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Four-loop on-shell integrals: $\overline{\text{MS}}$–on-shell relation and g-2

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We present first results towards a full four-loop calculation for both the anomalous magnetic moment of the muon and the $\overline{MS}$–on-shell relation. The calculation requires the detailed study of an up to now not considered class of diagrams, so-called on-shell diagrams, at four-loop order.

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1. Introduction

The anomalous magnetic moments of electron and muon have been measured with unrivaled precision. In the case of the muon the experimental value[1]

$$a_\mu^{exp} = 1.16592080(54)(33)[63] \cdot 10^{-3} \tag{1.1}$$

has to be compared with the theory prediction

$$a_\mu^{theo} = 1.16591790(65) \cdot 10^{-3} \,. \tag{1.2}$$

The discrepancy between the two values of about 3 standard deviations is approximately of the same order as the four-loop QED corrections. The four- and five-loop corrections have been calculated in Ref.[1] and have not been verified by an independent calculation. In this paper we will present first steps to such an independent calculation to verify the results in Ref.[1]. An object that is technically related is the $\overline{\text{MS}}$–on-shell relation for quark masses in QCD. One of the main motivations, why it is of importance to know the $\overline{\text{MS}}$–on-shell relation with four-loop accuracy, is the planned determination of the top-quark mass at a future linear collider. The precision reached at such an experiment requires an equally precise knowledge of the $\overline{\text{MS}}$–on-shell relation.

In the following we will review recent results for both the anomalous magnetic moment and the $\overline{\text{MS}}$–on-shell relation.

2. Calculation and Results

For both, the calculation of the $\overline{\text{MS}}$–on-shell relation and the anomalous magnetic moment of the muon, the evaluation of on-shell integrals is necessary. To be more precise, to obtain the $\overline{\text{MS}}$–on-shell relation the quark propagator has to be evaluated on its mass shell, while for the anomalous magnetic moment the magnetic form factor of the muon-photon vertex has to be calculated for vanishing photon momentum, thus also leading to an on-shell diagram. Typical diagrams appearing in the calculation are shown in Fig 1 and 2. On-shell integrals have not been studied at four loops in any detail. Thus one of the main obstacle for the calculation lies in the calculation of the missing master integrals.

The calculation is set up as follows, the Feynman diagrams are generated using `QGRAF` [2], its output is then converted into `FORM` [3] input using `q2e` and `exp` [4, 5]. Suitable projectors are applied and the resulting scalar integrals are reduced to master integrals using integration-by-parts identities implemented in `CRUSHER` [6] and `FIRE` [7]. The reduction leads to the master integrals shown in Figs. 3 and 4. The integrals in Fig. 3 can be expressed in closed form in terms of Gamma functions while the ones in Fig. 4 have been calculated in an expansion in $\varepsilon = (d-4)/2$ using the method of dimensional recurrence and analyticity [8]. As an example we show the result for master

[1]The errors indicated in brackets denote the statistical and systematic ones. The error given in square brackets is obtained by adding them in qudrature.

integral M_3 (cf. Fig. 4)

$$e^{4\varepsilon\gamma_E} M_3 = -\frac{1}{6\varepsilon^4} - \frac{7}{6\varepsilon^3} - \left(\frac{10}{3} + \frac{13\pi^2}{18}\right)\varepsilon^{-2} - \left(-\frac{61}{6} + \frac{73\pi^2}{18} + \frac{118\zeta_3}{9}\right)\varepsilon^{-1} - \left(-\frac{851}{4} + \frac{83\pi^2}{18} + \frac{637\zeta_3}{9} + \frac{37\pi^4}{10}\right)\varepsilon^0 - \left(-\frac{14861}{8} - \frac{3467\pi^2}{36} + \frac{1003\zeta_3}{18} + \frac{1121\pi^4}{60} + \frac{1894\pi^2\zeta_3}{27} + \frac{16018\zeta_5}{15}\right)\varepsilon^1 - \left(-\frac{613975}{48} - \frac{25981\pi^2}{24} - \frac{68293\zeta_3}{36} + \frac{83\pi^4}{24} + \frac{9559\pi^2\zeta_3}{27} + \frac{79891\zeta_5}{15} + \frac{59501\pi^6}{2835} + \frac{17704\zeta_3^2}{27}\right)\varepsilon^2 - \left(-\frac{7539347}{96} - \frac{382349\pi^2}{48} - \frac{482627\zeta_3}{24} - \frac{426659\pi^4}{720} + \frac{3757\pi^2\zeta_3}{54} + \frac{2525\zeta_5}{6} + \frac{43201\pi^6}{420} + \frac{88585\zeta_3^2}{27} + \frac{17204\pi^4\zeta_3}{45} + \frac{206434\pi^2\zeta_5}{45} + \frac{1267243\zeta_7}{21}\right)\varepsilon^3 + O\left(\varepsilon^4\right) . \tag{2.1}$$

