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The 3-Loop Non-Singlet Heavy Flavor Contributions to the Structure Function $g_1(x, Q^2)$ at Large Momentum Transfer

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Abstract

We calculate the massive flavor non-singlet Wilson coefficient for the heavy flavor contributions to the polarized structure function $g_1(x, Q^2)$ in the asymptotic region $Q^2 \gg m^2$ to 3-loop order in Quantum Chromodynamics at general values of the Mellin variable N and the momentum fraction x, and derive heavy flavor corrections to the Bjorken sum-rule. Numerical results are presented for the charm quark contribution. Results on the structure function $g_2(x, Q^2)$ in the twist-2 approximation are also given.

1 Introduction

Massless and massive contributions to the unpolarized and polarized structure functions in deep-inelastic scattering exhibit different scaling violations. For a precise determination of the QCD scale $\Lambda_{\rm QCD}$ or the strong coupling constant $\alpha_s(M_Z^2)$ their precise knowledge is therefore of importance [1]. In the case of the polarized structure function $g_1(x,Q^2)$ the complete heavy flavor corrections are only available at 1-loop order [2,3][1]. At higher orders in the coupling constant, the heavy flavor contributions were calculated in the asymptotic region $Q^2 \gg m^2$ based on the factorization derived in Ref. [5]. Here Q^2 denotes the virtuality of the exchanged gauge boson and m the heavy quark mass. The $O(\alpha_s^2)$ corrections in the polarized case were calculated in Refs. [6,7]. In the case of the structure function $g_1(x,Q^2)$, the 1-loop heavy flavor corrections have been accounted for at next-to-leading order (NLO) QCD analysis [8]. The corresponding flavor non-singlet corrections in the unpolarized case were calculated for pure photon exchange to $O(\alpha_s^2)$ in [5,9] and in Ref. [10] to $O(\alpha_s^3)$.

In the present paper we calculate the $O(\alpha_s^3)$ massive flavor non-singlet Wilson coefficient for the inclusive structure function $g_1(x,Q^2)$ in the asymptotic region $Q^2 \gg m^2$, and also present the corresponding $O(\alpha_s^2)$ result, extending Refs. [6,7].

The differential cross section for polarized deep-inelastic scattering [11–13] is given by

$$\frac{d^2\sigma_{\rm B}}{dxdy} = \frac{2\pi\alpha^2}{Q^4}\lambda_N^p f^p S\left[S_1^p(x,y)g_1(x,Q^2) + S_2^p(x,y)g_2(x,Q^2)\right], \tag{1.1}$$

with

$$\begin{aligned} f^L &= 1, \qquad f^T = \cos(\beta-\varphi)\frac{d\varphi}{2\pi}\sqrt{\frac{4M^2x}{Sy}\left[1-y-\frac{M^2xy}{S}\right]}, \\ S_1^L &= 2xy\left[(2-y)-2\frac{M^2}{S}xy\right], \qquad S_1^T = 2xy^2, \\ S_2^L &= -8x^2y\frac{M^2}{S}, \qquad S_2^T = 4xy\ . \end{aligned} \tag{1.2}$$

Here $\alpha = e^2/(4\pi)$ denotes the fine structure constant, M is the nucleon mass, $S=(p+l)^2$ is the center of mass energy of the lepton-nucleon system, with p and l the nucleon and lepton 4-momenta, respectively, $q = l - l'$ is the 4-momentum transfer and $Q^2 = -q^2$. $x = Q^2/(2p.q)$ and $y = p.q/p.l$ are the Bjorken variables. λ_N^p denotes the degree of the nucleon polarization. The spin 4–vectors in the longitudinal and transverse cases are given by

$$S_L = M(0,0,0;1) \tag{1.3}$$
$$S_T = M(0,\cos(\beta),\sin(\beta);0)\ , \tag{1.4}$$

and φ denotes the angle between the vectors of the spin and the outgoing lepton. It contributes in a non-trivial way in the case of transverse polarization.

The polarized structure functions are denoted by $g_1(x,Q^2)$ and $g_2(x,Q^2)$. In the leading twist approximation, the heavy flavor contributions to the structure function $g_1(x,Q^2)$ is given by, cf. [14],

$$g_1(x,Q^2) = \frac{1}{2}\Biggl\{\sum_{k=1}^{N_F} e_i^k\Biggl\{L_{q,g_1}^{\rm NS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right)\otimes\left[\Delta f_k(x,\mu^2,N_F) + \Delta f_{\bar{k}}(x,\mu^2,N_F)\right]$$

[1]For an implementation in Mellin space, see [4].

$$+\frac{1}{N_F} L_{q,g_1}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta\Sigma(x,\mu^2,N_F)$$
$$+\frac{1}{N_F} L_{g,g_1}^{\rm S}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta G(x,\mu^2,N_F)\Bigg\}$$
$$+e_Q^2\Bigg[H_{q,g_1}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta\Sigma(x,\mu^2,N_F)$$
$$+H_{g,g_1}^{\rm PS}\left(x, N_F+1, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \otimes \Delta G(x,\mu^2,N_F)\Bigg]\Bigg\} , \quad (1.5)$$

with $\Delta f_{k(\bar{k})}$ the N_F light flavor polarized (anti)quark densities, ΔG and $\Delta\Sigma = \sum_{l=1}^{N_F}[\Delta f_k + \Delta f_{\bar{k}}]$ the polarized gluon and singlet distributions, and e_i and e_Q the electric charges of the light quarks and the heavy quark Q, respectively. μ denotes the factorization scale and $\otimes$ the Mellin convolution

$$A(x) \otimes B(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) A(x_1) B(x_2) . \quad (1.6)$$

The actual flavor non-singlet distribution is defined by

$$\Delta^{\rm NS}(x,Q^2) = \sum_{k=1}^{N_F} e_k^2 \left[\Delta f_k(x,\mu^2,N_F) + \Delta f_{\bar{k}}(x,\mu^2,N_F) - \frac{1}{N_F}\Delta\Sigma(x,\mu^2,N_F)\right] . \quad (1.7)$$

However, according to the representation (1.5), we will consider its whole first term, depending on $L_{q,1}^{\rm NS}$ as the non-singlet contribution in what follows. The structure function $g_2(x,Q^2)$ can be obtained from $g_1(x,Q^2)$ using the Wandzura-Wilson relation [15].

The paper is organized as follows. In Section 2 we calculate the heavy flavor contributions to the non-singlet Wilson coefficient in the asymptotic region $Q^2 \gg m^2$ to the structure function $g_1(x,Q^2)$ to 3-loop order in the strong coupling constant. We present the results both in Mellin N and x–space. Numerical results are given in Section 3. Consequences for the polarized Bjorken sum rule are discussed in Section 4, and Section 5 contains the conclusions.

2 The Wilson Coefficient

The heavy flavor non-singlet Wilson coefficient contributing to the structure function $g_1(x,Q^2)$ in the asymptotic region $Q^2 \gg m^2$ receives its first contributions at $O(\alpha_s^2)$. In previous analyses [6,7] the tagged flavor case at $O(\alpha_s^2)$ has been considered. In what follows we will refer to the inclusive case, i.e. the complete contribution to the structure function $g_1(x,Q^2)$, and consider the terms due a single heavy quark.

The non-singlet heavy flavor Wilson coefficient contributing to the structure function $g_1(x,Q^2)$ in the asymptotic region $Q^2 \gg m^2$ is given by [16]

$$\begin{aligned} L_{q,g_1}^{\rm h,NS}(N_F+1) &= a_s^2\left[A_{qq,Q}^{(2),\rm NS}(N_F+1) + \hat{C}_{q,g_1}^{(2),\rm NS}(N_F)\right] \quad (2.1)\\ &+ a_s^3\left[A_{qq,Q}^{(3),\rm NS}(N_F+1) + A_{qq,Q}^{(2),\rm NS}(N_F+1)C_{q,g_1}^{(1),\rm NS}(N_F+1) + \hat{C}_{q,g_1}^{(3),\rm NS}(N_F)\right]. \end{aligned}$$

Here $A_{qq,Q}^{\rm NS}$ is the massive non-singlet operator matrix element (OME) and the label 'N_F+1' symbolically denotes that the OME is calculated at N_F massless and one massive flavor, $a_s = \alpha_s/(4\pi) \equiv g_s^2/(4\pi)^2$ parameterizes the strong coupling constant, and we use the convention

$$\hat{f}(N_F) = f(N_F+1) - f(N_F)\ . \tag{2.2}$$

The calculation of the different contributions to the Wilson coefficient is performed in $D = 4+\varepsilon$ dimensions to regulate the Feynman integrals. In the present polarized case the treatment of γ_5 has to be considered. In the flavor non-singlet case both for the massive OMEs and the massless Wilson coefficients γ_5 always appears in traces along one massless line and there is a Ward-Takahashi identity which implies the use of anti-commuting γ_5.

The inclusive massive OME $A_{qq,Q}^{\rm NS}$ to 3-loop order for even and odd moments N has been calculated in Ref. [10]. The corresponding diagrams have been reduced using integration-by-parts relations [17] applying an extension of the package `Reduze 2` [18][2]. The master integrals have been calculated using hypergeometric, Mellin-Barnes and differential equation techniques, mapping them to recurrences, which have been solved by modern summation technologies using extensively the packages `Sigma` [21, 22], `EvaluateMultiSums`, `SumProduction` [23], ρ`sum` [24], and `HarmonicSums` [25].

The massless Wilson coefficients $C_{q,g_1}(x,Q^2)$ from 1- to 3-loop order were calculated in Refs. [26–29]. At 3-loop order those of the structure function g_1 are obtained by that of F_3 [29], setting the d_{abc} terms in $\hat{C}_{q,g_1}^{(3),\rm NS}(N_F)$ to zero, cf. also [30, 31]. The non-singlet OMEs $A_{qq,Q}^{(k),\rm NS}$ at 2- and 3-loop order were calculated in [5, 9] and [10], respectively.

For comparison, the massless flavor non-singlet Wilson coefficient in Mellin space is given by [28, 29]

$$L_{q,g_1}^{\rm l,NS}(N_F) \quad = \quad 1 + \sum_{k=1}^{3} a_s^k C_{q,g_1}^{(k),\rm NS}(N_F)\ . \tag{2.3}$$

In Mellin N space the Wilson coefficient can be expressed by nested harmonic sums $S_{\vec{a}}(N)$ [32] which are defined by

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_\emptyset = 1,\ b, a_i \in \mathbb{Z}, b, a_i \neq 0, N > 0, N \in \mathbb{N}. \tag{2.4}$$

In the following, we drop the argument N of the harmonic sums and use the short-hand notation $S_{\vec{a}}(N) \equiv S_{\vec{a}}$. The Wilson coefficients depend on the logarithms

$$L_Q = \ln\left(\frac{Q^2}{\mu^2}\right) \qquad \text{and} \qquad L_M = \ln\left(\frac{m^2}{\mu^2}\right), \tag{2.5}$$

where the renormalization scale has been set equal to the factorization scale $\mu = \mu_R = \mu_F$.

As a short-hand notation we define the leading order splitting function $\Delta\gamma_{qq}^{(0)}$ up to its color factor

$$\Delta\gamma_{qq}^{(0)} = 4\left[2S_1 - \frac{3N^2+3N+2}{2N(N+1)}\right]\ . \tag{2.6}$$

The massive Wilson coefficient for the structure function $g_1(x,Q^2)$ in the asymptotic region in Mellin space in the on-shell scheme is given by

$$L_{q,g_1}^{\rm h,NS}(N) \quad = \quad \tfrac{1}{2}\left[1-(-1)^N\right]\Biggl\{a_s^2 C_F T_F \Biggl\{-\frac{1}{3}[L_M^2 + L_Q^2]\Delta\gamma_{qq}^{(0)} + L_M\Biggl[-\frac{2P_1}{9N^2(N+1)^2} - \frac{80}{9}S_1$$

[2]The package `Reduze 2` uses the packages `Fermat` [19] and `Ginac` [20].

