DESY 14-221, SFB/CPP-14-93, HU-EP-14/55

The hadronic vacuum polarization and automatic $\mathcal{O}(a)$ improvement for twisted mass fermions

Florian Burger,[1] Grit Hotzel,[1] Karl Jansen,[2] and Marcus Petschlies[3]

[1] *Humboldt-Universität zu Berlin, Institut für Physik, Newtonstr. 15, D-12489 Berlin, Germany*

[2] *NIC, DESY, Platanenallee 6, D-15738 Zeuthen, Germany*

[3] *The Cyprus Institute, P.O. Box 27456, 1645 Nicosia, Cyprus*

Abstract

The vacuum polarization tensor and the corresponding vacuum polarization function are the basis for calculations of numerous observables in lattice QCD. Examples are the hadronic contributions to lepton anomalous magnetic moments, the running of the electroweak and strong couplings and quark masses. Quantities which are derived from the vacuum polarization tensor often involve a summation of current correlators over all distances in position space leading thus to the appearance of short-distance terms. The mechanism of $\mathcal{O}(a)$ improvement in the presence of such short-distance terms is not directly covered by the usual arguments of on-shell improvement of the action and the operators for a given quantity. If such short-distance contributions appear, the property of $\mathcal{O}(a)$ improvement needs to be reconsidered. We discuss the effects of these short-distance terms on the vacuum polarization function for twisted mass lattice QCD and find that even in the presence of such terms automatic $\mathcal{O}(a)$ improvement is retained if the theory is tuned to maximal twist.

Keywords: lattice QCD, twisted mass fermions, hadronic vacuum polarization function, g-2, discretization effects, short-distance contributions, contact terms, Symanzik expansion

I. INTRODUCTION

The computation of hadronic contributions to observables derived from the vacuum polarization function, especially the muon anomalous magnetic moment, $a_\mu^{\rm had}$, have recently been a major target of the lattice community, see for instance [1–12]. The reason is that $a_\mu^{\rm had}$ is a prime candidate to find indications of physics beyond the standard model. One basic element to obtain the leading contribution to $a_\mu^{\rm had}$ and other quantities derived from the vacuum polarization function with good accuracy is the property of $\mathcal{O}(a)$ improvement which guarantees that physical quantities scale with a rate of $\mathcal{O}(a^2)$ towards the continuum limit.

For twisted mass fermions at maximal twist automatic $\mathcal{O}(a)$ improvement has been established for physical quantities without short-distance singularities [13] based on symmetry arguments of the lattice theory, see also [14] for a review. The hadronic vacuum polarization function in momentum space, $\Pi_{\mu\nu}(Q)$, however, also receives short-distance contributions arising from the Fourier summation of the 2-point vector current correlator $\langle J_\mu(x)\, J_\nu(y)\rangle$ for $x - y \to 0$.

Employing Symanzik's effective theory [15, 16], we show in the following that with our definition of the hadronic vacuum polarization function and at maximal twist these short-distance contributions do not spoil the automatic $\mathcal{O}(a)$-improvement of the vacuum polarization function in the twisted mass formulation of lattice QCD (tmLQCD). This finding is in accordance with a similar analysis performed for the chiral condensate and the topological susceptibility [17–19], which also involve summations over all lattice points with the corresponding appearance of short-distance contributions.

To demonstrate the $\mathcal{O}(a)$ improvement of the complete vacuum polarization function we will perform an operator product expansion (OPE) to find the operators appearing at small distances. An essential step is to also identify all operators that can mix with the ones in the OPE. This requires the investigation of the symmetry properties of all operators of equal and lower dimension. The classification of such operators up to mass dimension 6 are compiled in appendix C. This classification can also be useful to identify the renormalization pattern of other operators built from twisted mass fermions.

The paper is structured as follows. In Sect. II we state the momentum space definition of the hadronic vacuum polarization function whose short-distance contributions we investigate later on. We then briefly outline our strategy to prove automatic $\mathcal{O}(a)$ improvement in Sect. III. In Sect. IV and appendix A we discuss the position space properties of the definition given in Sect. II. Sect. V and the appendices B and C contain a list of the symmetries of the twisted mass lattice action and the corresponding classification of possible mixing operators. The Symanzik expansion constructed from these operators for the vacuum polarization function of the local vector current is presented in Sect. VI. In Sect. VII the discussion is extended to the case of the conserved vector current. Our conclusions follow in Sect. VIII.

II. DEFINITION OF THE VACUUM POLARIZATION FUNCTION

To keep the paper self-contained, we give here the expressions of the fermion actions used in our lattice calculation of the muon anomalous magnetic moment [11] in the twisted basis for a setup of active, mass-degenerate up and down and non-degenerate strange and charm quarks ($N_f = 2 + 1 + 1$). We will restrict the discussion to the valence quark sector. For details about the sea sector and the simulation setup for $N_f = 2 + 1 + 1$ twisted mass lattice QCD we refer to [20, 21].

In the valence sector we formally introduce three doublets of quarks: the light quark pair $\chi_l = (\chi_l^+, \chi_l^-) = (u, d)$, a strange quark pair $\chi_s = (\chi_s^+, \chi_s^-)$ and a charm quark pair $\chi_c = (\chi_c^+, \chi_c^-)$. The superscript sign refers to the sign of the twisted quark mass for the corresponding field in the valence Dirac operator. The action can be written concisely as a sum over standard twisted mass action terms for the fermion doublets [13],

$$S_F^{\rm val} = \sum_{q=l,s,c} \sum_x \bar{\chi}_q(x) \left[D_W + i\mu_q \gamma_5 \tau^3 \right] \chi_q(x) \,. \tag{1}$$

In the heavy sector, this procedure is based on the method introduced by Osterwalder and Seiler [22, 23]. μ_q denotes the bare twisted quark mass for flavor pair q (taken positive) and τ^3 is the third Pauli matrix acting in the flavor (sub-)space spanned by the quark doublet χ_q. Besides $S_F^{\rm val}$ we assume as usual an action term for the ghost fields corresponding to the valence sector. Contact to the physical quark content is made by identifying $(u, d) \leftrightarrow (\chi_l^+, \chi_l^-)$, $s \leftrightarrow \chi_s^-$ and $c \leftrightarrow \chi_c^+$. We thus choose these fields to initially construct the electromagnetic current operator as a Noether current resulting from the infinitesimal vector variation

$$\delta_V \chi = i\alpha(x)\, Q_{\rm em}\, \chi(x)$$
$$\delta_V \bar\chi = -i\alpha(x)\, \bar\chi(x)\, Q_{\rm em}\,,$$

with $Q_{\rm em} = \text{diag}\,(+2/3, -1/3, +2/3, 0, 0, -1/3)$ related to the electromagnetic charge matrix taking into account our choice of physical fields. This yields the point-split current operator

$$J^C_\mu = \frac{1}{2}\, \left[\bar\chi(x)\,(\gamma_\mu - r)\, U_\mu(x)\, Q_{\rm em}\, \chi(x+a\hat\mu) + \bar\chi(x+a\hat\mu)\,(\gamma_\mu + r)\, U_\mu(x)^\dagger\, Q_{\rm em}\chi(x)\right]\,. \tag{2}$$

The multiplet χ collects all flavor components of the three doublets. With this conserved current the full polarization tensor in position space is given by

$$\Pi^C_{\mu\nu}(x,y) = \langle J^C_\mu(x)\, J^C_\nu(y)\rangle - a^{-3}\,\delta_{\mu\nu}\,\delta_{xy}\,\langle S_\nu(y)\rangle\,. \tag{3}$$

Due to the usage of the point-split current operator on the lattice at the sink location y a contact term appears in Eq. (3) whose presence allows for exact current conservation at non-zero lattice spacing. The field S_ν in the contact term reads

$$S_\nu(y) = \frac{1}{2}\, \left[\bar\chi(y)\,(\gamma_\nu - r)\, U_\nu(y)\, {Q_{\rm em}}^2\, \chi(y+a\hat\nu) - \bar\chi(y+a\hat\nu)\,(\gamma_\nu + r)\, U_\nu(y)^\dagger\, {Q_{\rm em}}^2\, \chi(y)\right]\,. \tag{4}$$

Here, we will first investigate the local variant of the vector current and its correlation functions. Its interpolating field is given by the usual quark bilinear,

$$J^L_\mu(x) = \bar\chi(x)\,\gamma_\mu\, Q_{\rm em}\,\chi(x)\,. \tag{5}$$

and we define the bare polarization tensor in position space by the 2-point current correlator

$$\Pi^L_{\mu\nu}(x,y) = \langle J^L_\mu(x)\, J^L_\nu(y)\rangle\,. \tag{6}$$

In this case there is not any term analogous to $S_\nu(y)$ in (4) which could render an exact Ward identity to be satisfied. In the course of extracting a physical polarization function from the tensor in Eq. (6), the latter will have to be potentially additively and multiplicatively renormalized. This will be partly discussed later on.

The polarization tensor $\Pi_{\mu\nu}(Q)$ in momentum space at Euclidean momentum Q is obtained via the Fourier transform

$$\Pi_{\mu\nu}(Q) = a^4\, \sum_x\, {\rm e}^{iQ\cdot(x+a\hat\mu/2-y-a\hat\nu/2)}\,\Pi_{\mu\nu}(x,y) \tag{7}$$

with spacetime arguments in the Fourier phase shifted by half a lattice spacing. The polarization function $\Pi(Q)$ is derived from $\Pi_{\mu\nu}(Q)$ using the projector $P_{\mu\nu}(Q)$ on the transverse part of the tensor,

$$\begin{aligned} P_{\mu\nu}(Q) &= \hat{Q}_\mu \, \hat{Q}_\nu - \delta_{\mu\nu} \, \hat{Q}^2 \\ \Pi(Q) &= P_{\mu\nu}(Q) \, \Pi_{\mu\nu}(Q)/(3 \, (\hat{Q}^2)^2) \; . \end{aligned} \tag{8}$$

We note that this projection is non-trivial also for the polarization tensor constructed from conserved currents in Eq. (3), e.g. when stochastic methods are used and the Ward identity is not exactly fulfilled. Also the local $\Pi^L_{\mu\nu}$ will in general have a longitudinal component.

