DESY 14-006
January 2014
ISSN 0418-9833

Two-loop electroweak threshold corrections to the bottom and top Yukawa couplings

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Abstract

We study the relationship between the $\overline{\rm MS}$ Yukawa coupling and the pole mass for the bottom and top quarks at the two-loop electroweak order $\mathcal{O}(\alpha^2)$ in the gaugeless limit of the standard model. We also consider the $\overline{\rm MS}$ to pole mass relationships at this order, which include tadpole contributions to ensure the gauge independence of the $\overline{\rm MS}$ masses. In order to suppress numerically large tadpole contributions, we propose a redefinition of the running heavy-quark mass in terms of the $\overline{\rm MS}$ Yukawa coupling. We also present Δr in the $\overline{\rm MS}$ scheme at $\mathcal{O}(\alpha^2)$ in the gaugeless limit. As an aside, we also list the exact two-loop expression for the mass counterterms of the bottom and top quarks.

PACS numbers: 11.10.Gh, 12.15.Ff, 12.15.Lk, 14.65.Fy, 14.65.Ha

1 Introduction

The recent discovery of the Higgs boson [1] was a giant leap for particle physics. It confirmed that the concept of spontaneous symmetry braking in connection with the generation of masses by the Higgs mechanism could be realized in nature. This does not explain, however, the very large spread of the fermion masses and the values of the masses themselves. It is generally believed that some grand unified theories could provide a solution to this fundamental problem. An essential rôle in such analyses is played by the renormalization group (RG) equations, which determine the scale dependencies of the running parameters.

Due to the large values of their masses, the bottom quark and, even more so, the top quark attract great interest. Even disregarding the fact that quark masses require special consideration because quarks do not appear as free particles, one may introduce different parameters to describe the notion of quark mass. The most important definitions are those of the pole mass M, the running mass $m(\mu)$ in the $\overline{\text{MS}}$ scheme, and the running Yukawa coupling $y(\mu)$, defined in $\overline{\text{MS}}$ scheme as well. Here, μ is the 't Hooft mass of dimensional regularization. The relationships between these quantities can be obtained in perturbation theory as series in the strong-coupling constant α_s and Sommerfeld's fine-structure constant α. The terms containing $\ln \mu^2$ can be obtained using the RG beta functions and anomalous dimensions. In QCD, they have been computed in the three- [2] and four-loop [3] approximations. In the standard model (SM), the corresponding RG functions, known through two loops since Ref. [4], have recently been evaluated at the three-loop level [5].

The other aspect of the problem is the matching between the running parameters and the physical observables. These so-called threshold relationships not only include terms with $\ln \mu^2$, but also terms of non-logarithmic origin. The relation between the $\overline{\text{MS}}$ running mass and the pole mass of a quark were elaborated in QCD at one [6], two [7], and three [8] loops. The two-loop result in the supersymmetric extension of QCD was obtained in Ref. [9]. These corrections can be readily applied also to the Yukawa coupling of a quark. However, the situation becomes more complicated if electroweak corrections are taken into account. In this case, the relation between the $\overline{\text{MS}}$ mass and the $\overline{\text{MS}}$ Yukawa coupling of a fermion becomes nontrivial. The full one-loop corrections to the relationships between the $\overline{\text{MS}}$ Yukawa couplings and the pole masses of the fermions were derived in Ref. [10]. The mixed QCD-electroweak two-loop corrections were evaluated for the top quark in Ref. [11] and for the other fermions in Ref. [12] by using the method of large-mass expansion. In some earlier calculations, the tadpole contributions were omitted in the $\overline{\text{MS}}$ to pole mass relations of fermions, e.g. in the calculation of the electroweak parameter $\Delta\rho$ in the gaugeless limit in Ref. [13]. Expressed in terms of the top-quark and Higgs-boson pole masses, the result for $\Delta\rho$ thus obtained is correct, but some intermediate results differ from those in Ref. [11]. This situation was clarified in Ref. [14].

The aim of the present paper is to extend the theoretical knowledge of the relationships between the running $\overline{\text{MS}}$ masses and Yukawa couplings and the pole masses of the bottom and top quarks by calculating the two-loop electroweak corrections, at order $O(\alpha^2)$, in the approximation provided by the gaugeless limit.

This paper is organized as follows. In Sections 2 and 3, we give all the necessary definitions concerning the running mass and the Yukawa coupling, respectively. In Sections 4 and 5, we list analytical results for the bottom and top Yukawa couplings, respectively.

In Section 6, we present our numerical analysis. In Appendix A, we list two-loop expressions for the electroweak parameter $\Delta\bar{r}$, which enters the relationships between the running masses and the Yukawa couplings, both in the gaugeless and heavy-top-quark limits. Appendix B contains the exact two-loop renormalization constants of the bottom- and top-quark masses in $\overline{\rm MS}$ scheme. In Appendix C, we present the $\overline{\rm MS}$ to pole mass relation for the bottom quark in the heavy-top-quark limit.

2 Running mass

The pole mass M of a particle is determined by definition from the position of the pole of it's propagator.[1] We start from the general form of the inverse fermion propagator,

$$S^{-1}(\not{p}) = \not{p} - m_0 - \Sigma(\not{p})\,, \tag{1}$$

where p is the four-momentum and m_0 is the bare mass of the fermion. The self-energy function $\Sigma(\not{p})$ is given by the sum of all one-particle-irreducible Feynman diagrams pertaining to the transition of the fermion to itself and depends on further parameters of the theory, such as masses, couplings, and mixing parameters. An important comment should be made here. In the SM with the Higgs mechanism, the inclusion of tadpoles (see Fig. 1) is necessary to render the relationship between the pole and bare masses gauge independent [15]. Thus, we include in $\Sigma(\not{p})$ all Feynman diagrams which are one-particle irreducible with respect to all particles except for the Higgs boson.

In the electroweak theory, the left- and right-handed components of a fermion field propagate differently due to parity violation. On the other hand, we take the Cabibbo-Kobayashi-Maskawa quark mixing matrix to be unity and thus effectively turn off CP violation. In this case, the most general decomposition of $\Sigma(\not{p})$ reads [16]:

$$\Sigma(\not{p}) = \not{p}\,\omega_L A_L(p^2) + \not{p}\,\omega_R A_R(p^2) + m_0 B(p^2)\,, \tag{2}$$

where $\omega_{L,R} = (1 \mp \gamma_5)/2$ are the projectors on the left- and right-handed spinor components, respectively, and A_L, A_R, and B are dimensionless scalar functions of p^2 depending also on the SM parameters.

In order to invert the matrix in Eq. (1), we first decompose it into its left- and right-handed components. The inverse $S(\not{p})$ may be found in Eq. (10) of Ref. [16]. The poles of its left- and right-handed components coincide and are given by the solution $p^2 = M^2$ of the equation [16]

$$p^2[1 - A_L(p^2)][1 - A_R(p^2)] - m_0^2[1 + B(p^2)]^2 = 0\,. \tag{3}$$

We can solve Eq. (3) perturbatively by substituting the ansatz

$$M = m_0(1 + X_1 + X_2 + \ldots), \tag{4}$$

where X_i refers to the order i. Explicitly, through second order, we have

$$\begin{aligned} X_1 &= B_1 + \frac{1}{2}A_{L,1} + \frac{1}{2}A_{R,1}\,, \\ X_2 &= B_2 + \frac{1}{2}A_{L,2} + \frac{1}{2}A_{R,2} + X_1(A_{L,1} + A_{R,1} + A'_{L,1} + A'_{R,1} + 2B'_1) \\ &\quad -\frac{1}{2}X_1^2 - \frac{1}{2}A_{L,1}A_{R,1} + \frac{1}{2}B_1^2\,. \end{aligned} \tag{5}$$

[1] If the particle is unstable, then the pole position has a complex value. In this case, the pole mass is usually taken to be the real part of it.

In Eq. (5), all functions are to be evaluated at $p^2 = m_0^2$ and the prime stands for the derivative with respect to p^2/m_0^2. The self-energy function in Eq. (2) is gauge dependent and, in general, also infrared divergent. However, the coefficients X_i in Eq. (4) are manifestly gauge invariant [18], provided the tadpole diagrams are included, and infrared safe [19]. The generalizaion of Eqs. (4) and (5) to the case of intergeneration mixing is elaborated in Ref. [17].

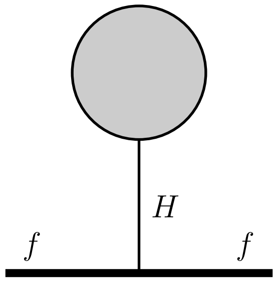


Figure 1: Tadpole contribution to the self-energy $\Sigma(\not{p})$ of a fermion f. H stands for the Higgs boson.

3 Yukawa coupling

We start from the SM renormalized in the on-shell scheme, in which Sommerfeld's fine-structure constant $\alpha = e^2/(4\pi)$ as measured in Thomson scattering and the pole masses M_X of the elementary particles $X = W, Z, H, t, b, \ldots$ serve as basic parameters, and consider a generic heavy quark with pole mass M. The sine S_w of the weak-mixing angle and the Yukawa coupling Y of the heavy quark are derived parameters, defined to all orders as

$$S_w = \sqrt{1 - \frac{M_W^2}{M_Z^2}}, \qquad Y = 2^{-1/2} \frac{eM}{S_w M_W}, \tag{6}$$

respectively. It is advantageaous to introduce the Fermi constant G_F, which is the fundamental coupling in the Fermi model of four-fermion interactions, by equating the total decay width of the muon evaluated in the QED-improved Fermi model and the SM [20]. This leads to the relationship

$$2^{5/2} G_F = \frac{e^2}{S_w^2 M_W^2}(1 + \Delta r), \tag{7}$$

where Δr accommodates all the radiative corections which the SM introduces beyond the QED-improved Fermi model [20].[2] Numerically, we have $G_F = 1.16637 \times 10^{-5}\mathrm{GeV}^{-2}$ [21].

