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Chiral Primaries in Strange Metals

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Abstract

It was suggested recently that the study of 1-dimensional QCD with fermions in the adjoint representation could lead to an interesting toy model for strange metals and their holographic formulation. In the high density regime, the infrared physics of this theory is described by a constrained free fermion theory with an emergent $\mathcal{N} = (2,2)$ superconformal symmetry. In order to narrow the choice of potential holographic duals, we initiate a systematic search for chiral primaries in this model. We argue that the bosonic part of the superconformal algebra can be extended to a coset chiral algebra of the form $\mathcal{W}_N = \mathrm{SO}(2N^2-2)_1/\mathrm{SU}(N)_{2N}$. In terms of this algebra the spectrum of the low energy theory decomposes into a finite number of sectors which are parametrized by special necklaces. We compute the corresponding characters and partition functions and determine the set of chiral primaries for $N \leq 5$.

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Contents

1 Introduction

Low dimensional examples of dualities between conformal field theories and gravitational models in Anti-deSitter (AdS) space provide an area of active research. There are several reasons why such developments are interesting. On the one hand, many low dimensional critical theories can actually be realized in condensed matter systems. As they are often strongly coupled, the AdS/CFT correspondence might provide intriguing new analytic tools to compute relevant physical observables. On the other hand, low dimensional incarnations of the AdS/CFT correspondence might also offer new views on the very working of dualities between conformal field theories and gravitational models in AdS backgrounds. This applies in particular to the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence since there exist many techniques to solve 2-dimensional models directly, without the use of a dual gravitational theory. Recent examples in this direction include the correspondence between certain 2-dimensional coset conformal field theories and higher spin gauge theories [1, 2], see also [3, 4, 5, 6] for examples involving supersymmetric conformal field theories and [8] for a more extensive list of the vast literature on the subject. It would clearly be of significant interest to construct new examples of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence which involve full string theories in AdS_3.

In 2012, Gopakumar, Hashimoto, Klebanov, Sachdev and Schoutens [7] studied a two-dimensional adjoint QCD in which massive Dirac fermions Ψ are coupled to an $\mathrm{SU}(N)$ gauge field. The fermions were assumed to transform in the adjoint rather than the fundamental representation of the gauge group. In the strongly coupled high density region of the phase space, the corresponding infrared fixed point is known to develop an $\mathcal{N} = (2,2)$ superconformal symmetry. For gauge groups SU(2) and SU(3) the fixed points possess Virasoro central charge $c_2 = 1$ and $c_3 = 8/3$, respectively. These central charges are smaller that the critical value of $c = 3$ below which one can only have a discrete set of $\mathcal{N} = (2,2)$ superconformal minimal models. Such theories are very well studied. Once we go beyond $N = 3$, however, the central charge $c_N = (N^2 - 1)/3$ of the infrared fixed point exceeds the critical value and the models are very poorly understood at present. Note that the central charge c_N of these models grows quadratically with the rank $N - 1$ of the gauge group. While this is very suggestive of a string theory dual, there exist very little further clues on the appropriate choice of the 7-dimensional compactification manifold M^7 of the relevant AdS background.

The most interesting structure inside any $\mathcal{N} = (2,2)$ superconformal field theory is its chiral ring. Recall that the $\mathcal{N} = (2,2)$ superconformal algebra contains a U(1) R-charge Q. The latter provides a lower bound on the conformal weights h in the theory, i.e. physical states ϕ in a unitary superconformal field theory obey the condition $h(\phi) \geq Q(\phi)$. States in the Neveu-Schwarz sector that saturate this bound, i.e. for which $h(\phi) = Q(\phi)$, are called *chiral primaries*. Since chiral primaries are protected by supersymmetry, they are expected to play a key role in discriminating between potential gravitational duals for the infrared fixed point of adjoint QCD. More concretely, the space of chiral primaries in the limit of large N should carry essential information on the compactification manifold M^7 of the dual AdS_3 background.

The goal of our work is to initiate a systematic study of the chiral ring for the models proposed by Gopakumar et al. In [7] the partition function of the infrared fixed point was studied for $N = 2, 3$. In these two cases the chiral ring is well understood through the relation with $\mathcal{N} = (2, 2)$ minimal models, as we mentioned above. The chiral primaries that are found in these two simple models are special representatives of a larger class of *regular* chiral primaries that can be constructed for all N. But once we pass the critical value of the central charge, i.e. for $c_N > 3$, additional chiral primaries start showing up. We shall find one example at $N = 4$ and three non-regular, or *exceptional*, chiral primaries for $N = 5$. In order to do so, we develop some technology that can be applied also to larger values of N and we hope that it will provide essential new tools in order to address the large N limit.

Let us briefly discuss the plan of this paper. In the next section, we shall describe the low energy theory, identify its chiral algebra, construct the relevant modular invariant partition function and finally discuss the emergent $\mathcal{N} = (2, 2)$ superconformal symmetry. Our discussion differs a bit from the one in [7] in that we work with a larger chiral algebra. Our algebra has the advantage that it contains the R-current of the model. This gives us more control over chiral primaries in the subsequent analysis. Section 3 is devoted to the representation theory of the chiral symmetry. There we shall explain how to label its representations and how to construct the corresponding characters. In doing so, we shall keep track of the R-charges. The section concludes with explicit lists of representations up to $N = 5$. In section 4 we turn to the main theme of this work, the set of chiral primaries. After explaining some general bounds on their conformal weights we describe the set of regular chiral primaries and study some of their properties. Finally, we construct all additional exceptional chiral primaries for $N = 4$ and $N = 5$. These were not known previously. Whether any of these additional chiral primaries survive in the large N limit remains an interesting issue for future research.

2 The model and its symmetries

The main purpose of this section is to review the setup described in [7]. Starting from 2-dimensional adjoint QCD we describe how the low energy description emerges in the limit of large density and strong coupling. Special attention is paid to the chiral symmetries of the theory which are identified at the end of the first subsection. The algebra we construct there is a bit larger than the one that was considered in [7]. In the second subsection we then describe how the state space of the low energy theory decomposes into representations of left- and right-moving chiral algebra. The section concludes with some comments on an emergent $\mathcal{N} = (2, 2)$ superconformal symmetry and the role of chiral primaries for future studies of AdS duals.

2.1 Review of the model

The model we start with is a 2-dimensional version of QCD with fermions in the adjoint representations, i.e.

$$\mathcal{L}(\Psi, A) = \operatorname{Tr}\left[\overline{\Psi}(i\gamma^\mu D_\mu - m - \mu\gamma^0)\Psi\right] - \frac{1}{2g_{\rm YM}^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu}\ . \tag{2.1}$$

Here, A denotes an SU(N) gauge field with field strength F and gauge coupling $g_{\rm YM}$. The complex Dirac fermions Ψ transform in the adjoint of the gauge group and D_μ denote the associated covariant derivatives. The two real parameters m and μ describe the mass and chemical potential of the fermions, respectively.

We are interested in the strongly coupled high density regime of the theory, i.e. in the regime of very large chemical potential $\mu \gg m$ and $g_{\rm YM}$. As is well known, we can approximate the excitations near the zero-dimensional Fermi surface by two sets of relativistic fermions, one from each component of the Fermi surface. These are described by the left- and right-moving components of massless Dirac fermions. At strong gauge theory coupling, the resulting (Euclidean) Lagrangian reads

$$\mathcal{L}_{\rm eff}(\psi, \overline{\psi}, A) = \operatorname{Tr}\left(\overline{\psi}^*\partial\,\overline{\psi} + \psi^*\overline{\partial}\,\psi + A_z[\,\psi^*, \psi\,] + A_{\bar z}[\,\overline{\psi}^*, \overline{\psi}\,]\right)\ . \tag{2.2}$$

Here we have dropped the term involving the field strength F, using that $g_{\rm YM} \to \infty$. Upon integrating out the two components A_z and $A_{\bar z}$ of the gauge field we obtain the constraints

$$J(z) \ := \ [\,\psi^*, \psi\,] \ \sim \ 0 \quad , \quad \bar{J}(\bar z) \ := \ [\,\overline{\psi}^*, \overline{\psi}\,] \ \sim \ 0\ . \tag{2.3}$$

These constraints are to be implemented on the state space of the $N^2 - 1$ components of the complex fermion ψ such that all the modes $J_n, n > 0$, of $J(z) = \sum J_n z^{-1-n}$ vanish on physical states, as is familiar from the standard Goddard-Kent-Olive coset construction [9].

In order to describe the chiral symmetry algebra of the resulting conformal field theory we shall start with the unconstrained model, which we refer to as the *numerator* theory. It is based on $M = N^2 - 1$ complex fermions $\psi_\nu, \nu = 1, \ldots, M$. These give rise to a Virasoro algebra with central charge $c_{\rm N} = N^2 - 1$, where the subscript N stands for numerator. We can decompose each complex fermion into two real components $\psi^n_\nu, n = 1, 2$, such that $\psi_\nu = \psi^1_\nu + i\psi^2_\nu$. From time to time we shall combine ν and n into a single index $\alpha = (\nu, n)$. Let us recall that the $2M$ real fermions ψ_α can be used to build SO($2M$) currents $K_{\alpha\beta}$ at level $k = 1$. The central charge of the associated Virasoro field coincides with the central charge $c_{\rm N}$ of the original fermions. The SO$(2M)_1$ current algebra generated by the modes of $K_{\alpha\beta}$ forms the numerator in the coset construction.

In order to describe the *denominator*, i.e. the algebra generated by the constraints (2.3), we need to recall a second way in which our fermions ψ^n_ν give rise to currents. According to the usual constructions, we can employ the representation matrices of the adjoint representation to build two sets of SU(N) currents at level $k = N$. These currents will be denoted by j^n_ν with $\nu = 1, \ldots, M$ and $n = 1, 2$. The currents J that were introduced in eq. (2.3) are obtained as $J_\nu = j^1_\nu + j^2_\nu$. The chiral SU($N$) currents J_ν form an affine algebra at level $k = 2N$. Through the Sugawara construction we obtain a Virasoro algebra with central charge $c_{\mathrm{D}} = 2(N^2 - 1)/3$, where the subscript D stands for denominator. Now we have assembled all the elements that are needed in defining the coset chiral algebra

$$\mathcal{W}_N := \mathrm{SO}(2N^2 - 2)_1/\mathrm{SU}(N)_{2N} \, . \tag{2.4}$$

The parameter N keeps track of the gauge group SU(N). The algebra $\mathcal{W}_N$ is a key element in our subsequent analysis. It is larger than the chiral symmetry considered in [7] which uses the subalgebra $\mathrm{SU}(N)_N \times \mathrm{SU}(N)_N \subset \mathrm{SO}(2N^2 - 2)_1$ to encode symmetries of the numerator theory.

2.2 Modular invariant partition function

The coset algebra $\mathcal{W}_N$ describes the chiral symmetries of our model. Consequently, the partition function must decompose into a sum of products of characters for the left- and right chiral symmetry. These characters will be discussed in much detail below. The aim of this section is to explain how they are put together in order to construct the partition function of the coset model.

We shall begin with a few simple comments on the numerator theory. As we reviewed above, its state space carries the action of a chiral SO($2M$) algebra at level $k = 1$. This current algebra possesses four sectors which are denoted by *id*,*v*,*sp* and *c*, respectively. When decomposed into the associated characters, the partition function takes the form

$$Z^{\mathrm{N}}(q, \bar{q}) = |\chi^{\mathrm{N}}_{id}|^2 + |\chi^{\mathrm{N}}_{v}|^2 + |\chi^{\mathrm{N}}_{sp}|^2 + |\chi^{\mathrm{N}}_{c}|^2 \; = \; M^{\mathrm{N}}_{AB}\, \chi^{\mathrm{N}}_A(q) \bar{\chi}^{\mathrm{N}}_B(\bar{q}) \, . \tag{2.5}$$

The labels A, B on the right hand side run through $A, B = id, v, sp$ and c and M^{N}_{AB} are integers which are defined through the expression on the left hand side. Explicitly, these integers are given by $M^{\mathrm{N}}_{AB} = \mathrm{diag}(1, 1, 1, 1)$.

Now we need to describe a similar set of integers M^{D}_{ab} for the denominator theory. This is obtained from the D-type modular invariant partition function for the $\mathrm{SU}(N)_{2N}$ Wess-Zumino-Witten model. Before we can spell it out, we need a bit of notation. To begin with, we introduce the set $\mathcal{J}_N$ of SU(N) weights $a = [\lambda_1, \ldots, \lambda_{N-1}]$ subject to the condition $\sum_{s=1}^{N-1} \lambda_s \leq 2N$. These label sectors of the SU(N) current algebra at level $k = 2N$.

On $\mathcal{J}_N$ we can define an action of $\mathbb{Z}_N$ such that

$$\gamma([\lambda_1, \ldots, \lambda_{N-1}]) = [2N - \sum_{s=1}^{N-1} \lambda_s, \lambda_1, \ldots, \lambda_{N-2}] \tag{2.6}$$

for the generator $\gamma \in \mathbb{Z}_N$. Obviously, γ maps elements $a \in \mathcal{J}_N$ back into $\mathcal{J}_N$ and it obeys $\gamma^N = id$.

In addition, we can also construct a map $h^{\mathrm{D}} : \mathcal{J}_N \to \mathbb{R}$ that assigns a conformal weight $h^{\mathrm{D}}(a)$ to each sector $a \in \mathcal{J}_N$. The weight is given by[1]

$$h^{\mathrm{D}}(a) = \frac{C_2(a)}{3N} . \tag{2.7}$$

With the help of the map $\gamma : \mathcal{J}_N \to \mathcal{J}_N$ and the weight $h^{\mathrm{D}} : \mathcal{J}_N \to \mathbb{R}$ we can finally define the so-called monodromy charge

$$Q_\gamma(a) \equiv h^{\mathrm{D}}(\gamma(a)) - h^{\mathrm{D}}(a) \bmod 1. \tag{2.8}$$

Now we have collected all the ingredients we need in order to spell out the desired D-type modular invariant partition function of the $\mathrm{SU}(N)_{2N}$ Wess-Zumino-Witten model,

$$Z^{\mathrm{D}}(q,\bar{q}) \;=\; \sum_{\{a\};Q_\gamma(a)\equiv 0} \frac{N}{N_a} |\sum_{b\in\{a\}} \chi_b|^2 \;=\; \sum_{ab} M^{\mathrm{D}}_{ab} \chi_a^{\mathrm{SU}(N)_{2N}}(q) \bar{\chi}_b^{\mathrm{SU}(N)_{2N}}(\bar{q}) . \tag{2.9}$$

The first summation is over orbits $\{a\}$ of weights for the affine $\mathrm{SU}(N)_{2N}$ under the action (2.6) of the identification current γ. The length of a generic orbit agrees with the size N of the gauge group $\mathrm{SU}(N)$. Some orbits $\{a\}$, however, possess fixed points so that their length $N_a = N_{\{a\}}$ can be a nontrivial divisor of N. For more details on simple current modular invariants see [10].

From eq. (2.9) we can read off the integer coefficients M^{D} of the decomposition [11],

$$M^{\mathrm{D}}_{ab} = \begin{cases} \sum_{p=1}^{N} \delta_{a,\sigma^p(b)} & \text{if } t(a) = 0 \mod N \\ 0 & \text{otherwise} \end{cases} , \tag{2.10}$$

where $t(a) = \sum_{s=1}^{N-1} s\,\lambda_s$ is the N-ality of an $\mathrm{SU}(N)_{2N}$ representation $a = [\lambda_1, ..., \lambda_{N-1}]$ and $\sigma^r(a) = 2Nw_r + c^r(a), r = 1, ..., N$, are associated with group automorphisms of $\mathrm{SU}(N)_{2N}$. They are defined in terms of the fundamental weights w_r and the Coxeter rotations $c^r(w_i) = w_{i+r} - w_r$, see [11, 12] for more details and [13, 14] for M^{D} at $N = 2, 3$. Note that the N-ality constraint coincides with the condition of vanishing monodromy charge that was built into eq. (2.9).

