

# Renormalization of local quark-bilinear operators for $N_f = 3$ flavors of SLiNC fermions

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## Abstract

The renormalization factors of local quark-bilinear operators are computed non-perturbatively for  $N_f = 3$  flavors of SLiNC fermions, with emphasis on the various procedures for the chiral and continuum extrapolations. The simulations are performed at a lattice spacing  $a = 0.074$  fm, and for five values of the pion mass in the range of 290-465 MeV, allowing a safe and stable chiral extrapolation. Emphasis is given in the subtraction of the well-known pion pole which affects the renormalization factor of the pseudoscalar current. We also compute the inverse propagator and the Green's functions of the local bilinears to one loop in perturbation theory. We investigate lattice artifacts by computing them perturbatively to second order as well as to all orders in the lattice spacing. The renormalization conditions are defined in the RI'-MOM scheme, for both the perturbative and non-perturbative results. The renormalization factors, obtained at different values of the renormalization scale, are translated to the  $\overline{\text{MS}}$  scheme and are evolved perturbatively to 2 GeV. Any residual dependence on the initial renormalization scale is eliminated by an extrapolation to the continuum limit. We also study the various sources of systematic errors.

Particular care is taken in correcting the non-perturbative estimates by subtracting lattice artifacts computed to one loop perturbation theory using the same action. We test two different methods, by subtracting either the  $\mathcal{O}(g^2 a^2)$  contributions, or the complete (all orders in  $a$ ) one-loop lattice artifacts.

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## I. INTRODUCTION

To make contact between lattice QCD simulation results and phenomenological numbers, which usually are given in the  $\overline{\text{MS}}$  scheme, we need to compute renormalization factors relating the bare lattice operators to renormalized ones. This requires a non-perturbative method, because low-order lattice perturbation theory is unreliable at present couplings. One such method is the ‘Regularization Independent Momentum’ (RI-MOM) renormalization scheme [1], which mimics the renormalization procedure used in continuum perturbation theory. In practical applications a variant of the RI-MOM scheme is preferred, the so-called RI’-MOM scheme, which differs from the RI-MOM scheme only in the quark field renormalization factor. The results can be converted to the  $\overline{\text{MS}}$  scheme using continuum perturbation theory.

The method involves comparing lattice calculations of off-shell Green’s functions with continuum perturbation theory results. This matching will work best at large virtualities or short distances, where the running coupling constant is small, and the effects of non-perturbative phenomena, such as chiral symmetry breaking, have died away. A drawback of the method is that discretization effects grow rapidly at short distances. It is thus desirable to remove the discretization errors in the off-shell lattice Green’s functions before making the comparison with the continuum.

Two approaches have been pursued. In [2] the full one-loop perturbative correction has been subtracted from the Green’s functions for operators with up to one covariant derivative, for the plaquette action and  $N_f = 2$  flavors of clover fermions. This procedure becomes very elaborate for higher dimensional operators and more complex actions. An alternative approach is to restrict oneself to one loop corrections of  $O(a^2)$ . The relevant expressions have been computed in [3] for a variety of operators and actions. In [4] we have compared the results of both methods using the Wilson action; we found agreement for that action to be better than 1% for local operators and better than 2% for operators with one covariant derivative.

In this work we compute renormalization factors for  $N_f = 3$  flavors of SLiNC (Stout Link Non-perturbative Clover) fermions [5, 6] and local quark-bilinear operators, studying the effects of one-loop  $O(a^2)$  subtraction, as well as the complete subtraction to one loop. The particular clover action used here has a single iterated mild stout smearing for the hopping

terms, together with thin links for the clover term. For the gauge fields we are using the tree-level Symanzik action.

The paper is organized as follows: In Sec. II we explain our method of non-perturbative renormalization and its numerical implementation, along with the data sets used in this work and a precise determination of the lattice scale. In Sec. III we briefly outline our perturbative computation and the different methods of subtraction. Then, in Sec. IV we present our non-perturbative results in the RI'-MOM scheme, the chiral extrapolation and the conversion to the  $\overline{\text{MS}}$ -scheme. The continuum extrapolation appears in Sec. V, where in Subsec. V A we give our final non-perturbative estimates in the schemes  $\overline{\text{MS}}$  at 2GeV, Renormalization Group Invariant (RGI), and RI'-MOM at a scale of  $1/a$ . The quality of the data upon subtraction of the perturbative terms is demonstrated in Subsec. V B, and finally, in Sec. VI we present our final results and conclude. We also include three Appendices: A. A summary of our perturbative results, B. The  $\beta$ -function and anomalous dimensions that are necessary for the conversion to the  $\overline{\text{MS}}$  and RGI schemes, and C. An alternative fitting for the pion pole subtraction.

## II. METHOD AND NUMERICAL IMPLEMENTATION

| Operator                 | Lorentz Structure             | Representation | Operator Basis                                                                                                           |
|--------------------------|-------------------------------|----------------|--------------------------------------------------------------------------------------------------------------------------|
| $\mathcal{O}^S$          | $\bar{q}q$                    | $\tau_1^{(1)}$ | $\mathcal{O}^S$                                                                                                          |
| $\mathcal{O}^P$          | $\bar{q}\gamma_5 q$           | $\tau_1^{(1)}$ | $\mathcal{O}^P$                                                                                                          |
| $\mathcal{O}_\mu^V$      | $\bar{q}\gamma_\mu q$         | $\tau_1^{(4)}$ | $\mathcal{O}_1^V, \mathcal{O}_2^V, \mathcal{O}_3^V, \mathcal{O}_4^V$                                                     |
| $\mathcal{O}_\mu^A$      | $\bar{q}\gamma_\mu\gamma_5 q$ | $\tau_1^{(4)}$ | $\mathcal{O}_1^A, \mathcal{O}_2^A, \mathcal{O}_3^A, \mathcal{O}_4^A$                                                     |
| $\mathcal{O}_{\mu\nu}^T$ | $\bar{q}\sigma_{\mu\nu} q$    | $\tau_1^{(6)}$ | $\mathcal{O}_{12}^T, \mathcal{O}_{13}^T, \mathcal{O}_{14}^T, \mathcal{O}_{23}^T, \mathcal{O}_{24}^T, \mathcal{O}_{34}^T$ |

TABLE I: The operators under study and their representations under the hypercubic group  $H(4)$  [7].

The operators which we study in this paper are listed in Table I, along with their representations under the hypercubic group  $H(4)$  [7]. We work on lattices with spacing  $a$  and volume  $V$ , gauge fixed to Landau gauge. Starting from the Green's function

$$G_{\alpha\beta}(p) = \frac{a^{12}}{V} \sum_{x,y,z} e^{-ip\cdot(x-y)} \langle q_\alpha(x) \mathcal{O}(z) \bar{q}_\beta(y) \rangle, \quad (1)$$

$q = u, d$  or  $s$ , with operator insertion  $\mathcal{O}$ , we obtain the vertex function (or amputated Green's

function)

$$\Gamma(p) = S^{-1}(p)G(p)S^{-1}(p), \quad (2)$$

where

$$S_{\alpha\beta}(p) = \frac{a^8}{V} \sum_{x,y} e^{-ip \cdot (x-y)} \langle q_\alpha(x) \bar{q}_\beta(y) \rangle \quad (3)$$

denotes the quark propagator. Following [1, 2, 4], we define the renormalized vertex function by

$$\Gamma_R(p) = Z_q^{-1} Z \Gamma(p) \quad (4)$$

and fix the renormalization factor  $Z$  by imposing the renormalization condition

$$\frac{1}{12} \text{Tr} [\Gamma_R(p) \Gamma_{\text{Born}}^{-1}(p)]_{p^2=\mu^2} = 1, \quad (5)$$

where  $\mu$  is the renormalization scale. The renormalization function of the fermion field ( $q_R = Z_q q$ ) is given by

$$Z_q(\mu) = \Lambda_q(p)|_{p^2=\mu^2}, \quad \Lambda_q(p) = \frac{\text{Tr} (-i \sum_\lambda \gamma_\lambda \sin(ap_\lambda) a S^{-1}(p))}{12 \sum_\lambda \sin^2(ap_\lambda)} \quad (6)$$

where  $\Lambda_q(p)$  is the projection of the fermion propagator onto the tree level or Born massless quark propagator. The renormalization factor  $Z$  is calculated from the condition

$$Z_q^{-1} Z \Lambda(p)|_{p^2=\mu^2} = 1, \quad \Lambda(p) = \frac{1}{12} \text{Tr} [\Gamma(p) \Gamma_{\text{Born}}^{-1}(p)] \quad (7)$$

where  $\Lambda(p)$  is the projected amputated Green's function and  $\Gamma_{\text{Born}}(p)$  is the Born term of the vertex function. Finally,  $Z$  has to be extrapolated to the chiral limit. Note that Eq. (7) is not afflicted with  $O(a)$  lattice artifacts, which are associated with operators of opposite chirality and drop out when the trace is taken.

The calculations are done for five sets of mass-degenerate quarks,  $m_u = m_d = m_s$ , on  $32^3 \times 64$  lattices at  $\beta = 5.50$  [6], with the SLiNC action. The clover parameter,  $c_{\text{sw}}$ , was set to 2.65, and the stout parameter,  $\omega$ , to 0.1 [5]. The parameters of the corresponding gauge field configurations are listed in Table II. In terms of the hopping parameter  $\kappa_q$ , the quark masses are given by

$$am_q = \frac{1}{2\kappa_q} - \frac{1}{2\kappa_{0c}} \quad (8)$$

with  $\kappa_{0c} = 0.121099(3)$ . We use momentum sources [8] to compute the Green's functions of Eq. (1) and discard quark-line disconnected contributions. Thus, unless stated otherwise, our renormalization factors refer to flavor nonsinglet operators.

To convert the renormalization factors to physical scales  $\mu$ , we need to know the lattice spacing  $a$ . Our strategy to set the scale is to use singlet quantities [9], which are flat in  $\delta m_q = m_q - \bar{m}$  up to corrections of  $O(\delta m_u^2 + \delta m_d^2 + \delta m_s^2)$  [6], where  $\bar{m}$  is the average quark mass ( $\bar{m} = (2m_\ell + m_s)/3$  in our case). We find  $a = 0.074(2)$  fm. With this choice of the lattice spacing, the pion masses in Table II range from 290 to 465 MeV, allowing a controlled extrapolation to the chiral limit. The renormalization factors are converted to the  $\overline{\text{MS}}$  scheme at  $\mu = 2$  GeV.

The lattice momenta are chosen according to

$$\Lambda_{\text{QCD}}^2 \ll p^2 \leq \left(\frac{\pi}{a}\right)^2. \quad (9)$$

On  $L^3 \times T$  lattices with periodic spatial and antiperiodic temporal boundary conditions

$$p = \left(\frac{2\pi}{L} n_1, \frac{2\pi}{L} n_2, \frac{2\pi}{L} n_3, \frac{2\pi}{T} \left(n_4 + \frac{1}{2}\right)\right). \quad (10)$$

To increase the number of momenta, we employ twisted boundary conditions to the quark fields,  $p \rightarrow p + B$ , with

$$B = \left(\frac{2\pi}{L} \theta_1, \frac{2\pi}{L} \theta_2, \frac{2\pi}{L} \theta_3, \frac{2\pi}{T} \theta_4\right). \quad (11)$$

Thus for lattices with  $T = 2L$  then

$$p = \frac{2\pi}{L} \left(n_1 + \theta_1, n_2 + \theta_2, n_3 + \theta_3, \frac{1}{2} \left(n_4 + \frac{1}{2} + \theta_4\right)\right). \quad (12)$$

We choose a fixed direction and then let  $p^2$  vary. The optimal choice is along the diagonal, which leaves us with overall  $O((ap)^2)$  corrections only, but no directional correction [10].

Our momenta and twist angles are listed in Table III.

| $\kappa_\ell$ | $\kappa_s$ | $am_\pi$   |
|---------------|------------|------------|
| 0.120900      | 0.120900   | 0.1757(10) |
| 0.120920      | 0.120920   | 0.1647(4)  |
| 0.120950      | 0.120950   | 0.1508(4)  |
| 0.120990      | 0.120990   | 0.1285(7)  |
| 0.121021      | 0.121021   | 0.1089(21) |

TABLE II: Parameters  $\kappa_l$  ( $l = u, d$ ),  $\kappa_s$  and pion masses of our lattice ensembles at  $\beta = 5.50$ .

| $\theta = (0, 0, 0, -\frac{1}{2})$ |          | $\theta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ |          | $\theta = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0)$ |          |
|------------------------------------|----------|------------------------------------------------------------------|----------|-------------------------------------------------------|----------|
| $n$                                | $(ap)^2$ | $n$                                                              | $(ap)^2$ | $n$                                                   | $(ap)^2$ |
| (1, 1, 1, 2)                       | 0.1542   | (0, 0, 0, 1)                                                     | 0.03855  | (0, 0, 0, 0)                                          | 0.009638 |
| (2, 2, 2, 4)                       | 0.6169   | (1, 1, 1, 3)                                                     | 0.3470   | (1, 1, 1, 2)                                          | 0.2410   |
| (3, 3, 3, 6)                       | 1.3879   | (2, 2, 2, 5)                                                     | 0.9638   | (2, 2, 2, 4)                                          | 0.7807   |
| (4, 4, 4, 8)                       | 2.4674   | (3, 3, 3, 7)                                                     | 1.8891   | (3, 3, 3, 6)                                          | 1.6289   |
| (5, 5, 5, 10)                      | 3.8553   | (4, 4, 4, 9)                                                     | 3.1228   | (4, 4, 4, 8)                                          | 2.7855   |
| (6, 6, 6, 12)                      | 5.5517   | (5, 5, 5, 11)                                                    | 4.6649   | (5, 5, 5, 10)                                         | 4.2505   |
| (7, 7, 7, 14)                      | 7.5564   | (6, 6, 6, 13)                                                    | 6.5155   | (6, 6, 6, 12)                                         | 6.0239   |
| (8, 8, 8, 16)                      | 9.8696   | (7, 7, 7, 15)                                                    | 8.6745   | (7, 7, 7, 14)                                         | 8.1058   |

TABLE III: Lattice momenta and twist angles in lattice units.