We now pass from the original on-shell scheme to the $\overline{\mathrm{MS}}$ scheme and denote the counterparts of e, M_W, M_Z, and M as $\bar{e}(\mu)$, $m_W(\mu)$, $m_Z(\mu)$, and $m(\mu)$, respectively, where μ is the 't Hooft mass of dimensional regularization. Accordingly, Eqs. (6) and (7) become

$$s_w(\mu) \;=\; \sqrt{1 - \frac{m_W^2(\mu)}{m_Z^2(\mu)}}, \qquad y(\mu) = 2^{-1/2} \frac{\bar{e}(\mu) m(\mu)}{s_w(\mu) m_W(\mu)}, \tag{8}$$

[2] In the literature, Eq. (7) is frequently written with $(1+\Delta r)$ replaced by $(1-\Delta r)^{-1}$, which is equivalent at one loop.

$$2^{5/2}G_F = \frac{\bar{e}^2(\mu)}{s_w^2(\mu)m_W^2(\mu)}[1+\Delta\bar{r}(\mu)], \tag{9}$$

respectively. An explicit analytic expression for $\Delta\bar{r}(\mu)$ through two loops may be found in Appendix A.[3] Substituting Eq. (9) into the second equality in Eq. (8), we find the relationship

$$y(\mu) = 2^{3/4}G_F^{1/2}M[1+\delta(\mu)], \tag{10}$$

with the radiative correction

$$\delta(\mu) = \frac{m(\mu)}{M}\frac{1}{\sqrt{1+\Delta\bar{r}(\mu)}} - 1. \tag{11}$$

In the context of RG analyses, Eq. (10) allows one to relate the running Yukawa coupling $y(\mu)$ at some matching scale $\mu = \mu_0$, which is typically chosen to be of the order of the Higgs vacuum expectation value $2^{-1/4}G_F^{-1/2} = 246$ GeV, to the physical SM parameters defined in the on-shell scheme, and Eq. (11) is frequently called threshold correction.

In order for the mass counterterms in the on-shell scheme to be gauge independent, the tadpole contributions, whose ultraviolet (UV) divergent and finite parts are both gauge dependent, must be properly included [15]. Since the pole and $\overline{\text{MS}}$ masses are related by the UV-finite parts of these mass counterterms, the tadpole contributions are indispensable in order for the gauge independence to be communicated from the pole masses [18] to the $\overline{\text{MS}}$ masses [10,12]. Since the Higgs boson happens to be considerably lighter than the top quark [1], the UV-finite parts of the tadpole contributions are numerically sizeable, as was observed for the top quark in Ref. [24]. In the case of the bottom quark, they even seriously jeopardize the perturbative stability of the $\overline{\text{MS}}$ mass, as will be demonstrated in Section 6.

On the other hand, it was noticed that $\delta(\mu)$ in Eq. (11) is devoid of tadpole contributions in $O(\alpha)$ [10] and $O(\alpha\alpha_s)$ [12]. In this paper, we investigate the situation at $O(\alpha^2)$ in the gaugeless limit. We find that the tadpoles contained in $m(\mu) - M$ and $\Delta\bar{r}(\mu)$ do not fully cancel in Eq. (11). While their leading contributions originating from genuine two-loop tadpoles and products of one-loop tadpoles drop out, subleading contributions arising from products of one-loop tadpoles and tadpole-free one-loop diagrams survive. In fact, the latter are indispensable to ensure that the well-known two-loop expression [4] for the RG beta function of $y(\mu)$,

$$\beta_y = \frac{\mu^2 d}{d\mu^2}y(\mu) = 2^{3/4}G_F^{1/2}M\frac{\mu^2 d}{d\mu^2}\delta(\mu), \tag{12}$$

is recovered. Specifically, the tadpole contributions to $\delta(\mu)$ at $O(\alpha^2)$ in the gaugeless limit cancel to such a degree that the latter is finite in the limit of massless Higgs boson. We verified this also for the full $O(\alpha^2)$ result for $\delta(\mu)$. In turn, this implies that the sizeable electroweak corrections that challenge the usefulness of $m(\mu)$ do not plague $\delta(\mu)$. This provides a strong motivation for us to introduce an alternative definition of running quark mass by rescaling $y(\mu)$ as[4]

$$m_Y(\mu) = 2^{-3/4}G_F^{-1/2}y(\mu). \tag{13}$$

[3]This electroweak parameter and its two-loop corrections significantly differ from the quantities $\Delta\hat{\rho}$, $\Delta\hat{r}$, and $\Delta\hat{r}_W$ introduced in Ref. [22] on the basis of a hybrid renormalization scheme and their two-loop corrections [23].

[4]In Ref. [24], it was denoted as $\hat{m}_f(\mu^2)$.

We call this the Yukawa mass. According to Eqs. (10) and (11), it is related to the pole and $\overline{\rm MS}$ masses as

$$m_Y(\mu) = M[1+\delta(\mu)] = \frac{m(\mu)}{\sqrt{1+\Delta\bar{r}(\mu)}}. \tag{14}$$

At tree level, we have $m_Y(\mu) = M = m(\mu)$.

If electroweak corrections are neglected and only the strong interactions are taken into account, then $\Delta\bar{r}(\mu) = 0$, so that $m_Y(\mu) = m(\mu)$. The pure QCD contribution to $\delta(\mu)$ in Eq. (11) reads [6,7]:

$$\begin{aligned}\delta_{\rm QCD}(\mu) &= \frac{\alpha_s(\mu)}{4\pi}\left(4L-\frac{16}{3}\right)+\left[\frac{\alpha_s(\mu)}{4\pi}\right]^2\left[\left(-14+\frac{4}{3}n_l\right)L^2+\left(\frac{314}{3}-\frac{52}{9}n_l\right)L\right.\\ &\quad\left.-\frac{3161}{18}-\frac{112}{3}\zeta_2-\frac{32}{3}\zeta_2\ln 2+\frac{8}{3}\zeta_3+\left(\frac{71}{9}+\frac{16}{3}\zeta_2\right)n_l\right]+\mathcal{O}(\alpha_s^3)\,, \end{aligned}\tag{15}$$

where $\alpha_s(\mu)$ is the strong-coupling constant in the $\overline{\rm MS}$ scheme, $L = \ln(M^2/\mu^2)$, and n_l is the number of light-quark flavors. The $\mathcal{O}(\alpha_s^3)$ term in Eq. (15) may be found in Ref. [8].

In Sections 4 and 5, we shall provide the electroweak corrections to Eq. (11) through two loops for the bottom and top quarks, respectively. The corresponding relationships between $m(\mu)$ and M may then be obtained via Eq. (14) in connection with the expression for $\Delta\bar{r}(\mu)$ listed in Appendix A.

We work in R_ξ gauge with four independent gauge parameters, associated with the electroweak gauge bosons and the gluon, and perform expansions in the ξ parameters as explained in Refs. [25,26]. All required $\overline{\rm MS}$ renormalization constants may be found through the appropriate orders in Refs. [25,26], except for two, namely the two-loop renormalization constants for the bottom- and top-quark masses. These missing results are presented in exact analytical forms in Appendix B.

4 Bottom quark

In the case of the bottom quark, the calculation of $\Sigma(\not{p})$ can essentially be reduced to the evaluation of vacuum bubble integrals. By explicit calculation, we find

$$\begin{aligned}\frac{m_{{\rm Y},b}(\mu)}{M_b} &= 1+\delta_{\rm QCD}(\mu)\\ &+\frac{\alpha}{4\pi S_W^2}\left\{\frac{M_t^2}{M_W^2}\left(-\frac{5}{16}+\frac{3}{8}L_t+N_c\left(\frac{1}{8}-\frac{1}{4}L_t\right)\right)-\frac{3}{8}\frac{M_W^2}{M_H^2-M_W^2}L_{wh}+\frac{1}{16}\frac{M_H^2}{M_W^2}\right.\\ &-\frac{3}{8}\frac{M_W^4}{(M_t^2-M_W^2)^2}L_{tw}+\frac{3}{8}\frac{M_W^2}{M_t^2-M_W^2}(1-L_{tw})+\frac{3}{8S_W^2}L_{wz}+S_W^2Q_b^2(-4+3L_b)\\ &+S_W^2v_b^2\left(-\frac{5}{2}+3L_z\right)+S_W^2a_b^2\left(-\frac{1}{2}+3L_z\right)-\frac{5}{8}-\frac{5}{16}\frac{M_Z^2}{M_W^2}+\frac{3}{8}L_w+\frac{3}{8}L_h\\ &+\frac{M_b^2}{M_W^2}\left(\frac{11}{48}+\frac{1}{4}L_b-\frac{1}{8}L_t-\frac{1}{8}L_z-\frac{3}{8}L_h\right.\\ &\left.\left.+v_b^2S_W^2C_W^2\left(-\frac{8}{3}-4L_b+4L_z\right)\right)+O\left(\frac{M_b^4}{M_W^4}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&+C_F\frac{\alpha_s}{4\pi}\frac{\alpha}{4\pi S_W^2}\bigg\{\frac{M_t^2}{M_W^2}\Big(-\frac{13}{4}-\frac{15}{16}L_b-\frac{3}{2}L_t+\frac{9}{8}L_tL_b+\frac{9}{8}L_t^2\\
&+N_c\Big(\frac{21}{16}-\frac{1}{2}\zeta_2+\frac{3}{8}(1-2L_t)L_b+\frac{7}{4}L_t-\frac{3}{4}L_t^2\Big)\Big)\\
&+\bigg(3\frac{M_t^2}{M_W^2}+3+\frac{21}{4}\frac{M_W^2}{M_t^2-M_W^2}+\frac{9}{4}\frac{M_W^4}{(M_t^2-M_W^2)^2}\bigg)\mathrm{Li}_2\Big(1-\frac{M_W^2}{M_t^2}\Big)\\
&+\frac{M_W^2}{M_t^2-M_W^2}\Big(-\frac{51}{8}+\frac{9}{8}(1-L_{tw})L_b+\frac{39}{8}L_{tw}\Big)+\frac{M_W^4}{(M_t^2-M_W^2)^2}\Big(\frac{33}{8}-\frac{9}{8}L_b\Big)L_{tw}\\
&+\frac{3}{8}\frac{M_W^2}{M_H^2-M_W^2}(4-3L_b)L_{wh}-\frac{1}{16}\frac{M_H^2}{M_W^2}(4-3L_b)-\frac{3}{8S_W^2}(4-3L_b)L_{wz}\\
&+S_W^2Q_b^2\Big(\frac{7}{4}-24\zeta_3-60\zeta_2+96\zeta_2\ln 2-21L_b+9L_b^2\Big)\\
&+S_W^2v_b^2\Big(\frac{23}{4}-\frac{15}{2}L_b-9L_z+9L_bL_z\Big)+S_W^2a_b^2\Big(\frac{55}{4}-\frac{3}{2}L_b-9L_z-9L_bL_z\Big)\\
&-\frac{23}{16}+\frac{5}{4}\frac{M_Z^2}{M_W^2}-\frac{15}{16}\frac{M_Z^2}{M_W^2}L_b-\frac{15}{8}L_b+\Big(-\frac{3}{2}+\frac{9}{8}L_b\Big)\Big(L_h+L_w+\frac{M_Z^2}{M_W^2}L_z\Big)-3L_w+\frac{3}{4}L_t\bigg\}\\
&+\left(\frac{\alpha M_t^2}{4\pi M_W^2S_W^2}\right)^2\left(A_{2,2}\ln^2\frac{M_t^2}{\mu^2}+A_{2,1}\ln\frac{M_t^2}{\mu^2}+B_2\right). \qquad (16)
\end{aligned}
$$

Here, $\delta_{\rm QCD}(\mu)$ is given by Eq. (15) with $M=M_b$, $L_X=\ln(M_X^2/\mu^2)$, $L_{XY}=L_X-L_Y$, $C_w=\sqrt{1-S_w^2}$, $I_b=-1/2$ and $Q_b=-1/3$ are the weak isospin and electric charge of the bottom quark, and $v_b=(I_b-2Q_bS_w^2)/(2S_wC_w)$ and $a_b=I_3/(2S_wC_w)$ are its vector and axial-vector couplings to the Z boson, respectively.