We are now prepared to construct a modular invariant that is associated with our coset model. In fact, following the standard procedures in coset conformal field theory we are led to

[1]The quadratic Casimir of an $\mathrm{SU}(N)$ representation a is given by

$$C_2(a) = \frac{1}{2}\left[-\frac{n^2}{N} + nN + \sum_{i=1}^{r}(l_i^2 + l_i - 2il_i)\right] ,$$

where $n = \sum_{i=1}^{r} l_i$ is the total number of boxes in the corresponding Young tableau, and $l_i = \sum_{s=i}^{N-1} \lambda_s$ $(i = 1, ..., r)$ denotes the length of the ith row.

consider

$$\tilde{Z}_N(q,\bar{q}) = \frac{1}{N^2}\sum_{ABab} M^{\mathrm{D}}_{ab} M^{\mathrm{N}}_{AB} \mathcal{X}^{\mathcal{W}}_{(A,a)}(q) \bar{\mathcal{X}}^{\mathcal{W}}_{(B,b)}(\bar{q}) \,. \tag{2.11}$$

The summation runs over the same range as in eqs. (2.5) and (2.9). The functions $\mathcal{X}$ are so-called branching functions. We will define and construct them in the next section. For most values of the label (A,a), the branching function $\mathcal{X}$ is a character χ of an irreducible representation of $\mathcal{W}_N$. More precisely, one finds that

$$\mathcal{X}^{\mathcal{W}}_{(A,a)}(q) = \chi^{\mathcal{W}}_{(A,a)}(q) \quad \text{when} \quad N_a = N \,, \tag{2.12}$$

i.e. when the orbit $\{a\}$ of a under the action of $\gamma \in \mathbb{Z}_N$ consists of N elements. The orbit $\{0\}$ of the vacuum representation is always such a long one. Consequently, in order for the vacuum to contribute with unit multiplicity, we had to divide the sum in eq. (2.11) by N^2. But this is a dangerous division. In order to see the problem, let us insert M^{N}_{AB} and M^{D}_{ab} as in eqs. (2.5) and (2.9), respectively. Then our modular invariant (2.11) reads

$$\tilde{Z}_N(q,\bar{q}) = \sum_A \sum_{\{a\},Q_\gamma(a)=0} \frac{N_a}{N} |\mathcal{X}^{\mathcal{W}}_{(A,a)}(q)|^2 \,. \tag{2.13}$$

Here, we sum over orbits $\{a\}$ instead of SU(N) representations a with vanishing monodromy charge. For short orbits we have $N_a < N$ so that the corresponding branching functions are divided by a non-trivial integer. Typically, one finds that these fractions are not compensated by corresponding multiplicities in the branching functions so that the modular invariant (2.11) possesses non-integer coefficients. This problem is of course well known and may be overcome by a process known as fixed point resolution, see [10, 15]. In the case of short orbits, i.e. when $N_a \neq N$, the branching function $\mathcal{X}^{\mathcal{W}}_{(A,a)}$ turns out to decompose into a sum of $\mathcal{W}_N$ characters $\chi^{\mathcal{W}}_{(A,a,m)}$ for irreducibles labeled by m. General formulas for such decompositions exist only for some coset chiral algebras, see e.g. [15]. Experience shows that the characters $\chi^{\mathcal{W}}_{(A,a,m)}$ can be used as building blocks for modular invariants Z^{res}_N such that

$$Z_N(q) = \tilde{Z}_N(q) + Z^{\mathrm{res}}_N(q) \tag{2.14}$$

has integer coefficients only. Z_N can therefore be interpreted as the partition function of the system. To spell out details, we shall mostly assume that N is a prime number. Under this condition, the sectors $(A,[2,2,\ldots,2])$ turn out to generate the only short orbits and the resolution process can be spelled out explicitly. Following a recipe first described in [10] we define

$$\chi^{\mathcal{W}}_{(A,a_*,m)}(q) = \frac{1}{N}\left(\mathcal{X}^{\mathcal{W}}_{(A,a_*)}(q) + d_{(A,m)}\right) \,, \tag{2.15}$$

$$\text{where} \quad d_{(A,m)} \in \mathbb{Z} \quad \text{with} \quad \sum_{m=1}^{N} d_{(A,m)} = 0 \,, \tag{2.16}$$

$m = 1, \ldots, N$ and A runs through its four possible values, as usual. We also introduced the shorthand $a_* = [2, 2, \ldots, 2]$. Note that the proposed characters indeed sum up to the branching functions. For $N > 2$ we propose the following values for $d_{(A,m)}$:[2]

$$\begin{aligned} d_{(id,1)} &= N-1\,, & d_{(id,p)} &= -1\,, \\ d_{(v,1)} &= 0\,, & d_{(v,p)} &= 0\,, \\ d_{(sp,1)} &= -\frac{(N-1)^2}{2}\,, & d_{(sp,p)} &= \frac{N-1}{2}\,, \\ d_{(c,1)} &= \frac{N^2-1}{2}\,, & d_{(c,p)} &= -\frac{N+1}{2}\,, \end{aligned} \tag{2.17}$$

$p = 2, ..., N$. Given these characters we can now construct Z^{res} through

$$Z_N^{\text{res}}(q) \;=\; \frac{1}{N^2} \sum_A \sum_{m=1}^{N} d^2_{(A,m)} = \frac{N^2+3}{2} \frac{N-1}{N} \tag{2.18}$$

for N prime. Since Z^{res} is a constant, it is obviously modular invariant. In addition, if we add this term to the modular invariant $\tilde{Z}_N$ we obtain an expression in which squares of characters are summed with integer coefficients,

$$Z_N(q) = \sum_A \sum_{\substack{\{a\}, a \neq a_* \\ Q(a)=0}} |\chi^{\mathcal{W}}_{(A,a)}(q)|^2 + \sum_A \sum_{m=1}^{N} |\chi^{\mathcal{W}}_{(A,a_*,m)}(q)|^2\,. \tag{2.19}$$

Here, the first summation is over all orbits of length N with vanishing monodromy charge. Of course our assumption that N be prime is crucial for the validity of the expression (2.19) for the partition function of our model.

2.3 Comments on superconformal symmetry

According to the usual Goddard-Kent-Olive (GKO) construction [9], the chiral algebra $\mathcal{W}_N$ contains a Virasoro field whose central charge is given by the difference of the central charges in the numerator and the denominator,

$$c = c(\mathcal{W}_N) = c_{\text{N}} - c_{\text{D}} = N^2 - 1 - \frac{2}{3}(N^2-1) = \frac{1}{3}(N^2-1)\,. \tag{2.20}$$

Of course, the coset chiral algebra contains many more fields. To be precise, any element of the numerator algebra that has trivial operator product with respect to the denominator currents makes it into our algebra $\mathcal{W}_N$. In the case at hand, the condition is also satisfied by the U(1) current

$$J(z) = \tfrac{1}{3} \textstyle\sum_{\nu,\mu} \psi^1_\nu(z) \psi^2_\mu(z) \kappa^{\nu\mu} \tag{2.21}$$

[2]For $N = 2$, we have $d_{(sp,1)} = -1$, $d_{(sp,2)} = 1$ and $d_{(c,1)} = 1$, $d_{(c,2)} = -1$, all others are zero.

where $\kappa^{\nu\mu}$ denotes the Killing form of SU(N). This U(1) current will play a very important role.

It was observed in [16] that the conformal symmetry is actually enhanced to a $\mathcal{N} = (2,2)$ superconformal one. This means that the state space admits the action of fermionic generators $G^{\pm}$ and an additional U(1) current J. While $G^{\pm}$ are not contained in our chiral algebra $\mathcal{W}_N$, the U(1) current is. In fact, it is precisely the current we found in the previous paragraph.

Let us recall that in models with $\mathcal{N} = 2$ supersymmetry there is an important subset of fields, namely the (anti-)chiral primaries. By definition, these correspond to states in the Neveu-Schwarz sector of the theory, i.e. with $A = id,v$, such that $h = |Q|$ where Q denotes the U(1) charge and h the conformal weight. Chiral primaries have many interesting properties. In particular, they give rise to the so-called chiral ring. In addition the space of chiral primaries is protected under deformations preserving the $\mathcal{N} = (2,2)$ superconformal symmetry. Therefore, it can serve as a "fingerprint" of our model.

As we discussed in the introduction, chiral primaries should play an important role when it comes to identifying the AdS dual of the superconformal field theory we are dealing with. As we pointed out in the introduction, AdS duals of 2-dimensional (super-)conformal field theories have recently attracted quite some attention. In the existing examples, the central charge is linear in N and the dual model is a higher spin theory in AdS_3. The case we are dealing with here is different: The central charge (2.20) is quadratic in N and hence standard arguments would suggest a richer dual model which is described by a full string theory in AdS_3 rather than a higher spin theory. The identification of this string theory would be significant progress. Clearly, the chiral primaries could play a central role in identifying the string dual.

3 Representations of the chiral symmetry

In the previous section we identified the chiral symmetry algebra $\mathcal{W}_N$ of the coset model. Our next aim is to develop the representation theory of this chiral symmetry. In the first subsection we shall provide several different ways to think about the pairs (A, a) that label non-trivial branching functions of our chiral algebra $\mathcal{W}_N$. Then we explain how to obtain the branching functions from the characters of the numerator and the denominator theory. We have worked out the first few terms in the expansion of these characters for all representations with $N \leq 5$. The results are sketched in the third subsection, at least to the extent to which they are needed later on. More details may be found in appendix B.

3.1 Labeling of orbits

The labels (A, a) of $\mathcal{W}_N$ branching functions have been described in the previous section already. Let us recall that A runs through the four values $A = id,v,sp,c$. The range of a was a little more difficult to state. It should be taken from the set $\mathcal{J}_N^0$ of $\mathrm{SU}(N)_{2N}$ labels a with vanishing monodromy charge $Q_\gamma(a) = 0$, see eq. (2.8). Since branching functions are invariant under the action (2.6) of the identification group $\mathbb{Z}_N$, we only need to pick one representative a from each orbit $\{a\} \in \mathcal{O}_N = \mathcal{J}_N^0/\mathbb{Z}_N$. In the next two subsections we will develop an approach that allows to enumerate the branching functions of $\mathcal{W}_N$ more systematically.

3.1.1 Solving the zero monodromy condition

Our first task is to describe those labels a that solve the $Q(a) = 0$ condition, i.e. elements $a \in \mathcal{J}_N^0$. Our claim is that elements a of $\mathcal{J}_N^0$ are in one-to-one correspondence with pairs of $\mathrm{SU}(N)$ Young diagrams Y' and Y'' with equal number $n' = |Y'| = |Y''|$ of boxes satisfying

$$r' + c'' \leq N \quad , \quad r'' + c' \leq 2N \tag{3.1}$$

where r', r'' and c', c'' denote the numbers of rows and columns of Y', Y'', respectively. Let us denote the row lengthes of the Young diagrams Y' and Y'' by[3]

$$Y' = (l'_1, \ldots, l'_{r'}) \quad , \quad Y'' = (l''_1, \ldots, l''_{r''}) \, .$$

Here, we arrange the l'_i and l''_i in decreasing order, i.e. $l'_i \geq l'_{i+1}$ etc. so that the largest entries are $l'_1 = c'$ and $l''_1 = c''$. From these two Young diagrams we can build a new diagram $Y = (l_1, \ldots, l_{N-1})$ through[4]

$$l_i = \begin{cases} r'' + l'_i & \text{for} \quad i = 1, ..., r' \\ r'' & \text{for} \quad i = r' + 1, ..., N - l''_1 \\ r'' - k & \text{for} \quad i = N - l''_k + 1, ..., N - l''_{k+1} \, , \quad k = 1, ..., r'' - 1 \\ 0 & \text{for} \quad i = N - l''_{r''} + 1, ..., N - 1 \, . \end{cases} \tag{3.2}$$

As will be shown in appendix C, the total number $|Y|$ of boxes in Y is $n = r''N$ and the value of the quadratic Casimir is

$$C_2(a) = C_2(Y) = n'N + C_2(Y') - C_2(Y'') \, , \tag{3.3}$$

where $C_2(Y')$ and $C_2(Y'')$ are the quadratic Casimirs of the $\mathrm{SU}(N)$ representations associated with Y' and Y'', respectively. Formula (3.3) follows directly by substituting eq. (3.2) into the definition of the quadratic Casimir invariant of $\mathrm{SU}(N)$.

We will not prove the parametrization of $\mathcal{J}_N^0$ through pairs (Y', Y'') here. But let us make a few comments at least. To begin with, the first constraint in eq. (3.1) is a necessary condition for $Y = (l_1, ..., l_{N-1})$ to be a representation of $\mathrm{SU}(N)$ while the second constraint ensures that

[3] The row lengths l'_i $(i = 1, ..., r')$ are related to the Dynkin labels λ'_s by $l'_i = \sum_{s=i}^{N-1} \lambda'_s$ (and similarly for l''_i).
[4] This extends the construction of [17].

it is also a representation of the affine group $\mathrm{SU}(N)_{2N}$. So, the two constraints together ensure that Y corresponds to a representation of the SU(N) current algebra at level $k = 2N$, i.e. to an element of our set $\mathcal{J}_N$. It is not difficult to show that the representations Y also obey the zero monodromy condition or, equivalently, the N-ality condition $\sum_i^{N-1} i\lambda_i = \sum_i l_i = 0 \mod N$. Since $|Y| = n = r''N$ for any pair (Y', Y''), $|Y| = \sum_i l_i$ is always a multiple of N and the N-ality condition trivially holds true, so $Y \in \mathcal{J}_N^0$. Checking that the representations Y give the complete set $\mathcal{J}_N^0$ would require some more work. We have checked with computer algebra up to $N = 7$ that the prescription (3.2) indeed provides the complete set of $\mathrm{SU}(N)_{2N}$ representations with vanishing monodromy charge.

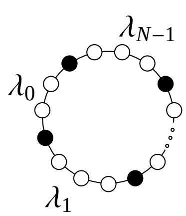
Once we accept this parametrization of sectors with vanishing monodromy charge through pairs (Y', Y'') of Young diagrams, it is easy to count. Indeed, we have checked up to $N = 9$ that the total number of elements in $\mathcal{J}_N^0$ is given by the series A082936 in [18],

$$|\mathcal{J}_N^0| = \frac{1}{3N} \sum_{n|N} \varphi(N/n) \binom{3n}{n}, \tag{3.4}$$

where $\varphi(n)$ is Euler's phi function. When $N = 7$, for example, we obtain $|\mathcal{J}_7^0| = 5538$ representations of the affine algebra $\mathrm{SU}(7)_{14}$ with vanishing monodromy charge.

3.1.2 Necklace representation of orbits

The partition function Z_N involves a summation over orbits $\{a\}$ of weights $a \in \mathcal{J}_N^0$ for the affine algebra $\mathrm{SU}(N)_{2N}$ under the action (2.6) of the identification current γ. The right way to proceed is therefore to group the elements of $\mathcal{J}_N^0$ into orbits $\{a\}$. It turns out that the orbits possess a nice representation in terms of *necklaces* with N black and $2N$ white beads. A necklace is constructed from the affine Dynkin labels $[\lambda_0, \lambda_1, ..., \lambda_{N-1}]$ of any representation within a given orbit. We stated the relation between the row lengthes l_i and the Dynkin labels λ_i in the previous section. The additional entry λ_0 of the affine Dynkin label is simply given by $\lambda_0 = 2N - \sum_{i=1}^{N-1} \lambda_i$. Necklaces are direct graphical representations of the affine Dynkin labels. The entries of the affine Dynkin label determine the number of white beads which are separated by the black beads, i.e. the structure of a necklace is: λ_0 white beads, black bead, λ_1 white beads, black bead, etc.