### III. PERTURBATIVE RESULTS

Our perturbative computation for the renormalization factors is performed in one loop perturbation theory using a variety of fermionic and gluonic actions. In this paper we focus on the SLiNC action [11] providing the results for general gauge,  $\alpha$ , ( $\alpha = 0$  (1) for Landau (Feynman) gauge) and action parameters ( $c_{\text{sw}}$ ,  $\omega$ ). The Feynman diagrams entering this computation and the procedure for their evaluation are extensively described in Ref. [12]. For comparison with the non-perturbative renormalization factors, in the results shown in this section we employ  $\omega = 0.1$  for the parameter appearing in the stout links of the fermion part, and for the clover parameter we test both the tree-level value suggested by one-loop perturbation theory,  $c_{\text{sw}} = 1$  and the value employed in the simulations, that is [5]:  $c_{\text{sw}} = 2.65$ . We have performed two separate computations to one loop, as described below:

- A. up to second order in the lattice spacing,  $a$ , for which the results are given in a closed form as a function of the external momentum  $p$  and well as  $a$ ,  $\alpha$ ,  $c_{\text{sw}}$  and  $\omega$ .
- B. to all orders in  $a$ , for general values of  $a$ ,  $\alpha$ ,  $c_{\text{sw}}$  and  $\omega$ , but for specific choices for the external momentum; we have computed these terms for all the momenta employed in the non-perturbative computation.

Our results for each computation are discussed below.

### A. Subtraction of $\mathcal{O}(g^2 a^2)$ contributions

We compute to one loop the inverse propagator,  $S^{-1}(p)$ , and the amputated two-point Green's functions of the local operators (scalar, pseudoscalar, vector, axial-vector, tensor),  $\Gamma_{\mathcal{O}}(p)$ . The  $Z$ -factors are extracted by applying the renormalization conditions given in Eqs. (6) - (7). The tree-level values for the fermion operators  $\mathcal{O} = S, P, V, A$  and  $T$  are

$$\Gamma_{\text{Born}}(p) = -i \mathbb{1}, \quad -i \gamma_5, \quad -i \gamma_\mu, \quad -i \gamma_5 \gamma_\mu, \quad -i \gamma_5 \sigma_{\mu\nu}, \quad (13)$$

respectively. In the mass-independent schemes which we consider, the bare quark masses must be set to zero. For the appropriate evaluation of the  $Z$ -factor, the  $\mathcal{O}(g^2 a^2)$  terms are omitted from the Green's function entering Eqs. (6) - (7), obtaining:

$$Z_q = 1 + \frac{g^2 C_F}{16 \pi^2} \left( -13.0233 + 4.79201 \alpha + c_{\text{sw}}(2.01543 - 4.67344 \omega) + 1.24220 c_{\text{sw}}^2 + 152.564 \omega - 541.381 \omega^2 - \alpha \log(a^2 \mu^2) \right), \quad (14)$$

$$Z_S = 1 + \frac{g^2 C_F}{16 \pi^2} \left( -13.6067 - \alpha + c_{\text{sw}}(18.2213 \omega - 6.83528) + 1.36741 c_{\text{sw}}^2 + 140.264 \omega - 481.361 \omega^2 + 3 \log(a^2 \mu^2) \right), \quad (15)$$

$$Z_P = 1 + \frac{g^2 C_F}{16 \pi^2} \left( -21.7334 - \alpha + c_{\text{sw}}(2.01543 - 4.67344 \omega) - 1.74485 c_{\text{sw}}^2 + 201.198 \omega - 649.867 \omega^2 + 3 \log(a^2 \mu^2) \right), \quad (16)$$

$$Z_V = 1 + \frac{g^2 C_F}{16 \pi^2} \left( -16.5029 + \frac{\alpha}{2} + c_{\text{sw}}(4.2281 - 10.3971 \omega) + 0.46414 c_{\text{sw}}^2 + 168.263 \omega - 584.846 \omega^2 \right), \quad (17)$$

$$Z_A = 1 + \frac{g^2 C_F}{16 \pi^2} \left( -12.5396 + \frac{\alpha}{2} + c_{\text{sw}}(1.05025 \omega - 0.19725) + 2.02027 c_{\text{sw}}^2 + 137.796 \omega - 500.593 \omega^2 \right), \quad (18)$$

$$Z_T = 1 + \frac{g^2 C_F}{16 \pi^2} \left( -13.5383 + \alpha + c_{\text{sw}}(3.49054 - 8.48923 \omega) + 1.71918 c_{\text{sw}}^2 + 147.129 \omega - 535.088 \omega^2 - \log(a^2 \mu^2) \right), \quad (19)$$

where  $C_F = (N^2 - 1)/(2N)$ .

The  $\mathcal{O}(g^2 a^2)$  contributions of the Green's functions are useful in non-perturbative computations of the  $Z$ -factors, since they may be subtracted from the non-perturbative values;



these terms are reliable up to a limited range of the lattice spacing.

As an example we show the expression for the projected quark-antiquark Green's function  $\Lambda_q$  defined in Eq. (6):

$$\Lambda_q = Z_q + \Lambda_q^{(2)} \quad (20)$$

using tree-level Symanzik gluons, Landau gauge  $c_{\text{sw}} = 2.65$  and  $\omega = 0.1$ . The  $\mathcal{O}(g^2 a^0)$  contribution gives  $Z_q$ , while the  $\mathcal{O}(g^2 a^2)$  terms are denoted by  $\Lambda_q^{(2)}$ ; the latter is given by:

$$\Lambda_q^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} \left[ p^2 (-0.8825 + 0.3972 \log(a^2 p^2)) + \frac{p^4}{p^2} \left( 1.9141 - \frac{157}{180} \log(a^2 p^2) \right) \right]. \quad (21)$$

Beyond  $\mathcal{O}(a^0)$ , the terms depend not only on the length, but also on the direction of the four-vector  $p$ , due to the appearance of the Lorentz noninvariant terms  $p^4 \equiv \sum_\rho p_\rho^4$ .

In the left panel of Fig. 1 we plot  $\Lambda_q^{(2)}$  of Eq. (21). In the figure we highlight (in green) the values of the momenta which we actually employ in the non-perturbative evaluation of the  $Z$ -factors. An immediate observation from the plot is that  $\Lambda_q^{(2)}$  is significantly large (especially for the diagonal momenta) as compared to the one-loop perturbative estimate of  $Z_q$  at the same action parameters ( $Z_q = 1.148$ ). This is in contrary to the case of  $N_f = 2$  calculated for Wilson fermions and plaquette action, in which  $c_{\text{sw}} = 1$ , and thus, to understand better the behavior of  $\Lambda_q^{(2)}$  we plot it in the right panel of Fig. 1 for  $c_{\text{sw}} = 1$  and  $\omega = 0.1$ . Indeed, the  $\mathcal{O}(g^2 a^2)$  terms in this case are one order of magnitude smaller compared to the case of  $c_{\text{sw}} = 2.65$ , and thus under control ( $Z_q = 0.99401$  for  $c_{\text{sw}} = 1$ ). By analogy with Eq. (21) and Fig. 1, we also plot in Fig. 2 the  $\mathcal{O}(g^2 a^2)$  terms for  $Z_A$ ,  $\Lambda_A^{(2)}$  ( $\Lambda_A \equiv Z_A + \Lambda_A^{(2)}$ ):

$$\Lambda_A^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} \left[ p^2 (-0.8337 + 0.5806 \log(a^2 p^2)) + \frac{p^4}{p^2} \left( 2.3481 - \frac{157}{180} \log(a^2 p^2) \right) \right]. \quad (22)$$

In this case, the effect of  $\Lambda_A^{(2)}$  on the one-loop perturbative value of  $Z_A=1.156$ , is even more pronounced at  $c_{\text{sw}}=2.65$ . Another difference between the  $N_f=2$  and  $N_f=3$  cases is that in the latter the diagonal momenta do not lead to the smallest  $\mathcal{O}(g^2 a^2)$  effect. As a consequence, the pure non-perturbative data (estimated with diagonal momenta) and the  $\mathcal{O}(a^2)$  subtracted data have a significant numerical difference. On the other hand, for Wilson fermions, the diagonal momenta lead to suppressed  $\mathcal{O}(g^2 a^2)$  effects; we had taken that observation on the  $N_f=2$  case as supporting evidence for the choice of diagonal momenta.

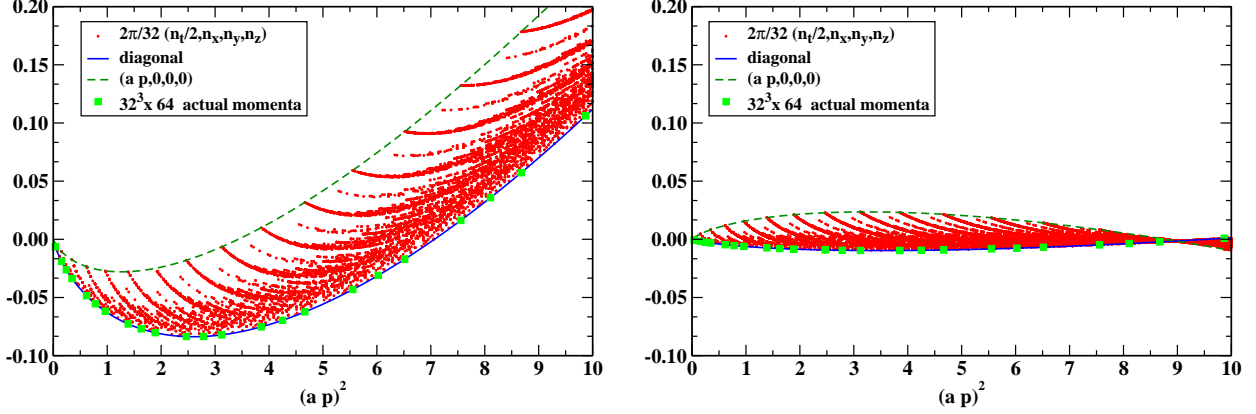


FIG. 1:  $\Lambda_q^{(2)}$  as a function of  $(ap)^2$  for  $\omega = 0.1$  and: i)  $c_{\text{sw}} = 2.65$  (left); ii)  $c_{\text{sw}} = 1$  (right). For comparison, to one loop:  $Z_q = 1.14807$  for  $c_{\text{sw}} = 2.65$  and  $Z_q = 0.99401$  for  $c_{\text{sw}} = 1$ .

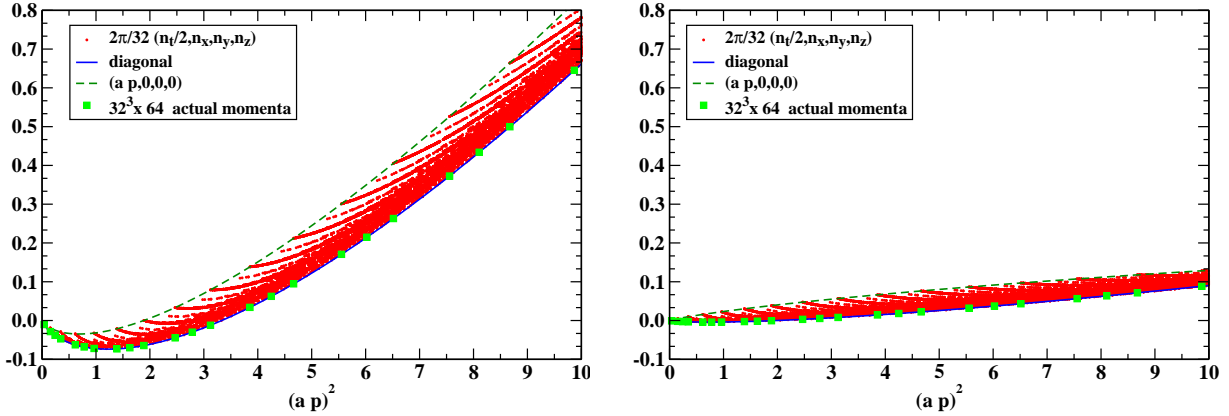


FIG. 2: Similar to Fig. 1 for  $\Lambda_A^{(2)}$ . For comparison, to one loop:  $Z_A = 1.15623$  for  $c_{\text{sw}} = 2.65$  and  $Z_A = 0.97179$  for  $c_{\text{sw}} = 1$ .

