DESY 14-229
NORDITA-2015-140

Spectra of Sigma Models on Semi-Symmetric Spaces

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Abstract

Sigma models on semi-symmetric spaces provide the central building block for string theories on AdS backgrounds. Under certain conditions on the global supersymmetry group they can be made one-loop conformal by adding an appropriate fermionic Wess-Zumino term. We determine the full one-loop dilation operator of the theory. It involves an interesting new XXZ-like interaction term. Eigenvalues of our dilation operator, i.e. the one-loop anomalous dimensions, are computed for a few examples.

Contents

1 Introduction

Non linear sigma models (NLSM), such as the famous $O(N)$ or $\mathbb{CP}^N$ models, play an important role in high and low energy physics as well as mathematics. For a long time, research focused on cases in which the target is a symmetric space, i.e. can be written as a quotient G/H of a (super)group G by a subgroup $H \subset G$ that is invariant under the action of an involution $\sigma : G \to G$, i.e. by an automorphism of order two. The AdS/CFT correspondence has brought more general homogenous spaces into the spotlight, including

$$PSU(2,2|4)\big/SO(1,4)\times SO(5) \quad \text{and} \quad SPO(2,2|6)\big/SO(1,3)\times U(3)$$

which describe strings moving on $AdS_5 \times S^5$ and $AdS_4 \times \mathbb{CP}^3$, respectively. These spaces are examples of so-called generalized symmetric spaces G/H. By definition, the stabilizer sub(super)group H of a generalized symmetric space is left invariant by the action of an automorphism σ of order $M > 2$. Sigma models on such generalized symmetric spaces are not uniquely fixed by the target space manifold since the G symmetry leaves us with some $M-1$ dimensional space of metrics. Additionally, it is possible to add θ or Wess-Zumino (WZ) terms.

Since all the known relevant examples, such as the two displayed above, involve automorphisms or order four, we shall restrict to $M = 4$. The corresponding coset spaces G/H are often referred to as semi-symmetric and their sigma models appear as a part of the world-sheet action for strings in homogeneous AdS backgrounds, regardless of whether one works within the Green-Schwarz [1–5], pure spinor [6] or hybrid formalism [7, 8]. Only the choice of couplings depends on which formulation of superstring theory is actually being used. In this paper we are not concerned with the relations between the different approaches [9–14] and simply pick the couplings in the action such that we recover the NLSM of the hybrid and pure spinor models. In these cases, the metric conspires with a fermionic WZ term in order to make the action classically integrable and the one-loop beta-function vanish [15].

The aim of this work is to study the spectrum of one-loop anomalous dimensions for one-loop conformal NLSMs on semi-symmetric superspaces. Our analysis is valid for all such models, compact and non-compact. In order to fully control the one-loop spectrum, one also needs to enumerate possible vertex operators in the free theory. This is a problem of harmonic analysis which will not be addressed in the present work, see [16], however, where the corresponding issue has been solved for compact supercoset geometries. The analysis of [16] carries over to non-compact cosets as long as the stabilizer subgroup H is compact and a generalization to noncompact H is possible at least on a case-by-case basis.

Our results for the full one-loop dilation operator of one-loop conformal NLSMs on semi-symmetric spaces will be given in section 4. For non-derivative fields (zero modes) the one-loop dilation operator is given by the Bochner Laplacian, just as in the case of symmetric superspaces, see [16]. For operators including world-sheet derivatives the results becomes more interesting. In this case, the one-loop dilation operator turns out to involve an interesting new XXZ-like interaction term that we introduce in section 4 and derive in section 5. The interacting spins take values in space of tangent vectors to the semi-symmetric background. We shall also evaluate the general formulas for fields involving two derivatives. In this case the dilation operator can be diagonalized easily so that we can read off the anomalous dimensions.

The structure of the paper is as follows. In section 2 we will recall some facts of sigma models on semi-symmetric spaces, focusing on the one-loop conformal case that appears as part of the action in hybrid and pure spinor models. We will also present the one-loop expansion of the action. In section 3 we briefly describe how to build fields for the model and we spell out the leading terms of these vertex operators in the background field expansion. Section 4 contains the main result of this work. There we describe the full one-loop dilation operator and we analyze the anomalous dimensions for a particular subset of fields. The results are then derived in the section 5.

Auxiliary integral formulas, finally, are collected in an appendix at the end of the paper.

2 Sigma Models on semi-symmetric spaces

The goal of this section is to introduce and discuss the models we are dealing with. In the first subsection we construct their action. Their background field expansion is the subject of the second subsection where we present all terms that are relevant for our one-loop computation of anomalous dimensions.

2.1 Semi-symmetric spaces and the coset action

The models we are about to discuss belong to the class of sigma models on coset superspaces G/H with couplings chosen such that the beta-function vanishes, at least to leading order. Symmetric spaces, i.e. coset spaces G/H for which the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is fixed by a Lie algebra homomorphism σ of order two, are the most commonly considered cases, at least for bosonic groups G. When G is a supergroup, it is natural to consider so-called semi-symmetric spaces in which the denominator Lie (super-)algebra $\mathfrak{h} \subset \mathfrak{g}$ is kept fixed by a homomorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ of order four, i.e. $\sigma^4 = \mathbf{1}$. Such a homomorphism determines a decomposition of the form

$$\mathfrak{g} = \bigoplus_{\alpha=0}^{3} \mathfrak{g}_\alpha , \qquad \text{where} \qquad \sigma|_{\mathfrak{g}_\alpha} = \exp\left(\frac{\pi i}{2}\alpha\right) \mathbf{1} \ . \tag{1}$$

We shall also assume that $\mathfrak{g}_\alpha$ is contained in the bosonic subalgebra $\mathfrak{g}_{\bar{0}}$ for $\alpha = 0, 2$ while the subspaces $\mathfrak{g}_\alpha$ for $\alpha = 1, 3$ are contained in the odd part $\mathfrak{g}_{\bar{1}}$ of the algebra. As we have reviewed in the introduction, semi-symmetric superspaces of this type play an important role, in particular in the AdS/CFT correspondence.

