

Proceedings of the Second Annual LHCP  
DESY 14-147  
August 27, 2014

## PRECISION DETERMINATION OF THE TOP-QUARK MASS

SVEN-OLAF MOCH

*II. Institut für Theoretische Physik  
Universität Hamburg  
Luruper Chaussee 149  
D-22761 Hamburg, Germany*

### ABSTRACT

Precision determinations of the top-quark mass require theory predictions with a well-defined mass parameter in a given renormalization scheme. The top-quark's running mass in the  $\overline{\text{MS}}$  scheme can be extracted with good precision from the total cross section at next-to-next-to-leading order in QCD. The Monte Carlo top-quark mass parameter measured from comparison to events with top-quark decay products is not identical with the pole mass. Its translation to the pole mass scheme introduces an additional uncertainty of the order of 1 GeV.

### PRESENTED AT

The Second Annual Conference  
on Large Hadron Collider Physics  
Columbia University, New York, U.S.A  
June 2-7, 2014

# 1 Introduction

Since the discovery of the top-quark almost 20 years ago the mass of the heaviest elementary particle currently known has been measured with an ever increasing and, by now, with unprecedented precision. The top-quark mass is a fundamental parameter of the Standard Model (SM) and the precise value is indispensable for predictions of cross sections at the Large Hadron Collider (LHC). Moreover, in the absence of direct evidence for new physics beyond the SM, precision theory predictions confronted with precision measurements have become an important area of research for self-consistency tests of the SM or in searching for new physics phenomena. This has been the motivation for significant progress, both on the theoretical and the experimental side, in addressing issues arising in precision top-quark mass determinations, see, e.g., [1, 2] for reviews of recent activities.

Here, two examples are given, where the numerical value of the top-quark mass  $m_t$  directly affects relevant physics interpretations. On the left in Fig. 1, the current experimental results for the  $W$ -boson mass  $M_W$  and the top-quark mass  $m_t$  are shown in comparison with the theory predictions of the SM and its minimal supersymmetric extension (MSSM) for a range of Higgs boson masses  $M_H$ , see, e.g., [3]. The plot indicates consistency of the values for the various mass parameters  $M_W$ ,  $m_t$  and  $M_H$  at the level of  $1\sigma$  uncertainties within the SM. On the right in Fig. 1 the direct impact of the top-quark mass on the Higgs sector is illustrated. Regions of stability of the electroweak vacuum in the  $m_t$  and  $M_H$  plane are plotted, which can be obtained from extrapolating the SM up to the Planck scale, see, e.g., [4–8]. Thus, at high scales the existence of a well-defined minimum of the Higgs potential that can induce breaking of the electroweak symmetry, depends crucially on the precise numerical value of  $m_t$ .