It is also interesting to investigate the  $c_{\text{sw}}$  dependence of  $\Lambda_q^{(2)}$  for various values of  $(ap)^2$ . For this testing we set  $\omega = 0.1$  and the dependence is shown in Fig. 3. The left panel corresponds to diagonal momenta (equal components), while the right panel to momenta with a nonzero component in a single direction. We observe that for  $c_{\text{sw}} < 1.5$  the  $\mathcal{O}(g^2 a^2)$  terms are very small, while for  $c_{\text{sw}} > 2$  these contributions increase very fast.

Although we have tested both  $c_{\text{sw}} = 1$  and  $c_{\text{sw}} = 2.65$ , we choose to employ the tree-level value  $c_{\text{sw}} = 1$  for consistency to one loop perturbation theory. This will be the value used in the subtraction of lattice artifacts from the non-perturbative estimates (Subsec. VB).

A collection of the perturbative results for the  $\mathcal{O}(g^2 a^2)$  terms is given in Appendix A.

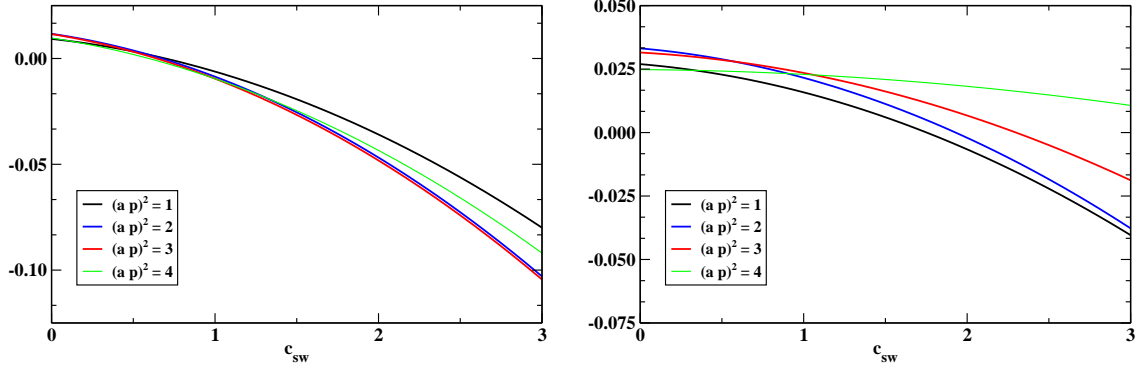


FIG. 3:  $\Lambda_q^{(2)}$  as a function of  $c_{\text{sw}}$  for  $\omega = 0.1$  and for representative values of diagonal  $((a p)/2 (1, 1, 1, 1)$ , left) and non-diagonal momenta  $((a p, 0, 0, 0)$ , right).

### B. Complete subtraction of one-loop lattice artifacts

Here we present our results for the one-loop computation including all orders in the lattice spacing. The main motivation for such a calculation comes from the observation that at high values of  $(a p)^2$  the  $\mathcal{O}(g^2 a^2)$  terms are no longer under control and become large. This does not necessarily indicate large artifacts, since there might be cancellations with higher order artifacts. We illustrate that this is indeed the case in this work, demonstrating the possible complicated dependence of observables on the lattice spacing and showing the intrinsic  $\mathcal{O}(a)$  improvement of the SLiNC action. Since the  $\mathcal{O}(g^2 a^2)$  artifacts cannot be trustworthy for the whole range of employed momenta, it is unnatural to subtract them from the non-perturbative estimates. Due to the nature of the computation to all orders in  $a$ , the dependence on the external momentum cannot be given in a closed form since it is included in the propagators. Thus, we compute the one-loop expression separately for each value of the external momentum used in the simulations. From the latter expression one must omit the  $\mathcal{O}(a^0)$  contributions; this is achieved by subtracting the  $\mathcal{O}(g^2 a^0)$  terms, computed analytically in Subsection III A (Eqs. (14) - (19)). It is interesting to compare the  $\mathcal{O}(a^2)$  terms and the total one-loop lattice artifacts, and thus we plot the two contributions in Figs. 4 - 6 for  $c_{\text{sw}} = 1$ , and for two values of the stout parameter,  $\omega = 0, 0.1$ . To match our non-perturbative computation we choose diagonal momenta. A comparison of the lattice artifacts to all orders in  $a$  for  $\omega = 0$  and  $\omega = 0.1$  confirms that the presence of the stout parameter suppresses them. Concentrating on  $\omega = 0.1$ , one observes that for small values of  $(a p)^2$  the two computations (blue line, green points) lead to compatible results, as

expected. The  $\mathcal{O}(a^2)$  contribution increases with  $(ap)^2$ , while the complete one-loop artifacts are much smaller (except for  $Z_q$ ). Nevertheless, both results are relatively small compared to the  $N_f = 2$  case presented in Ref. [4], and this is presumably due to the improvement of the action using the stout smearing. This is an indication of suppressed lattice artifacts.

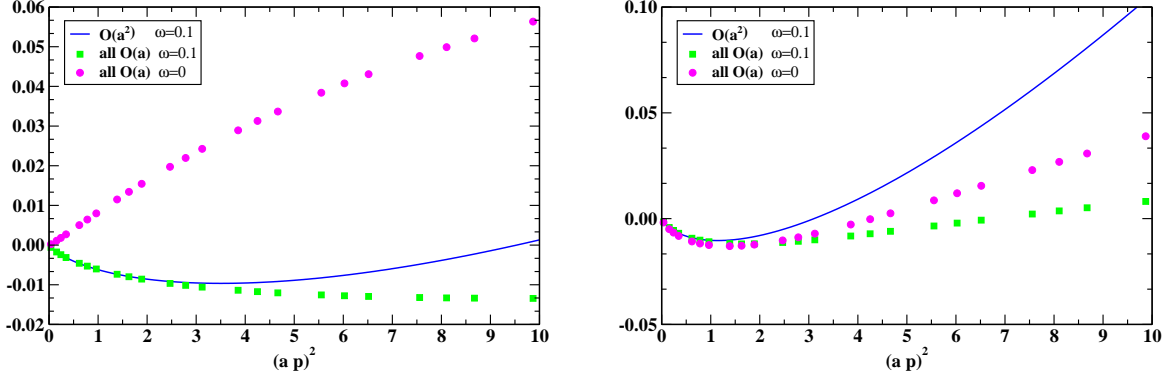


FIG. 4: Terms of all  $\mathcal{O}(a)$  and  $\mathcal{O}(a^2)$  for  $Z_q$  (left) and  $Z_T$  (right) as a function of  $(ap)^2$ .

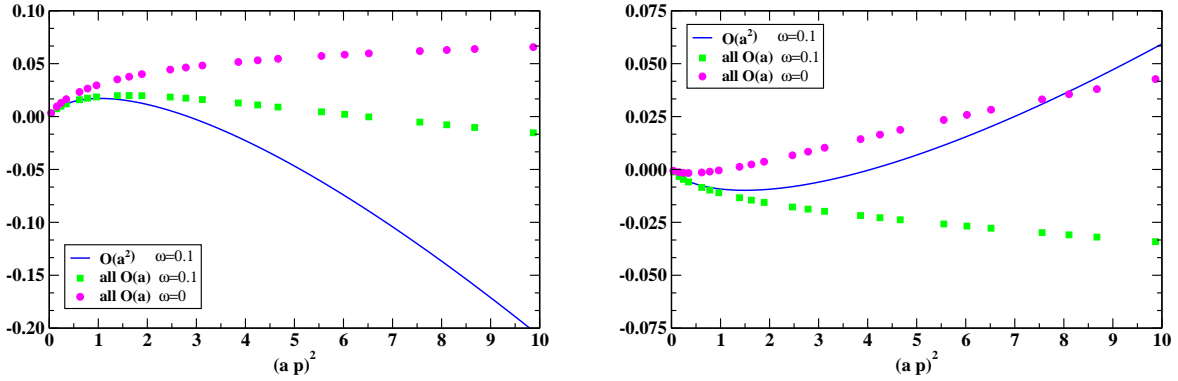


FIG. 5: Similar to Fig. 4 for  $Z_S$  (left) and  $Z_P$  (right).

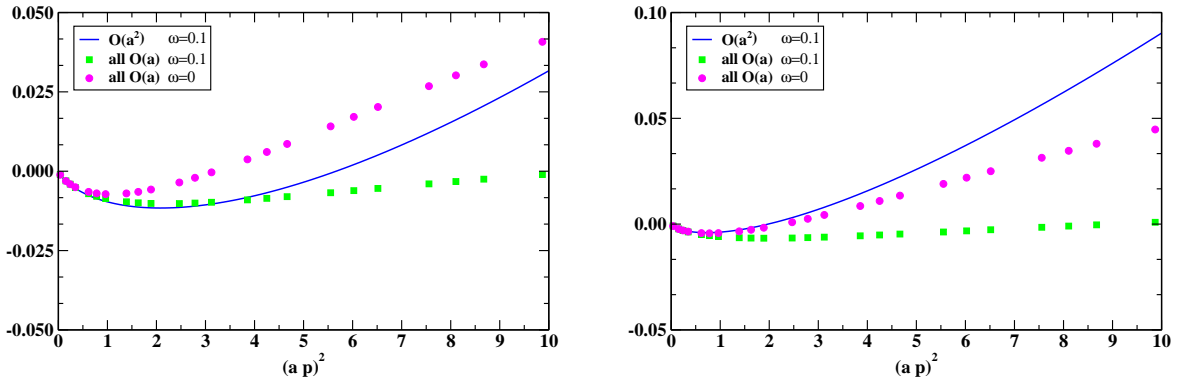


FIG. 6: Similar to Fig. 4 for  $Z_V$  (left) and  $Z_A$  (right).

## IV. NON-PERTURBATIVE RESULTS

We shall present our raw results now, including the extrapolation to the chiral limit, the conversion to the  $\overline{\text{MS}}$  scheme and subtraction of the pion pole in case of the pseudoscalar density.

### A. RI'-MOM and chiral extrapolation

In Fig. 7 we show  $Z_q, Z_S, Z_V, Z_A$  and  $Z_T$  as a function of the renormalization scale for our various pion masses. The dependence on the pion mass is found to be very weak, as shown in Fig. 8 for a particular value of the scale, and well represented by a linear curve. In the following we extrapolate the  $Z$ -factors linearly to the chiral limit for each value of  $(ap)^2$ . The  $Z$ -factor of the pseudoscalar density,  $Z_P$ , needs to be treated separately, because it suffers from the pion pole in the Green's function of Eq. (1). This we will deal with in Sec. IV C.

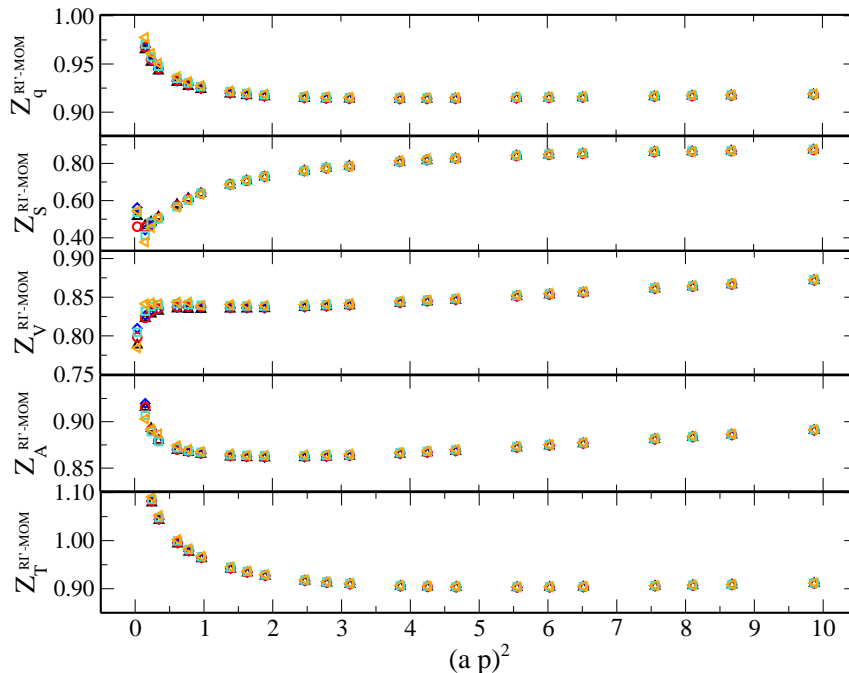


FIG. 7: Results on the  $Z$ -factors for all pion masses versus the renormalization scale in lattice units. Different colors and shapes denote different ensembles (see also legend of Fig. 8): black up triangles, blue diamonds, red circles, green squares and yellow left triangles stand for  $m_\pi = 465, 439, 402, 345,$  and  $290$  MeV, respectively.

