

# Torus partition functions and spectra of gauged linear sigma models

Stefan Groot Nibbelink

*Arnold-Sommerfeld-Center for Theoretical Physics, Fakultät für Physik,  
Ludwig-Maximilians-Universität München, Theresienstraße 37, 80333 München, Germany*

Fabian Ruehle

*Deutsches Elektronen-Synchrotron DESY, Notkestrasse 85, 22607 Hamburg, Germany*

Worldsheet (0,2) gauged linear sigma models are often used to study supersymmetric heterotic string compactifications with non-trivial vector bundles. We make use of supersymmetric localization techniques to determine their one-loop partition functions. In particular we derive conditions which ensure that the full partition function is modular invariant and we propose a method to determine the massless and massive target space matter spectrum.

## INTRODUCTION

Ever since its introduction, string theory has inspired interesting progress, ranging from abstract mathematical insights to concrete phenomenological motivations for cosmology and particle physics of the standard model (SM) and beyond. String theory may be defined as a 2D conformal field theory (CFT) on the string worldsheet (WS) that maps into a target space, which is assumed to be 4D Minkowski space times a compact geometry (we focus on the heterotic string [1]). As this leads to complicated non-linear sigma models, string phenomenology is often studied in its supergravity limit, compactified on special holonomy manifolds (e.g. Calabi-Yau (CY)) with vector bundles [2], yielding supersymmetric grand unified or SM gauge groups and matter spectra.

As supergravity only constitutes the low energy effective field theory target space description of string theory, important WS quantum effects (e.g. Wilson line consistency conditions [3]) may be missed. Therefore, it is not obvious that every supergravity background has a lift to a full string theory. Due to their implicit definition (e.g. the metrics are unknown), descriptions of CYs with vector bundles mostly rely on topological methods. This makes even computing the massless matter spectra a challenging task. Such complications are avoided by using orbifolds which admit exact CFT descriptions [4, 5]. However, these geometries only account for special points in the moduli spaces of a minor set of all known CYs.

Gauged linear sigma models (GLSMs) [6, 7] provide an approximate WS description for CY compactifications. Albeit not CFTs by themselves, GLSMs are commonly assumed to flow to genuine CFTs in the infrared [8]. Given that these are again interacting WS theories, exact string results seem once more out of reach. However, using localization techniques [9] it is possible to determine their torus partition functions and the elliptic genera [10–13]. (Similarly, sphere partition functions [14, 15] are related to exact moduli Kähler potentials [16, 17].)

We use and extend these results to derive one-loop

modular invariance conditions necessary to ensure a string lift of the supergravity theory. Moreover, we propose a method to determine not only the massless but the full target space spectrum. As such our results may be relevant for applications of string theory to physics beyond the SM. A detailed exposition of the derivation of our results with applications will be provided elsewhere.

## (0,2) GAUGED LINEAR SIGMA MODELS

A GLSM is characterized in terms of matter superfields coupled to gauge multiplets [6, 7]. In the (0,2) superspace formalism these superfields are functions of the WS coordinates  $\sigma, \bar{\sigma}$  and the Grassmann variables  $\theta^+, \bar{\theta}^+$ . The minimal characterization of these superfields is given by their gauge, global left-moving non-R- and right-moving R-symmetry charges. The normalization of the R-charge is chosen such that  $\theta^+$  has R-charge 1. We consider the following (0,2) matter superfields

Super-field	Chiral		Fermi	
	$\mathcal{Z}^a, z^a$	$\psi^a$	$\Lambda^\alpha, \lambda^\alpha$	$f^\alpha$
$U(1)$	$q_a$	$q_a$	$Q_\alpha$	$Q_\alpha$
$U(1)_R$	$r_a$	$r_a - 1$	$R_\alpha$	$R_\alpha - 1$
$U(1)_L$	$l_a$	$l_a$	$L_\alpha$	$L_\alpha$

where  $a = 1, \dots, d$  and  $\alpha = 1, \dots, D$  label the chiral and Fermi superfields, respectively. In addition, we introduce the following bosonic and fermionic gauge multiplets

Super-field	Bos. gauge			Fermi gauge	
	$\mathcal{V}, \mathcal{A}, \mathbf{a}$	$\chi$	$D$	$\Sigma, \varphi$	$\mathfrak{s}$
$U(1)_R$	0	-1	0	0	-1
$U(1)_L$	0	0	0	-1	-1

where  $\mathbf{a}$  is a gauge field one-form. The fields  $f^\alpha$  and  $D$  are auxiliary. The fermionic gauge transformations read

$$\delta \Lambda^\alpha = \Xi N^\alpha(\mathcal{Z}), \quad \delta \Sigma = \Xi, \quad (1)$$

with the Fermi superfield  $\Xi$  as gauge parameter. In the following, we allow for  $D_b$  bosonic and  $D_f$  fermionic gaugings labeled by  $k$  and  $\kappa$ , respectively, and introduce the shorthand notation  $q \cdot \mathbf{a} = q_k \mathbf{a}_k$ .