Starting from Eq. (8) we define the real and momentum-averaged polarization function

$$\Pi^{(\mathrm{av})}(\hat{Q}^2) = \mathrm{Re}\left(\frac{1}{\#\mathcal{G}(Q)} \sum_{Q' \in \mathcal{G}(Q)} \Pi(Q') \right) \; . \tag{9}$$

$\hat{Q}$ are the lattice momenta and component-wise related to Q via $\hat{Q}_\mu = 2\sin(aQ_\mu/2)/a$. $\mathcal{G}(Q)$ is the set which contains all momenta obtained from Q using discrete rotations and reflections of the 4-dimensional lattice. We include rotations mixing time and spatial coordinates, where they are possible, although our configurations feature $T = 2\,L$ for the lattice time and spatial extent T and L, respectively. Moreover, in practice we also average over momenta with the same $\hat{Q}^2$ which are only connected by a spacetime transformation in the continuum. Correspondingly, $\#\mathcal{G}(Q)$ denotes the number of elements of this set. This defines our method to extract the scalar vacuum polarization function as a function of the squared lattice 4-momentum.

Relations Eq. (14) and Eq. (15) to be given below show that it is not necessary to calculate the polarization tensor for all combinations of single quark currents as suggested by Eq. (2) and (3). It is sufficient to restrict to combinations of single quark currents with, say, plus components of the quark doublets.

In the following we restrict the discussion to the light valence quark sector. In the heavy valence sector analogous arguments are used and the latter will be covered in a more general framework in [24].

III. PROCEDURE

Many applications of the vacuum polarization function require the vacuum polarization function in momentum space $\Pi(\hat{Q}^2)$. In the following we investigate the scaling properties of the polarization tensor implied by relation (9) in position space. This will enable us to draw conclusions on $\Pi(\hat{Q}^2)$.

Given the on-shell $\mathcal{O}(a)$ improvement of the vector current correlator at physical distances in the continuum limit [13] we focus on the impact of contributions to the Fourier sum from small and zero distance. Formally, we are interested in the quantity

$$\Pi^{(\mathrm{av})}(\hat{Q}^2) = \left[P_{\mu\nu}(Q)\, a^4 \sum_{x \in V} \mathrm{e}^{iQ\cdot(x+a\hat{\mu}/2-y-a\hat{\nu}/2)}\, \Pi_{\mu\nu}(x,y) \right]^{\mathrm{av}} \tag{10}$$

for a physical 4-volume V. This $\Pi^{(\mathrm{av})}$ can be expanded in the continuum limit as

$$\Pi^{(\mathrm{av})} = \sum_{k \geq -6} C_k\, a^k\,, \tag{11}$$

such that

$$\begin{aligned} C_0 = & \frac{1}{3(Q^2)^2} P_{\mu\nu}(Q) \int_V d^4x\, \langle J_\mu(x)\, J_\nu(y) \rangle\, \mathrm{e}^{iQ(x-y)} \\ & + \text{ contributions from operators of dimension } 6\,. \end{aligned} \tag{12}$$

We will argue that $C_1 = 0$ automatically in tmLQCD at maximal twist, irrespective of the remaining C_k for $k \neq 1$. A similar statement holds for all operators sharing the symmetry class of the polarization tensor with which it can mix such that the polarization function can be obtained by subtractions without $\mathcal{O}(a)$ terms.

To that end we proceed in two steps.

1. We examine the possible mixing of the polarization tensor in position space with operators of equal and lower dimension; the occurrence of such a mixing requires the definition of a subtracted operator. The mixing pattern of operators is investigated for zero distance.

2. We use the Symanzik expansion technique with reference to the twisted mass lattice action and the subtracted operator to show that all contributions to C_1 vanish at maximal twist.

At the non-perturbative level the identification of the mixing pattern and of the terms in the Symanzik expansion relies on the symmetries of the lattice and the continuum theory. We emphasize, that in the context of tmLQCD automatic $\mathcal{O}(a)$ improvement requires the absence of $\mathcal{O}(a)$ terms irrespective of a particular choice of mixing and improvement coefficients. The only parameter ultimately assumed to be tuned is the twist angle such that maximal twist is realized. See Refs. [20, 21, 25] for details how this has been achieved for the $N_f = 2+1+1$ setup we are interested in here. For our purposes, we only need to recall that maximal twist corresponds to a vanishing bare quark mass $m_q = 0$ in the Wilson Dirac operator such that the twisted mass μ_q takes the role of the physical mass.

We do not go into detail about the vacuum expectation values of operators in the continuum but assume that a regularization can be chosen (independent of the lattice spacing) such that they can be calculated as finite quantities.

IV. SYMMETRY PROJECTIONS

Our discussion of operator mixing and the Symanzik expansion proceeds in position space, yet the position space current correlators given in Eq. (3) and (6) do not have a definite transformation behavior under the relevant symmetries of the lattice theory. Our definition of the hadronic vacuum polarization function in momentum space given in Eq. (8) on the other hand incorporates projections to spacetime as well as light, strange or charm isospin symmetry sectors. Since it is desirable to have these transformation properties in position space as well, we show that these projections in momentum space automatically imply the required properties for the correlators in position space.

a. Spacetime transformation group The momentum projector $P_{\mu\nu}(Q)$ given in Eq. (8) transforms like a rank-2 tensor. Restricting the set of momenta to a representative set we can extend the average over $\mathcal{G}(Q)$ to the complete spacetime transformation group. As outlined in appendix A we can realize this average equivalently in position space. This amounts to defining the projected polarization tensor

$$[\Pi_{\mu'\nu'}(x',y')]^{(av)} = \frac{1}{N_{\mathcal{G}}} \sum_{R\in\mathcal{G}} \Lambda(R)_{\mu}^{\mu'}\, \Lambda(R)_{\nu}^{\nu'}\, \Pi_{\mu\nu}(\Lambda(R)x', \Lambda(R)y')\,, \tag{13}$$

where $\Lambda(R)$ are the representation matrices of the lattice rotations and reflections. In this form the vacuum polarization tensor in position space exhibits the transformation behavior of a rank-2 tensor. We will leave out the brackets $[\quad]^{(\mathrm{av})}$ from position space operators and assume this exact rank-n tensor transformation behavior for all operators in the following sections.

In anticipation of the following discussion we note, that in particular we have invariance of the tensor under spacetime inversion $Q \to -Q$ or $x \to -x$. This is one of the key transformations in the discussion of automatic $\mathcal{O}(a)$ improvement. Moreover, with the definition in Eq. (13) the average over momentum orbits becomes trivial as in the continuum.

b. Isospin For $SU(2)$ isospin relations we use the flavor matrices $\tau^{\pm}$, τ^3 based on the Pauli matrices, and $\tau^0 = \mathbb{1}$. Correspondingly, with $J^{\tau} = \bar{\chi}\,\gamma_{\mu}\tau\,\chi$ we denote the isospin component of the current for any of the three doublets.

The implications of taking the real part of the polarization tensor in momentum space can be immediately seen by using the relation

$$\langle J_{\mu}^{f_1}(x)\,J_{\nu}^{f_2}(y)\rangle^{*} = \langle J_{\mu}^{\bar{f}_2}(x)\,J_{\nu}^{\bar{f}_1}(y)\rangle \tag{14}$$

of the current correlator in position space and the corresponding relation

$$\Pi_{\mu\nu}^{f_1 f_2\,*}(Q) = \Pi_{\mu\nu}^{\bar{f}_2\bar{f}_1}(-Q) \tag{15}$$

for the polarization tensor in momentum space. Here (f_1, f_2) denotes a pair of quark flavor indices and $\bar{f}_{1/2}$ is the flavor with opposite sign of the twisted mass parameter compared to flavor $f_{1/2}$.

Given the electromagnetic charge matrix we can split the electromagnetic current of the light quarks into its isospin components

$$J_l^{\mathrm{em}} = \frac{2}{3}\,J^{\mathrm{up}} - \frac{1}{3}\,J^{\mathrm{down}} = \frac{1}{6}\,J^{\tau^0} + \frac{1}{2}\,J^{\tau^3}\,. \tag{16}$$

Hence, we only need the components with flavor structure τ^0 and τ^3. Using the relation (14) the correlator of two such isospin currents $J^{a,b} = J^{f} + \sigma_{a,b}\,J^{\bar{f}}$ with $\sigma_{a,b} \in \{\pm 1\}$ in momentum space can be decomposed according to

$$\begin{aligned}
\Pi^{ab}_{\mu\nu}(Q) &= \langle J^a J^b \rangle \\
&= \Pi^{ff}_{\mu\nu}(Q) + \sigma_b\,\Pi^{f\bar f}_{\mu\nu}(Q) + \sigma_a\,\Pi^{\bar f f}_{\mu\nu}(Q) + \sigma_a\sigma_b\,\Pi^{\bar f\bar f}_{\mu\nu}(Q) \\
&= \Pi^{ff}_{\mu\nu}(Q) + \sigma_b\,\Pi^{f\bar f}_{\mu\nu}(Q) + \sigma_a\,\Pi^{f\bar f\,*}_{\mu\nu}(-Q) + \sigma_a\sigma_b\,\Pi^{ff\,*}_{\mu\nu}(-Q) \\
&\xrightarrow{[\quad]^{(\mathrm{av})}} 2\,\mathrm{Re}\left(\left[\Pi^{ff}_{\mu\nu}(Q)\right]^{(\mathrm{av})}\right)(1+\sigma_a\sigma_b) + 2\,\mathrm{Re}\left(\left[\Pi^{f\bar f}_{\mu\nu}(Q)\right]^{(\mathrm{av})}\right)(\sigma_a+\sigma_b) \\
&\qquad + 2i\,\mathrm{Im}\left(\left[\Pi^{ff}_{\mu\nu}(Q)\right]^{(\mathrm{av})}\right)(1-\sigma_a\sigma_b) + 2i\,\mathrm{Im}\left(\left[\Pi^{f\bar f}_{\mu\nu}(Q)\right]^{(\mathrm{av})}\right)(-\sigma_a+\sigma_b)\;. \qquad (17)
\end{aligned}$$

As before, $[\quad]^{(\mathrm{av})}$ denotes the average over equivalent momenta, in particular averaging over Q and $-Q$. From Eq. (17) we find that the contributions from the current-current correlator with equal isospin components for both currents are purely real ($\sigma_a = \sigma_b$), whereas the mixed isospin combinations are purely imaginary ($\sigma_a = -\sigma_b$). These latter contributions are isospin symmetry breaking lattice artefacts in tmLQCD as can be checked by symmetry arguments along the lines of the following sections. Retaining only the real part of the averaged momentum space correlator removes these terms explicitly. We thus only need to consider the correlators $\langle J^\tau\, J^\tau \rangle$ with $\tau \in \{1, \tau^3\}$.

Knowing that we only need to consider correlators of same isospin, we can infer, that in position space we always get correlators for flavor pairs $(f_1,\, f_2)$, which are symmetrized in the indices $(1,2)$ and the bar operation. These combinations, too, are manifestly real.