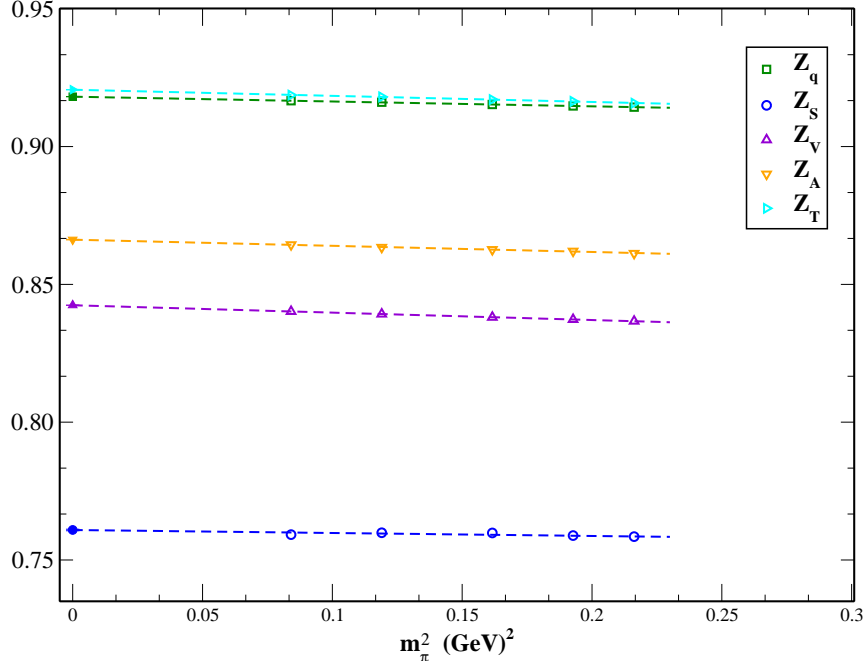


FIG. 8: The dependence of the  $Z$ -factors on the pion mass at  $(ap)^2 \sim 2.5$ , together with a linear fit.

### B. Conversion to $\overline{\text{MS}}$

Although RI'-MOM is a convenient scheme to compute the renormalization factors, our aim is to present them in the  $\overline{\text{MS}}$  scheme. Starting from the results in RI'-MOM at various scales,  $\mu$ , we convert them to the  $\overline{\text{MS}}$  scheme at a reference scale; we will set that scale to 2 GeV. The conversion factors do not depend on the regularization scheme (and, thus, they are independent of the lattice discretization) when expressed in terms of the renormalized coupling constant. However, expressing them in terms of the bare coupling constant introduces a dependence on the action. The conversion factors,  $C_{\mathcal{O}}^{\text{RI}'\text{-MOM}, \overline{\text{MS}}}$ , and the expression for running the scale to 2 GeV in  $\overline{\text{MS}}$ ,  $R(\mu, 2 \text{ GeV})$ , are defined such that:

$$Z_{\mathcal{O}}^{\overline{\text{MS}}}(2 \text{ GeV}) = R(\mu, 2 \text{ GeV}) C_{\mathcal{O}}^{\text{RI}'\text{-MOM}, \overline{\text{MS}}} Z_{\mathcal{O}}^{\text{RI}'\text{-MOM}}(\mu), \quad (23)$$

and their perturbative expressions are available up to three loops. For the relation between the renormalized coupling constant,  $g_R$ , and the bare one,  $g$ :  $g_R = Z_g^{-1} g$ , we use the two-loop results of Ref. [13] for  $Z_g$  corresponding to clover fermions and Wilson gluons.  $R(\mu, 2 \text{ GeV})$  is expressed in terms of  $\Lambda_{\overline{\text{MS}}}$  which for  $N_f=3$  was estimated to be 339 MeV [14]. The conversion factors we use are adapted from Ref. [15] and are applicable to the naive dimensional regularization (NDR) of the  $\overline{\text{MS}}$  scheme [16], in which  $C_P = C_S$ . Moreover, the conversion

factor from the RI'-MOM to the  $\overline{\text{MS}}$  scheme for the vector and axial-vector operators is 1.

In an alternative procedure the RGI scheme is used as an intermediate scheme to obtain the conversion factors for the operators. Those factors  $C^{\text{RI}'\text{-MOM},\overline{\text{MS}}}$  are found from relating the renormalization function in  $\overline{\text{MS}}$  (at 2 GeV) and RI'-MOM (at a scale  $\mu$ ):

$$Z_{\mathcal{O}}^{\text{RGI}} = Z_{\mathcal{O}}^{\overline{\text{MS}}}(2 \text{ GeV}) \Delta Z_{\mathcal{O}}^{\overline{\text{MS}}}(2 \text{ GeV}) = Z_{\mathcal{O}}^{\text{RI}'\text{-MOM}}(\mu) \Delta Z_{\mathcal{O}}^{\text{RI}'\text{-MOM}}(\mu), \quad (24)$$

and thus

$$\begin{aligned} Z_{\mathcal{O}}^{\overline{\text{MS}}}(2 \text{ GeV}) &= C_{\mathcal{O},\text{RGI}}^{\text{RI}'\text{-MOM},\overline{\text{MS}}}(\mu, 2 \text{ GeV}) Z_{\mathcal{O}}^{\text{RI}'\text{-MOM}}(\mu), \\ C_{\mathcal{O},\text{RGI}}^{\text{RI}'\text{-MOM},\overline{\text{MS}}}(\mu, 2 \text{ GeV}) &= \frac{\Delta Z_{\mathcal{O}}^{\text{RI}'\text{-MOM}}(\mu)}{\Delta Z_{\mathcal{O}}^{\overline{\text{MS}}}(2 \text{ GeV})}. \end{aligned} \quad (25)$$

The quantity  $\Delta Z_{\mathcal{O}}^S(\mu)$  in scheme  $S$  is expressed in terms of the  $\beta$ -function and the anomalous dimension of the operator under study,  $\gamma_{\mathcal{O}}^S \equiv \gamma^S$ :

$$\Delta Z_{\mathcal{O}}^S(\mu) = \left( 2\beta_0 \frac{g^S(\mu)^2}{16\pi^2} \right)^{-\frac{\gamma_0}{2\beta_0}} \exp \left\{ \int_0^{g^S(\mu)} dg' \left( \frac{\gamma^S(g')}{\beta^S(g')} + \frac{\gamma_0}{\beta_0 g'} \right) \right\}. \quad (26)$$

To three-loop approximation  $\Delta Z_{\mathcal{O}}^S(\mu)$  takes the form:

$$\begin{aligned} \Delta Z_{\mathcal{O}}^S(\mu) &= \left( 2\beta_0 \frac{g^S(\mu)^2}{16\pi^2} \right)^{-\frac{\gamma_0}{2\beta_0}} \left( 1 + \frac{g^S(\mu)^2}{16\pi^2} \frac{\beta_1 \gamma_0 - \beta_0 \gamma_1^S}{2\beta_0^2} + \right. \\ &\quad \left. \frac{g^S(\mu)^4}{(16\pi^2)^2} \frac{-2\beta_0^3 \gamma_2^S + \beta_0^2 (\gamma_1^S (2\beta_1 + \gamma_1^S) + 2\beta_2 \gamma_0) - 2\beta_0 \beta_1 \gamma_0 (\beta_1 + \gamma_1^S) + \beta_1^2 \gamma_0^2}{8\beta_0^4} \right). \end{aligned} \quad (27)$$

The expressions for the coupling  $g^S(\mu)$  in the  $\overline{\text{MS}}$  and in the RI'-MOM schemes coincide to three loops and read [17]:

$$\frac{g^S(\mu)^2}{16\pi^2} = \frac{1}{\beta_0 L} - \frac{\beta_1 \log L}{\beta_0^3 L^2} + \frac{1}{\beta_0^5} \frac{\beta_1^2 \log^2 L - \beta_1^2 \log L + \beta_2 \beta_0 - \beta_1^2}{L^3}, \quad L = \log \frac{\mu^2}{\Lambda_{\overline{\text{MS}}}^2}. \quad (28)$$

In Appendix B we give the definitions of the  $\beta$ -function and the anomalous dimension for the fermion field and local operators, as well as their perturbative coefficients to three loops. In Fig. 9 we demonstrate the effects in the  $Z$ -factors resulting from the use of the two- and three-loop expressions for  $C_{\mathcal{O}}^{\text{RI}'\text{-MOM},\overline{\text{MS}}}$  and  $R(\mu, 2 \text{ GeV})$ , and the corresponding expressions for the alternative conversion (via RGI) as given by Eq. (27). The ensemble used for this Figure corresponds to  $m_\pi = 465 \text{ MeV}$ . We find that the discrepancies are at 8% maximum for  $Z_S$  (and consequently  $Z_P$  discussed later). For the standard conversion factors  $C_{\mathcal{O}}^{\text{RI}'\text{-MOM},\overline{\text{MS}}}$

the difference in the two-loop values of  $Z_g$  between the Wilson and tree-level Symanzik gluons is expected to be (based on their one-loop difference) within this systematic error. The conversion via the RGI scheme has the property that it uses continuum results; in the rest of the paper we use the intermediate RGI scheme and employ the three-loop result of Eq. (27) for all the conversions. In the final results presented in Sec. V we also give a systematic error due to differences in the conversion factor.

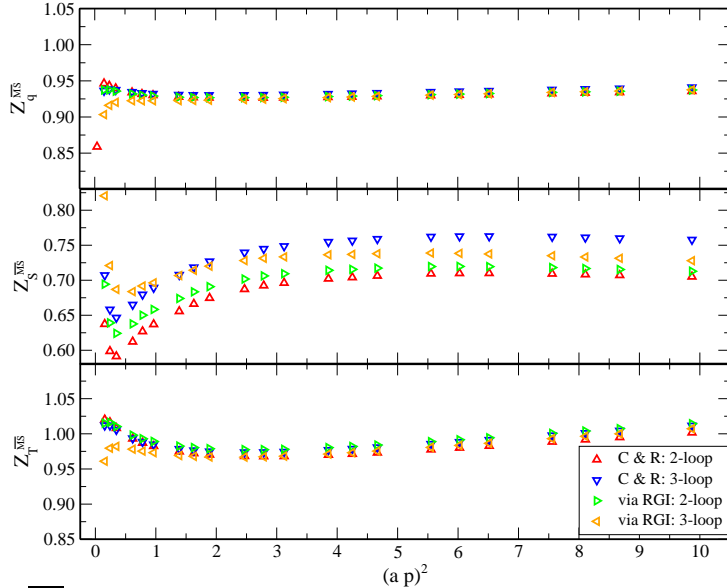


FIG. 9:  $Z_q^{\overline{\text{MS}}}$ ,  $Z_S^{\overline{\text{MS}}}$ ,  $Z_T^{\overline{\text{MS}}}$  ( $m_\pi=465$  MeV) using two-loop (red up triangles) and three-loop (blue down triangles) expressions for  $C_{\mathcal{O}}^{\text{RI}'\text{-MOM},\overline{\text{MS}}}R(\mu, 2\text{ GeV})$ . Green right triangles (yellow left triangles) correspond to the two- (three-) loop expressions of Eq. (27) using the intermediate RGI scheme.

Another systematic error could result from the ambiguity in the value of  $\Lambda_{\overline{\text{MS}}}$  needed in Eq. (28). Having this in mind, it is interesting to see how the conversion factors are affected by any variation of the value of  $\Lambda_{\overline{\text{MS}}}$ . This is demonstrated in Fig. 10. We observed that the conversion factors of  $Z_q$  and  $Z_T$  are not affected by variations of  $\Lambda_{\overline{\text{MS}}}$ . On the other hand,  $Z_S$  shows some sensitivity on  $\Lambda_{\overline{\text{MS}}}$  in the range from 275 to 375 MeV. Nevertheless, this dependence on  $\Lambda_{\overline{\text{MS}}}$  is smaller than the one due to the order of  $g^S(\mu)$  used in the conversion factors.