The matter and gauge kinetic actions are normalized with factors  $1/g^2$  and  $1/e^2$ , respectively. The coupling constant  $g$  is dimensionless while the coupling  $e$  has mass-dimension one. Also the (0,2) superpotential

$$\mathcal{W} = \mu G_\alpha(\mathcal{Z}) \Lambda^\alpha \quad (2)$$

contains a mass scale  $\mu$ .  $\mathcal{W}$  has to be gauge and  $U(1)_L$  invariant, have R-charge 1 and be subject to  $G_\alpha N^\alpha = 0$ . To encode target space Kähler deformations, Fayet-Iliopoulos (FI) parameters  $\xi$  are included for the gauge multiplets. Absence of one-loop FI divergences requires

$$\sum_a (q_k)_a = e_d^T q_k = 0, \quad e_d^T = (1, \dots, 1). \quad (3)$$

### FULL ONE-LOOP PARTITION FUNCTION

We define the GLSM path integral

$$Z = \int \mathcal{D}(\mathcal{V}, \mathcal{A}) \mathcal{D}\Sigma \mathcal{D}Z \mathcal{D}\Lambda e^{-S}. \quad (4)$$

On the one-loop WS torus  $T^2$  a gauge background  $\mathbf{a}$  is fully determined by its holonomies

$$\nu = \tau a + a', \quad a = \oint_{\mathcal{C}_\tau} \mathbf{a}, \quad a' = \oint_{\mathcal{C}_1} \mathbf{a}, \quad (5)$$

w.r.t. the cycles  $\mathcal{C}_\tau$  and  $\mathcal{C}_1$  in the fundamental directions of the WS torus with complex structure  $\tau$ . The holonomies take values on the same  $T^2$ , i.e. they are defined modulo the periodicities  $\nu \equiv \nu + 1 \equiv \nu + \tau$ . Similarly, the zero modes of the external gauge fields  $\mathbf{a}_I$ , for  $I = L, R$ , can be characterized by their holonomies  $\nu_I = \tau a_I + a'_I$ . The spin structures can be introduced by fixing the L- and R-symmetry holonomies to

$$\nu_R = \tau \frac{s}{2} + \frac{s'}{2}, \quad \nu_L = \tau \frac{t}{2} + \frac{t'}{2}, \quad (6)$$

labeled by  $s, t, s', t' = 0, 1$  for  $Spin(32)/\mathbb{Z}_2$ . (The  $E_8 \times E_8$  theory requires two L-symmetries.)

The computation of path integrals of supersymmetric theories simplifies enormously due to localization [9]. The crucial observation is that the kinetic actions of both the matter and gauge multiplets are Q-exact, where Q is a combination of the supercharges. As supersymmetry is a symmetry of the theory, Q-exact terms do not modify the path integral. Thus, one can consider limits, like  $g, e \rightarrow 0$ , in which the kinetic terms of the non-zero modes completely dominate over their interactions. Concretely, one expands *all* fields as

$$\Phi = \Phi_0 + \delta\Phi, \quad \delta\Phi = g \sum_{m,n} \Phi_n^m Y_n^m, \quad (7)$$

in zero modes  $\Phi_0$  and quantum fluctuations  $\Phi_n^m$  with the torus mode functions  $Y_n^m$ . The path integral then reduces to a finite-dimensional zero mode integral of the expectation value evaluated in the free field theory of the non-zero modes:

$$Z = \int_{T^2} d^2\nu \int d\Phi_0 \langle e^{-S_0} \rangle_{\text{free}}. \quad (8)$$

$S_0$  contains the interactions of zero modes with themselves and the other modes. Throughout this work we implicitly use modular invariant integration measures.