A necklace represents the whole orbit $\{a\}$, since the action of the identification current γ corresponds to a rotation of the necklace but not a modification of the necklace itself. The identification of orbits with necklaces enables us to find a simple formula for the number

$$|\mathcal{O}_N| = \sum_{N_a|N} \frac{1}{3N_a^2} \sum_{n|N_a} \mu(N_a/n) \binom{3n}{n} \tag{3.5}$$

of orbits $\mathcal{O}_N = \mathcal{J}_N^0/\mathbb{Z}_N$. Here, μ denotes the classical Möbius function.

Let us stress that the number $4|\mathcal{O}_N|$ counts the number of inequivalent branching functions and not the number of representations of our chiral algebra. If we assume that a branching function associated to an orbit of length N_a can be decomposed into characters of N/N_a inequivalent representations, then the number of $\mathcal{W}_N$ sectors is given by

$$|\mathcal{R}_N^{\mathcal{W}}| = 4 \sum_{N_a|N} \frac{N}{N_a} \frac{1}{3N_a^2} \sum_{n|N_a} \mu(N_a/n) \binom{3n}{n} .$$

This formula produces the correct results at least when N is prime. The factor of 4 in front of the sum stems from the summation over A.

3.2 Representations and characters

Having parametrized and counted the orbits of (A, a) we will discuss the associated branching functions and the closely related characters of the chiral algebra $\mathcal{W}_N$ in more detail. By definition, the character of a $\mathcal{W}_N$ representation R is obtained through

$$\chi_R^{\mathcal{W}}(q, x) = \operatorname{tr}_R \left(q^{L_0^{\mathcal{W}} - \frac{c}{24}} x^{2J_0} \right) \tag{3.6}$$

where $L_0^{\mathcal{W}}$ denotes the zero mode of the coset Virasoro algebra and J_0 is the zero mode of the current (2.21). The subscript R refers to the choice of a representation of the chiral algebra $\mathcal{W}_N$.

As usual in coset conformal field theory we can obtain branching functions by decomposing the characters $\chi^{\mathrm{N}}(q, x, z_i)$ of the numerator $\mathrm{N} \equiv \mathrm{SO}(2M)_1$ into characters $\chi^{\mathrm{D}}(q, z_i)$ of the denominator $\mathrm{D} \equiv \mathrm{SU}(N)_{2N}$,

$$\chi_A^{\mathrm{N}}(q, x, z_i) = \operatorname{tr}_A \left(q^{L_0^{\mathrm{N}} - \frac{c}{24}} x^{2J_0} \prod_{i=1}^{N-1} z_i^{H_{i0}} \right) = \sum_{A,a} \mathcal{X}_{(A,a)}^{\mathcal{W}}(q, x) \chi_a^{\mathrm{D}}(q, z_i) \, . \tag{3.7}$$

Here we have twisted the characters of the numerator free fermion model with the zero modes H_{i0} of the Cartan currents of D in Chevalley basis (constructed from fermions). By the very definition of the coset chiral algebra $\mathcal{W}_N$ this implies that states of the coset algebra possess vanishing H_{i0} charge. In other words, all the dependence on the variables z_i on the right hand side of the previous equation is contained in the characters χ^{D} of the denominator algebra $\mathrm{SU}(N)_{2N}$.

The summation in eq. (3.7) runs over representations a of the denominator algebra, i.e. over weights $a = [\lambda_1, \ldots, \lambda_{N-1}]$ of SU(N) subject to the condition $\sum_{s=1}^{N-1} \lambda_s \leq 2N$. Note that the generator of the latter carry no charge with respect to the U(1) current $J(z)$ so that the corresponding characters are independent of x. The label A runs through the four sectors $A = id, v, sp, c$ of the SO($2M$) current algebra at level $k = 1$, as before.

Let us note the following fundamental properties of the branching functions introduced in eq. (3.7),

$$\mathcal{X}^{\mathcal{W}}_{(A,a)}(q, x) = 0 \qquad\qquad \text{if } Q_\gamma(a) \neq 0 \, , \tag{3.8}$$

$$\mathcal{X}^{\mathcal{W}}_{(A,a)}(q, x) = \mathcal{X}^{\mathcal{W}}_{(B,b)}(q, x) \qquad \text{if } \ B = A \, , \ b = \gamma^n(a) \tag{3.9}$$

for some choice of n. Using these two properties, we can rewrite eq. (3.7) in the form

$$\chi^{\mathrm{N}}_A(q, x, z_i) = \sum_{\{a\}, Q_\gamma(a)=0} \mathcal{X}^{\mathcal{W}}_{(A,a)}(q, x) S_{\{a\}}(q, z_i) \tag{3.10}$$

where the sum extends over orbits $\{a\}$ of denominator labels under the identification current γ whose monodromy charge vanishes and we defined

$$S_{\{a\}}(q, z_i) = \sum_{b \in \{a\}} \chi^{\mathrm{D}}_b(q, z_i) \, . \tag{3.11}$$

In order to progress, we must now insert explicit formulas for the various characters. The functions on the left hand side of eq. (3.10) are actually very easy to construct from the free fermion representation which gives

$$\begin{aligned} &\chi^{\mathrm{N}}_{id}(q, x, z_i) \pm \chi^{\mathrm{N}}_{v}(q, x, z_i) \\ &\qquad = \prod_{X \in \mathrm{SU}(N)} \left(q^{-1/24} \prod_{n=1}^{\infty} (1 \pm x^{1/3} z^{\alpha(X)} q^{n-1/2})(1 \pm x^{-1/3} z^{\alpha(X)} q^{n-1/2}) \right) , \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\chi^{\mathrm{N}}_{sp}(q, x, z_i) \pm \chi^{\mathrm{N}}_{c}(q, x, z_i) \\ &\qquad = \prod_{X \in \mathrm{SU}(N)} \left(q^{1/12} x^{1/6} \prod_{n=1}^{\infty} (1 \pm x^{1/3} z^{\alpha(X)} q^{n})(1 \pm x^{-1/3} z^{\alpha(X)} q^{n-1}) \right) , \end{aligned} \tag{3.13}$$

where

$$z^{\alpha(X)} \equiv z_1^{\alpha_1(X)} z_2^{\alpha_2(X)} \cdots z_{N-1}^{\alpha_{N-1}(X)} \tag{3.14}$$

and $\alpha(X) = (\alpha_1(X), ..., \alpha_{N-1}(X))$ is a root vector of SU(N). Of course, we can obtain explicit formulas for the characters $\chi^{\mathrm{N}}_A, A = id, v$ by taking the sum and difference of the expressions in the first line.

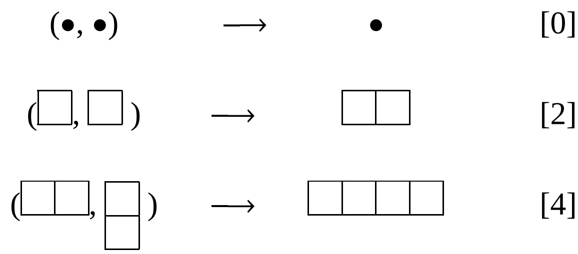


Figure 1: Pairs of Young diagrams (Y', Y'') inducing $Y \in \mathcal{J}_N^0$ for $N = 2$.

Characters of the denominator algebra $\mathrm{SU}(N)_{2N}$ are a little bit more complicated but of course also well known. In terms of the string functions $c_\lambda^b(q)$ of the denominator theory, the characters can be written as

$$\chi_b^{\mathrm{D}}(q, z_i) = \sum_{\lambda \in P/kM} c_\lambda^b(q) \Theta_\lambda(q, z_i)\,, \tag{3.15}$$

$$\Theta_\lambda(q, z_i) = \sum_{\beta^\vee \in M^\vee} q^{\frac{k}{2}|\beta^\vee + \lambda/k|^2} \prod_{i=1}^{N-1} z_i^{k(\beta^\vee + \lambda/k, \alpha_i^\vee)}\,, \tag{3.16}$$

where P and M ($M^\vee$) denote the weight and (co)root lattice of $\mathrm{SU}(N)$, respectively, and $\alpha_i^\vee$ are the simple coroots of $\mathrm{SU}(N)$. By comparing the q-expansion of the right-hand side of equation (3.10) with that of expressions (3.12,3.13), we find the x-dependence of the branching functions order by order in x and q.

3.3 Examples with small N

In order to illustrate the constructions we outlined above and to prepare for our search of chiral primaries, we want to work out some explicit results with $N \leq 5$. Let us recall that the central charge of the models with $N = 2$ and $N = 3$ satisfies $c_N < 3$ so that these two models are part of the minimal series of $\mathcal{N} = (2, 2)$ superconformal theories. The other two cases, $N = 4$ and $N = 5$, however, are outside this range and hence our results here are new.

N = 2

For $N = 2$, there are $|\mathcal{J}_2^0| = 3$ representations of $\mathrm{SU}(2)_4$ with vanishing monodromy charge. Such representations can be constructed from pairs of Young diagrams (Y', Y'') by eq. (3.2), as shown in figure 1. Under the action of $\gamma \in \mathbb{Z}_2$ these representations form two orbits. The first one is long, i.e. $N_{\{0\}} = N = 2$ and it consists of $\{[0], [4]\}$. There is a second orbit of length $N_{\{2\}} = 1$ which is given by $\{[2]\}$.

In the case at hand, it is actually possible to derive explicit expressions for the branching functions $\mathcal{X}^{\mathcal{W}}$ from the general decomposition formula (3.10), see appendix A,

$$
\begin{aligned}
\mathcal{X}^{\mathcal{W}}_{(id\pm v,[0])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm1)^n q^{\frac{3}{2}n^2}x^n\,,\\
\mathcal{X}^{\mathcal{W}}_{(id\pm v,[2])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm1)^{n+1} q^{\frac{3}{2}\left(n+\frac{1}{3}\right)^2}\left(x^{n+\frac{1}{3}}+x^{-(n+\frac{1}{3})}\right),\\
\mathcal{X}^{\mathcal{W}}_{(sp\pm c,[0])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm1)^n q^{\frac{3}{2}\left(n+\frac{1}{2}\right)^2}\,,\\
\mathcal{X}^{\mathcal{W}}_{(sp\pm c,[2])}(q,x) &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}(\pm1)^{n+1} q^{\frac{3}{2}\left(n+\frac{1}{6}\right)^2}\left(x^{n+\frac{1}{6}}\pm x^{-(n+\frac{1}{6})}\right).
\end{aligned}
\tag{3.17}
$$

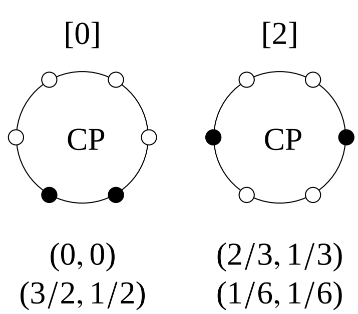
From these expressions we can read off the conformal weights of the ground states in all 8 sectors. Similarly, we can also determine the maximal value Q the U(1) charge can assume among the ground states of these sectors. In particular there are two sectors with $A{=}id$. The sector $(id,[0])$ is the vacuum sector with $h=0$ and $Q=0$.

As discussed at the end of section 2.2, the branching functions associated with the fixed points $(A,a_*)=(A,[2])$ can be decomposed into two characters of our algebra $\mathcal{W}_2$. For $A=id$ and v, for example, these characters read

$$
\begin{aligned}
\chi^{\mathcal{W}}_{(id,[2],1)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{3}\right)^2}x^{2(n+\frac{1}{3})}\,,\\
\chi^{\mathcal{W}}_{(id,[2],2)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{3}\right)^2}x^{-2(n+\frac{1}{3})}\,,\\
\chi^{\mathcal{W}}_{(v,[2],1)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{6}\right)^2}x^{2(n+\frac{1}{6})}\,,\\
\chi^{\mathcal{W}}_{(v,[2],2)} &= \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{6\left(n+\frac{1}{6}\right)^2}x^{-2(n+\frac{1}{6})}\,.
\end{aligned}
\tag{3.18}
$$

It is easy to check that these formulas agree with the expressions (2.15) and (2.16) when $x=1$.

Let us also display the necklace patterns for the two orbits $\{[0],[4]\}$ and $\{[2]\}$. The affine Dynkin labels for these orbits are $[4,0]$ (or $[0,4]$) and $[2,2]$. These correspond to the following two necklaces,



In the two lines below the necklace we use tuples (h, Q) to display the ground state energy h and maximal U(1) charge Q among the ground states of the sectors with A=id (first line) and A=v (second line). In principle, there are also two sectors with A=sp, c which we do not show here. The label 'CP' we placed in the center of the two necklaces will be explained in the next section.

Let us finally also spell out the full partition function of the model. In the case at hand, our general expression (2.19) reads as follows

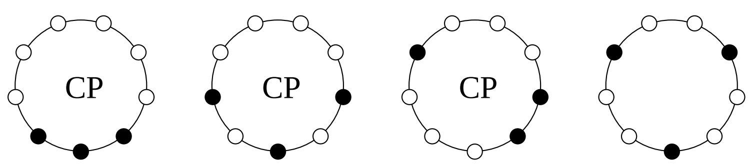
$$\begin{aligned} Z_2 &= \sum_A \left(|\chi^{\mathcal{W}}_{(A,[0])}|^2 + |\chi^{\mathcal{W}}_{(A,[2],1)}|^2 + |\chi^{\mathcal{W}}_{(A,[2],2)}|^2 \right) \\ &= \frac{1}{|\eta(q)|^2} \sum_{n,w\in\mathbb{Z}} q^{\frac{k_L^2}{2}} \bar{q}^{\frac{k_R^2}{2}} x^{rk_L} \bar{x}^{rk_R} \quad \text{with} \quad k_{L,R} = \frac{n}{r} \pm \frac{wr}{2}, \quad r = 2\sqrt{3} \end{aligned} \tag{3.19}$$

see appendix A.3 for a few more details. The resummation leading to the second line is in principle straight-forward. The final result coincides with the usual partition function of a free boson compactified on a circle of radius $2\sqrt{3}$.

N = 3

After having gone through the example of $N = 2$ quite carefully, we can now be a bit more sketchy with $N = 3$. In this case we obtain $|\mathcal{J}^0_3| = 10$ representations of $SU(3)_6$ with vanishing monodromy charge. They can be grouped into four orbits, three of which have length $N = 3$ while the last one has length $N_{[2,2]} = 1$. More explicitly, the orbits are given by $\{[0,0], [6,0], [0,6]\}$,$\{[1,1], [4,1], [1,4]\}$, $\{[3,0], [0,3], [3,3]\}$ and $\{[2,2]\}$. The reader is invited to recover this list from pairs of Young diagrams Y' and Y'', as explained in section 3.1.1 above. The four orbits are associated with the following four necklaces

[0,0]	[1,1]	[3,0]	[2,2]



$(0, 0)$	$(2/3, 1/3)$	$(1/3, 1/3)$	$(1/9, 0)$
$(3/2, 1/2)$	$(1/6, 1/6)$	$(5/6, 1/2)$	$(11/18, 1/2)$

The lines below these diagrams display again some information about the associated branching functions $\mathcal{X}^{\mathcal{W}}_{(A,a)}$ for A=id and A=v, namely the ground state energy h and the maximum Q of the U(1) charge. These results can be read off from the branching functions which we computed numerically, see appendix B.1 for the first few terms.