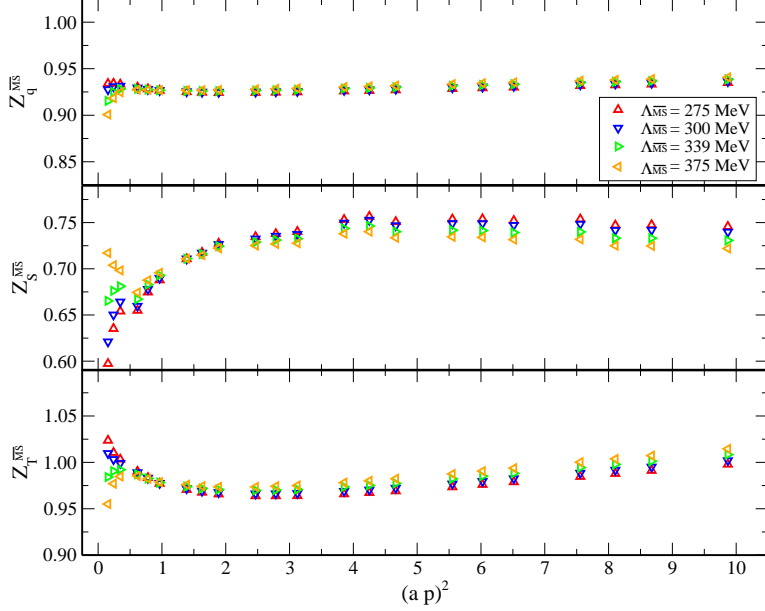


FIG. 10:  $Z_q$ ,  $Z_S$ ,  $Z_T$  in the  $\overline{\text{MS}}$  scheme using different values of  $\Lambda_{\overline{\text{MS}}}$ .

### C. Chiral extrapolation of $Z_P$ and $Z_P/Z_S$

To obtain  $Z_P$  in the chiral limit, we must subtract the pion pole from the pseudoscalar vertex function  $\Lambda_P(p, m_\pi)$ . To do so we consider as ansatz a two-parameter fit function for each momentum  $p$

$$f^{(2)}(p, m_\pi) = a_P(p) + \frac{c_P(p)}{m_\pi^2} \quad (29)$$

or a three-parameter fit function

$$f^{(3)}(p, m_\pi) = a_P(p) + b_P(p) m_\pi^2 + \frac{c_P(p)}{m_\pi^2}, \quad (30)$$

and fit Eq. (29) or Eq. (30) to the ratio

$$R(p, m_\pi) = \frac{\Lambda_P(p, m_\pi)}{Z_q(p, m_\pi) C_{P,\text{RGI}}^{\text{RI}'\text{-MOM}, \overline{\text{MS}}}(p, 2 \text{ GeV})}. \quad (31)$$

The coefficient  $a_P(p)$  is the number we are looking for, the continuum limit of which corresponds to  $(Z_P^{\overline{\text{MS}}})^{-1}$ . In Fig. 11 we show the result of both local fits. The coefficient  $b_P(p)$  turns out to be rather small. In fact, it is compatible with zero. The two-parameter fit of Eq. (29) leads to significantly smaller errors on  $a_P(p)$ . In the following we shall employ the local two-parameter fit to subtract the pion pole and extrapolate  $Z_P$  to the chiral limit. The results will be shown in Sec. V. The stability of the fit is discussed in Appendix C using alternatives for the pion pole subtraction, along with an assessment of systematic errors.

In some applications a precise value of  $Z_P/Z_S$  is needed, which suggests a direct fit of the ratio. In this ratio the factors  $Z_q$  and  $C_{\mathcal{O}}^{\text{RI-MOM}, \overline{\text{MS}}}$  drop out. To subtract the pion pole from  $\Lambda_P(p, m_\pi)$ , we proceed as before and fit Eq. (29) and Eq. (30), respectively, with coefficients  $a_{PS}(p)$ ,  $b_{PS}(p)$  and  $c_{PS}(p)$  to

$$R(p, m_\pi) = \frac{\Lambda_P(p, m_\pi)}{\Lambda_S(p, m_\pi)}. \quad (32)$$

In the chiral limit  $Z_S/Z_P = a_{PS}(p)$ . Again, the parameter  $b_{PS}(p)$  has little effect on the result. As before, we shall adopt the local two-parameter fit and show results of the continuum extrapolation in Sec. V.

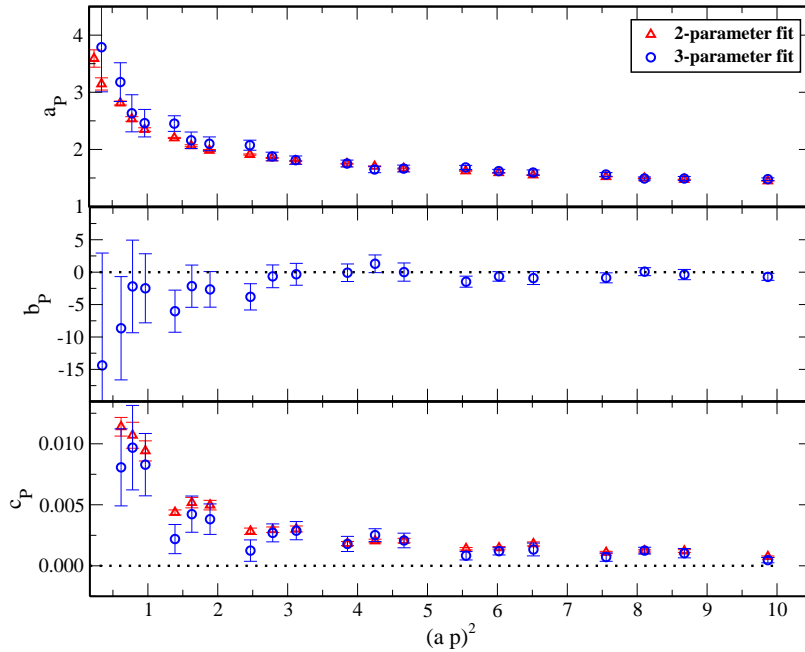


FIG. 11: Red triangles: The parameters  $a_P$ ,  $c_P$  extracted from the 2-parameter fit. Blue circles: The parameters  $a_P$ ,  $b_P$ ,  $c_P$  extracted from the 3-parameter fit.

## V. CONTINUUM EXTRAPOLATIONS

We now come to the main topic of this paper, the subtraction of lattice artifacts. Clearly, the renormalization factors  $Z_V$ ,  $Z_A$  and  $Z_S^{\overline{\text{MS}}}$ ,  $Z_P^{\overline{\text{MS}}}$  and  $Z_T^{\overline{\text{MS}}}$  show some residual dependence on  $(ap)^2$ , which we will address now. The final aim is to extrapolate the data to  $(ap)^2 = 0$ .

### A. Unsubtracted data

Let us first look at the raw data, extrapolated to the chiral limit and converted to the  $\overline{MS}$  scheme in Sec. IV. In Figs. 12 - 13 we plot  $Z_V$ ,  $Z_A$  and  $Z_P/Z_S$  as a function of  $(ap)^2$ . Similarly, in Figs. 14, 15, 16 and 17 we plot  $Z_q$ ,  $Z_S$ ,  $Z_P$  and  $Z_T$  in the RI'-MOM and  $\overline{MS}$  scheme, respectively, at  $\mu = 2 \text{ GeV}$ . We find  $Z_V$ ,  $Z_A$  and  $Z_P/Z_S$  and  $Z_S^{\overline{MS}}$ ,  $Z_P^{\overline{MS}}$  and  $Z_T^{\overline{MS}}$  to lie approximately on a linear curve for  $(ap)^2 \gtrsim 2$ , which allows a fit to a straight line. The dashed lines show a fit to the interval  $(ap)^2 \in [2, 10]$ .

While statistical errors are small, there are some systematic errors, which should be carefully examined. One source of error is the accuracy of the conversion factors. Another source arises from the choice of fit interval. We have also done fits to the intervals  $[1, 10]$ ,  $[3.7, 10]$  and  $[2, 6]$ . The difference in results will give us an estimate of the systematic error.

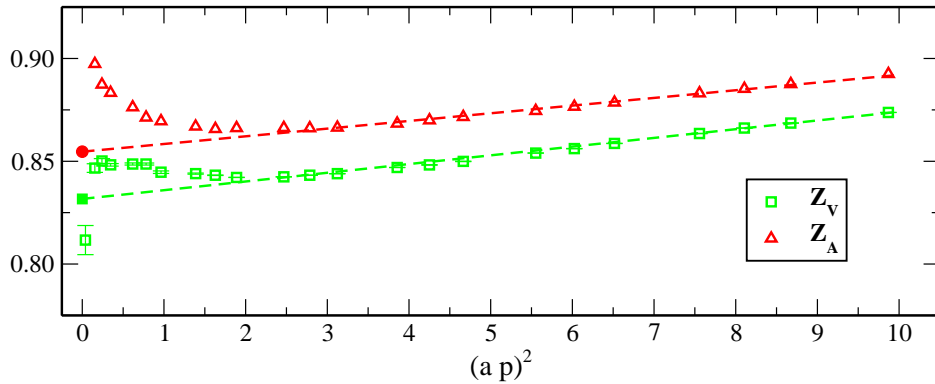


FIG. 12: The dependence of  $Z_V$  and  $Z_A$  on the renormalization scale, for the chirally extrapolated data. Dashed lines represent the continuum extrapolation and filled points the extrapolated value.

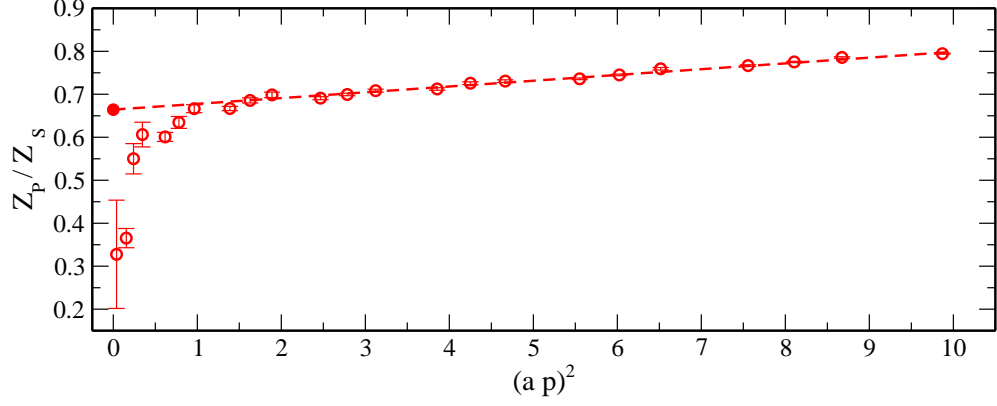


FIG. 13: Similar to Fig. 12 for  $Z_P$ .

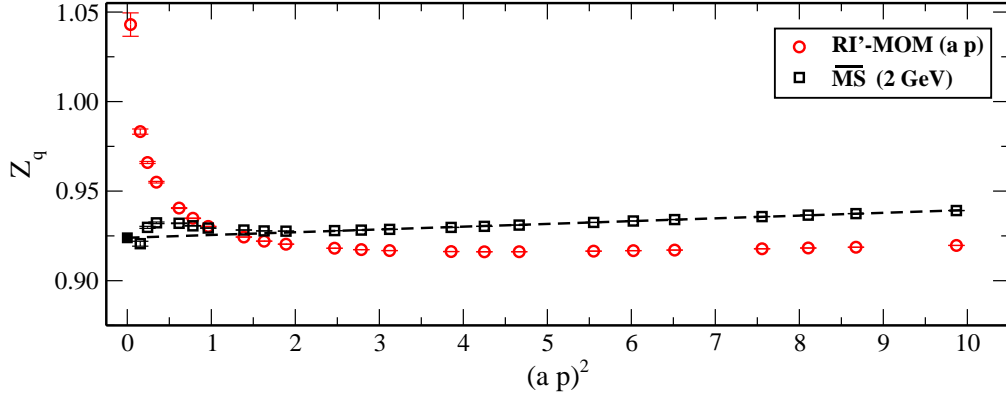


FIG. 14: The dependence of  $Z_q^{\text{RI}'\text{-MOM}}$  and  $Z_q^{\overline{\text{MS}}}$  on the momentum scale, for the chirally extrapolated non-perturbative data. Dashed lines represent the continuum extrapolation and filled points the extrapolated value.

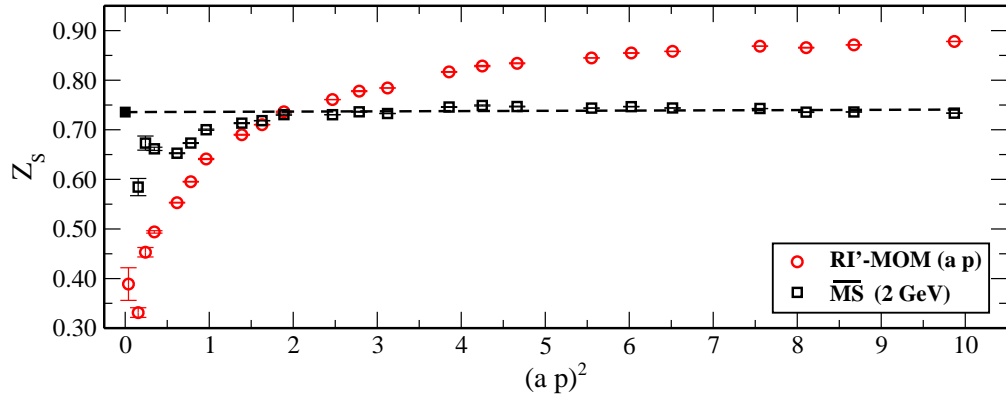


FIG. 15: Similar to Fig. 14 for  $Z_S$ .