$$
+\frac{16}{3}S_2\Big] + L_Q\Big[-\frac{2P_6}{9N^2(N+1)^2} + \frac{4\big(29N^2+29N-6\big)}{9N(N+1)}S_1 + \frac{8}{3}S_1^2 - 8S_2\Big] + \frac{16}{3}S_{2,1}
$$

$$
+\frac{P_{34}}{27N^3(N+1)^3} + \Big(-\frac{2P_{11}}{27N^2(N+1)^2} + \frac{8}{3}S_2\Big)S_1 - \frac{2\big(29N^2+29N-6\big)}{9N(N+1)}S_1^2
$$

$$
-\frac{8}{9}S_1^3 + \frac{2\big(35N^2+35N-2\big)}{3N(N+1)}S_2 - \frac{112}{9}S_3\Big\}
$$

$$
+a_s^3\Big\{C_F^2T_F\Big[\frac{1}{6}\left[L_Q^3 + L_M^2L_Q\right]\Delta\gamma_{qq}^{(0)\,2} + L_M^2\Big[-\frac{2P_{28}}{3N^3(N+1)^3} - \frac{16}{3}S_1^3
$$

$$
+\frac{2P_5}{3N^2(N+1)^2}S_1 - \frac{4(N-1)(N+2)}{N(N+1)}S_1^2 + \frac{64}{3}S_3 + \frac{64}{3}S_{-3} - \frac{128}{3}S_{-2,1}
$$

$$
+\Big(-\frac{64}{3N(N+1)} + \frac{128}{3}S_1\Big)S_{-2} + \frac{10}{3}\Delta\gamma_{qq}^{(0)}S_2\Big] + L_Q^2\Big[-\frac{2P_{30}}{9N^3(N+1)^3}
$$

$$
+\frac{2P_{12}}{9N^2(N+1)^2}S_1 - \frac{4\big(107N^2+107N-54\big)}{9N(N+1)}S_1^2 - 16S_1^3 + \frac{64}{3}\left[S_3 + S_{-3}\right]
$$

$$
+\Big(-\frac{64}{3N(N+1)} + \frac{128}{3}S_1\Big)S_{-2} - \frac{128}{3}S_{-2,1} + \frac{22}{3}\Delta\gamma_{qq}^{(0)}S_2\Big]
$$

$$
+L_ML_Q\Delta\gamma_{qq}^{(0)}\Big[\frac{P_1}{9N^2(N+1)^2} + \frac{40}{9}S_1 - \frac{8}{3}S_2\Big] + L_M\Big[\frac{P_{40}}{9N^4(N+1)^4}
$$

$$
+\Big(\frac{2P_{31}}{9N^3(N+1)^3} + \frac{16\big(59N^2+59N-6\big)}{9N(N+1)}S_2 - \frac{256}{3}S_3 - \frac{256}{3}S_{-2,1}\Big)S_1
$$

$$
+\Big(-\frac{4P_3}{3N^2(N+1)^2} + \frac{32}{3}S_2\Big)S_1^2 - \frac{160}{9}S_1^3 - \frac{4P_8}{9N^2(N+1)^2}S_2 - 32S_2^2
$$

$$
+\frac{32\big(29N^2+29N+12\big)}{9N(N+1)}S_3 - \frac{256}{3}S_4 + \Big(-\frac{64\big(16N^2+10N-3\big)}{9N^2(N+1)^2} - \frac{128}{3}S_2
$$

$$
+\frac{1280}{9}S_1\Big)S_{-2} + \Big(\frac{64\big(10N^2+10N+3\big)}{9N(N+1)} - \frac{128}{3}S_1\Big)S_{-3} - \frac{128}{3}S_{-4}
$$

$$
+\frac{128}{3}S_{3,1} - \frac{128\big(10N^2+10N-3\big)}{9N(N+1)}S_{-2,1} - \frac{128}{3}S_{-2,2} + \frac{512}{3}S_{-2,1,1}
$$

$$
+8\Delta\gamma_{qq}^{(0)}\zeta_3\Big] + L_Q\Big[\frac{4P_{48}}{27N^4(N+1)^4(N+2)^3} + \Big(-\frac{4P_{36}}{27N^3(N+1)^3} + \frac{640}{9}S_3
$$

$$
-\frac{32\big(67N^2+67N-21\big)}{9N(N+1)}S_2 + \frac{64}{3}S_{2,1} + \frac{512}{3}S_{-2,1}\Big)S_1 + \Big(\frac{2P_{15}}{27N^2(N+1)^2}
$$

$$
-\frac{224}{3}S_2\Big)S_1^2 + \frac{32(4N-1)(4N+5)}{9N(N+1)}S_1^3 + \frac{80}{9}S_1^4 + \frac{2P_{14}}{9N^2(N+1)^2}S_2 + 48S_2^2
$$

$$
-\frac{32\big(53N^2+53N+16\big)}{9N(N+1)}S_3 + \frac{352}{3}S_4 + \Big(-\frac{64P_{27}}{9(N-1)N^2(N+1)^2(N+2)}
$$

$$
-\frac{128\big(10N^2+10N-3\big)}{9N(N+1)}S_1 - \frac{256}{3}S_1^2 + \frac{256}{3}S_2\Big)S_{-2} + 64S_{-2}^2 + \frac{448}{3}S_{-4}
$$

$$
+\Big(-\frac{64\big(10N^2+10N+9\big)}{9N(N+1)} + \frac{256}{3}S_1\Big)S_{-3} + \frac{16\big(9N^2+9N-2\big)}{3N(N+1)}S_{2,1}
$$

$$+64S_{3,1}+\frac{128\big(10N^2+10N-3\big)}{9N(N+1)}S_{-2,1}-\frac{256}{3}S_{-3,1}-64S_{2,1,1}$$

$$-\frac{512}{3}S_{-2,1,1}+\Bigg(-\frac{16\big(9N^2+9N-2\big)}{N(N+1)}+64S_1\Bigg)\zeta_3\Bigg]+\frac{P_{46}}{162N^5(N+1)^5}$$

$$-\frac{128\big(112N^3+112N^2-39N+18\big)}{81N^2(N+1)}S_{-2,1}+\Bigg(\frac{P_{45}}{162N^4(N+1)^4}-\frac{64}{9}S_2^2$$

$$+\frac{8P_{16}}{81N^2(N+1)^2}S_2-\frac{8\big(347N^2+347N+54\big)}{27N(N+1)}S_3+\frac{128}{9N(N+1)}S_{2,1}$$

$$+\frac{704}{9}S_4-\frac{320}{9}S_{3,1}-\frac{256\big(10N^2+10N-3\big)}{27N(N+1)}S_{-2,1}+\frac{1024}{9}S_{-2,1,1}$$

$$-\frac{256}{9}S_{-2,2}\Bigg)S_1+\Bigg(\frac{P_{25}}{9N^3(N+1)^3}+\frac{16\big(5N^2+5N-4\big)}{9N(N+1)}S_2-\frac{128}{9}S_{2,1}$$

$$+16S_3-\frac{256}{9}S_{-2,1}\Bigg)S_1^2+\Bigg(-\frac{16P_4}{27N^2(N+1)^2}+\frac{128}{27}S_2\Bigg)S_1^3+\Bigg(\frac{400}{27}S_3$$

$$+\frac{P_{24}}{81N^3(N+1)^3}+\frac{256}{3}S_{-2,1}\Bigg)S_2-\frac{32\big(23N^2+23N-3\big)}{27N(N+1)}S_2^2+\frac{512}{9}S_5$$

$$+\frac{8P_{17}}{81N^2(N+1)^2}S_3-\frac{176\big(17N^2+17N+6\big)}{27N(N+1)}S_4+\Bigg(-\frac{64P_9}{81N^3(N+1)^3}$$

$$+\frac{128P_7}{81N^2(N+1)^2}S_1-\frac{128}{9N(N+1)}S_1^2+\frac{256}{27}S_1^3-\frac{1280}{27}S_2+\frac{512}{27}S_3$$

$$-\frac{512}{9}S_{2,1}\Bigg)S_{-2}+\Bigg(\frac{64\big(112N^3+224N^2+169N+39\big)}{81N(N+1)^2}+\frac{128}{9}S_1^2$$

$$+\frac{128}{9}S_2-\frac{128\big(10N^2+10N+3\big)}{27N(N+1)}S_1\Bigg)S_{-3}+\Bigg(-\frac{128\big(10N^2+10N+3\big)}{27N(N+1)}$$

$$+\frac{256}{9}S_1\Bigg)S_{-4}+\frac{256}{9}S_{-5}+\frac{16P_2}{9N^2(N+1)^2}S_{2,1}+\frac{256}{9}S_{2,3}-\frac{512}{9}S_{2,-3}$$

$$+\frac{16\big(89N^2+89N+30\big)}{27N(N+1)}S_{3,1}-\frac{512}{9}S_{4,1}-\frac{128\big(10N^2+10N-3\big)}{27N(N+1)}S_{-2,2}$$

$$+\frac{512}{9}S_{-2,3}+\frac{512}{9}S_{2,1,-2}+\frac{256}{9}S_{3,1,1}+\frac{512\big(10N^2+10N-3\big)}{27N(N+1)}S_{-2,1,1}$$

$$+\frac{512}{9}S_{-2,2,1}-\frac{2048}{9}S_{-2,1,1,1}+\frac{16\big(2N^3+2N^2+2N+1\big)}{3N^3(N+1)^3}\zeta_2+\Bigg(-\frac{64}{3}S_2$$

$$+\frac{2P_{13}}{9N^2(N+1)^2}-\frac{1208}{9}S_1\Bigg)\zeta_3+\Bigg(\frac{8}{3}S_{2,1,1}-\frac{8}{3}B_4+12\zeta_4\Bigg)\Delta\gamma_{qq}^{(0)}$$

$$+(-1)^N\Bigg(-L_M^2\frac{64}{3(N+1)^3}-L_Q^2\frac{64}{3(N+1)^3}+L_M\Bigg[-\frac{256(4N+1)}{9(N+1)^4}$$

$$+\frac{128}{3(N+1)^3}S_1\Bigg]+L_Q\frac{64P_{39}}{9(N-1)N^2(N+1)^4(N+2)^3}+\frac{16P_{41}}{81N^5(N+1)^5}$$

$$-\frac{32P_{26}}{27N^4(N+1)^4}S_1+\frac{64\big(2N^2+2N+1\big)}{9N^3(N+1)^3}S_1^2+\frac{64\big(2N^2+2N+1\big)}{9N^3(N+1)^3}S_2$$

$$
+\frac{16\big(2N^3+2N^2+2N+1\big)}{3N^3(N+1)^3}\zeta_2\Big)\Big]+C_AC_FT_F\Big[L_M^3\frac{22}{27}\Delta\gamma_{qq}^{(0)}+L_Q^3\frac{44}{27}\Delta\gamma_{qq}^{(0)}
$$

$$
+L_M^2\Big[\frac{2P_{21}}{9N^3(N+1)^2}-\frac{184}{9}S_1+\Big(\frac{32}{3N(N+1)}-\frac{64}{3}S_1\Big)S_{-2}-\frac{32}{3}\big[S_3+S_{-3}\big]
$$

$$
+\frac{64}{3}S_{-2,1}\Big]+L_Q^2\Big[\frac{2P_{23}}{27N^3(N+1)^2}-\frac{16\big(194N^2+194N-33\big)}{27N(N+1)}S_1-\frac{176}{9}S_1^2
$$

$$
+\frac{176}{3}S_2-\frac{32}{3}S_3+\Big(\frac{32}{3N(N+1)}-\frac{64}{3}S_1\Big)S_{-2}-\frac{32}{3}S_{-3}+\frac{64}{3}S_{-2,1}\Big]
$$

$$
+L_M\Big[\frac{P_{38}}{81N^4(N+1)^3}+\Big(-\frac{8P_{29}}{81N^3(N+1)^3}+32S_3+\frac{128}{3}S_{-2,1}\Big)S_1
$$

$$
+\frac{1792}{27}S_2-\frac{16\big(31N^2+31N+9\big)}{9N(N+1)}S_3+\frac{160}{3}S_4+\Big(\frac{32\big(16N^2+10N-3\big)}{9N^2(N+1)^2}
$$

$$
-\frac{640}{9}S_1+\frac{64}{3}S_2\Big)S_{-2}+\Big(-\frac{32\big(10N^2+10N+3\big)}{9N(N+1)}+\frac{64}{3}S_1\Big)S_{-3}+\frac{64}{3}S_{-4}
$$

$$
-\frac{128}{3}S_{3,1}+\frac{64\big(10N^2+10N-3\big)}{9N(N+1)}S_{-2,1}+\frac{64}{3}S_{-2,2}-\frac{256}{3}S_{-2,1,1}-8\Delta\gamma_{qq}^{(0)}\zeta_3\Big]
$$

$$
+L_Q\Big[-\frac{16\big(230N^3+460N^2+213N-11\big)}{9N(N+1)^2}S_2-\frac{4P_{49}}{81N^4(N+1)^4(N+2)^3}
$$

$$
+\Big(\frac{4P_{37}}{81N^3(N+1)^3}-\frac{32\big(11N^2+11N+3\big)}{9N(N+1)}S_2-\frac{128}{3}S_{2,1}-\frac{256}{3}S_{-2,1}
$$

$$
+32S_3\Big)S_1+\Big(\frac{16\big(194N^2+194N-33\big)}{27N(N+1)}+\frac{32}{3}S_2\Big)S_1^2+\frac{352}{27}S_1^3-\frac{32}{3}S_2^2
$$

$$
+\frac{16\big(368N^2+368N-9\big)}{27N(N+1)}S_3-\frac{224}{3}S_4+\Big(\frac{32P_{27}}{9(N-1)N^2(N+1)^2(N+2)}
$$

$$
+\frac{64\big(10N^2+10N-3\big)}{9N(N+1)}S_1+\frac{128}{3}S_1^2-\frac{128}{3}S_2\Big)S_{-2}-32S_{-2}^2+\Big(-\frac{128}{3}S_1
$$

$$
+\frac{32\big(10N^2+10N+9\big)}{9N(N+1)}\Big)S_{-3}-\frac{224}{3}S_{-4}-\frac{64\big(11N^2+11N-3\big)}{9N(N+1)}S_{2,1}
$$

$$
-\frac{64}{3}S_{3,1}-\frac{64\big(10N^2+10N-3\big)}{9N(N+1)}S_{-2,1}+\frac{128}{3}S_{-3,1}+64S_{2,1,1}+\frac{256}{3}S_{-2,1,1}
$$

$$
+\Big(96-64S_1\Big)\zeta_3\Big]+\frac{64\big(112N^3+112N^2-39N+18\big)}{81N^2(N+1)}S_{-2,1}
$$

$$
+\frac{P_{47}}{729N^5(N+1)^5}+\Big(-\frac{16(N-1)\big(2N^3-N^2-N-2\big)}{9N^2(N+1)^2}S_2+\frac{112}{9}S_2^2
$$

$$
-\frac{4P_{44}}{729N^4(N+1)^4}+\frac{80(2N+1)^2}{9N(N+1)}S_3-\frac{208}{9}S_4-\frac{8\big(9N^2+9N+16\big)}{9N(N+1)}S_{2,1}
$$

$$
+\frac{64}{3}S_{3,1}+\frac{128\big(10N^2+10N-3\big)}{27N(N+1)}S_{-2,1}+\frac{128}{9}S_{-2,2}-\frac{512}{9}S_{-2,1,1}\Big)S_1
$$