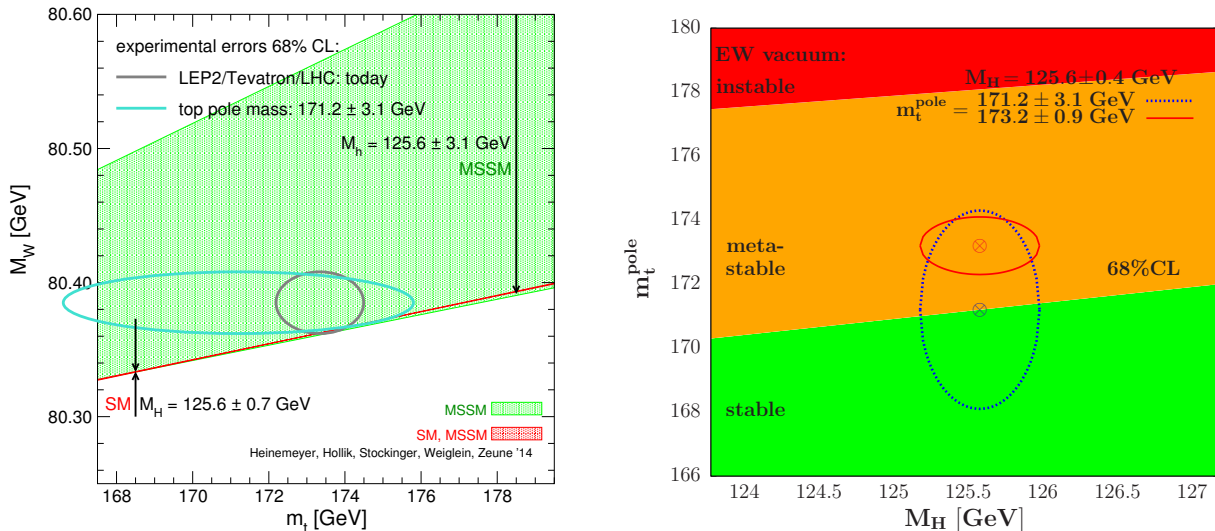


Figure 1: Left: Current experimental results for  $M_W$  and  $m_t^{\text{pole}}$  and their  $1\sigma$  uncertainties in comparison with the SM (red band) and the MSSM prediction (light-shaded green band). (Figure courtesy S.Heinemeyer, cf. Ref. [3]). Right: Ellipses for the  $1\sigma$  uncertainties in the  $[M_H, m_t^{\text{pole}}]$  plane with Higgs mass  $M_H = 125.6 \pm 0.4$  GeV and  $\alpha_s(M_Z) = 0.1187$  confronted with the areas in which the SM vacuum is absolutely stable, meta-stable and unstable up to the Planck scale. (Figure from Ref. [9]).

## 2 Top-quark mass definition

Quark masses are not physical observables. This implies, first of all, that the determination of  $m_t$  relies on the comparison of theory predictions  $\sigma_{\text{th}}(m_t)$  for cross sections with the experimentally measured values  $\sigma_{\text{exp}}$  for a given observable and kinematics as the best fit solution to the equation  $\sigma_{\text{exp}} = \sigma_{\text{th}}(m_t)$ . The accuracy

of this approach is intrinsically limited by the sensitivity  $\mathcal{S}$  of  $\sigma_{\text{th}}(m_t)$  to  $m_t$ ,

$$\left| \frac{\Delta\sigma}{\sigma} \right| = \mathcal{S} \times \left| \frac{\Delta m_t}{m_t} \right|. \quad (1)$$

Thus, for a given experimental error or a theoretical uncertainty  $\Delta\sigma$  on the cross section, the greater the sensitivity  $\mathcal{S}$  the better the accuracy for  $m_t$  can be achieved.

In Quantum Chromodynamics (QCD), quark masses are simply parameters of the Lagrangian. They appear in the theory predictions  $\sigma_{\text{th}}(m_t)$  and, as such, they are subject to the definition of a renormalization scheme once quantum corrections at higher orders are included. In many QCD applications the pole mass is the conventional scheme choice. The top-quark’s pole mass  $m_t^{\text{pole}}$  is introduced in a gauge invariant and well-defined way at each finite order of perturbation theory as the location of the single pole in the two-point correlation function. The pole mass scheme is, in fact, inspired by the definition of the electron mass in Quantum Electrodynamics. For heavy quarks, however, this has its short-comings [10, 11], because due to confinement quarks do not appear as free particles in asymptotic states in the  $S$ -matrix. Therefore, the pole mass  $m_t^{\text{pole}}$  must acquire non-perturbative corrections, because in the full theory the quark two-point function does not display any pole. This leads to an intrinsic uncertainty in the definition of  $m_t^{\text{pole}}$  of the order of  $\Lambda_{\text{QCD}}$  related to the renormalon ambiguity [12].

Fortunately, one can consider alternative definitions based on the (modified) minimal subtraction in the  $\overline{\text{MS}}$  scheme, which realizes the concept of a running mass  $m_t(\mu)$  at a scale  $\mu$ . More generally, one can define so-called short-distance masses  $m_t(R, \mu)$ , where  $R$  is a scale associated with the scheme. The  $\overline{\text{MS}}$  mass is then just one example of a short-distance mass  $m_t(R, \mu)$  with  $R$  taken at the scale  $R \sim m_t$ . Other schemes define a so-called  $1S$  mass [13, 14] through the perturbative contribution to the mass of a hypothetical  $^3S_1$  toponium bound state or a “potential-subtracted” (PS) mass [15].

As alternative renormalization schemes, all short-distance masses  $m_t(R, \mu)$  can be related to the pole mass  $m_t^{\text{pole}}$  through a perturbative series,

$$m_t^{\text{pole}} = m_t^{\text{MSR}}(R, \mu) + \delta m_t(R, \mu), \quad \delta m_t(R, \mu) = R \sum_{n=1}^{\infty} \sum_{k=0}^n a_{nk} \alpha_s^n(\mu) \ln^k \left( \frac{\mu^2}{R^2} \right), \quad (2)$$

with coefficients  $a_{nk}$  known to three loops in QCD [16, 17].