According to the general formula (2.13), the function $\tilde{Z}_N$ takes the form

$$\tilde{Z}_3(q) = \sum_A \left(|\mathcal{X}^{\mathcal{W}}_{(A,[0,0])}|^2 + |\mathcal{X}^{\mathcal{W}}_{(A,[1,1])}|^2 + |\mathcal{X}^{\mathcal{W}}_{(A,[3,0])}|^2 + \tfrac{1}{3} |\mathcal{X}^{\mathcal{W}}_{(A,[2,2])}|^2 \right) . \tag{3.20}$$

Once again, the x-dependent branching functions for the short orbit can be decomposed into a sum of $\mathcal{W}_N$ characters. For instance, the branching function[5]

$$\mathcal{X}^{\mathcal{W}}_{(id,[2,2])}(q,x) = 1 + (2x^{4/3} + 3x^{2/3} + 5 + 3x^{-2/3} + 2x^{-4/3})\, q + O(q^2)\,, \tag{3.21}$$

can be written as the sum of three characters,

$$\begin{aligned} \chi^{\mathcal{W}}_{(id,[2,2],1)}(q,x) &= \tfrac{1}{2}\Big(\mathrm{ch}^{\mathrm{NS,ext}}_{1/9} + \widetilde{\mathrm{ch}}^{\mathrm{NS,ext}}_{1/9}\Big) = 1 + (x^{2/3} + 3 + x^{-2/3})\, q + O(q^2)\,, \\ \chi^{\mathcal{W}}_{(id,[2,2],p)}(q,x) &= \tfrac{1}{2}\Big(\mathrm{ch}^{\mathrm{NS,ext}}_{11/18} + \widetilde{\mathrm{ch}}^{\mathrm{NS,ext}}_{11/18}\Big) = (x^{4/3} + x^{2/3} + 1 + x^{-2/3} + x^{-4/3})\, q + O(q^2) \end{aligned} \tag{3.22}$$

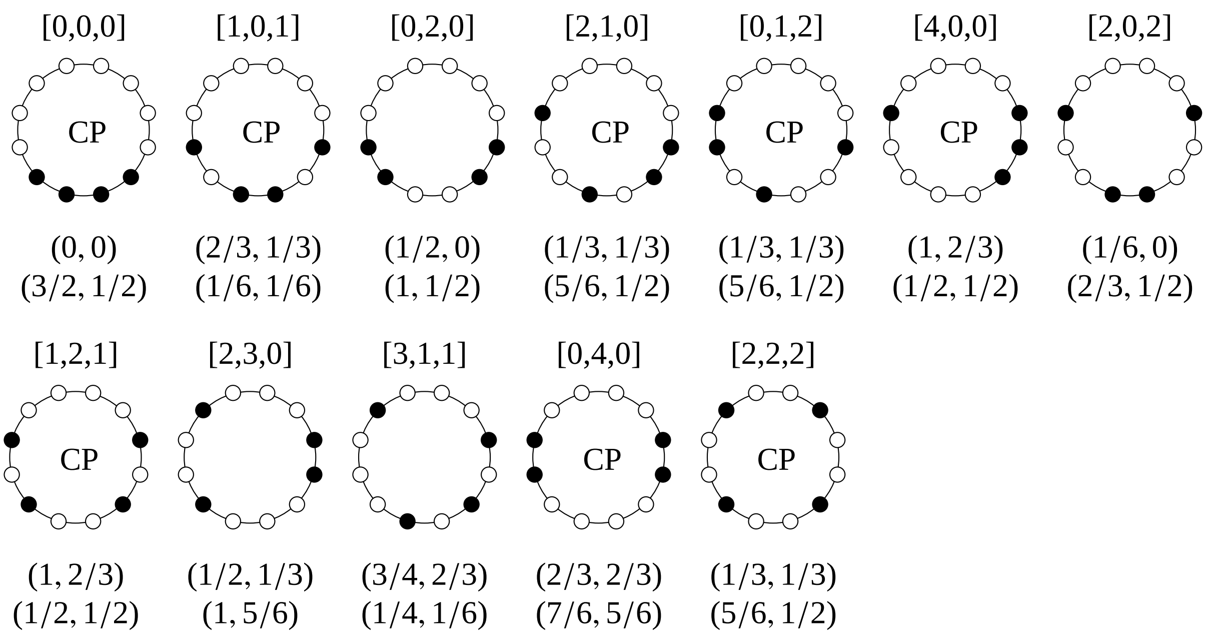
for $p = 2, 3$. Here, the $\mathrm{ch}^{\mathrm{NS,ext}}$ are extended $\mathcal{N} = 2$ characters, as defined in [7]. After the resolution of the fixed point in the sectors $(A, [2, 2])$, we obtain the partition function

$$Z_3(q,x) = \sum_A \Big(|\chi^{\mathcal{W}}_{(A,[0,0])}|^2 + |\chi^{\mathcal{W}}_{(A,[1,1])}|^2 + |\chi^{\mathcal{W}}_{(A,[3,0])}|^2 + |\chi^{\mathcal{W}}_{(A,[2,2],1)}|^2 + 2|\chi^{\mathcal{W}}_{(A,[2,2],2)}|^2 \Big)\,, \tag{3.23}$$

where all summands are considered as functions of both q and x. Of course, for $x = 1$ we recover the expression (2.19) for the modular invariant partition function we described above.

N = 4

For $N = 4$ there exist twelve different orbits which are labeled by the following necklaces



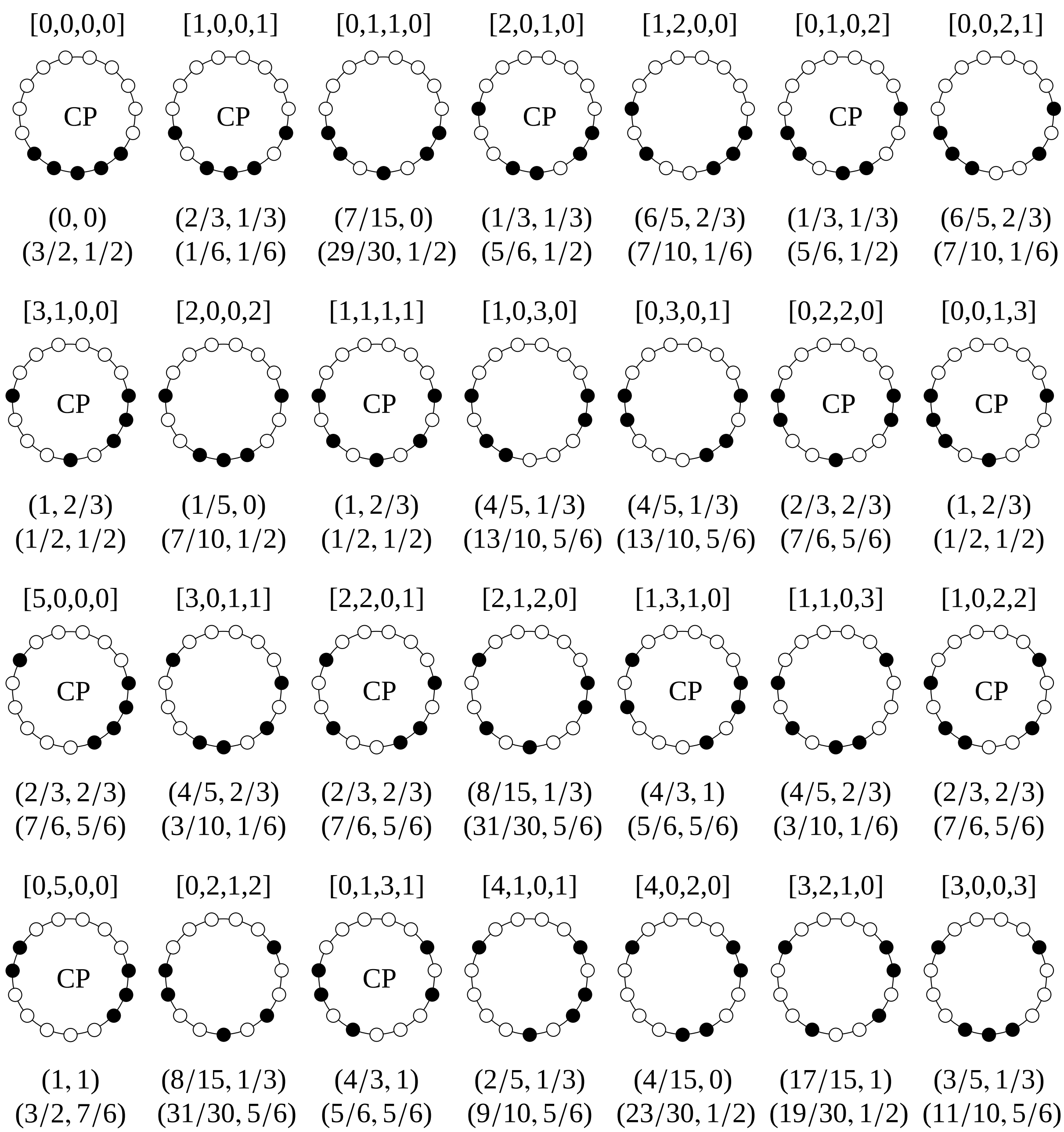
Note that in this case there are two short orbits. While the orbit $\{[0,4,0], [4,0,4]\}$ has length $N_{[0,4,0]} = 2$, the element $a_* = \{[2,2,2]\}$ is fixed under the action of γ and hence gives an orbit of length $N_{[2,2,2]} = 1$. The branching functions of all orbits are displayed in appendix B.2. Note that

[5] In comparison with appendix B.1, we have reintroduced the factor $q^{-c_3/24}$ in $\mathcal{X}^{\mathcal{W}}$ ($c_3 = 8/3$).

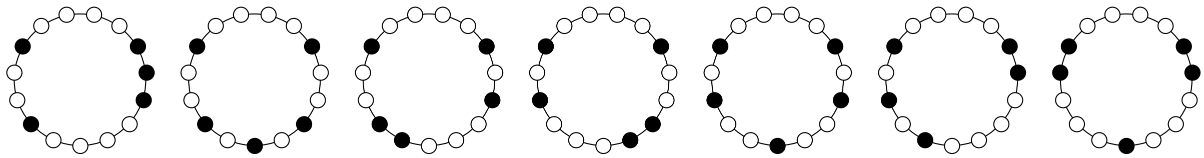
4×2 of these branching functions should be decomposed into the sum of $\mathcal{W}_N$ characters since they are associated to short orbits. For the remaining ones, the branching functions coincide with the characters. We shall not discuss the resolution of fixed points and the partition function of the system in any more detail.

N = 5

Since $N = 5$ is the first prime number beyond the minimal model bound, the final case in our discussion is the most important one. For $N = 5$ there are 41 different orbits which are labeled by the following necklaces:

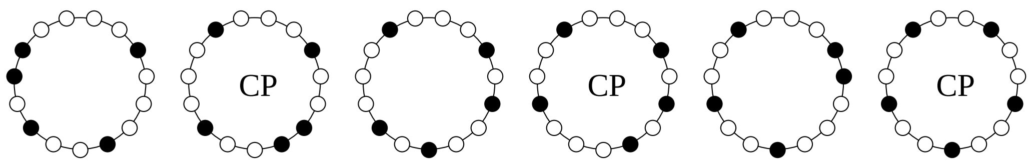


[2,4,0,0] [2,1,1,2] [2,0,3,1] [1,3,0,2] [1,2,2,1] [1,1,4,0] [0,3,3,0]



[2,4,0,0]	[2,1,1,2]	[2,0,3,1]	[1,3,0,2]	[1,2,2,1]	[1,1,4,0]	[0,3,3,0]
$(13/15, 2/3)$	$(2/5, 1/3)$	$(6/5, 1)$	$(6/5, 1)$	$(16/15, 1)$	$(4/5, 2/3)$	$(3/5, 0)$
$(41/30, 7/6)$	$(9/10, 5/6)$	$(7/10, 1/2)$	$(7/10, 1/2)$	$(17/30, 1/2)$	$(13/10, 7/6)$	$(11/10, 5/6)$

[0,1,2,3] [3,2,0,2] [3,1,2,1] [2,3,1,1] [2,2,3,0] [2,2,2,2]



[0,1,2,3]	[3,2,0,2]	[3,1,2,1]	[2,3,1,1]	[2,2,3,0]	[2,2,2,2]
$(17/15, 1)$	$(1, 2/3)$	$(13/15, 2/3)$	$(2/3, 2/3)$	$(7/15, 1/3)$	$(1/3, 0)$
$(19/30, 1/2)$	$(1/2, 1/2)$	$(11/30, 1/6)$	$(7/6, 5/6)$	$(29/30, 5/6)$	$(5/6, 5/6)$

As usual, the length of the orbit $a_* = \{[2,2,2,2]\}$ is $N_{a_*} = 1$. All other orbits are of maximal length N. The first few terms of the branching functions are displayed in appendix B.3.

Let us briefly describe how to resolve the fixed point when we work with x-dependent branching functions and characters. One may find the following expression for the branching function

$$\mathcal{X}^{\mathcal{W}}_{(id,[2,2,2,2])} = 1 + (4y^2 + 19y^{4/3} + 36y^{2/3} + 47)\, q + O(q^2)\,, \tag{3.24}$$

in appendix B.3. It can be written as a sum of five functions,

$$\begin{aligned} \chi^{\mathcal{W}}_{(id,[2,2,2,2],1)} &= 1 + (3y^{4/3} + 8y^{2/3} + 11)\, q + O(q^2)\,, \\ \chi^{\mathcal{W}}_{(id,[2,2,2,2],p)} &= (y^2 + 4y^{4/3} + 7y^{2/3} + 9)\, q + O(q^2) \end{aligned} \tag{3.25}$$

for $p = 2, ..., 5$, which we propose for the characters. Here we have introduced the shorthand $y^n \equiv x^n + x^{-n}$. Note that for $x = 1$ the coefficients of q in the characters must equal $165/5 = 33$. Then, after resolution of the fixed point, we get

$$Z_5 = \sum_A \Big(\sum_{\{a\} \in O_5/\{[2,2,2,2]\}} |\chi^{\mathcal{W}}_{(A,a)}|^2 + |\chi^{\mathcal{W}}_{(A,[2,2,2,2],1)}|^2 + 4|\chi^{\mathcal{W}}_{(A,[2,2,2,2],2)}|^2 \Big)\,. \tag{3.26}$$

This concludes our brief discussion of branching functions, characters and partition functions for the examples with $N \leq 5$.

4 Chiral primary fields

In this section we will describe the main results of this work. We have described the chiral symmetry $\mathcal{W}_N$ and the complete modular invariant partition function Z_N for a family of field theories with $\mathcal{N} = (2,2)$ superconformal symmetry. Our goal now is to determine the chiral primaries of these models. Since we know how the spectrum of the model is built from the various representations of the chiral algebra $\mathcal{W}_N$ all that is left to do is to find (anti-)chiral primaries in the individual sectors. In principle this is straightforward once the characters of the chiral algebra are known. Indeed, for $N \leq 5$, the chiral primaries can be read off from the q-expanded branching functions listed in appendix B. In section 4.1, we show that there exists an upper bound on the conformal weight of a chiral primary. In order to organize the chiral primaries, we will then define and discuss in section 4.2 the class of *regular* chiral primaries. In section 4.3, we discuss a few examples and show that for $N \leq 3$ there are no other chiral primaries besides the regular ones. This will change for theories with $N \geq 4$, as we shall show in section 4.4.