$$
+\Big(\frac{4P_{18}}{9N^3(N+1)^3}+\frac{32}{9N(N+1)}S_2-\frac{80}{9}S_3+\frac{128}{9}S_{2,1}+\frac{128}{9}S_{-2,1}\Big)S_1^2
$$

$$
\begin{aligned}
&+\left(\frac{4P_{35}}{81N^3(N+1)^3}+\frac{496}{27}S_3-\frac{64}{3}S_{2,1}-\frac{128}{3}S_{-2,1}\right)S_2-\frac{64}{27}S_1^3S_2\\
&-\frac{4\left(15N^2+15N+14\right)}{9N(N+1)}S_2^2-\frac{8P_{20}}{81N^2(N+1)^2}S_3+\frac{4\left(443N^2+443N+78\right)}{27N(N+1)}S_4\\
&-\frac{224}{9}S_5+\Biggl(\frac{32P_9}{81N^3(N+1)^3}-\frac{64P_7}{81N^2(N+1)^2}S_1+\frac{64}{9N(N+1)}S_1^2-\frac{128}{27}S_1^3\\
&+\frac{640}{27}S_2-\frac{256}{27}S_3+\frac{256}{9}S_{2,1}\Biggr)S_{-2}+\Biggl(-\frac{32\left(112N^3+224N^2+169N+39\right)}{81N(N+1)^2}\\
&+\frac{64\left(10N^2+10N+3\right)}{27N(N+1)}S_1-\frac{64}{9}S_1^2-\frac{64}{9}S_2\Biggr)S_{-3}+\Biggl(\frac{64\left(10N^2+10N+3\right)}{27N(N+1)}\\
&-\frac{128}{9}S_1\Biggr)S_{-4}-\frac{128}{9}S_{-5}-\frac{8P_{19}}{9N^2(N+1)^2}S_{2,1}-\frac{8(13N+4)(13N+9)}{27N(N+1)}S_{3,1}\\
&+\frac{256}{9}\left[S_{2,-3}+S_{4,1}-S_{-2,3}-S_{2,1,-2}-S_{3,1,1}-S_{-2,2,1}\right]-\frac{128}{3}S_{2,3}\\
&+\frac{64}{3}S_{2,2,1}+\frac{64\left(10N^2+10N-3\right)}{27N(N+1)}S_{-2,2}-\frac{256\left(10N^2+10N-3\right)}{27N(N+1)}S_{-2,1,1}\\
&+\frac{224}{9}S_{2,1,1,1}+\frac{1024}{9}S_{-2,1,1,1}-\frac{8\left(2N^3+2N^2+2N+1\right)}{3N^3(N+1)^3}\zeta_2\\
&+\left(\frac{P_{22}}{27N^2(N+1)^2}+\frac{4\left(593N^2+593N+108\right)}{27N(N+1)}S_1-16S_1^2+16S_2\right)\zeta_3\\
&+\left(\frac{4B_4}{3}-4S_{2,1,1}-12\zeta_4\right)\Delta\gamma_{qq}^{(0)}+(-1)^N\Biggl(L_M^2\frac{32}{3(N+1)^3}+L_Q^2\frac{32}{3(N+1)^3}\\
&+L_M\left[\frac{128(4N+1)}{9(N+1)^4}-\frac{64}{3(N+1)^3}S_1\right]-L_Q\frac{32P_{39}}{9(N-1)N^2(N+1)^4(N+2)^3}\\
&-\frac{8P_{41}}{81N^5(N+1)^5}+\frac{16P_{26}}{27N^4(N+1)^4}S_1-\frac{32\left(2N^2+2N+1\right)}{9N^3(N+1)^3}\left[S_1^2+S_2\right]\\
&-\frac{8\left(2N^3+2N^2+2N+1\right)}{3N^3(N+1)^3}\zeta_2\Biggr)\Biggr]+C_FT_F^2\Biggl[-L_M^3\frac{16}{27}\Delta\gamma_{qq}^{(0)}-L_Q^3\frac{8}{27}\Delta\gamma_{qq}^{(0)}\\
&+L_M^2\left[-\frac{8P_1}{27N^2(N+1)^2}-\frac{320}{27}S_1+\frac{64}{9}S_2\right]+L_Q^2\Biggl[\frac{16\left(29N^2+29N-6\right)}{27N(N+1)}S_1\\
&-\frac{8P_6}{27N^2(N+1)^2}+\frac{32}{9}S_1^2-\frac{32}{3}S_2\Biggr]-L_M\frac{248}{81}\Delta\gamma_{qq}^{(0)}+L_Q\Biggl[\frac{8P_{33}}{81N^3(N+1)^3}\\
&+\left(-\frac{16P_{10}}{81N^2(N+1)^2}+\frac{64}{9}S_2\right)S_1-\frac{16\left(29N^2+29N-6\right)}{27N(N+1)}S_1^2-\frac{64}{27}S_1^3\\
&+\frac{16\left(35N^2+35N-2\right)}{9N(N+1)}S_2-\frac{896}{27}S_3+\frac{128}{9}S_{2,1}\Biggr]-\frac{2P_{43}}{729N^4(N+1)^4}+\frac{64}{81}S_2\\
&+\frac{12064}{729}S_1+\frac{320}{81}S_3-\frac{64}{27}S_4-\frac{112}{27}\Delta\gamma_{qq}^{(0)}\zeta_3\Biggr]+C_FN_FT_F^2\Biggl[-L_M^3\frac{8}{27}\Delta\gamma_{qq}^{(0)}\\
&-L_Q^3\frac{16}{27}\Delta\gamma_{qq}^{(0)}+L_M\left[\frac{4P_{32}}{81N^3(N+1)^3}-\frac{2176}{81}S_1-\frac{320}{27}S_2+\frac{64}{9}S_3\right]
\end{aligned}
$$

$$
+L_Q^2\Biggl[-\frac{16P_6}{27N^2(N+1)^2}+\frac{32\bigl(29N^2+29N-6\bigr)}{27N(N+1)}S_1+\frac{64}{9}S_1^2-\frac{64}{3}S_2\Biggr]
$$
$$
+L_Q\Biggl[\Biggl(-\frac{32P_{10}}{81N^2(N+1)^2}+\frac{128}{9}S_2\Biggr)S_1-\frac{32\bigl(29N^2+29N-6\bigr)}{27N(N+1)}S_1^2-\frac{128}{27}S_1^3
$$
$$
+\frac{16P_{33}}{81N^3(N+1)^3}+\frac{32\bigl(35N^2+35N-2\bigr)}{9N(N+1)}S_2-\frac{1792}{27}S_3+\frac{256}{9}S_{2,1}\Biggr]
$$
$$
+\frac{4P_{42}}{729N^4(N+1)^4}-\frac{24064}{729}S_1+\frac{128}{81}S_2+\frac{640}{81}S_3-\frac{128}{27}S_4+\frac{64}{27}\Delta\gamma_{qq}^{(0)}\zeta_3\Biggr]
$$
$$
+\hat{C}_{q,g_1}^{\mathrm{NS},(3)}(N_F)\Biggr\}\Biggr\}\;. \tag{2.7}
$$

Here the color factors are given by $C_A = N_c, C_F = (N_c^2-1)/(2N_c), T_F = 1/2$ in $SU(N_c)$, and $N_c = 3$ in the case of Quantum Chromodynamics. $\hat{C}_{q,g_1}^{\mathrm{NS},(3)}(N_F)$ denotes the massless Wilson coefficient at 3-loop order, cf. (2.2), and the polynomials P_i are given by

$$P_1 = -3N^4-6N^3-47N^2-20N+12 \tag{2.8}$$
$$P_2 = 7N^4+14N^3+3N^2-4N-4 \tag{2.9}$$
$$P_3 = 19N^4+38N^3-9N^2-20N+4 \tag{2.10}$$
$$P_4 = 28N^4+56N^3+28N^2+2N+1 \tag{2.11}$$
$$P_5 = 33N^4+54N^3+9N^2-52N-28 \tag{2.12}$$
$$P_6 = 57N^4+96N^3+65N^2-10N-24 \tag{2.13}$$
$$P_7 = 112N^4+224N^3+121N^2+9N+9 \tag{2.14}$$
$$P_8 = 141N^4+246N^3+241N^2-8N-84 \tag{2.15}$$
$$P_9 = 181N^4+266N^3+82N^2-3N+18 \tag{2.16}$$
$$P_{10} = 235N^4+524N^3+211N^2+30N+72 \tag{2.17}$$
$$P_{11} = 359N^4+772N^3+335N^2+30N+72 \tag{2.18}$$
$$P_{12} = 501N^4+894N^3+541N^2-116N-204 \tag{2.19}$$
$$P_{13} = 561N^4+1122N^3+767N^2+302N+48 \tag{2.20}$$
$$P_{14} = 1131N^4+2118N^3+1307N^2+32N-276 \tag{2.21}$$
$$P_{15} = 1139N^4+2710N^3+635N^2+216N+828 \tag{2.22}$$
$$P_{16} = 1199N^4+2398N^3+1181N^2+18N+90 \tag{2.23}$$
$$P_{17} = 1220N^4+2359N^3+1934N^2+357N-138 \tag{2.24}$$
$$P_{18} = 3N^5+11N^4+10N^3+3N^2+7N+8 \tag{2.25}$$
$$P_{19} = 12N^5+16N^4+18N^3-15N^2-5N-8 \tag{2.26}$$
$$P_{20} = 27N^5+863N^4+1573N^3+1151N^2+144N-36 \tag{2.27}$$
$$P_{21} = 51N^5+102N^4+121N^3+118N^2+48N+48 \tag{2.28}$$
$$P_{22} = 648N^5-2103N^4-4278N^3-3505N^2-682N-432 \tag{2.29}$$
$$P_{23} = 1407N^5+2418N^4+1793N^3+134N^2-384N+144 \tag{2.30}$$
$$P_{24} = -11145N^6-32355N^5-37523N^4-14329N^3+2392N^2+120N-1512 \tag{2.31}$$
$$P_{25} = -151N^6-469N^5-181N^4+305N^3+208N^2+40N+8 \tag{2.32}$$
$$P_{26} = 3N^6+9N^5+70N^4+77N^3+39N^2-10N-12 \tag{2.33}$$
$$P_{27} = 6N^6+18N^5-N^4-20N^3+46N^2+29N-6 \tag{2.34}$$

$$
\begin{aligned}
P_{28} &= 15N^6+36N^5+30N^4+8N^3+3N^2+16N+20 && (2.35)\\
P_{29} &= 155N^6+465N^5+465N^4+371N^3+108N^2+108N+54 && (2.36)\\
P_{30} &= 216N^6+567N^5+687N^4+381N^3+37N^2-44N+12 && (2.37)\\
P_{31} &= 309N^6+807N^5+693N^4-271N^3-638N^2+68N+216 && (2.38)\\
P_{32} &= 525N^6+1575N^5+1535N^4+973N^3+536N^2+48N-72 && (2.39)\\
P_{33} &= 609N^6+1485N^5+1393N^4+83N^3-422N^2+156N+216 && (2.40)\\
P_{34} &= 795N^6+2043N^5+2075N^4+517N^3-298N^2+156N+216 && (2.41)\\
P_{35} &= 868N^6+2469N^5+2487N^4+940N^3+27N^2+63N+72 && (2.42)\\
P_{36} &= 1770N^6+4671N^5+4765N^4+1205N^3-227N^2+1044N+756 && (2.43)\\
P_{37} &= 7531N^6+23673N^5+23055N^4+7375N^3+1614N^2+936N-324 && (2.44)\\
P_{38} &= -4785N^7-14355N^6-4399N^5+10327N^4+3548N^3+3000N^2\\
&\quad +1080N-1728 && (2.45)\\
P_{39} &= 25N^7+138N^6+311N^5+464N^4+672N^3+670N^2+264N+48 && (2.46)\\
P_{40} &= -45N^8-162N^7-858N^6-1960N^5-1885N^4-1094N^3-804N^2\\
&\quad -40N+192 && (2.47)\\
P_{41} &= 39N^8+138N^7+847N^6+1371N^5+1283N^4+485N^3+101N^2\\
&\quad +132N+72 && (2.48)\\
P_{42} &= 3549N^8+14196N^7+23870N^6+25380N^5+15165N^4+1712N^3-2016N^2\\
&\quad +144N+432 && (2.49)\\
P_{43} &= 5487N^8+21948N^7+36370N^6+28836N^5+11943N^4+4312N^3+2016N^2\\
&\quad -144N-432 && (2.50)\\
P_{44} &= 10807N^8+43228N^7+62898N^6+39178N^5+7027N^4+702N^3+3240N^2\\
&\quad +3456N+1620 && (2.51)\\
P_{45} &= 42591N^8+166764N^7+245088N^6+128254N^5-26735N^4-40762N^3\\
&\quad -3928N^2-1272N-2160 && (2.52)\\
P_{46} &= -18351N^{10}-89784N^9-208773N^8-267222N^7-192265N^6-46700N^5\\
&\quad +14565N^4+7730N^3+1240N^2+1464N+144 && (2.53)\\
P_{47} &= 165N^{10}+825N^9+106856N^8+321746N^7+396657N^6+247433N^5\\
&\quad +126914N^4+51804N^3+6336N^2+4752N+5184 && (2.54)\\
P_{48} &= 828N^{11}+7632N^{10}+29217N^9+59592N^8+66844N^7+35738N^6+7405N^5\\
&\quad +16688N^4+27880N^3+11552N^2-3312N-2304 && (2.55)\\
P_{49} &= 8274N^{11}+78519N^{10}+313841N^9+686295N^8+881001N^7+638778N^6\\
&\quad +204948N^5+7992N^4+32296N^3+26544N^2-10656N-8640\ . && (2.56)
\end{aligned}
$$

We would like to note that we disagree with the $O(a_s^2\ln(Q^2/\mu^2))$ terms given in [28], but agree with the representation in [29, 51].

