c;z_1,z_2,z_3) = \left[\frac{z_1-1}{z_2-1} + \frac{z_1-1}{(a-1)(z_2-1)}\theta_1 + \frac{z_1(a-c+(b_2-1)z_2)-(a-c+b_2-1)z_2}{(a-1)(b_2-1)(z_2-1)z_2}\theta_2 + \frac{z_1-1}{(a-1)(z_2-1)}\theta_3\right] F_D^{(3)}(a-1;b_1+1,b_2-1,b_3;c;z_1,z_2,z_3). \quad (93)$$

Example 3. Reduction of hypergeometric function $F_D^{(5)}$ of five variables.

FdIndexChange[{−1,{0,1,0,0,−1},0}, {a,{b_1,b_2,b_3,b_4,b_5},c,{z_1,z_2,z_3,z_4,z_5}}]

$$\left\{\left\{\frac{z_2-1}{z_5-1}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2-1}{(a-1)(z_5-1)}, \frac{z_2(a+(b_5-1)z_5-c)-z_5(a+b_5-c-1)}{(a-1)(b_5-1)(z_5-1)z_5}\right\},\right.$$
$$\left.\{a-1,\{b_1,b_2+1,b_3,b_4,b_5-1\},c,\{z_1,z_2,z_3,z_4,z_5\}\}\right\}$$

This has the explicit form:

$$F_D^{(5)}(a;b_1,b_2,b_3,b_4,b_5;c;z_1,z_2,z_3,z_4,z_5) = \left[\frac{z_2-1}{z_5-1} + \frac{z_2-1}{(a-1)(z_5-1)}\theta_1 + \frac{z_2-1}{(a-1)(z_5-1)}\theta_2 + \frac{z_2-1}{(a-1)(z_5-1)}\theta_4 + \frac{z_2(a+(b_5-1)z_5-c)-z_5(a+b_5-c-1)}{(a-1)(b_5-1)(z_5-1)z_5}\theta_5\right] F_D^{(5)}(a-1;b_1+1,b_2-1,b_3;c;z_1,z_2,z_3,z_4,z_5).$$

The hypergeometric function F_D is not built into the current version of `Mathematica`. The series representation of the hypergeometric function $F_D^{(r)}$, Eq. (39), is implemented in our package. The functions **FdDiffSeries**[] and **FdSeries**[] allow to make numerical cross-checks of the results of the differential reduction. The corresponding examples for using these functions are all collected in the file `example-FdFunction.m`, which is available in [34].

4.2 Package FsFunction

Again, the program can be loaded in a standard way:

<< "FsFunction.m"

and its structure and output are similar to the **FdFunction** package. The package**FsFunction** includes the following basic routines:

$$\textbf{FsIndexChange}[\text{changingVector}, \text{parameterVector}], \quad (94)$$

Here, again, "changingVector" is the list of integers by which the values of the parameters of function F_S are to be changed, i.e., the vectors $\vec{m}_a, \vec{m}_b, m_c$ in Eq. (85), while the set of initial parameters of the function F_S is defined in the list "parameterVector" corresponding to the vector $\vec{a}+\vec{m}_a;\vec{b}+\vec{m}_b, c+m_c$ and the arguments $\vec{z}$ in the l.h.s. of Eq. (85).

The structure of the output of **FsIndexChange**[] is the following:

$$\{A, B, C, D, E, F\}, \{\text{parameterVectorNew}\}\}, \tag{95}$$

where

(i) "parameterVectorNew" is the set of new parameters of function F_S;

(ii) A, B, C, D, E, F are the rational functions corresponding to the ratios of P_0/R P_1/R, P_2/R, P_3/R, P_{12}/R and P_{13}/R entering Eq. (85).

Example 4: Reduction of F_S.

FsIndexChange[$\{-1,1,0,0,0,0\}, \{a_1,a_2,b_1,b_2,b_3,c,z_1,z_2,z_3\}$]

$$\begin{aligned}
&\Bigg\{\Bigg\{\frac{a_1+b_1z_1+b_2+b_3-c+1}{a_1+b_2+b_3-c+1}, -\frac{1-z_1}{a_1+b_2+b_3-c+1},\\
&-\frac{z_2\left(-a_1-b_2-b_3+c-1\right)-b_1z_1\left(z_2-1\right)}{\left(a_2-1\right)z_2\left(a_1+b_2+b_3-c+1\right)},\\
&-\frac{z_3\left(-a_1-b_2-b_3+c-1\right)-b_1z_1\left(z_3-1\right)}{\left(a_2-1\right)z_3\left(a_1+b_2+b_3-c+1\right)},\\
&-\frac{z_2-z_1\left(z_2-1\right)}{\left(a_2-1\right)z_2\left(a_1+b_2+b_3-c+1\right)}, -\frac{z_3-z_1\left(z_3-1\right)}{\left(a_2-1\right)z_3\left(a_1+b_2+b_3-c+1\right)}\Bigg\},\\
&\Bigg\{a_1+1, a_2-1, b_1, b_2, b_3, c, z_1, z_2, z_3\Bigg\}\Bigg\}
\end{aligned} \tag{96}$$