A variety of methods for top-quark mass extractions has been proposed thus far, see, e.g., [1, 2], which use a number of distinct observables. Examples include determinations of  $m_t$  from the total cross section, or its extraction from the distribution of the invariant mass of a lepton and a  $b$ -jet, see, e.g., [18, 19] and [20, 21], respectively.

With enough statistics, as expected from the LHC runs at increased collision energy, also exclusive observables with reconstructed top-quarks come into focus. The (normalized) differential distribution of the  $t\bar{t} + 1$ -jet cross section with respect to the invariant mass of the  $t\bar{t} + 1$ -jet system displays very good sensitivity to  $m_t$ ,  $\mathcal{S} \sim 10 \dots 20$  in Eq. (1) depending on the kinematical region and can, potentially, lead to very precise values for  $m_t$ , see [22].

All those methods employ mostly the pole mass scheme. The  $1S$  mass and the PS mass have been considered in applications to hadro-production of top-quark pairs in [23, 24]. In the sequel we will discuss the determination of the running mass in the  $\overline{\text{MS}}$  scheme and  $m_t$  from reconstructed kinematics as well as the relation of those mass parameters to the pole mass  $m_t^{\text{pole}}$ .

### 3 Running top-quark mass

The running mass in the  $\overline{\text{MS}}$  scheme has so far been used in theory predictions for the inclusive cross section [28, 29] or for differential distributions in [27]. Such (semi-)inclusive observables are known with good precision, i.e., to next-to-leading order (NLO) in perturbative QCD in the case of differential distributions [30, 31] or even to next-to-next-to-leading order (NNLO) in the case of the inclusive cross section [32–35], see also [36–38] for approximate NNLO differential cross sections. These computations are typically carried

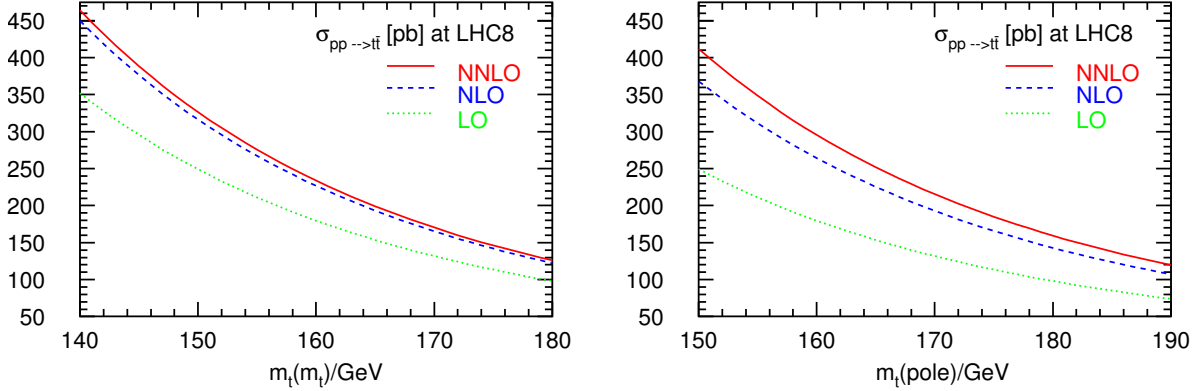


Figure 2: The LO, NLO and NNLO QCD predictions for the  $t\bar{t}$  total cross section at the LHC ( $\sqrt{s} = 8$  TeV) as a function of the top-quark mass in the  $\overline{\text{MS}}$  scheme  $m_t(m_t)$  at the scale  $\mu = m_t(m_t)$  (left) and in the on-shell scheme  $m_t^{\text{pole}}$  at the scale  $\mu = m_t^{\text{pole}}$  (right) with the ABM12 PDFs. (Figure from Ref. [25]).