4.1 Bound on the dimension of chiral primaries

There are a few general results on the dimension of chiral primaries that are useful to discuss before we get into concrete examples. In any $\mathcal{N} = 2$ superconformal field theory, the conformal weight of chiral primaries is bounded from above by

$$h(\phi_{\rm cp}) \leq \frac{c}{6} \tag{4.1}$$

where c is the central charge of the Virasoro algebra [19]. This bound is independent of the sector in which the chiral primary resides.

In order to derive stronger sector dependent bounds, we recall that the fields in the numerator theory satisfy $h^{\rm N} \geq 3|Q^{\rm N}|$. States that make it into the coset sector $(A, \{a\})$ contain the highest weight vector of a $\mathrm{SU}(N)_{2N}$ representation $b \in \{a\}$ in the orbit of a. The latter has weight $h^{\rm D}_b$ and charge $Q^{\rm D}_b = 0$. For the dimension h and charge Q of the coset fields we obtain the constraint $h + h^{\rm D}_b = h^{\rm N} \geq 3|Q^{\rm N}| = 3|Q|$ and consequently for coset states ϕ in the sector $(A, \{a\})$,

$$h(\phi) \geq 3|Q(\phi)| - min_{b\in\{a\}}(h^{\rm D}_b)\ .$$

For (anti-)chiral primaries $\phi_{\rm cp}$ with $h(\phi_{\rm cp}) = |Q(\phi_{\rm cp})|$ this inequality implies that

$$2h(\phi_{\rm cp}) \leq min_{b\in\{a\}}(h^{\rm D}_b) \quad \text{ or } \quad h(\phi_{\rm cp}) \leq min_{b\in\{a\}}\left(\frac{C_2(b)}{6N}\right)\ . \tag{4.2}$$

In addition to this constraint, the U(1) charges must also satisfy $|Q| = k/6$ ($k \in \mathbb{N}_0$). It is easy to see that this implies

$$min_{b\in\{a\}}\left(\frac{C_2(b)}{N}\right) \in \mathbb{Z}$$

if the sector $(A, \{a\})$ is to contain a chiral primary and N is odd. For even N a similar condition holds with N replaced by $N/2$.

As we shall see below there exist some important sectors for which this bound is so strong that it does not permit chiral primaries above the ground states. In other sectors, however, our bound (4.2) is much less powerful. This applies in particular to those that are associated with the fixed point a_*. In fact, in the representation a_*, the quadratic Casimir assumes its largest eigenvalue

$$C_2(a_*) = \frac{1}{3}N(N^2-1) \quad \text{or} \quad \frac{C_2(a_*)}{6N} = \frac{c_N}{6}\ .$$

Hence, in the fixed point sectors our bound (4.2) coincides with the universal bound (4.1). This appears to leave a lot of room for chiral primaries.

4.2 Regular chiral primaries

As we stressed before, there exists a large set of chiral primaries that may be constructed very explicitly for any value of N. Their description is particularly simple when we use our parametrization of orbits in terms of two Young diagrams Y' and Y'', see section 3.1. We will determine $\mathcal{W}$ sectors containing regular chiral primaries in the first subsection and then count regular chiral primaries of the full (non-chiral) conformal field theory in the second.

4.2.1 Parametrization and properties

In section 3 we constructed all solutions $a \in \mathcal{J}_N$ of the vanishing monodromy condition $Q(a) = 0$ in terms of a pair of Young diagrams $Y' = Y'(a)$ and $Y'' = Y''(a)$. As indicated in our notation we now think of these Young diagrams as functions of the sector label a. This map is obtained by reversing our formula (3.2) for the construction of $Y(a)$ from Y' and Y''. As we shall show below, for elements of the following subset

$$\mathcal{S}_N = \{a = (l_1, \dots, l_r) | Y'(a) = Y''(a); Q(a) = 0\} \subset \mathcal{J}^0_N \tag{4.3}$$

we can find a regular chiral primary in the coset sectors $(A, \{a\})$ with $A = id$ if $|Y'|$ is even and $A = v$ otherwise.

There is a relatively simple geometrical construction of the Young diagrams $Y(a)$ that can be obtained with $Y' = Y''$. In fact, it follows from eq. (3.2) that a Young diagram $Y(a), a \in \mathcal{S}_N$ is obtained from Y' by first completing Y' to an $r' \times N$ rectangular Young diagram and then attaching the (rotated) 'complementary' diagram $(N - l'_{r'}, ..., N - l'_1)$ from the left to the original Young diagram Y'. An example is shown in figure 2.

Note that the condition $Y'(a) = Y''(a)$ is not invariant under the action of the identification current. So, in order to find out whether a sector $(A, \{a\})$ contains a regular chiral primary, one has to check the condition $Y'(b) = Y''(b)$ for all $b \in \{a\}$.

There are two important remarks we have to make concerning the precise relation between coset sectors containing regular chiral primaries and elements of the set $\mathcal{S}_N$. The first one

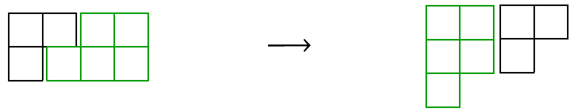


Figure 2: Generation of a representation $Y \in \mathcal{S}_4$. The Young diagram Y' with lengths $(l'_1, l'_2) = (2, 1)$ (black) induces the SU(4) representation Y with lengths $(l_1, l_2, l_3) = (4, 3, 1)$.

concerns the sectors obtained from $a_* = [2, 2, \ldots, 2]$ that give rise to the unique fixed points for N prime. A moment of thought about the construction we sketched above shows that a_* can never be in $\mathcal{S}_N$ unless $N = 2$. Hence the distinction between coset sectors and labels $(A, \{a\})$, as well as all our discussion of fixed point resolutions, is not relevant for the discussion of regular chiral primaries when N is a prime number.

More importantly, we want to stress that there exist orbits $\{a\} \in \mathcal{O}_N$ that contain two elements from $\mathcal{S}_N$. It is not too difficult to list these orbits explicitly. From the general construction of $\mathcal{S}_N$ we can infer that

$$a_\nu = [0, ..., 0, N, 0, ..., 0] \in \mathcal{S}_N$$

for $\nu = 1, \ldots, N-1$. Here, the only non-zero entry N can appear in any position ν, i.e. $\lambda_\nu = N$. Field identifications can map a_ν to $a_{N-\nu}$ so that we have now found $\lfloor (N-1)/2 \rfloor$ orbits that contain two elements of $\mathcal{S}_N$. One may also argue that these are the only ones that contain more than one element of $\mathcal{S}_N$.

For coset sectors $(A, \{a\})$ with $a \in \mathcal{S}_N$ there exists a simple formula to compute their exact conformal weight. By eq. (3.3) the quadratic Casimir of a representation $a \in \mathcal{S}_N$ is simply $C_2(a) = n'N$. This implies that the conformal weight of the ground states is

$$h(\psi_{(A,\{a\})}) = \frac{C_2(a)}{6N} = \frac{n'}{6}\,, \tag{4.4}$$

for $a \in \mathcal{S}_N$. Let us add that the number $n' = |Y'| = |Y''|$ of boxes in Y' is given in terms of the representation labels $(l_1, ..., l_r)$ of a by

$$n' = \sum_{i=1}^{r'} l'_i = \sum_{i=1}^{r'} (l_i - r') \tag{4.5}$$

with $r' = n/N$ and where $n = \sum_{i=1}^{r} l_i$ is the number of boxes of $Y = Y(a)$. With the help or sector dependent bound from the previous subsection, see eq. (4.2), we can now show that in all the sectors associated with $a \in \mathcal{S}_N$, chiral primaries must be ground states of the $\mathcal{W}$ algebra. In fact, by combining the conformal weight (4.4) with the bound (4.2), we find

$$\frac{n'}{6} = h(\psi_{(A,\{a\})}) \leq h(\phi_{\text{cp}}) \leq min_{b\in\{a\}} \left(\frac{C_2(b)}{6N}\right) = \frac{n'}{6}\,.$$

Hence, these sectors cannot contain any chiral primaries in addition to the ones we will find among their ground states.

4.2.2 Counting of regular chiral primaries

Before we look into examples let us count the regular chiral primaries along with their conformal weight. This will proceed in several steps. First we shall count the number of elements in $\mathcal{S}_N$, then we employ the result to count the number representations of our chiral algebra $\mathcal{W}$ that contain a regular chiral primary and finally we determine the counting function for regular chiral primaries from the full partition function of the model, at least for N prime.

Our description of the set $\mathcal{S}_N$ in terms of Young diagrams $Y' = Y''$ makes it an easy task to determine $|\mathcal{S}_N|$. The conditions for the choice of Y' and Y'' we spelled out before eq. (3.2). They imply that diagrams Y' corresponding to elements in $\mathcal{S}_N$ must fit into a rectangle of size $r' \times c'$ with $r' + c' = N$. Such Young diagrams are counted through the series

$$\tilde{T}_N(\mathfrak{q}) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_{\mathfrak{q}} - \sum_{k=0}^{N-1} \begin{bmatrix} N-1 \\ k \end{bmatrix}_{\mathfrak{q}} = \sum_{k=0}^{N-1} \mathfrak{q}^k \begin{bmatrix} N-1 \\ k \end{bmatrix}_{\mathfrak{q}} , \tag{4.6}$$

which is denoted by A161161 in [18]. The $\mathfrak{q}$-binomial coefficient that multiplies $\mathfrak{q}^k$ counts all Young diagrams Y' that fit into a rectangle with $(N-1-k)\times k$ boxes. The factor $\mathfrak{q}^k$ corresponds to attaching to each of these Young diagrams from the left a single column of k boxes. As a consequence, the individual summands in eq. (4.6) count the number of Young diagrams fitting into a rectangle with $(N-k)\times k$ boxes. By the binomial theorem we find

$$|\mathcal{S}_N| = \tilde{T}_N(1) = \sum_{k=0}^{N-1} \binom{N-1}{k} = 2^{N-1} .$$

Let us list also the coefficients $\tilde{t}_n^N$ of the function $\tilde{T}_N(\mathfrak{q}) = \sum_n \tilde{t}_n^N \mathfrak{q}^n$ for all values with $N \leq 7$,

$$\begin{aligned}
N = 2 :& \quad 1, 1\\
N = 3 :& \quad 1, 1, 2\\
N = 4 :& \quad 1, 1, 2, 3, 1\\
N = 5 :& \quad 1, 1, 2, 3, 5, 2, 2\\
N = 6 :& \quad 1, 1, 2, 3, 5, 7, 5, 4, 3, 1\\
N = 7 :& \quad 1, 1, 2, 3, 5, 7, 11, 8, 9, 7, 6, 2, 2 .
\end{aligned}$$

Let us note in passing that, at large N, the coefficients $\tilde{t}_n^N$ of $\tilde{T}_N(\mathfrak{q})$ coincide with the number p_n of partitions of n, i.e.

$$\lim_{N\to\infty} \tilde{T}_N(\mathfrak{q}) = \prod_{k=1}^{\infty} \frac{1}{1-\mathfrak{q}^k} = \sum_{n=0}^{\infty} p(n)\mathfrak{q}^n . \tag{4.7}$$

In order to count the number of $\mathcal{W}$ representations that contain a regular chiral primary we recall two facts discussed above. The first one concerns the fixed point resolution. When $N > 2$

is prime, there is only one short orbit a_* and since a_* is not a element of $\mathcal{S}_N$ the counting of *regular* chiral primaries is not affected by the fixed point resolution. On the other hand, there are a few orbits that contain two elements of $\mathcal{S}_N$. These need to be subtracted from the counting function $\tilde{T}_N(\mathfrak{q})$ in order to obtain a counting function $T_N(\mathfrak{q})$ for $\mathcal{W}$ sectors containing regular chiral primaries

$$T_N(\mathfrak{q}) := \tilde{T}_N(\mathfrak{q}) - \sum_{n=1}^{\lfloor\frac{N-1}{2}\rfloor} \mathfrak{q}^{n(N-n)/6}\,. \tag{4.8}$$

As explained above, the over-counting we are trying make up for is associated with the Dynkin labels $a_v \in \mathcal{S}_N$. Since the sector (A, a_v) containing the associated regular chiral primary has conformal weight $h_{(A,[0\ldots N\ldots 0])} = n(N-n)/6$, we have included the appropriate power of $\mathfrak{q}$ in our subtraction.

After this preparation we can now turn to the counting of (regular) chiral primaries. By their very definition, (anti-)chiral primaries are fields in the Neveu-Schwarz sector of the theory for which the conformal weight h and the U(1) charge Q satisfy $h = \pm Q$. Here, the upper sign applies to chiral primaries while the lower one is relevant for anti-chiral primaries. It is then obvious that chiral primaries are counted by

$$Z_N^{\mathrm{cp}}(q,\bar{q}) = \frac{1}{(2\pi i)^2}\oint\frac{dx}{x}\oint\frac{d\bar{x}}{\bar{x}}\, Z_N^{\mathrm{NS}}(qx^{-2},\overline{qx}^{-2},x,\bar{x}) \tag{4.9}$$

where Z_N^{NS} denotes the contribution from the NS sector of the model, i.e. the summands $A{=}id$ and $A = v$, to the full (resolved) partition function. For anti-chiral primaries, the first two arguments of the partition function in the integrand must be replaced by qx^2 and $\overline{qx}^2$, respectively. We know that this counting function for chiral primaries receives contributions from the regular ones. The latter have been determined above so that

$$Z_N^{\mathrm{cp}}(q,\bar{q}) = Z_N^{\mathrm{cp,reg}}(q,\bar{q}) + Z_N^{\mathrm{cp,exc}}(q,\bar{q}) = T_N\left((q\bar{q})^{1/6}\right) + Z_N^{\mathrm{cp,exc}}(q,\bar{q})\,. \tag{4.10}$$

The counting function T_N for regular chiral primaries has been constructed in eqs. (4.8) and (4.6) above. If all chiral primaries were regular, there would be no additional contributions. But we shall see below that this is not the case. Starting from $N = 4$ not all chiral primaries are regular. The additional *exceptional* chiral primaries are counted by $Z_N^{\mathrm{cp,exc}}$.

4.3 Examples with $N \leq 3$: Minimal models

The aim of this and the following subsection is to illustrate our general constructions through the first two examples, namely $N = 2$ and $N = 3$. These possess central charge $c_N < 3$ and hence they belong to the minimal series of $\mathcal{N} = (2,2)$ superconformal minimal models. For models from this series the chiral primaries are well known. Our only task is therefore to show that the general constructions of regular chiral primaries outlined in the previous subsection allows us to recover all known chiral primaries.

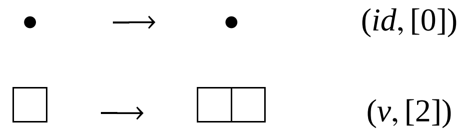


Figure 3: The sets of Young diagrams Y' and Y for $N = 2$.

N = 2

Let us start by reviewing briefly the case of $N = 2$ which gives a CFT with Virasoro central charge $c_2 = 1 \leq 3$. In section 3.3 we have listed all the sectors of this model along with the conformal weight and maximal R-charge of their ground states. From the results we can easily deduce that there are only two sectors containing chiral primaries, namely the sectors $(id, [0])$ and $(v, [2], 1)$. Recall that the label $(v, [2])$ labels a branching function that can be decomposed into a sum of two characters. These characters, which were displayed in eq. (3.18), show that only one of the corresponding sectors contains a chiral primary. Moreover, since the conformal weight of all chiral primaries is bounded by $c_2/6 = 1/6$ there can be no chiral primaries among the excited states of the model. Hence, we conclude that model contains two chiral primaries. One is the identity field, the other one a chiral primary of weight $h = 1/6$.