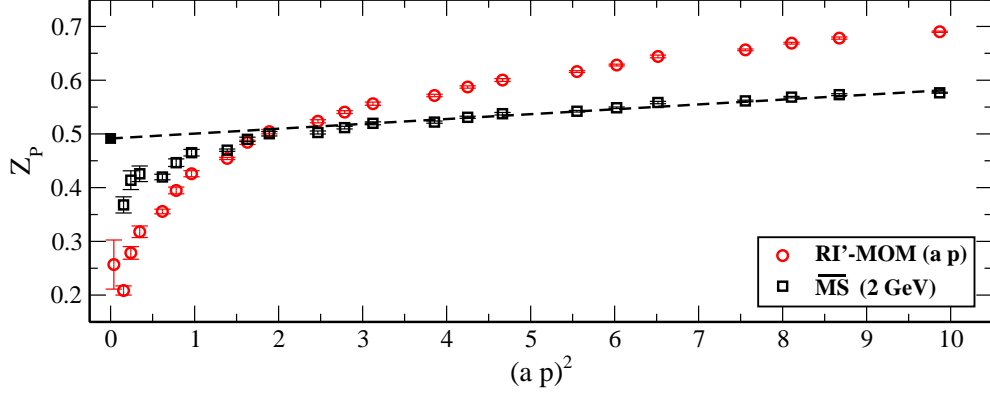


FIG. 16: Similar to Fig. 14 for  $Z_P$ .

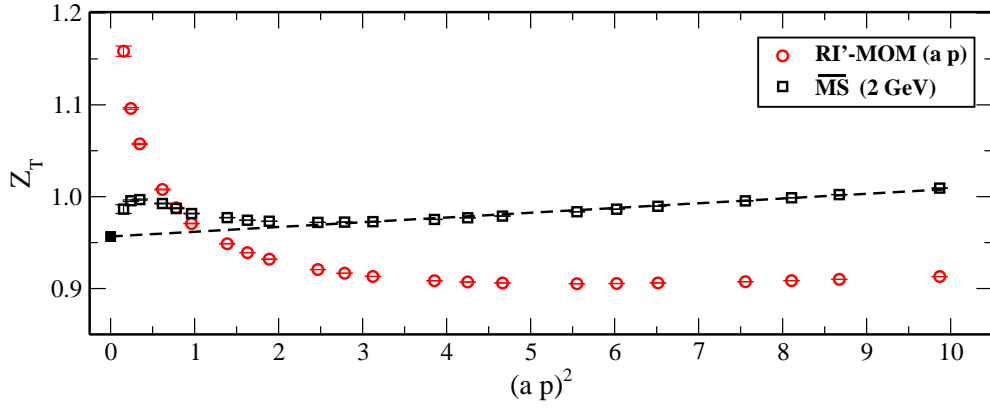


FIG. 17: Similar to Fig. 14 for  $Z_T$ .

In Table IV we present the  $Z$ -factors, after chiral and continuum extrapolation, from a fit to the interval  $(ap)^2 \in [2, 10]$ . The number in the first bracket indicates the purely statistical error. The number in the second bracket states the systematic error, which is taken from the difference of the fit to  $(ap)^2 \in [2, 10]$  and  $[2, 6]$ . The third number, wherever it applies, reflects the difference of two- and three-loop conversion factors.

|                                             |                       |
|---------------------------------------------|-----------------------|
| $Z_q^{\overline{\text{MS}}}(2 \text{ GeV})$ | 0.9239(001)(003)(028) |
| $Z_S^{\overline{\text{MS}}}(2 \text{ GeV})$ | 0.7356(053)(155)(277) |
| $Z_P^{\overline{\text{MS}}}(2 \text{ GeV})$ | 0.4915(025)(125)(184) |
| $Z_P/Z_S$                                   | 0.6643(032)(023)      |
| $Z_V$                                       | 0.8317(005)(021)      |
| $Z_A$                                       | 0.8547(0008)(036)     |
| $Z_T^{\overline{\text{MS}}}(2 \text{ GeV})$ | 0.9566(013)(054)(101) |

TABLE IV: Continuum extrapolated values on  $Z_q^{\overline{\text{MS}}}$ ,  $Z_S^{\overline{\text{MS}}}$ ,  $Z_P^{\overline{\text{MS}}}$ ,  $Z_T^{\overline{\text{MS}}}$ ,  $Z_V$ ,  $Z_A$  and  $Z_P/Z_S$ . The number in the first (second) bracket is the statistical (systematic) error, and where applicable, the one in the third bracket comes from the difference in using the two- and three-loop results for the conversion factor via RGI.

### B. Subtraction of one-loop perturbative lattice artifacts

We now turn to the subtraction of lattice artifacts. There are two kinds of subtractions we employ (see Sec. III), the subtraction of one-loop  $\mathcal{O}(a^2)$  corrections and the complete subtraction of one-loop lattice artifacts. For each case we use both the bare,  $g$ , and boosted coupling,

$$g_b^2 = \frac{g^2}{P(g)}, \quad (33)$$

where  $P(g)$  is the plaquette at  $\beta = 5.50$  and for the ensembles used in this work we find:  $P(\beta = 5.5) \sim 0.52$ . For the improvement coefficient  $c_{\text{sw}}$  we take the tree-level value,  $c_{\text{sw}} = 1$ . The effect of the subtraction is demonstrated as examples in Figs. 18 - 20 for the chirally extrapolated data. For  $Z_q$  (Fig. 18) we find no clear preference for any kind of subtraction. In case of the quark-bilinear operators (Figs. 19 -20) the  $\mathcal{O}(g^2 a^2)$  corrections are very small for  $(ap)^2 < 4$ , but beyond that they get out of control and show significant  $O((ap)^4)$  effects. Complete subtraction of one-loop lattice artifacts, on the other hand, has a small, albeit appreciable, effect on the  $Z$ -factors. In case of  $Z_V$ ,  $Z_A$  (not shown here) and  $Z_T$  the data is brought on a perfectly straight line for  $(ap)^2 \in [2, 10]$ , using the boosted coupling  $g_b^2$ .

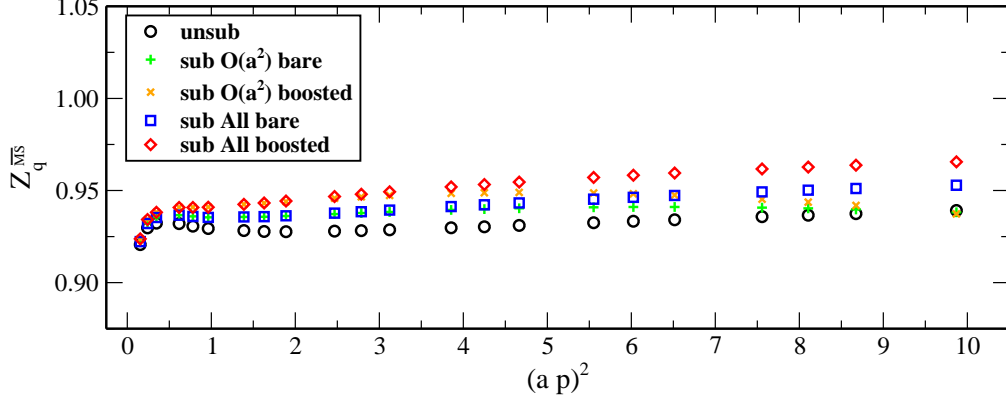


FIG. 18: Chirally extrapolated values for  $Z_q^{\overline{\text{MS}}}$  before the perturbative subtraction (black circles) and after subtraction of: a.  $\mathcal{O}(g^2 a^2)$  terms using bare coupling,  $g$  (green plus points), b.  $\mathcal{O}(g_b^2 a^2)$  terms using boosted coupling,  $g_b$  (orange crosses), c. complete subtraction using  $g$  (blue squares), and d. complete subtraction using  $g_b$  (red diamonds).

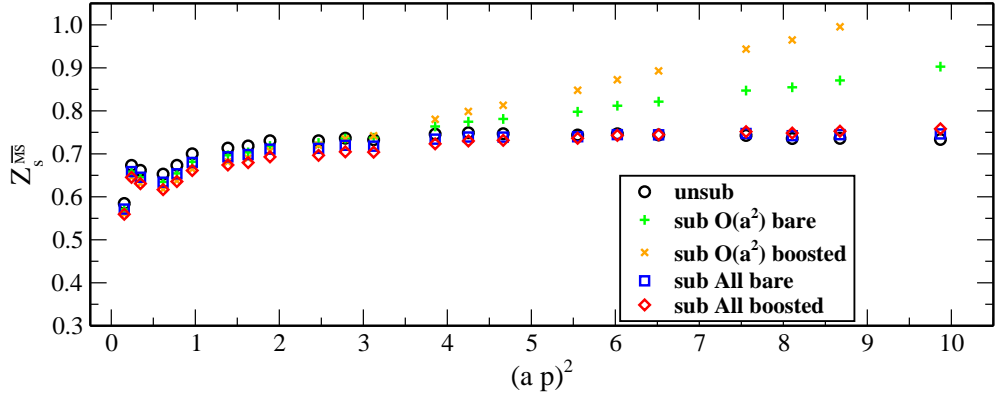


FIG. 19: Similar to Fig. 18 for  $Z_S$ .

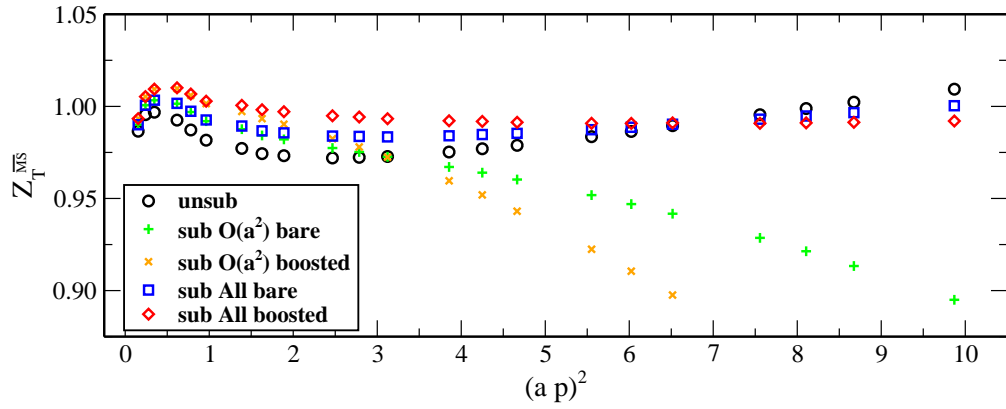


FIG. 20: Similar to Fig. 18 for  $Z_T$ .

## VI. FINAL RESULTS AND DISCUSSION

We take the lattice data of Sec. IV, improved by complete subtraction of one-loop lattice artifacts with boosted coupling  $g_b$ , as our final result. In Figs. 21 - 27 we show the  $Z$ -factors before and after the subtraction, together with a linear extrapolation to the continuum.

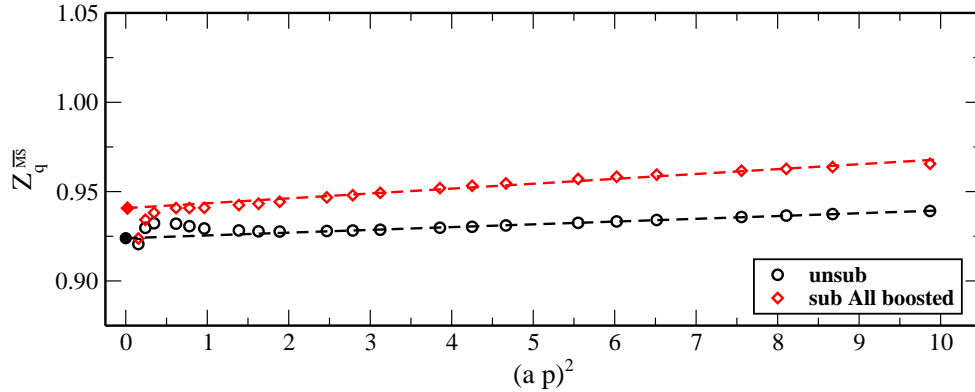


FIG. 21: Chirally extrapolated values for  $Z_q^{\overline{\text{MS}}}$  prior the perturbative subtraction (black circles) and after the complete subtraction of one-loop lattice artifacts using  $g_b$  (red diamonds).

