

Ordering variable for parton showers

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ABSTRACT: The parton splittings in a parton shower are ordered according to an ordering variable, for example the transverse momentum of the daughter partons relative to the direction of the mother, the virtuality of the splitting, or the angle between the daughter partons. We analyze the choice of the ordering variable and conclude that one particular choice has the advantage of factoring softer splittings from harder splittings graph by graph in a physical gauge.

KEYWORDS: perturbative QCD, parton shower

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1 Introduction

In a companion paper [1], we have introduced a parton shower event generator, DEDUCTOR [2], based on our earlier work [3, 4]. The corresponding shower algorithm is designed to be suitable for an improved treatment of spin, as described in ref. [5], and for an improved treatment of color, as described in ref. [6]. This shower generator contains features that differ from other parton shower event generators even when one uses the leading color approximation and averages over spins, as we do in ref. [1]. One of these features is that the algorithm uses non-zero masses for initial state partons, which requires modified evolution equations for the parton distribution functions, as described in a separate companion paper [7]. The second feature is the choice of shower evolution variable, which is the subject of this paper.

In a parton shower event generator, when a parton labelled i splits to two partons, one typically assigns a variable V_i^2 to the splitting, where V_i^2 is a function of the momenta of the mother parton and its daughters. The purpose of defining V_i^2 is to order splittings within the shower: if the splitting of parton i comes before the splitting of parton j then $V_i^2 > V_j^2$. When V_i^2 has dimensions of mass squared, one can define a dimensionless shower time $t_i \propto \log(Q_0^2/V_i^2)$ where Q_0^2 denotes the scale of the hard interaction that initiates the shower. Then t increases as the shower progresses.

In most cases (k_T -ordering, virtuality ordering, but *not* angular ordering), V_i^2 is a measure of the hardness of the splitting: $V_i^2 \rightarrow 0$ when the angle between the daughter partons approaches zero or when the momentum of one of the daughter partons approaches zero. The ordering variable in DEDUCTOR is hardness based in this sense.

There is a physics reason for choosing a hardness based ordering parameter. The reason is simple to state if we are not too precise about it. We think of the simulated event as measured by possible physical observables. We can imagine a class of observables that have a resolution scale corresponding to a certain value of the shower time. With observables in this class, we see what happens at space-time separations from the hard interaction that are smaller than some value $1/V_i$. Anything that happens at larger space-time separations is “unresolvable.” To measure an observable in this class, we need to run the parton shower to time $t_i \propto \log(Q_0^2/V_i^2)$, but no further. If we use an observable that has a resolution down to softer interactions, we see more detail, corresponding to greater space-time separations. The parton shower algorithm allows us to see more detail if we generate more splittings that are softer and softer.

What does this look like as realized in the parton shower event generator? The parton shower generates final states starting with some hard process in hadron-hadron scattering. The shower includes both final state splittings and initial state splittings from the incoming partons. For each event, there is a shower history, H , as in figure 1. The event generator generates a given shower history with a probability P_H .

Now, the shower history has the appearance of a Feynman diagram, with its corresponding amplitude \mathcal{M}_H . Let’s assume that we use a physical gauge for evaluating \mathcal{M}_H . One might imagine that P_H is approximately proportional to the corresponding $|\mathcal{M}_H|^2$, $P_H \approx \mathcal{N}|\mathcal{M}_H|^2$. (The normalization factor \mathcal{N} need not concern us here.) That can’t be quite right because the event generator needs to account for interference graphs, in which a soft gluon is emitted from one parton in \mathcal{M}_H and from a different parton in an amplitude $\mathcal{M}_{H'}$. For that reason, we really need to discuss amplitudes \mathcal{M}_H . We will, in fact, discuss amplitudes, but let us ignore this issue for a first orientation. Then it is possible, but not necessary, that the P_H is an approximation to $\mathcal{N}|\mathcal{M}_H|^2$ for the corresponding Feynman diagram. It is not necessary that $P_H \approx \mathcal{N}|\mathcal{M}_H|^2$ because all that one really needs is that $\sum_H P_H$ approximates $\mathcal{N} \sum_H |\mathcal{M}_H|^2$.

Nevertheless, we believe that it is desirable that P_H approximates $\mathcal{N}|\mathcal{M}_H|^2$ (after accounting for interference graphs) and we used that idea as a design goal in constructing the parton shower algorithm used in DEDUCTOR. Maintaining this diagram by diagram correspondence constrains the choice of the ordering variable for shower evolution. Our purpose in this paper is to study this constraint and explain why we made the choice of

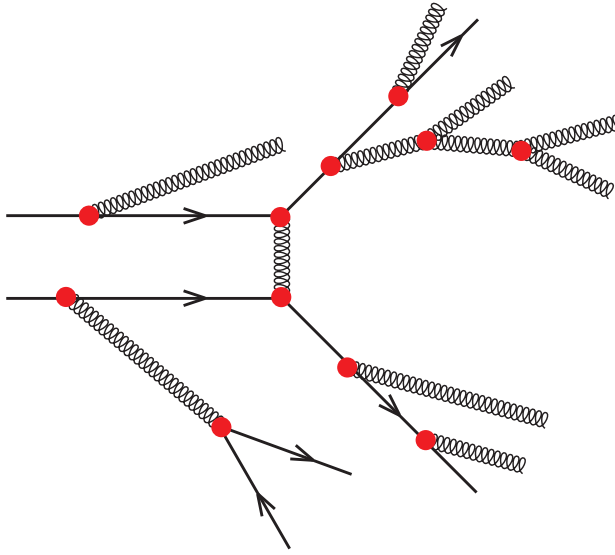


Figure 1. A possible shower history. The hard interaction is quark-quark scattering via gluon exchange. Both the initial state quarks and the final state quarks can emit gluons. Shower time increases as one moves away from the hard interaction. Parton splittings at later shower time are softer.

shower time used in DEDUCTOR.

2 Shower time for final state splittings

In order to see why a correspondence between shower splitting probabilities and Feynman diagrams constrains the choice of shower time, it is easiest to start with final state splittings. Suppose that, in a shower history H of interest, a final state parton 0 splits into partons 1 and 2, each of which splits further. This continues through some (finite) number of splittings.

In the Feynman diagram corresponding to history H , the momenta of the mother parton and of the two daughters are related by $p_0 = p_1 + p_2$. Let the partons have masses m_i , $i = 0, 1, 2$. We define corresponding virtualities v_i^2 by $v_i^2 = p_i^2 - m_i^2$. With final state splittings, we have $v_i^2 \geq 0$. In our Feynman diagram, the propagator for the mother parton has a denominator $1/v_0^2$. The daughter partons have propagators with denominators $1/v_1^2$ and $1/v_2^2$.

With the idea that the shower evolves from hard interactions to softer interactions, the picture for the Feynman diagram is that the splitting $0 \rightarrow 1 + 2$ should be relatively hard compared to the subsequent splittings of the daughter partons 1 and 2. In particular, we ought to be able to set v_1^2 and v_2^2 to 0 when calculating the propagator denominator $1/[(p_1 + p_2)^2 - m_0^2] = 1/v_0^2$ that controls the probability of the splitting $0 \rightarrow 1 + 2$.¹

¹Large logarithms are a feature of parton showers. In this situation we generate a large logarithm from an integration $\int dv_0^2/v_0^2$, integrated over a wide range of v_0^2 . A second power of the logarithm comes from integrating over the momentum fraction z in the splitting.

This statement is of some practical significance. When the shower algorithm generates the splitting $0 \rightarrow 1 + 2$, the splittings of partons 1 and 2 have not yet been generated. Thus v_1^2 and v_2^2 are unknown. Knowing only that the daughter virtualities should be small, we set them to zero.

This qualitative argument suggests a design feature for a parton shower. The definition of the shower ordering variable should be such that one can neglect v_1^2 and v_2^2 when calculating v_0^2 .

We now need to make more precise the proposition that one should be able to “neglect v_1^2 and v_2^2 when calculating v_0^2 .” We make use of the idea that parton showers are about the development of jets: groups of particles with large momenta approximately collinear to a certain direction. Letting Q_0 denote the total momentum of the final state partons created by the hard process that initiates the shower, we define components of vectors using a reference frame in which Q_0 has only a time component. We align the z -axis of our reference frame along the direction of the jet of interest.

We suppose that the mother parton is part of the jet of interest and we describe its momentum p_0 using $(+, -, \perp)$ components² in our reference frame. In this frame, partons in the jet have momenta with large plus components and small minus components: $p^+ \gg |\mathbf{p}| \gg p^-$, where the transverse components are denoted by a boldface symbol, \mathbf{p} . The components of p_0 are

$$p_0 = \left(P, \frac{\mathbf{p}_0^2 + m_0^2 + v_0^2}{2P}, \mathbf{p}_0 \right) . \quad (2.1)$$

Here m_0 is the mass of the mother parton, \mathbf{p}_0 is its momentum transverse to the jet direction, and v_0^2 is its virtuality. With our choice of reference frame, P is large: $P^2 \gg \mathbf{p}_0^2$, $P^2 \gg m_0^2$, $P^2 \gg v_0^2$. We could have chosen the reference frame so that $\mathbf{p}_0 = 0$, but we leave the choice general. Parton 0 splits into two partons with momenta

$$\begin{aligned} p_1 &= \left(zP, \frac{\mathbf{p}_1^2 + m_1^2 + v_1^2}{2zP}, \mathbf{p}_1 \right) , \\ p_2 &= \left((1-z)P, \frac{\mathbf{p}_2^2 + m_2^2 + v_2^2}{2(1-z)P}, \mathbf{p}_2 \right) . \end{aligned} \quad (2.2)$$

Here z and $(1-z)$ are the momentum fractions carried by the daughter partons. Their transverse momenta are related by

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_0 . \quad (2.3)$$

Partons 1 and 2 are off shell with virtualities v_1^2 and v_2^2 , respectively. Setting $p_0^- = p_1^- + p_2^-$ and solving for v_0 , we find

$$v_0^2 = \frac{((1-z)\mathbf{p}_1 - z\mathbf{p}_2)^2}{z(1-z)} + \frac{m_1^2}{z} + \frac{m_2^2}{1-z} - m_0^2 + \frac{v_1^2}{z} + \frac{v_2^2}{(1-z)} . \quad (2.4)$$

If we neglect v_1^2 and v_2^2 when calculating v_0^2 , we obtain \tilde{v}_0^2 , where

$$\tilde{v}_0^2 = \frac{((1-z)\mathbf{p}_1 - z\mathbf{p}_2)^2}{z(1-z)} + \frac{m_1^2}{z} + \frac{m_2^2}{1-z} - m_0^2 . \quad (2.5)$$

²Specifically, $v^\pm = (v^0 \pm v^3)/\sqrt{2}$. Then $v^2 = 2v^+v^- - \mathbf{v}^2$.