Equation (16) is expanded in powers of M_b/M_W, through terms of $O(M_b^2/M_W^2)$ in $O(\alpha)$ and through terms of $O(M_b^0/M_W^0)$ beyond that. In fact, the $O(\alpha M_b^4/M_W^4)$ terms in Eq. (16) are already negligibly small. The exact analytical result of $O(\alpha)$ may be found in Ref. [10].[5]

The $O(\alpha^2)$ term in Eq. (16) is calculated adopting the approximation by the gaugeless limit. Since its leading behavior in the mass of the heaviest SM particle is of $O(M_t^4)$, we pull out the factor M_t^4/M_W^4 so as to minimize the M_t dependence of the coefficients $A_{2,2}$, $A_{2,1}$, and B_2, for which we obtain

$$
\begin{aligned}
A_{2,2}^{\rm gl} &= \frac{63}{128}-\frac{45}{64}H_t-\frac{45}{128}H_t^2\,,\\
A_{2,1}^{\rm gl} &= -\frac{23}{128}+\left(\frac{33}{16}-\frac{75}{64}\ln H_t\right)H_t+\left(\frac{45}{32}-\frac{45}{64}\ln H_t\right)H_t^2+\frac{3}{32}(4-H_t)\,I_1(H_t)\,,\\
B_2^{\rm gl} &= -\frac{45}{128}H_t^2\ln^2 H_t+\left(-\frac{5}{16}+\frac{241}{128}H_t+\frac{45}{32}H_t^2\right)\ln H_t+\frac{5}{64}(4-H_t)I_1(H_t)\\
&\quad+\frac{3}{32}(4-H_t)I_2(H_t)+\left(-\frac{17}{16}+\frac{9}{32}H_t-\frac{9}{64}H_t^2\right)\Phi(H_t/4)\\
&\quad+\left(-\frac{93}{64}+\frac{11}{16H_t}+\frac{9}{8}H_t-\frac{9}{32}H_t^2\right)\left(\mathrm{Li}_2(1-H_t)-\zeta_2\right)\\
&\quad+\frac{1491}{512}-\frac{13}{8}\zeta_2+\Big(-\frac{695}{256}+\frac{9}{8}\zeta_2\Big)H_t
\end{aligned}
$$

[5] Note, however, that the result in Ref. [10] is written in terms of G_F and needs to be rewritten in terms of α if it is to used in lieu of the $O(\alpha)$ term in Eq. (16).

$$+\Big(-\frac{751}{512}-\frac{9}{128}S_1+\frac{243}{64}S_2-\frac{13}{32}\zeta_2\Big)H_t^2\,, \tag{17}$$

where $H_t = M_H^2/M_t^2$, $S_1 = \pi/\sqrt{3}$, $S_2 = 4/(9\sqrt{3})\mathrm{Cl}_2(\pi/3)$ with $\mathrm{Cl}_2(\theta)$ being Clausen's function defined in Eq. (43), $\Phi(z)$ is related to a two-loop self-energy integral and defined in Eq. (42), and

$$\begin{aligned} I_1(z) &= \int_0^1 dt\, \ln[z(1-t)+t^2]\,, \\ I_2(z) &= \int_0^1 dt\, \ln\left[\frac{1}{z}-t(1-t)\right] \end{aligned} \tag{18}$$

are related to one-loop self-energy integrals and may be expressed in terms of logarithms.

As is evident from Eq. (13), the running bottom Yukawa coupling $y_b(\mu)$ simply emerges from Eq. (16) by multiplication with the μ-independent overall factor $2^{3/4}G_F^{1/2}M_b$. The ratio $m_b(\mu)/M_b$ may be obtained from Eq. (16) and the formula for $\Delta\bar{r}$ given in Appendix A using Eq. (14). For the reader's convenience, we shall present a formula for $m_b(\mu)/M_b$ in the heavy-top-quark limit in Appendix C. We shall use it in Section 6 for comparisons with existing $O(\alpha\alpha_s)$ results and for testing the relevance of the tadpole contributions at two loops.

5 Top quark

The counterpart of Eq. (16) for the top quark reads

$$\begin{aligned} \frac{m_{\mathrm{Y},t}(\mu)}{M_t} &= 1+\delta_{\mathrm{QCD}}(\mu)+\delta_\alpha(\mu)+\delta_{\alpha\alpha_s}(\mu) \\ &+\left(\frac{\alpha M_t^2}{4\pi M_W^2 S_W^2}\right)^2\left(A_{2,2}\ln^2\frac{M_t^2}{\mu^2}+A_{2,1}\ln\frac{M_t^2}{\mu^2}+B_2\right)\,, \end{aligned} \tag{19}$$

where pure QCD correction, $\delta_{\mathrm{QCD}}(\mu)$, is given through two loops by Eq. (15) with $M = M_t$, the one-loop electroweak correction, $\delta_\alpha(\mu)$, may be found in Ref. [10], and the mixed two-loop correction $\delta_{\alpha\alpha_s}(\mu)$ was evaluated in Ref. [27]. Here, we independently obtain this correction by combining $m_t(\mu)/M_t$ evaluated exactly for non-vanishing gauge-boson masses in Ref. [11] with our result for $\Delta\bar{r}$ in Appendix A according to Eq. (14) establishing full agreement with Ref. [27]. Furthermore, we present here the two-loop electroweak correction, parametrized by $A_{2,2}$, $A_{2,1}$, and B_2.

The case of the top quark is more complicated because a heavy particle now propagates on external lines. In such a situation, the result cannot be reduced to vacuum bubble diagrams any more. The heavy-top-quark limit leads to non-euclidean expansions, which are rather involved. In the gaugeless limit, we find it convenient to introduce the variable

$$\Delta_H = 1-\frac{M_H^2}{M_t^2}\,. \tag{20}$$

Assuming the weak neutral scalar resonance recently discovered at the CERN Large Hadron Collider [1] to be the SM Higgs boson, we have $M_H \approx 125$ GeV, so that Δ_H

is close to 0.5. Nevertheless, it appears to be a good expansion parameter. After the expansion in Δ_H, the resulting integrals correspond to two-loop self-energies with one mass and on-shell kinematics. These integrals may be evaluated with the help of the ONSHELL2 program [28].

There is, however, one problem related to this procedure. In fact, such a naïve expansion can break down if certain threshold singularities are present. In the case under consideration, this occurs, for example, when a Feynman diagram has a unitary cut intersecting one Higgs boson line and one or two massless lines. Taken off-shell, such diagrams produce terms of the type $(p^2 - M_H)^n \ln(p^2 - M_H^2)$, which disappear in the limit $p^2 \to M_H^2$. The presence of such terms tells us that the expansion in the variable $p^2 - M_H^2$ breaks down at some order n. This issue may be discussed in connection with pole masses of gauge bosons on the basis of the results obtained in Ref. [26]. Detailed analysis reveals that the first five coefficients of the naïive expansion yield the correct result. In the present case, we find that the first six coefficient of the expansion in Δ_H give the correct result, but, starting from $O(\Delta_H^6)$, threshold singularities appear. In dimensional regularization, they manifest themselves as poles in $\varepsilon = 2 - d/2$ with d being the space-time dimensionality, which are not compensated by renormalization. Detailed inspection reveals that, among the more than two hundred diagrams contributing to $\Sigma(\not{p})$, only four are plagued by threshold singularities. They are shown in Fig. 2. These diagrams were evaluated without expansions, using the results for the master integrals from Ref. [26].

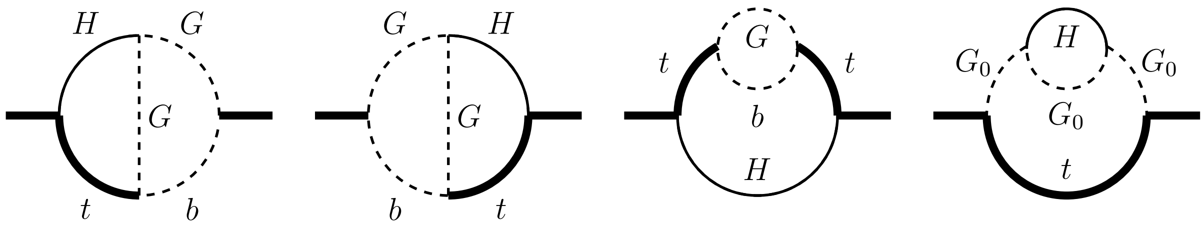


Figure 2: Four Feynman diagrams whose expansions in Δ_H suffer from threshold singularities. G_0 and G denote the neutral and the charged Goldstone bosons, respectively. Their masses vanish in the gaugeless limit.