One obtains the analytic continuation of the harmonic sums to complex values of N by performing their asymptotic expansion analytically, cf. [33, 34].[3] Furthermore, the nested harmonic sums obey the shift relations

$$
S_{b,\vec{a}}(N) = S_{b,\vec{a}}(N-1) + \frac{\mathrm{sign}(b)^N}{N^{|b|}} S_{\vec{a}}(N)\ , \tag{2.57}
$$

[3] These expansions can now be obtained automatically using the package `HarmonicSums` [25].

through which any regular point in the complex plane can be reached using the analytic asymptotic representation as input. The poles of the nested harmonic sums $S_{\vec{a}}(N)$ are located at the non-positive integers. In data analyses, one may thus encode the QCD evolution [35] together with the Wilson coefficient for complex values of N analytically and finally perform one numerical contour integral around the singularities of the problem.[4]

In x-space the Wilson coefficient is represented in terms of harmonic polylogarithms [37] over the alphabet $\{f_0, f_1, f_{-1}\}$, which were again reduced applying the shuffle relations [38]. They are defined by

$$H_{b,\vec{a}}(x) = \int_0^x dy f_b(y) H_{\vec{a}}(y), \quad H_{\underbrace{0,\ldots,0}_{k}}(x) = \frac{1}{k!}\ln^k(x), \quad H_{\emptyset} = 1, \tag{2.58}$$

$$f_0(x) = \frac{1}{x}, \qquad f_1(x) = \frac{1}{1-x}, \qquad f_{-1}(x) = \frac{1}{1+x} \,. \tag{2.59}$$

The Wilson coefficient is represented by three contributions, the $(\ldots)_+$-function term, the $\delta(1-x)$-term, and the regular term. Here the $+$-distribution is defined by

$$\int_0^1 dy\, [F(y)]_+\, g(y) = \int_0^1 dy F(y)\, [g(y) - g(1)] \,. \tag{2.60}$$

One obtains

$$\begin{aligned}
L_{q,g_1}^{\rm NS}(x) = {} & a_s^2 \Biggl\{ \Biggl(\frac{1}{1-x} C_F T_F \Biggl[\frac{8}{3}\left[L_Q^2 + L_M^2\right] + L_M \Biggl[\frac{80}{9} + \frac{16}{3} H_0\Biggr] + L_Q \Biggl[-\frac{116}{9} - \frac{32}{3} H_0 \\
& - \frac{16}{3} H_1 \Biggr] + \frac{718}{27} + \frac{268}{9} H_0 + 8 H_0^2 + \left(\frac{116}{9} + \frac{16}{3} H_0\right) H_1 + \frac{8}{3} H_1^2 + \frac{16}{3} H_{0,1} \\
& - \frac{32}{3}\zeta_2 \Biggr] \Biggr)_+ + \delta(1-x) \Biggl(C_F T_F \Biggl[2\left[L_M^2 + L_Q^2\right] + L_M \frac{2}{3} - L_Q \frac{38}{3} + \frac{265}{9} \Biggr] \Biggr) \\
& + C_F T_F \Biggl[-\frac{4}{3}(x+1)\left[L_Q^2 + L_M^2\right] + L_M \Biggl[-\frac{8}{9}(11x-1) - \frac{8}{3}(x+1) H_0 \Biggr] \\
& + L_Q \Biggl[\frac{8}{9}(14x+5) + \frac{16}{3}(x+1) H_0 + \frac{8}{3}(x+1) H_1 \Biggr] - \frac{4}{27}(218x+47) \\
& - \frac{8}{9}(28x+13) H_0 - 4(x+1) H_0^2 + \left(-\frac{8}{9}(14x+5) - \frac{8}{3}(x+1) H_0 \right) H_1 \\
& - \frac{4}{3}(x+1) H_1^2 - \frac{8}{3}(x+1) H_{0,1} + \frac{16}{3}(x+1)\zeta_2 \Biggr] \Biggr\} \\
& + a_s^3 \Biggl\{ \Biggl(\frac{1}{(1-x)^2} C_A C_F T_F \Biggl[-\frac{4}{81}(800x - 773) H_0^2 + \frac{32}{81}(94x - 121)\zeta_2 \\
& + \frac{32}{9}(x+2) H_{0,1} \Biggr] + \frac{1}{1-x} \Biggl(C_A C_F T_F \Biggl[-L_M^3 \frac{176}{27} - L_Q^3 \frac{352}{27} + L_M^2 \Biggl[\frac{184}{9} \\
& + \frac{16}{3} H_0^2 - \frac{32}{3}\zeta_2 \Biggr] + L_Q^2 \Biggl[\frac{3104}{27} + \frac{704}{9} H_0 + \frac{16}{3} H_0^2 + \frac{352}{9} H_1 - \frac{32}{3}\zeta_2 \Biggr]
\end{aligned}$$

[4]For precise numerical implementations of the analytic continuation of harmonic sums see [36].

$$+L_M\Big[\frac{1240}{81}+\frac{1792}{27}H_0+\frac{248}{9}H_0^2+\frac{32}{9}H_0^3-16H_0^2H_1+32H_0H_{0,1}-\frac{64}{3}H_{0,0,1}$$

$$+\Big(-\frac{320}{9}-\frac{64}{3}H_0\Big)\zeta_2+96\zeta_3\Big]+L_Q\Big[-\frac{80}{9}H_0^3-\frac{30124}{81}-\frac{14144}{27}H_0-\frac{1216}{9}H_0^2$$

$$+\Big(-\frac{6208}{27}-\frac{704}{9}H_0-\frac{16}{3}H_0^2\Big)H_1+\Big(-\frac{352}{9}+\frac{32}{3}H_0\Big)H_1^2-64H_0H_{0,-1}$$

$$+\Big(-\frac{704}{9}+\frac{32}{3}H_0-\frac{128}{3}H_1\Big)H_{0,1}-\frac{128}{3}H_{0,0,1}+128H_{0,0,-1}+64H_{0,1,1}$$

$$+\Big(192+\frac{128}{3}H_0+64H_1\Big)\zeta_2-\frac{256}{3}\zeta_3\Big]+\frac{43228}{729}+\frac{3256}{81}H_0+\frac{496}{81}H_0^3+\frac{16}{27}H_0^4$$

$$+\Big(\frac{32}{3}-\frac{32}{9}H_0-\frac{160}{9}H_0^2-\frac{112}{27}H_0^3\Big)H_1+\frac{8}{9}H_0^2H_1^2-\frac{64}{27}H_0H_1^3+\Big(\frac{368}{9}H_0$$

$$+\frac{16}{3}H_0^2+\Big(-8-\frac{128}{9}H_0\Big)H_1+\frac{128}{9}H_1^2\Big)H_{0,1}-\frac{32}{9}H_{0,1}^2+\Big(-\frac{1072}{27}+\frac{32}{9}H_0$$

$$+\frac{320}{9}H_1\Big)H_{0,0,1}+\frac{224}{9}\big[H_{0,1,1,1}-H_{0,0,1,1}\big]+\Big(\frac{160}{9}H_0-32H_1+24\Big)H_{0,1,1}$$

$$+\Big(-\frac{496}{27}H_0-\frac{112}{9}H_0^2+\Big(8-\frac{160}{9}H_0\Big)H_1-\frac{128}{9}H_1^2+\frac{32}{9}H_{0,1}\Big)\zeta_2+\Big(-\frac{1196}{27}$$

$$+\frac{160}{9}H_0-\frac{32}{9}H_1\Big)\zeta_3+\frac{296}{3}\zeta_4-\frac{32}{3}B_4\Big]+C_F^2T_F\Big[L_Q^3\Big[16-\frac{32}{3}H_0-\frac{64}{3}H_1\Big]$$

$$+L_M^2L_Q\Big[16-\frac{32}{3}H_0-\frac{64}{3}H_1\Big]+L_M^2\Big[-22-16H_0+\Big(8+\frac{128}{3}H_0\Big)H_1$$

$$+\frac{16}{3}H_0^2+16H_1^2-\frac{64}{3}\zeta_2\Big]+L_Q^2\Big[-\frac{334}{3}+\frac{32}{9}H_0+\Big(\frac{856}{9}+\frac{320}{3}H_0\Big)H_1$$

$$+\frac{80}{3}H_0^2+48H_1^2-\frac{256}{3}\zeta_2\Big]+L_ML_Q\Big[\frac{88}{3}-\frac{176}{9}H_0+\Big(-\frac{640}{9}-\frac{64}{3}H_0\Big)H_1$$

$$-\frac{32}{3}H_0^2+\frac{64}{3}\zeta_2\Big]+L_M\Big[-\frac{206}{3}-\frac{112}{3}H_0+\frac{88}{9}H_0^2+\Big(\frac{160}{3}+\frac{32}{3}H_0\Big)H_1^2$$

$$+\frac{64}{9}H_0^3+\Big(\frac{152}{3}+\frac{1424}{9}H_0+\frac{160}{3}H_0^2\Big)H_1-\frac{64}{3}H_0H_{0,1}+\Big(-\frac{784}{9}-\frac{128}{3}H_0$$

$$-\frac{64}{3}H_1\Big)\zeta_2-64\zeta_3\Big]+L_Q\Big[\frac{2360}{9}+\frac{4508}{27}H_0-\frac{160}{3}H_0^2-\frac{224}{9}H_0^3-\frac{320}{9}H_1^3$$

$$+\Big(-\frac{4556}{27}-\frac{3680}{9}H_0-128H_0^2\Big)H_1+\Big(-\frac{512}{3}-128H_0\Big)H_1^2+128H_0H_{0,-1}$$

$$+\Big(48-\frac{64}{3}H_0+\frac{64}{3}H_1\Big)H_{0,1}-256H_{0,0,-1}-64H_{0,1,1}+\Big(\frac{2608}{9}+\frac{832}{3}H_0$$

$$+\frac{448}{3}H_1\Big)\zeta_2+320\zeta_3\Big]-\frac{14197}{54}-\frac{3262}{27}H_0+\frac{4}{3}H_0^4+\frac{196}{27}H_0^2+\frac{380}{81}H_0^3$$

$$+\Big(\frac{302}{9}+\frac{13624}{81}H_0+\frac{1628}{27}H_0^2+\frac{304}{27}H_0^3\Big)H_1+\Big(\frac{448}{9}+\frac{80}{9}H_0-\frac{8}{9}H_0^2\Big)H_1^2$$

$$+\frac{128}{27}H_0H_1^3+\Big(\frac{112}{9}-\frac{1304}{27}H_0-\frac{32}{3}H_0^2+\frac{160}{9}H_0H_1-\frac{128}{9}H_1^2\Big)H_{0,1}$$