Before we move on to the action of our sigma models, let us introduce some more notation. It will be convenient to define the shorthands $\mathfrak{h} := \mathfrak{g}_0$ and $\mathfrak{m} := \bigoplus_{\alpha=1}^{3} \mathfrak{g}_\alpha$. Furthermore, to each component $\mathfrak{g}_\alpha$ we can associate a projector $P_\alpha : \mathfrak{g} \to \mathfrak{g}_\alpha$. For the projector onto the complement of $\mathfrak{h}$ we introduce the symbol

$$P = 1 - P_0 = \sum_{\alpha=1}^{3} P_\alpha \tag{2}$$

in accordance with [16]. Let us finally fix an invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$. We note that any such form satisfies

$$(X, Y) = 0 \quad \text{for} \quad X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta \quad \text{and} \quad \alpha + \beta \neq 0_{\text{mod}4} \ . \tag{3}$$

This fundamental property of the invariant form will be used frequently in what follows.

The action of our sigma models is defined on maps $g : \Sigma \to G$ from the 2-dimensional world-sheet Σ to the supergroup G [17]

$$\mathcal{S}[g] = \frac{R^2}{2} \int_\Sigma \frac{d^2z}{\pi} \sum_{\alpha=1}^{3} (p_\alpha + q_\alpha)\,(P_\alpha \jmath, P_{\alpha'} \bar{\jmath}) \ , \tag{4}$$

where $\alpha' := 4 - \alpha$ and the currents $\jmath$ and $\bar{\jmath}$ are defined as

$$\jmath = g^{-1}\partial g \quad , \quad \bar{\jmath} = g^{-1}\bar{\partial} g \ . \tag{5}$$

Note that the action $\mathcal{S}$ is invariant under right multiplication $g \to gh$ by an H-valued function h since $P_\alpha \mathfrak{h} = 0$ for $\alpha = 1, 2, 3$. By construction, the model is also invariant under a global left action of G, regardless of how we choose the coefficients (couplings) p_α and q_α. Reality of the action requires that

$$p_\alpha = p_{\alpha'} \quad \text{and} \quad q_\alpha = -q_{\alpha'} \ . \tag{6}$$

Integrability and the vanishing of the one-loop beta-function imposes strong constraints on the couplings. These have been worked out in [7, 8, 17]. In our analysis, we shall adopt the following solution

$$\begin{gathered} p_0 = q_0 = 0, \\ p_\alpha = 1 \ , \quad q_\alpha = 1 - \tfrac{\alpha}{2} \ , \quad \text{for } \alpha = 1, 2, 3 \end{gathered} \tag{7}$$

This choice reproduces the supercoset sigma models that appear in the pure spinor and hybrid formalisms for string theory on AdS spaces [7].

2.2 One-loop action in background field expansion

For our one-loop computation of anomalous dimensions we need to expand the action to leading order in the background field expansion. In order to do so, we introduce the coordinates

$$\imath : G/H \to G \ , \qquad g_0 e^\phi \chi \mapsto g_0 e^\phi \ , \tag{8}$$

where $\phi \in \mathfrak{m}$. The expansion of the currents $\jmath$ in these coordinates is

$$\jmath = P e^{-\phi} \partial e^\phi = P \left[\partial\phi - \frac{1}{2}\,[\phi\,,\,\partial\phi] + \frac{1}{6}\,[\phi\,,\,[\phi\,,\,\partial\phi]] \right] + \cdots \tag{9}$$

and similarly for $\bar{\jmath}$. Let us introduce the notation $\phi_\alpha := P_\alpha \phi = t_a^\alpha \phi_\alpha^a$, where t_a^α with $a = 1, \ldots, \dim \mathfrak{g}_\alpha$ denotes a basis of $\mathfrak{g}_\alpha$. Note that the objects ϕ_α are Grassmann even by construction. Hence, in working with ϕ_α we do not have to worry about the grading.

The projected currents $P_{\alpha}J$ that appear in the action can now be rewritten as

$$P_{\alpha}J = \partial\phi_\alpha - \frac{1}{2}\sum_{\substack{\beta+\gamma\equiv\alpha\\ \beta,\gamma\neq 0}} [\phi_\beta\,,\,\partial\phi_\gamma] + \frac{1}{6}\sum_{\substack{\beta+\gamma+\delta\equiv\alpha\\ \beta,\gamma,\delta\neq 0}} [\phi_\beta\,,\,[\phi_\gamma\,,\,\partial\phi_\delta]] + \cdots \tag{10}$$

Inserting this expression into the action (4) and expanding $\mathcal{S} \sim \mathcal{S}_0 + \mathcal{S}_1$ up to the leading non-trivial order in the coupling we obtain

$$\mathcal{S}_0 = \frac{R^2}{2}\int_\Sigma \frac{d^2z}{\pi}\sum_{\alpha=1}^{3} p_\alpha \left(\partial\phi_\alpha, \bar\partial\phi_{\alpha'}\right) \tag{11}$$

for the tree level (free) action and $\mathcal{S}_1 = \mathcal{S}_1^s + \mathcal{S}_1^a$ where the symmetric part of the one-loop interaction is given by