Let us reproduce this simple conclusion from the construction of regular chiral primaries. The construction we sketched above instructs us to list all Young diagrams Y' that can fit into a rectangle of size $r' \times c'$ where $r' + c' = N = 2$. Obviously, there are only two such Young diagrams, namely the trivial one and the single box. These are depicted in the leftmost column of figure 3. Applying the general prescription (3.2) (with $Y'' = Y'$) we obtain two Young diagrams Y in the second column. From the two columns we can read off the labels of the corresponding coset sectors $(A, \{a\})$. These are displayed in the third column. As we explained above, the first label A is determined by the number of boxes n' of the Young diagram Y' in the first column. It is $A = id$ if n' is even and $A = v$ otherwise. The second entry contains the orbit $\{a\}$ of the $SU(2)_4$ representation a that is associated with the Young diagram Y in the second column. According to eq. (4.4), the conformal weights of the corresponding chiral primaries are given by $h(\psi_{(A,a)}) = |Y'|/6$. In this case, we recovered all chiral primaries through the construction of the regular ones. The counting function for chiral primaries is given by

$$Z_2^{\rm cp}(q, \bar{q}) = 1 + (q\bar{q})^{1/6} \; = T_2\left((q\bar{q})^{1/6}\right) \tag{4.11}$$

and it obviously coincides with the counting function for regular chiral primaries we stated in the previous subsection.

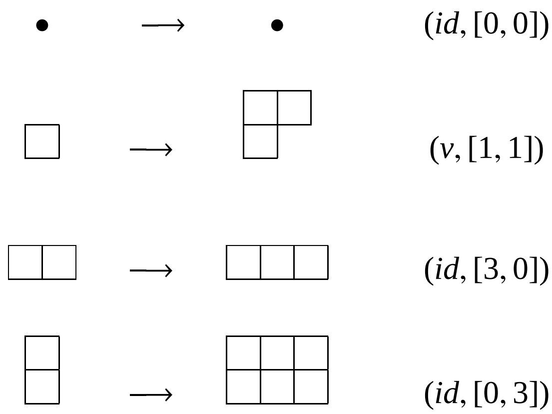


Figure 4: The sets of Young diagrams Y' and Y for $N = 3$.

N = 3

For $N = 3$ we can proceed similarly. In this case, the model has central charge $c_3 = 8/3$, still below the critical value $c = 3$. It is well known to possess 3 chiral primaries of conformal weights $h = 0, 1/6, 1/3$ [7] and one can verify this statement through a quick glance at the data we provided with the list of necklaces in section 3.3. Note that the three necklaces that are associated with chiral primaries have been marked with the letters 'CP' in the center.

Let us now apply our general construction of regular chiral primaries to the case $N = 3$. To begin with, we must list all the Young diagrams Y' that fit into a rectangle of size 1×2 or 2×1. There are four such diagrams which are depicted in the leftmost column of figure 4. Application of the construction (3.2) (with $Y'' = Y'$) gives four representations of SU(3), as shown in the second column of figure 4. The corresponding coset sectors are listed in the right column. In this case two of the obtained sectors coincide since the $\mathrm{SU}(3)_6$ sectors $[0, 3]$ and $[3, 0]$ are related by the simple current automorphism. Hence we end up with three inequivalent coset representations whose ground states can provide a regular chiral primary. Their labels are displayed in the third column of figure 4.

We can easily scan the partition function Z_3 given in eq. (3.23) for chiral primaries from the list in the third column of figure 4 to obtain

$$Z_3^{\mathrm{cp}}(q, \bar{q}) = 1 + (q\bar{q})^{1/6} + (q\bar{q})^{1/3} \,. \tag{4.12}$$

The answer agrees with the counting function for regular chiral primaries we proposed in the previous subsection, i.e.

$$Z_3^{\mathrm{cp}}(q, \bar{q}) = Z_3^{\mathrm{cp,reg}}(q, \bar{q}) = \tilde{T}_3((q\bar{q})^{1/6}) - (q\bar{q})^{1/3} \,.$$

The subtraction of $(q\bar{q})^{1/3}$ is explained by the field identification $(id, [3, 0]) = (id, [0, 3])$.

4.4 Exceptional chiral primaries for $N \geq 4$

For $N = 2, 3$ the complete set of chiral primaries is given by the class of regular chiral primaries. While this class still plays a role for higher N we shall find additional chiral primaries when $N \geq 4$. We refer to them as *exceptional* chiral primaries.

N = 4

For $N = 4$ the central charge $c_4 = 5$ exceeds the bound $c = 3$ that can be reached with supersymmetric minimal models. Therefore we can no longer rely on known results on the set of chiral primaries. Let us therefore first apply our general constructions of regular chiral primaries and then check whether they provide the complete set of chiral primaries.

The analysis is summarized in figure 5. In the first column we list all the Young diagrams Y' which can fit into rectangles of size 1×3 or 3×1 or 2×2. From these we build Young diagrams for representations of SU(4) with the help of eq. (3.2). The results are shown in the second column. Taking the first two columns together we determine the list of coset sectors shown in the third column. Note that $(v, [4, 0, 0])$ and $(v, [0, 0, 4])$ refer to the same sector of the model since $[4, 0, 0]$ may be obtained from $[0, 0, 4]$ by applying the simple current automorphism. Hence, our construction gives seven different coset sectors whose ground states are chiral primary.

In order to check whether we are missing any chiral primaries of the model, we must scan the space of states with conformal weight $h \leq c_4/6 = 5/6$, or a little less if we used the sector dependent bound (4.2). Since $5/6 < 1$, all chiral primaries must be ground states. Hence we can perform the scan by looking through the pairs (h, Q) we displayed when we listed the necklaces for $N = 4$ in section 3.3. Those necklaces that give rise to chiral primaries have already been marked by a 'CP' in the center. Not surprisingly we find all the seven regular chiral primaries from the third column of figure 5.

On the other hand, the scan we just performed gives one more chiral primary that does not appear in the right column of figure 5, namely a ground state of the coset sector $(id, [2, 2, 2])$. This new chiral primary has conformal weight $h = Q = 1/3$ and it is our first example of an exceptional chiral primary. Our findings may be summarized in the following expression

$$Z_4^{\text{cp}}(q, \overline{q}) = 1 + (q\overline{q})^{1/6} + 3(q\overline{q})^{1/3} + 2(q\overline{q})^{1/2} + (q\overline{q})^{2/3} \tag{4.13}$$

which is equal to

$$Z_4^{\text{cp}}(q, \overline{q}) = Z_4^{\text{cp,reg}}(q, \overline{q}) + (q\overline{q})^{1/3} .$$

The additional term $(q\overline{q})^{1/3}$ counts the exceptional chiral primary. A word of caution is in order. In general, chiral primaries can appear in coset sectors which are fixed points of the theory. As we discussed before, such fixed points must be resolved, and it is not *a priori* clear whether this

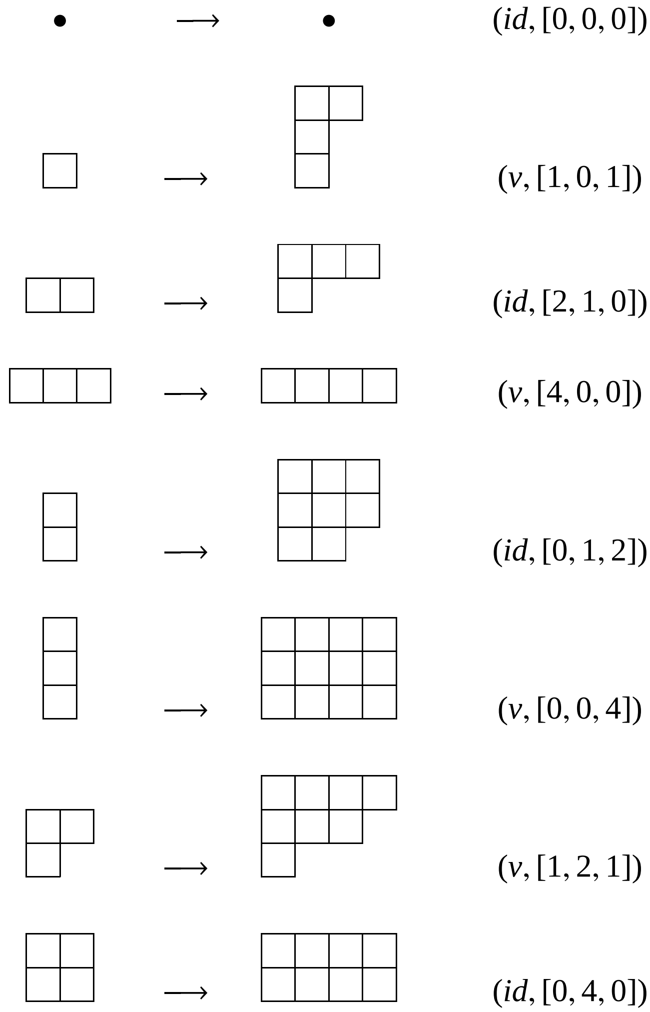


Figure 5: The sets of Young diagrams Y' and Y for $N = 4$.

changes the multiplicity of the chiral primaries or not. For $N = 4$ both the sector $(id, [0, 4, 0])$ and $(id, [2, 2, 2])$ are fixed points and their ground states are chiral primary. The chiral primaries appear with multiplicity one in both $\mathcal{X}^{\mathcal{W}}_{(id+v,[0,4,0])}$ and $\mathcal{X}^{\mathcal{W}}_{(id+v,[2,2,2])}$, as can be seen from the their q-expansions in appendix B.2. The result (4.13) for $Z_4^{\rm cp}$ holds true provided that the fixed point resolution does *not* change the multiplicities. Otherwise the counting of chiral primaries would need to be modified accordingly.

N = 5

As in the previous discussion we shall begin by listing all the regular chiral primaries for $N = 5$. After constructing all Young diagrams Y' which fit into rectangles of size 2×3, 3×2, 1×4 or 4×1 we apply eq. (3.2) to obtain the 16 Young diagrams Y. It would take quite a bit of space to display all of them. So, let us simply produce a list of the corresponding Dynkin labels,

$$\begin{aligned}\mathcal{S}_5 = \{&(id,[0,0,0,0]),(v,[1,0,0,1]),(id,[2,0,1,0]),(id,[0,1,0,2]),(v,[0,0,1,3]),\\ &(v,[1,1,1,1]),(v,[3,1,0,0]),(id,[5,0,0,0]),(id,[2,2,0,1]),(id,[0,2,2,0]),\\ &(id,[1,0,2,2]),(id,[0,0,0,5]),(v,[0,1,3,1]),(v,[1,3,1,0]),(id,[0,5,0,0]),\\ &(id,[0,0,5,0])\}\,.\end{aligned}$$

Note that both $(id,[5,0,0,0])$ and $(id,[0,0,0,5])$ as well as $(id,[0,5,0,0])$ and $(id,[0,0,5,0])$ are identified by the simple current automorphism. Hence we would expect 14 coset sectors whose ground states are (regular) chiral primary.

Let us now look for the complete set of chiral primaries. In this case, the sector independent bound (4.1) restricts the conformal weight of chiral primaries to satisfy $h \leq c_5/6 = 4/3$. Here we inserted the central charge $c_5 = 8$. One can again do a little better using the sector dependent bound (4.2), but in the case at hand we also listed all contributions to branching functions up to weight $h = 4/3$, see appendix B.3. The results show that once more all chiral primaries are $\mathcal{W}_N$ ground states so that we can detect chiral primaries from the data that were provided in section 3.3 where we listed the necklaces for $N = 5$. We see 17 sectors of our $\mathcal{W}_N$ algebra contain a chiral primary among its ground states. This is three more that the 14 regular chiral primaries we described in the previous paragraph. The exceptional chiral primaries correspond to ground states of the sectors $(v,[3,2,0,2])$, $(id,[2,3,1,1])$ and $(v,[2,2,2,2])$ and they possess conformal weights $h = 1/2$, $h = 2/3$ and $h = 5/6$, respectively.

From the resolved partition function (3.26), we thus find

$$Z_5^{\rm cp} = 1 + (q\bar{q})^{1/6} + 2(q\bar{q})^{1/3} + 4(q\bar{q})^{1/2} + 5(q\bar{q})^{2/3} + 3(q\bar{q})^{5/6} + q\bar{q} \tag{4.14}$$

$$= T_5\left((q\bar{q})^{1/6}\right) + (q\bar{q})^{1/2} + (q\bar{q})^{2/3} + (q\bar{q})^{5/6}\,. \tag{4.15}$$

The last three terms give the counting function $Z_5^{\rm cp,exc}$ for exceptional chiral primaries. Let us point out that one of the new chiral primaries is sitting inside the fixed point sector (v, a_*), something that could not happen with the regular chiral primaries for N prime.

5 Conclusions

In this paper we described the chiral symmetry $\mathcal{W}_N$ and the complete modular invariant partition function Z_N for a family of field theories with $\mathcal{N} = (2,2)$ superconformal symmetry that arise in the low energy limit of 1-dimensional adjoint QCD. We developed techniques to study these theories for $N \geq 4$, where the theory does not correspond to a supersymmetric minimal model. Special attention was payed to the set of chiral primaries which are counted by a function $Z_N^{\rm cp}(q)$ which we introduced in eq. (4.10).

One of our main results is the discovery of exceptional chiral primaries for $N = 4, 5$, which lie outside the set of regular chiral primaries. In fact, we found one such chiral primary for $N = 4$ and three of them for $N = 5$. Regular chiral primaries were described in some detail in section 4.2. These fields are counted by a function $T_N(\mathfrak{q})$ which we defined in eq. (4.8).

Our research is motivated by the desire to constrain the holographic dual of the superconformal field theory under consideration. To this end, one would like to find all chiral primaries in the limit of a large number N of colors. As we discussed in section 4, the counting function T_N for regular chiral primaries has a well-defined and simple limiting behavior (4.7). A similar analysis for the exceptional chiral primaries has not been performed yet. It is possible that such chiral primaries do not survive the large N limit. We will return to this problem in a forthcoming publication.

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Appendix

A Branching functions and fixed-point resolution for $N = 2$

In this appendix we explain how to compute the branching functions for $N = 2$, how to resolve the fixed point and how to recover the partition function of a compactified free boson. The first subsection contains a list of relevant functions and identities. Branching functions of the model are computed in the second subsection before we discuss the partition function in the final part.

A.1 Notations

Throughout this appendix, we use the following notation for theta functions

$$\Psi_k(a|b)_x := \sum_{n\in\mathbb{Z}} q^{\frac{a}{2}(n+\frac{1}{k})^2} x^{b(n+\frac{1}{k})}$$
$$\tilde{\Psi}_k(a|b)_x := \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{a}{2}(n+\frac{1}{k})^2} x^{b(n+\frac{1}{k})}\,.$$

For $k = 1, 2$, these reduce to ordinary Jacobi theta functions,

$$\theta_3(a|b)_x := \Psi_1(a|b)_x\,, \quad \theta_4(a|b)_x := \tilde{\Psi}_1(a|b)_x\,,$$
$$\theta_2(a|b)_x := \Psi_2(a|b)_x\,, \quad \theta_1(a|b)_x := -i\tilde{\Psi}_2(a|b)_x.$$

Whenever we set $x = 1$, we omit the second parameter in the brackets and the small subscript.