$$
+\left(\frac{1184}{27}+\frac{128}{9}H_0-\frac{256}{9}H_1\right)H_{0,0,1}+\left(-16-\frac{32}{3}H_0+\frac{64}{3}H_1\right)H_{0,1,1}
-\frac{16}{9}H_{0,1}^2-\frac{128}{9}H_{0,0,0,1}+\frac{64}{3}H_{0,0,1,1}+\left(-\frac{2488}{27}-\frac{1192}{27}H_0-\frac{80}{9}H_0^2+\left(-\frac{160}{9}
+\frac{32}{3}H_0\right)H_1+\frac{128}{9}H_1^2-\frac{32}{9}H_{0,1}\right)\zeta_2+\left(\frac{3088}{27}-\frac{128}{9}H_0+\frac{160}{9}H_1\right)\zeta_3+\frac{64}{3}B_4
-\frac{664}{9}\zeta_4\Bigg]+C_FT_F^2\Bigg[L_M^3\frac{128}{27}+L_Q^3\frac{64}{27}+L_M^2\left[\frac{320}{27}+\frac{64}{9}H_0\right]+L_Q^2\Bigg[-\frac{464}{27}
-\frac{128}{9}H_0-\frac{64}{9}H_1\Bigg]+L_M\frac{1984}{81}+L_Q\Bigg[\frac{3760}{81}+\frac{2144}{27}H_0+\left(\frac{928}{27}+\frac{128}{9}H_0\right)H_1
+\frac{64}{3}H_0^2+\frac{64}{9}H_1^2+\frac{128}{9}H_{0,1}-\frac{256}{9}\zeta_2\Bigg]-\frac{12064}{729}+\frac{64}{81}H_0-\frac{160}{81}H_0^2-\frac{32}{81}H_0^3
+\frac{896}{27}\zeta_3\Bigg]+C_FN_FT_F^2\Bigg[L_M^3\frac{64}{27}+L_Q^3\frac{128}{27}+L_Q^2\left[-\frac{928}{27}-\frac{256}{9}H_0-\frac{128}{9}H_1\right]
+L_M\left[\frac{2176}{81}-\frac{320}{27}H_0-\frac{32}{9}H_0^2\right]+L_Q\Bigg[\frac{7520}{81}+\frac{4288}{27}H_0+\frac{128}{3}H_0^2+\Bigg(\frac{1856}{27}
+\frac{256}{9}H_0\Bigg)H_1+\frac{128}{9}H_1^2+\frac{256}{9}H_{0,1}-\frac{512}{9}\zeta_2\Bigg]+\frac{24064}{729}+\frac{128}{81}H_0-\frac{320}{81}H_0^2
-\frac{64}{81}H_0^3-\frac{512}{27}\zeta_3\Bigg]\Bigg)\Bigg)_+ +\delta(1-x)\Bigg(C_AC_FT_F\Bigg[-L_M^3\frac{44}{9}-L_Q^3\frac{88}{9}+L_M^2\Bigg[\frac{34}{3}
-\frac{16}{3}\zeta_3\Bigg]+L_Q^2\left[\frac{938}{9}-\frac{16}{3}\zeta_3\right]+L_M\left[-\frac{1595}{27}+\frac{272}{9}\zeta_3+\frac{68}{3}\zeta_4\right]+L_Q\Bigg[-\frac{11032}{27}
-\frac{32}{3}\zeta_2+\frac{1024}{9}\zeta_3-\frac{196}{3}\zeta_4\Bigg]+\frac{55}{243}-\frac{10045}{81}\zeta_3-\frac{16}{9}\zeta_2\zeta_3+\frac{2624}{27}\zeta_4-\frac{176}{9}\zeta_5
-8B_4\Bigg]+C_F^2T_F\Bigg[L_Q^3 6+L_M^2L_Q 6+L_M^2\left[-10+\frac{32}{3}\zeta_3\right]+L_Q^2\left[-48+\frac{32}{3}\zeta_3\right]
+L_ML_Q 2+L_M\left[-5-\frac{112}{9}\zeta_3-\frac{136}{3}\zeta_4\right]+L_Q\Bigg[\frac{368}{3}+\frac{64}{3}\zeta_2-\frac{1616}{9}\zeta_3
+\frac{392}{3}\zeta_4\Bigg]-\frac{2039}{18}+\frac{13682}{81}\zeta_3+\frac{32}{9}\zeta_2\zeta_3-\frac{3304}{27}\zeta_4+16B_4+\frac{352}{9}\zeta_5\Bigg]
+C_FT_F^2\Bigg[L_M^3\frac{32}{9}+L_Q^3\frac{16}{9}+L_M^2\frac{8}{9}-L_Q^2\frac{152}{9}+L_M\frac{496}{27}+L_Q\frac{1624}{27}-\frac{3658}{243}
+\frac{224}{9}\zeta_3\Bigg]+C_FN_FT_F^2\Bigg[L_M^3\frac{16}{9}+L_Q^3\frac{32}{9}-L_Q^2\frac{304}{9}+L_M\frac{700}{27}+L_Q\frac{3248}{27}+\frac{4732}{243}
-\frac{128}{9}\zeta_3\Bigg]\Bigg)+C_AC_FT_F\Bigg[L_M^3\frac{88}{27}(x+1)+L_Q^3\frac{176}{27}(x+1)+L_M^2\Bigg[-\frac{4}{9}(83x-37)
+\frac{32}{3}(x+1)H_0+\frac{32}{3}\frac{x^2+1}{x+1}\left[H_{0,-1}-H_{-1}H_0\right]+\frac{16}{3}\frac{x}{x+1}\left[2\zeta_2-H_0^2\right]\Bigg]
$$

$$
\begin{aligned}
&+L_Q^2\Big[-\frac{4}{27}\big(865x+109\big)-\frac{256}{9}\big(x+1\big)H_0+\frac{32}{3}\frac{x^2+1}{x+1}\big[H_{0,-1}-H_{-1}H_0\big]\\
&-\frac{176}{9}\big(x+1\big)H_1+\frac{16}{3}\frac{x}{x+1}\big[2\zeta_2-H_0^2\big]\Big]+L_M\Big[-\frac{4}{81}\big(4577x-4267\big)\\
&-\frac{16}{27}\big(29x-109\big)H_0+\frac{4}{9}\frac{19x^2+4x+25}{x+1}H_0^2-\frac{32}{9}\frac{x}{x+1}H_0^3+\Big(\frac{32}{3}\big(x-1\big)\\
&+8\big(x+1\big)H_0^2\Big)H_1+\frac{128}{9}\frac{4x^2+3x+4}{x+1}H_{0,-1}+\Big(-\frac{16}{3}-16H_0\Big)\big(x+1\big)H_{0,1}\\
&+\Big(-\frac{128}{9}\frac{4x^2+3x+4}{x+1}H_0+\frac{16}{3}\frac{x^2+1}{x+1}\big[4H_{0,1}-H_0^2\big]\Big)H_{-1}+\frac{64}{3}\frac{x}{x+1}H_{0,0,1}\\
&+\frac{32}{3}\frac{x^2+1}{x+1}\big[H_{0,0,-1}-2H_{0,-1,1}-2H_{0,1,-1}\big]+\Big(\frac{16}{9}\frac{3x^2+14x-9}{x+1}\\
&-\frac{64}{3}\frac{x^2+1}{x+1}H_{-1}+\frac{16}{3}\frac{3x^2+4x+3}{x+1}H_0\Big)\zeta_2-32\frac{x^2+3x+1}{x+1}\zeta_3\Big]\\
&+L_Q\Big[\frac{4}{81}\big(12329x-577\big)+\Big(\frac{64}{27}\frac{181x^2+239x+49}{x+1}-\frac{32}{3}\frac{\big(x-1\big)^2}{x+1}H_{-1}^2\\
&+\frac{32}{9}\frac{6x^4+25x^3+18x^2+25x+6}{\big(x+1\big)x}H_{-1}\Big)H_0+\Big(-\frac{8}{9}\frac{12x^3-21x^2-77x-24}{x+1}\\
&+\frac{16}{3}\frac{5x^2-2x+5}{x+1}H_{-1}\Big)H_0^2+\frac{80}{9}\frac{x}{x+1}H_0^3+\Big(\frac{8}{27}\big(703x+253\big)-\frac{8}{3}\big(x-3\big)H_0^2\\
&+\frac{352}{9}\big(x+1\big)H_0\Big)H_1+\Big(\frac{176}{9}\big(x+1\big)-\frac{16}{3}\big(x+1\big)H_0\Big)H_1^2+\Big(\frac{64}{3}\frac{3x+1}{x+1}H_0\\
&-\frac{32}{9}\frac{6x^4+25x^3+18x^2+25x+6}{\big(x+1\big)x}+\frac{64}{3}\frac{\big(x-1\big)^2}{x+1}H_{-1}\Big)H_{0,-1}+\Big(\frac{208}{9}\big(x+1\big)\\
&+\frac{16}{3}\big(x-3\big)H_0+\frac{64}{3}\big(x+1\big)H_1\Big)H_{0,1}+\frac{32}{3}\big(x+3\big)H_{0,0,1}-32\big(x+1\big)H_{0,1,1}\\
&-\frac{64}{3}\frac{\big(x-1\big)^2}{x+1}H_{0,-1,-1}-\frac{32}{3}\frac{5x^2+10x+9}{x+1}H_{0,0,-1}+\Big(-\frac{64}{3}\big(x+2\big)H_1\\
&+\frac{16}{9}\frac{12x^3-23x^2-72x-17}{x+1}-\frac{32}{3}\frac{\big(x-1\big)^2}{x+1}H_{-1}-\frac{16}{3}\frac{3x^2+8x+3}{x+1}H_0\Big)\zeta_2\\
&+\frac{64}{3}\frac{3x^2+3x+2}{x+1}\zeta_3\Big]-\frac{2}{729}\big(108295x-86681\big)+\Big(-\frac{4}{81}\big(995x-2807\big)\\
&-\frac{32}{81}\frac{199x^2+174x+199}{x+1}H_{-1}+\frac{32}{9}\big(x+1\big)H_{-1}^2-\frac{64}{27}\frac{x^2+1}{x+1}H_{-1}^3\Big)H_0\\
&+\Big(\frac{4}{81}\frac{253x^2+391x+586}{x+1}-\frac{16}{27}\frac{19x^2+18x+19}{x+1}H_{-1}+\frac{16}{9}\frac{x^2+1}{x+1}H_{-1}^2\Big)H_0^2\\
&+\Big(\frac{8}{81}\frac{22x^2+7x+25}{x+1}-\frac{32}{27}\frac{x^2+1}{x+1}H_{-1}\Big)H_0^3-\frac{16}{27}\frac{x}{x+1}H_0^4+\Big(\frac{8}{9}\big(9x+4\big)H_0\\
&-\frac{8}{27}\big(65x-29\big)+\frac{8}{9}\big(14x+3\big)H_0^2+\frac{56}{27}\big(x+1\big)H_0^3\Big)H_1+\Big(-\frac{4}{9}\big(43x-46\big)\\
&-\frac{8}{9}\big(2x+5\big)H_0-\frac{4}{9}\big(x+1\big)H_0^2\Big)H_1^2+\frac{32}{27}\big(x+1\big)H_0H_1^3+\Big(-\frac{64}{9}\big(x+1\big)H_{-1}
\end{aligned}
$$

$$
+\frac{32}{81}\frac{199x^2+174x+199}{x+1}+\frac{64}{9}\frac{x^2+1}{x+1}H_{-1}^2\Big)H_{0,-1}+\Big(\frac{256}{27}\frac{4x^2+3x+4}{x+1}H_{-1}
-\frac{8}{27}(143x+2)-\frac{16}{9}(13x+6)H_0-\frac{8}{3}(x+1)H_0^2+\Big(\frac{8}{9}(11x+20)
+\frac{64}{9}(x+1)H_0\Big)H_1-\frac{64}{9}(x+1)H_1^2\Big)H_{0,1}+\frac{16}{9}(7x+1)H_{0,1}^2+\Big(\frac{64}{9}(x+1)
-\frac{128}{9}\frac{x^2+1}{x+1}H_{-1}\Big)H_{0,-1,-1}-\frac{256}{27}\frac{4x^2+3x+4}{x+1}H_{0,-1,1}+\Big(-\frac{64}{9}\frac{x^2+1}{x+1}H_{-1}
+\frac{32}{27}\frac{19x^2+18x+19}{x+1}\Big)H_{0,0,-1}+\Big(\frac{8}{27}\frac{9x^2+101x+12}{x+1}+\frac{64}{9}\frac{x^2+1}{x+1}H_{-1}
-\frac{16}{9}(7x+1)H_0-\frac{160}{9}(x+1)H_1\Big)H_{0,0,1}-\frac{256}{27}\frac{4x^2+3x+4}{x+1}H_{0,1,-1}
+\Big(-\frac{16}{9}(x+7)-\frac{128}{9}\frac{x^2+1}{x+1}H_{-1}-\frac{16}{9}(11x+5)H_0+16(x+1)H_1\Big)H_{0,1,1}
+\frac{64}{9}\frac{x^2+1}{x+1}\big[2H_{0,-1,-1,-1}+2H_{0,-1,1,1}+H_{0,0,-1,-1}-H_{0,0,-1,1}+H_{0,0,0,-1}
-H_{0,0,1,-1}+2H_{0,1,-1,1}+2H_{0,1,1,-1}\big]+\frac{64}{9}\frac{5x^2+6x-1}{x+1}H_{0,0,0,1}
-\frac{16}{9}\frac{x^2-2x-11}{x+1}H_{0,0,1,1}-\frac{112}{9}(x+1)H_{0,1,1,1}+\Big(\frac{16}{81}\frac{174x^2+209x-189}{x+1}
-\frac{32}{27}\frac{29x^2+18x+29}{x+1}H_{-1}+\Big(-\frac{8}{9}(3x+14)+\frac{80}{9}(x+1)H_0\Big)H_1
+\Big(\frac{8}{27}\frac{63x^2+29x+6}{x+1}-\frac{32}{9}\frac{x^2+1}{x+1}H_{-1}\Big)H_0-\frac{32}{9}\frac{x^2+1}{x+1}H_{-1}^2+\frac{64}{9}(x+1)H_1^2
+\frac{8}{9}\frac{3x^2+8x+9}{x+1}H_0^2-\frac{16}{9}(7x+1)H_{0,1}\Big)\zeta_2+\Big(\frac{2}{27}\frac{497x^2+1102x+1085}{x+1}
+\frac{128}{9}\frac{x^2+1}{x+1}H_{-1}+\frac{32}{9}\frac{6x^2+4x-3}{x+1}H_0+\frac{16}{9}(x+1)H_1\Big)\zeta_3+\frac{16}{3}(x+1)B_4
-\frac{8}{3}\frac{36x^2+51x+22}{x+1}\zeta_4\Big]+C_F^2T_F\Big[\big[L_Q^3+L_M^2L_Q\big]\Big[-\frac{8}{3}(x+5)+\frac{32}{3}(x+1)H_1
+8(x+1)H_0\Big]+L_M^2\Big[28(2x-1)+\Big(-\frac{8}{3}(11x+5)+\frac{64}{3}\frac{x^2+1}{x+1}H_{-1}\Big)H_0
-\frac{4}{3}\frac{9x^2+10x+9}{x+1}H_0^2+\Big(-\frac{16}{3}(2x+1)-\frac{64}{3}(x+1)H_0\Big)H_1-\frac{8}{3}(x+1)H_{0,1}
-8(x+1)H_1^2-\frac{64}{3}\frac{x^2+1}{x+1}H_{0,-1}+\frac{8}{3}\frac{9x^2+10x+9}{x+1}\zeta_2\Big]+L_Q^2\Big[\frac{4}{9}(161x+130)
+\Big(-\frac{16}{3}(15x+4)+\frac{64}{3}\frac{x^2+1}{x+1}H_{-1}\Big)H_0-24(x+1)H_1^2+\Big(-\frac{16}{9}(50x+17)
-\frac{160}{3}(x+1)H_0\Big)H_1-\frac{4}{3}\frac{21x^2+34x+21}{x+1}H_0^2+\frac{8}{3}\frac{23x^2+38x+23}{x+1}\zeta_2
-\frac{64}{3}\frac{x^2+1}{x+1}H_{0,-1}-8(x+1)H_{0,1}\Big]+L_ML_Q\Big[\frac{4}{9}(19x-85)+\frac{8}{3}(13x+1)H_0
$$