Finally, the operator in the contact term Eq. (4) contains the squared electromagnetic charge matrix. Thus, it also consists of two isospin components given by τ^0 and τ^3. Again the isospin component τ^3 is purely imaginary whereas the component with τ^0 is purely real. Thus, for the contact term we limit our considerations to the component with $\tau^0 = \mathbb{1}$.

Having shown that the definitions in momentum space imply position space operators with equivalent transformation behavior under the symmetries of the lattice theory enables us to study their mixing and the application of the Symanzik program.

V. MIXING OF THE POLARIZATION TENSOR

We start our considerations with the local vector current correlator, which is symmetry projected as described in the previous section. When renormalizing the correlator it will in general mix with operators of equal and lower dimension which have otherwise the same transformation properties. Moreover, in the Fourier sum we include the contribution where the spacetime points in the 2-point correlator coincide. These points give rise to expectation values of four-quark operators and need to be handled separately. They are accounted for by allowing additional contributions of contact terms, again of equal and lower dimension and with same transformation properties.

The polarization tensor in position space is of mass dimension 6. We thus write a subtracted polarization tensor in position space as

$$\begin{aligned}\left[J^\tau_\mu(x)\, J^\tau_\nu(y)\right]_{\text{sub}} = &\sum_{k=0}^{6}\sum_{i\geq 0}\frac{Z^{(0)}_{ki}}{a^{6-k}}\, O_{ki\,\mu\nu}(x,y) + a^{-4}\,\delta_{xy}\sum_{k=0}^{6}\sum_{i\geq 0}\frac{Z^{(1)}_{ki}}{a^{2-k}}\, B^{(1)}_{ki\,\mu\nu}(y)\\ &+ a^{-4}\,\bar\partial^{(x)}_\mu\,\delta_{xy}\sum_{k=0}^{6}\sum_{i\geq 0}\frac{Z^{(2)}_{ki}}{a^{1-k}}\, B^{(2)}_{ki\,\nu}(y)\\ &+ a^{-4}\,\bar\partial^{(x)}_\kappa\,\bar\partial^{(x)}_\lambda\,\delta_{xy}\sum_{k=0}^{6}\sum_{i\geq 0}\frac{Z^{(3)}_{ki}}{a^{-k}}\, B^{(3)}_{ki\,\mu\nu\kappa\lambda}(y)\\ &+ \ldots\,. \end{aligned} \tag{18}$$

With index k we label the dimension of the operators and index i runs over the possible operators within each dimension. As a lattice version of the Dirac δ function we use $a^{-4}\,\delta_{xy} \xrightarrow{a\to 0} \delta(x-y)$. The parity-odd first lattice derivative $\bar\partial_\mu$ is given by $\left(\partial^f_\mu + \partial^b_\mu\right)/2$ with ∂^f_μ and ∂^b_μ being the lattice forward and backward partial derivatives, respectively. For definiteness we can set $O_{ki\,\mu\nu} = J^\tau_\mu\, J^\tau_\nu$ for $k=6$, $i=0$.

When enumerating the operators O_{ki}, $B^{(n)}_{ki}$, we keep explicit factors of Wilson and twisted quark mass, m_q and μ_q, respectively, as well as of the dimensionless Wilson parameter r at zeroth and first power. With the parametrization in Eq. (18), i.e. the explicit factoring out of powers of the lattice spacing and of quark masses, the dimensionless coefficients $Z^{(n)}_{ki}$ do not have a power dependence on the lattice spacing [26, 27]. The detailed form of these factors would be fixed by a proper set of renormalization conditions. We will not formulate such conditions, but stay on the level of a general subtracted operator. This is sufficient for our purposes, since we are primarily interested in the transformation properties of the operators with the attributed coefficients.

Taking the Fourier transform of Eq. (18), the contributions from the operators $B^{(1)}$ are momentum independent, while those from $B^{(2)}$ and $B^{(3)}$ generate terms that depend on the external momentum. For $B^{(2)}$ there are no operators to give rise to $\mathcal{O}(a)$ terms.

The general notation for $B^{(3)}_{ki\,\mu\nu\kappa\lambda}$ is meant to include various Lorentz structures, $B^{(3)}_{ki\,\mu\nu\kappa\lambda} \propto B^{(3)}_{ki}\,\delta_{\mu\nu}\,\delta_{\kappa\lambda}$, $B^{(3)}_{ki}\,\delta_{\mu\kappa}\,\delta_{\nu\lambda}$, $B^{(3)}_{ki\,\mu\kappa}\,\delta_{\nu\lambda}$, etc.. The sets of operators for the $B^{(n)}$ that can mix with the polarization tensor via short-distance contributions can be constructed from the mass parameters, the Wilson parameter r, quark bilinears and products of those as well as the lattice covariant derivative and the lattice field strength tensor $C_{\mu\nu}$ as known e.g. from the Sheikholeslami-Wohlert term [28]. The set is restricted by the symmetries of the lattice theory. For twisted mass lattice QCD we use the following list of symmetry transformations,

- twisted time reversal
- twisted parity
- charge conjugation
- $\mathcal{P} \times \mathcal{D} \times [m_0 \to -m_0] \times [r \to -r]$
- $\mathcal{R}_5^{1/2} \times \mathcal{D} \times [\mu_q \to -\mu_q]$

The details of these transformations are described in [14, 29] and for completeness a brief listing is given in appendix B.

To investigate the mixing pattern for $\Pi^L_{\mu\nu}(x,y)$ obtained from the correlator of two local vector currents in the continuum limit, we distinguish the two cases $x = y$ and $x \neq y$ for the spacetime arguments in the Fourier sum Eq. (7).

$$\begin{aligned}\Pi^L_{\mu\nu}(Q) &= a^4 \sum_{x\neq y} \langle \left[J^L_\mu(x)\right]_R \, \left[J^L_\nu(y)\right]_R\rangle \, \mathrm{e}^{iQ(x-y)} + a^4 \, \langle \left[J^L_\mu(y)\, J^L_\nu(y)\right]_R\rangle \\ &= \Pi^{(2)}_{\mu\nu}(Q) + \Pi^{(4)}_{\mu\nu}\,, \end{aligned} \tag{19}$$

where $[\quad]_R$ denotes a given renormalization scheme. The two terms in Eq. (19) have to be considered individually due to their different behavior under renormalization in the continuum limit.

a. $\quad x = y \quad$ $\Pi^{(4)}_{\mu\nu}$ is the lattice vacuum expectation value of a four-quark operator of mass dimension 6. We recall, that τ is either τ^0 or τ^3. Problematic terms can also arise through singularities in the limit $x \to y$ when performing the continuum limit. In the continuum these terms can be identified by expanding the operator product to have the form of a ratio $\langle O^{(k)}(y)\rangle/||x-y||^k$ of a condensate over a power of the distance $||x-y||^k$ with k a positive integer (up to logarithms). Here difficulties are rooted in the integration in the Fourier transform over a region of extension of the lattice spacing around y.

We capture these short-distance contributions by subtracting from the current-current correlator in position space all possible local operators of equal and lower dimension, which are allowed to appear constrained by the lattice symmetries. This involves contributions in the form of the $B_{ki}^{(n)}$ given in Eq. (18). The candidate mixing operators $B_{ki}^{(n)}$ have been separated into those that include and do not include covariant derivatives. They are listed in tables {I}, {II}, {III} and {IV} in appendix C.

b. $x \neq y$ $\Pi^{(2)}_{\mu\nu}(Q)$ is composed of a product of two vector currents in position space at non-zero distance $x \neq y$. This makes the situation rather definite here. For this operator there is neither mixing nor additive renormalization. The local current operators are normalized multiplicatively with a factor Z_V, which can be determined non-perturbatively [30] in a lattice calculation. Thus

$$\left[J^L_\mu(x)\right]_R = Z_V\, J^L_\mu(x) \tag{20}$$

$$\left[\Pi^{(2)}_{\mu\nu}\right]_R = \sum_{x \neq y} \left[J^L_\mu(x)\right]_R \left[J^L_\nu(y)\right]_R \,. \tag{21}$$

In the language of Eq. (18) we have $Z_{ki} \neq 0$ only for $(k = 6,\, i = 0)$ and zero else. For automatic $\mathcal{O}(a)$ improvement of the latter correlator for physical distances $x \neq y$ the on-shell improvement conditions are sufficient within tmLQCD at maximal twist [13].

VI. SYMANZIK EXPANSION FOR THE LOCAL CASE

The operators allowed in the mixing pattern when using the local light quark current $J^L_\mu(x)$ are listed in tables {I}, {II}, {III} and {IV} in appendix C. According to this collection the subtracted operator reads

$$\begin{aligned}
\left[J^\tau_\mu(x)\, J^\tau_\nu(y)\right]_{\rm sub} &= J^\tau_\mu(x)\, J^\tau_\nu(y) + \frac{Z^1}{a^6}\,\delta_{\mu\nu}\,\delta_{xy} + \frac{Z^{rm}\, rm_q}{a^5}\,\delta_{\mu\nu}\,\delta_{xy} + \frac{Z^{m^2}\, m_q^2 + Z^{\mu^2}\, \mu_q^2}{a^4}\,\delta_{\mu\nu}\,\delta_{xy} \\
&+ \frac{Z^{r\bar\chi\chi}}{a^3}\, r\,\bar\chi\chi\,\delta_{\mu\nu}\,\delta_{xy} + \frac{Z^{rm^3}\, rm_q^3}{a^3}\,\delta_{\mu\nu}\,\delta_{xy} \\
&+ \frac{1}{a^4}\left(Z^{Q^2}\,\delta_{\mu\nu}\,\bar\partial^2 + Z^{QQ}\,\bar\partial_\mu\,\bar\partial_\nu\right)\delta_{xy} + \frac{rm_q}{a^3}\left(Z^{rmQ^2}\,\delta_{\mu\nu}\,\bar\partial^2 + Z^{rmQQ}\,\bar\partial_\mu\,\bar\partial_\nu\right)\delta_{xy} \\
&+ \text{ operators of dimension } \geq 4\,.
\end{aligned} \tag{22}$$

The expansion of the lattice action close to the continuum limit follows from the local effective action

$$S_{eff} = S_4 + aS_5 + a^2 S_6 + a^3 S_7 + \dots\,, \tag{23}$$

where $S_k \equiv \int \mathcal{L}_k \, d^4x$ and the terms $\mathcal{L}_k$ contain linear combinations of fields with mass dimension k. We expand its exponential up to $\mathcal{O}(a^3)$. The corrections to the gauge field Lagrangian in the continuum limit start with $\mathcal{O}\left(a^2\right)$ and in fact contain only even powers of the lattice spacing [31]. We thus concentrate on the corrections to the fermion action. The operators that can appear in $\mathcal{L}_5$ and $\mathcal{L}_6$ have been listed in Refs. [26, 28].