The approximation $v_0^2 \approx \tilde{v}_0^2$ is a good approximation provided

$$\frac{v_1^2}{z} \ll v_0^2, \quad \frac{v_2^2}{1-z} \ll v_0^2. \quad (2.6)$$

That is

$$\frac{v_1^2}{2p_1 \cdot n} \ll \frac{v_0^2}{2p_0 \cdot n}, \quad \frac{v_2^2}{2p_2 \cdot n} \ll \frac{v_0^2}{2p_0 \cdot n}. \quad (2.7)$$

where n is the lightlike vector $n = (0, Q_0^-, \mathbf{0})$. This definition gives n dimensions of momentum.

We want eq. (2.7) to hold whenever the splittings of the two daughter partons come after the splitting $0 \rightarrow 1 + 2$. To guarantee that, we want to arrange the definition of shower time such that the splitting of parton 1 comes after the splitting of parton 0 when the first of conditions (2.7) holds and such that the splitting of parton 2 comes after the splitting of parton 0 when second of conditions (2.7) holds.

These ordering conditions are a simple restatement of the requirement that $v_0^2 \approx \tilde{v}_0^2$. But what is \tilde{v}_0^2 ? It is the limit of v_0^2 as $v_1^2 \rightarrow 0$ and $v_2^2 \rightarrow 0$ with fixed $\mathbf{p}_1, \mathbf{p}_2$, and z . This was a deliberate choice. It was motivated as follows. We define $\tilde{v}_0^2 = (\tilde{p}_1 + \tilde{p}_2)^2 - m_0^2$, where \tilde{p}_1 and \tilde{p}_2 are on-shell approximations to p_1 and p_2 : $\tilde{p}_i^2 = m_i^2$ for $i = 1, 2$. We want the components of $\delta p_i = p_i - \tilde{p}_i$ to be small. Since $v_i^2 = (\tilde{p}_i^2 + \delta p_i)^2 - m_i^2 = 2\tilde{p}_i \cdot \delta p_i + (\delta p_i)^2$, we have, neglecting $(\delta p_i)^2$,

$$v_i^2/2 \approx \tilde{p}_i^+ \delta p_i^- - \tilde{\mathbf{p}}_i \cdot \delta \mathbf{p}_i + \tilde{p}_i^- \delta p_i^+. \quad (2.8)$$

Now, $\tilde{p}_i^+ \gg |\tilde{\mathbf{p}}_i| \gg \tilde{p}_i^-$. Thus we can make the components of δp_i small by choosing

$$\delta p^- = \frac{v_i^2}{2\tilde{p}_i^+} \quad (2.9)$$

and letting $\delta \mathbf{p}_i = 0$ and $\delta p_i^+ = 0$. One could also let $\delta \mathbf{p}_i$ and δp_i^+ be small, of a similar size to δp^- , but then δp^- is still determined approximately by eq. (2.9). Such a choice is equivalent to the simple choice that we make here.³

We need one more step in order to turn eq. (2.7) into a definition of shower time. Since the plus components of the parton momenta p_0, p_1 and p_2 in this frame are all much larger than their transverse and minus components, we have

$$p_i \cdot n \approx p_i \cdot Q_0. \quad (2.10)$$

Then we can write the ordering conditions (approximately) as

$$\frac{v_1^2}{2p_1 \cdot Q_0} \ll \frac{v_0^2}{2p_0 \cdot Q_0}, \quad \frac{v_2^2}{2p_2 \cdot Q_0} \ll \frac{v_0^2}{2p_0 \cdot Q_0}. \quad (2.11)$$

³For instance, when the partons are part of a high energy jet, if we define \tilde{v}_0^2 by taking the limit $v_1^2 \rightarrow 0$ and $v_2^2 \rightarrow 0$ with the three-momenta of the partons held constant, adjusting the daughter parton energies a little to put them on-shell, we obtain the same final definition of shower time.

This leads us to the definition of shower time for a final state splitting. For the splitting of parton i , we take

$$e^{-t_i} = \frac{\Lambda_i^2}{Q_0^2} \quad (2.12)$$

where we define the ordering variable Λ by

$$\Lambda_i^2 = \frac{p_i^2 - m_i^2}{2 p_i \cdot Q_0} Q_0^2. \quad (2.13)$$

With this definition, the splittings of partons 1 and 2 come after the splitting of parton 0 when the conditions (2.11) hold. Of course, within a parton shower we calculate p_i^2 as the square of the sum of the momenta of the daughter partons to parton i , using the approximation that the daughter partons are on shell. Additionally, at each splitting a small amount of momentum needs to be taken from the existing final state partons in the event, as described in section 4.1 of ref. [3]. Thus, there is an ambiguity in how to calculate $p_i \cdot Q_0$. We calculate $p_i \cdot Q_0$ defining p_i to be the momentum of the parton i as it existed when the parton was created, before it splits.

In the shower algorithm, we take “after” to mean $t_1 > t_0$ and $t_2 > t_0$. This replaces \ll in eq. (2.11) by simply $<$. This does not treat the regions $t_1 \approx t_0$ and $t_2 \approx t_0$ very exactly. We recognize this as a shortcoming of a leading order shower that can be fixed if we move on to a next-to-leading order shower.

3 Shower time and physical time

Using $(+, -, \perp)$ components as in eq. (2.1), the shower time for the splitting of the mother parton 0 is

$$e^{-t} = \frac{v_0^2}{2 p_0 \cdot Q_0} \approx \sqrt{\frac{2}{Q_0^2}} \frac{v_0^2}{2P} = \sqrt{\frac{2}{Q_0^2}} \left[p_0^- - \frac{\mathbf{p}_0^2 + m_0^2}{2P} \right]. \quad (3.1)$$

Here $(\mathbf{p}_0^2 + m_0^2)/(2P)$ is what the minus component of the momentum p_0 would be if the mother parton were on shell. When p_0^- does not equal this value, there is a deficit of minus momentum and the parton can only exist in this state for an interval in x^+ given by the inverse of the minus-momentum deficit. That is, the interval in x^+ between the vertex where the mother parton was created and where it decays can be estimated by

$$\Delta x^+ \sim \left[p_0^- - \frac{\mathbf{p}_0^2 + m_0^2}{2P} \right]^{-1}. \quad (3.2)$$

The interval in space-time, Δx^μ , between the two vertices is approximately in the plus-direction, so the corresponding time interval is $\Delta t = \Delta x^0 \approx \Delta x^+/\sqrt{2}$. That is,

$$\Delta t \sim \frac{1}{\sqrt{2}} \left[p_0^- - \frac{\mathbf{p}_0^2 + m_0^2}{2P} \right]^{-1}. \quad (3.3)$$

This gives

$$e^{-t} \sim \frac{1}{\sqrt{Q_0^2} \Delta t} \quad (3.4)$$

or

$$t \sim \log\left(\sqrt{Q_0^2} \Delta t\right). \quad (3.5)$$

That is, the shower time at which a parton splits is an estimate of the logarithm of the coordinate time interval needed for the splitting, normalized by $\sqrt{Q_0^2}$ to make it dimensionless.

4 Shower time for initial state splittings

We now repeat the derivation of section 2, this time for initial state splittings in collisions of two hadrons A and B. We consider splitting of a parton from hadron A. We define components of vectors using a reference frame in which Q_0 is predominantly in the time direction. We approximate the hadron momenta p_A and p_B to be lightlike, with $2p_A \cdot p_B = s$. We will use $(+, -, \perp)$ components of vectors and choose the axes so that p_A has only a plus component and p_B has only a minus component. In this frame, initial state partons from hadron A have large plus components, small transverse components, and very small minus components.

We start with an initial state parton with momentum

$$p_0 = \left(P, \frac{\mathbf{p}_0^2 + m_0^2 + v_0^2}{2P}, \mathbf{p}_0 \right). \quad (4.1)$$

Here $v_0^2 = p_0^2 - m_0^2$, the virtuality of the initial state parton, is negative. This parton splits (in backward evolution) into two partons, 1 and 2. Parton 1 is the new initial state parton, while parton 2 is radiated. Thus

$$p_0 = p_1 - p_2. \quad (4.2)$$

We define

$$\begin{aligned} p_1 &= \left(\frac{1}{z} P, z \frac{\mathbf{p}_1^2 + m_1^2 + v_1^2}{2P}, \mathbf{p}_1 \right), \\ p_2 &= \left(\frac{1-z}{z} P, \frac{z}{1-z} \frac{\mathbf{p}_2^2 + m_2^2 + v_2^2}{2P}, \mathbf{p}_2 \right). \end{aligned} \quad (4.3)$$

Here $v_1^2 \leq 0$ and $v_2^2 \geq 0$.