$$
\begin{aligned}
&+8(x+1)H_0^2+\left(\frac{128}{9}(4x+1)+\frac{32}{3}(x+1)H_0\right)H_1-\frac{32}{3}(x+1)\zeta_2\Bigg]\\
&+L_M\Bigg[-\frac{4}{9}(337x+235)H_0-\frac{4}{9}\frac{195x^2+238x+123}{x+1}H_0^2-\frac{32}{9}\frac{3x^2+4x+3}{x+1}H_0^3\\
&+\left(-\frac{4}{9}(287x-113)-\frac{224}{9}(5x+2)H_0-\frac{80}{3}(x+1)H_0^2\right)H_1+\frac{4}{3}(178x-125)\\
&+\left(-\frac{16}{3}(7x+3)-\frac{16}{3}(x+1)H_0\right)H_1^2+\left(\frac{184}{9}(x+1)+\frac{32}{3}(x+1)H_0\right)H_{0,1}\\
&-\frac{256}{9}\frac{4x^2+3x+4}{x+1}H_{0,-1}+\frac{16}{3}\frac{3x^2-2x+3}{x+1}H_{0,0,1}+\Bigg(\frac{256}{9}\frac{4x^2+3x+4}{x+1}H_0\\
&+\frac{32}{3}\frac{x^2+1}{x+1}\left[H_0^2-4H_{0,1}\right]\Bigg)H_{-1}+\frac{64}{3}\frac{x^2+1}{x+1}\left[2H_{0,-1,1}-H_{0,0,-1}+2H_{0,1,-1}\right]\\
&+\Bigg(\frac{8}{9}\frac{117x^2+118x+81}{x+1}+\frac{128}{3}\frac{x^2+1}{x+1}H_{-1}+\frac{16}{3}\frac{(x+3)(3x+1)}{x+1}H_0\\
&+\frac{32}{3}(x+1)H_1\Bigg)\zeta_2+\frac{16}{3}\frac{x^2+14x+1}{x+1}\zeta_3\Bigg]+L_Q\Bigg[-\frac{8}{27}(557x+652)\\
&+\Bigg(\frac{8}{9}\frac{115x^2+99x+32}{x+1}-\frac{64}{9}\frac{6x^4+25x^3+18x^2+25x+6}{(x+1)x}H_{-1}\\
&+\frac{64}{3}\frac{(x-1)^2}{x+1}H_{-1}^2\Bigg)H_0+\frac{32}{9}\frac{9x^2+13x+9}{x+1}H_0^3+\Bigg(-\frac{32}{3}\frac{5x^2-2x+5}{x+1}H_{-1}\\
&+\frac{4}{9}\frac{48x^3+519x^2+706x+315}{x+1}\Bigg)H_0^2+\Bigg(\frac{8}{27}(908x-19)+\frac{16}{9}(169x+97)H_0\\
&+\frac{32}{3}(7x+5)H_0^2\Bigg)H_1+\left(\frac{32}{3}(13x+6)+64(x+1)H_0\right)H_1^2+\frac{160}{9}(x+1)H_1^3\\
&+\left(\frac{64}{9}(13x+1)+\frac{16}{3}(x+9)H_0-\frac{32}{3}(x+1)H_1\right)H_{0,1}+\Bigg(-\frac{128}{3}\frac{3x+1}{x+1}H_0\\
&+\frac{64}{9}\frac{6x^4+25x^3+18x^2+25x+6}{(x+1)x}-\frac{128}{3}\frac{(x-1)^2}{x+1}H_{-1}\Bigg)H_{0,-1}\\
&+\frac{128}{3}\frac{(x-1)^2}{x+1}H_{0,-1,-1}+\frac{64}{3}\frac{5x^2+10x+9}{x+1}H_{0,0,-1}+\frac{16}{3}(5x-3)H_{0,0,1}\\
&+48(x+1)H_{0,1,1}+\Bigg(-\frac{16}{9}\frac{24x^3+245x^2+318x+137}{x+1}+\frac{64}{3}\frac{(x-1)^2}{x+1}H_{-1}\\
&-\frac{16}{3}\frac{35x^2+66x+35}{x+1}H_0-\frac{32}{3}(9x+5)H_1\Bigg)\zeta_2-\frac{32}{3}\frac{21x^2+30x+17}{x+1}\zeta_3\Bigg]\\
&+\frac{1}{27}(12332x-4905)+\Bigg(-\frac{64}{9}(x+1)H_{-1}^2+\frac{64}{81}\frac{199x^2+174x+199}{x+1}H_{-1}\\
&+\frac{1}{81}(-10999x-8399)+\frac{128}{27}\frac{x^2+1}{x+1}H_{-1}^3\Bigg)H_0+\Bigg(\frac{32}{27}\frac{19x^2+18x+19}{x+1}H_{-1}\\
&-\frac{32}{9}\frac{x^2+1}{x+1}H_{-1}^2-\frac{2}{81}\frac{4179x^2+5255x+2868}{x+1}\Bigg)H_0^2+\Bigg(\frac{64}{27}\frac{x^2+1}{x+1}H_{-1}\\
&-\frac{10}{81}\frac{177x^2+218x+105}{x+1}\Bigg)H_0^3-\frac{1}{27}\frac{51x^2+70x+51}{x+1}H_0^4+\Bigg(-\frac{152}{27}(x+1)H_0^3
\end{aligned}
$$

$$
+\frac{1}{27}\big(-3457x+1951\big)-\frac{16}{81}\big(593x+335\big)H_0-\frac{8}{27}\big(146x+71\big)H_0^2\Big)H_1
$$

$$
+\Big(-\frac{8}{9}\big(3x+55\big)-\frac{8}{9}\big(9x+1\big)H_0+\frac{4}{9}\big(x+1\big)H_0^2\Big)H_1^2-\frac{64}{27}\big(x+1\big)H_0H_1^3
$$

$$
+\Big(-\frac{64}{81}\frac{199x^2+174x+199}{x+1}+\frac{128}{9}\big(x+1\big)H_{-1}-\frac{128}{9}\frac{x^2+1}{x+1}H_{-1}^2\Big)H_{0,-1}
$$

$$
+\Big(\frac{4}{27}\big(251x+407\big)+\frac{16}{27}\big(10x+43\big)H_0+\frac{16}{3}\big(x+1\big)H_0^2+\Big(\frac{64}{9}\big(x-1\big)
$$

$$
-\frac{80}{9}\big(x+1\big)H_0\Big)H_1-\frac{512}{27}\frac{4x^2+3x+4}{x+1}H_{-1}+\frac{64}{9}\big(x+1\big)H_1^2\Big)H_{0,1}
$$

$$
+\frac{8}{9}\big(x+1\big)H_{0,1}^2+\frac{512}{27}\frac{4x^2+3x+4}{x+1}H_{0,-1,1}+\Big(\frac{256}{9}\frac{x^2+1}{x+1}H_{-1}
$$

$$
-\frac{128}{9}\big(x+1\big)\Big)H_{0,-1,-1}+\Big(-\frac{64}{27}\frac{19x^2+18x+19}{x+1}+\frac{128}{9}\frac{x^2+1}{x+1}H_{-1}\Big)H_{0,0,-1}
$$

$$
+\Big(\frac{4}{27}\frac{357x^2+130x+93}{x+1}-\frac{128}{9}\frac{x^2+1}{x+1}H_{-1}+\frac{64}{9}\big(x+1\big)\big[2H_1-H_0\big]\Big)H_{0,0,1}
$$

$$
+\frac{512}{27}\frac{4x^2+3x+4}{x+1}H_{0,1,-1}+\Big(-\frac{32}{9}\big(13x+1\big)+\frac{16}{3}\big(x+1\big)\big[H_0-2H_1\big]
$$

$$
+\frac{256}{9}\frac{x^2+1}{x+1}H_{-1}\Big)H_{0,1,1}+\frac{128}{9}\frac{x^2+1}{x+1}\big[H_{0,0,-1,1}-2H_{0,-1,-1,-1}-2H_{0,-1,1,1}
$$

$$
-H_{0,0,-1,-1}-H_{0,0,0,-1}+H_{0,0,1,-1}-2H_{0,1,-1,1}-2H_{0,1,1,-1}\big]
$$

$$
+\frac{8}{9}\frac{21x^2+10x+21}{x+1}H_{0,0,0,1}-\frac{32}{9}\frac{7x^2+6x+7}{x+1}H_{0,0,1,1}+\Big(\frac{64}{9}\frac{x^2+1}{x+1}H_{-1}^2
$$

$$
+\frac{4}{81}\frac{1619x^2+1338x+1511}{x+1}+\Big(\frac{4}{27}\frac{147x^2+298x-9}{x+1}+\frac{64}{9}\frac{x^2+1}{x+1}H_{-1}\Big)H_0
$$

$$
+\frac{64}{27}\frac{29x^2+18x+29}{x+1}H_{-1}+\frac{4}{9}\frac{\big(x+5\big)\big(5x+1\big)}{x+1}H_0^2-\frac{64}{9}\big(x+1\big)H_1^2
$$

$$
+\Big(\frac{16}{9}\big(x+9\big)-\frac{16}{3}\big(x+1\big)H_0\Big)H_1+\frac{16}{9}\big(x+1\big)H_{0,1}\Big)\zeta_2+\Big(-\frac{80}{9}\big(x+1\big)H_1
$$

$$
-\frac{8}{27}\frac{235x^2+404x+409}{x+1}-\frac{256}{9}\frac{x^2+1}{x+1}H_{-1}+\frac{8}{9}\frac{15x^2+22x+15}{x+1}H_0\Big)\zeta_3
$$

$$
+\frac{4}{9}\frac{131x^2+178x+131}{x+1}\zeta_4-\frac{32}{3}\big(x+1\big)B_4\Big]+C_FT_F^2\Big[-L_M^3\frac{64}{27}\big(x+1\big)
$$

$$
-L_Q^3\frac{32}{27}\big(x+1\big)+L_M^2\Big[-\frac{32}{27}\big(11x-1\big)-\frac{32}{9}\big(x+1\big)H_0\Big]+L_Q^2\Big[\frac{32}{27}\big(14x+5\big)
$$

$$
+\frac{64}{9}\big(x+1\big)H_0+\frac{32}{9}\big(x+1\big)H_1\Big]-L_M\frac{992}{81}\big(x+1\big)+L_Q\Big[-\frac{32}{81}\big(187x+16\big)
$$

$$
-\frac{64}{27}\big(28x+13\big)H_0-\frac{32}{3}\big(x+1\big)H_0^2+\Big(-\frac{64}{27}\big(14x+5\big)-\frac{64}{9}\big(x+1\big)H_0\Big)H_1
$$

$$
-\frac{32}{9}\big(x+1\big)H_1^2-\frac{64}{9}\big(x+1\big)H_{0,1}+\frac{128}{9}\big(x+1\big)\zeta_2\Big]+\frac{16}{729}\big(431x+323\big)
$$

$$+\frac{64}{81}(6x-7)H_0+\frac{16}{81}(11x-1)H_0^2+\frac{16}{81}(x+1)H_0^3-\frac{448}{27}(x+1)\zeta_3\Biggr]$$
$$+C_F N_F T_F^2\Biggl[-\left[L_M^3+2L_Q^3\right]\frac{32}{27}(x+1)+L_Q^2\left[\frac{64}{27}(14x+5)+\frac{128}{9}(x+1)H_0\right.$$
$$\left.+\frac{64}{9}(x+1)H_1\right]+L_M\left[\frac{32}{81}(5x-73)+\frac{32}{27}(11x-1)H_0+\frac{16}{9}(x+1)H_0^2\right]$$
$$+L_Q\Biggl[-\frac{64}{81}(187x+16)-\frac{128}{27}(28x+13)H_0+\frac{64}{9}(x+1)\left[4\zeta_2-2H_{0,1}-H_1^2\right]$$
$$-\frac{64}{3}(x+1)H_0^2+\left(-\frac{128}{9}(x+1)H_0-\frac{128}{27}(14x+5)\right)H_1\Biggr]+\frac{32}{81}(x+1)H_0^3$$
$$-\frac{64}{729}(161x+215)+\frac{128}{81}(6x-7)H_0+\frac{32}{81}(11x-1)H_0^2+\frac{256}{27}(x+1)\zeta_3\Biggr]$$
$$+\hat{C}_{q,g_1}^{\mathrm{NS},(3)}(N_F)\Biggr\}\,. \tag{2.61}$$

Again, we used the short hand notation $H_{\vec{a}}(x) \equiv H_{\vec{a}}$ also here. The transformation of the Wilson coefficient to the $\overline{\mathrm{MS}}$ scheme for the heavy quark mass affects the massive OME at 3-loops and was given in Ref. [10]; the terms are the same in the unpolarized and polarized case.

The non-singlet contributions to the structure function $g_2(x,Q^2)$ can be obtained via the Wandzura-Wilczek relation [15]

$$g_2(x,Q^2) = -g_1(x,Q^2) + \int_x^1 \frac{dy}{y} g_1(y,Q^2), \tag{2.62}$$

where both structure functions refer to the twist-2 contributions. This relation is implied by a relation of the OMEs in the light-cone expansion, cf. [39]. The relation has also been proven in the covariant parton model in Refs. [40–42]. For gluonic initial states, it was derived in [43]. Eq. (2.62) also holds including target mass corrections [44,45] and finite light quark contributions [45]. Furthermore, it holds in non-forward [46] and diffractive scattering, including target mass corrections [47,48].

3 Numerical Results

In what follows, we will choose the factorization and renormalization scale $\mu^2 = Q^2$. We first study the behaviour of the massive and massless Wilson coefficients in the small and large x region and then give numerical illustrations in the whole x-region.

