Non-planar Feynman diagrams and Mellin-Barnes representations with `AMBRE 3.0`

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Abstract. We introduce the Mellin-Barnes representation of general Feynman integrals and discuss their evaluation. The Mathematica package `AMBRE` has been recently extended in order to cover consistently non-planar Feynman integrals with two loops. Prospects for the near future are outlined. This write-up is an introduction to new results which have also been presented elsewhere.

1. Introduction

The evaluation of Feynman integrals is a central numerical problem of perturbative quantum field theory and is not solved in generality beyond the one-loop case. One has to consider arbitrary L-loop integrals $G(X)$ with loop momenta k_l , with E external legs with momenta p_e, and with N internal lines with masses m_i and propagators $1/D_i$,

$$G(X) = \frac{1}{(i\pi^{d/2})^L} \int \frac{d^d k_1 \ldots d^d k_L \, X(k_1, \ldots, k_L)}{D_1^{n_1} \ldots D_i^{n_i} \ldots D_N^{n_N}}, \tag{1}$$

with $d = 4 - 2\epsilon$, $D_i = q_i^2 - m_i^2 = [\sum_{l=1}^{L} c_i^l k_l + \sum_{e=1}^{E} d_i^e p_e] - m_i^2$, and some tensors $X(k_1, \ldots, k_L)$ in the loop momenta. For several reasons, one is interested in analytical solutions either:

- Analytical results are well suited for the exact cancellation of certain intermediate terms, arising either from the renormalization procedure, from regularization, or just from the organization of the calculation.
- Analytical results may stabilize the numerics.
- With analytical results, analytical continuations of representations of Feynman integrals into regions of physical interest may be performed. Typically, it is easier to determine a Feynman integral in the Euclidean region, but often it is needed in the Minkowskian.

There are quite different approaches to tackle multi-loop Feynman integrals. Without aiming at completeness, we like to mention:

- Direct evaluation of a Feynman parameter representation (see e.g. the seminal articles [1, 2], and for later improvements [3, 4, 5] and references therein)

- Solving systems of differential or difference equations (see e.g. [6, 7, 8] and refs. therein)
- Expansion by regions (see e.g. [9] and refs. therein)
- Sector decomposition (see e.g. [10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and refs. therein)
- Mellin-Barnes (MB-) representations (see e.g. [20, 21, 22, 23, 24, 25, 26] and refs. therein)

For an overview of most recent developments we recommend [27]. We are concentrating here on the Mellin-Barnes approach. Quite some work has been devoted to an automation of it, see [28, 29, 30, 31, 32, 33, 34] and software at the webpage http://projects.hepforge.org/mbtools. Nevertheless, the formalism is much less developed than e.g. that with differential equations, where algorithms have been worked out for solving them in terms of certain classes of functions. An analogue idea would be here using Cauchy's theorem in order to derive from a Mellin-Barnes representation multiple sums of residues, and then to sum up these sums analytically [35]. How this might work with the software of the RISC (Linz) group [36, 37, 38, 39, 40, 41, 42, 43, 44] around the packages `SIGMA` [45, 46], `EvaluateMultiSums`, and `SumProduction` [47, 48, 49] is indicated in [50]. For not too involved classes of functions, one may try to apply the packages `SUMMER` (http://www.nikhef.nl/∼t68/) or `XSUMMER` [51]. For automation, several problems have to be solved:

- Determine if the topology is planar or non-planar – Mathematica package `PlanarityTest` [34], see http://prac.us.edu.pl/∼gluza/ambre/planarity/.
- Construct the appropriate Mellin-Barnes representations – Mathematica package `AMBRE` [29, 30, 31, 50], see http://prac.us.edu.pl/∼gluza/ambre/. Here, also the Mathematica packages `MB` [28] and `MBresolve` [32] are integrated – for the analytic continuation of Mellin-Barnes integrals in ϵ; as well as `MBasymptotics` (2005, http://projects.hepforge.org/mbtools/) – for the parametric expansions of Mellin-Barnes integrals and `barnesroutines` by D. Kosower (2008, http://projects.hepforge.org/mbtools/) – for the automatic application of the first and second Barnes lemmas.
- Change the MB-integrals into multiple nested sums – Mathematica package `MBsums`, under development by M. Ochman (DESY) et al., see [50].
- Finally, try to perform the multiple sums analytically, see [50, 52].
- Alternatively, a purely numerical evaluation of the multi-dimensional Mellin-Barnes integral may be envisaged. Here it is wishful to cover not only Euclidean cases, but also the Minkowskian kinematics; this has not been studied so far.

Of course, a solution of the general case is not to be expected. The limitations of the method have several reasons; we mention here:

- The number of loops;
- The number of different scales due to internal masses and the kinematics;
- The number of external legs;
- A planar or non-planar topology.

A Feynman parameter integral with N internal legs has, essentially, a dimensionality $N-1$. For the corresponding MB-integral, the dimensionality will be different, and for complex problems it often will come out much higher.

2. Mellin-Barnes representations

For the Feynman integral (1) one may derive the following Feynman parameter integral:

$$G(X) = \frac{(-1)^{N_\nu}\Gamma\left(N_\nu - \frac{d}{2}L\right)}{\prod\limits_{i=1}^{N}\Gamma(n_i)} \int \prod_{j=1}^{N} dx_j \; x_j^{n_j-1} \delta(1-\sum_{i=1}^{N} x_i) \frac{U(x)^{N_\nu - d(L+1)/2}}{F(x)^{N_\nu - dL/2}} \tag{2}$$

The functions U and F are called graph or Symanzik polynomials.