We can easily find the virtuality v_0^2 by comparing the minus momentum of parton 0 to the difference of the minus momenta of partons 1 and 2:

$$v_0^2 = z v_1^2 - \frac{z}{1-z} v_2^2 - \frac{1}{1-z} (\mathbf{p}_2 - (1-z)\mathbf{p}_1)^2 + z m_1^2 - \frac{z}{1-z} m_2^2 - m_0^2. \quad (4.4)$$

We note that we can approximate v_0^2 by its value for on-shell daughter partons,

$$v_0^2 \approx -\frac{1}{1-z} (\mathbf{p}_2 - (1-z)\mathbf{p}_1)^2 + z m_1^2 - \frac{z}{1-z} m_2^2 - m_0^2, \quad (4.5)$$

provided that

$$z |v_1^2| \ll |v_0^2|, \quad \frac{z}{1-z} v_2^2 \ll |v_0^2|. \quad (4.6)$$

That is

$$\frac{|v_1^2|}{2 p_1 \cdot n} \ll \frac{|v_0^2|}{2 p_0 \cdot n} \quad (4.7)$$

and

$$\frac{v_2^2}{2 p_2 \cdot n} \ll \frac{|v_0^2|}{2 p_0 \cdot n}, \quad (4.8)$$

where n is the lightlike vector $n = (0, Q_0^-, \mathbf{0})$.

For the parton “2” that was radiated into the final state, since $p_2^+ \gg p_2^\perp \gg p_2^-$, we have $2 p_2 \cdot n \approx 2 p_2 \cdot Q_0$. For the initial state partons “0” and “1,” this same approximation is valid. However, later formulas are nicer if we stick with $p_0 \cdot n$ and $p_1 \cdot n$ using the lightlike vector n . A useful notation for this is provided by writing

$$\begin{aligned} p_0 \cdot n &= \eta_0 p_A \cdot Q_0, \\ p_1 \cdot n &= \eta_1 p_A \cdot Q_0, \end{aligned} \quad (4.9)$$

where the momentum fractions η_0 and η_1 are defined by

$$\begin{aligned} \eta_0 &= 2 p_0 \cdot p_B / s, \\ \eta_1 &= 2 p_1 \cdot p_B / s. \end{aligned} \quad (4.10)$$

With this notation, we can write the conditions for neglecting v_1^2 and v_2^2 as

$$\frac{|v_1^2|}{2 \eta_1 p_A \cdot Q_0} \ll \frac{|v_0^2|}{2 \eta_0 p_A \cdot Q_0}, \quad \frac{v_2^2}{2 p_2 \cdot Q_0} \ll \frac{|v_0^2|}{2 \eta_0 \cdot Q_0}. \quad (4.11)$$

This leads us to define the shower time for an initial state splitting that is analogous to eq. (2.13) for a final state splitting,

$$\begin{aligned} e^{-t_i} &= \frac{\Lambda_i^2}{Q_0^2} = \frac{p_i^2 - m_i^2}{2 p_i \cdot Q_0} && \text{final state parton,} \\ e^{-t_i} &= \frac{\Lambda_i^2}{Q_0^2} = \frac{|p_i^2 - m_i^2|}{2 \eta_i p_A \cdot Q_0} && \text{initial state parton.} \end{aligned} \quad (4.12)$$

with an analogous equation for an initial state parton from hadron B. Because at each splitting a small amount of momentum needs to be taken from the existing final state partons in the event, as described in section 7.3 below, there is an ambiguity in how exactly to calculate η_i . We define η_i to be the momentum fraction of the initial state parton as it existed when the parton was created, before it splits.

5 Interference graphs

A parton with label l can emit a gluon, giving an amplitude \mathcal{M}_l (in a physical gauge). We have, so far, analyzed the choice of ordering for squared amplitudes, $|\mathcal{M}_l|^2$. However, we need to account for quantum interference. A parton with label k , moving in a different direction, can emit the gluon, giving an amplitude \mathcal{M}_k . When the gluon is soft and its

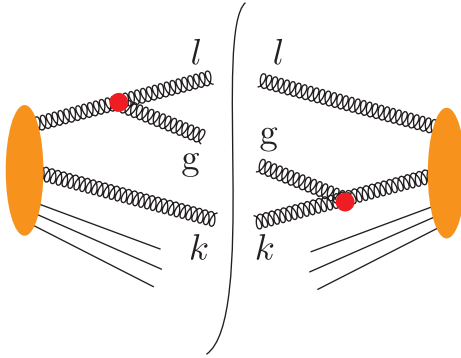


Figure 2. Emission of a soft gluon g from parton l in \mathcal{M} and from parton k in \mathcal{M}^* .

direction is not highly collinear with either parton l or parton k , both processes are important and should be included in a parton shower. Thus we need to account for interference contributions $\mathcal{M}_l \mathcal{M}_k^*$ and $\mathcal{M}_k \mathcal{M}_l^*$, as illustrated in figure 2

The parton shower algorithm of [1, 3, 4] takes such interference contributions into account, at least approximately. To understand how this works, consider emission of a gluon with momentum \hat{p}_g from parton l with interference from emission from parton k . The momenta of the partons l and k are p_l and p_k before the emission and \hat{p}_l and \hat{p}_k after the emission. The probability for this emission in the limit that the gluon is soft, $\hat{p}_g \rightarrow 0$, is proportional to the dipole splitting function

$$\bar{w}_{lk}^{\text{dipole}} = 4\pi\alpha_s \frac{-(\hat{p}_g \cdot \hat{p}_l \hat{p}_k - \hat{p}_g \cdot \hat{p}_k \hat{p}_l)^2}{(\hat{p}_g \cdot \hat{p}_k \hat{p}_g \cdot \hat{p}_l)^2} . \quad (5.1)$$

This splitting function is described in more detail in sections (5.3) and (5.5) of ref. [6]. In eq. (5.1), partons l and k can have nonzero masses. The expression for $\bar{w}_{lk}^{\text{dipole}}$ is simpler in the massless limit, $\hat{p}_l^2 \rightarrow 0$ and $\hat{p}_k^2 \rightarrow 0$, where it becomes

$$\bar{w}_{lk}^{\text{dipole}} \rightarrow 4\pi\alpha_s \frac{2\hat{p}_k \cdot \hat{p}_l}{\hat{p}_g \cdot \hat{p}_k \hat{p}_g \cdot \hat{p}_l} . \quad (5.2)$$

This splitting function multiplies an appropriate matrix C_{lk} in color space, with $C_{lk} = C_{kl}$. Eq. (5.1) includes all four diagrams for emission from either parton l or parton k in the amplitude and the conjugate amplitude, calculated in the limit $\hat{p}_g = \lambda P_g$ with $\lambda \rightarrow 0$.

We introduce a partitioning function A'_{lk} with the properties that $A'_{lk} > 0$ and $A'_{lk} + A'_{kl} = 1$ as follows⁴

$$\begin{aligned} \frac{1}{2} \sum_l \sum_{k \neq l} \bar{w}_{lk}^{\text{dipole}} C_{lk} &= \frac{1}{2} \sum_l \sum_{k \neq l} [A'_{lk} + A'_{kl}] \bar{w}_{lk}^{\text{dipole}} C_{lk} \\ &= \sum_l \sum_{k \neq l} A'_{lk} \bar{w}_{lk}^{\text{dipole}} C_{lk} . \end{aligned} \quad (5.3)$$

⁴The color factors C_{lk} are the factors in square brackets in eq. (5.7) of ref. [6]. Equation (5.3) appeared as eq. (5.8) of that paper, but there we inadvertently left out the color factors C_{lk} .

We define A'_{lk} as in eq. (7.12) of ref. [5]:

$$A'_{lk} = \frac{\hat{p}_g \cdot \hat{p}_k \hat{p}_l \cdot \hat{Q}}{\hat{p}_g \cdot \hat{p}_k \hat{p}_l \cdot \hat{Q} + \hat{p}_g \cdot \hat{p}_l \hat{p}_k \cdot \hat{Q}}. \quad (5.4)$$

Here \hat{Q} is the total momentum of the final state particles just after the splitting. We see that $A'_{lk} \rightarrow 1$ and $A'_{kl} \rightarrow 0$ when \hat{p}_g becomes collinear with \hat{p}_l . We treat the term $A'_{lk} \bar{w}_{lk}^{\text{dipole}} C_{lk}$ as primarily describing the emission of the gluon from parton l , with parton k playing a passive role as a spectator. For instance, there is a momentum mapping that takes a small amount of momentum from the rest of the partons in the event and delivers it to the partons involved in the splitting so that they can be on shell both before and after the splitting. We use the momentum mapping associated with $p_l \rightarrow \hat{p}_l + \hat{p}_g$ and simply ignore the small momentum transfer to parton k . More importantly for the topic of this paper, we define the shower time from the splitting $p_l \rightarrow \hat{p}_l + \hat{p}_g$.

This seems rather crude. Can it be sensible? Note that the shower time associated with gluon emission from parton l is $t = -\log(\Lambda_l^2/Q_0^2)$, where

$$\Lambda_l^2 \approx \frac{|(\hat{p}_l \pm \hat{p}_g)^2 - m_l^2|}{2(\hat{p}_l \pm \hat{p}_g) \cdot Q_0} Q_0^2 = \frac{2\hat{p}_l \cdot \hat{p}_g}{2(\hat{p}_l \pm \hat{p}_g) \cdot Q_0} Q_0^2. \quad (5.5)$$

Here the $+$ sign is for a final state splitting while the $-$ sign is for an initial state splitting. Of course, the same definition with $l \rightarrow k$ applies for Λ_k^2 . For small \hat{p}_g , this becomes

$$\Lambda_l^2 \approx \frac{\hat{p}_l \cdot \hat{p}_g}{\hat{p}_l \cdot Q_0} Q_0^2. \quad (5.6)$$

Compare this to the same function using \hat{Q} in place of Q_0 :

$$\tilde{\Lambda}_l^2 \approx \frac{\hat{p}_l \cdot \hat{p}_g}{\hat{p}_l \cdot \hat{Q}} \hat{Q}^2. \quad (5.7)$$

We have

$$\tilde{\Lambda}_l^2 \approx \alpha_l \Lambda_l^2, \quad (5.8)$$

where

$$\alpha_l = \frac{\hat{Q}^2}{Q_0^2} \frac{\hat{p}_l \cdot Q_0}{\hat{p}_l \cdot \hat{Q}}. \quad (5.9)$$

Thus

$$A'_{lk} \approx \frac{\alpha_k \Lambda_k^2}{\alpha_l \Lambda_l^2 + \alpha_k \Lambda_k^2}. \quad (5.10)$$

Note that only the ratio of α_l to α_k , not their individual values, matters in A'_{lk} . Furthermore, α_l/α_k is typically neither much larger than 1 nor much smaller than 1.