$$\begin{aligned}\mathcal{S}_1^s \;=\;& \frac{R^2}{2}\int_\Sigma \frac{d^2z}{\pi}\Big[\sum_{\alpha+\beta+\gamma\equiv 0}\frac{p_\alpha}{2}\big\{-\big(\partial\phi_\alpha, [\phi_\beta\,,\,\bar\partial\phi_\gamma]\big) - \big([\phi_\beta\,,\,\partial\phi_\gamma]\,,\bar\partial\phi_\alpha\big)\big\}\\ &+ \sum_{\alpha+\beta+\gamma+\delta\equiv 0}\frac{p_\alpha}{6}\big\{\big(\partial\phi_\alpha, [\phi_\beta\,,\,[\phi_\gamma\,,\,\bar\partial\phi_\delta]]\big) + \big([\phi_\beta\,,\,[\phi_\gamma\,,\,\partial\phi_\delta]]\,,\bar\partial\phi_\alpha\big)\big\}\\ &+\sum_\alpha\sum_{\substack{\beta+\gamma\equiv\alpha\\ \delta+\epsilon\equiv\alpha'}}\frac{p_\alpha}{4}\big([\phi_\beta\,,\,\partial\phi_\gamma]\,,[\phi_\delta\,,\,\bar\partial\phi_\epsilon]\big)\Big]\,,\end{aligned} \tag{12}$$

while the antisymmetric part takes the form

$$\begin{aligned}\mathcal{S}_1^a \;=\;& \frac{R^2}{2}\int_\Sigma \frac{d^2z}{\pi}\Big[\sum_{\alpha+\beta+\gamma\equiv 0}\frac{q_\alpha}{2}\big\{-\big(\partial\phi_\alpha, [\phi_\beta\,,\,\bar\partial\phi_\gamma]\big) + \big([\phi_\beta\,,\,\partial\phi_\gamma]\,,\bar\partial\phi_\alpha\big)\big\}\\ &+ \sum_{\alpha+\beta+\gamma+\delta\equiv 0}\frac{q_\alpha}{6}\big\{\big(\partial\phi_\alpha, [\phi_\beta\,,\,[\phi_\gamma\,,\,\bar\partial\phi_\delta]]\big) - \big([\phi_\beta\,,\,[\phi_\gamma\,,\,\partial\phi_\delta]]\,,\bar\partial\phi_\alpha\big)\big\}\\ &+\sum_\alpha\sum_{\substack{\beta+\gamma\equiv\alpha\\ \delta+\epsilon\equiv\alpha'}}\frac{q_\alpha}{4}\big([\phi_\beta\,,\,\partial\phi_\gamma]\,,[\phi_\delta\,,\,\bar\partial\phi_\epsilon]\big)\Big]\,.\end{aligned} \tag{13}$$

From the tree level action $\mathcal{S}_0$ in eq. (11) we read off that the free two-point correlation function is given by

$$\langle\,\phi_\alpha(z,\bar z)\otimes\phi_{\alpha'}(w,\bar w)\,\rangle_0 = -\frac{(-1)^{|a|}}{R^2 p_\alpha}\log\left|\frac{z-w}{\epsilon}\right|^2\sum_{a=1}^{\dim\mathfrak{g}_\alpha} t_a^\alpha\otimes t_{a'}^{\alpha'}\eta^{aa'}\;. \tag{14}$$

In our conventions $(t^b, t^a) = \eta^{ab}$ and $t^a = \eta^{ab} t_b$. In passing we note that the two point function between a holomorphic and anti-holomorphic derivative of the fundamental fields contains a contact term,

$$\left\langle\, \partial\phi_\alpha(z,\bar z) \otimes \bar\partial\phi_{\alpha'}(w,\bar w)\,\right\rangle_0 = \frac{(-1)^{|a|}}{R^2 p_\alpha} \delta^2(z-w) \sum_{a=1}^{\dim \mathfrak{g}_\alpha} t_a^\alpha \otimes t_{a'}^{\alpha'} \eta^{aa'} \ . \tag{15}$$

These need to be taken into account if one or both of the insertion points are integrated over, see below.

In our analysis we shall split the one-loop terms of the interaction into three vertices and four vertices,

$$\mathcal{S}_1 = \int \frac{d^2 z}{\pi} \left(\Omega_3(z,\bar z) + \Omega_4(z,\bar z)\right) \ . \tag{16}$$

Once again, we can then split these vertices into a symmetric and an anti-symmetric part, i.e. $\Omega_p = \Omega_p^s + \Omega_p^a$. If we consider the one-loop conformal case (7), the previous expressions simplify drastically. In particular the first row of eq. (12) cancels so that

$$\Omega_3^s(z,\bar z) = 0 \tag{17}$$

and the first row of eq. (13) is

$$\Omega_3^a = -\sum_{\alpha+\beta+\gamma=0} \frac{R^2}{4} (q_\alpha + q_\gamma)(\partial\phi_\alpha, [\phi_\beta, \bar\partial\phi_\gamma]) \ . \tag{18}$$

For some of the calculation we go through later is will be useful to see this interaction explicitly,

$$\begin{aligned} \Omega_3^a(z,\bar z) \ &= \ \frac{R^2}{2}\Big\{ -\frac{1}{2}(\partial\phi_1, [\phi_2, \bar\partial\phi_1]) + \frac{1}{2}(\partial\phi_3, [\phi_2, \bar\partial\phi_3]) \\ &\quad -\frac{1}{4}(\partial\phi_2, [\phi_1, \bar\partial\phi_1]) + \frac{1}{4}(\partial\phi_2, [\phi_3, \bar\partial\phi_3]) \\ &\quad -\frac{1}{4}(\partial\phi_1, [\phi_1, \bar\partial\phi_2]) + \frac{1}{4}(\partial\phi_3, [\phi_3, \bar\partial\phi_2])\Big\} \ . \end{aligned} \tag{19}$$