For the coefficients $A_{2,2}$, $A_{2,1}$, and B_2 in Eq. (19), we obtain

$$\begin{aligned}
A_{2,2}^{\text{gl}} &= \frac{243}{128}, \qquad (21)\\
A_{2,1}^{\text{gl}} &= -\frac{297}{128} - \frac{117}{128}\Delta_H + \Big(\frac{3}{8} - \frac{27}{64}S_1\Big)\Delta_H^2 + \Big(\frac{9}{64} - \frac{3}{16}S_1\Big)\Delta_H^3 + \Big(\frac{9}{256} - \frac{3}{32}S_1\Big)\Delta_H^4 \\
&\quad + \Big(\frac{9}{640} - \frac{1}{16}S_1\Big)\Delta_H^5 + \Big(\frac{3}{640} - \frac{13}{288}S_1\Big)\Delta_H^6 + \Big(\frac{3}{4480} - \frac{5}{144}S_1\Big)\Delta_H^7 \\
&\quad + \Big(-\frac{3}{2240} - \frac{1}{36}S_1\Big)\Delta_H^8 + \Big(-\frac{3}{1280} - \frac{89}{3888}S_1\Big)\Delta_H^9 + O(\Delta_H^{10}), \qquad (22)\\
B_2^{\text{gl}} &= -\frac{2147}{512} - \zeta_3 + \frac{593}{192}\zeta_2 - \frac{1}{2}S_1 + \frac{405}{128}S_2 + \frac{81}{16}S_2S_1 + \Delta_H\Big(\frac{275}{256} + \frac{1}{4}\zeta_3 - \frac{121}{48}\zeta_2 \\
&\quad + \frac{559}{128}S_1 - \frac{1107}{64}S_2 - \frac{9}{4}S_2S_1\Big) + \Delta_H^2\Big(\frac{637}{512} + \frac{1}{4}\zeta_3 - \frac{1961}{384}\zeta_2 + \frac{383}{64}S_1 - \frac{3051}{256}S_2 \\
&\quad - \frac{9}{16}S_2S_1\Big) + \Delta_H^3\Big(\frac{11}{72} - \frac{2765}{576}\zeta_2 + \frac{18925}{3456}S_1 - \frac{987}{128}S_2\Big) + \Delta_H^4\Big(\frac{2743}{4608} - \frac{5275}{1152}\zeta_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8213}{1728}S_1-\frac{1537}{256}S_2\Big)+\Delta_H^5\Big(\frac{18437}{23040}-\frac{2713}{576}\zeta_2+\frac{79807}{17280}S_1-\frac{3283}{640}S_2\Big)\\
&+\Delta_H^6\Big(\frac{367913}{414720}-\frac{19789}{4320}\zeta_2+\frac{1017707}{233280}S_1-\frac{26267}{5760}S_2-\frac{1}{768}\ln\Delta_H\Big)\\
&+\Delta_H^7\Big(\frac{302152}{297675}-\frac{47041}{10080}\zeta_2+\frac{14051993}{3265920}S_1-\frac{125141}{30240}S_2-\frac{9}{4480}\ln\Delta_H\Big)\\
&+\Delta_H^8\Big(\frac{1313737501}{1219276800}-\frac{47467}{10368}\zeta_2+\frac{108071203}{26127360}S_1-\frac{26375}{6912}S_2-\frac{127}{53760}\ln\Delta_H\Big)\\
&+\Delta_H^9\Big(\frac{6463766951}{5486745600}-\frac{144331}{31104}\zeta_2+\frac{288955985}{70543872}S_1-\frac{1549721}{435456}S_2-\frac{607}{241920}\ln\Delta_H\Big)\\
&+O(\Delta_H^{10})\,. \qquad (23)
\end{aligned}
$$

We observe that, in Eq. (23), the coefficients of the expansion in Δ_H contain terms proportional to $\ln\Delta_H$ starting from $O(\Delta_H^6)$. This is how the threshold singularities mentioned above manifest themselves. Fortunately, this does not spoil the convergence property of Eq. (23). In fact, for $M_H = 125$ GeV, the terms of $O(\Delta_H^n)$ with $n = 0,\ldots,9$ in Eq. (23) add up as

$$
\begin{aligned}
B_2^{\mathrm{gl}} &\approx 1.9936-0.2011+0.1455+0.0201+0.0064+0.0024+0.0010+0.0004\\
&\quad+0.0002+0.0001\\
&= 1.9685\,, \qquad (24)
\end{aligned}
$$

exhibiting rapid convergence. For comparison, we also display the convergence property of the power series in Eq. (22), which is not challenged by $\ln\Delta_H$ terms:

$$
\begin{aligned}
A_{2,1}^{\mathrm{gl}} &\approx -2.3203-0.4396-0.0903-0.0222-0.0072-0.0026-0.0010-0.0004\\
&\quad-0.0001-0.0001\\
&= -2.8837\,. \qquad (25)
\end{aligned}
$$

6 Discussion

We are now in a position to present our numerical analysis. For this, we adopt the following values for the input parameters from Ref. [21]:

$$
\begin{aligned}
M_Z &= 91.1876\text{ GeV}\,, \qquad M_W = 80.385\text{ GeV}\,, \qquad M_t = 173.5\text{ GeV}\,,^{6}\\
G_F &= 1.16637\times10^{-5}\text{ GeV}^{-2}\,, \qquad \alpha^{-1}=137.035999\,, \qquad \alpha_s^{(5)}(M_Z)=0.1184\,. \qquad (26)
\end{aligned}
$$

Furthermore, we take the effective fine-structure constant at the scale of the Z-boson mass to be $\alpha^{-1}(M_Z) = 127.944$. We neglect the masses M_f of all light fermions $f \neq b, t$, since their effects are negligible and do not play any rôle in our considerations.

Through the three-loop order, the QCD relation between the running and pole masses is given by [8]

$$
[m_t(M_t)-M_t]_{\mathrm{QCD}} = M_t\left[-\frac{4}{3}\frac{\alpha_s^{(6)}(M_t)}{\pi}-9.125\left(\frac{\alpha_s^{(6)}(M_t)}{\pi}\right)^2\right.
$$

[6] The values of the top-quark mass quoted by the experimental collaborations correspond to parameters in Monte Carlo event generators in which, apart from parton showering, the partonic subprocesses are calculated at the tree level, so that a rigorous theoretical definition of the top-quark mass is lacking [21,29]. For definiteness, we take the value from Ref. [21] to be the pole mass M_t.

$$- 80.405 \left(\frac{\alpha_s^{(6)}(M_t)}{\pi} \right)^3 \Bigg] . \tag{27}$$

These corrections carry over to the mass difference $m_{Y,t}(M_t) - M_t$ because $\Delta\bar{r}$ in Eq. (14) does not receive pure QCD corrections, as is evident from Eq. (37). Using $\alpha_s^{(6)}(M_t) = 0.1079$ [30], which follows from the value of $\alpha_s^{(5)}(M_Z)$ in Eq. (26) via four-loop evolution and three-loop matching [31], we obtain the numerical result

$$[m_t(M_t) - M_t]_{\rm QCD} = -7.95 \text{ GeV} - 1.87 \text{ GeV} - 0.57 \text{ GeV} = -10.38 \text{ GeV} . \tag{28}$$

Let us now estimate how well the approximations by the gaugeless limit work for the relationships of the $\overline{\rm MS}$ masses $m(\mu)$ and Yukawa couplings $y(\mu)$ to the pole masses M for the bottom and top quarks. Since, at two loops, the exact SM results are not yet available for comparison, we are led to do this at one loop, using the exact results from Ref. [10]. Let us first consider the $\overline{\rm MS}$ masses. In the gaugeless limit, the one-loop corrections are given by

$$\begin{aligned} \frac{m_b(\mu)}{M_b} &= 1 + \frac{\alpha M_t^2}{4\pi M_W^2 S_W^2} \left\{ \left(-\frac{N_c}{H_t} + \frac{3}{8} + \frac{3H_t}{8} \right) \ln \frac{M_t^2}{\mu^2} + \frac{N_c}{H_t} - \frac{5}{16} - \frac{3H_t}{8} \right. \\ &\quad \left. + \frac{3H_t}{8} \ln H_t \right\} + \cdots , \\ \frac{m_t(\mu)}{M_t} &= 1 + \frac{\alpha M_t^2}{4\pi M_W^2 S_W^2} \left\{ \left(-\frac{N_c}{H_t} - \frac{3}{8} + \frac{3H_t}{8} \right) \ln \frac{M_t^2}{\mu^2} + \frac{N_c}{H_t} + 1 - \frac{H_t}{2} \right. \\ &\quad \left. + \frac{H_t^2}{16} \ln H_t - \frac{H_t^2}{8} \left(\frac{4}{H_t} - 1 \right)^{3/2} \arccos \left(\frac{\sqrt{H_t}}{2} \right) \right\} + \cdots , \end{aligned} \tag{29}$$

where H_t is defined below Eq. (17). With the input parameters in Eq. (26), we obtain the following values (in GeV) in the gaugeless limit (g.l.) and the full SM (full):

$$[m_b(\mu) - M_b]_{O(\alpha)} = \begin{cases} -2.64 + 0.56 \times \ln(\mu[\text{GeV}]) & \text{g.l.} \\ -2.75 + 0.59 \times \ln(\mu[\text{GeV}]) & \text{full} \end{cases} , \tag{30}$$

$$[m_t(\mu) - M_t]_{O(\alpha)} = \begin{cases} -123.85 + 26.52 \times \ln(\mu[\text{GeV}]) & \text{g.l.} \\ -118.92 + 25.35 \times \ln(\mu[\text{GeV}]) & \text{full} \end{cases} . \tag{31}$$

Evaluating Eq. (30) at $\mu = M_b$, we find shifts of -1.81 GeV and -1.87 GeV, differing by only 3%. Similarly, Eq. (31) at $\mu = M_t$ yields shifts of 12.91 GeV and 11.78 GeV, with 9% difference. We thus conclude that the gaugeless limit provides a reasonable approximation for the $\overline{\rm MS}$ masses of the bottom and top quarks. This may be partially traced to the fact that the major contributions arise from the top-quark tadpole, which is preserved by the gaugeless limit.

Table 1: The various loop contributions to $m_t(M_t) - M_t$ in GeV.

M_H [GeV]	QCD	$O(\alpha)$	$O(\alpha\alpha_s)$	$O(\alpha^2)$	total
124	-10.38	12.11	-0.39	-0.51	0.83
125	-10.38	11.91	-0.39	-0.49	0.65
126	-10.38	11.71	-0.38	-0.48	0.46

A detailed analysis of the relationship $m_t(M_t) - M_t$, including the contributions of orders $O(\alpha_s^n)$ with $n = 1, 2, 3$ and $O(\alpha\alpha_s^n)$ with $n = 0, 1$, has recently been presented in Ref. [24]. We are now in the position to extend this analysis to order $O(\alpha^2)$. Table 1 corresponds to Table 1 in Ref. [24] with the corresponding column added. For $M_H \approx 125$ GeV, the $O(\alpha^2)$ shift in $m_t(M_t) - M_t$ estimated in the gaugeless limit is about -0.5 GeV, bringing $m_t(M_t)$ even closer to M_t. In view of the above discussion, we expect that this shift will receive a correction of order 10% once the residual $O(\alpha^2)$ terms will become available. The $O(\alpha_s^4)$ QCD correction to $m_t(M_t) - M_t$ was estimated in Ref. [32] to be about 20 MeV. It is plausible to assign a theoretical uncertainty of order 100 MeV, i.e. twice the magnitude of the $O(\alpha^2)$ shift, to the values in the rightmost column in Table 1.