What happens? We can consider three cases.

First, one can have $\Lambda_l^2 \ll \Lambda_k^2$. Then $A'_{lk} \approx 1$ and $A'_{kl} \approx 0$, so the splitting is treated almost entirely as gluon emission from parton l , for which Λ_l^2 defines the shower time. Thus Λ_l^2 must be smaller than the Λ^2 of the previous splitting in the shower. In this case, Λ_k^2 is much larger than Λ_l^2 and may be larger than the Λ^2 of the previous splitting, so that the

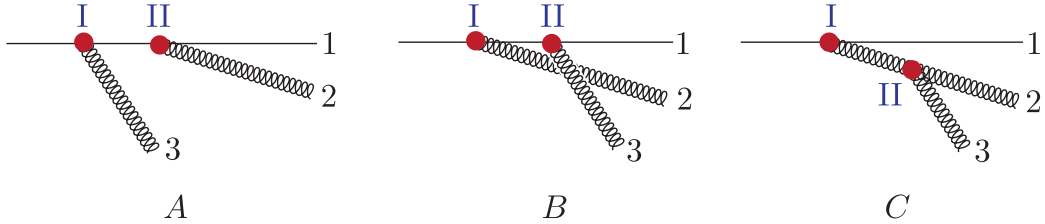


Figure 3. Three possible shower histories.

approximations needed to neglect $(\hat{p}_k \pm \hat{p}_g)^2 - m_k^2$ in the previous splitting that produced parton k may not be valid. However, in this case, if we examine the graphs that go into $\bar{w}_{lk}^{\text{dipole}}$, using a physical gauge $\hat{Q} \cdot A = 0$, we find that the dominant graph is the one in which the soft gluon is emitted from parton l both in the amplitude and in the conjugate amplitude. Graphs involving emission from parton k are suppressed. Thus it does not matter if the approximations do not work well for emission from parton k .

Second, one can have $\Lambda_k^2 \ll \Lambda_l^2$. This is the same as the previous case but with $l \leftrightarrow k$.

In the third case, Λ_k^2 and Λ_l^2 are of a similar size. In this case, we really do have substantial quantum interference in a physical gauge. The shower algorithm sometimes uses Λ_l^2 to define the shower time and sometimes uses Λ_k^2 . However, it does not much matter which Λ^2 value is used because the two are of similar size.

One could, of course, use Q_0 in place of \hat{Q} in the definition of A'_{lk} . Then the argument given above would be simpler. Our only reason for not doing that is that the code for generating parton splittings is somewhat simpler with A'_{lk} defined using \hat{Q} .

6 Connection to k_T ordering

To see if the choice of shower time makes a difference, we consider an example in which a final state quark splits into a quark, labelled 1, and two gluons, labelled 2 and 3. For simplicity, in this section we take the quark to be massless. There are three possible shower histories, illustrated in figure 3. Which histories can be generated in a parton shower depends on the momenta p_1 , p_2 and p_3 and on the choice of the shower time variable.

6.1 Kinematics

Consider a final state splitting of a massless mother parton to massless daughter partons, i and j . We think of the partons as part of a jet of momentum p_J , approximated as lightlike. We define $(+, -, \perp)$ components of vectors so that the plus direction is in the direction of p_J :

$$p_J = (P, 0, \mathbf{0}) . \quad (6.1)$$

As in eq. (2.2), we take the $(+, -, \perp)$ components of the momenta of partons to be

$$\begin{aligned} p_i &= \left(x_i P, \frac{\mathbf{p}_i^2}{2x_i P}, \mathbf{p}_i \right) , \\ p_j &= \left(x_j P, \frac{\mathbf{p}_j^2}{2x_j P}, \mathbf{p}_j \right) . \end{aligned} \quad (6.2)$$

Partons i and j may have daughter partons and thus have nonzero virtualities, but we are neglecting their virtualities here.

For the splitting of the mother parton into partons i and j , we can define a squared transverse momentum k_{T}^2 by

$$k_{\text{T}}^2 = \frac{1}{(x_i + x_j)^2} (x_j \mathbf{p}_i - x_i \mathbf{p}_j)^2 . \quad (6.3)$$

One justifies the name k_{T}^2 by noting that if $\mathbf{p}_i + \mathbf{p}_j = 0$, then $k_{\text{T}}^2 = \mathbf{p}_i^2$. Furthermore, if we boost the momenta using a null plane boost $\mathbf{p}_i \rightarrow \mathbf{p}_i + x_i \mathbf{v}$ and $\mathbf{p}_j \rightarrow \mathbf{p}_j + x_j \mathbf{v}$, we can reach a frame in which $\mathbf{p}_i + \mathbf{p}_j = 0$ while leaving k_{T}^2 invariant. In a k_{T} ordered shower, one orders splittings so that k_{T}^2 for a daughter splitting is smaller than k_{T}^2 for the mother splitting.

The virtuality of the mother parton is related to k_{T}^2 by

$$(p_i + p_j)^2 = \frac{(x_i + x_j)^2}{x_i x_j} k_{\text{T}}^2 . \quad (6.4)$$

Ordering according to shower time as defined in eq. (2.13) amounts to ordering in the variable

$$\Lambda^2 = \frac{(p_i + p_j)^2}{x_i + x_j} \frac{Q_0^2}{2p_J \cdot Q_0} . \quad (6.5)$$

The two ordering variables are related by

$$\Lambda^2 = \frac{x_i + x_j}{x_i x_j} k_{\text{T}}^2 \frac{Q_0^2}{2p_J \cdot Q_0} . \quad (6.6)$$

Consider now the shower history A in figure 3. We take P to represent the plus-momentum of the mother parton, so that $x_1 + x_2 + x_3 = 1$. There are two splittings, which we can call I and II. The k_{T} values for the two splittings are

$$\begin{aligned} k_{\text{T}}^2(\text{I}) &= ((x_1 + x_2)\mathbf{p}_3 - x_3(\mathbf{p}_1 + \mathbf{p}_2))^2 , \\ k_{\text{T}}^2(\text{II}) &= \frac{1}{(x_1 + x_2)^2} (x_1 \mathbf{p}_2 - x_2 \mathbf{p}_1)^2 . \end{aligned} \quad (6.7)$$

It will prove useful to exchange the transverse momentum variables for two dimensional angular variables, defined by

$$\boldsymbol{\theta}_i = \frac{\sqrt{2}}{P} \frac{\mathbf{p}_i}{x_i} . \quad (6.8)$$

We define differences in angles by

$$\boldsymbol{\theta}_{ij} = \boldsymbol{\theta}_i - \boldsymbol{\theta}_j . \quad (6.9)$$

Using these angular variables, the k_{T}^2 values for the two splittings are

$$\begin{aligned} k_{\text{T}}^2(\text{I}) &= \frac{(p_J \cdot Q_0)^2}{Q_0^2} x_3^2 (x_1 \boldsymbol{\theta}_{31} + x_2 \boldsymbol{\theta}_{32})^2 , \\ k_{\text{T}}^2(\text{II}) &= \frac{(p_J \cdot Q_0)^2}{Q_0^2} \frac{(x_1 x_2)^2}{(x_1 + x_2)^2} \boldsymbol{\theta}_{12}^2 . \end{aligned} \quad (6.10)$$

The corresponding Λ^2 values for ordering according to eq. (2.13) are

$$\begin{aligned}\Lambda^2(\text{I}) &= \frac{p_J \cdot Q_0}{2} \frac{x_3}{x_1 + x_2} (x_1 \boldsymbol{\theta}_{31} + x_2 \boldsymbol{\theta}_{32})^2, \\ \Lambda^2(\text{II}) &= \frac{p_J \cdot Q_0}{2} \frac{x_1 x_2}{x_1 + x_2} \boldsymbol{\theta}_{12}^2.\end{aligned}\tag{6.11}$$

Thus history A is a valid shower history for partons 1, 2, and 3 according to k_T ordering if $k_T^2(\text{I}) > k_T^2(\text{II})$ and it is a valid history according to Λ ordering if $\Lambda^2(\text{I}) > \Lambda^2(\text{II})$. The same analysis applies to histories B and C. We simply have to permute the labels 1,2,3 in eqs. (6.10) and (6.11).