In some calculations we will modify the vertex Ω_3 by adding a total derivative. This can simplify the expression significantly. One example is given by the following modification of the three vertex,

$$\begin{aligned} \Omega_3'^a \equiv \Omega_3(z,\bar z)^a &+ \frac{R^2}{8}\partial\big((\phi_2, [\phi_1, \bar\partial\phi_1]) - (\phi_2, [\phi_3, \bar\partial\phi_3])\big) \\ &+ \frac{R^2}{8}\bar\partial\big((\partial\phi_1, [\phi_1, \phi_2]) - (\partial\phi_3, [\phi_3, \phi_2])\big) \\ &= -\frac{R^2}{2}(\partial\phi_1, [\phi_2, \bar\partial\phi_1]) + \frac{R^2}{2}(\partial\phi_3, [\phi_2, \bar\partial\phi_3]) \ . \end{aligned} \tag{20}$$

In order to spell out the four vertex Ω_4 we make use of the following simple equality

$$(\partial\phi_\alpha, [\phi_\beta, [\phi_\gamma, \bar{\partial}\phi_\delta]]) = ([\phi_\gamma, [\phi_\beta, \partial\phi_\alpha]], \bar{\partial}\phi_\delta) = -([\phi_\beta, \partial\phi_\alpha], [\phi_\gamma, \bar{\partial}\phi_\delta])\,, \tag{21}$$

This allows to bring Ω_4 into the form

$$\Omega_4 = \frac{R^2}{2} \sum_{\alpha+\cdots+\delta=0} \left(\frac{p_{\alpha+\beta}}{4} - \frac{p_\alpha}{3} - \frac{q_\alpha - q_\delta}{6} + \frac{q_{\alpha+\beta}}{4} \right) ([\phi_\beta, \partial\phi_\alpha], [\phi_\gamma, \bar{\partial}\phi_\delta]) \tag{22}$$

which concludes our discussion of the action and in particular the various interaction terms.

3 Coset fields and their expansion

Our discussion of fields splits into two parts again. First we will review the construction of vertex operators for coset spaces G/H, see [16,18]. Then, we expand these operators up to the first non-trivial order in the background field expansion.

3.1 Vertex operators in coset models

As explained in [16, 18], vertex operators for coset models are composed of a zero mode contribution and a tail of currents. The latter contains all the derivatives.

The zero mode factor of a vertex operator is associated with a square integrable section of a homogeneous vector bundle over the coset. The bundle depends on the tail of currents, see below. Following the notation used in [16], we will denote the space of sections by Γ_λ, where λ labels a representation of the denominator subgroup $H \subset G$ and hence a homogeneous vector bundle over G/H. The space of such sections can be obtained as

$$\Gamma_\lambda = \Gamma_\lambda(G/H) = \{F \in L_2(G) \otimes \mathcal{S}_\lambda : F(gh) = S_\lambda(h^{-1})F(g)\ \forall h \in H\} \tag{23}$$

where $\mathcal{S}_\lambda$ denotes the carrier space of the representation S_λ of H. For the associated vertex operators we use the symbol

$$V_{\Lambda\lambda}(z,\bar{z}) \stackrel{\text{def}}{=} V[\mathcal{D}_{\Lambda\lambda}](z,\bar{z}) \quad \text{for} \quad \mathcal{D}_{\Lambda\lambda} \in \Gamma_\lambda\ . \tag{24}$$

Let us note that the space of sections Γ_λ of a homogenous vector bundle on G/H carries an action of the global symmetry (super-)group G. Under this action, the space Γ_λ decomposes into irreducible representations. The latter are labeled by Λ. How many times any given Λ appears within Γ_λ can only be determined with the tools of harmonic analysis. When both G and H are compact, there exists a universal formula for the multiplicities, see [16].

However, the counting of fields is an issue that can be treated separately from the computation of their anomalous dimensions.

In addition to the zero mode factor, vertex operators also contain a tail factor composed of currents and their derivatives. For these we introduce the compact notation

$$\jmath_{\mathbf{m}}(z,\bar{z}) = \bigotimes_{\rho=1}^{r} \partial^{m_\rho - 1} \jmath(z,\bar{z}) \ , \tag{25}$$

with a multi-index $\mathbf{m} = \{m_1, \ldots, m_r\}$ such that $m_\rho \geq m_{\rho+1} \geq 1$. The definition of $\bar{\jmath}_{\bar{\mathbf{m}}}$ is analogous. Note that these composite fields transform in (a subrepresentation of) the tensor product of the r-fold tensor product of the $\mathfrak{h}$ representation $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$.

Let us now assemble the complete vertex operators from the zero mode factor and the tail of currents as follows

$$\Phi_{\mathbf{\Lambda}}(z,\bar{z}) = \mathsf{d}_{\lambda\mu\bar{\mu}}(V_{\Lambda\lambda} \otimes \jmath_{\mathbf{m}} \otimes \bar{\jmath}_{\bar{\mathbf{m}}})(z,\bar{z}) \ , \qquad \mathbf{\Lambda} \stackrel{\mathrm{def}}{=} (\Lambda, \lambda, \mu, \bar{\mu}) \ . \tag{26}$$

This formula contains one new ingredient, namely the intertwiner d which ensures that $\Phi_{\mathbf{\Lambda}}$ is actually invariant under global H transformations, as it has to in order for it to represent a physical operator of the G/H coset model. The intertwiner d is itself composed of several pieces,

$$\mathsf{d}_{\lambda\mu\bar{\mu}} \stackrel{\mathrm{def}}{=} \mathsf{c}_{\lambda\mu\bar{\mu}}(1 \otimes \mathsf{p}_\mu \otimes \mathsf{p}_{\bar{\mu}}) \ , \tag{27}$$

Here, c denotes an intertwiner between the triple tensor product of the representations $[\lambda]$, $[\mu]$ and $[\bar{\mu}]$ of the denominator algebra $\mathfrak{h}$ and the trivial representation on the complex numbers, i.e.

$$\mathsf{c}^{\lambda\mu\bar{\mu}} : [\lambda] \otimes [\mu] \otimes [\bar{\mu}] \ \rightarrow \ \mathbb{C} \ . \tag{28}$$

The maps p project from a (partially symmetrized) tensor power of the representation $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ down to the irreducible representations μ and $\bar{\mu}$ of the denominator algebra $\mathfrak{h}$.