Next, let us introduce Ramanujan's theta function,

$$f(a,b) := \sum_{n\in\mathbb{Z}} (ab)^{\frac{n^2}{2}} \left(\frac{a}{b}\right)^{\frac{n}{2}} = \prod_{n\in\mathbb{N}} \left(1 + \frac{1}{a}(ab)^n\right)\left(1 + \frac{1}{b}(ab)^n\right)\left(1 - (ab)^n\right).$$

It is related to theta functions through

$$f(a,b) = q^{-\frac{\beta^2}{8\alpha}} u^{-\frac{\beta}{2\alpha}} \Psi_{\frac{2\alpha}{\beta}}(\alpha|1)_u$$
$$f(-a,-b) = q^{-\frac{\beta^2}{8\alpha}} u^{-\frac{\beta}{2\alpha}} \tilde{\Psi}_{\frac{2\alpha}{\beta}}(\alpha|1)_u$$

where the variables u and q on the right hand side are related to a and b through $ab = q^\alpha$ and $a/b = q^\beta u^2$. Ramanujan's theta function obeys Weierstrass' three-term relation [21],

$$f(a,b)f(c,d) = f(ad,bc)f(ac,bd) + a\, f(c/a, a^2bd)f(d/a, a^2bc)$$

whenever $ab = cd$, and Hirschhorn's generalized quintuple product identity [22],

$$f(a,b)f(c,d) = f(ac,bd)f(ac\cdot b^3, bd\cdot a^3)$$
$$+ a\, f\left(\frac{d}{a}, \frac{a}{d}(abcd)\right) f\left(\frac{bc}{a}, \frac{a}{bc}(abcd)^2\right) + b\, f\left(\frac{c}{b}, \frac{b}{c}(abcd)\right) f\left(\frac{ad}{b}, \frac{b}{ad}(abcd)^2\right)$$

for $(ab)^2 = cd$. This concludes our brief list of mathematical functions and identities.

A.2 Branching functions

In order to illustrate how branching functions are computed, let us focus on the decomposition of the $id + v$ sector of the $\mathrm{SO}(6)_1$ WZW model. According to our general prescription (3.12), the corresponding character reads

$$\begin{aligned}
&\chi^{\mathrm{SO}(6)_1}_{id+v}(q,x,z)\\
&= q^{-1/8}\prod_{n\in\mathbb{N}}(1+x^{1/3}q^{n-1/2})(1+x^{-1/3}q^{n-1/2})(1+x^{1/3}z^2q^{n-1/2})\\
&\times(1+x^{-1/3}z^{-2}q^{n-1/2})(1+x^{1/3}z^{-2}q^{n-1/2})(1+x^{-1/3}z^2q^{n-1/2}) \qquad \text{(A.1)}\\
&= \eta^{-3}f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})f(q^{1/2}x^{1/3}z^2,q^{1/2}x^{-1/3}z^{-2})f(q^{1/2}x^{1/3}z^{-2},q^{1/2}x^{-1/3}z^2)\\
&= \eta^{-3}f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})\left(f(qx^{2/3},qx^{-2/3})f(qz^4,qz^{-4})+q^{1/2}x^{1/3}z^2f(q^2x^{2/3},x^{-2/3})f(q^2z^4,z^{-4})\right)
\end{aligned}$$

where $\eta\equiv\eta(q)$ is the Dedekind eta function. In the final step we inserted Weierstrass' three-term relation. Using Hirschhorn's quintuple product, one can then write

$$\begin{aligned}
&f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})f(qx^{2/3},qx^{-2/3})\\
&\quad= f(q^{3/2}x,q^{3/2}x^{-1})f(q^3,q^3)+q^{1/2}f(q,q^5)\left(x^{1/3}f(q^{5/2}x,q^{1/2}x^{-1})+x^{-1/3}f(q^{5/2}x^{-1},q^{1/2}x)\right)
\end{aligned}$$

for $a=q^{1/2}x^{1/3}$, $b=q^{1/2}x^{-1/3}$, $c=qx^{2/3}$, $d=qx^{-2/3}$ and

$$\begin{aligned}
&x^{1/3}f(q^{1/2}x^{1/3},q^{1/2}x^{-1/3})f(q^2x^{2/3},x^{-2/3})\\
&\quad= q^{1/2}f(q^{3/2}x,q^{3/2}x^{-1})f(1,q^6)+f(q^2,q^4)\left(x^{1/3}f(q^{5/2}x,q^{1/2}x^{-1})+x^{-1/3}f(q^{5/2}x^{-1},q^{1/2}x)\right)
\end{aligned}$$

for $a=q^{1/2}x^{1/3}$, $b=q^{1/2}x^{-1/3}$, $c=q^2x^{2/3}$, $d=x^{-2/3}$. In the second case we employed the following obvious symmetry property of f,

$$\frac{1}{\sqrt{u}}f\left(u,\frac{A}{u}\right)=\sqrt{u}f\left(Au,\frac{1}{u}\right).$$

Substituting these products back and simplifying, one arrives at

$$\begin{aligned}
&\chi^{\mathrm{SO}(6)_1}_{id+v}(q,x,z)\\
&\quad=\eta^{-1}\Psi_1(3|1)_x\left(\frac{\theta_3(6)}{\eta^2}\theta_3(2|4)_z+\frac{\theta_2(6)}{\eta^2}\theta_2(2|4)_z\right)\\
&\qquad+\eta^{-1}\left(\Psi_3(3|1)_x+\Psi_3(3|-1)_x\right)\left(\frac{\Psi_3(6)}{\eta^2}\theta_3(2|4)_z+\frac{\Psi_6(6)}{\eta^2}\theta_2(2|4)_z\right). \qquad \text{(A.2)}
\end{aligned}$$

In this formula $\theta_3(6)/\eta^2$, $\theta_2(6)/\eta^2$, $\Psi_3(6)/\eta^2$ and $\Psi_6(6)/\eta^2$ are combinations of $\mathrm{SU}(2)_4$ string functions $c^0_0+c^0_4$, $2c^0_2$, c^2_0 and c^2_2, respectively (see e.g. [23] for more information). Expressions in round brackets can be then recognized as $S_{\{0\}}$ and $S_{\{2\}}$ which were defined in eq. (3.11). Indeed, taking into account the symmetry properties of the SU(2)$_k$ string functions,

$$c^{\Lambda}_{\lambda} = c^{J(\Lambda)}_{J(\lambda)} = c^{\Lambda}_{\lambda+2k} = c^{\Lambda}_{-\lambda}$$

where J is the SU(2)$_k$ simple current, one readily sees from

$$S^{N=2}_{\{a\}} = \sum_{\Lambda\in\{a\}} \sum_{\lambda\in\{-2,0,2,4\}} c^{\Lambda}_{\lambda} \sum_{n\in\mathbb{Z}} q^{4\left(n+\frac{\lambda}{8}\right)^2} z^{8\left(n+\frac{\lambda}{8}\right)}$$

that

$$\begin{aligned} S_{\{0\}} &= (c^0_0 + c^0_4)\,\theta_3(2|4)_z + 2c^0_2\,\theta_2(2|4)_z \\ S_{\{2\}} &= c^2_0\,\theta_3(2|4)_z + c^2_2\,\theta_2(2|4)_z. \end{aligned}$$

The decomposition (A.2) of the SO(6)$_1$ character (A.1) thus leads to the following branching functions

$$\mathcal{X}^{\mathcal{W}}_{(id+v,[0])}(q,x) = \eta^{-1}\Psi_1(3|1)_x\,, \qquad \mathcal{X}^{\mathcal{W}}_{(id+v,[2])}(q,x) = \eta^{-1}\left(\Psi_3(3|1)_x + \Psi_3(3|-1)_x\right)\,.$$

Going along the same lines, one may compute the six remaining branching functions. These are given by

$$\begin{aligned} \mathcal{X}^{\mathcal{W}}_{(id-v,[0])}(q,x) &= \eta^{-1}\tilde{\Psi}_1(3|1)_x\,, & \mathcal{X}^{\mathcal{W}}_{(id-v,[2])}(q,x) &= -\eta^{-1}\left(\tilde{\Psi}_3(3|1)_x + \tilde{\Psi}_3(3|-1)_x\right)\,, \\ \mathcal{X}^{\mathcal{W}}_{(sp+c,[0])}(q,x) &= \eta^{-1}\Psi_2(3|1)_x\,, & \mathcal{X}^{\mathcal{W}}_{(sp+c,[2])}(q,x) &= \eta^{-1}\left(\Psi_6(3|1)_x + \Psi_6(3|-1)_x\right)\,, \\ \mathcal{X}^{\mathcal{W}}_{(sp-c,[0])}(q,x) &= \eta^{-1}\tilde{\Psi}_2(3|1)_x\,, & \mathcal{X}^{\mathcal{W}}_{(sp-c,[2])}(q,x) &= -\eta^{-1}\left(\tilde{\Psi}_6(3|1)_x - \tilde{\Psi}_6(3|-1)_x\right)\,. \end{aligned} \tag{A.3}$$

A.3 The partition function

The 'unresolved' partition function $\tilde{Z}_2$ of the $N = 2$ model is constructed from the branching functions (A.3) according to the general prescription (2.13),

$$\begin{aligned} \tilde{Z}_2 &= \sum_{A=id,v,sp,c} \left(|\mathcal{X}^{\mathcal{W}}_{(A,[0])}|^2 + \tfrac{1}{2}|\mathcal{X}^{\mathcal{W}}_{(A,[2])}|^2\right) = \frac{1}{2} \sum_{B=id+v,id-v,sp+c,sp-c} \left(|\mathcal{X}^{\mathcal{W}}_{(B,[0])}|^2 + \tfrac{1}{2}|\mathcal{X}^{\mathcal{W}}_{(B,[2])}|^2\right) \\ &= \frac{1}{2|\eta|^2}\Big\{|\Psi_1(3|1)_x|^2 + |\tilde{\Psi}_1(3|1)_x|^2 + |\Psi_2(3|1)_x|^2 + |\tilde{\Psi}_2(3|1)_x|^2 + \frac{1}{2}[|\Psi_3(3|1)_x + \Psi_3(3|-1)_x|^2 \\ &\qquad + |\tilde{\Psi}_3(3|1)_x + \tilde{\Psi}_3(3|-1)_x|^2 + |\Psi_6(3|1)_x + \Psi_6(3|-1)_x|^2 + |\tilde{\Psi}_6(3|1)_x - \tilde{\Psi}_6(3|-1)_x|^2]\Big\}\,. \end{aligned}$$

As we explained before, the model suffers from a fixed point in the sectors $(A,[2])$ so that $\tilde{Z}_2$ does not describe the partition function of a well-defined CFT: The multiplicities of some states inside the square brackets are non-integer. In order to cure the issue, let us add the following modular-invariant contribution

$$Z_2^{\mathrm{res}} := \frac{1}{4|\eta|^2}[|\Psi_3(3|1)_x - \Psi_3(3|-1)_x|^2 + |\tilde{\Psi}_3(3|1)_x - \tilde{\Psi}_3(3|-1)_x|^2 + |\Psi_6(3|1)_x - \Psi_6(3|-1)_x|^2 + |\tilde{\Psi}_6(3|1)_x + \tilde{\Psi}_6(3|-1)_x|^2]\,. \tag{A.4}$$

Note that this expression reduces to $Z_2^{res}(x=1) = 1$ due to Euler's pentagonal number theorem. Regrouping terms, we end up with

$$\begin{aligned} Z_2 &:= \tilde{Z}_2 + Z_2^{\mathrm{res}} \qquad\qquad (A.5)\\ &= \frac{1}{2|\eta|^2}\Big\{|\Psi_1(3|1)_x|^2 + |\tilde{\Psi}_1(3|1)_x|^2 + |\Psi_2(3|1)_x|^2 + |\tilde{\Psi}_2(3|1)_x|^2 + |\Psi_3(3|1)_x|^2 + |\Psi_3(3|-1)_x|^2 \\ &\quad + |\tilde{\Psi}_3(3|1)_x|^2 + |\tilde{\Psi}_3(3|-1)_x|^2 + |\Psi_6(3|1)_x|^2 + |\Psi_6(3|-1)_x|^2 + |\tilde{\Psi}_6(3|1)_x|^2 + |\tilde{\Psi}_6(3|-1)_x|^2\Big\}\,. \end{aligned}$$

With a little bit of additional effort, this expression may be resummed into a more compact form

$$Z_2 = \frac{1}{|\eta|^2}\sum_{n,w\in\mathbb{Z}} q^{\frac{k_L^2}{2}}\bar{q}^{\frac{k_R^2}{2}} x^{rk_L}\bar{x}^{rk_R} \qquad \text{with} \qquad k_{L,R} = \frac{n}{r} \pm \frac{wr}{2}, \quad r = 2\sqrt{3} \tag{A.6}$$

which is the well known partition function of a free boson that has been compactified on a circle of radius $r = 2\sqrt{3}$.

B Branching functions for $N = 3, 4, 5$

In this appendix we give the q-expansions of the branching functions $\mathcal{X}^{\mathcal{W}}$ up to order $O(q^{c_N/6})$. In order to better read off the conformal weights h, we omit the overall factor $q^{-c_N/24}$ in the $\mathcal{X}^{\mathcal{W}}$'s. As shown in section 4.1, there are no chiral primaries with conformal weight larger than $h{=}c_N/6$. Chiral primary fields (with $h = Q$) are marked by CP_h. We restrict to the NS sector, i.e. we only display the branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ ($a \in \mathcal{J}_N^0$). Similar expansions exist for all the branching functions $\mathcal{X}^{\mathcal{W}}_{(sp+c,a)}$ in the R sector.

B.1 N = 3

The central charge is $c_3 = 8/3$. Chiral primaries exist only for $h \leq 4/9$. The expansion of the branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ ($a \in \mathcal{J}_3^0$) is given by

$$\mathcal{X}^{\mathcal{W}}_{(id+v,[0,0])} = 1 + O(q^1) \qquad \boxed{\mathrm{CP}_0}$$

$$\mathcal{X}^{\mathcal{W}}_{(id+v,[1,1])} = \left(x^{-1/3} + x^{1/3}\right)q^{1/6} + O(q^{2/3}) \qquad \boxed{\mathrm{CP}_{1/6}}$$

$$\mathcal{X}^{\mathcal{W}}_{(id+v,[3,0])} = \left(x^{-2/3} + 1 + x^{2/3}\right)q^{1/3} + O(q^{5/6}) \qquad \boxed{\mathrm{CP}_{1/3}}$$

$$\mathcal{X}^{\mathcal{W}}_{(id+v,[2,2])} = q^{1/9} + O(q^{11/18})\,.$$

B.2 N = 4

The central charge is $c_4 = 5$. The sector independent bound on the conformal weight of a chiral primary state is therefore $h \leq 5/6$.