6.2 A specific case

Mostly, Λ ordering and k_T ordering are equivalent for splittings to make three partons. Here is one case where they differ. Suppose that $x_3 \ll x_2 \ll 1$. Then we take $x_1 \approx 1$ since $x_1 + x_2 + x_3 = 1$. Define $\boldsymbol{\theta} = (\boldsymbol{\theta}_{31} + \boldsymbol{\theta}_{32})/2$ and suppose that $\boldsymbol{\theta}_{12}^2 \ll \boldsymbol{\theta}^2$. Then $\boldsymbol{\theta}_{31}^2 \approx \boldsymbol{\theta}_{32}^2 \approx \boldsymbol{\theta}^2$. Under these circumstances, for history A we have

$$\begin{aligned}k_T^2(\text{I}) &= \frac{(p_J \cdot Q_0)^2}{Q_0^2} x_3^2 \boldsymbol{\theta}^2, \\ k_T^2(\text{II}) &= \frac{(p_J \cdot Q_0)^2}{Q_0^2} x_2^2 \boldsymbol{\theta}_{12}^2\end{aligned}\tag{6.12}$$

and

$$\begin{aligned}\Lambda^2(\text{I}) &= \frac{p_J \cdot Q_0}{2} x_3 \boldsymbol{\theta}^2, \\ \Lambda^2(\text{II}) &= \frac{p_J \cdot Q_0}{2} x_2 \boldsymbol{\theta}_{12}^2.\end{aligned}\tag{6.13}$$

Thus history A is allowed for k_T ordering if

$$\frac{\boldsymbol{\theta}_{12}^2}{\boldsymbol{\theta}^2} < \frac{x_3^2}{x_2^2}\tag{6.14}$$

while history A is allowed for Λ ordering if

$$\frac{\boldsymbol{\theta}_{12}^2}{\boldsymbol{\theta}^2} < \frac{x_3}{x_2}.\tag{6.15}$$

Suppose that

$$\frac{x_3^2}{x_2^2} \ll \frac{\boldsymbol{\theta}_{12}^2}{\boldsymbol{\theta}^2} \ll \frac{x_3}{x_2}.\tag{6.16}$$

Then history A is allowed for Λ ordering. However, history A is not generated with k_T ordering. With the same sort of analysis, we find instead that with k_T ordering we generate histories B and C, while with Λ ordering histories B and C are forbidden.

How can we interpret this result? The region in which $x_3 \ll x_2 \ll 1$ and $\boldsymbol{\theta}_{12}^2 \ll \boldsymbol{\theta}^2 \ll 1$ with $\boldsymbol{\theta}_{12}^2/\boldsymbol{\theta}^2$ limited by eq. (6.16) is important. It can generate four large logarithms, two from two angle integrations and two from two momentum fraction integrations. The

analysis of section 2 indicates that both denominators in the Feynman graph corresponding to history A become independently small in this region, so that the square of graph A (in a physical gauge) gives a leading contribution. However, this does not work for graphs B and C. For instance, in graph B, the $(p_1 + p_3)^2$ is so large that it dominates the denominator proportional to $(p_1 + p_2 + p_3)^2$ as represented in eq. (2.4). Then the needed sensitivity to p_2 is absent. Thus the leading Feynman graph in the amplitude for $0 \rightarrow 1 + 2 + 3$ is that corresponding to history A.

What happens, then, if one uses a k_T ordered shower, which generates the shower using histories B and C but not A? Surprisingly, we get the right answer. There are two reasons for this.

The first reason concerns the propagator denominator corresponding to splitting I in graphs B and C. Consider graph B. The momentum $p_1 + p_3$ is really significantly off shell, considering that the corresponding parton subsequently splits to partons 1 and 3. However, when we generate it, we treat it as exactly on shell. Thus when we generate splitting I in history B we use essentially the same propagator denominator as when we generate splitting II in history A. (However, depending on how the shower algorithm treats momentum conservation, the angle between partons 1 and 2 may be adjusted significantly to account for the recoil from parton 3.)

The second reason concerns the amplitude for splitting II in graph B, in which parton 3 is emitted. This is an emission of a soft, wide angle gluon. The amplitude can be approximated using the eikonal approximation,

$$\mathcal{A} = \frac{g^n \cdot \epsilon_3}{n \cdot p_3}, \quad (6.17)$$

where n is a lightlike vector in the direction of the jet, $n = (1, 0, 0, 0)$. Here ϵ_3 is the polarization vector of gluon 3. The same approximation applies to splitting II in graph C. Finally, this same approximation applies to splitting I in graph A. Thus the kinematic factor describing the emission of parton 3 is the same in graphs A, B, and C. The three graphs have different color factors, which we may call \mathcal{C}_A , \mathcal{C}_B , and \mathcal{C}_C , respectively. Color invariance implies that

$$\mathcal{C}_B + \mathcal{C}_C = \mathcal{C}_A. \quad (6.18)$$

Putting these two arguments together, we see that when we add the parton shower approximations corresponding to histories B and C, we get the parton shower approximation corresponding to history A. That is, even though using a k_T ordered shower generates the given three parton configuration according to histories B and C instead of A, the result is approximately the same.

An analogous argument shows that one can also use angular ordering, in which history A applies whenever $\theta_{12} \ll \theta$. This, of course, is the basic physics argument behind angle ordered showers [8].

7 Consequences for initial state splittings

We now turn to the effect of using Λ ordering in the initial state shower. We consider the case in which all of the partons are gluons. We seek insight into what initial state emissions

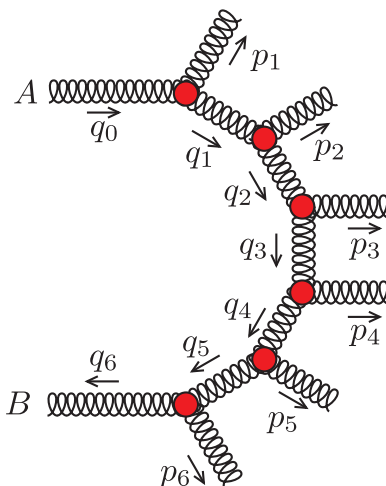


Figure 4. Initial state shower history showing initial state gluons with momenta $\{q_0, \dots, q_N\}$ and emitted gluons with momenta $\{p_1, \dots, p_N\}$ with $N = 6$.

are allowed by Λ ordering.

7.1 Kinematics

The kinematics for the initial state shower is illustrated in figure 4. We have a sequence of initial state parton momenta, which we denote by

$$\{q_0, q_1, \dots, q_{h-1}, q_h, q_{h+1}, \dots, q_{N-1}, q_N\}.$$

Gluon 0 is the constituent of hadron A at the very end of the shower as we move from hard to soft interactions and gluon N is the constituent of hadron B at the soft end of the initial state shower. The hard interaction is associated with the exchange of a gluon with momentum q_h .

As in section 4, we treat the hadron momenta as being lightlike: $p_A^2 = p_B^2 = 0$. Then $s = (p_A + p_B)^2$ is given by $s = 2p_A \cdot p_B$. We use a reference frame in which p_A has only a plus component, $p_A^+ > 0$, with $p_A^- = 0$ and $p_A^\perp = 0$. Similarly, p_B has only a minus component, $p_B^- > 0$, with $p_B^+ = 0$ and $p_B^\perp = 0$.

We treat all of the momenta p_i of emitted gluons as being on shell: $p_i^2 = 0$. We take initial state gluon 0 to be on shell and collinear with hadron A: $q_0 = \eta_0 p_A$. We take initial state gluon N to be on shell and collinear with hadron B: $-q_N = \xi_N p_B$. Momentum is conserved at each vertex: $q_{i-1} = q_i + p_i$. Then the q_i for $0 < i < N$ are spacelike and generally have non-zero transverse components.

This is not the way that the shower is generated. In actually generating the initial state shower, at each splitting stage, we treat the new initial state gluon as being on shell with zero transverse momentum. This involves some approximations, which we outline in section 7.3. In this section, we ignore these complications and simply use spacelike initial state partons with non-zero transverse components and exact momentum conservation.

For each initial state gluon, decompose q_i according to

$$q_i = \eta_i p_A - \xi_i p_B + q_i^\perp, \quad (7.1)$$

where $q_i^\perp \cdot p_A = q_i^\perp \cdot p_B = 0$. Because $p_i \cdot p_A > 0$ and $p_i \cdot p_B > 0$, we have $\eta_i > \eta_{i+1}$ and $\xi_i > \xi_{i-1}$. Also $\xi_0 = 0$ and $\eta_N = 0$. That is, the momentum fractions η_i decrease to zero as we move from hadron A to hadron B and the momentum fractions ξ_i decrease to zero as we move from hadron B to hadron A. The initial state gluon momenta for $0 < i < N$ are all spacelike:

$$q_i^2 = -\eta_i \xi_i s - \mathbf{q}_i^2 < 0. \quad (7.2)$$

Here and in the following, we denote the two components of q_i^\perp with a boldface symbol \mathbf{q}_i , using the euclidian inner product in the transverse space so that $\mathbf{q}_i^2 > 0$.

7.2 Evolution of the gluon virtualities

There is an instructive relation between q_i^2 and q_{i-1}^2 . To derive this relation, we start with the identity

$$0 < [(\eta_{i-1} - \eta_i)\mathbf{q}_i - \eta_i \mathbf{p}_i]^2. \quad (7.3)$$

Using $\mathbf{q}_{i-1} = \mathbf{q}_i + \mathbf{p}_i$, this identity is equivalent to

$$\frac{\mathbf{q}_{i-1}^2}{\eta_{i-1}} < \frac{\mathbf{q}_i^2}{\eta_i} + \frac{\mathbf{p}_i^2}{\eta_{i-1} - \eta_i}. \quad (7.4)$$

The condition that p_i is lightlike is

$$\mathbf{p}_i^2 = (\eta_{i-1} - \eta_i)(\xi_i - \xi_{i-1})s. \quad (7.5)$$

Thus eq. (7.4) is equivalent to

$$\frac{\mathbf{q}_{i-1}^2 + \eta_{i-1}\xi_{i-1}s}{\eta_{i-1}} < \frac{\mathbf{q}_i^2 + \eta_i\xi_i s}{\eta_i}. \quad (7.6)$$

Using eq. (7.2), this is

$$\frac{-q_{i-1}^2}{\eta_{i-1}} < \frac{-q_i^2}{\eta_i}. \quad (7.7)$$

This is the Λ ordering condition (4.7) for an initial state shower starting at the hard interaction and moving toward hadron A except that “ \ll ” is replaced by “ $<$.” However, the actual shower generation uses “ $<$.”