In our discussion below we shall often suppress the $\mathfrak{g}$ and $\mathfrak{h}$ representation labels $\Lambda, \lambda, \mu, \bar{\mu}$ and simply write

$$\Phi_{\mathbf{\Lambda}}(z,\bar{z}) = \mathsf{d} \circ V \otimes \jmath_{\mathbf{m}} \otimes \jmath_{\bar{\mathbf{m}}} \ . \tag{29}$$

3.2 Background field expansion of vertex operators

The one-loop expansion of a general coset field around an arbitrary point $g_0 H$ may be written schematically as

$$\Phi_\Lambda(z,\bar{z} \mid g_0) = \mathsf{d} \circ \big(V^{(0)} + V^{(1)} \ldots\big) \otimes \big(\jmath_{\mathbf{m}}^{(0)} + \jmath_{\mathbf{m}}^{(1)} \ldots\big) \otimes \big(\bar{\jmath}_{\bar{\mathbf{m}}}^{(0)} + \bar{\jmath}_{\bar{\mathbf{m}}}^{(1)} \cdots\big) \ , \tag{30}$$

Let us spell out concrete expressions for the various terms in the expansion. For the zero mode contribution $V_{\Lambda\lambda} = V$ the leading terms in the background field expansion read

$$
\begin{aligned}
V &= \mathcal{D}_{\Lambda\lambda}(g_0) - L_\Lambda(\mathrm{Ad}_{g_0}\phi(z,\bar{z}))\mathcal{D}_{\Lambda\lambda}(g_0) + \ldots && (31)\\
&= V^{(0)} + \sum_{\alpha=1,2,3} V^{(1)}(t_a)\phi^a_\alpha + \ldots\ . && (32)
\end{aligned}
$$

Here, $L_\Lambda(X)$ denotes the right invariant vector field that is associated with an element $X \in \mathfrak{g}$, see [16] for more details. By definition, $V^{(0)}$ is the constant term in the expansion around g_0.

Similarly, we can also expand the tail of currents. For the current $\jmath$ one finds that

$$
\jmath = \partial\phi - \frac{1}{2}\sum_{\substack{\beta+\gamma\neq 0\\ \beta,\gamma\neq 0}} [\phi_\beta\, ,\, \partial\phi_\gamma] + \ldots\ .
$$

Note that in the case of symmetric spaces, i.e. when the index α runs over $\alpha = 0,1$ only, the leading nontrivial term vanishes because the sum of $\beta = 1$ and $\gamma = 1$ is $\beta+\gamma = 0$ mod 2. Hence, while this term did not appear in [16], we need to consider it in dealing with semi-symmetric spaces. When the expansion of the current is inserted into our expression (25) it gives

$$
\begin{aligned}
\jmath_{\mathbf{m}} := \overset{(0)}{\jmath}_{\mathbf{m}} + \overset{(1)}{\jmath}_{\mathbf{m}} + \cdots = \bigotimes_{\rho=1}^{r} \partial^{m_\rho}\phi + \\
-\sum_{k=1}^{r}\bigotimes_{\rho=1}^{k-1}\partial^{m_\rho}\phi\otimes\partial^{m_k-1}\left[\frac{1}{2}\sum_{\substack{\beta+\gamma\neq 0\\ \beta,\gamma\neq 0}}[\phi_\beta\, ,\, \partial\phi_\gamma]\right]\otimes\bigotimes_{\rho=k+1}^{r}\partial^{m_\rho}\phi\ldots
\end{aligned}
$$

and similarly for the tail factors $\bar{\jmath}_{\bar{\mathbf{m}}}$ in which all the unbarred labels m are replaced by barred ones and antiholomorphic derivatives $\bar{\partial}$ appear instead of the holomorphic ones.

4 One-loop dilation operator: Summary of results

Since the computation of the one-loop dilation operator is a bit involved, we shall state our results beforehand and also apply the formula to one set of operators that includes the sigma model interaction. This will allow us to re-derive the vanishing of the beta-function for the models under consideration.

4.1 The one-loop dilation operator

It is useful to begin with a short description of the space of states which our dilation operator acts upon. As one may infer from the discussion in the previous section, this space is constructed out of two building blocks. To begin with, the zero mode term in our vertex operators contributes a factor $L_2(G)$ of square integrable functions on G. Recall that the space $L_2(G)$ carries two commuting actions of $\mathfrak{g}$. After restricting the right action from $\mathfrak{g}$ to the subalgebra $\mathfrak{h}$ we obtain

$$L_2(G) \cong \bigoplus_{\lambda} \Gamma_\lambda(G/H) \otimes \mathcal{S}_\lambda$$

which describes the decomposition of $L_2(G)$ into $\mathfrak{g}$ left and $\mathfrak{h}$ right modules. The second building block of the state space is the familiar $\mathfrak{h}$ module $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ which comes with the currents. From these two elements we can construct

$$\mathcal{H}_{r,\bar{r}} \equiv \left(L_2(G) \otimes \mathfrak{m}^{\otimes r} \otimes \mathfrak{m}^{\otimes \bar{r}}\right)^{\mathfrak{h}} = \bigoplus_{\lambda} \Gamma_\lambda(G/H) \otimes \left(\mathcal{S}_\lambda \otimes \mathfrak{m}^{\otimes r} \otimes \mathfrak{m}^{\otimes \bar{r}}\right)^{\mathfrak{h}} \quad (33)$$

where the superscript $\mathfrak{h}$ means that we project to the space of $\mathfrak{h}$ invariants. The space of all vertex operators with naive scaling dimension $(h_0, \bar{h}_0) = (n, \bar{n})$ has the form

$$\mathcal{H}^{(n,\bar{n})} = \bigoplus_{r,\bar{r}} \left[\sum_{l=0}^{n-r} p_l(n-r) + \sum_{l=0}^{\bar{n}-\bar{r}} p_l(\bar{n}-\bar{r})\right] \mathcal{H}_{r,\bar{r}} \quad (34)$$

where $p_l(m)$ denotes the number of partitions of m that have length l. These multiplicities arise from the placement of derivatives in the tail of currents.