At small x, the pure massive Wilson coefficient behaves like

$$L_{q,g_1}^{\mathrm{h,NS}}(N_F+1) - \hat{C}_{q,g_1}^{\mathrm{NS},(3)}(N_F) \quad\propto\quad a_s^2 4C_F T_F \ln^2(x) + a_s^3\left[\frac{16}{27}C_A C_F T_F - \frac{5}{9}C_F^2 T_F\right]\ln^4(x), \tag{3.1}$$

while in the region $x \to 1$ one obtains

$$L_{q,g_1}^{\mathrm{h,NS}}(N_F+1) - \hat{C}_{q,g_1}^{\mathrm{NS},(3)}(N_F) \quad\propto\quad a_s^2 C_F T_F \frac{8}{3}\left(\frac{\ln^2(1-x)}{1-x}\right)_+ + a_s^3 C_F^2 T_F\Biggl[16\ln^2\left(\frac{Q^2}{m^2}\right)$$

$$+\frac{160}{3}\ln^2\left(\frac{Q^2}{m^2}\right)+\frac{448}{9}\Bigg]\left(\frac{\ln^2(1-x)}{1-x}\right)_+ . \tag{3.2}$$

There is a term $\propto \ln^3(1-x)/(1-x)$ at $O(\ln(Q^2/\mu^2))$, being of relevance for different choices of the factorization scale.

The above results can be compared with the case of the massless Wilson coefficient

$$\hat{C}_{q,g_1}^{\rm NS,(2)}(N_F) \propto a_s^2\frac{10}{3}C_FT_F\ln^2(x) \tag{3.3}$$

$$\hat{C}_{q,g_1}^{\rm NS,(3)}(N_F) \propto a_s^3\left[\frac{92}{27}C_FC_AT_F-\frac{31}{9}C_F^2T_F\right]\ln^4(x) \tag{3.4}$$

$$\hat{C}_{q,g_1}^{\rm NS,(2)}(N_F) \propto a_s^2\frac{8}{3}C_FT_F\left(\frac{\ln^2(1-x)}{1-x}\right)_+ \tag{3.5}$$

$$\hat{C}_{q,g_1}^{\rm NS,(3)}(N_F) \propto a_s^3\frac{80}{9}C_F^2T_F\left(\frac{\ln^4(1-x)}{1-x}\right)_+ . \tag{3.6}$$

The small x behaviour can be compared with leading order predictions for the non-singlet evolution kernel in Refs. [49, 50]. Indeed both the massive and massless contributions follow the principle pattern $\sim c_k a_s^{k+1}\ln^{2k}(x)$. However, as is well known [49], less singular terms widely cancel the numerical effect of these leading terms. For the large x terms the massless terms exhibit a stronger soft singularity than the massive ones.

In the following numerical illustrations we use the polarized parton distributions of Ref. [8], which are of next-to-leading order (NLO), since no next-to-next-to-leading order (NNLO) data analysis based on the anomalous dimensions calculated in Ref. [51] has been performed yet. The values of α_s correspond to those of the unpolarized NNLO analysis [52]. The heavy and light flavor Wilson coefficients being discussed in the following are given in Eqs. (2.2) and (2.3).

In Figure 1, the 2- and 3-loop heavy flavor corrections to the non-singlet term of the structure function $xg_1(x,Q^2)$ are calculated in the case of charm, assuming $m_c = 1.59$ GeV [53], using the formula for the Wilson coefficient Eq. (2.61), and setting $\mu^2 = Q^2$. With growing Q^2, the distribution diminishes at larger values of x and grows towards medium values. The $O(\alpha_s^3)$ corrections lead to stronger effects if compared to those at $O(\alpha_s^2)$. We have applied the asymptotic Wilson coefficients for all the Q^2 values given here, which only holds for values $Q^2/m^2 \gtrsim 10$. For the heavy quark distributions we formally show also the result at $Q^2 = 4$ GeV^2, outside this region, indicated by dotted ($O(a_s^2)$) and dash-dotted lines ($O(a_s^3)$).

Figure 2 shows the effect of the Wilson coefficients comparing the contributions from $O(\alpha_s^0)$ to $O(\alpha_s^3)$ at $Q^2 = 4$ GeV^2 as an example, where a depletion is obtained with growing order. The 3-loop light flavor contributions to $xg_1(x,Q^2)$ ($N_F = 3$) are illustrated in Figure 3. Here the evolution is strengthened by growing Q^2 in the large x region and depleted for lower values of Q^2, considering only the effects due to the Wilson coefficient.

In Figures 4 and 5 we illustrate the ratio of the flavor non-singlet charm corrections to those by the light quarks given in Eq. (2.3) up to $O(\alpha_s^2)$ and $O(\alpha_s^3)$, respectively. At $O(\alpha_s^2)$ the effect is of $O(1\%)$ and below, for the lower scales Q^2, but higher values are obtained for very large scales as $Q^2 \simeq 1000$ GeV^2 in the region $x \sim 0.003$. A qualitatively similar picture is obtained including the $O(\alpha_s^3)$ corrections. The effect on the ratio $g_1^{\rm heavy}/g_1^{\rm light}|_{\rm NS}$ is about doubled. To resolve relative effects of $O(2\%)$ requires higher luminosities than available in present day experiments. They may become available in the planned experiments at a future EIC [54].

Figure 6 shows the 2- and 3-loop charm flavor non-singlet contributions to the structure function $xg_2(x,Q^2)$ according to the Wandzura-Wilczek relation (2.62) implying the oscillatory behaviour. In size these effects are comparable to those of the structure function $xg_1(x,Q^2)$ shown in Figure 1. With growing Q^2 the effects become somewhat smaller. In Figure 7 we show the corresponding massless contributions to the structure function $g_2(x,Q^2)$ at $Q^2 = 4$ GeV^2 for the different orders in a_s, which slightly diminish adding higher order contributions. Taking into account the $O(a_s^3)$ corrections, the light flavor corrections to $g_2(x,Q^2)$ (1.5,2.62) grow somewhat in size with larger values of Q^2, see Figure 8. Similar to the case of the structure function xg_1 the $O(a_s^3)$ charm flavor non-singlet corrections to the structure function $xg_2(x,Q^2)$ amount to $O(1\%)$.

4 The Bjorken Sum Rule

The polarized Bjorken sum rule [55] refers to the first moment of the flavor non-singlet combination

$$\int_0^1 dx \left[g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)\right] = \frac{1}{6}\left|\frac{g_A}{g_V}\right| C_{\rm BJ}(\hat{a}_s), \tag{4.1}$$

with $g_{A,V}$ the neutron decay constants, $g_A/g_V \approx -1.2767 \pm 0.0016$ [56] and $\hat{a}_s = \alpha_s/\pi$. The 1- [57], 2- [58], 3- [30] and 4-loop QCD corrections [31] in the massless case are given by

$$\begin{aligned} C_{\rm BJ}(\hat{a}_s), \quad = \quad & 1 - \hat{a}_s + \hat{a}_s^2(-4.58333 + 0.33333N_F) + \hat{a}_s^3(-41.4399 + 7.60729N_F - 0.17747N_F^2) \\ & +\hat{a}_s^4(-479.448 + 123.472N_F - 7.69747N_F^2 + 0.10374N_F^3)\ , \end{aligned} \tag{4.2}$$

choosing the renormalization scale $\mu^2 = Q^2$, cf. [28] for $SU(3)_c$. Here N_F denotes the number of active light flavors. The expression for general color factors was given in Ref. [31].

For the asymptotic massive corrections (2.2) only the first moments of the massless Wilson coefficients $\hat{C}_{q,g_1}^{(2,3),\rm NS}(N_F)$ contribute, since the first moments of the massive non-singlet OMEs vanish due to fermion number conservation, a property holding even at higher order. Therefore, any new heavy quark changes Eq. (4.2) by a shift in $N_F \rightarrow N_F + 1$ only, for the asymptotic corrections. Different results are obtained in the tagged flavor case [5, 7] at $O(\alpha_s^2)$, where no inclusive structure functions are considered. Corresponding power corrections were derived in [59, 60].

5 Conclusions

We calculated the heavy flavor non-singlet Wilson coefficients of the polarized inclusive structure function $g_1(x,Q^2)$ to $O(\alpha_s^3)$ in the asymptotic region $Q^2 \gg m^2$. The first contributions of this kind are of $O(\alpha_s^2)$. In the case of twist-2 operators the corresponding contributions to the structure function $g_2(x,Q^2)$ can be obtained using the Wandzura-Wilczek relation (2.62) [15], cf. [39–42, 45]. The asymptotic Wilson coefficient is obtained by using the factorization formula [5], Eq. (2.2), based on the massive OME [10] and the massless Wilson coefficient [29] to 3-loop order. The heavy flavor Wilson coefficient can be thoroughly represented by nested harmonic sums in Mellin-N space and by harmonic polylogarithms in x-space. We presented numerical results corresponding to the charge weighted polarized parton contributions $\propto \Delta f(x,Q^2) + \Delta\bar{f}(x,Q^2)$, cf. (1.5), referring to the polarized parton distribution functions at NLO [8] for an illustration. Comparing with the corresponding massless cases the heavy flavor corrections in case of charm are of $O(1-2\%)$, requiring high luminosity experiments to be resolved, which are planned for the future electron-ion collider EIC [54]. We also considered the contribution of the asymptotic Wilson coefficient to the polarized Bjorken sum-rule. Due to fermion number conservation for the massive flavor non-singlet OME in all orders in α_s, only the first moment of the massless Wilson coefficient contributes and the effect of each heavy flavor results in a shift of N_F by one unit in the expression for the massless polarized Bjorken sum-rule. The results of the present calculation could be easily applied to derive the asymptotic heavy flavor corrections to the neutral current structure function xG_3, [61]. However, the corresponding massless Wilson coefficient to 3-loop order has not been calculated yet.

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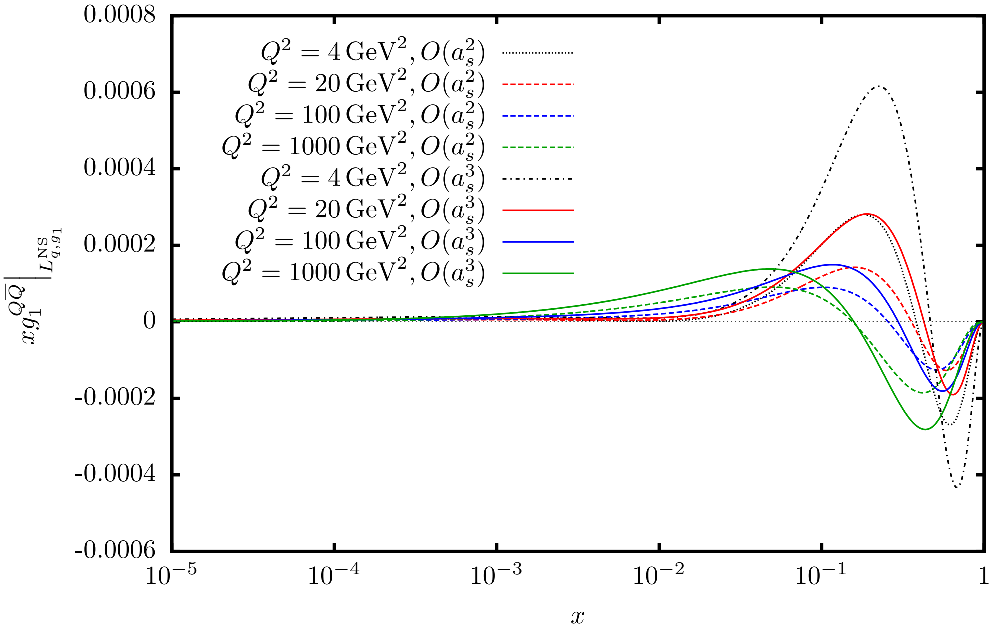

Figure 1: The 2- and 3-loop non-singlet charm contributions to the structure function $xg_1(x, Q^2)$ by the asymptotic heavy flavor Wilson coefficients in the on-shell scheme for $m_c = 1.59$ GeV. Here we used the value of $\alpha_s(M_Z^2) = 0.1132$ and the NLO parton distribution [8] as reference. Figures 2–8 below are calculated using the same setting.

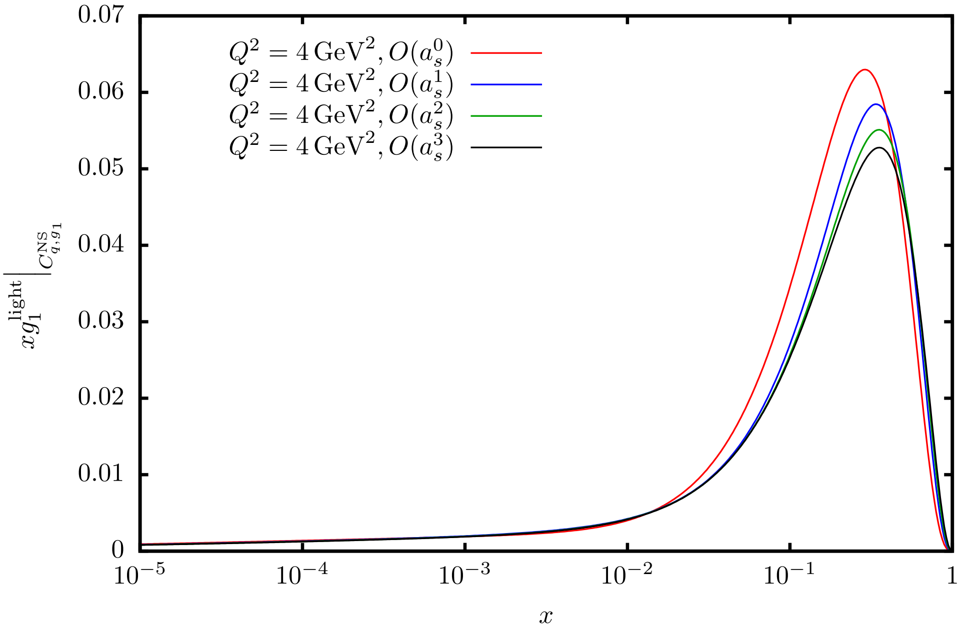

Figure 2: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_1(x, Q^2)$ at $Q^2 = 4$ GeV^2 illustrating the contributions for the different orders in a_s.