The situation is different for the bottom quark. For $\mu = M_b$, the shifts of orders $O(\alpha)$, $O(\alpha\alpha_s)$, and $O(\alpha^2)$ in $m_b(M_b) - M_b$ read

$$[m_b(M_b) - M_b]_{O(\alpha,\alpha\alpha_s,\alpha^2)} = -1.81 \text{ GeV} - 0.83 \text{ GeV} + 1.55 \text{ GeV}\,, \tag{32}$$

respectively. We observe that the two-loop electroweak correction as estimated in the gaugeless limit is uncomfortably large. Based on the one-loop consideration above, we expect the error due to the lack of knowledge of the residual $O(\alpha^2)$ terms to be of order 5%. The abnormal size of the $O(\alpha^2)$ term in Eq. (32) is related to the tadpole contribution as will become apparent below.

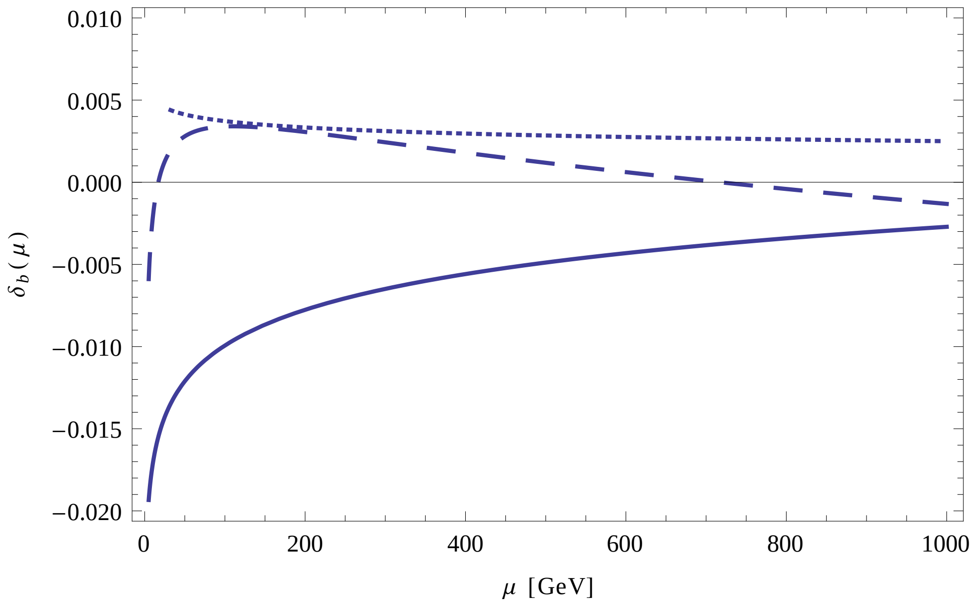
We now turn to the $\overline{\text{MS}}$ Yukawa couplings or, equivalently, the Yukawa masses $m_Y(\mu)$. In Figs. 3(a) and (b), we display the corrections of orders $O(\alpha)$, $O(\alpha\alpha_s)$, and $O(\alpha^2)$ to $m_{Y,b}(\mu)/M_b$ and $m_{Y,t}(\mu)/M_t$ given in Eqs. (16) and (19), respectively, as functions of μ. We observe that, in the case of the bottom quark, the $O(\alpha^2)$ correction exceeds the $O(\alpha\alpha_s)$ one in size over a large μ range, while, in the case of the top quark, the $O(\alpha^2)$ correction is always much smaller than the $O(\alpha\alpha_s)$ one. At the thresholds $\mu = M_b, M_t$, where Eqs. (16) and (19) act as matching conditions for the RG evolution of the $\overline{\text{MS}}$ Yukawa couplings, we have[7]

$$\frac{m_{Y,b}(\mu)}{M_b} = 1 + \delta_{\text{QCD}}(M_b) - 0.0197 - 0.0068 + 0.0057\,, \tag{33}$$

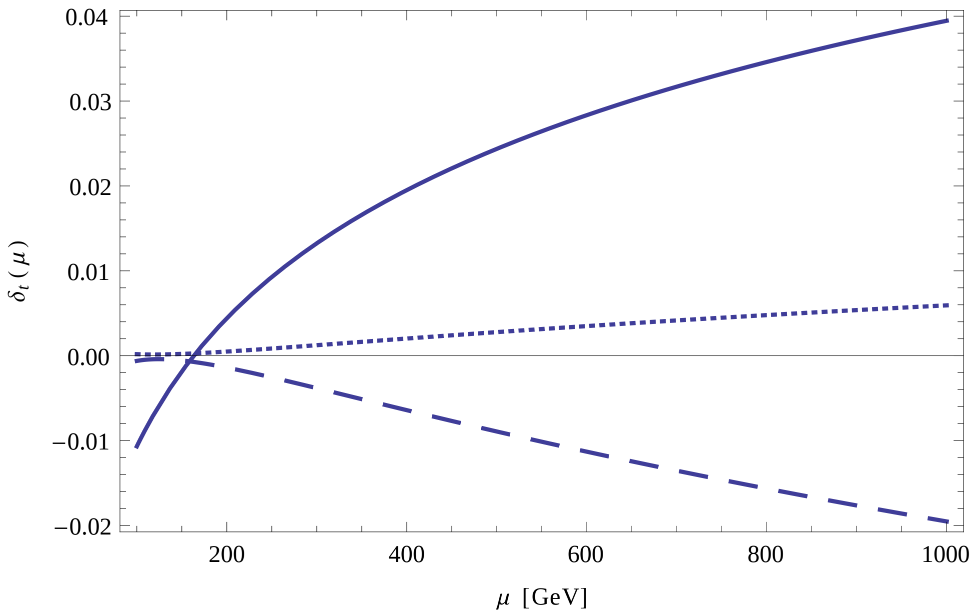
$$\frac{m_{Y,t}(\mu)}{M_t} = 1 + \delta_{\text{QCD}}(M_t) + 0.0013 - 0.0009 + 0.0003\,, \tag{34}$$

where the last three numbers on the right-hand sides represent the $O(\alpha)$, $O(\alpha\alpha_s)$, and $O(\alpha^2)$ corrections. In contrast to $m(\mu)$, $m_Y(\mu)$ is devoid of leading tadpole contributions as per construction in Eq. (14). As a consequence, the electroweak corrections to $m_Y(\mu)/M$ are considerably smaller in size than those to $m(\mu)/M$, as is evident from the comparisons of Eq. (32) with Eq. (33) for the bottom quark and of Table 1 with Eq. (34) for the top quark. By the same token, the gaugeless-limit approximation is expected to be less reliable for $m_Y(\mu)/M$ than for $m(\mu)/M$, in the sense that its relative deviation from the full SM result is larger, while the absolute deviation may be similar. In the remainder of this section, we investigate this issue quantitatively, using again the one-loop results as

[7] In Eq. (2.49) of Ref. [33], a numerical interpolation formula for $y_t(\mu)$ including also the $O(\alpha^2)$ correction may be found. Unfortunately, the $O(\alpha)$ and $O(\alpha^2)$ corrections are not distinguished there. Since their sum is greatly dominated by the $O(\alpha)$ correction, the $O(\alpha^2)$ one cannot be extracted reliably enough to allow for any meaningful comparison with our result in Eq. (19).



(a)



(b)

Figure 3: Corrections of orders $O(\alpha)$ (solid lines), $O(\alpha\alpha_s)$ (long-dashed lines), and $O(\alpha^2)$ (short-dashed lines) to (a) $m_{Y,b}(\mu)/M_b$ and (b) and $m_{Y,t}(\mu)/M_t$ given in Eqs. (16) and (19), respectively, as functions of μ. The $O(\alpha^2)$ corrections are evaluated in the gaugeless-limit approximation. The QCD corrections are not shown.

benchmarks. From the full-SM and gaugeless-limit formulas in Ref. [10], we obtain

$$\left[\frac{m_{Y,b}(\mu)}{M_b}-1\right]_{O(\alpha)} = \begin{cases} -0.0490+0.0097\times\ln(\mu[\mathrm{GeV}]) & \text{g.l.} \\ -0.0244+0.0031\times\ln(\mu[\mathrm{GeV}]) & \text{full} \end{cases}, \tag{35}$$

$$\left[\frac{m_{Y,t}(\mu)}{M_t} - 1\right]_{O(\alpha)} = \begin{cases} -0.141 + 0.029 \times \ln(\mu[\mathrm{GeV}]) & \text{g.l.} \\ -0.111 + 0.022 \times \ln(\mu[\mathrm{GeV}]) & \text{full} \end{cases} . \tag{36}$$

At first sight, the agreement still seems to be good, especially for the top quark. However, evaluating Eq. (35) at $\mu = M_b$, we obtain -0.0346 versus -0.0197, i.e. a relative deviation of more than 70%, and evaluating Eq. (36) at $\mu = M_t$, we obtain 0.0098 versus 0.0013, i.e. a relative deviation of more than 700%. Obviously, the $O(\alpha^2)$ corrections displayed in Fig. 3 have to be taken with a grain of salt. While they allow us to estimate the residual theoretical uncertainties more reliably, their numerical values are only indicative. We conclude that the electroweak perturbative expansions of $m_Y(\mu)/M$ for the bottom and top quarks exhibit useful convergence behaviors, while the $O(\alpha^2)$ terms require more work.

7 Conclusion

In this paper, we calculated the electroweak corrections to the relationships between the $\overline{\mathrm{MS}}$ running masses $m(\mu)$ and Yukawa couplings $y(\mu)$ of the bottom and top quarks and their pole masses M at two loops in the gaugeless limit of the SM.

We verified at one loop that the gaugeless limit provides a reliable approximation for the $\overline{\mathrm{MS}}$ masses $m(\mu)$. In the top-quark case, the new $O(\alpha^2)$ correction induces a shift of about -0.5 GeV in mass difference $m_t(M_t) - M_t$ and reduces its theoretical uncertainty down to the order of 100 MeV. In the bottom-quark case, however, the new $O(\alpha^2)$ correction is nearly as large in size as the $O(\alpha)$ one, indicating that the convergence property of the perturbative expansion is jeopardized by the tadpole contributions.