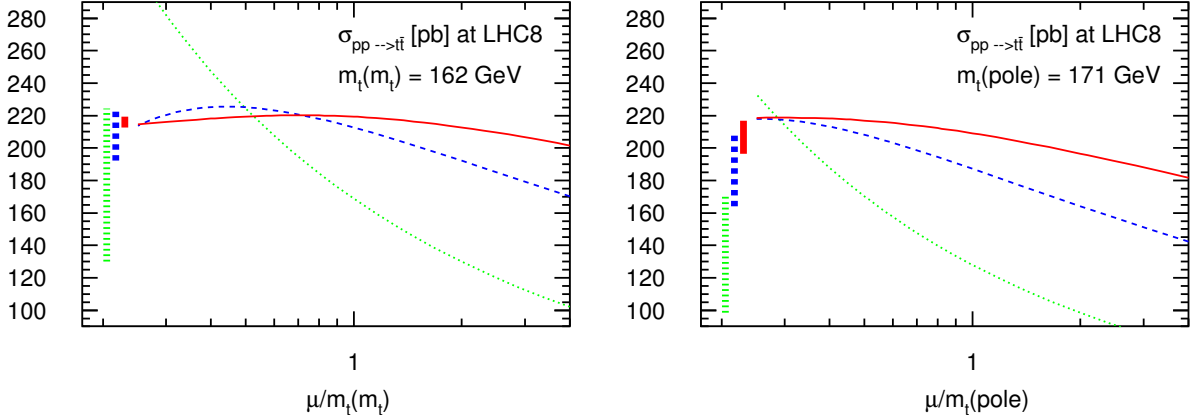


Figure 3: The scale dependence of the LO, NLO and NNLO QCD predictions for the  $t\bar{t}$  total cross section at the LHC ( $\sqrt{s} = 8$  TeV) for a top-quark mass  $m_t(m_t) = 162$  GeV in the  $\overline{\text{MS}}$  scheme (left) and  $m_t^{\text{pole}} = 171$  GeV in the on-shell scheme (right) with the ABM12 PDFs and the choice  $\mu = \mu_r = \mu_f$ . The vertical bars indicate the size of the scale variation in the standard range  $\mu/m_t^{\text{pole}} \in [1/2, 2]$  and  $\mu/m_t(m_t) \in [1/2, 2]$ , respectively. (Figure from Ref. [25]).

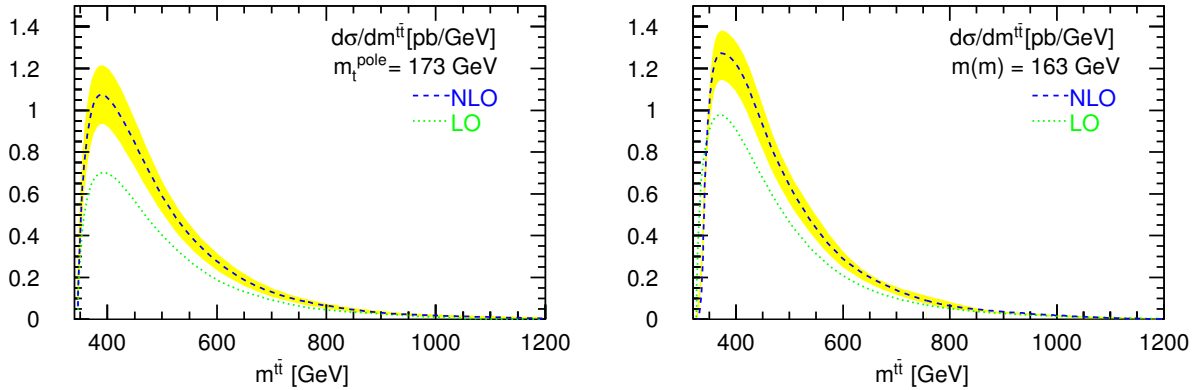


Figure 4: The differential cross section versus the invariant mass  $m^{\bar{t}t}$  of the top-quark pair in the pole (left) and the  $\overline{\text{MS}}$  (right) mass scheme at the LHC with  $\sqrt{s} = 8$  TeV. The dotted (green) curves are the LO contributions, the dashed (blue) curves include NLO corrections and are obtained with the PDF set CT10 [26]. The scale dependence in the ranges  $\mu/m_t^{\text{pole}}$  or  $\mu/m(m) \in [1/2, 2]$  is shown as a band around the NLO curve. (Figure from Ref. [27]).

out in the pole mass scheme so that Eq. (2) can be employed to relate  $m_t^{\text{pole}}$  to the  $\overline{\text{MS}}$  mass. For theory predictions in terms of the  $\overline{\text{MS}}$  mass the perturbative expansion in the strong coupling converges significantly faster. At the same time, the residual scale dependence as a measure of the remaining theoretical uncertainty is much improved when using the  $\overline{\text{MS}}$  mass in contrast to the pole mass  $m_t^{\text{pole}}$ .