From the expansion of the operator Eq. (22) and $\exp\left(-S_{eff}\right)$ in Eq. (23) the full Symanzik expansion in momentum space is obtained and reads

$$\begin{aligned}
\Pi^{\tau}_{\mu\nu}(Q) = {} & a^4 \, \langle J^{\tau}_{\mu}(y)\, J^{\tau}_{\nu}(y) \rangle_0 + \frac{\tilde{Z}^1}{a^2}\, \delta_{\mu\nu} \\
& + \frac{\tilde{Z}^1}{a}\, \langle -S_5 \rangle_0 \, \delta_{\mu\nu} + \frac{\tilde{Z}^{rm}\, rm_q}{a}\, \delta_{\mu\nu} \\
& + \tilde{Z}^1 \, \langle -S_6 + \frac{1}{2} S_5^2 \rangle_0 \, \delta_{\mu\nu} + \tilde{Z}^{rm} \, \langle -rm_q \, S_5 \rangle_0 \, \delta_{\mu\nu} + \left(\tilde{Z}^{m^2}\, m_q^2 + \tilde{Z}^{\mu^2}\, \mu_q^2 \right) \delta_{\mu\nu} \\
& + a\, \tilde{Z}^1 \, \langle -S_7 + S_5 S_6 - \frac{1}{6} S_5^3 \rangle_0 \, \delta_{\mu\nu} + a \, \langle \left(\tilde{Z}^{rm}\, rm_q \right) \left(-S_6 + \frac{1}{2} S_5^2 \right) \rangle_0 \, \delta_{\mu\nu} \\
& + a \, \langle - \left(\tilde{Z}^{m^2}\, m_q^2 + \tilde{Z}^{\mu^2}\, \mu_q^2 \right) S_5 \rangle_0 \, \delta_{\mu\nu} + a\, \tilde{Z}^{r\bar{\chi}\chi} \, \langle r\, \bar{\chi}\chi \rangle_0 \, \delta_{\mu\nu} + a\, \tilde{Z}^{rm^3}\, r\, m_q^3 \, \delta_{\mu\nu} \\
& + \left(\tilde{Z}^{Q^2}\, \delta_{\mu\nu}\, \hat{Q}^2 + \tilde{Z}^{QQ}\, \hat{Q}_{\mu}\, \hat{Q}_{\nu} \right) + arm_q \left(\tilde{Z}^{rmQ^2}\, \delta_{\mu\nu}\, \hat{Q}^2 + \tilde{Z}^{rmQQ}\, \hat{Q}_{\mu}\, \hat{Q}_{\nu} \right) \\
& + \left\{ \mathcal{O}\left(a^2\right) \,,\ \text{operators of higher dimension} \right\} .
\end{aligned} \tag{24}$$

Since we are working at maximal twist $m_q \to 0$, we may drop all terms involving the untwisted quark mass. Using the $\mathcal{R}_5^{1/2}$-symmetry [14] we see that the vacuum expectation values of S_5, $S_5 S_6$ as well as of $\mu_q^2 S_5$ and $\bar{\chi}\chi$ vanish as these merely contain $\mathcal{R}_5^{1/2}$-odd operators. Similarly all terms in S_7 disappear by either the $\mathcal{R}_5^{1/2}$- or the $\mathcal{P} \times [\mu_q \to -\mu_q]$ symmetry as is demonstrated in appendix D.

We may then conclude that at maximal twist there are no $\mathcal{O}(a)$ lattice artefacts stemming from the contributions in Eq. (24) to $\Pi^{\tau}_{\mu\nu}$, whose Symanzik expansion we write again for this case,

$$\begin{aligned}
\Pi^{\tau}_{\mu\nu}(Q) = {} & a^4 \, \langle J^{\tau}_{\mu}(y)\, J^{\tau}_{\nu}(y) \rangle_0 + \frac{\tilde{Z}^1}{a^2}\, \delta_{\mu\nu} + \tilde{Z}^1 \, \langle -S_6 + \frac{1}{2} S_5^2 \rangle_0 \, \delta_{\mu\nu} + \left(\tilde{Z}^{\mu^2}\, \mu_q^2 \right) \delta_{\mu\nu} \\
& + \left(\tilde{Z}^{Q^2}\, \delta_{\mu\nu}\, Q^2 + \tilde{Z}^{QQ}\, Q_{\mu}\, Q_{\nu} \right) + \left\{ \mathcal{O}\left(a^2\right) \,,\ \text{operators of higher dimension} \right\} .
\end{aligned} \tag{25}$$

VII. APPLICATION TO THE CONSERVED CURRENT CORRELATOR

The conserved current can be written in the following form

$$J_\mu^C(x) = J_\mu^L(x) + \frac{a}{2}\left[\bar\chi\,\gamma_\mu\,\tau\,\left(\overrightarrow{\nabla}_\mu^f + \overleftarrow{\nabla}_\mu^f\right)\,\chi\right](x) - \frac{ar}{2}\left[\bar\chi\,\tau\,\left(\overrightarrow{\nabla}_\mu^f - \overleftarrow{\nabla}_\mu^f\right)\,\chi\right](x)\,, \qquad (26)$$

where $\overrightarrow{\nabla}_\mu^f$ is the covariant lattice derivative acting to the right. It is a sum of the local current operator and two local operators of mass dimension 4. This constitutes already the beginning of its Symanzik expansion. Similarly, for the lattice contact term we have

$$S_\mu^\tau(x) = \delta_{xy}\,\delta_{\mu\nu}\,S_\nu^\tau(y)$$
$$S_\nu^\tau(y) = \frac{a}{2}\left[\bar\chi\,\tau\,\gamma_\nu\,\left(\overrightarrow{\nabla}_\nu^f - \overleftarrow{\nabla}_\nu^f\right)\,\chi\right](y) - \frac{ar}{2}\left[\bar\chi\,\tau\,\left(\overrightarrow{\nabla}_\nu^f + \overleftarrow{\nabla}_\nu^f\right)\,\chi\right](y) - r\,\bar\chi\,\tau\,\chi(y)\,. \qquad (27)$$

Hence, both the conserved current as well as the lattice contact term are a sum of local quark-bilinear operators for which we can use the Symanzik expansion.

Having written the conserved current as the local current plus two operators containing derivatives that are of dimension 4 implies that there is no principle alteration of the mixing with lower dimensional operators for $\langle J_\mu^C\,J_\nu^C\rangle$ compared to the local case, since $J_\mu^C\,J_\nu^C$ can be expressed as a sum of the product of local currents and additional terms of dimension 7 and 8. Moreover, for the short-distance part of the vacuum polarization tensor formed from the conserved current the appearance of mixing operators is further constrained by the vector Ward identity. Thus, the considerations for the occurrence of $\mathcal{O}\,(a)$ terms are basically the same as for local case.

The only addition is the lattice contact term where we have $r\bar\chi\,\tau\,\chi$. As stated earlier, due to the symmetry projections discussed in Sect. IV, $\bar\chi\,\tau\,\chi$ with $\tau = \tau^3$ is excluded and only $\tau = 1$ needs to be considered. At maximal twist, when $\mathcal{R}_5^{1/2}$ is a symmetry of the continuum theory, this term will vanish, since it is odd under $\mathcal{R}_5^{1/2}$.

Combining the above arguments, the hadronic vacuum polarization function formed from the conserved vector current according to Eq. (3) and Eq. (8) exhibits no $\mathcal{O}(a)$ contributions.

VIII. CONCLUSIONS

A crucial element in obtaining accurate results from lattice QCD calculations is the suppression of lattice spacing artefacts and a controlled approach towards the continuum limit. The lattice community has therefore developed a number of actions and improved operators that guarantee that physical quantities scale with a rate of $\mathcal{O}\left(a^2\right)$ to the continuum limit.

One particular lattice QCD formulation, which we have investigated here, is the twisted mass formulation [13, 23, 32, 33]. When tuning the twisted mass lattice action to maximal twist physical quantities are automatically $\mathcal{O}\left(a\right)$ improved [13]. Indeed, in numerical computations with two dynamical quarks the $\mathcal{O}\left(a^2\right)$ scaling of many physical quantities could be demonstrated [34–36] showing also that these remaining $\mathcal{O}\left(a^2\right)$ lattice artefacts are often very small as can be deduced from [25].

However, the arguments that lead to $\mathcal{O}\left(a\right)$ improvement for twisted mass fermions do not immediately cover quantities that involve summations over all lattice points leading thus to short-distance contributions.

Here, we have examined the behavior of the hadronic vacuum polarization function which serves as a most important basic quantity to compute hadronic contributions to electroweak observables, quark masses and also the strong coupling constant. In order to see whether short-distance contributions affect the rate of the continuum limit scaling, we have constructed the Symanzik expansion for these short-distance contributions.

We have found that when the theory is tuned to maximal twist, automatic O(a) improvement prevails for the complete vacuum polarization function provided that is is defined as eigenstate of the symmetry transformations of the lattice action. Thus, continuum limit extrapolations of our lattice results can safely be performed employing fit functions without linear terms in the lattice spacing as has been done in [11]. In the course of this work, we have established the classification of the twisted mass symmetry properties of operators up to dimension 6, see appendix C for a complete list.

In this paper, we have concentrated on the twisted mass formulation of lattice QCD. However, it would be important to extend the analysis to other lattice formulations of QCD to ensure that the short-distance contributions do not spoil the desired $\mathcal{O}\left(a\right)$ improvement of the corresponding vacuum polarization function.

Another extension of the present work, which however goes substantially beyond the scope of this paper, is a potentially generalized analysis of short-distance contributions to a larger class of operators in twisted mass lattice QCD, which is currently under investigation [24].

Acknowledgments

We are most grateful to K. Cichy and E. Garcia Ramos for very constructive discussions on the $\mathcal{O}(a)$ improvement of quantities which involve short-distance contributions. Special thanks goes to the referee of [11] for raising the question about the $\mathcal{O}(a)$ improvement of the vacuum polarization function and thus motivating this work. This work has been supported in part by the DFG Corroborative Research Center SFB/TR9. G.H. gratefully acknowledges the support of the German Academic National Foundation (Studienstiftung des deutschen Volkes e.V.) and of the DFG-funded Graduate School GK 1504. K.J. was supported in part by the Cyprus Research Promotion Foundation under contract ΠΡΟΣΕΛΚΥΣΗ/ΕΜΠΕΙΡΟΣ/0311/16.