Detailed inspection revealed that the bulk of the corrections to the mass difference $m(M) - M$ originates from the tadpole contributions, which are shared by the quantity $\Delta\bar{r}(\mu)$ defined in Eq. (9). Owing to cancellations on the right-hand side of Eq. (14), the relationship between $y(\mu)$ and M is devoid of leading tadpole contributions, which motivated our proposition to measure the running of the heavy-quark masses by using $y(\mu)$ after appropriate rescaling, as in Eq. (13). We called this new mass parameter Yukawa mass $m_Y(\mu)$. In fact, the electroweak corrections to the mass difference $m_Y(M) - M$ are typically one order of magnitude smaller than those to $m(M) - M$. While this considerably improves the convergence properties of the electroweak perturbation expansions, especially for the bottom quark, it greatly reduces the usefulness of the gaugeless-limit approximation, as we demonstrated at one loop. In the bottom-quark case, we found the $O(\alpha^2)$ correction thus estimated to be small against the $O(\alpha)$ one, but comparable to the $O(\alpha\alpha_s)$ one. In the top-quark case, the hierarchy among the $O(\alpha)$, $O(\alpha\alpha_s)$, and $O(\alpha^2)$ corrections turned out as naïvely expected.

Acknowledgements

The authors thank M.Yu. Kalmykov for fruitful discussions. This work was supported in part by the German Federal Ministry for Education and Research BMBF through Grant No. 05H12GUE and by the German Research Foundation DFG through the Collaborative Research Centre No. 676 *Particles, Strings and the Early Universe: the Structure of Matter and Space-Time.*

A $\Delta\bar{r}(\mu)$ in the $\overline{\rm MS}$ scheme

In this appendix, we present the electroweak parameter $\Delta\bar{r}$ appearing in Eq. (9) through two loops in the SM. In the following, all masses and couplings are the running quantities defined in $\overline{\rm MS}$ scheme. To simplify the notation, the argument μ is always omitted. We thus need to specify the coefficients $x_{1,0}$, $x_{1,1}$, and $x_{2,0}$ in the perturbative expansion

$$\Delta\bar{r} = \frac{g^2}{16\pi^2}\,x_{1,0} + C_F\frac{g_s^2}{16\pi^2}\frac{g^2}{16\pi^2}\,x_{1,1} + \left(\frac{g^2}{16\pi^2}\right)^2 x_{2,0} + \cdots\,, \tag{37}$$

where $g = \bar{e}/s_w$ and g_s are the electroweak and strong-coupling constants in the $\overline{\rm MS}$ scheme.

The coefficients $x_{1,0}$ and $x_{1,1}$ may be given in exact form:

$$\begin{aligned}
x_{1,0} &= N_c\frac{2m_t^4}{m_W^2m_H^2}(1-l_t) - N_c\frac{m_t^2}{m_W^2}\Big(\frac{1}{4}-\frac{1}{2}l_t\Big) + \frac{3}{4}\frac{m_W^2}{m_H^2-m_W^2}\ln\frac{m_W^2}{m_H^2}\\
&\quad +\frac{m_H^2}{m_W^2}\Big(-\frac{7}{8}+\frac{3}{4}l_h\Big) + \frac{m_W^2}{m_H^2}\Big(-1-\frac{1}{2c_W^4}+3l_w+\frac{3}{2c_w^4}l_h\Big)\\
&\quad +\frac{5}{4}+\frac{5}{8c_W^2}-\frac{3}{4s_W^2}\ln c_W^2-\frac{3}{4}l_h-\frac{3}{4}l_w-\frac{3}{4c_W^2}l_z\,,\\
x_{1,1} &= N_c\frac{m_t^4}{m_W^2m_H^2}(20-20l_t+12l_t^2) + N_c\frac{m_t^2}{m_W^2}\Big(-\frac{13}{8}+\zeta_2+l_t-\frac{3}{2}l_t^2\Big)\,,
\end{aligned} \tag{38}$$

where, in analogy with the definitions introduced below Eq. (16), $l_x = \ln(m_x^2/\mu^2)$, $c_w = m_W/m_Z$, and $s_w = \sqrt{1-c_w^2}$.

The two-loop electroweak correction $x_{2,0}$ in the full SM with arbitrary particle masses may be expressed in terms of dilogarithms. However the result is too lengthy to be presented here. Instead, we evaluate $x_{2,0}$ in two different approximations: the gaugeless limit of the SM, $x_{2,0}^{\rm gl}$, and the heavy-top-quark limit, $x_{2,0}^{\rm ht}$. In the former case, we have

$$\begin{aligned}
x_{2,0}^{\rm gl} &= \frac{4N_c^2m_t^8}{m_H^4m_W^4}(1-l_t) + \frac{N_cm_t^6}{m_H^2m_W^4}\bigg\{\frac{N_c}{2}(1-l_t)(-3+2l_t) - \frac{45}{32}l_t^2 + \frac{3}{32}l_h^2\\
&\quad -\frac{3}{16}l_tl_h + \frac{13}{2}l_t - \frac{5}{8}\zeta_2 - \frac{15}{2} - \frac{3}{16}\mathcal{H}(h_t) + \frac{9}{2}\Phi(h_t/4)\bigg\}\\
&\quad +N_c\frac{m_t^4}{m_W^4}\bigg\{\Big(\frac{141}{64}-\frac{3}{16}h_t-\frac{3}{64}h_t^2+\frac{N_c}{4}\Big)l_t^2 + \Big(-\frac{15}{64}+\frac{3}{16}h_t-\frac{3}{64}h_t^2\Big)l_h^2\\
&\quad +\Big(-\frac{81}{32}+\frac{3}{4}h_t+\frac{3}{32}h_t^2\Big)l_tl_h + \Big(-\frac{5}{4}-\frac{11}{16}h_t-\frac{N_c}{4}\Big)l_t\\
&\quad +\Big(\frac{43}{8}-\frac{25}{16}h_t\Big)l_h + \frac{N_c}{16} + \frac{7}{16}\zeta_2 - \frac{295}{64} + \frac{17}{8}h_t\\
&\quad +\Big(\frac{15}{32}-\frac{3}{8}h_t+\frac{3}{32}h_t^2\Big)\mathcal{H}(h_t) + \Big(-\frac{15}{8}-\frac{3}{16}h_t+\frac{3}{32}h_t^2\Big)\Phi(h_t/4)\bigg\}\\
&\quad +\frac{m_H^4}{m_W^4}\bigg\{\frac{27}{32}l_h^2 - \frac{27}{8}l_h + \frac{1}{4}\zeta_2 - \frac{243}{32}S_2 + \frac{457}{128}\bigg\}\,,
\end{aligned} \tag{39}$$

where $h_t = m_H^2/m_t^2$. In the latter case, we obtain

$$x_{2,0}^{\rm ht} = N_c\bigg\{N_c\frac{4m_t^8}{m_W^4m_H^2}(1-l_t) + \frac{m_t^6}{m_W^4m_H^2}\bigg(-\frac{15}{2}-\zeta_2+\frac{13}{2}l_t-\frac{3}{2}l_t^2$$

$$+\frac{N_c}{2}(1-l_t)(-3+2l_t)\Big) + \frac{m_t^4}{(m_H^2-m_W^2)^2}\Big(\frac{3}{4}l_h^2 - \frac{3}{4}l_w^2 + 3l_{wh}l_t\Big)$$
$$+\frac{m_t^4}{m_W^2(m_H^2-m_W^2)}\Big(\frac{3}{2}l_w(1-l_t) + \frac{3}{2}(l_t-l_h) + \frac{3}{8}(l_t+l_h)^2\Big)$$
$$+\frac{m_t^4}{m_H^4}\Big(-8 - \frac{4}{c_W^4} + 6l_w + 6l_t + \frac{3}{c_W^4}(l_t+l_z)\Big)$$
$$+\frac{m_t^4}{m_W^2 m_H^2}\left(\frac{4}{3}\frac{s_W^2}{c_W^2}l_t^2 + (1-l_t)\Big(\frac{59}{18} + \frac{179}{36c_W^2} - \frac{3}{2s_W^2}\ln c_W^2 - \frac{3}{2c_W^2}l_z - 3l_w\Big)\right)$$
$$+\frac{m_t^4}{m_W^4}\left(N_c\Big(\frac{1}{16} - \frac{1}{4}l_t(1-l_t)\Big) + \frac{305}{64} + \frac{11}{8}\zeta_2 + \frac{1}{2}l_h + \frac{9}{8}l_h^2\right.$$
$$\left.-\frac{21}{4}l_hl_t + \frac{29}{8}l_t + \frac{57}{16}l_t^2\right)\Bigg\} + O(m_t^2)\,. \tag{40}$$

In Eq. (39), two new functions have been introduced. They correspond to two-loop vacuum bubble integrals with two different masses, namely $J(0, m_1^2, m_2^2)$ and $J(m_1^2, m_2^2, m_2^2)$. Adopting the notations from Ref. [34] with $z = m_1^2/m_2^2$ and from Ref. [35] with $z = m_1^2/(4m_2^2)$, they are given by

$$\mathcal{H}(z) = 2\mathrm{Li}_2(1-z) + \frac{1}{2}\ln^2 z\,, \tag{41}$$
$$\Phi(z) = 4\sqrt{\frac{z}{1-z}}\mathrm{Cl}_2(2\arcsin\sqrt{z}), \tag{42}$$

respectively, where

$$\mathrm{Cl}_2(\theta) = -\int_0^\theta d\theta' \ln\left(2\sin\frac{\theta'}{2}\right) \tag{43}$$

is Clausen's integral [36].