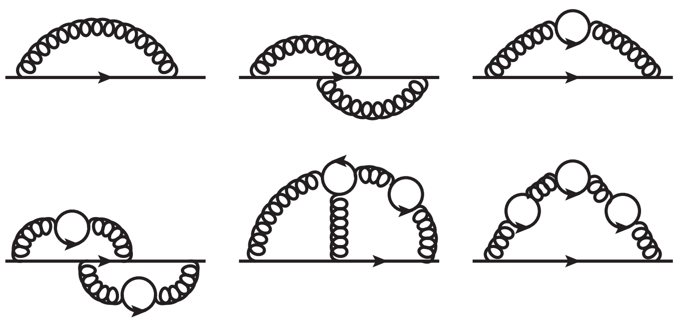

Figure 1: Feynman diagrams for the calculation of the $\overline{\text{MS}}$–on-shell relation.

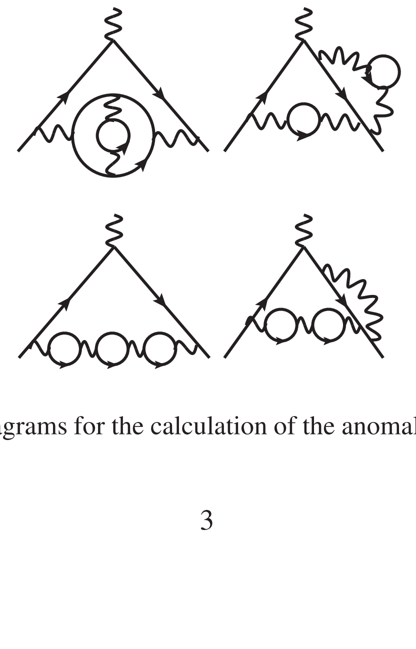

Figure 2: Feynman diagrams for the calculation of the anomalous magnetic moment.

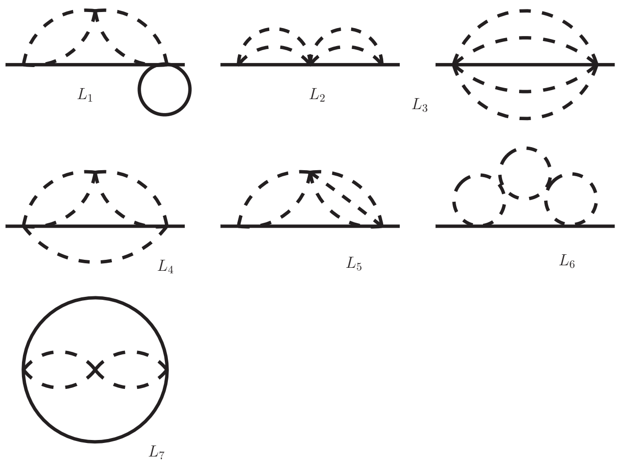

Figure 3: Simple master integrals. They can be expressed in closed form in terms of Gamma functions.

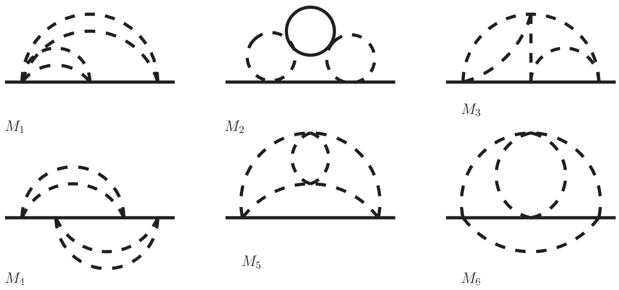

Figure 4: Non-trivial master integrals. They can be calculated in an expansion in $\varepsilon = (4-d)/2$.

2.1 $\overline{\text{MS}}$–on-shell relation

The $\overline{\text{MS}}$–on-shell relation can be written as a power series in the strong coupling constant α_s

$$
\begin{aligned}
z_m^{\text{OS}}(\mu) &= \frac{\bar{m}_q(\mu)}{M_q} = \frac{Z_m^{\text{OS}}}{Z_m^{\overline{\text{MS}}}} \\
&= 1 + \frac{\alpha_s(\mu)}{\pi}\delta z_m^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \delta z_m^{(2)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \delta z_m^{(3)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^4 \delta z_m^{(4)} \\
&\quad + \mathcal{O}\left(\alpha_s^5\right)
\end{aligned}
\tag{2.2}
$$

and labeling contributions from massless and massive quark loops by n_l and n_h, respectively, we obtain the result for contributions from diagrams with at least two massless quark loops

$$
\begin{aligned}
z_m^{\text{OS}} &= 1 - A_s 1.333 + A_s^2\left(-14.229 - 0.104\, n_h + 1.041\, n_l\right) \\
&\quad + A_s^3\left(-197.816 - 0.827\, n_h - 0.064\, n_h^2 + 26.946\, n_l - 0.022\, n_h n_l - 0.653\, n_l^2\right) \\
&\quad + A_s^4\left(-43.465\, n_l^2 - 0.017\, n_h n_l^2 + 0.678\, n_l^3 + \ldots\right) + \mathcal{O}\left(A_s^5\right),
\end{aligned}
\tag{2.3}
$$

with $A_s = \alpha_s(m_q)/\pi$.