The gauge holonomies  $\nu$  are always zero modes. For generic values of the holonomies  $\nu$  and  $\nu_I$  there are no additional ones, but for specific values further zero modes  $\Phi_0$  occur:

$\Phi_0$	Zero mode conditions
$\chi_0$	$\nu_R = 0$
$\varphi_0$	$\nu_L = 0$
$\mathfrak{s}_0$	$\nu_L = \nu_R = 0$
$z_0^a$	$q_a \cdot \nu + l_a \nu_L + r_a \nu_R = 0$
$\psi_0^a$	$q_a \cdot \nu + l_a \nu_L + (r_a - 1) \nu_R = 0$
$\lambda_0^\alpha$	$Q_\alpha \cdot \nu + L_\alpha \nu_L + R_\alpha \nu_R = 0$

We return to some consequences of the zero modes after computing the path integrals over the other modes.

### ONE-LOOP DETERMINANT FACTORS

In the limit  $e, g \rightarrow 0$  all non-zero modes decouple so that their one-loop determinant factors become infinite products, like  $\prod_{m,n} [m\tau + n + \omega]$  with  $\omega = \nu\tau + \nu'$ . Such infinite products can be  $\zeta$ -regularized, yielding expressions involving Dedekind  $\eta$  and higher genus theta functions

$$Z_d \left[ \begin{smallmatrix} v \\ v' \end{smallmatrix} \right] = \frac{\theta_1(\omega|\tau)}{\eta^d} = \frac{\theta \left[ \begin{smallmatrix} \frac{1}{2}e_d - v \\ \frac{1}{2}e_d - v' \end{smallmatrix} \right]}{\eta^d}. \quad (9)$$

The resulting partition functions for chiral and Fermi superfields read

$$Z_{\text{chiral}} = \eta_{s'}^s \frac{\overline{Z_d \left[ \begin{smallmatrix} v - e_d a_R \\ v' - e_d a'_R \end{smallmatrix} \right]}}{\left| Z_d \left[ \begin{smallmatrix} v \\ v' \end{smallmatrix} \right] \right|^2}, \quad Z_{\text{fermi}} = Z_D \left[ \begin{smallmatrix} V \\ V' \end{smallmatrix} \right], \quad (10a)$$

with  $v = q \cdot a + l a_L + r a_R$  and  $V = Q \cdot a + L a_L + R a_R$  (and similarly for the primed versions). The appropriate GSO-projection that ensures the interpretation of having target space bosons and fermions is implemented by the factor  $\eta_{s'}^s = (-)^{s+s'+s's}$ . For the bosonic and fermionic gauge superfields we have

$$Z_{\text{bos.g.}} = Z_1^{D_b} \left[ \begin{smallmatrix} -a_R \\ -a'_R \end{smallmatrix} \right], \quad Z_{\text{fer.g.}} = \frac{\overline{Z_1^{D_f} \left[ \begin{smallmatrix} -a_L \\ -a'_L \end{smallmatrix} \right]}}{\left| Z_1^{D_f} \left[ \begin{smallmatrix} -a_R - a'_L \\ -a'_R - a'_L \end{smallmatrix} \right] \right|^2}. \quad (10b)$$

## EFFECTIVE TARGET SPACE GEOMETRY

The target space geometry  $\mathcal{M}$  is parameterized by the scalar zero modes  $z_0$ . Even though by supersymmetric localization the partition function is independent of the parameters  $e, g, \mu$ , we would like to define a free limit for the non-zero modes, that nevertheless mimics the conformal limit ( $e, \mu \rightarrow \infty$ ) for the zero modes, so that we can determine the effective geometry from the path integral. After integrating out the auxiliary fields  $f$  and  $D$ ,  $S_0$  contains the potential

$$V = \frac{e^2}{2g^4} \left( \sum_a q_a |z_0^a|^2 - \xi_{\text{ren}} \right)^2 + g^2 \mu^2 \sum_\alpha \left| G_\alpha(z_0^a) \right|^2 + \frac{1}{e^2} \left| \mathfrak{s}_0 \cdot N^\alpha(z_0) \right|^2, \quad (11)$$

where  $\xi_{\text{ren}} = g^2 \xi$  are the physical FI parameters. An appropriate choice for this limit is  $g \rightarrow 0$  such that  $e^2/g^4, g^2 \mu^2 \sim 1/\sqrt{g}$ . Hence, one obtains the well-known conditions on the target space geometry [6].