These findings are illustrated in Figs. 2 and 3. The theory predictions for inclusive top-quark pair production with the  $\overline{\text{MS}}$  and the pole mass are compared in Fig. 2. The result in terms of the  $\overline{\text{MS}}$  mass  $m_t(m_t)$  displays a much improved convergence as the higher order corrections are successively added. The corresponding scale dependence is shown in Fig. 3 and the predictions with the  $\overline{\text{MS}}$  mass exhibit a much better scale stability of the perturbative expansion. It is also interesting to observe, that the point of minimal sensitivity where  $\sigma_{\text{LO}} \simeq \sigma_{\text{NLO}} \simeq \sigma_{\text{NNLO}}$  is located at scales  $\mu = \mathcal{O}(m_t(m_t))$ , i.e., it coincides with the natural hard scale of the process for the  $\overline{\text{MS}}$  mass (Fig. 3, left), whereas it resides at fairly low scales,  $\mu \simeq m_t^{\text{pole}}/4 \simeq 45$  GeV for the pole mass predictions (Fig. 3, right).

For the distribution in the invariant mass  $m^{t\bar{t}}$  of the top quark pair the same findings can be seen in Fig. 4. For the  $\overline{\text{MS}}$  mass predictions the convergence is improved. Also the overall shape of the distribution changes in comparison to case of the pole mass, the peak becomes more pronounced, while the position of the peak remains stable against radiative corrections. This is essential for precision determinations of the  $\overline{\text{MS}}$  mass in specific kinematic regions of the invariant mass distribution from LHC data in the upcoming high-energy runs.

The results for the running mass imply, that experimental determinations of the mass parameter from the measured cross section can be performed with very good accuracy and a small residual theoretical uncertainty. This has been done in [25], where a fully correlated fit of the running mass from data for the total cross section at Tevatron and the LHC has given the value for the  $\overline{\text{MS}}$  mass at NNLO to

$$m_t(m_t) = 162.3 \pm 2.3 \text{ GeV}, \quad (3)$$

with an error in  $m_t(m_t)$  due the experimental data, the PDFs and the value of  $\alpha_s(M_Z)$ . An additional theoretical uncertainty from the variation of the factorization and renormalization scales in the usual range ( $\mu/m_t(m_t) \in [1/2, 2]$ ) is small,  $\Delta m_t(m_t) = \pm 0.7$  GeV. Eq. (3) is equivalent to the pole mass value of

$$m_t^{\text{pole}} = 171.2 \pm 2.4 \pm 0.7 \text{ GeV}, \quad (4)$$

using the known perturbative conversion Eq. (2) at two loops. This is the value displayed in both plots of Fig. 1, which show good consistency of the procedure and also with the top-quark mass values obtained from other determinations within the current uncertainties. The accuracy of a mass determination in this way is limited to order 1%, though, by the overall sensitivity of the total cross section to the mass parameter,  $\mathcal{S} \sim 5$  in Eq. (1).

## 4 Monte Carlo mass

The currently most precise measurement of the top-quark mass has been reported in [39] as the world combination of the experiments ATLAS, CDF, CMS and D0,

$$m_t = 173.34 \pm 0.76 \text{ GeV}. \quad (5)$$

This combination is based on determinations of  $m_t$  as a best fit to the mass parameter implemented in the respective Monte Carlo program used to generate the theory input. It is referred to as Monte Carlo (MC) top-quark mass definition and is, therefore, lacking a direct relation to a mass parameter in a well-defined renormalization scheme.

Nonetheless, the MC mass definition can be translated to a theoretically well-defined short-distance mass definition at a low scale with an uncertainty currently estimated to be of the order of 1 GeV, see [1, 40]. This translation uses the fact that multi-observable analyses like in [39] effectively assign a high statistical weight to the invariant mass distribution of the reconstructed boosted top-quarks, because of the large sensitivity of the system on the mass parameter, especially around the peak region.

The top-quark invariant mass distribution can be computed to higher orders in perturbative QCD, cf., Fig. 3, and its peak position can also be described in an effective theory approach based on a factorization [41, 42] into a hard, a soft non-perturbative and a universal jet function. Each of those functions depends in a fully coherent and transparent way on the mass at a particular scale. The reconstructed top object largely corresponds to the jet function which is governed by a short-distance mass  $m_t^{\text{MRS}}$  at the scale of the top quark width  $\Gamma_t$ , see, e.g., [1, 40]. This line of arguments allows one to systematically implement proper short-distance mass schemes for the description of the MC mass in Eq. (5), which can then indeed be converted to the pole mass.