We will now construct the 1-loop dilation operator from an $\mathfrak{h}$ invariant operator on the space $L_2(G) \otimes \mathfrak{m}^r \otimes \mathfrak{m}^{\bar{r}}$. It consists of three separate pieces,

$$D^{\text{1-loop}}_{r,\bar{r}} = H^{\text{h}} + H^{\text{ht}} + H^{\text{tt}} \ . \quad (35)$$

The first term acts on the zero mode factor only and it is identical to the so-called Bochner Laplacian for sections of vector bundles,

$$H^{\text{h}} = \mathbf{Cas}_{\mathfrak{g}} - \mathbf{Cas}_{\mathfrak{h}} \ .$$

Let us stress that this operator acts only on $L_2(G) \cong \bigoplus_\lambda \Gamma_\lambda(G/H) \otimes \mathcal{S}_\lambda$ where $\cong$ is an isomorphism of a $\mathfrak{g}$ left and an $\mathfrak{h}$ right module. In particular, the second term acts through the right action of $\mathfrak{h}$ on functions.

The second term in our dilation operator (35) describes an interaction between the tail of currents and the head. It reads

$$H^{\text{ht}} = -\sum_{\rho=1}^{r} ((-1)^c + Q_{\alpha\gamma}) R(t_c)\, (t^c)^{\rho}_{\alpha} - \sum_{\bar{\rho}=1}^{\bar{r}} ((-1)^c + Q_{\alpha\gamma}) R(t_c)\, (t^c)^{\bar{\rho}}_{\alpha}$$

where $R(t_c)$ denotes the right action of elements $t_c \in \mathfrak{g}$ on $L_2(G)$. The indices on $t_c \in \mathfrak{g}_\gamma$ indicate that the generator acts on a component of the ρ^{th} current that lies in the subspace $\mathfrak{g}_\alpha \subset \mathfrak{m}$. Finally, we have also introduced

$$Q_{\alpha\beta} \equiv q_\alpha + q_\beta - q_{\alpha+\beta} \ . \tag{36}$$

Note that the action of H^{ht} on vertex operators induces transitions between different vector bundles on G/H.

The most interesting contribution is the tail-tail interaction term. It consists of a sum of pairwise interactions between left- and right moving currents. The interaction terms resemble the XXZ Hamiltonian for spin-spin interaction, at least when written in an appropriate form. Let us recall that the XXZ Hamiltonian can be written as

$$H^{\text{XXZ}}_{\text{pair}} = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \Delta \sigma^z \otimes \sigma^z = (1+\Delta)(I+P) + (1-\Delta)K \ .$$

Here, the σ^a are Pauli-matrices, I is the identity on the tensor product of two spin representations, P the permutation and K the projection to the spin zero subspace. Similarly, we now define three operators on the tensor product $\mathfrak{m} \otimes \mathfrak{m}$ that are somewhat similar to I, P and K,

$$X^{abcd}_{\alpha\beta\gamma\delta} \ = \ (-1)^{|c||d|} f^{bai} f^{cd}{}_i \ , \tag{37}$$

$$A^{abcd}_{\alpha\beta\gamma\delta} \ = \ (-1)^{|c|} f^{adi} f^{cb}{}_i \ , \tag{38}$$

$$H^{abcd}_{\alpha\beta\gamma\delta} \ = \ (-1)^{|b|+|b||d|} f^{aci} f^{bd}{}_i \ . \tag{39}$$

Here, the Greek letters run through $\alpha, \dots = 1,2,3$ and the corresponding Roman letters $a, \dots$ run through a basis of $\mathfrak{g}_\alpha, \dots$.[1] Let us also stress that the index i runs over a basis of $\mathfrak{g}$. In terms of the operators X, A and H, we can write tail-tail interaction terms as

$$\begin{aligned} H^{\text{tt}} \ = \ & \sum_{\rho,\bar\rho=1}^{r,\bar r} \sum_{\alpha,\dots,\delta=1}^{3} \frac{1}{2}\left(p_{\alpha+\beta} + (-1)^{|a|+|c|} Q_{\alpha\beta} Q_{\gamma\delta}\right) X^{\rho\bar\rho}_{\alpha\beta\gamma\delta} + \frac{1}{2}\left(p_{\alpha+\delta} + \right. \\ & \left. +(-1)^{|a|+|c|} Q_{\alpha\delta} Q_{\gamma\beta}\right) A^{\rho\bar\rho}_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(p_{\alpha+\gamma} + (-1)^{|a|+|c|} Q_{\alpha\gamma} Q_{\beta\delta}\right) H^{\rho\bar\rho}_{\alpha\beta\gamma\delta} \\ & + (2 + Q_{\alpha\gamma} - Q_{\beta\delta})\, H^{\rho\bar\rho}_{\alpha\beta\gamma\delta} \end{aligned}$$

Note that all three contributions to the operator (35) commute with the action of $\mathfrak{h}$ on $L_2(G) \otimes \mathfrak{m}^{r+\bar r}$ and hence the sum descends to a well-defined operator on the space (34) of vertex operators in the sigma model.