Appendix A: Spacetime symmetry projections in position space

The momentum projector $P_{\mu\nu}(Q)$ given in (8) transforms like a rank-2-tensor, that is for any discrete spacetime transformation Λ we have

$$P_{\mu\nu}(\Lambda Q) = \Lambda_\mu^{\mu'}\, \Lambda_\nu^{\nu'}\, P_{\mu'\nu'}(Q)\,.$$

Λ denotes a representation of the essentially hypercubic lattice symmetry group. We can restrict the set of momenta to a representative set and translate the average over $\mathcal{G}(Q)$ to position space. Moreover, instead of averaging over $\mathcal{G}(Q)$ for a specific momentum Q we can average over the complete spacetime transformation group $\mathcal{G}$ [1]and define

$$\begin{aligned}
[\Pi(Q)]^{(\mathrm{av})} &= \frac{1}{N_{\mathcal{G}(Q)}} \sum_{Q\in\mathcal{G}(Q)} P_{\mu\nu}(Q)\, a^4 \sum_x \Pi_{\mu\nu}(x,y)\, \mathrm{e}^{iQ(x+a\hat{\mu}/2-y-a\hat{\nu}/2)} \\
&= \frac{1}{N_{\mathcal{G}}} \sum_{\Lambda\in\mathcal{G}} P_{\mu\nu}(\Lambda Q_{\mathrm{fix}})\, a^4 \sum_x \Pi_{\mu\nu}(x,y)\, \mathrm{e}^{i(\Lambda Q_{\mathrm{fix}})(x+a\hat{\mu}/2-y-a\hat{\nu}/2)} \\
&= P_{\mu'\nu'}(Q_{\mathrm{fix}})\, a^4 \sum_x \frac{1}{N_{\mathcal{G}}} \sum_{\Lambda\in\mathcal{G}} \Lambda_\mu^{\mu'}\, \Lambda_\nu^{\nu'}\, \Pi_{\mu\nu}(x,y)\, \mathrm{e}^{iQ_{\mathrm{fix}}\,\Lambda^{-1}(x+a\hat{\mu}/2-y-a\hat{\nu}/2)} \qquad \text{(A1)}
\end{aligned}$$

[1] For any momentum Q the number of elements $N_{\mathcal{G}(Q)}$ divides the number of elements in the whole group $N_{\mathcal{G}}$.

where Q_{fix} is some fixed reference momentum. We can rewrite the transformed spacetime argument in the Fourier phase in Eq. (A1) as

$$\begin{aligned}\Lambda^{-1}(x + a\hat{\mu}/2) &= x' + a\hat{\mu}'/2 \\ \mu' &= \sigma_\Lambda(\mu) \\ x' &= \begin{cases} \Lambda^{-1}x & \mu - \text{direction not reflected} \\ \Lambda^{-1}(x + a\hat{\mu}) & \mu - \text{direction reflected} \end{cases} ,\end{aligned} \tag{A2}$$

where σ_Λ is the permutation generated by Λ. Hence, we obtain

$$\begin{aligned}[\Pi(Q)]^{(\mathrm{av})} &= \frac{1}{3(Q^2)^2} P_{\mu'\nu'}(Q_{\text{fix}})\, a^4 \sum_{x'} \frac{1}{N_{\mathcal{G}}} \sum_{\Lambda \in \mathcal{G}} \Lambda_\mu^{\mu'}\, \Lambda_\nu^{\nu'}\, \Pi_{\mu\nu}(\Lambda x', \Lambda y')\, \mathrm{e}^{iQ_{\text{fix}}\,(x' + a\hat{\mu}'/2 - y' - a\hat{\nu}'/2)} \\ &= \frac{1}{3(Q^2)^2} P_{\mu'\nu'}(Q_{\text{fix}})\, a^4 \sum_{x'} \left[\Pi_{\mu'\nu'}(x', y')\right]^{(av)} \mathrm{e}^{iQ_{\text{fix}}\,(x' + a\hat{\mu}'/2 - y' - a\hat{\nu}'/2)} \,.\end{aligned} \tag{A3}$$

By construction the operator

$$\left[\Pi_{\mu'\nu'}(x', y')\right]^{(av)} = \frac{1}{N_{\mathcal{G}}} \sum_{\Lambda \in \mathcal{G}} \Lambda_\mu^{\mu'}\, \Lambda_\nu^{\nu'}\, \Pi_{\mu\nu}(\Lambda x', \Lambda y') \tag{A4}$$

has the same transformation behavior as the projector $P_{\mu\nu}$; it transforms like a true rank-2 tensor in position space and the trace of the tensor, $\sum_{\mu'} \left[\Pi_{\mu'\mu'}(x', y')\right]^{(av)}$, is a scalar.

Appendix B: Symmetry transformations

$$
\begin{array}{lll}
\mathcal{T}_{1/2}: & & x \to Tx = (-x_0,\, \vec{x}) \\
& & \chi(x) \to i\tau^{1/2}\,\gamma_0\,\gamma_5\,\chi(Tx) \\
& & \bar{\chi}(x) \to -i\bar{\chi}(Tx)\,\tau^{1/2}\,\gamma_5\,\gamma_0 \\
& & U_0(x) \to U_0(Tx - a\hat{0})^\dagger\,,\; U_i(x) \to U_i(Tx) \\
\mathcal{T} \times [\mu_q \to -\mu_q]: & & \\
& \text{with } \mathcal{T}: & x \to Tx = (-x_0,\, \vec{x}) \\
& & \chi(x) \to i\,\gamma_0\,\gamma_5\,\chi(Tx) \\
& & \bar{\chi}(x) \to -i\bar{\chi}(Tx)\,\gamma_5\,\gamma_0 \\
& & U_0(x) \to U_0(Tx - a\hat{0})^\dagger\,,\; U_i(x) \to U_i(Tx) \\
\hline
\mathcal{P}_{1/2}: & & x \to Px = (x_0,\, -\vec{x}) \\
& & \chi(x) \to i\tau^{1/2}\,\gamma_0\,\chi(Px) \\
& & \bar{\chi}(x) \to -i\bar{\chi}(Px)\,\tau^{1/2}\,\gamma_0 \\
& & U_0(x) \to U_0(Px)\,,\; U_i(x) \to U_i(Px - a\hat{i})^\dagger \\
\mathcal{P} \times [\mu_q \to -\mu_q]: & & \\
& \text{with } \mathcal{P}: & x \to Px = (x_0,\, -\vec{x}) \\
& & \chi(x) \to i\,\gamma_0\,\chi(Px) \\
& & \bar{\chi}(x) \to -i\bar{\chi}(Px)\,\gamma_0 \\
& & U_0(x) \to U_0(Px)\,,\; U_i(x) \to U_i(Px - a\hat{i})^\dagger \\
\hline
\mathcal{C}: & & \chi(x) \to C^{-1}\,\bar{\chi}(x)^T \\
& & \bar{\chi}(x) \to -\chi(x)^T\,C \\
& & U_\mu(x) \to U_\mu(x)^* \\
& \text{with} & C = i\gamma_0\gamma_2 \text{ in representation of [14]} \\
\hline
\end{array}
\tag{B1}
$$

$\mathcal{P} \times \mathcal{D} \times [m_0 \to -m_0] \times [r \to -r]$:

	with $\mathcal{D}$:	$U_\mu(x) \to U_\mu(-x - a\hat{\mu})^\dagger$
		$\chi(x) \to -i\,\chi(-x)$
		$\bar{\chi}(x) \to -i\,\bar{\chi}(-x)$
$\mathcal{R}_5^{1/2} \times \mathcal{D} \times [\mu_q \to -\mu_q]$:		
	with $\mathcal{R}_5^{1/2}$:	$\chi(x) \to i\,\gamma_5\,\tau^{1/2}\,\chi(x)$
		$\bar{\chi}(x) \to i\,\bar{\chi}(x)\,\gamma_5\,\tau^{1/2}$

Appendix C: Operator listings

The relevant lattice operators which potentially mix with $\Pi_{\mu\nu}$ at short distances are listed in the following tables {I}, {II}, {III} and {IV}. The first one contains operators not involving derivatives whereas the second accommodates the derivative operators. We note that for obtaining a complete set of operators for any operator $O_{\mu\nu}$ appearing in the tables the diagonal part $\delta_{\mu\nu}\,O_{\mu\mu}$ (without summation over μ) and the trace $\delta_{\mu\nu}\,O_{\lambda\lambda}$ must be included separately. Since these have the same quantum numbers as $O_{\mu\nu}$ given in the table (with $I_{\mu\mu} = 1$), we do not repeat those quantum numbers.

Furthermore, to save space the common prefactor $r^k\,m_q^{n_m}\,\mu_q^{n_\mu}$ ($k \in \{0, 1\}$, n_m, $n_\mu \in \mathbb{N}_0$), which is essential for counting the dimension of the operator, is omitted for all but the first operator. Its quantum numbers can be inferred from the first line of each table and have to be multiplied with the quantum numbers in the respective column. For the reader's convenience we have added an expanded list of non-derivative operators, which contain the operators up to dimension three relevant for the discussion of $\mathcal{O}(a)$ improvement, as an ancillary file to the arXiv submission.