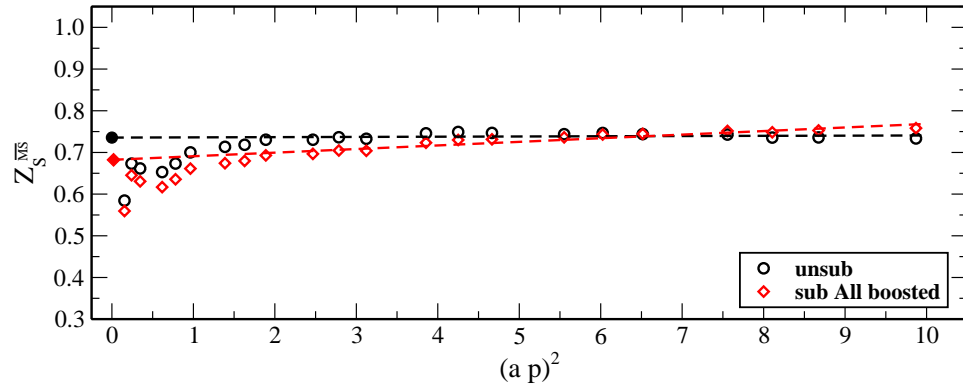


FIG. 22: Similar to Fig. 21 for  $Z_S$ .

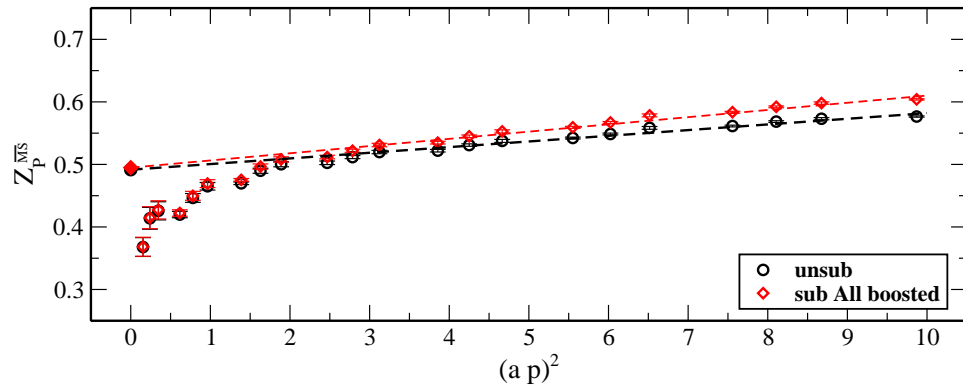


FIG. 23: Similar to Fig. 21 for  $Z_P$ .



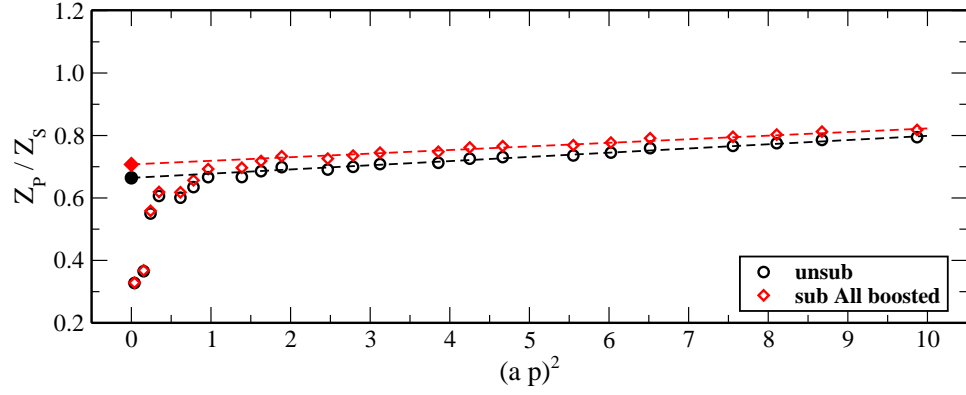


FIG. 24: Similar to Fig. 21 for  $Z_P/Z_S$ .

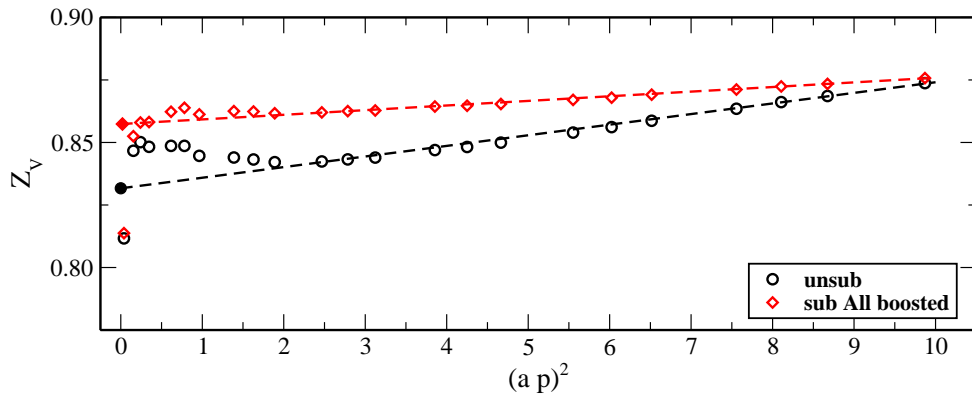


FIG. 25: Similar to Fig. 21 for  $Z_V$ .

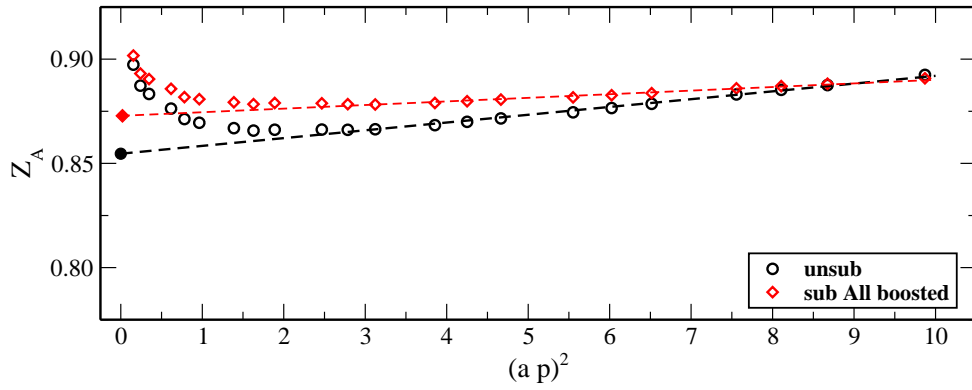


FIG. 26: Similar to Fig. 21 for  $Z_A$ .

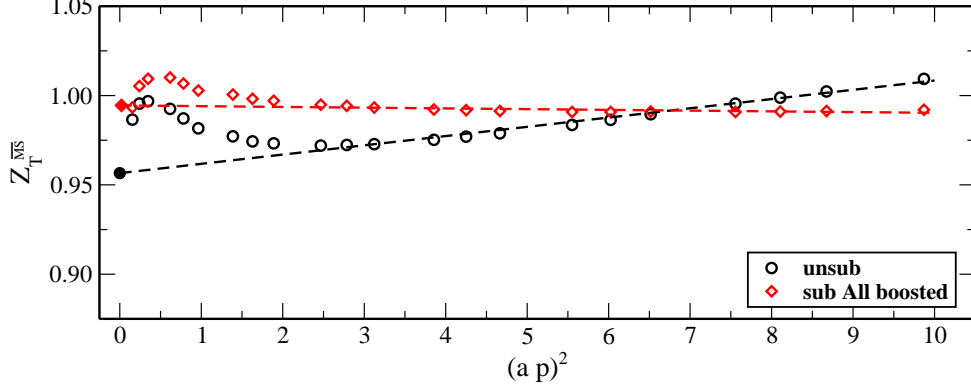


FIG. 27: Similar to Fig. 21 for  $Z_T$ .

In Table V we give our final numbers, corresponding to the solid diamonds in Figs. 21 - 27. The corrected numbers differ by up to 8% from the unsubtracted results in Table IV.

|                              |                  |
|------------------------------|------------------|
| $Z_q^{\overline{\text{MS}}}$ | 0.9408(008)(024) |
| $Z_S^{\overline{\text{MS}}}$ | 0.6822(061)(176) |
| $Z_P^{\overline{\text{MS}}}$ | 0.4948(026)(128) |
| $Z_P/Z_S$                    | 0.7075(036)(068) |
| $Z_V$                        | 0.8574(002)(007) |
| $Z_A$                        | 0.8728(006)(027) |
| $Z_T^{\overline{\text{MS}}}$ | 0.9945(010)(035) |

TABLE V: Continuum results for  $Z_q$ ,  $Z_S$ ,  $Z_P$ ,  $Z_P/Z_S$ ,  $Z_V$ ,  $Z_A$  and  $Z_T$  in the  $\overline{\text{MS}}$  scheme at  $\mu = 2 \text{ GeV}$ , where it applies. The numbers refer to the fit interval  $(ap)^2 \in [2, 10]$ . The first number in brackets is the statistical error, the second number the systematic error due to the fit range.

The conversion factors to the RGI scheme and the RI'-MOM scheme at  $\mu = 1/a$  are given by

$$\begin{aligned}
 Z_q^{\text{RGI}} &= 0.9461 Z_q^{\overline{\text{MS}}}, & Z_q^{\text{RI}'\text{-MOM}} &= 1.0004 Z_q^{\overline{\text{MS}}}, \\
 Z_S^{\text{RGI}} &= 0.7503 Z_S^{\overline{\text{MS}}}, & Z_S^{\text{RI}'\text{-MOM}} &= 0.9211 Z_S^{\overline{\text{MS}}}, \\
 Z_P^{\text{RGI}} &= 0.7503 Z_P^{\overline{\text{MS}}}, & Z_P^{\text{RI}'\text{-MOM}} &= 0.9211 Z_P^{\overline{\text{MS}}}, \\
 Z_T^{\text{RGI}} &= 1.0604 Z_T^{\overline{\text{MS}}}, & Z_T^{\text{RI}'\text{-MOM}} &= 0.9870 Z_T^{\overline{\text{MS}}}.
 \end{aligned} \tag{34}$$

To recapitulate, in a previous work [4] we have computed renormalization factors of quark-bilinear operators for  $N_f = 2$  flavors of dynamical clover fermions. We have refined the original procedure [1] in several aspects, including the use of momentum sources and

the perturbative subtraction of lattice artifacts. In this work we extended the calculation to  $N_f = 3$  flavors of SLiNC fermions [6]. Using twisted boundary conditions, the lattice momenta were chosen to lie strictly on the diagonal,  $p_\mu = \sqrt{p^2}/4, \forall \mu$ .

Complete subtraction of one-loop lattice artifacts has brought  $Z_V, Z_A, Z_S^{\overline{MS}}, Z_P^{\overline{MS}}$  and  $Z_T^{\overline{MS}}$  onto a straight line for  $(ap)^2 \in [2, 10]$ ,

$$Z(p) = Z(0) + Z'(0) (ap)^2, \quad (35)$$

with the continuum value being given by  $Z(0)$ . The origin of the remaining  $(ap)^2$ -dependence is not completely clear to us. The corrections are found to be substantial. In case of  $Z_S$  they amount to 8%. The renormalization factor of the local vector current,  $Z_V$ , can be determined independently from the Dirac form factor of the proton at zero momentum transfer by demanding that  $F_1^p(0) = 1$ . In [18] we found  $Z_V = 0.857(1)$  at a pion mass of  $m_\pi = 220$  MeV, which is in perfect agreement with our final number in Table V. This gives support for our procedure of subtracting lattice artifacts.

It helped that the unsubtracted data were approximately linear in  $(ap)^2$  already for  $(ap)^2 \gtrsim 2$ , in contrast to the case of  $N_f = 2$  flavors of clover fermions and the plaquette action [4]. We attribute that to the Symanzik and stout-link improved action employed here, which appears to suppress lattice artifacts in quark Green's functions; see e.g., Figs. 4 - 6 demonstrating that the stout smearing leads to smaller lattice artifacts compared to the non-smearred case ( $\omega = 0$ ).