[1] To be fully precise we should actually let Greek indices run over irreducible sub-representations of the $\mathfrak{h}$ action on $\mathfrak{m}$ and the corresponding Latin indices over a basis of these irreducible subspaces. We think of these additional (finer) projections as being implemented implicitly.

Let us conclude with two remarks on the validity of our formula (35) of the dilation operator. We want to stress that the expressions we provided also hold for non-compact semi-symmetric spaces, such as AdS geometries. In those cases, the zero mode spectrum is continuous and that allows us to analytically continue the results beyond $L_2(G)$ to some non-normalizable functions on the target space. Thereby, the formulas can also be applied, e.g. to the Noether currents of the global $\mathfrak{g}$ symmetry which involve functions on the target space that transform in the adjoint representation.

Finally, while the formulas are derived for semi-symmetric spaces, they can also be applied to symmetric spaces in which the denominator algebra $\mathfrak{h}$ is left invariant under an automorphism of order two rather than four. In that case, only the zero model Hamiltonian H^{h} and one term from the tail-tail interaction H^{tt}, namely the term $2H^{\rho\bar{\rho}}_{\alpha\beta\gamma\delta}$, can contribute, and we recover what has been found in [16]. Let us stress that our analysis here assumes that all elements in a subspace $\mathfrak{g}_\alpha$ are either bosonic or fermionic. The investigations in [16] were a bit more general in that they did not impose such a condition. In fact, for many interesting symmetric spaces the subspace $\mathfrak{g}_1$ contains both fermionic and bosonic directions.

4.2 Example: Marginal operators

As an example, let us consider the subspace of operators of naive dimension $(h, \bar{h}) = (1, 1)$ whose head is a constant function on the semi-symmetric target space. In this case, the contributions from H^{h} and H^{ht} vanish trivially. This implies that the space of such operators does not mix with operators whose zero mode factors are non-trivial. To be even more concrete let us pick the example of $AdS_5 \times S^5$ in which the space $\mathfrak{m}$ decomposes into four irreducible representations of the denominator subgroup. These are the fermionic subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_3$ along with the direct summands of $\mathfrak{g}_2 = \mathfrak{g}_{2^+} \oplus \mathfrak{g}_{2^-}$ that are associated with tangent vectors of the sphere S^5 and of AdS_5, respectively. It follows that the space of $\mathfrak{h}$ invariant operators is spanned by the following four basis elements

$$\left((\jmath_1, \bar{\jmath}_3) \,, (\jmath_{2^+}, \bar{\jmath}_{2^+}) \,, (\jmath_{2^-}, \bar{\jmath}_{2^-}) \,, (\jmath_3, \bar{\jmath}_1) \right) \tag{40}$$

From these fields of naive scaling dimension $(h_0, \bar{h}_0) = (1, 1)$ one can build the kinetic operator

$$\mathcal{L} \equiv \frac{3}{2} (\jmath_1, \bar{\jmath}_3) + (\jmath_{2^+}, \bar{\jmath}_{2^+}) + (\jmath_{2^-}, \bar{\jmath}_{2^-}) + \frac{1}{2} (\jmath_3, \bar{\jmath}_1) \ . \tag{41}$$

This is the combination of marginal operators that appears in the integrand of the action. In order for the $AdS_5 \times S^5$ sigma model to be one-loop conformal, the operator $\mathcal{L}$ must possess vanishing one-loop anomalous dimension. Using the results we sketched in the previous subsection we shall verify that $\mathcal{L}$ is an eigenvector of our one-loop dilation operator with zero eigenvalue.

In the basis (40), our one loop dilation operator acts as a 4×4 matrix whose elements are quadratic in the structure constants. The matrix elements can easily be worked out from the general formula for H^{tt} with $\rho = 1 = \bar{\rho}$. Each matrix element corresponds to a specific configuration of the parameters $\alpha, \beta, \gamma, \delta$. After inserting the values of p_α and q_α from eq. (7) it turns out that the matrix elements are all given by $\pm C$ where C is the value of the Casimir element $\mathbf{Cas}_{\mathfrak{h}}$ in the representation $\mathfrak{g}_{2+}$ of the denominator algebra $\mathfrak{h}$. More precisely, using the relation

$$f_{ad}{}^{e} f_{bc}{}^{d} \eta^{ab} = 0 \tag{42}$$

which holds for all Lie superalgebras $\mathfrak{g}$ with vanishing dual Coxeter number, one finds that

$$D_{1,1}^{\text{1-loop}} = C \cdot \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} .$$

It is now easy to verify that the vector $(3/2, 1, 1, 1/2)$ is indeed an eigenvector of the dilation operator with vanishing eigenvalue, as we anticipated.

Similarly one can also analyze the anomalous dimensions of operators $V_{\Lambda \mathfrak{m} \mathcal{J}}$ with naive scaling weight $(h_0, \bar{h}_0) = (1, 0)$ with Λ the adjoint representation of $\mathfrak{g}$. Once again there are four vertex operators of this type and the dilation operator can be worked out easily. In this case it receives contributions from the Bochner Laplacian and the head-tail interaction H^{ht}. The dilation operator can be shown to possess an eigenvector with zero eigenvalue, corresponding to a component of the conserved Noether current of the theory.

5 One-loop dilation operator: The derivation

Having stated the main results we now want to go through our calculations. These are a lot more involved than in the case of symmetric superspaces since the action we consider here contains many new terms that have no analogue for NLSM on symmetric spaces. The section begins with a brief outline of the computation. Then we dive into the details and evaluate all the various contributing terms.