The powers of τ^3 and γ_5 appearing in fermion bilinears such as

$$r^k\,m_q^{n_m}\,\mu_q^{n_\mu}\bar{\chi}(\tau^3)^m(\gamma_5)^l\Gamma\chi$$

with $\Gamma \in \{1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5\}$ and four-quark operators

$$r^k\,m_q^{n_m}\,\mu_q^{n_\mu}\bar{\chi}(\tau^3)^m(\gamma_5)^l\Gamma\chi\bar{\chi}(\tau^3)^{m'}(\gamma_5)^{l'}\Gamma\chi$$

can take the values m, m', l, $l' \in \{0, 1\}$.

operator	$\mathcal{P}_{1/2}$	$\mathcal{P}\,[-\mu]$	$\mathcal{T}_{1/2}$	$\mathcal{T}\,[-\mu]$
$\delta_{\mu\nu}\, r^k\, m^{n_m}\, \mu^{n_\mu}$	$I_{\mu\nu}$	$(-)^{n_\mu}\, I_{\mu\nu}$	$I_{\mu\nu}$	$(-)^{n_\mu}\, I_{\mu\nu}$
$\delta_{\mu\nu}\, \bar\chi\, (\tau^3)^m\, \gamma_5^l\, \chi(x)$	$(-)^{m+l}\, I_{\mu\nu}$	$(-)^l\, I_{\mu\nu}$	$(-)^{m+l}\, I_{\mu\nu}$	$(-)^l\, I_{\mu\nu}$
$\mathrm{tr}\,[C_{\mu\lambda}\, C_{\lambda\nu}(x)]$	$I_{\mu\nu}$	$I_{\mu\nu}$	$I_{\mu\nu}$	$I_{\mu\nu}$
$\mathrm{tr}\left[C_{\mu\lambda}\, \tilde C_{\lambda\nu}(x)\right] + [\mu \leftrightarrow \nu]$	$-I_{\mu\nu}$	$-I_{\mu\nu}$	$-I_{\mu\nu}$	$-I_{\mu\nu}$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, C_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+m}\, I_{\mu\nu}$	$(-)^l\, I_{\mu\nu}$	$(-)^{l+m}\, I_{\mu\nu}$	$(-)^l\, I_{\mu\nu}$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, \tilde C_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+m+1}\, I_{\mu\nu}$	$(-)^{l+1}\, I_{\mu\nu}$	$(-)^{l+m+1}\, I_{\mu\nu}$	$(-)^{l+1}\, I_{\mu\nu}$
$\delta_{\mu\nu}\, \bar\chi\, (\tau^3)^m\, \gamma_5^l\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \chi(x)$	$(-)^{l+l'+m+m'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'+m+m'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \gamma_\mu\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \gamma_\nu\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+l'+m+m'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'+m+m'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+l'+m+m'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'+m+m'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$
$\delta_{\mu\nu}\, \bar\chi\, (\tau^3)^m\, \gamma_5^l\, t^a\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, t^a\, \chi(x)$				
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \gamma_\mu\, t^a\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \gamma_\nu\, t^a\, \chi(x) + [\mu \leftrightarrow \nu]$				
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, t^a\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, t^a\, \chi(x) + [\mu \leftrightarrow \nu]$				
$\delta_{\mu\nu}\, \bar\chi\, \tau^b\, \gamma_5^l\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \chi(x)$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$
$\bar\chi\, \tau^b\, \gamma_5^l\, \gamma_\mu\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \gamma_\nu\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$
$\bar\chi\, \tau^b\, \gamma_5^l\, \sigma_{\mu\lambda}\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{l+l'}\, I_{\mu\nu}$
$\delta_{\mu\nu}\, \bar\chi\, \tau^b\, \gamma_5^l\, t^a\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, t^a\, \chi(x)$				
$\bar\chi\, \tau^b\, \gamma_5^l\, \gamma_\mu\, t^a\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \gamma_\nu\, t^a\, \chi(x) + [\mu \leftrightarrow \nu]$				
$\bar\chi\, \tau^b\, \gamma_5^l\, \sigma_{\mu\lambda}\, t^a\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, t^a\, \chi(x) + [\mu \leftrightarrow \nu]$				

Table I: Transformation properties of operators without covariant derivatives up to mass dimension 6 for $\mathcal{P}_{1/2}$, $\mathcal{P}\,[-\mu]$, $\mathcal{T}_{1/2}$ and $\mathcal{T}\,[-\mu]$; $[-A]$ is short-hand for $[A \to -A]$. $C_{\mu\nu}$ is a version of the lattice field strength tensor as appearing in the Sheikholeslami-Wohlert term [28]. $I_{\mu\nu} = (-1)^{\delta_{\mu 0} + \delta_{\nu 0}}$.

operator	$\mathcal{C}$	$\mathcal{P}\,\mathcal{D}\,[-m]\,[-r]$	$\mathcal{R}_5^{1/2}\,\mathcal{D}\,[-\mu]$
$\delta_{\mu\nu}\, r^k\, m^{n_m}\, \mu^{n_\mu}$	$+1$	$(-)^{n_m+k}\, I_{\mu\nu}$	$(-)^{n_\mu}$
$\delta_{\mu\nu}\, \bar\chi\, (\tau^3)^m\, \gamma_5^l\, \chi(x)$	$+1$	$(-)^{l+1}\, I_{\mu\nu}$	$(-)^m$
$\mathrm{tr}\,[C_{\mu\lambda}\, C_{\lambda\nu}(x)]$	$+1$	$I_{\mu\nu}$	$+1$
$\mathrm{tr}\left[C_{\mu\lambda}\, \tilde C_{\lambda\nu}(x)\right] + [\mu \leftrightarrow \nu]$	$+1$	$-I_{\mu\nu}$	$+1$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, C_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$+1$	$(-)^{l+1} I_{\mu\nu}$	$(-)^m$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, \tilde C_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$+1$	$(-)^l I_{\mu\nu}$	$(-)^m$
$\delta_{\mu\nu}\, \bar\chi\, (\tau^3)^m\, \gamma_5^l\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \chi(x)$	$+1$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{m+m'}$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \gamma_\mu\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \gamma_\nu\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+l'}$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{m+m'}$
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$+1$	$(-)^{l+l'}\, I_{\mu\nu}$	$(-)^{m+m'}$
$\delta_{\mu\nu}\, \bar\chi\, (\tau^3)^m\, \gamma_5^l\, t^a\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, t^a\, \chi(x)$			
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \gamma_\mu\, t^a\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \gamma_\nu\, t^a\, \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\, (\tau^3)^m\, \gamma_5^l\, \sigma_{\mu\lambda}\, t^a\, \chi\, \bar\chi\, (\tau^3)^{m'}\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, t^a\, \chi(x) + [\mu \leftrightarrow \nu]$			
$\delta_{\mu\nu}\, \bar\chi\, \tau^b\, \gamma_5^l\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \chi(x)$	$+1$	$(-)^{l+l'}\, I_{\mu\nu}$	$+1$
$\bar\chi\, \tau^b\, \gamma_5^l\, \gamma_\mu\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \gamma_\nu\, \chi(x) + [\mu \leftrightarrow \nu]$	$(-)^{l+l'}$	$(-)^{l+l'}\, I_{\mu\nu}$	$+1$
$\bar\chi\, \tau^b\, \gamma_5^l\, \sigma_{\mu\lambda}\, \chi\, \bar\chi\, \tau^b\, \gamma_5^{l'}\, \sigma_{\lambda\nu}\, \chi(x) + [\mu \leftrightarrow \nu]$	$+1$	$(-)^{l+l'}\, I_{\mu\nu}$	$+1$

$\delta_{\mu\nu}\,\bar{\chi}\,\tau^b\,\gamma_5^l\,t^a\,\chi\,\bar{\chi}\,\tau^b\,\gamma_5^{l'}\,t^a\,\chi(x)$			
$\bar{\chi}\,\tau^b\,\gamma_5^l\,\gamma_\mu\,t^a\,\chi\,\bar{\chi}\,\tau^b\,\gamma_5^{l'}\,\gamma_\nu\,t^a\,\chi(x)+[\mu\leftrightarrow\nu]$			
$\bar{\chi}\,\tau^b\,\gamma_5^l\,\sigma_{\mu\lambda}\,t^a\,\chi\,\bar{\chi}\,\tau^b\,\gamma_5^{l'}\,\sigma_{\lambda\nu}\,t^a\,\chi(x)+[\mu\leftrightarrow\nu]$			

Table II: Transformation properties of operators without covariant derivatives for $\mathcal{C}$, $\mathcal{P}\,\mathcal{D}\,[-m]\,[-r]$ and $\mathcal{R}_5^{1/2}\,\mathcal{D}\,[-\mu]$.

operator	$\mathcal{P}_{1/2}$	$\mathcal{P}\,[-\mu]$	$\mathcal{T}_{1/2}$	$\mathcal{T}\,[-\mu]$
$r^k m^{n_m} \mu^{n_\mu}$	$+1$	$(-)^{n_\mu}$	$+1$	$(-)^{n_\mu}$
$\mathrm{Tr}\left(\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\lambda\,C_{\lambda\nu}(x)+\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\lambda\,C_{\lambda\mu}(x)\right)$	$I_{\mu\nu}$	$I_{\mu\nu}$	$I_{\mu\nu}$	$I_{\mu\nu}$
$\mathrm{Tr}\left(\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\lambda\,\tilde{C}_{\lambda\nu}(x)+\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\lambda\,\tilde{C}_{\lambda\mu}(x)\right)$	$-I_{\mu\nu}$	$-I_{\mu\nu}$	$-I_{\mu\nu}$	$-I_{\mu\nu}$
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l\pm\gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}_\mu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$	$(-)^{l+m}I_{\mu\nu}$	$(-)^l I_{\mu\nu}$	$(-)^{l+m}I_{\mu\nu}$	$(-)^l I_{\mu\nu}$
$\bar{\chi}\,(\tau^3)^m\,\gamma_5^l\,\left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}_\nu\pm\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\nu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l\pm\gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\gamma_\nu\,\gamma_5^l\pm\gamma_5^l\,\gamma_\nu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\gamma_5^l\,\left(\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\nu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\not\nabla}\,\gamma_\nu\,\gamma_5^l\,\overrightarrow{\nabla}_\mu\mp\overleftarrow{\nabla}_\mu\,\gamma_5^l\,\gamma_\nu\,\overrightarrow{\not\nabla}\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}_\nu\,\overleftarrow{\not\nabla}\,\gamma_5^l\mp\gamma_5^l\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$	$(-)^{l+m}I_{\mu\nu}$	$(-)^l I_{\mu\nu}$	$(-)^{l+m}I_{\mu\nu}$	$(-)^l I_{\mu\nu}$
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\nu\,\gamma_5^l\mp\gamma_5^l\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}_\nu\,\gamma_5^l\mp\gamma_5^l\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}^2\,\overleftarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l\mp\gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}^2\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}\,\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}\,\gamma_\nu\,\gamma_5^l\mp\gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}^2\,\gamma_\nu\,\gamma_5^l\mp\gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}^2\,\overrightarrow{\nabla}_\mu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}^2\,\gamma_\nu\,\gamma_5^l\mp\gamma_5^l\,\gamma_\nu\,\overrightarrow{\not\nabla}^2\,\overrightarrow{\nabla}_\mu\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\gamma_\nu\,\gamma_5^l\mp\gamma_5^l\,\gamma_\nu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				
$\bar{\chi}\,(\tau^3)^m\,\left(\overleftarrow{\not\nabla}^2\,\overleftarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l\mp\gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}^2\right)\,\chi(x)+[\mu\leftrightarrow\nu]$				