B Renormalization constants of the quark masses

The calculations presented in this paper require the renormalizations of the bottom- and top-quark masses through two loops and of all other masses and couplings at one loop. For our purposes, knowledge of the corresponding renormalization constants in the approximations of the gaugeless and heavy-top-quark limits is sufficient. However, we also calculated the mass renormalization constants Z_m of the bottom and top quarks exactly at two loops in the SM. For future applications and checks, we present our results in the following. In the $\overline{\mathrm{MS}}$ scheme, Z_m is defined as

$$Z_m = \left[\frac{m^0}{m(\mu)}\right]^2, \tag{44}$$

and may be written in the form

$$Z_m = 1 + \frac{g^2}{16\pi^2}\frac{Z_\alpha^{(1,1)}}{\varepsilon} + C_F\frac{g_s^2}{16\pi^2}\frac{Z_{\alpha_s}^{(1,1)}}{\varepsilon}$$
$$+C_F\frac{g^2}{16\pi^2}\frac{g_s^2}{16\pi^2}\left(\frac{Z_{\alpha\alpha_s}^{(2,2)}}{\varepsilon^2} + \frac{Z_{\alpha\alpha_s}^{(2,1)}}{\varepsilon}\right) + \left(\frac{g^2}{16\pi^2}\right)^2\left(\frac{Z_{\alpha^2}^{(2,2)}}{\varepsilon^2} + \frac{Z_{\alpha^2}^{(2,1)}}{\varepsilon}\right) + \cdots\,. \tag{45}$$

Using the same notation as in Appendix A, we have

$$
\begin{aligned}
Z_{\alpha}^{(1,1)} &= -\frac{1}{3}-\frac{3}{4}\frac{m_H^2}{m_W^2}-\frac{3}{4}\frac{m_t^2}{m_W^2}-3\frac{m_W^2}{m_H^2}+\frac{1}{3}\frac{m_Z^2}{m_W^2}-\frac{3}{2}\frac{m_Z^4}{m_H^2m_W^2}+2N_c\frac{m_t^4}{m_H^2m_W^2}\,, &(46)\\
Z_{\alpha_s}^{(1,1)} &= -3\,, &(47)\\
Z_{\alpha\alpha_s}^{(2,2)} &= 2+\frac{9}{2}\frac{m_H^2}{m_W^2}+\frac{27}{4}\frac{m_t^2}{m_W^2}+18\frac{m_W^2}{m_H^2}-2\frac{m_Z^2}{m_W^2}+9\frac{m_Z^4}{m_H^2m_W^2}-24N_c\frac{m_t^4}{m_H^2m_W^2}\,, &(48)\\
Z_{\alpha\alpha_s}^{(2,1)} &= \frac{25}{12}-3\frac{m_t^2}{m_W^2}+\frac{31}{24}\frac{m_Z^2}{m_W^2}+4N_c\frac{m_t^4}{m_H^2m_W^2}\,, &(49)\\
Z_{\alpha^2}^{(2,2)} &= \frac{83}{24}+\frac{9}{16}\frac{m_H^2m_t^2}{m_W^4}-\frac{9}{32}\frac{m_H^4}{m_W^4}+\frac{9}{32}\frac{m_Z^2m_t^2}{m_W^4}+\frac{5}{16}\frac{m_Z^2m_H^2}{m_W^4}+\frac{9}{8}\frac{m_Z^4}{m_W^4}\frac{m_t^2}{m_H^2}\\
&\quad+\frac{85}{48}\frac{m_Z^4}{m_W^4}-\frac{15}{8}\frac{m_Z^6}{m_W^4m_H^2}+\frac{9}{4}\frac{m_Z^8}{m_W^4m_H^4}+\frac{9}{16}\frac{m_t^2}{m_W^2}+\frac{11}{8}\frac{m_H^2}{m_W^2}-\frac{1}{6}\frac{m_Z^2}{m_W^2}\\
&\quad-\frac{5}{4}\frac{m_Z^4}{m_W^2m_H^2}+\frac{9}{4}\frac{m_t^2}{m_H^2}+\frac{29}{4}\frac{m_Z^2}{m_H^2}+9\frac{m_Z^4}{m_H^4}+18\frac{m_W^2}{m_H^2}+9\frac{m_W^4}{m_H^4}\\
&\quad+n_G\left(\frac{1}{6}+\frac{1}{6}\frac{m_Z^4}{m_W^4}-\frac{3}{2}\frac{m_Z^6}{m_W^4m_H^2}-\frac{1}{3}\frac{m_Z^2}{m_W^2}+3\frac{m_Z^4}{m_W^2m_H^2}-2\frac{m_Z^2}{m_H^2}-\frac{m_W^2}{m_H^2}\right)\\
&\quad+N_c\left(-\frac{39}{16}\frac{m_t^4}{m_W^4}-\frac{3}{8}\frac{m_H^2m_t^2}{m_W^4}-\frac{2}{3}\frac{m_Z^2m_t^4}{m_W^4m_H^2}-6\frac{m_Z^4m_t^4}{m_W^4m_H^4}+\frac{3}{4}\frac{m_Z^4m_t^2}{m_W^4m_H^2}\right.\\
&\quad\left.+\frac{2}{3}\frac{m_t^4}{m_W^2m_H^2}-12\frac{m_t^4}{m_H^4}+\frac{3}{2}\frac{m_t^2}{m_H^2}\right)+N_cn_G\left(\frac{11}{162}+\frac{11}{162}\frac{m_Z^4}{m_W^4}-\frac{11}{18}\frac{m_Z^6}{m_W^4m_H^2}\right.\\
&\quad\left.-\frac{11}{81}\frac{m_Z^2}{m_W^2}+\frac{11}{9}\frac{m_Z^4}{m_W^2m_H^2}-\frac{10}{9}\frac{m_Z^2}{m_H^2}-\frac{m_W^2}{m_H^2}\right)+4N_c^2\frac{m_t^8}{m_W^4m_H^4}\,, &(50)\\
Z_{\alpha^2}^{(2,1)} &= \frac{103}{144}+\frac{11}{32}\frac{m_t^4}{m_W^4}+\frac{33}{32}\frac{m_H^4}{m_W^4}-\frac{79}{192}\frac{m_Z^2m_t^2}{m_W^4}-\frac{3}{8}\frac{m_Z^2m_H^2}{m_W^4}-\frac{1019}{576}\frac{m_Z^4}{m_W^4}+\frac{59}{24}\frac{m_Z^6}{m_W^4m_t^2}\\
&\quad+\frac{53}{96}\frac{m_t^2}{m_W^2}-\frac{3}{4}\frac{m_H^2}{m_W^2}+\frac{4}{9}\frac{m_Z^2}{m_W^2}+\frac{31}{12}\frac{m_Z^4}{m_W^2m_H^2}-\frac{17}{2}\frac{m_Z^2}{m_H^2}-\frac{176}{3}\frac{m_W^2}{m_H^2}\\
&\quad+n_G\left(-\frac{37}{72}-\frac{47}{144}\frac{m_Z^4}{m_W^4}+2\frac{m_Z^6}{m_W^4m_H^2}+\frac{47}{72}\frac{m_Z^2}{m_W^2}-4\frac{m_Z^4}{m_W^2m_H^2}+\frac{8}{3}\frac{m_Z^2}{m_H^2}+\frac{4}{3}\frac{m_W^2}{m_H^2}\right)\\
&\quad+N_c\left(-\frac{5}{2}\frac{m_t^6}{m_W^4m_H^2}-\frac{11}{32}\frac{m_t^4}{m_W^4}+\frac{3}{8}\frac{m_H^2m_t^2}{m_W^4}+\frac{4}{9}\frac{m_Z^2m_t^4}{m_W^4m_H^2}+\frac{19}{12}\frac{m_Z^4}{m_W^4}\frac{m_t^2}{m_H^2}\right.\\
&\quad\left.-\frac{4}{9}\frac{m_t^4}{m_W^2m_H^2}-\frac{20}{3}\frac{m_Z^2m_t^2}{m_W^2m_H^2}+\frac{35}{6}\frac{m_t^2}{m_H^2}\right)+N_cn_G\left(-\frac{623}{1944}-\frac{517}{3888}\frac{m_Z^4}{m_W^4}\right.\\
&\quad\left.+\frac{22}{27}\frac{m_Z^6}{m_W^4m_H^2}+\frac{517}{1944}\frac{m_Z^2}{m_W^2}-\frac{44}{27}\frac{m_Z^4}{m_W^2m_H^2}+\frac{40}{27}\frac{m_Z^2}{m_H^2}+\frac{4}{3}\frac{m_W^2}{m_H^2}\right), &(51)
\end{aligned}
$$

for the bottom quark and

$$
\begin{aligned}
Z_{\alpha}^{(1,1)} &= \frac{2}{3}-\frac{3}{4}\frac{m_H^2}{m_W^2}+\frac{3}{4}\frac{m_t^2}{m_W^2}-3\frac{m_W^2}{m_H^2}-\frac{2}{3}\frac{m_Z^2}{m_W^2}-\frac{3}{2}\frac{m_Z^4}{m_H^2m_W^2}+2N_c\frac{m_t^4}{m_H^2m_W^2}\,, &(52)\\
Z_{\alpha_s}^{(1,1)} &= -3\,, &(53)\\
Z_{\alpha\alpha_s}^{(2,2)} &= -4+18\frac{m_W^2}{m_H^2}+4\frac{m_Z^2}{m_W^2}+9\frac{m_Z^4}{m_H^2m_W^2}+\frac{9}{2}\frac{m_H^2}{m_W^2}-\frac{27}{4}\frac{m_t^2}{m_W^2}-24N_c\frac{m_t^4}{m_H^2m_W^2}\,, &(54)
\end{aligned}
$$

$$Z^{(2,1)}_{\alpha\alpha_s} = \frac{31}{12} + 3\frac{m_t^2}{m_W^2} + \frac{19}{24}\frac{m_Z^2}{m_W^2} + 4N_c\frac{m_t^4}{m_H^2 m_W^2}\,, \tag{55}$$

$$\begin{aligned}
Z^{(2,2)}_{\alpha^2} = {} & \frac{85}{24} + \frac{9}{16}\frac{m_t^4}{m_W^4} - \frac{9}{16}\frac{m_H^2 m_t^2}{m_W^4} - \frac{9}{32}\frac{m_H^4}{m_W^4} - \frac{33}{32}\frac{m_t^2 m_Z^2}{m_W^4} + \frac{17}{16}\frac{m_Z^2 m_H^2}{m_W^4} - \frac{9}{8}\frac{m_Z^2 m_t^4}{m_W^4 m_H^2} \\
& + \frac{89}{48}\frac{m_Z^4}{m_W^4} - \frac{3}{8}\frac{m_Z^6}{m_W^4 m_H^2} + \frac{9}{4}\frac{m_Z^8}{m_W^4 m_H^4} + \frac{3}{16}\frac{m_t^2}{m_W^2} + \frac{5}{8}\frac{m_H^2}{m_W^2} \\
& - \frac{1}{3}\frac{m_Z^2}{m_W^2} - \frac{11}{4}\frac{m_Z^4}{m_W^2 m_H^2} - \frac{9}{4}\frac{m_t^2}{m_H^2} + \frac{41}{4}\frac{m_Z^2}{m_H^2} + 9\frac{m_Z^4}{m_H^4} + 15\frac{m_W^2}{m_H^2} + 9\frac{m_W^4}{m_H^4} \\
& + n_G\left(-\frac{1}{3} - \frac{1}{3}\frac{m_Z^4}{m_W^4} - \frac{3}{2}\frac{m_Z^6}{m_W^4 m_H^2} + \frac{2}{3}\frac{m_Z^2}{m_W^2} + 3\frac{m_Z^4}{m_W^2 m_H^2} - 2\frac{m_Z^2}{m_H^2} - \frac{m_W^2}{m_H^2}\right) \\
& + N_c\left(3\frac{m_t^6}{m_W^4 m_H^2} - \frac{33}{16}\frac{m_t^4}{m_W^4} - \frac{3}{8}\frac{m_H^2 m_t^2}{m_W^2} - \frac{8}{3}\frac{m_Z^2 m_t^4}{m_W^4 m_H^2} - 6\frac{m_Z^4 m_t^4}{m_W^4 m_H^4} + \frac{3}{4}\frac{m_Z^4 m_t^2}{m_W^4 m_H^2} \right. \\
& \left. + \frac{8}{3}\frac{m_t^4}{m_W^2 m_H^2} - 12\frac{m_t^4}{m_H^4} + \frac{3}{2}\frac{m_t^2}{m_H^2}\right) + N_c n_G\left(-\frac{11}{81} - \frac{11}{81}\frac{m_Z^4}{m_W^4} - \frac{11}{18}\frac{m_Z^6}{m_W^4 m_H^2} \right. \\
& \left. + \frac{22}{81}\frac{m_Z^2}{m_W^2} + \frac{11}{9}\frac{m_Z^4}{m_W^2 m_H^2} - \frac{10}{9}\frac{m_Z^2}{m_H^2} - \frac{m_W^2}{m_H^2}\right) + 4N_c^2\frac{m_t^8}{m_W^4 m_H^4}\,,
\end{aligned} \tag{56}$$