When the effective target space is compact the zero modes  $z_0$  need not be constant but only harmonic, i.e.

$$z_0(\sigma, \bar{\sigma}) = h_w(\sigma) + \tilde{h}_w(\bar{\sigma}). \quad (12)$$

The holomorphic functions  $h_w(\sigma)$  define non-trivial mappings  $T^2 \rightarrow \mathcal{M}$  characterized by sets of integers  $w$ , which encode the Kaluza-Klein and winding numbers when the target space contains a torus. In the path integral we need to integrate over all possible field configurations, which leads to a lattice sum over the various instanton sectors labeled by  $w$ :

$$Z_{\text{inst}} = \text{Vol}(\mathcal{M}) \sum_w e^{-S_0^{\text{kin}}(h_w, \tilde{h}_w)}, \quad (13)$$

where  $S_0^{\text{kin}}$  is the kinetic energy of the zero modes. The volume  $\text{Vol}(\mathcal{M})$  of the target space manifold arises from the integral over the constant zero modes in  $z_0$ . In general, such lattice sums can be cast in the form

$$Z_{\text{inst}} = \sum_{W \in \Gamma} q^{\frac{1}{2}} W_L^2 \bar{q}^{\frac{1}{2}} W_R^2 \quad (14)$$

via a partial Poisson resummation, where  $q = e^{2\pi i \tau}$  and  $W_{L,R}$  encode the mappings  $h_w$ .

## GAUGINO ZERO MODES

The partition function in the sector  $\nu_R = 0$  computes the elliptic genus. In order to absorb the resulting gaugino zero modes  $\chi_0$ , one expands the action  $S_0$  to an appropriate order in  $\chi_0$ . As observed in [11], the resulting expression can be written as  $\bar{\mathcal{D}}$ -derivatives; the gauge holonomy integrals become contour integrals around the points  $q_a \cdot \nu + l_a \nu_L = 0$  in  $T^2$  fundamental domains. Using a residue theorem, the holonomy integrals reduce to finite sums in this sector.

## MODULAR INVARIANCE

Next we investigate the consequences of one-loop modular invariance of the full partition function

$$Z = \int_{T^2} d^2\nu Z_{\text{dets}} Z_{\text{inst}}. \quad (15)$$

For simplicity we assume that the instanton partition function  $Z_{\text{inst}}$  is modular invariant by itself, i.e. the lattice  $\Gamma$  is even and self-dual. Simply taking  $Z_{\text{dets}}$  as the product of the expressions in (10) does not produce a modular invariant result in general. However, it is well-known [18–20] that by using so-called vacuum phases one can obtain expressions for partition functions,

$$\widehat{Z}_d[v'] = e^{-\pi i v^T (v' - e_d)} Z_d[v'], \quad (16)$$

that are modular invariant up to phases:

$$\widehat{Z}_d[v'](\tau + 1) = e^{2\pi i \frac{d}{12}} \widehat{Z}_d[v'+v](\tau), \quad (17a)$$

$$\widehat{Z}_d[v']\left(\frac{1}{\tau}\right) = e^{-2\pi i \frac{d}{4}} \widehat{Z}_d[v'](\tau). \quad (17b)$$

Hence, by including such phase factors, we have a modular invariant partition function

$$\widehat{Z}_{\text{dets}} = \widehat{Z}_{\text{chiral}} \widehat{Z}_{\text{fermi}} \widehat{Z}_{\text{bos.g.}} \widehat{Z}_{\text{fer.g.}}, \quad (18)$$

provided that e.g.  $D - D_f = 12 + d - D_b$ . In order to reduce to the standard heterotic string in the absence of gaugings, we take  $d = 4 + D_b$  and  $D = 16 + D_f$ .

This partition function is well-defined provided that the WS torus periodicities of the L-, R-, and gauge holonomies are respected. For  $k, k' \in \mathbb{Z}^d$  we have

$$\widehat{Z}_d[v'+k] = e^{\pi i k^T (e_d - v')} \widehat{Z}_d[v'], \quad (19a)$$

$$\widehat{Z}_d[v'+k'] = e^{\pi i v^T k'} \widehat{Z}_d[v']. \quad (19b)$$

Since two different types of phases appear in these equations, requiring invariance under the periodicities of the holonomies,  $a_{L,R}$  and  $a$ , leads to both linear

$$\frac{1}{2} e_D^T Q_k \equiv \frac{1}{2} e_d^T q_k, \quad (20a)$$

$$\frac{1}{2} (e_D^T L - e_d^T l + D_f) \equiv \frac{1}{2} (e_D^T R - e_d^T r) \equiv 0, \quad (20b)$$