Thus, the top-quark mass parameter  $m_t^{\text{MC}}$  is identified with a scale-dependent short-distance mass  $m_t^{\text{MSR}}(R)$  at low scales, cf. [40],

$$m_t^{\text{MC}} = m_t^{\text{MRS}}(3_{-2}^{+6} \text{ GeV}), \quad (6)$$

with an uncertainty  $\Delta m_t$  originating from the range of possible scales,  $R \simeq 1 \dots 9 \text{ GeV}$ . The value of  $\Delta m_t$  can be read off from Tab. 1 as  $\Delta m_t = {}_{-0.62}^{+0.32} \text{ GeV}$ . It should be emphasized, though, that this uncertainty is only an estimate of the conceptual uncertainty that is currently inherent in Eq. (6). Very likely, the true corrections are not exactly calculable since a complete analytic control of the MC machinery is not feasible and the exact definition of the MC mass also depends on details of the parton shower, the shower cut and the hadronization model, see, e.g., [43].

Subsequently, there are two choices to convert  $m_t^{\text{MSR}}$  in Eq. (6) to the pole mass  $m_t^{\text{pole}}$ . The first possibility applies the renormalization group to run  $m_t^{\text{MSR}}$  from the low scales,  $R \simeq 1 \dots 9 \text{ GeV}$ , up to  $R = m_t$  in order to obtain the corresponding value for the  $\overline{\text{MS}}$  mass  $m_t(m_t)$ . This procedure effectively resums large logarithms. Afterwards,  $m_t(m_t)$  is then converted to the pole mass at a given order in perturbation theory. Tab. 1 illustrates this procedure for  $m_t^{\text{MSR}}(3\text{GeV}) = 173.40 \text{ GeV}$ , see [1] for a extensive documentation.

$m_t^{\text{MSR}}(1)$	$m_t^{\text{MSR}}(3)$	$m_t^{\text{MSR}}(9)$	$m_t(m_t)$	$m_{1\text{lp}}^{\text{pole}}$	$m_{2\text{lp}}^{\text{pole}}$	$m_{3\text{lp}}^{\text{pole}}$
173.72	173.40	172.78	163.76	171.33	172.95	173.45

Table 1: Columns 1-3: Top-quark MSR masses at different scales. Column 4:  $\overline{\text{MS}}$  mass  $m_t(m_t)$  converted at  $\mathcal{O}(\alpha_s^3)$  for  $\alpha_s(M_Z) = 0.1185$  from the MSR mass  $m_t^{\text{MRS}}(3 \text{ GeV})$ . Columns 5-7: Pole masses at 1, 2 and 3 loop converted from the  $\overline{\text{MS}}$  mass  $m_t(m_t)$ . All numbers are given in GeV units.

The second choice converts the short distance mass  $m_t^{\text{MSR}}$  at the low scales directly to the pole mass as shown in Tab. 2. This leads to relatively small corrections, however, the convergence of the perturbative expansion is poor and it is therefore disfavored. In the application of the one-, two- or three-loop conversion formula, the value of the mass parameter shifts by roughly  $\Delta m_t \sim 0.15\text{GeV}$  with every additional order. This is due to large logarithms which need to be resummed via the renormalization group equation [44].

$m_t^{\text{MSR}}(3)$	$m_{1\text{lp}}^{\text{pole}}$	$m_{2\text{lp}}^{\text{pole}}$	$m_{3\text{lp}}^{\text{pole}}$
173.40	173.72	173.87	173.98

Table 2: Column 1: Top-quark MSR mass at  $R = 3 \text{ GeV}$ . Columns 2-4 show the 1, 2 and 3 loop pole masses converted from the MSR mass  $m_t^{\text{MRS}}(3 \text{ GeV})$ . All numbers are given in GeV units.