2.2 Anomalous magnetic moment of the muon

In the approximation of a massless electron only the leading term including the logarithms can be obtained. For the sub-leading contributions a proper asymptotic expansion has to be performed. Expanding a_μ in a power series in the fine structure constant α

$$a_\mu = 1+\frac{\alpha}{\pi}a_\mu^{(1)}+\left(\frac{\alpha}{\pi}\right)^2 a_\mu^{(2)}+\left(\frac{\alpha}{\pi}\right)^3 a_\mu^{(3)}+\left(\frac{\alpha}{\pi}\right)^4 a_\mu^{(4)} + \mathscr{O}\left(\alpha^5\right) \tag{2.4}$$

and marking contributions from electron loops by n_l

$$a_\mu^{(4)} = n_l^3 a_\mu^{(43)} + n_l^2 a_\mu^{(42)} + \cdots \tag{2.5}$$

we obtain the result for contributions with three electron loops

$$a_\mu^{(43)} = \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324}+\frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \approx 7.19666\,, \tag{2.6}$$

where $L_{\mu e} = \ln(M_\mu^2/M_e^2)$. The result for diagrams with two electron loops can be further split into a contribution with and without an additional muon loop, $a_\mu^{(42)b}$ and $a_\mu^{(42)a}$, respectively,

$$a_\mu^{(42)} = a_\mu^{(42)a} + a_\mu^{(42)b},$$

with

$$\begin{aligned} a_\mu^{(42)a} &= L_{\mu e}^2\left[\pi^2\left(\frac{5}{36}-\frac{a_1}{6}\right)+\frac{\zeta_3}{4}-\frac{13}{24}\right] + L_{\mu e}\left[-\frac{a_1^4}{9}+\pi^2\left(-\frac{2a_1^2}{9}+\frac{5a_1}{3}-\frac{79}{54}\right)\right.\\ &\left. -\frac{8a_4}{3}-3\zeta_3+\frac{11\pi^4}{216}+\frac{23}{6}\right] - \frac{2a_1^5}{45}+\frac{5a_1^4}{9}+\pi^2\left(-\frac{4a_1^3}{27}+\frac{10a_1^2}{9}\right.\\ &\left.-\frac{235a_1}{54}-\frac{\zeta_3}{8}+\frac{595}{162}\right)+\pi^4\left(-\frac{31a_1}{540}-\frac{403}{3240}\right)+\frac{40a_4}{3}+\frac{16a_5}{3}-\frac{37\zeta_5}{6}\\ &+\frac{11167\zeta_3}{1152}-\frac{6833}{864}\\ &\approx -3.62427\,, \end{aligned} \tag{2.7}$$

$$a_\mu^{(42)b} = \left(\frac{119}{108}-\frac{\pi^2}{9}\right)L_{\mu e}^2 + \left(\frac{\pi^2}{27}-\frac{61}{162}\right)L_{\mu e} - \frac{4\pi^4}{45}+\frac{13\pi^2}{27}+\frac{7627}{1944} \approx 0.49405\,. \tag{2.8}$$

Our results for $a_\mu^{(43)}$ and $a_\mu^{(42)b}$ agree with the results given in Refs. [12, 9]. The result for $a_\mu^{(42)a}$ can be compared with the result from Refs. [1, 10]

$$a_\mu = -3.64204(112)\,. \tag{2.9}$$

Our new result confirms the previously obtained results, the small discrepancy is due to missing terms in the expansion in m_e/m_μ.

To finish, let us also present the result of a recent calculation of the contribution from τ-leptons. Using asymptotic expansion in Ref. [11] the four-loop corrections due to τ-leptons was obtained. The new result

$$A_{2,\mu}^{(8)}(M_\mu/M_\tau) = 0.0421670 + 0.0003257 + 0.0000015 = 0.0424941(2)(53) \tag{2.10}$$

is more precise and in full agreement with previous evaluations [1].

3. Conclusions

We presented results for both the $\overline{\mathrm{MS}}$–on-shell relation and the anomalous magnetic moment of the muon at four-loop order. These result comprise a first step towards the full four-loop calculation and confirm the results known in the literature.

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