With an analogous derivation, we have

$$\frac{-q_{i+1}^2}{\xi_{i+1}} < \frac{-q_i^2}{\xi_i}. \quad (7.8)$$

This is the Λ ordering condition for an initial state shower starting at the hard interaction and moving toward hadron B.

We thus see that shower splittings that violate Λ ordering in the form (4.7) are impossible if we use exact kinematics in the initial state shower. However, we do not use

exact kinematics. Instead, we approximate initial state partons as being exactly on shell, then adjust them to have spacelike momenta by taking momentum from elsewhere in the event, as described in the following subsection. When we make these approximations, it is important to impose Λ ordering so that we do not generate splittings that invalidate the approximations.

7.3 Momentum conservation

The initial state shower does not appear all at once as depicted in figure 4. Rather, it is generated step by step, starting from the hard interaction. To understand the relationship between the final result depicted in figure 4 and the steps in generating the shower, let us suppose that the shower has been partly generated. At the current stage of shower generation, the latest parton to be generated on the hadron A side is parton i , with $i < h$. The latest parton to be generated on the hadron B side is parton j , with $j > h$. We work in a reference frame, “frame 0,” in which these partons have zero transverse momenta. We approximate them as being on shell. That is, we are neglecting their virtualities as having negligible effect on the calculation of all harder interactions. Thus partons i and j have momentum components

$$\begin{aligned} q_i &= (\eta_i p_A^+, 0, \mathbf{0}) && \text{frame 0 ,} \\ -q_j &= (0, \xi_j p_B^-, \mathbf{0}) && \text{frame 0 .} \end{aligned} \quad (7.9)$$

Now we let parton i split. Unfortunately, a parton with zero virtuality cannot split. Thus we need to change q_i to a new momentum \tilde{q}_i with

$$\tilde{q}_i = \left(\tilde{\eta}_i p_A^+, \frac{\tilde{q}_i^2}{2\tilde{\eta}_i p_A^+}, \mathbf{0} \right) \quad \text{frame 0 .} \quad (7.10)$$

We want the shower to exactly conserve momentum, so we will have to take the needed momentum $\tilde{q}_i - q_i$ from elsewhere in the event. We get the needed momentum by applying a small Lorentz transformation $\Lambda_1(\omega)$ to every final state parton that exists at this stage of the evolution,

$$p_k \rightarrow \tilde{p}_k = \Lambda_1(\omega) p_k \quad k = i + 1, \dots, j . \quad (7.11)$$

Then if we define

$$Q_{ij} = \sum_{k=i+1}^j p_k , \quad (7.12)$$

we have

$$Q_{ij} \rightarrow \tilde{Q}_{ij} = \Lambda_1(\omega) Q_{ij} \quad (7.13)$$

The original Q_{ij} obeys

$$q_i = Q_{ij} + q_j . \quad (7.14)$$

If we now use momentum conservation with the shifted final state momenta, we have a modified q_i ,

$$\tilde{q}_i = \tilde{Q}_{ij} + q_j . \quad (7.15)$$

That is

$$\begin{aligned}
\tilde{q}_i &= \Lambda(\omega)Q_{ij} + q_j \\
&= \Lambda_1(\omega)[q_i - q_j] + q_j \\
&= \Lambda_1(\omega)q_i + (1 - \Lambda_1(\omega))q_j .
\end{aligned} \tag{7.16}$$

We choose the Lorentz transformation $\Lambda_1(\omega)$ to be a boost in the z -direction with boost angle ω . Then the components of \tilde{q}_i are

$$\tilde{q}_i = (e^\omega \eta_i p_A^+, -(1 - e^{-\omega}) \xi_j p_B^-, \mathbf{0}) . \tag{7.17}$$

With this, we obtain the square of \tilde{q}_i as a function of ω ,

$$\begin{aligned}
-\tilde{q}_i^2 &= (e^\omega - 1)2\eta_i \xi_j p_A \cdot p_B \\
&= (e^\omega - 1)Q_{ij}^2 ,
\end{aligned} \tag{7.18}$$

where we have used eq. (7.15). Thus

$$e^\omega = 1 + \frac{-\tilde{q}_i^2}{Q_{ij}^2} . \tag{7.19}$$

Since the virtuality of any exchanged gluon is typically much smaller than the square of the momenta of all of the final state particles created between i and j , we need only a small boost: $\omega \ll 1$. We see that the momentum fraction $\tilde{\eta}_i$ in eq. (7.10) is slightly different from η_i :

$$\tilde{\eta}_i = e^\omega \eta_i . \tag{7.20}$$

Now that gluon i has a nonzero virtuality, it can split in backward evolution to a new initial state gluon with momentum q_{i-1} and a final state gluon with momentum p_i , with

$$q_{i-1} - p_i = \tilde{q}_i , \tag{7.21}$$

where \tilde{q}_i is given by eq. (7.10). We take q_{i-1} and p_i to be lightlike with components

$$\begin{aligned}
q_{i-1} &= \left(\eta_{i-1} p_A^+, \frac{\mathbf{k}_i^2}{2\tilde{z}_i^2 \eta_{i-1} p_A^+}, \frac{1}{\tilde{z}_i} \mathbf{k}_i \right) && \text{frame 0 ,} \\
p_i &= \left((1 - \tilde{z}_i) \eta_{i-1} p_A^+, \frac{\mathbf{k}_i^2}{2\tilde{z}_i^2 (1 - \tilde{z}_i) \eta_{i-1} p_A^+}, \frac{1}{\tilde{z}_i} \mathbf{k}_i \right) && \text{frame 0 .}
\end{aligned} \tag{7.22}$$

Here q_{i-1} has a new momentum fraction η_{i-1} . We define

$$\tilde{z}_i = \frac{\tilde{\eta}_i}{\eta_{i-1}} \tag{7.23}$$

and assign a momentum fraction $(1 - \tilde{z}_i)\eta_{i-1}$ to the emitted gluon. Then the plus component of momentum is conserved according to eq. (7.21). We assign transverse momentum \mathbf{k}_i/\tilde{z}_i to both q_{i-1} and p_i . Then the transverse components of momentum are also conserved. With a small calculation, we see that the minus component of momentum is conserved as long as \mathbf{k}_i is chosen so that

$$-\tilde{q}_i^2 = \frac{\mathbf{k}_i^2}{1 - \tilde{z}_i} . \tag{7.24}$$

We need one more step. We should change to a new reference frame, “frame 1,” so that the transverse components of q_{i-1} are zero. This is simple with a null-plane boost $\Lambda_2(\mathbf{v})$ such that a vector with components P in frame 0 has components $P' = \Lambda_2(\mathbf{v})P$ in frame 1, with

$$P' = \left(P^+, P^- + \frac{1}{2} P^+ \mathbf{v}^2 + \mathbf{P} \cdot \mathbf{v}, \mathbf{P} + P^+ \mathbf{v} \right). \quad (7.25)$$

If we choose

$$\mathbf{v} = -\frac{1}{\tilde{\eta}_i p_A^+} \mathbf{k}_i, \quad (7.26)$$

we have

$$\begin{aligned} q_{i-1} &= \left(\eta_{i-1} p_A^+, 0, \mathbf{0} \right) && \text{frame 1,} \\ p_i &= \left((1 - \tilde{z}_i) \eta_{i-1} p_A^+, \frac{\mathbf{k}_i^2}{2(1 - \tilde{z}_i) \eta_{i-1} p_A^+}, \mathbf{k}_i \right) && \text{frame 1,} \\ \tilde{q}_i &= \left(\tilde{\eta}_i p_A^+, \frac{\tilde{q}_i^2}{2\eta_{i-1} p_A^+}, -\mathbf{k}_i \right) && \text{frame 1,} \\ -q_j &= \left(0, \xi_j p_B^-, \mathbf{0} \right) && \text{frame 1.} \end{aligned} \quad (7.27)$$

The components of the final state momenta \tilde{p}_i for $i = i + 1, \dots, j$ change according to the boost defined by eq. (7.25). However, the boost does not change the components of a vector that lies entirely in the minus direction, so the components of q_j are not changed.

The practical effect of this is that q_{i-1} and p_i are given by eq. (7.27), while the components of the momenta p_k of final state gluons are modified by the net Lorentz transformation $\Lambda_2(\mathbf{v})\Lambda_1(\omega)$. This analysis has been for massless partons. We state the needed Lorentz transformation for the general case of massive partons in appendix A.

The important point of this subsection is not the precise form of the Lorentz transformations, but rather the idea that, in generating an initial state splitting, we first take a small amount of momentum from the final state that allows us to conserve momentum in the splitting, then we change reference frames so that the new initial state parton has zero transverse momentum. Thus the momenta of final state partons shift at each emission so as to recoil against the transverse momentum of a newly emitted parton.

7.4 Dynamical regimes

Now, consider possible dynamical regimes for the initial state shower of figure 4. One regime is the standard one with soft or collinear gluons emitted from the initial state gluons. For the hadron A side of the shower, the momentum fractions $z_i = \eta_i/\eta_{i-1}$ are either close to 1 for a soft gluon emission or finite, close to neither 0 or 1, for a collinear splitting. Then all of the η_i are roughly the same size. Then eq. (7.7) implies that the virtualities $-q_i^2$ decrease as we move from the hard interaction toward hadron A. If the z_i are not close to 1, the splitting transverse momenta,

$$\mathbf{k}_i = (1 - z_i) \mathbf{q}_{i-1} - \mathbf{p}_i \quad (7.28)$$

also decrease since, from eq. (4.5),

$$\mathbf{k}_i^2 \approx (1 - z_i)(-q_i^2). \quad (7.29)$$

There is another regime, which one can call the cut pomeron regime. All of the \mathbf{k}_i can be of roughly the same size, but the z_i can be small: $\eta_{i-1} \gg \eta_i$. In this regime, also $\xi_{i+1} \gg \xi_i$. The cut pomeron regime can be important when $s = 2p_A \cdot p_B$ is much larger than the \mathbf{k}_i^2 . This regime has been extensively studied [9–12] and there are computer codes that generate events in the cut pomeron regime [13–17]. Of course, there are possible regimes intermediate between the standard regime and the cut pomeron regime, but we content ourselves here with just these two idealized cases.