In summary, this leads to the following result for the pole mass, which corresponds to the MC mass in Eq. (5),

$$m_t^{\text{pole}} = 173.39 \pm 0.76 \text{ GeV (exp)} + \Delta m_{\text{th}}, \quad (7)$$

where the small increase by  $0.05 \text{ GeV}$  in the central value compared to Eq. (5), is due to the shift of the three-loop pole mass with respect to  $m_t^{\text{MSR}}(3 \text{ GeV})$  in Tab. 1. The theoretical uncertainty can be estimated to

$$\Delta m_{\text{th}} = {}_{-0.62}^{+0.32} \text{ GeV} (m_t^{\text{MC}} \rightarrow m_t^{\text{MSR}}(3\text{GeV})) + 0.50 \text{ GeV} (m_t(m_t) \rightarrow m_t^{\text{pole}}), \quad (8)$$

where, as indicated, the first part of the uncertainty is due to the scale choices when relating the MC mass to the short-distance mass and is subject to the qualifications mentioned above. The second part of the

uncertainty,  $\Delta m_t = +0.50\text{GeV}$ , estimates the unknown higher order corrections in the conversion of the  $\overline{\text{MS}}$  to the pole mass. Those corrections are positive and the quoted value for  $\Delta m_t$  is taken as the difference between the two-loop and the three-loop conversion, see column 6 and 7 in Tab. 1. This part can definitely be diminished once the relation of the pole to the  $\overline{\text{MS}}$  mass, i.e., the respective coefficients  $a_{nk}$  in Eq. (2), are known to four loops in QCD.

Altogether, the additional uncertainties in Eq. (8) are sizeable and have not been addressed in [39] when interpreting the experimental measurement of the top-quark mass in Eq. (5). The theory uncertainties are not uncorrelated, i.e., the linear sum  $\Delta m_{\text{th}} = {}^{+0.82}_{-0.62}\text{ GeV}$  in Eq. (8) should be combined in quadrature with the experimental error in Eq. (7) leading to  $m_t^{\text{pole}} = 173.39^{+1.12}_{-0.98}\text{ GeV}$  for the MC mass in Eq. (5).

## 5 Summary

The top-quark mass is an outstanding parameter in the SM. Its numerical value is important for many precision tests of the model at current collider energies as well as for possible extrapolations to high energies.

In QCD an unambiguous definition of the mass parameter requires the choice of a renormalization scheme, which is conventionally taken to be the pole mass, although this has its short-comings due to the renormalon ambiguity. A theoretically well-defined determination of the top-quark mass as a short-distance mass is possible in QCD even to NNLO by using inclusive observables like the total cross section for hadro-production of top-quark pairs. This has the advantage that the theory predictions in terms of the  $\overline{\text{MS}}$  mass converge faster at higher orders and are less affected by scale variations. Results for the determination of the top-quark mass in this way have been presented in Eqs. (3) and (4).

The top-quark mass parameter measured via kinematical reconstruction from the top-quark decay products by comparison to MC simulations, termed the MC mass, is not identical to the pole mass. However, the measured values can be converted to the pole mass provided certain assumption on the relation of the MC mass to a short-distance mass at a low scale are made. This conversion leads to an additional uncertainty of the order of 1 GeV as quantified in Eqs. (6)-(8). Within the current accuracies, all those determinations show good consistency. Further efforts both in theory and experiment are required though, to reduce the uncertainty.

## ACKNOWLEDGEMENTS

This work is partially supported by Deutsche Forschungsgemeinschaft in Sonderforschungsbereich 676 and by the European Commission through contract PITN-GA-2010-264564 (*LHCPhenoNet*).

## References

- [1] S. Moch *et al.*, (2014), arXiv:1405.4781.
- [2] A. Juste *et al.*, (2013), arXiv:1310.0799.
- [3] S. Heinemeyer *et al.*, JHEP 1312 (2013) 084 [arXiv:1311.1663].
- [4] M. Holthausen, K.S. Lim and M. Lindner, JHEP 1202 (2012) 037 [arXiv:1112.2415].
- [5] F. Bezrukov *et al.*, JHEP 1210 (2012) 140 [arXiv:1205.2893].
- [6] G. Degrossi *et al.*, JHEP 1208 (2012) 098 [arXiv:1205.6497].
- [7] D. Buttazzo *et al.*, JHEP 1312 (2013) 089 [arXiv:1307.3536].
- [8] A. Andreassen, W. Frost and M.D. Schwartz, (2014), arXiv:1408.0292.
- [9] S. Alekhin, A. Djouadi and S. Moch, Phys. Lett. B716 (2012) 214 [arXiv:1207.0980].