The expansion of the branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ $(a \in \mathcal{J}^0_4)$ is given by

$$
\begin{aligned}
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,0,0])} &= 1 + O(q^1) && \boxed{\mathrm{CP}_0} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[1,0,1])} &= \left(x^{1/3} + x^{-1/3}\right) q^{1/6} + \left(x^{2/3} + 1 + x^{-2/3}\right) q^{2/3} + O(q^{7/6}) && \boxed{\mathrm{CP}_{1/6}} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,2,0])} &= q^{1/2} + O(q^1) \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,1,0])} &= \mathcal{X}^{\mathcal{W}}_{(id+v,[0,1,2])} = \left(x^{2/3} + 1 + x^{-2/3}\right) q^{1/3} \\
&\quad + \left(x + 2x^{1/3} + 2x^{-1/3} + x^{-1}\right) q^{5/6} + O(q^{4/3}) && \boxed{\mathrm{CP}_{1/3}} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[4,0,0])} &= \left(x + x^{1/3} + x^{-1/3} + x^{-1}\right) q^{1/2} + O(q) && \boxed{\mathrm{CP}_{1/2}} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,0,2])} &= q^{1/6} + \left(x + 2x^{1/3} + 2x^{-1/3} + x^{-1}\right) q^{2/3} + O(q^{7/6}) \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[1,2,1])} &= \left(x + 2x^{1/3} + 2x^{-1/3} + x^{-1}\right) q^{1/2} + O(q) && \boxed{\mathrm{CP}_{1/2}} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,3,0])} &= \left(x^{2/3} + 1 + x^{-2/3}\right) q^{1/2} + O(q) \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[3,1,1])} &= \left(x^{1/3} + x^{-1/3}\right) q^{1/4} + O(q^{5/4}) \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,4,0])} &= \left(x^{4/3} + x^{2/3} + 2 + x^{-2/3} + x^{-4/3}\right) q^{2/3} + O(q^{7/6}) && \boxed{\mathrm{CP}_{2/3}} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,2,2])} &= \left(x^{2/3} + 2 + x^{-2/3}\right) q^{1/3} \\
&\quad + \left(3x + 5x^{1/3} + 5x^{-1/3} + 3x^{-1}\right) q^{5/6} + O(q^{4/3}) && \boxed{\mathrm{CP}_{1/3}} .
\end{aligned}
$$

B.3 N = 5

The central charge is $c_5 = 8$, and we expand up to order $O(q^{4/3})$ in order to capture all contributions from chiral primaries with conformal weight $h \leq c_5/6$. In order to write the expansion of branching functions $\mathcal{X}^{\mathcal{W}}_{(id+v,a)}$ $(a \in \mathcal{J}^0_5)$ we shall introduce the shorthand $y^n \equiv x^n + x^{-n}$.

$$
\begin{aligned}
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,0,0,0])} &= 1 + q + O(q^{3/2}) && \boxed{\mathrm{CP}_0} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[1,0,0,1])} &= y^{1/3} q^{1/6} + (y^{2/3} + 1)\, q^{2/3} + (y + 3y^{1/3})\, q^{7/6} + O(q^{5/3}) && \boxed{\mathrm{CP}_{1/6}} \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[0,1,1,0])} &= q^{7/15} + (y + 2y^{1/3})\, q^{29/30} + O(q^{22/15}) \\
\mathcal{X}^{\mathcal{W}}_{(id+v,[2,0,1,0])} &= (y^{2/3} + 1)\, q^{1/3} + (y + 2y^{1/3})\, q^{5/6}
\end{aligned}
$$

$$+ (y^{4/3} + 5y^{2/3} + 6)\, q^{4/3} + O(q^{11/6}) \qquad \boxed{\mathrm{CP}_{1/3}}$$

$$\mathcal{X}^{W}_{(id+v,[1,2,0,0])} = y^{1/3} q^{7/10} + (y^{4/3} + 3y^{2/3} + 4)\, q^{6/5} + O(q^{17/10})$$

$$\mathcal{X}^{W}_{(id+v,[0,1,0,2])} = \mathcal{X}^{W}_{(id+v,[2,0,1,0])} \qquad \boxed{\mathrm{CP}_{1/3}}$$

$$\mathcal{X}^{W}_{(id+v,[0,0,2,1])} = \mathcal{X}^{W}_{(id+v,[1,2,0,0])}$$

$$\mathcal{X}^{W}_{(id+v,[3,1,0,0])} = (y + y^{1/3})\, q^{1/2} + (y^{4/3} + 2y^{2/3} + 3)\, q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[2,0,0,2])} = q^{1/5} + (y + 2y^{1/3})\, q^{7/10} + (2y^{4/3} + 4y^{2/3} + 7)\, q^{6/5} + O(q^{17/10})$$

$$\mathcal{X}^{W}_{(id+v,[1,1,1,1])} = (y + 2y^{1/3})\, q^{1/2} + (2y^{4/3} + 6y^{2/3} + 8)\, q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[1,0,3,0])} = \mathcal{X}^{W}_{(id+v,[0,3,0,1])}$$

$$\mathcal{X}^{W}_{(id+v,[0,3,0,1])} = (y^{2/3} + 1)\, q^{4/5} + (y^{5/3} + 4y + 6y^{1/3})\, q^{13/10} + O(q^{9/5})$$

$$\mathcal{X}^{W}_{(id+v,[0,2,2,0])} = (y^{4/3} + y^{2/3} + 2)\, q^{2/3} + (y^{5/3} + 3y + 5y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[0,0,1,3])} = \mathcal{X}^{W}_{(id+v,[3,1,0,0])} \qquad \boxed{\mathrm{CP}_{1/2}}$$

$$\mathcal{X}^{W}_{(id+v,[5,0,0,0])} = (y^{4/3} + y^{2/3} + 1)\, q^{2/3} + (y^{5/3} + y + 2y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[3,0,1,1])} = y^{1/3} q^{3/10} + (y^{4/3} + 3y^{2/3} + 4)\, q^{4/5}$$

$$+ (2y^{5/3} + 7y + 13y^{1/3})\, q^{13/10} + O(q^{41/30})$$

$$\mathcal{X}^{W}_{(id+v,[2,2,0,1])} = (y^{4/3} + 2y^{2/3} + 3)\, q^{2/3} + (2y^{5/3} + 6y + 10y^{1/3})\, q^{7/6} + O(q^{5/3}) \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[2,1,2,0])} = (y^{2/3} + 1)\, q^{8/15} + (y^{5/3} + 4y + 7y^{1/3})\, q^{31/30} + O(q^{23/15})$$

$$\mathcal{X}^{W}_{(id+v,[1,3,1,0])} = (y^{5/3} + 2y + 3y^{1/3})\, q^{5/6}$$

$$+ (2y^{2} + 6y^{4/3} + 11y^{2/3} + 13)\, q^{4/3} + O(q^{11/6}) \qquad \boxed{\mathrm{CP}_{5/6}}$$

$$\mathcal{X}^{W}_{(id+v,[1,1,0,3])} = \mathcal{X}^{W}_{(id+v,[3,0,1,1])}$$

$$\mathcal{X}^{W}_{(id+v,[1,0,2,2])} = \mathcal{X}^{W}_{(id+v,[2,2,0,1])} \qquad \boxed{\mathrm{CP}_{2/3}}$$

$$\mathcal{X}^{W}_{(id+v,[0,5,0,0])} = (y^{2} + y^{4/3} + 2y^{2/3} + 2)\, q + O(q^{3/2}) \qquad \boxed{\mathrm{CP}_{1}}$$

$$\mathcal{X}^{W}_{(id+v,[0,2,1,2])} = (y^{2/3} + 1)\, q^{8/15} + (y^{5/3} + 4y + 7y^{1/3})\, q^{31/30} + O(q^{23/15})$$

$$\mathcal{X}^{W}_{(id+v,[0,1,3,1])} = \mathcal{X}^{W}_{(id+v,[1,3,1,0])} \qquad \boxed{\mathrm{CP}_{5/6}}$$

$$\mathcal{X}^{W}_{(id+v,[4,1,0,1])} = (y^{2/3} + 1)\, q^{2/5} + (y^{5/3} + 3y + 5y^{1/3})\, q^{9/10} + O(q^{7/5})$$

$$\mathcal{X}^{W}_{(id+v,[4,0,2,0])} = q^{4/15} + (y + 2y^{1/3})\, q^{23/30} + (3y^{4/3} + 6y^{2/3} + 10)\, q^{19/15} + O(q^{7/5})$$

$$
\begin{aligned}
\mathcal{X}^{W}_{(id+v,[3,2,1,0])} &= (y+2y^{1/3})\,q^{19/30} + (y^2+4y^{4/3}+9y^{2/3}+11)\,q^{17/15} + O(q^{49/30})\\
\mathcal{X}^{W}_{(id+v,[3,0,0,3])} &= (y^{2/3}+1)\,q^{3/5} + (y^{5/3}+3y+5y^{1/3})\,q^{11/10} + O(q^{8/5})\\
\mathcal{X}^{W}_{(id+v,[2,4,0,0])} &= (y^{4/3}+y^{2/3}+2)\,q^{13/15} + O(q^{41/30})\\
\mathcal{X}^{W}_{(id+v,[2,1,1,2])} &= (y^{2/3}+2)\,q^{2/5} + (y^{5/3}+5y+9y^{1/3})\,q^{9/10} + O(q^{7/5})\\
\mathcal{X}^{W}_{(id+v,[2,0,3,1])} &= (y+2y^{1/3})\,q^{7/10} + (y^2+5y^{4/3}+10y^{2/3}+13)\,q^{6/5} + O(q^{17/10})\\
\mathcal{X}^{W}_{(id+v,[1,3,0,2])} &= \mathcal{X}^{W}_{(id+v,[2,0,3,1])}\\
\mathcal{X}^{W}_{(id+v,[1,2,2,1])} &= (y+2y^{1/3})\,q^{17/30} + (y^2+5y^{4/3}+11y^{2/3}+14)\,q^{16/15} + O(q^{47/30})\\
\mathcal{X}^{W}_{(id+v,[1,1,4,0])} &= (y^{4/3}+2y^{2/3}+2)\,q^{4/5}\\
&\quad + (y^{7/3}+4y^{5/3}+9y+13y^{1/3})\,q^{13/10} + O(q^{9/5})\\
\mathcal{X}^{W}_{(id+v,[0,3,3,0])} &= q^{3/5} + (y^{5/3}+3y+5y^{1/3})\,q^{11/10} + O(q^{8/5})\\
\mathcal{X}^{W}_{(id+v,[0,1,2,3])} &= \mathcal{X}^{W}_{(id+v,[3,2,1,0])}\\
\mathcal{X}^{W}_{(id+v,[3,2,0,2])} &= (y+2y^{1/3})\,q^{1/2} + (3y^{4/3}+7y^{2/3}+9)\,q + O(q^{3/2}) && \boxed{\mathrm{CP}_{1/2}}\\
\mathcal{X}^{W}_{(id+v,[3,1,2,1])} &= y^{1/3}q^{11/30} + (2y^{4/3}+6y^{2/3}+8)\,q^{13/15} + O(q^{41/30})\\
\mathcal{X}^{W}_{(id+v,[2,3,1,1])} &= (y^{4/3}+3y^{2/3}+4)\,q^{2/3}\\
&\quad + (4y^{5/3}+12y+19y^{1/3})\,q^{7/6} + O(q^{5/3}) && \boxed{\mathrm{CP}_{2/3}}\\
\mathcal{X}^{W}_{(id+v,[2,2,3,0])} &= (y^{2/3}+1)\,q^{7/15} + (y^{5/3}+4y+7y^{1/3})\,q^{29/30} + O(q^{22/15})\\
\mathcal{X}^{W}_{(id+v,[2,2,2,2])} &= q^{1/3} + (y^{5/3}+5y+9y^{1/3})\,q^{5/6}\\
&\quad + (4y^2+19y^{4/3}+36y^{2/3}+47)\,q^{4/3} + O(q^{11/16}) && \boxed{\mathrm{CP}_{5/6}}\,.
\end{aligned}
$$

C $\mathrm{SU}(N)_{2N}$ representations with zero monodromy charge

In this appendix we will prove the formula (3.3) for the quadratic Casimir $C_2(Y)$ of a representation $a \in \mathcal{J}^0_N$ associated with a Young diagram Y. As described in section 3 we pick up a pair of SU(N) Young diagrams Y' and Y'' satisfying the conditions listed in the first paragraph of section 3.1.1. From these two Young diagrams we build a new diagram $Y = (l_1, \ldots, l_{N-1})$ through our prescription (3.2).

We now claim that the resulting Young diagram Y possesses

$$|Y| = r''N \tag{C.1}$$

boxes and that the eigenvalue of the SU(N) quadratic Casimir on Y takes the value

$$C_2(Y) = n'N + C_2(Y') - C_2(Y'') \,. \tag{C.2}$$

In order to prove these two statements, we use eq. (3.2) to obtain

$$|Y| = \sum_i l_i = \sum_{i=1}^{r'} (r'' + l'_i) + r''(N - r' - l''_1) + \sum_{i=1}^{r''-1} (r'' - i)(l''_i - l''_{i+1}) \,. \tag{C.3}$$

Since

$$\sum_{i=1}^{r''-1} i(l''_i - l''_{i+1}) = \sum_{i=1}^{r''-1} i\, l''_i - \sum_{i=2}^{r''} (i-1) l''_i = n' - r'' l''_{r''} \,, \tag{C.4}$$

we arrive at

$$|Y| = r'r'' + n' + r''(N - r' - l''_1) + r''(l''_1 - l''_{r''}) - (n' - r'' l''_{r''}) = r''N \,, \tag{C.5}$$

which proves eq. (C.1).

The quadratic Casimir on Y is therefore given by

$$C_2(Y) = \frac{1}{2}\left[N r''(N + 1 - r'') + \sum_i l_i (l_i - 2i) \right].$$

Let us compute the last term in the brackets,

$$\begin{aligned}
&\sum_i l_i(l_i - 2i) \\
&\quad = \sum_{i=1}^{r'} (r'' + l'_i)(r'' + l'_i - 2i) + \sum_{i=1}^{N - l''_1 - r'} r''(r'' - 2(r' + i)) \\
&\qquad + \sum_{i=1}^{l''_1 - l''_2} (r'' - 1)(r'' - 1 - 2(N - l''_1 + i) + \cdots + \sum_{i=1}^{l''_{r''-1} - l''_{r''}} 1 \cdot (1 - 2(N - l''_{r''-1} + i)) \\
&\quad = \sum_{i=1}^{r'} (r'' + l'_i)(r'' + l'_i - 2i) - \sum_{i=1}^{N - l''_1 - r'} r''(r'' + 2i) + \sum_{i=1}^{r''-1} (r'' - i)(r'' - i - 2N)(l''_i - l''_{i+1}) \\
&\qquad + 2 \sum_{i=1}^{r''-1} (r'' - i) l''_i (l''_i - l''_{i+1}) - \sum_{i=1}^{r''-1} (r'' - i)(l''_i - l''_{i+1} + 1)(l''_i - l''_{i+1}) \\
&\quad =: \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 \,.
\end{aligned} \tag{C.6}$$

Using the identities analogous to eq. (C.4),

$$
\begin{aligned}
\sum_{i=1}^{r''-1} i(l_i''^2 - l_{i+1}''^2) &= \sum_{i=1}^{r''} l_i''^2 - r'' l_{r''}''^2 \,, \\
\sum_{i=1}^{r''-1} i^2(l_i'' - l_{i+1}'') &= 2\sum_{i=1}^{r''} i l_i'' - n' - (r''^2 - r'') l_{r''}'' \,,
\end{aligned}
\tag{C.7}
$$

we can conclude

$$
\begin{aligned}
\Sigma_1 + \Sigma_2 &= \sum_{i=1}^{r'} l_i'(l_i' - 2i) + 2r''n' - r''(N - l_1'')(N - l_1'' - r'' + 1) \,, \\
\Sigma_3 &= 2\sum_{i=1}^{r''} i l_i'' + (2N - 2r'' - 1)n' + (r''^2 - 2r''N) l_1'' \,, \\
\Sigma_4 + \Sigma_5 &= \sum_{i=1}^{r''-1} (r'' - i)((l_i''^2 - l_{i+1}''^2) - (l_i'' - l_{i+1}'')) \\
&= -\sum_{i=1}^{r''} l_i''^2 + n' + r''(l_1''^2 - l_1'') \,.
\end{aligned}
\tag{C.8}
$$

When summed up, this contributions give

$$
\sum_i l_i(l_i - 2i) = -Nr''(N + 1 - r'') + 2n'N + \sum_{i=1}^{r'} l_i'(l_i' - 2i) - \sum_{i=1}^{r''} l_i''(l_i'' - 2i) \tag{C.9}
$$

and thus

$$
C_2(Y) = Nn' + \frac{1}{2}\left(\sum_{i=1}^{r'} l_i'(l_i' - 2i) - \sum_{i=1}^{r''} l_i''(l_i'' - 2i)\right) . \tag{C.10}
$$

Since $|Y'| = |Y''| = n'$ holds by construction, this expression is equivalent to eq. (C.2).

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