Since neither transverse momenta nor virtualities need to decrease during shower generation if splitting momentum fractions z_i are small, DEDUCTOR can generate events in the cut pomeron regime. However, there is one issue that we need to consider. We start with a hard interaction, but in figure 4, which is the hard interaction? It seems that we can start anywhere. We adopt the following procedure. Of all the initial state propagators in figure 4, one has the largest $|q_i^2|$. We take that to be the hard interaction, labelled h . Then we develop initial state showers on both the A side and the B side of the hard interaction. We want the gluon with $i = h$ to be the one with the largest virtuality, so we require

$$|q_i^2| < |q_h^2| \quad (7.30)$$

in the initial state shower splittings. In figure 4 as drawn, q_h^2 is the invariant \hat{t} of the parton-parton scattering that constitutes the hard interaction. However, in generating the hard interaction, there is no distinction between the two final state partons. If we exchange them, q_h^2 is the invariant \hat{u} . Thus we define

$$|q_h^2| = \min(|\hat{t}|, |\hat{u}|). \quad (7.31)$$

Note that applying the cut eq. (7.30) is by no means the same as imposing k_T ordering or virtuality ordering on the initial state shower. No $|q_i^2|$ can be larger than $|q_h^2|$, but as we move from the hard interaction towards hadron A or towards hadron B, the $|q_i^2|$ can decrease, then increase again. Thus we can have a gluon emitted with high transverse momentum and large negative rapidity, another gluon emitted with high transverse momentum and large positive rapidity, and several gluons emitted with smaller transverse momenta and intermediate rapidities.

Although DEDUCTOR does generate events in the cut pomeron regime, it was not designed with this regime in mind. To adapt it for cut pomeron physics, one would need to incorporate suitable Sudakov factors for each propagator in figure 5. These Sudakov factors should represent unresolved real radiation together with virtual radiation. In the current version of DEDUCTOR, there is no Sudakov factor associated with the hard interaction and the factors associated with initial state gluon emissions are merely those that keep the hard scattering cross section from being changed by shower evolution. It would be fairly straightforward to create extra Sudakov factors as weights that would differ from 1 in the event of large rapidity separations. Investigation of what extra Sudakov factors to incorporate remains as a topic for future investigation.

7.5 A numerical investigation

In order to understand the effect in DEDUCTOR of initial state parton splittings with $z \ll 1$ and of very forward gluon scattering in the hard interaction, we examine an observable that was calculated in ref. [17] using the generator HIGH ENERGY JETS. We examine multijet production in 7 TeV proton-proton collisions. The jets are defined using the anti- k_T algorithm [18] with $R = 0.6$, found with the aid of FASTJET [19]. Only jets with $|\mathbf{p}_{i\perp}| > 30$ GeV and rapidity $|y_j| < 4.5$ are considered. Define $\Delta y = y_f - y_b$, where y_f is the rapidity of the jet with the greatest rapidity and y_b is the rapidity of the jet with the smallest rapidity. We consider only events in which the average of the transverse momenta of the most forward and backward jets is sufficiently large: $(|\mathbf{p}_{f\perp}| + |\mathbf{p}_{b\perp}|)/2 > 60$ GeV. In these events, we measure the total number N of jets. In standard events, the hard scattering is gluon-gluon scattering to produce jets with $|\mathbf{p}_\perp| > 60$ GeV. Then N should usually be 2. Initial state radiation can produce a 30 GeV jet in the rapidity range between the jets produced by the hard scattering, but this should be rather rare. In the cut pomeron picture, when Δy is large, there is the possibility of producing extra jets in the rapidity range between the jets with extremal rapidities. Thus if we plot the average value $\langle N \rangle$ of the number of jets versus Δy , we can expect to see a rising curve.

Plots of $\langle N \rangle$ versus Δy are shown in figure 5. There are five curves. The lowest was generated with DEDUCTOR. With DEDUCTOR, the production of extra jets between the jets with extremal rapidities is not rare: the average number, $\langle N \rangle - 2$, of extra jets rises with Δy to a value of more than 1/2. The next higher curve in this range was generated with PYTHIA (version 8.176) [20]. The PYTHIA curve is somewhat higher than the DEDUCTOR curve, although, since the PYTHIA shower is k_T ordered, one would think that the PYTHIA curve should be lower. One effect that tends to raise the PYTHIA curve is that the value of α_s for initial state radiation is large: $\alpha_s(M_Z^2) = 0.137$ compared to $\alpha_s(M_Z^2) = 0.118$ in DEDUCTOR.⁵ The next three curves were generated by HIGH ENERGY JETS and were taken from figure 11 of ref. [17]. The three curves correspond to three different choices of parameters within HIGH ENERGY JETS. We see that the HIGH ENERGY JETS curves lie substantially above the PYTHIA and DEDUCTOR curves.

By construction, DEDUCTOR allows the transverse momentum of produced jets to decrease and then increase as the initial state shower proceeds over a large rapidity range. The probability that this happens is controlled by Sudakov factors that are constructed to preserve the cross section for the hard event. To represent cut pomeron physics, this may not be what one wants. We see in figure 5 that although the Λ ordering in DEDUCTOR allows the production of extra high k_T jets, the probability for doing so is not as high as it is in HIGH ENERGY JETS. We expect that with implementation of more physically motivated Sudakov factors, the DEDUCTOR curve may be modified for large Δy .

⁵In DEDUCTOR, α_s is evaluated at a scale $\lambda_R k_T^2$, where $\lambda_R \approx 0.4$ [1]. Thus a more accurate comparison is to $\alpha_s(\lambda_R M_Z^2) = 0.126$ for DEDUCTOR.

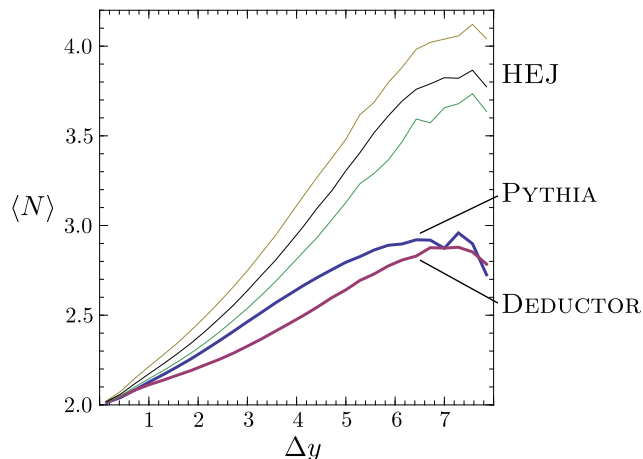


Figure 5. Average number $\langle N \rangle$ of jets versus the rapidity difference Δy between the jet with the greatest rapidity and the jet with the least rapidity. All jets must have $|\mathbf{p}_{i\perp}| > 30$ GeV and the average of the transverse momentum of the two jets with extremal rapidity must be at least 60 GeV. The upper three curves are $\langle N \rangle$ according to HIGH ENERGY JETS, with three choices within this program, as reported in figure 11 of ref. [17]. The middle curve in the range $5 < \Delta y < 7$ is the result from PYTHIA. The bottom curve in this range is the result from DEDUCTOR.

8 Ending the shower

The shower time that we use, eqs. (2.12) and (2.13), is well suited for determining the relative ordering of two splitting vertices. However, at some point in shower evolution, the splittings are too soft for perturbation theory to be reliable. Thus one needs to end the perturbative shower and substitute a non-perturbative model.⁶ The simplest thing to do would be to stop the shower at a fixed value of Λ . However, that is not sensible because of the factor $1/p_i \cdot Q_0$ in Λ^2 . This factor is not invariant under boosts in the direction of p_i . Consider potential splittings of two partons with the same virtuality $|p_i^2 - m_i^2|$ and the same splitting variable z . With a fixed cutoff on Λ , a parton that is part of a moderate momentum jet might be allowed to split while a parton that is part of a very high momentum jet is not allowed to split.

Instead, we veto all splittings in which $k_T^2 < k_{T,\min}^2$ where $k_{T,\min}^2$ is, say, 1 GeV.⁷ Neglecting all masses, $k_T^2 = z(1-z)|p_i^2|$ for final state splittings, so this cut keeps the virtuality, z , and $(1-z)$ from being too small. For initial state splittings with zero masses, $k_T^2 = (1-z)|q_i^2|$, so this cut keeps the virtuality and $(1-z)$ from being too small. Since the end of PYTHIA showers is based on k_T^2 , this choice facilitates matching to the PYTHIA

⁶The shower in ref. [1] does not have a non-perturbative hadronization model. We anticipate providing an interface to the string model using PYTHIA.

⁷Here, to be precise, for a final state splitting, \mathbf{k}_T is the part of the momentum of one daughter parton relative to the mother parton. For an initial state splitting, \mathbf{k}_T is part of the momentum of the parton 2 in section 4 relative to the direction of initial state parton 1. In each case, we use a reference frame in which Q_0 has only a time component.

model of hadronization.

9 Conclusions

In a parton shower event generator, one assigns a variable V_i^2 to parton splittings for the purpose of ordering the splittings. If the splitting of parton i comes before the splitting of parton j then $V_i^2 > V_j^2$. Often, as in DEDUCTOR, V_i^2 is related to the hardness of the splitting in the sense that $V_i^2 \rightarrow 0$ when the daughter partons become collinear or when one becomes soft.