- [10] I.I. Bigi *et al.*, Phys. Rev. D50 (1994) 2234 [hep-ph/9402360].
- [11] M. Beneke and V.M. Braun, Nucl. Phys. B426 (1994) 301 [hep-ph/9402364].
- [12] M.C. Smith and S. Willenbrock, Phys. Rev. Lett. 79 (1997) 3825 [hep-ph/9612329].
- [13] A.H. Hoang, Z. Ligeti and A.V. Manohar, Phys. Rev. D59 (1999) 074017 [hep-ph/9811239].
- [14] A. Hoang and T. Teubner, Phys. Rev. D60 (1999) 114027 [hep-ph/9904468].
- [15] M. Beneke, Phys. Lett. B434 (1998) 115 [hep-ph/9804241].
- [16] K. Chetyrkin and M. Steinhauser, Nucl. Phys. B573 (2000) 617 [hep-ph/9911434].
- [17] K. Melnikov and T.v. Ritbergen, Phys. Lett. B482 (2000) 99 [hep-ph/9912391].
- [18] M. Beneke *et al.*, Nucl. Phys. B855 (2012) 695 [arXiv:1109.1536].
- [19] S. Chatrchyan *et al.* [CMS Collaboration], Phys. Lett. B728 (2014) 496 [arXiv:1307.1907].
- [20] S. Biswas, K. Melnikov and M. Schulze, JHEP 1008 (2010) 048 [arXiv:1006.0910].
- [21] G. Heinrich *et al.*, JHEP 1406 (2014) 158 [arXiv:1312.6659].
- [22] S. Alioli *et al.*, Eur. Phys. J. C73 (2013) 2438 [arXiv:1303.6415].
- [23] V. Ahrens *et al.*, Phys. Lett. B703 (2011) 135 [arXiv:1105.5824].
- [24] P. Falgari, A. Papanastasiou and A. Signer, JHEP 1305 (2013) 156 [arXiv:1303.5299].
- [25] S. Alekhin, J. Blümlein and S. Moch, Phys. Rev. D89 (2014) 054028 [arXiv:1310.3059].
- [26] J. Gao *et al.*, Phys. Rev. D89 (2014) 033009 [arXiv:1302.6246].
- [27] M. Dowling and S. Moch, (2013), arXiv:1305.6422.
- [28] U. Langenfeld, S. Moch and P. Uwer, Phys. Rev. D80 (2009) 054009 [arXiv:0906.5273].
- [29] M. Aliev *et al.*, Comput. Phys. Commun. 182 (2011) 1034 [arXiv:1007.1327].
- [30] J.M. Campbell and R. Ellis, Nucl. Phys. Proc. Suppl. 205-206 (2010) 10 [arXiv:1007.3492].
- [31] J.M. Campbell and R.K. Ellis, (2012), arXiv:1204.1513.
- [32] P. Bärnreuther, M. Czakon and A. Mitov, Phys. Rev. Lett. 109 (2012) 132001 [arXiv:1204.5201].
- [33] M. Czakon and A. Mitov, JHEP 1212 (2012) 054 [arXiv:1207.0236].
- [34] M. Czakon and A. Mitov, JHEP 1301 (2013) 080 [arXiv:1210.6832].
- [35] M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. 110 (2013) 252004 [arXiv:1303.6254].
- [36] N. Kidonakis and B.D. Pecjak, Eur. Phys. J. C72 (2012) 2084 [arXiv:1108.6063].
- [37] M. Guzzi, K. Lipka and S. Moch, (2014), arXiv:1406.0386.
- [38] A. Broggio, A. Papanastasiou and A. Signer, (2014), arXiv:1407.2532.
- [39] ATLAS, CDF, CMS & D0 Collaborations, (2014), arXiv:1403.4427.
- [40] A.H. Hoang and I.W. Stewart, Nucl. Phys. Proc. Suppl. 185 (2008) 220 [arXiv:0808.0222].
- [41] S. Fleming *et al.*, Phys. Rev. D77 (2008) 074010 [hep-ph/0703207].
- [42] S. Fleming *et al.*, Phys. Rev. D77 (2008) 114003 [arXiv:0711.2079].
- [43] P.Z. Skands and D. Wicke, Eur. Phys. J. C52 (2007) 133 [hep-ph/0703081].
- [44] A.H. Hoang *et al.*, Phys. Rev. Lett. 101 (2008) 151602 [arXiv:0803.4214].