This has the explicit form:

$$\begin{aligned}
&F_S(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) =\\
&\Bigg[\frac{a_1+b_1z_1+b_2+b_3-c+1}{a_1+b_2+b_3-c+1} - \frac{1-z_1}{a_1+b_2+b_3-c+1}\theta_1\\
&-\frac{z_2\left(-a_1-b_2-b_3+c-1\right)-b_1z_1\left(z_2-1\right)}{\left(a_2-1\right)z_2\left(a_1+b_2+b_3-c+1\right)}\theta_2\\
&-\frac{z_3\left(-a_1-b_2-b_3+c-1\right)-b_1z_1\left(z_3-1\right)}{\left(a_2-1\right)z_3\left(a_1+b_2+b_3-c+1\right)}\theta_3\\
&-\frac{z_2-z_1\left(z_2-1\right)}{\left(a_2-1\right)z_2\left(a_1+b_2+b_3-c+1\right)}\theta_1\theta_2 - \frac{z_3-z_1\left(z_3-1\right)}{\left(a_2-1\right)z_3\left(a_1+b_2+b_3-c+1\right)}\theta_1\theta_3\Bigg]\\
&\times F_S(a_1+1, a_2-1; b_1, b_2, b_3; c; z_1, z_2, z_3).
\end{aligned} \tag{97}$$

Example 5: Reduction of F_S

FsIndexChange[{1,0,0,0,1,2}, {a_1,a_2,b_1,b_2,b_3,c,z_1,z_2,z_3}]

$\{\{-\frac{b_1 z_1(z_2-1)(-a_2 z_3-a_1+c)-z_2(a_2(z_3(b_3-c)+b_2(z_3-1))+c(c+1))+a_2 b_3 z_3-a_2 c z_3+c^2+c}{c(c+1)(z_2-1)}, \frac{z_1(a_2 z_3+a_1-c)-a_2 z_3+c}{c(c+1)},$
$-\frac{z_2(a_2-b_2 z_3-b_3 z_3+b_2+c z_3-2c-1)-a_2 z_3-b_1 z_1(z_2-1)(z_3-1)+b_3 z_3+c+z_3}{c(c+1)(z_2-1)},$
$\frac{z_2(z_3(b_3-c)+b_2(z_3-1)+c)+b_1 z_1(z_2-1)(z_3-1)-b_3 z_3+c z_3-c}{c(c+1)(z_2-1)}, \frac{(z_1(z_2-1)-z_2)(z_3-1)}{c(c+1)(z_2-1)}, -\frac{-z_3 z_1+z_1+z_3}{c^2+c}\},$
$\{a_1+1, a_2, b_1, b_2, b_3+1, c+2, z_1, z_2, z_3\}\}$

This has the explicit form:

$$
\begin{aligned}
&F_S(a_1, a_2; b_1, b_2, b_3; c; z_1, z_2, z_3) = \\
&\Bigg[-\frac{b_1 z_1 (z_2-1)(-a_2 z_3 - a_1 + c) - z_2 (a_2 (z_3 (b_3 - c) + b_2 (z_3 - 1)) + c(c+1))}{c(c+1)(z_2-1)} \\
&+\frac{a_2 b_3 z_3 - a_2 c z_3 + c^2 + c}{c(c+1)(z_2-1)} + \frac{z_1 (a_2 z_3 + a_1 - c) - a_2 z_3 + c}{c(c+1)}\theta_1 \\
&-\frac{z_2 (a_2 - b_2 z_3 - b_3 z_3 + b_2 + c z_3 - 2c - 1) - a_2 z_3 - b_1 z_1 (z_2-1)(z_3-1)}{c(c+1)(z_2-1)}\theta_2 \\
&+\frac{b_3 z_3 + c + z_3}{c(c+1)(z_2-1)}\theta_2 \\
&+\frac{z_2 (z_3 (b_3 - c) + b_2 (z_3 - 1) + c) + b_1 z_1 (z_2-1)(z_3-1) - b_3 z_3 + c z_3 - c}{c(c+1)(z_2-1)}\theta_3 \\
&\frac{(z_1 + (z_2 - 1) - z_2)(z_3 - 1)}{c(c+1)(z_2-1)}\theta_1\theta_2 - \frac{-z_3 z_1 + z_1 + z_3}{c^2 + c}\theta_1\theta_3\Bigg] \\
&\times F_S(a_1+1, a_2; b_1, b_2, b_3+1; c+2; z_1, z_2, z_3).
\end{aligned}
\tag{98}
$$

Also the hypergeometric function F_S is not built into the current version of `Mathematica`, whereas our package implements the series representation of the hypergeometric function F_S, Eq. (54). Again, this series representation is suitable for numerical checks of the results of the differential reduction and the corresponding examples are gathered in the file `example-FsFunction.m` available from [34].

5 Conclusion

The differential-reduction algorithm for Horn-type hypergeometric functions allows one to compare different functions in this class whose values for the parameters differ by integers. This proceeds in an entirely algorithmic manner suitable for automation in a computer algebra system. In this paper, we have presented the `Mathematica`-based programs **FdFunction** and **FsFunction** for the differential reduction of the generalized hypergeometric function F_D of r variables and the Lauricella-Saran hypergeometric function F_S of three variables.

Both functions are related with one-loop massive Feynman diagrams and both belong to the class of Horn-type hypergeometric function of order two.

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