(where  $a \equiv b$  means  $a - b \in \mathbb{Z}$ ) and quadratic conditions

$$\frac{a_L}{2} (L^2 - l^2 + D_f) \equiv \frac{a_R}{2} (R^2 - (r - e_d)^2 + D_b) \equiv 0, \quad (21a)$$

$$\frac{a_{L,R}}{2} (L^T R - l^T r) \equiv \frac{a_k}{2} (Q_k^T Q_l - q_k^T q_l) \equiv 0, \quad (21b)$$

$$\frac{a_{k,L}}{2} (Q_k^T L - q_k^T l) \equiv \frac{a_{k,R}}{2} (Q_k^T R - q_k^T r) \equiv 0. \quad (21c)$$

In order to allow for a consistent string theory interpretation of the GLSM, these conditions have to be fulfilled in all sectors of the theory. Moreover, if we assume that

the GLSM flows to a (0,2) CFT which possesses a  $U(1)_R$  symmetry, these requirements have to be fulfilled for all possible gauge holonomies  $a_k, a_R \in [0, 1[$ . This means that all the conditions that contain either  $a_k$  or  $a_R$  have to be strictly enforced, i.e. “ $\equiv$ ” turns into “ $=$ ”. Since for a generic (0,2) heterotic string model we only need the  $U(1)_L$  to introduce the spin structures  $t, t'$ , these conditions only have to be enforced for  $a_L = 1/2$ .

In summary, the conditions (20) and (21) are the fundamental consistency requirements of the GLSM: They encoded gauge anomaly cancellation and reduce to the orbifold modular invariance conditions in the  $\nu_R = 0$  sector.

## SPECTRA

The conventional massless spectrum computation of GLSMs makes use of  $\overline{Q}$ -cohomology (see e.g. [21–23]). We outline an alternative way to determine not only the massless but the *full* spectrum of GLSMs. This method follows the calculation of the (orbifold) CFT spectrum by making a  $q$ -expansion of the full partition function. Depending on whether theta functions appear in the numerator or denominator, one uses their sum or product representations, respectively. However, contrary to orbifold theories we have sectors with continuous integrals over the gauge holonomies  $\nu$  rather than discrete sums over the twisted sectors.

In detail, the left- and right-moving masses read

$$M_R^2 = \frac{1}{2}W_R^2 + \frac{1}{2}p_R^2 - \frac{1}{2} + \delta c + N_R, \quad (22a)$$

$$M_L^2 = \frac{1}{2}W_L^2 + \frac{1}{2}P_L^2 - 1 + \delta c + N_L, \quad (22b)$$

in terms of the so-called shifted momenta

$$p_R = \begin{pmatrix} p_d + v - (a_R + \frac{1}{2})e_d \\ p_{D_f} - (a_L + \frac{1}{2})e_{D_f} \end{pmatrix}, \quad P_L = \begin{pmatrix} P_D + V - \frac{1}{2}e_D \\ P_{D_b} - (a_R + \frac{1}{2})e_{D_b} \end{pmatrix} \quad (23)$$

with  $p_d \in \mathbb{Z}^d$  and  $v, V$  defined below (10a). We also introduced the number operators  $N_{L,R}$  and the vacuum shift

$$\delta c = \frac{1}{2}\tilde{v}^T(e_d - \tilde{v}) + \frac{D_f}{2}\tilde{v}_f(1 - \tilde{v}_f) - \frac{D_b + D_f}{8}, \quad (24)$$

where  $\tilde{v} \equiv v$  and  $\tilde{v}_f \equiv a_L + a_R$  such that all components satisfy  $0 \leq \tilde{v}_a < 1$ . The spectrum is subject to various projection conditions:

The sums over the spin structures  $s', t'$  lead to GSO-projections. As discussed above in the sector  $\nu_R = 0$  the integral over  $\nu$  reduces to a finite sum. In particular, the sum over the remaining values of  $a'$  induces a projection which is very similar to the orbifold projection. Both projections identify a sublattice within the lattice spanned by  $p$  and  $P$  of the same dimension. In contrast, in the  $\nu_R \neq 0$  sectors the integral over  $a'$  only leads to a projection onto a sublattice of lower dimension, provided (3),

(20) and (21) are fulfilled. As usual, the level matching condition  $M_L^2 = M_R^2$  is obtained from the  $\tau_1$ -integral.

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