$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^3_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overrightarrow{\nabla}^3_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$	$(-)^{l+m}I_{\mu\nu}$ $(-)^l I_{\mu\nu}$	$(-)^{l+m}I_{\mu\nu}$ $(-)^l I_{\mu\nu}$
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\,\overleftarrow{\nabla}_\nu\,\gamma_\mu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\mu\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}^2_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}_\nu\,\overleftarrow{\nabla}_\mu\,\gamma_\mu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\mu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\nu\,\overleftarrow{\nabla}^2_\mu\,\gamma_\mu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\mu\,\overrightarrow{\nabla}^2_\mu\,\overrightarrow{\nabla}_\nu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\,\overleftarrow{\not\nabla}\,\gamma_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\gamma_\mu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}^2_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\gamma_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\gamma_\mu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}^2_\mu\,\gamma_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\gamma_\mu\,\overrightarrow{\nabla}^2_\mu\,\overrightarrow{\not\nabla}\right)\chi(x)+[\mu\leftrightarrow\nu]$		

$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}_\nu\,\overrightarrow{\not\nabla}\,\gamma_5^l \mp \gamma_5^l\,\overleftarrow{\not\nabla}\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$	$(-)^{l+m}\,I_{\mu\nu}$ $(-)^l\,I_{\mu\nu}$	$(-)^{l+m}\,I_{\mu\nu}$ $(-)^l\,I_{\mu\nu}$
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\overrightarrow{\nabla}_\nu\,\gamma_5^l \mp \gamma_5^l\,\overleftarrow{\nabla}_\nu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\nu\,\gamma_5^l \mp \gamma_5^l\,\overleftarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2\,\overrightarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}^2\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\nabla}\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}\,\overrightarrow{\nabla}\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\nabla}\,\overrightarrow{\nabla}\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\overrightarrow{\not\nabla}\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\not\nabla}\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\not\nabla}\,\overrightarrow{\not\nabla}_\mu\,\overrightarrow{\not\nabla}\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}^2\,\overrightarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\not\nabla}^2\,\overrightarrow{\not\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\,\overrightarrow{\nabla}_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}^2_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$		
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\,\overrightarrow{\nabla}_\nu\,\gamma_\mu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\mu\,\overleftarrow{\nabla}_\nu\,\overrightarrow{\nabla}^2_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\,\gamma_\mu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\mu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\nu\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\nu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\mu\,\gamma_\mu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\mu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\nu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\,\overrightarrow{\not\nabla}\,\gamma_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\gamma_\mu\,\overleftarrow{\not\nabla}\,\overrightarrow{\nabla}^2_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\,\overleftarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\,\gamma_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\gamma_\mu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\,\overrightarrow{\nabla}_\mu\right)\chi(x)+[\mu\leftrightarrow\nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\mu\,\gamma_\mu\,\gamma_\nu\,\gamma_5^l \mp \gamma_5^l\,\gamma_\nu\,\gamma_\mu\,\overleftarrow{\nabla}_\mu\,\overrightarrow{\nabla}_\mu\,\overrightarrow{\not\nabla}\right)\chi(x)+[\mu\leftrightarrow\nu]$		

Table III: Transformation properties of operators including derivatives for $\mathcal{P}_{1/2}$, $\mathcal{P}\,[-\mu]$, $\mathcal{T}_{1/2}$ and $\mathcal{T}\,[-\mu]$.

operator	$\mathcal{C}$	$\mathcal{P}\,\mathcal{D}\,[-m]\,[-r]$	$\mathcal{R}_5^{1/2}\,\mathcal{D}\,[-\mu]$
$r^k m^{n_m} \mu^{n_\mu}$	$+1$	$(-)^{n_m}(-)^k$	$(-)^{n_\mu}$
$\mathrm{Tr}\left(\overrightarrow{\nabla}_\mu \overrightarrow{\nabla}_\lambda C_{\lambda\nu}(x) + \overrightarrow{\nabla}_\nu \overrightarrow{\nabla}_\lambda C_{\lambda\mu}(x)\right)$	$+1$	$I_{\mu\nu}$	$+1$
$\mathrm{Tr}\left(\overrightarrow{\nabla}_\mu \overrightarrow{\nabla}_\lambda \tilde{C}_{\lambda\nu}(x) + \overrightarrow{\nabla}_\nu \overrightarrow{\nabla}_\lambda \tilde{C}_{\lambda\mu}(x)\right)$	N	$-I_{\mu\nu}$	$+1$
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu \gamma_\nu \gamma_5^l \pm \gamma_5^l \gamma_\nu \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$	± 1	$(-)^l I_{\mu\nu}$	$(-)^m$
$\bar{\chi}\,(\tau^3)^m \gamma_5^l \left(\overleftarrow{\nabla}_\mu \overleftarrow{\nabla}_\nu \pm \overrightarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu \gamma_\nu \gamma_5^l \pm \gamma_5^l \gamma_\nu \overrightarrow{\nabla}_\mu \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu \overleftarrow{\not\nabla}\, \gamma_\nu \gamma_5^l \pm \gamma_5^l \gamma_\nu \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \gamma_5^l \left(\overleftarrow{\nabla}_\mu \overrightarrow{\nabla}_\nu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \gamma_\nu \gamma_5^l \overrightarrow{\nabla}_\mu \mp \overleftarrow{\nabla}_\mu \gamma_5^l \gamma_\nu \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu \overleftarrow{\nabla}_\nu \overleftarrow{\not\nabla}\, \gamma_5^l \mp \gamma_5^l \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\nu \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$	± 1	$(-)^l I_{\mu\nu}$	$(-)^m$
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu \overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\nu \gamma_5^l \mp \gamma_5^l \overrightarrow{\nabla}_\nu \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu \overleftarrow{\nabla}_\nu \gamma_5^l \mp \gamma_5^l \overrightarrow{\nabla}_\nu \overrightarrow{\nabla}_\mu \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}^2 \overleftarrow{\nabla}_\mu \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\nabla}_\mu \overrightarrow{\nabla}^2\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}\, \overleftarrow{\nabla}_\mu \overleftarrow{\nabla}\, \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\nabla}\, \overrightarrow{\nabla}_\mu \overrightarrow{\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu \overleftarrow{\nabla}^2 \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\nabla}^2 \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu \overleftarrow{\not\nabla}^2 \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\not\nabla}^2 \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu \overleftarrow{\not\nabla}\, \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\not\nabla}^2 \overleftarrow{\nabla}_\mu \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\nabla}_\mu \overrightarrow{\not\nabla}^2\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar{\chi}\,(\tau^3)^m \left(\overleftarrow{\nabla}^3_\mu \gamma_\nu \gamma_5^l \mp \gamma_5^l \gamma_\nu \overrightarrow{\nabla}^3_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$	± 1	$(-)^l I_{\mu\nu}$	$(-)^m$

$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\, \overleftarrow{\nabla}_\nu\, \gamma_\mu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\mu\, \overrightarrow{\nabla}_\nu\, \overrightarrow{\nabla}^2_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\nabla}_\nu\, \overleftarrow{\nabla}_\mu\, \gamma_\mu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\mu\, \overrightarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\nu\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\nu\, \overleftarrow{\nabla}^2_\mu\, \gamma_\mu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\mu\, \overrightarrow{\nabla}^2_\mu\, \overrightarrow{\nabla}_\nu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\, \overleftarrow{\not\nabla}\, \gamma_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \gamma_\mu\, \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}^2_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu\, \gamma_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \gamma_\mu\, \overrightarrow{\nabla}_\mu\, \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}^2_\mu\, \gamma_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \gamma_\mu\, \overrightarrow{\nabla}^2_\mu\, \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\nabla}_\nu\, \overrightarrow{\not\nabla}\, \gamma_5^l \mp \gamma_5^l\, \overleftarrow{\not\nabla}\, \overrightarrow{\nabla}_\nu\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$	± 1	$(-)^l\, I_{\mu\nu}$	$(-)^m$
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\not\nabla}\, \overrightarrow{\nabla}_\nu\, \gamma_5^l \mp \gamma_5^l\, \overleftarrow{\nabla}_\nu\, \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\nu\, \gamma_5^l \mp \gamma_5^l\, \overleftarrow{\nabla}_\nu\, \overrightarrow{\nabla}_\mu\, \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2\, \overrightarrow{\nabla}_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}^2\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\nabla}\, \overrightarrow{\nabla}_\mu\, \overrightarrow{\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\nabla}\, \overrightarrow{\nabla}\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\nabla}\, \overrightarrow{\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\not\nabla}\, \overrightarrow{\not\nabla}\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\not\nabla}\, \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\not\nabla}\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\not\nabla}\, \overrightarrow{\not\nabla}_\mu\, \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}^2\, \overrightarrow{\nabla}_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\not\nabla}^2\, \overrightarrow{\not\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\, \overrightarrow{\nabla}_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}^2_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$			
$\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\, \overrightarrow{\nabla}_\nu\, \gamma_\mu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\mu\, \overleftarrow{\nabla}_\nu\, \overrightarrow{\nabla}^2_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\nabla}_\nu\, \overrightarrow{\nabla}_\mu\, \gamma_\mu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\mu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\nu\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\nu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\mu\, \gamma_\mu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\mu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\nu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}^2_\mu\, \overrightarrow{\not\nabla}\, \gamma_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \gamma_\mu\, \overleftarrow{\not\nabla}\, \overrightarrow{\nabla}^2_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\nabla}_\mu\, \overleftarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\, \gamma_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \gamma_\mu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\not\nabla}\, \overrightarrow{\nabla}_\mu\right) \chi(x) + [\mu \leftrightarrow \nu]$ $\bar\chi\,(\tau^3)^m \left(\overleftarrow{\not\nabla}\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\mu\, \gamma_\mu\, \gamma_\nu\, \gamma_5^l \mp \gamma_5^l\, \gamma_\nu\, \gamma_\mu\, \overleftarrow{\nabla}_\mu\, \overrightarrow{\nabla}_\mu\, \overrightarrow{\not\nabla}\right) \chi(x) + [\mu \leftrightarrow \nu]$			

Table IV: Transformation properties of operators including derivatives for $\mathcal{C}$, $\mathcal{P}\,\mathcal{D}\,[-m]\,[-r]$ and $\mathcal{R}_5^{1/2}\,\mathcal{D}\,[-\mu]$.

Appendix D: Symmetry properties of S_7

In Table D we list all possible terms of mass dimension 7 appearing in an expansion of the effective action to order a^3. We discuss their transformation properties under the $\mathcal{R}_5^{1/2}$ and $\mathcal{P}\times[\mu_q \to -\mu_q]$ symmetries which are symmetries of the continuum twisted mass action. We restrict the discussion to operators involving the twisted mass μ_q only since the bare quark mass $m_q = 0$ at maximal twist. We note further that neither $\mathcal{R}_5^{1/2}$ nor $\mathcal{P} \times [\mu_q \to -\mu_q]$ is affected by commuting two different derivative operators in a given expression such that we omit the commuted expressions. $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ denote the continuum field strength tensor and its dual, respectively.