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## Appendix A: Perturbative Results

In this section we summarize the perturbative results for the  $\mathcal{O}(g^2 a^2)$  contributions  $\Lambda_q^{(2)}$  and  $\Lambda_{\mathcal{O}}^{(2)}$  as defined in Sec. III (see Eq. (20)). For simplicity, we restrict to the case of the SLiNC action, which uses tree-level Symanzik gluons. The expressions, evaluated at  $\omega = 0$ , correspond to the usual clover action. Our results are given as a function of the momentum  $p$  and general values for the action parameters  $c_{\text{sw}}, \omega$ . The gauge fixing parameter  $\alpha$  is 0(1) for the Landau (Feynman) gauge.

$$\begin{aligned} \Lambda_q^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} & \left( p^2 \left( 1.14717 - 1.51605 \alpha + c_{\text{sw}}(-0.653431 + 0.505886 \omega) - 0.497834 c_{\text{sw}}^2 \right. \right. \\ & \left. \left. - 12.0983 \omega + 28.0799 \omega^2 + \log(a^2 p^2) \left( -\frac{73}{360} + \frac{3\alpha}{8} + \frac{c_{\text{sw}}^2}{4} + \frac{c_{\text{sw}}}{4} + \omega \right) \right) \right. \\ & + \frac{p^4}{p^2} \left( 2.10650 + 0.395834 \alpha + c_{\text{sw}}(0.284537 - 0.362507 \omega) + 0.128381 c_{\text{sw}}^2 \right. \\ & \left. \left. - 4.08165 \omega - 16.0889 \omega^2 - \frac{157}{180} \log(a^2 p^2) \right) \right) \end{aligned} \quad (\text{A1})$$

$$\begin{aligned} \Lambda_S^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} & \left( p^2 \left( -1.20757 + 0.75755 \alpha + c_{\text{sw}}(3.19935 - 4.79168 \omega) - 0.69430 c_{\text{sw}}^2 \right. \right. \\ & \left. \left. - 0.64987 \omega + 0.71144 \omega^2 + \log(a^2 p^2) \left( \frac{17}{360} - \frac{3\alpha}{8} - \frac{5 c_{\text{sw}}}{4} + \frac{c_{\text{sw}}^2}{4} + \omega \right) \right) \right. \\ & + \frac{p^4}{p^2} \left( 1.6065 + 0.52083 \alpha + c_{\text{sw}}(0.28454 - 0.36251 \omega) + 0.12838 c_{\text{sw}}^2 \right. \\ & \left. \left. - 4.08165 \omega - 16.0889 \omega^2 - \frac{157}{180} \log(a^2 p^2) \right) \right) \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \Lambda_P^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} & \left( p^2 \left( 0.44076 - 0.67794 \alpha + c_{\text{sw}}(0.50589 \omega - 0.65343) - 0.22227 c_{\text{sw}}^2 \right. \right. \\ & \left. \left. - 5.59237 \omega - 1.17320 \omega^2 + \log(a^2 p^2) \left( \frac{17}{360} + \frac{\alpha}{8} + \frac{c_{\text{sw}}}{4} + \frac{c_{\text{sw}}^2}{4} + \omega \right) \right) \right. \\ & + \frac{p^4}{p^2} \left( 1.6065 + 0.52083 \alpha + c_{\text{sw}}(0.28454 - 0.36251 \omega) + 0.12838 c_{\text{sw}}^2 \right. \\ & \left. \left. - 4.08165 \omega - 16.0889 \omega^2 - \frac{157}{180} \log(a^2 p^2) \right) \right) \end{aligned} \quad (\text{A3})$$

$$\begin{aligned}
\Lambda_V^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} & \left( p^2 \left( 0.44724 - 0.79717 \alpha + c_{\text{sw}}(1.83028 \omega - 1.61663) - 0.14183 c_{\text{sw}}^2 \right. \right. \\
& \left. \left. - 1.02045 \omega + 1.46637 \omega^2 + \log(a^2 p^2) \left( \frac{47}{360} + \frac{3\alpha}{16} + \frac{5 c_{\text{sw}}}{8} - \frac{c_{\text{sw}}^2}{8} - \frac{\omega}{2} \right) \right) \right. \\
& \left. + \frac{p^4}{p^2} \left( 2.54053 + 0.23958 \alpha + c_{\text{sw}}(0.28454 - 0.36251 \omega) + 0.12838 c_{\text{sw}}^2 \right. \right. \\
& \left. \left. - 4.08165 \omega - 16.0889 \omega^2 - \frac{157}{180} \log(a^2 p^2) \right) \right) \quad (\text{A4})
\end{aligned}$$

$$\begin{aligned}
\Lambda_A^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} & \left( p^2 \left( -0.37692 - 0.07942 \alpha + c_{\text{sw}}(0.30976 - 0.81851 \omega) - 0.85384 c_{\text{sw}}^2 \right. \right. \\
& \left. \left. + 1.45079 \omega + 2.40868 \omega^2 + \log(a^2 p^2) \left( \frac{47}{360} - \frac{\alpha}{16} - \frac{c_{\text{sw}}}{8} + \frac{5 c_{\text{sw}}^2}{8} - \frac{\omega}{2} \right) \right) \right. \\
& \left. + \frac{p^4}{p^2} \left( 2.54053 + 0.23958 \alpha + c_{\text{sw}}(0.28454 - 0.36251 \omega) + 0.12838 c_{\text{sw}}^2 \right. \right. \\
& \left. \left. - 4.08165 \omega - 16.0889 \omega^2 - \frac{157}{180} \log(a^2 p^2) \right) \right) \quad (\text{A5})
\end{aligned}$$

$$\begin{aligned}
\Lambda_T^{(2)} = a^2 \frac{g^2 C_F}{16 \pi^2} & \left( p^2 \left( 0.17468 - 0.59766 \alpha + c_{\text{sw}}(1.38881 \omega - 1.29556) - 0.51102 c_{\text{sw}}^2 \right. \right. \\
& \left. \left. + 1.32727 \omega + 2.66032 \omega^2 + \log(a^2 p^2) \left( \frac{19}{120} + \frac{\alpha}{8} + \frac{c_{\text{sw}}}{2} + \frac{c_{\text{sw}}^2}{4} - \omega \right) \right) \right. \\
& \left. + \frac{p^4}{p^2} \left( 2.85187 + 0.14583 \alpha + c_{\text{sw}}(0.28454 - 0.36251 \omega) + 0.12838 c_{\text{sw}}^2 \right. \right. \\
& \left. \left. - 4.08165 \omega - 16.0889 \omega^2 - \frac{157}{180} \log(a^2 p^2) \right) \right) \quad (\text{A6})
\end{aligned}$$

## Appendix B: $\beta$ -function and anomalous dimensions

In this Appendix we give the definitions of the  $\beta$ -function and the anomalous dimension for the fermion field and the local operators. The perturbative coefficients up to three loops are given in  $SU(3)$  and in the Landau gauge.

The scale dependence of the renormalized operator is encoded in the anomalous dimension and is defined as:

$$\gamma^{\mathcal{S}} = -\mu \frac{d}{d\mu} \log Z_{\mathcal{S}} = \gamma_0 \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} + \gamma_1^{\mathcal{S}} \left( \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^2 + \gamma_2^{\mathcal{S}} \left( \frac{g^{\mathcal{S}}(\mu)^2}{16\pi^2} \right)^3 + \dots \quad (\text{B1})$$

where  $\mathcal{S}$  is the renormalization scheme. The  $\beta$ -function is defined as:

$$\beta^{\mathcal{S}} = \mu \frac{d}{d\mu} g^{\mathcal{S}}(\mu) = -\beta_0 \frac{g^{\mathcal{S}}(\mu)^3}{16\pi^2} - \beta_1 \frac{g^{\mathcal{S}}(\mu)^5}{(16\pi^2)^2} - \beta_2 \frac{g^{\mathcal{S}}(\mu)^7}{(16\pi^2)^3} + \dots \quad (\text{B2})$$

The coefficients of the  $\beta$ -function in the  $\overline{\text{MS}}$  and the RI'-MOM schemes coincide up to three loops and are given by [15, 22]:

$$\beta_0 = 11 - \frac{2}{3}N_f, \quad (\text{B3})$$

$$\beta_1 = 102 - \frac{38}{3}N_f, \quad (\text{B4})$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2. \quad (\text{B5})$$

The coefficients of the anomalous dimension for the quark field in the  $\overline{\text{MS}}$  and RI'-MOM schemes are [23]:

$$\gamma_0 = 0, \quad (\text{B6})$$

$$\gamma_1 = \frac{134}{3} - \frac{8}{3}N_f, \quad (\text{B7})$$

$$\gamma_2^{\overline{\text{MS}}} = \frac{20729}{18} - 79\zeta_3 - \frac{1100}{9}N_f + \frac{40}{27}N_f^2, \quad (\text{B8})$$

$$\gamma_2^{\text{RI'-MOM}} = \frac{52321}{18} - 79\zeta_3 - \frac{1100}{9}N_f + \frac{40}{27}N_f^2, \quad (\text{B9})$$

for the scalar/pseudoscalar operators [24, 25]:

$$\gamma_0 = -8, \quad (\text{B10})$$

$$\gamma_1^{\overline{\text{MS}}} = -\frac{404}{3} + \frac{40}{9}N_f, \quad (\text{B11})$$

$$\gamma_1^{\text{RI'-MOM}} = -252 + \frac{104}{9}N_f, \quad (\text{B12})$$

$$\gamma_2^{\overline{\text{MS}}} = -2498 + \left( \frac{4432}{27} + \frac{320}{3}\zeta_3 \right) N_f + \frac{280}{81}N_f^2, \quad (\text{B13})$$

$$\gamma_2^{\text{RI'-MOM}} = -\frac{40348}{3} + \frac{6688}{3}\zeta_3 + \left( \frac{35176}{27} - \frac{256}{9}\zeta_3 \right) N_f - \frac{1712}{81}N_f^2, \quad (\text{B14})$$

( $\zeta_3 = 1.20206\dots$ ) and for the tensor [15, 26]:

$$\gamma_0 = \frac{8}{3}, \quad (\text{B15})$$

$$\gamma_1 = \frac{724}{9} - \frac{104}{27}N_f, \quad (\text{B16})$$

$$\gamma_2^{\overline{\text{MS}}} = \frac{105110}{81} - \frac{1856}{27}\zeta_3 - \left( \frac{10480}{81} + \frac{320}{9}\zeta_3 \right) N_f - \frac{8}{9}N_f^2, \quad (\text{B17})$$

$$\gamma_2^{\text{RI'-MOM}} = \frac{359012}{81} - \frac{26144}{27}\zeta_3 + \left( -\frac{39640}{81} + \frac{512}{27}\zeta_3 \right) N_f + \frac{2288}{243}N_f^2. \quad (\text{B18})$$

### Appendix C: An alternative for the pion pole subtraction

Here we mention an alternative in handling the pion pole subtraction to determine the renormalization factors for  $Z_P$ . We can perform a global fit to the ratio of Eq. (31) taking into account the data at all pion masses and all scales simultaneously. Since in the same fit we combine both correlated and uncorrelated data extracted from different ensembles, we use the super-jackknife procedure [27] for the error estimation of the fit parameters. We employ a fit function with four, five and six parameters, of the form:

$$f^{(4)}(p, m_\pi) = a_0 + a_2 p^2 + \frac{c_0 + c_2 p^2}{m_\pi^2}, \quad (\text{C1})$$

$$f^{(5)}(p, m_\pi) = a_0 + a_2 p^2 + b_0 m_\pi^2 + \frac{c_0 + c_2 p^2}{m_\pi^2}, \quad (\text{C2})$$

$$f^{(6)}(p, m_\pi) = a_0 + a_2 p^2 + (b_0 + b_2 p^2) m_\pi^2 + \frac{c_0 + c_2 p^2}{m_\pi^2}. \quad (\text{C3})$$

In the above functions the parameters  $a_i$ ,  $b_i$ ,  $c_i$  are constants, but their estimation depends on the range of momenta that we take into account. Since the data at  $(ap)^2 < 1$  do not exhibit any plateau behavior, they are entirely excluded from the fit. In fact, in the application of each fit shown in Eqs. (C1) - (C3) we use data corresponding to various momentum ranges,  $(ap)^2$  (see text below). Similarly to the case of the two- and three-parameter fit, we are interested in extracting  $Z_P$  in  $\overline{\text{MS}}$  with a single fit which is related to  $a_0$ , through  $Z_P^{\overline{\text{MS}}} = (a_0)^{-1}$ . To summarize, we have a total of 12 estimates for  $Z_P^{\overline{\text{MS}}}$  extracted from the global fits, which correspond to all combinations between the three fit functions (Eqs. (C1) - (C3)) and the four momentum ranges in  $(ap)^2 \in [1 : 10]$ ,  $[2 : 6]$ ,  $[2 : 10]$  and  $[3.7 - 10]$ . We find that the six-parameter fit is not very stable, while the four- and five-parameter global fits give compatible results with the local two- and three-parameter local fits applied to each momentum squared individually. An interesting observation is that the results obtained using the interval  $[2 : 10]$  are compatible for all types of fits discussed here (see Table VI). The same global fitting has been tested to find  $Z_P/Z_S$ .

|                              | 4-parameter fit | 5-parameter fit | 6-parameter fit |
|------------------------------|-----------------|-----------------|-----------------|
| $Z_P^{\overline{\text{MS}}}$ | 0.4917(022)     | 0.4921(078)     | 0.4852(176)     |

TABLE VI: Estimates for  $Z_P$  in the  $\overline{\text{MS}}$  scheme using the global fits of Eq. (C1) - (C3) and the data in the momentum range [2 : 10]. Statistical errors are shown in parentheses.

We find that the four-, five- and six-parameter global fits are, in general, more unstable compared to the two- and three-parameter local fits and will not be used as final estimates. Moreover, the restriction of the global fitting functions to have quadratic dependence with respect to the momentum (Eqs. (C1) - (C3)) is not based on any theoretical arguments. Nevertheless, agreement within errorbars between different fits gives confidence on the final estimate.

As résumé of these studies we use in Sec. V the local two-parameter fit to remove the pion pole and combine it with a linear fit in the  $(ap)^2$  interval [2 : 10] counting for a remaining momentum dependence in the non-pole term after chiral extrapolation. The resulting constant is then the inverse of the renormalization function in the  $\overline{\text{MS}}$  scheme.

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