The repeated application of the Mellin-Barnes representation [20]

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(\lambda+z)\Gamma(-z)\frac{B^z}{A^{\lambda+z}} \tag{3}$$

to the Symanzik polynomials U and F allows to replace the sums of monomials in their definitions (here: $A(x)$ and $B(x)$) by products of these monomials and Γ-functions and, subsequently, to perform the x-integrations. The integration path separates poles of $\Gamma[\lambda+z]$ and $\Gamma[-z]$. `AMBRE 1.1, 1.2, 2.0` perform so-called loop-by-loop integrations with $U=1$ and are efficient enough for many applications (figure 1, left). For the massless, on-shell non-planar diagram of figure 1, right, which was first solved in [24], one would have to treat:

$$\begin{aligned} U(x) &= +\,x[1]x[2]+x[1]x[4]+x[2]x[4]+x[1]x[5]+x[2]x[5]+x[2]x[6] \qquad (4)\\ &\quad +\,x[4]x[6]+x[5]x[6]+x[1]x[7]+x[4]x[7]+x[5]x[7]+x[6]x[7], \\ F(x) &= -s\;x[1]x[2]x[5]-s\;x[1]x[3]x[5]-s\;x[2]x[3]x[5]-u\;x[2]x[4]x[6] \qquad (5)\\ &\quad -\,s\;x[3]x[5]x[6]-t\;x[1]x[4]x[7]-s\;x[3]x[5]x[7]-s\;x[3]x[6]x[7]. \end{aligned}$$

Here it becomes operational to apply the Cheng-Wu theorem [53, 54] which states that (2) holds also with a modified delta function $\delta(1-\sum_{i\in\Omega}x_i)$ where Ω is an arbitrary subset of the lines $1,\ldots,N$, when the integration over the rest of the variables, i.e. for $i\notin\Omega$, is extended to the integration from 0 to ∞. With `AMBRE 3.0`, non-planar Feynman integrals with two loops may be efficiently represented by a direct approach with use of the Cheng-Wu theorem. Planar integrals are treated by the loop-by-loop ansatz with the earlier `AMBRE` versions. For the example of the *massless non-planar double box* mentioned, `AMBRE 3.0` derives without manual interaction the 4-dimensional Mellin-Barnes representation of [24]. It might look more promising to apply here instead the loop-by-loop approach, but experience shows that the dimensionality becomes, without further interventions, higher. At the other hand, for the *massive non-planar double box*, the loop-by-loop approach gives the MB-presentation derived first in [25]. We mention here shortly that 3-loop integrals may be treated in a hybrid way by a loop-by-loop approach, subsequently using `AMBRE 1.2` and `AMBRE 3.0` [50]; see the example files `MB_hybrid_3loopNP_massless.nb, MB_hybrid_3loopNP_massless.m, out_MB_hybrid_3loopNP_massless` at the webpage http://prac.us.edu.pl/∼gluza/ambre/.

Finally we would like to mention that the mathematica package `AMBRE 3.0` is released for public use at the webpage http://prac.us.edu.pl/∼gluza/ambre/. There is no source file made publicly available so far, but it may be made available on request. We consider this to be appropriate; for closer information we refer to the webpages http://fh.desy.de/projekte/-gfitter01/Gfitter01.htm (March 2013) and http://zfitter-gfitter.desy.de/ (April 2014). For a collection of experts' views on proper distribution of scientific software in basic research, we would like to refer to [55, 56], a summary of a round table discussion at ACAT 2014.

3. Summary

We scetched essential features of the Mellin-Barnes approach to Feynman integrals and its implementation in `AMBRE`. In the new version `AMBRE 3.0` the Cheng-Wu theorem is implemented and the replacement of the Symanzik polynomials by MB-integrals is performed globally. The alternative loop-by-loop approach is implemented in `AMBRE 2.0` or older versions. Which of the approaches is more appropriate for a given problem has to be investigated.

A treatment of non-planar three-loop integrals deserves the combined application of `AMBRE 1.2` and of `AMBRE 3.0` in a hybrid approach. This will be improved in the nearest future. The *analytical* summation of series of MB-residues is under study. Concerning the alternative to analytical summation, namely a *numerical* evaluation of the MB-integrals, we are restricted with `AMBRE` to the Euclidean kinematics so far. The Minkowskian numerics is on our to-do list.

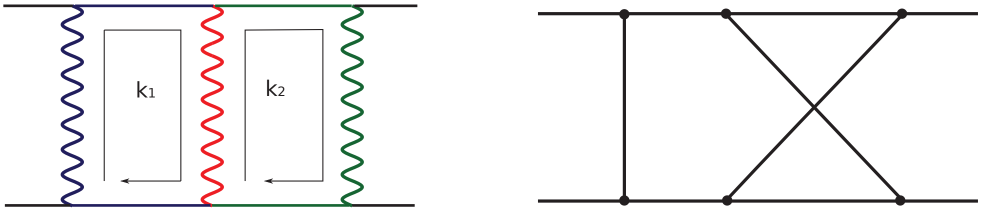

Figure 1. Left: The planar double box with loop momenta as used for the loop-by-loop representation. Right: The non-planar double box.

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