In the four fermion operators we have included a generic transformation matrix $T^A = \tau^\mu \times t^a \times \Gamma$ where $\tau \in \{\tau^0, \tau^1, \tau^2, \tau^3\}$, $\Gamma \in \{1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_5\gamma_\mu, \gamma_5\}$ and t^a are acting in flavor-, Dirac- and color-space, respectively. Their index A used as a short-hand notation for flavor-, Dirac- and color-indices is summed over in the fermion bilinear product. Different Dirac structures are related via Fierz-identities and have the same transformation properties under the symmetries. Since T^A is appearing twice in all products this introduces an even number of both flavor- and Dirac-matrices such that the symmetry transformation is the same as for the trivial product with all matrices equal to the identity.

operator	$\mathcal{R}_5^{1/2}$	$\mathcal{P}\,[-\mu_q]$	operator	$\mathcal{R}_5^{1/2}$	$\mathcal{P}\,[-\mu_q]$
$\mu_q^4\bar{\chi}\chi$	-1	+1	$\mu_q^4\bar{\chi}\gamma_5\tau^3\chi$	+1	-1
$\mu_q^3\bar{\chi}\not{D}\chi$	+1	-1	$\mu_q^3\bar{\chi}\gamma_5\tau^3\not{D}\chi$	-1	+1
$\mu_q^3\mathrm{tr}\,[G_{\mu\nu}G_{\mu\nu}]$	+1	-1	-	-	-
$\mu_q^2\bar{\chi}D^2\chi$	-1	+1	$\mu_q^2\bar{\chi}\gamma_5\tau^3D^2\chi$	+1	-1
$\mu_q^2\bar{\chi}\sigma_{\mu\nu}G_{\mu\nu}\chi$	-1	+1	$\mu_q^2\bar{\chi}\gamma_5\tau^3\sigma_{\mu\nu}G_{\mu\nu}\chi$	+1	-1
$\mu_q(\bar{\chi}T^A\chi)^2$	+1	-1	$\mu_q(\bar{\chi}\gamma_5\tau^3T^A\chi)\ (\bar{\chi}T^A\chi)$	-1	+1
$\mu_q(\bar{\chi}\gamma_5\tau^3T^A\chi)^2$	+1	-1	-	-	-
$\mu_q\bar{\chi}\not{D}\sigma_{\mu\nu}G_{\mu\nu}\chi$	+1	-1	$\mu_q\bar{\chi}\gamma_5\tau^3\not{D}\sigma_{\mu\nu}G_{\mu\nu}\chi$	-1	+1
$\mu_q\bar{\chi}\not{D}D^2\chi$	+1	-1	$\mu_q\bar{\chi}\gamma_5\tau^3\not{D}D^2\chi$	-1	+1
$\mu_q\bar{\chi}\gamma_\mu D_\mu^3\chi$	+1	-1	$\mu_q\bar{\chi}\gamma_5\tau^3\gamma_\mu D_\mu^3\chi$	-1	+1
$\mu_q\bar{\chi}\gamma_\mu[D_\nu, G_{\mu\nu}]\chi$	+1	-1	$\mu_q\bar{\chi}\gamma_5\tau^3\gamma_\mu[D_\nu, G_{\mu\nu}]\chi$	-1	+1
$(\bar{\chi}T^A\chi)(\bar{\chi}\not{D}T^A\chi)$	-1	+1	$(\bar{\chi}\gamma_5\tau^3T^A\chi)(\bar{\chi}\not{D}T^A\chi)$	+1	-1
$(\bar{\chi}\gamma_5\tau^3T^A\chi)(\bar{\chi}\gamma_5\tau^3T^A\not{D}\chi)$	-1	+1	-	-	-
$\bar{\chi}\not{D}\gamma_\mu D_\mu^3\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3\not{D}\gamma_\mu D_\mu^3\chi$	+1	-1
$\bar{\chi}\not{D}\gamma_\mu[D_\nu, G_{\mu\nu}]\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3\not{D}\gamma_\mu[D_\nu, G_{\mu\nu}]\chi$	+1	-1
$\bar{\chi}D^2\sigma_{\mu\nu}G_{\mu\nu}\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3D^2\sigma_{\mu\nu}G_{\mu\nu}\chi$	+1	-1
$\bar{\chi}\sigma_{\kappa\lambda}G_{\kappa\lambda}\sigma_{\mu\nu}G_{\mu\nu}\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3\sigma_{\kappa\lambda}G_{\kappa\lambda}\sigma_{\mu\nu}G_{\mu\nu}\chi$	+1	-1
$\bar{\chi}(D^2)^2\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3(D^2)^2\chi$	+1	-1
$\bar{\chi}D^4\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3D^4\chi$	+1	-1
$\bar{\chi}\gamma_5G_{\mu\nu}\tilde{G}_{\mu\nu}\chi$	-1	+1	$\bar{\chi}\tau^3G_{\mu\nu}\tilde{G}_{\mu\nu}\chi$	+1	-1
$\bar{\chi}G_{\mu\nu}G_{\mu\nu}\chi$	-1	+1	$\bar{\chi}\gamma_5\tau^3G_{\mu\nu}G_{\mu\nu}\chi$	+1	-1
$\bar{\chi}\chi\mathrm{tr}\,[G_{\mu\nu}G_{\mu\nu}]$	-1	+1	$\bar{\chi}\gamma_5\tau^3\chi\mathrm{tr}\,[G_{\mu\nu}G_{\mu\nu}]$	+1	-1
$\bar{\chi}G_{\mu\nu}\tilde{G}_{\mu\nu}\chi$	+1	-1	$\bar{\chi}\gamma_5\tau^3G_{\mu\nu}\tilde{G}_{\mu\nu}\chi$	-1	+1
$\bar{\chi}\chi\mathrm{tr}\left[G_{\mu\nu}\tilde{G}_{\mu\nu}\right]$	+1	-1	$\bar{\chi}\gamma_5\tau^3\chi\mathrm{tr}\left[G_{\mu\nu}\tilde{G}_{\mu\nu}\right]$	-1	+1

Table V: Transformation properties of operators appearing in S_7.

[1] T. Blum, Phys. Rev. Lett. **91**, 052001 (2003), hep-lat/0212018.

[2] QCDSF, M. Gockeler *et al.*, Nucl.Phys. **B688**, 135 (2004), hep-lat/0312032.

[3] C. Aubin and T. Blum, Phys.Rev. **D75**, 114502 (2007), hep-lat/0608011.

[4] X. Feng, K. Jansen, M. Petschlies, and D. B. Renner, Phys.Rev.Lett. **107**, 081802 (2011), 1103.4818.

[5] P. Boyle, L. Del Debbio, E. Kerrane, and J. Zanotti, Phys.Rev. **D85**, 074504 (2012), 1107.1497.

[6] M. Della Morte, B. Jager, A. Juttner, and H. Wittig, JHEP **1203**, 055 (2012), 1112.2894.

[7] D. Bernecker and H. B. Meyer, Eur.Phys.J. **A47**, 148 (2011), 1107.4388.

[8] C. Aubin, T. Blum, M. Golterman, and S. Peris, Phys.Rev. **D86**, 054509 (2012), 1205.3695.

[9] D. B. Renner, X. Feng, K. Jansen, and M. Petschlies, PoS **LATTICE2011**, 022 (2012), 1206.3113.

[10] X. Feng *et al.*, Phys.Rev. **D88**, 034505 (2013), 1305.5878.

[11] F. Burger *et al.*, (2013), 1308.4327.

[12] T. Blum *et al.*, (2013), 1311.2198.

[13] R. Frezzotti and G. C. Rossi, JHEP **08**, 007 (2004), hep-lat/0306014.

[14] A. Shindler, Phys.Rept. **461**, 37 (2008), 0707.4093.

[15] K. Symanzik, Nucl.Phys. **B226**, 187 (1983).

[16] K. Symanzik, Nucl.Phys. **B226**, 205 (1983).

[17] K. Cichy, E. Garcia-Ramos, K. Jansen, and A. Shindler, PoS **LATTICE2013**, 128 (2013), 1312.3534.

[18] K. Cichy, E. Garcia-Ramos, K. Jansen, and A. Shindler, PoS **LATTICE2013**, 129 (2013), 1312.3535.

[19] K. Cichy, E. Garcia-Ramos, and K. Jansen, in preparation (2014).

[20] ETMC, R. Baron *et al.*, JHEP **1006**, 111 (2010), arXiv:1004.5284.

[21] ETMC, R. Baron *et al.*, Comput.Phys.Commun. **182**, 299 (2011), 1005.2042.

[22] K. Osterwalder and E. Seiler, Annals Phys. **110**, 440 (1978).

[23] R. Frezzotti and G. Rossi, JHEP **0410**, 070 (2004), hep-lat/0407002.

[24] M. Petschlies et al., in preparation .

[25] R. Frezzotti, G. Martinelli, M. Papinutto, and G. Rossi, JHEP **0604**, 038 (2006), hep-

lat/0503034.

[26] M. Luscher, S. Sint, R. Sommer, and P. Weisz, Nucl.Phys. **B478**, 365 (1996), hep-lat/9605038.

[27] P. Weisz, p. 93 (2010), 1004.3462.

[28] B. Sheikholeslami and R. Wohlert, Nucl.Phys. **B259**, 572 (1985).

[29] M. Constantinou *et al.*, (2013), 1311.5057.

[30] G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa, and A. Vladikas, Nucl.Phys. **B445**, 81 (1995), hep-lat/9411010.

[31] M. Luscher and P. Weisz, Commun.Math.Phys. **97**, 59 (1985).

[32] Alpha, R. Frezzotti, P. A. Grassi, S. Sint, and P. Weisz, JHEP **0108**, 058 (2001), hep-lat/0101001.

[33] R. Frezzotti and G. C. Rossi, Nucl. Phys. Proc. Suppl. **128**, 193 (2004), hep-lat/0311008.

[34] ETMC, R. Baron *et al.*, JHEP **1008**, 097 (2010), 0911.5061.

[35] ETMC, P. Dimopoulos, R. Frezzotti, G. Herdoiza, C. Urbach, and U. Wenger, PoS **LAT2007**, 102 (2007), 0710.2498.

[36] ETMC, C. Alexandrou *et al.*, Phys.Rev. **D78**, 014509 (2008), 0803.3190.