This hardness ordering criterion allows a rather wide range of choices for V_i^2 . We propose a more restrictive criterion based on factoring soft interactions from hard interactions. Recall that when one wants to calculate an infrared safe inclusive cross section, one can factor the hardest interaction from parton distribution functions and from softer interactions. In our view, a parton shower event generator works the same way, but on many scales. First the hardest interaction is factored from the rest, then the next hardest interaction, and so forth. We ask that this work on a graph-by-graph basis in a physical gauge (to leading order in the splitting functions). With that demand, we find that V_i^2 should be Λ_i^2 , where

$$\begin{aligned} \Lambda_i^2 &= \frac{p_i^2 - m_i^2}{2p_i \cdot Q_0} Q_0^2 && \text{final state parton ,} \\ \Lambda_i^2 &= \frac{|p_i^2 - m_i^2|}{2\eta_i p_A \cdot Q_0} Q_0^2 && \text{initial state parton .} \end{aligned} \tag{9.1}$$

Here η_i is the momentum fraction for an initial state parton from hadron A and Q_0 is a fixed timelike vector, which we take to be the sum of the momenta of the final state particles at the hardest interaction. We use this ordering parameter in DEDUCTOR.

We investigated in section 6 the relation of Λ ordering to k_T ordering, at least for the case of massless final state partons and for just two successive splittings. The ordering of vertices according to Λ and k_T can be different in the case of wide angle emission of soft gluons. Of course, wide angle, soft gluon emission is important in a gauge theory like QCD. However, wide angle, soft gluon emission has important color coherence properties. Taking color coherence into account, we find that k_T ordered emissions, treated with the parton shower approximation of on-shell daughter partons and summed over graphs, gives approximately the same result as Λ ordered emissions. Indeed, angle ordered parton showers, by design, also reorder emissions to get an equivalent result to hardness ordered emissions. Thus parton shower algorithms are surprisingly robust against changes of the ordering prescription, at least for final state splittings to the level that we have investigated the question. Nevertheless, we prefer the ordering parameter of eq. (9.1) because, with this choice, factorization works graph by graph.

We investigated in section 7 the relation of Λ ordering to k_T ordering for the initial state shower in hadron-hadron collisions. We found that Λ ordering makes available a wider phase space in the case of a series of splittings with small values of the momentum fraction variable z . The wider phase space incorporates the phase space associated with

cut pomeron exchange. We have not investigated the Sudakov factors that should be associated with such emissions. Currently in DEDUCTOR we use only the standard, probability conserving, Sudakov factors. Then we find that the additional phase space is not often filled.

Acknowledgments

This work was supported in part by the United States Department of Energy and by the Helmholtz Alliance “Physics at the Terascale.” We thank Jennifer Smillie and Jeppe Andersen for sending us results from HIGH ENERGY JETS. We thank Hannes Jung for helpful discussions about small x effects.

A Momentum conservation for initial state splitting

We saw in section 7.3 how momentum conservation is maintained in generating the initial state shower when all partons are massless. In this appendix, we record the needed Lorentz transformation when the partons are allowed to have non-zero masses.

Let the momenta of the incoming hadrons, treated as massless, be p_A and p_B , defined so that $s = 2p_A \cdot p_B$. Let the momentum fractions of the incoming partons be η_a and η_b . The incoming partons, with flavors a and b , may have non-zero masses, so we define their momenta to be

$$\begin{aligned} p_a &= \eta_a p_A + \frac{m(a)^2}{\eta_a \eta_b s} \eta_b p_B , \\ p_b &= \eta_a p_B + \frac{m(b)^2}{\eta_a \eta_b s} \eta_a p_A . \end{aligned} \tag{A.1}$$

We can simplify our notation by defining massless vectors

$$\begin{aligned} n_a &= \eta_a p_A , \\ n_b &= \eta_a p_B , \end{aligned} \tag{A.2}$$

and dimensionless mass squared variables

$$\nu(f) = \frac{m(f)^2}{\eta_a \eta_b s} . \tag{A.3}$$

Then

$$\begin{aligned} p_a &= n_a + \nu(a) n_b , \\ p_b &= n_b + \nu(b) n_a . \end{aligned} \tag{A.4}$$

We can use the vectors n_a and n_b as two basis vectors. The projection operator onto the space transverse to n_a and n_b is given by

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - \frac{n_a^\mu n_b^\nu + n_b^\mu n_a^\nu}{n_a \cdot n_b} . \tag{A.5}$$

Now suppose that parton “a” splits, in the sense of backward evolution. The new incoming parton has flavor \hat{a} , mass $m(\hat{a})$, and momentum fraction $\hat{\eta}_a$. We define the momentum fraction of the splitting as $z = \eta_a/\hat{\eta}_a$.⁸ The momentum of the new initial state parton is then

$$\hat{p}_a = \frac{1}{z} n_a + z\nu(\hat{a}) n_b . \quad (\text{A.6})$$

A new final state particle, labelled $m + 1$, is emitted. Its momentum has the form

$$\hat{p}_{m+1} = x_a n_a + x_b n_b + k_\perp , \quad (\text{A.7})$$

where the magnitude of the transverse momentum is fixed by

$$|k_\perp^2| = 2x_a x_b n_a \cdot n_b - m(f_{m+1})^2 . \quad (\text{A.8})$$

The azimuthal angle, ϕ , of k_\perp is another of the splitting variables besides z . The third splitting variable is the dimensionless virtuality

$$y = -\frac{(\hat{p}_a - \hat{p}_{m+1})^2 - m^2(a)}{2n_a \cdot n_b} . \quad (\text{A.9})$$

Typically, $y \ll 1$.

In order to conserve momentum, we need to adjust the momenta of all of the previously created final state particles,

$$p_i^\mu \rightarrow \hat{p}_i^\mu = \Lambda^\mu{}_\nu p_i^\nu \quad i = 1, \dots, m . \quad (\text{A.10})$$

This works provided

$$\begin{aligned} x_a &= \frac{1/z - 1 - y(1 + z\nu(b)) - (1 - z)\nu(a)\nu(b) - z\nu(f_{m+1})\nu(b)}{1 - z^2\nu(\hat{a})\nu(b)} , \\ x_b &= z \frac{y + \nu(f_{m+1}) - \nu(a) + (1 + y)z\nu(\hat{a}) + z\nu(\hat{a})\nu(b)(\nu(a) - z\nu(\hat{a}))}{1 - z^2\nu(\hat{a})\nu(b)} . \end{aligned} \quad (\text{A.11})$$

The needed Lorentz transformation starts with a small boost along the z -axis

$$\Lambda_{\parallel}^{\mu\nu}(\omega) = g_{\perp}^{\mu\nu} + \frac{e^\omega n_a^\mu n_b^\nu + e^{-\omega} n_b^\mu n_a^\nu}{n_a \cdot n_b} , \quad (\text{A.12})$$

with boost angle ω given by

$$e^\omega = [1 + \nu(b)]^{-1} \left[\frac{1}{z} + \nu(b) - x_a \right] . \quad (\text{A.13})$$

Then we apply a transverse null plane boost $\Lambda_\perp(v_\perp)$,

$$\Lambda_\perp^{\mu\nu}(v_\perp) = g^{\mu\nu} + \sqrt{\frac{2}{n_a \cdot n_b}} [v_\perp^\mu n_b^\nu - n_b^\mu v_\perp^\nu] - v_\perp^2 \frac{n_b^\mu n_b^\nu}{n_a \cdot n_b} , \quad (\text{A.14})$$

⁸In generating the shower, we define the splitting variable $z = \eta_a/\hat{\eta}_a$. Elsewhere in this paper, we use z to denote a momentum fraction ratio in several contexts. The precise value of z depends on which partons are approximated as being on shell and which are allowed to be off shell. We hope that this does not cause confusion.

with boost velocity

$$v_{\perp} = -\frac{e^{-\omega}}{[1 + \nu(b)]\sqrt{2n_{\mathbf{a}} \cdot n_{\mathbf{b}}}} k_{\perp} . \quad (\text{A.15})$$

Our normalization convention for v_{\perp} here is different from that used in section 7.3. We also note that $\Lambda(v_{\perp})$ leaves $n_{\mathbf{b}}$ unchanged, but it changes $p_{\mathbf{b}}$ by a small amount, proportional to $\nu(b)$. Thus the interpretation is a little different from that described in section 7.3 for massless partons. However, the difference in interpretation is of no real consequence.

The complete Lorentz transformation is

$$\begin{aligned} \Lambda^{\mu\nu}(\omega, v_{\perp}) = & g_{\perp}^{\mu\nu} + \frac{e^{\omega} n_{\mathbf{a}}^{\mu} n_{\mathbf{b}}^{\nu} + e^{-\omega} n_{\mathbf{b}}^{\mu} n_{\mathbf{a}}^{\nu}}{n_{\mathbf{a}} \cdot n_{\mathbf{b}}} \\ & + \sqrt{\frac{2}{n_{\mathbf{a}} \cdot n_{\mathbf{b}}}} [e^{\omega} v_{\perp}^{\mu} n_{\mathbf{b}}^{\nu} - n_{\mathbf{b}}^{\mu} v_{\perp}^{\nu}] - e^{\omega} v_{\perp}^2 \frac{n_{\mathbf{b}}^{\mu} n_{\mathbf{b}}^{\nu}}{n_{\mathbf{a}} \cdot n_{\mathbf{b}}} . \end{aligned} \quad (\text{A.16})$$

The analysis of this appendix follows from applying the choice y, z, ϕ of splitting variables to the analysis of section 4.4 of ref. [3] except that we choose the Lorentz transformation Λ_{ν}^{μ} differently. As noted at eq. (3.9) of ref. [21], the choice that we make here is better adapted to summing logs to evaluate the transverse momentum of Z-bosons produced in the Drell-Yan process.

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