$$\begin{aligned}
Z^{(2,1)}_{\alpha^2} = {} & +\frac{11}{48} + \frac{3}{16}\frac{m_t^4}{m_W^4} - \frac{3}{8}\frac{m_t^2 m_H^2}{m_W^4} + \frac{33}{32}\frac{m_H^4}{m_W^4} + \frac{223}{192}\frac{m_Z^2 m_t^2}{m_W^4} - \frac{3}{8}\frac{m_Z^2 m_H^2}{m_W^4} - \frac{289}{192}\frac{m_Z^4}{m_W^4} \\
& + \frac{59}{24}\frac{m_Z^6}{m_W^4 m_H^2} + \frac{91}{96}\frac{m_t^2}{m_W^2} - \frac{3}{4}\frac{m_H^2}{m_W^2} + \frac{2}{3}\frac{m_Z^2}{m_W^2} + \frac{31}{12}\frac{m_Z^4}{m_W^2 m_H^2} - \frac{17}{2}\frac{m_Z^2}{m_H^2} - \frac{176}{3}\frac{m_W^2}{m_H^2} \\
& + n_G\left(-\frac{7}{72} + \frac{13}{144}\frac{m_Z^4}{m_W^4} + 2\frac{m_Z^6}{m_W^4 m_H^2} - \frac{13}{72}\frac{m_Z^2}{m_W^2} - 4\frac{m_Z^4}{m_W^2 m_H^2} + \frac{8}{3}\frac{m_Z^2}{m_H^2} + \frac{4}{3}\frac{m_W^2}{m_H^2}\right) \\
& + N_c\left(-\frac{5}{2}\frac{m_t^6}{m_W^4 m_H^2} - \frac{25}{32}\frac{m_t^4}{m_W^4} + \frac{3}{8}\frac{m_H^2 m_t^2}{m_W^4} + \frac{4}{9}\frac{m_Z^2 m_t^4}{m_W^4 m_H^2} + \frac{19}{12}\frac{m_Z^4 m_t^2}{m_W^4 m_H^2} \right. \\
& \left. - \frac{4}{9}\frac{m_t^4}{m_W^2 m_H^2} - \frac{20}{3}\frac{m_Z^2 m_t^2}{m_W^2 m_H^2} + \frac{35}{6}\frac{m_t^2}{m_H^2}\right) + N_c n_G\left(-\frac{293}{1944} + \frac{143}{3888}\frac{m_Z^4}{m_W^4} \right. \\
& \left. + \frac{22}{27}\frac{m_Z^6}{m_W^4 m_H^2} - \frac{143}{1944}\frac{m_Z^2}{m_W^2} - \frac{44}{27}\frac{m_Z^4}{m_W^2 m_H^2} + \frac{40}{27}\frac{m_Z^2}{m_H^2} + \frac{4}{3}\frac{m_W^2}{m_H^2}\right),
\end{aligned} \tag{57}$$

for the top quark. Here, all the masses and couplings are defined in the $\overline{\rm MS}$ scheme scheme at renormalization scale μ, and $n_G = 3$ is the number of fermion generations. Equations (54) and (55) agree with Ref. [11].

C $\overline{\rm MS}$ mass of the bottom quark

As anticipated at the end of Section 4, we present here a closed expression for the ratio $m_b(\mu)/M_b$, in terms of on-shell renormalized parameters. At the two-loop level, we use the large-mass expansion with respect to the top-quark mass. Using the notation introduced in Section 4, the result reads

$$\frac{m_b(\mu)}{M_b} = 1 + \delta_{\rm QCD}(\mu) + \frac{\alpha}{4\pi}\left\{ N_c\frac{M_t^4}{M_W^2 M_H^2 S_W^2}\left(1 - L_t\right)\right.$$

$$
\begin{aligned}
&+\frac{M_t^2}{M_W^2 S_W^2}\Big(-\frac{5}{16}+\frac{3}{8}L_t\Big)+\frac{M_t^2 M_W^2}{(M_t^2-M_W^2)^2 S_W^2}\Big(-\frac{3}{8}L_{tw}\Big)+\frac{3}{8}\frac{M_W^2}{(M_t^2-M_W^2)S_W^2}\\
&+a_v^2\frac{M_Z^2}{M_H^2}\Big(-4+12L_z\Big)+\frac{M_H^2}{M_W^2 S_W^2}\Big(-\frac{3}{8}-\frac{3}{8}L_h\Big)+\frac{M_W^2}{M_H^2 S_W^2}\Big(-\frac{1}{2}+\frac{3}{2}L_w\Big)\\
&+Q_b^2\Big(-4+3L_b\Big)+v_b^2\Big(-\frac{5}{2}+3L_z\Big)+a_v^2\Big(-\frac{1}{2}-3L_z\Big)\\
&+\frac{M_b^2}{M_W^2 S_W^2}\Bigg(\frac{11}{48}+\frac{1}{4}L_b-\frac{1}{8}L_t-\frac{1}{8}L_z-\frac{3}{8}L_h\\
&+v_b^2 S_W^2 C_W^2\Big(-\frac{8}{3}-4L_b+4L_z\Big)+O\left(\frac{M_b^4}{M_W^4}\right)\Bigg)\Bigg\}\\
&+C_F\frac{\alpha_s(\mu)}{4\pi}\frac{\alpha}{4\pi}\Bigg\{N_c\frac{M_t^4}{M_W^2 M_H^2 S_W^2}\Big(-2+16L_t+3L_b-3L_bL_t-6L_t^2\Big)\\
&+\frac{M_t^2}{M_W^2 S_W^2}\Big(-\frac{13}{4}-\frac{3}{2}L_t-\frac{15}{16}L_b+\frac{9}{8}L_bL_t+\frac{9}{8}L_t^2\Big)\\
&+\frac{1}{S_W^2}\frac{M_t^2 M_W^2}{(M_t^2-M_W^2)^2}L_{tw}\Big(\frac{33}{8}-\frac{9}{8}L_b\Big)+\frac{1}{S_W^2}\frac{M_W^2}{M_t^2-M_W^2}\Big(-\frac{51}{8}+\frac{9}{8}L_b+\frac{3}{4}L_{tw}\Big)\\
&+\frac{1}{S_W^2}\Big(-\frac{63}{16}+\frac{3}{4}L_t-3L_w\Big)+\frac{1}{S_W^2}\Bigg(3\frac{M_t^2}{M_W^2}+3+\frac{21}{4}\frac{M_W^2}{M_t^2-M_W^2}\\
&+\frac{9}{4}\frac{M_W^4}{(M_t^2-M_W^2)^2}\Bigg)\mathrm{Li}_2\Big(1-\frac{M_W^2}{M_t^2}\Big)+\frac{M_H^2}{M_W^2 S_W^2}(1-L_h)\Big(\frac{3}{2}-\frac{9}{8}L_b\Big)\\
&+a_b^2\frac{M_Z}{M_H}(1-3L_z)\Big(16-12L_b\Big)+\frac{M_W^2}{M_H^2 S_W^2}(1-3L_w)\Big(2-\frac{3}{2}L_b\Big)\\
&+Q_b^2\Big(\frac{7}{4}-24\zeta_3-60\zeta_2+96\zeta_2\log 2-21L_b+9L_b^2\Big)\\
&+v_b^2\Big(\frac{23}{4}-\frac{15}{2}L_b-9L_z+9L_bL_z\Big)+a_b^2\Big(\frac{55}{4}-\frac{3}{2}L_b-9L_z-9L_bL_z\Big)\Bigg\}\\
&+\left(\frac{\alpha}{4\pi}\right)^2\Bigg\{\left(N_c\frac{M_t^4}{M_W^2 M_H^2 S_W^2}\right)^2\frac{(1-L_t)(3+L_t)}{2}\\
&-\frac{N_c M_t^6}{16M_W^4 M_H^2 S_W^4}(77-75L_t+18L_t^2+8\zeta_2)+\frac{N_c M_t^4}{M_H^4}\Bigg(\frac{-7+5L_t+3L_z+3L_tL_z}{4C_W^4}\\
&+\frac{-7+5L_t+3L_z+3L_tL_z}{2C_W^2}+\frac{3(-7+5L_t+2L_w+2L_tL_w+L_z+L_tL_z)}{4S_W^4}\\
&+\frac{-7+5L_t+3L_z+3L_tL_z}{2S_W^2}\Bigg)\\
&+\frac{N_c M_t^4}{M_W^2 M_H^2}\Bigg(\frac{67-67L_t+32L_t^2-8L_z+8L_tL_z}{48C_W^2}+\frac{3(-1+L_t)}{16S_W^4})\\
&-\frac{-59-16L_b+59L_t+16L_bL_t-32L_t^2+24L_z-24L_tL_z}{48S_W^2}\Bigg)\\
&+\frac{N_c M_t^4}{M_W^4}\frac{469-16L_h+136L_t-144L_hL_t+156L_t^2+8\zeta_2}{128M_W^4 S_W^4}
\end{aligned}
$$

$$+\frac{M_t^4}{M_W^4 S_W^4}\left(-\frac{29}{512}-\frac{53}{128}L_t+\frac{27}{128}L_t^2+\frac{1}{4}\zeta_2\right)+O(M_t^2)\Bigg\}\,. \tag{58}$$

Expanding the $O(\alpha\alpha_s)$ term of Eq. (58) in powers of M_W^2/M_t^2, we find agreement with Ref. [12].

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