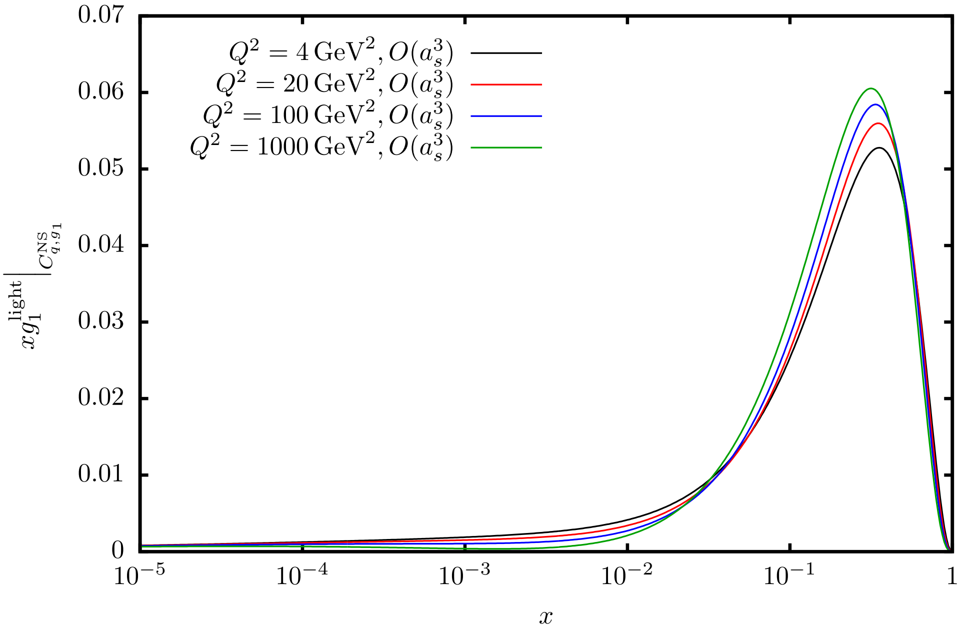

Figure 3: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_1(x, Q^2)$ at $O(a_s^3)$ for different values of Q^2.

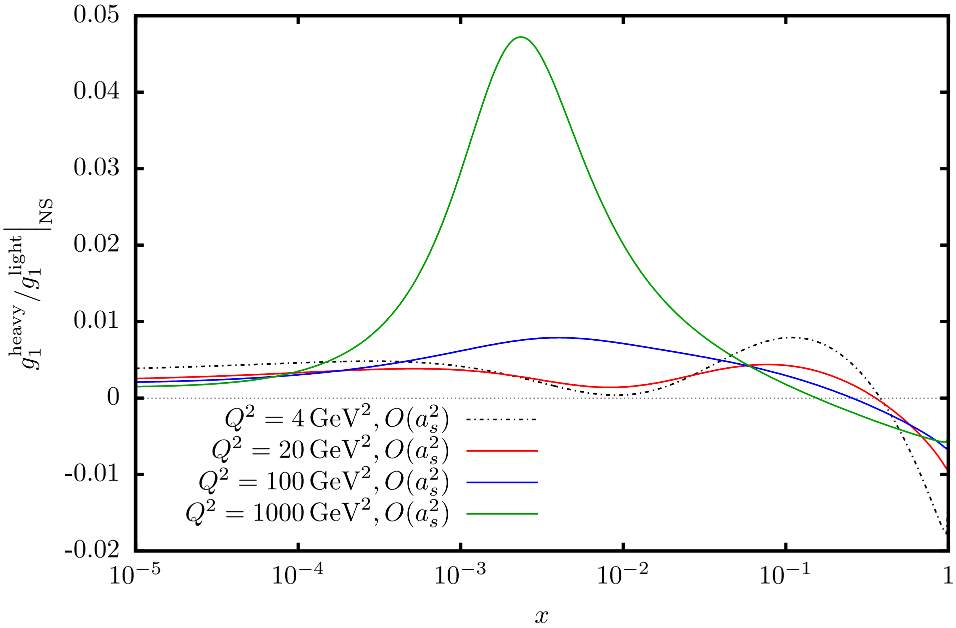

Figure 4: The ratio $g_1^{\rm charm}/g_1^{\rm light}|_{\rm NS}$ in the non-singlet case at $O(a_s^2)$ for different values of Q^2.

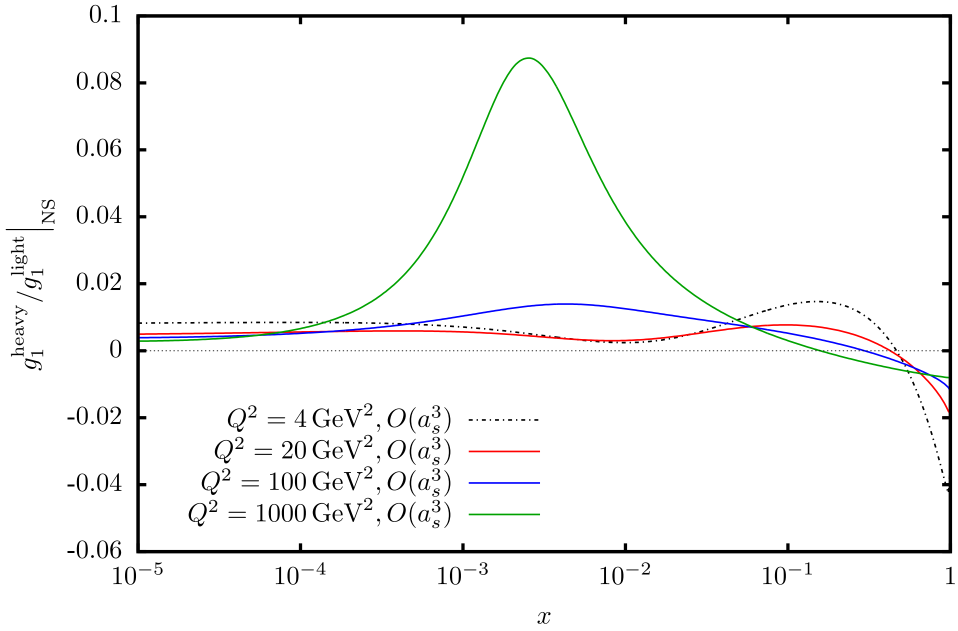

Figure 5: The ratio $g_1^{\text{charm}}/g_1^{\text{light}}|_{\text{NS}}$ in the non-singlet case at $O(a_s^3)$ for different values of Q^2.

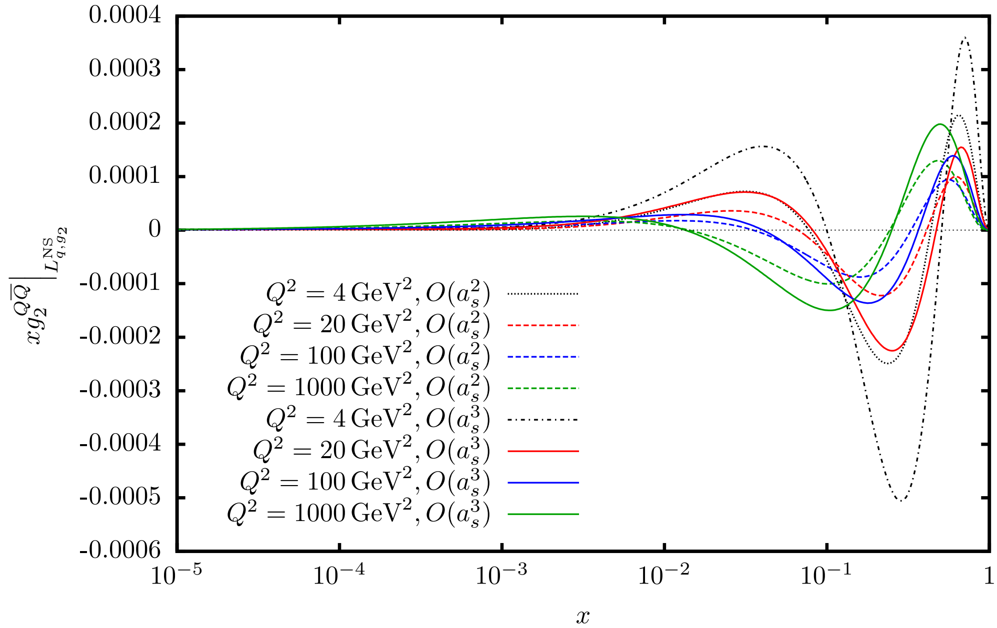

Figure 6: The 2- and 3-loop non-singlet charm contributions to the twist 2 contributions of the structure function $xg_2(x, Q^2)$ by the asymptotic heavy flavor Wilson coefficients in the on-shell scheme.

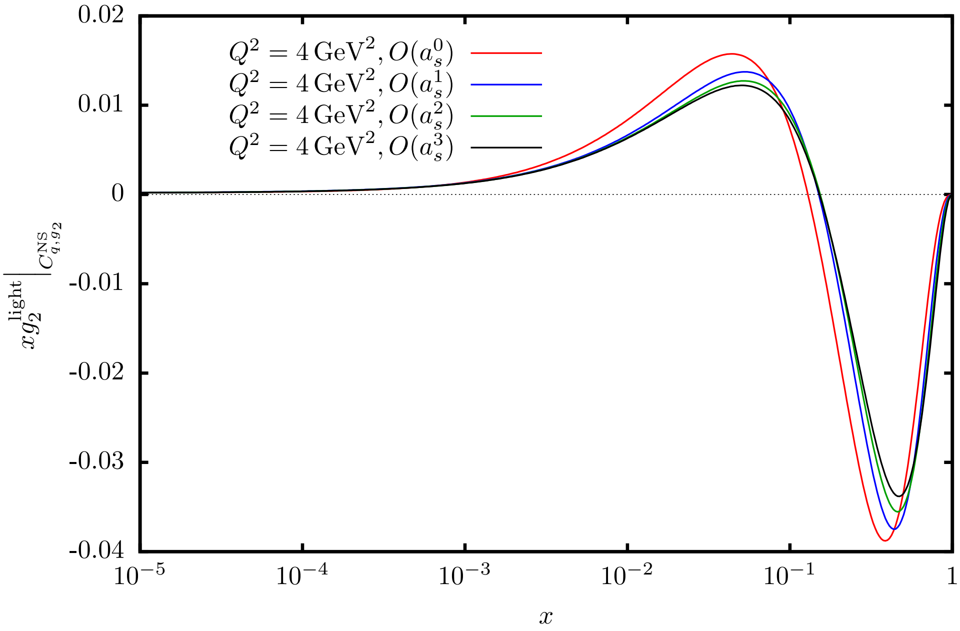

Figure 7: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_2(x, Q^2)$ at $Q^2 = 4$ GeV^2 illustrating the contributions for the different orders in a_s.

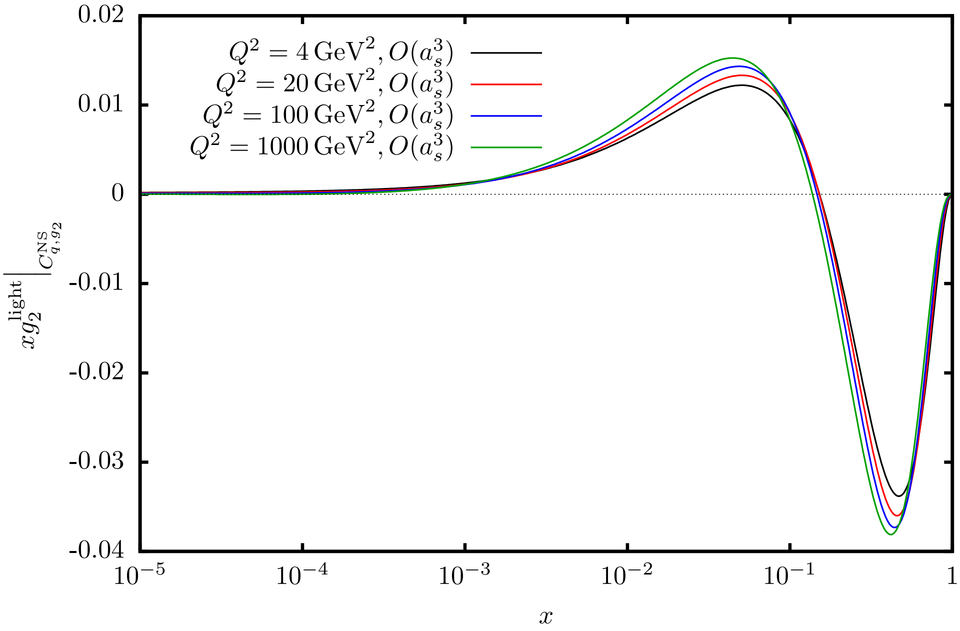

Figure 8: The light flavor contributions ($N_F = 3$) to the non-singlet charm contributions to the structure function $xg_2(x, Q^2)$ at $O(a_s^3)$ for different values of Q^2.

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