5.1 Outline of the computation

The one-loop anomalous dimension $\delta \mathbf{h} = \delta \mathbf{h}_\Phi$ of a field Φ appears in the coefficient of the logarithmic singularity of the two point function at one-loop, see e.g. [16] for a detailed discussion,

$$\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle_1 = \langle\, 2\delta\mathbf{h}\cdot \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle_0 \, \log \left|\frac{\epsilon}{u-v}\right|^2 + \cdots \ .$$

The correlation function on the right hand side is evaluated in the free theory, i.e. by performing Wick contractions with the propagator (14). By definition, the one-loop correlation function on the left hand side is obtained as the leading non-trivial term in

$$\begin{aligned} &\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle = \\ &\qquad = \int_{G/H} d\mu(g_0 H)\, \langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}\mid g_0) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v}\mid g_0) e^{-\mathcal{S}_{\text{int}}}\,\rangle_{0,c} \ , \end{aligned} \tag{43}$$

where the subscript c stands for 'connected'. When counting loops, recall that each propagator carries a factor $1/R^2$ and each insertion of the interaction produces a factor R^2. The one-loop contribution contains all terms that are suppressed by a factor $1/R^2$ relative to the tree level.

Expanding the two point correlation function (43) to one-loop, we have

$$\langle\, \Phi_{\mathbf{\Lambda}}(u,\bar{u}) \otimes \Phi_{\mathbf{\Xi}}(v,\bar{v})\,\rangle = \int_{G/H} d\mu(g_0 H)\, \mathsf{d}\otimes\mathsf{d}\big(I_0 + I_1 + \dots\big) \ , \tag{44}$$

where

$$I_0 = \Big\langle\, V^{(0)} \otimes \jmath^{(0)}_{\mathbf{m}}(u) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{m}}}(\bar{u}) \otimes V^{(0)} \otimes \jmath^{(0)}_{\mathbf{m}}(v) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{m}}}(\bar{v})\,\Big\rangle_0 \ . \tag{45}$$

Here we used the schematic representation (30) and the fact that derivatives of the fundamental field are (anti)holomorphic in the free theory. In order to have a non-vanishing tree level result, the total $\mathbb{Z}_4$ grading of all currents $\jmath$ and their derivatives has to vanish. The same condition must be satisfied for $\bar{\jmath}$ and all their derivatives. One may note that this simple selection rule ceases to hold at loop level, at least on the face of it.

Let us now turn to I_1. As we noted above, the R^{-2} corrections to the correlation functions are collected in I_1. There are a variety of different terms. To begin with, there are three different terms in which no interaction term appears. In order to produce the desired factor $1/R^2$, these must involve one additional Wick contraction compared to the tree-level term. This contraction can either involve the two zero mode factors (case A), or two fields from the tails (case C) or one field from the head and one from the tail (case G). Next, there exist several terms that involve one interaction term. If the latter is given by the three vertex Ω_3, then we must expand either a zero mode factor V (case F) or a tail $\jmath$ (case D) to the leading non-trivial order. Terms involving a single interaction term Ω_4 contain two additional Wick contractions compared to the tree-level computation and hence also contain one factor $1/R^2$ (case B). Finally, we also have to consider one type of contribution with two interaction three-vertices Ω_3 since these contain three additional Wick contractions compared to tree-level (case E).

The quantity I_1 is obtained by summing all these different contributions, i.e. $I_1 = \sum I_K^\nu$ where

$$I_A = \left\langle V_u^{(1)} \otimes \jmath_{\mathbf{m},u}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(1)} \otimes \jmath_{\mathbf{m},v}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \right\rangle \tag{46}$$

$$I_B = \left\langle V_u^{(0)} \otimes \jmath_{\mathbf{m},u}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(0)} \otimes \jmath_{\mathbf{m},v}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \ \mathcal{O}_4 \right\rangle \tag{47}$$

$$I_C^1 = \left\langle V_u^{(0)} \otimes \jmath_{\mathbf{m},u}^{(1)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(0)} \otimes \jmath_{\mathbf{m},v}^{(1)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \right\rangle \tag{48}$$

$$I_D^1 = \left\langle V_u^{(0)} \otimes \jmath_{\mathbf{m},u}^{(1)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(0)} \otimes \jmath_{\mathbf{m},v}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \ \mathcal{O}_3 \right\rangle \tag{49}$$

$$I_E = \left\langle V_u^{(0)} \otimes \jmath_{\mathbf{m},u}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(0)} \otimes \jmath_{\mathbf{m},v}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \ \frac{1}{2}\mathcal{O}_3^2 \right\rangle \tag{50}$$

$$I_F^1 = \left\langle V_u^{(1)} \otimes \jmath_{\mathbf{m},u}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(0)} \otimes \jmath_{\mathbf{m},v}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \ \mathcal{O}_3 \right\rangle \tag{51}$$

$$I_G^1 = \left\langle V_u^{(1)} \otimes \jmath_{\mathbf{m},u}^{(0)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},u}^{(0)} \otimes V_v^{(0)} \otimes \jmath_{\mathbf{m},v}^{(1)} \otimes \bar{\jmath}_{\bar{\mathbf{m}},v}^{(0)} \right\rangle \tag{52}$$

Here, the subscripts u and v label fields that are inserted at $(u, \bar{u})$ and $(v, \bar{v})$, respectively. We have introduced the following shorthand notation for integrated interaction vertices

$$\mathcal{O}_3 = -\int_{\mathbb{C}} \frac{d^2 z}{\pi}\ \Omega_3(z, \bar{z}) \quad , \quad \mathcal{O}_4 = -\int_{\mathbb{C}} \frac{d^2 z}{\pi}\ \Omega_4(z, \bar{z}) \ .$$

Note that we included the minus sign from the $\exp(-S)$ into our definition of $\mathcal{O}_{3,4}$. Similarly, we also put a factor 1/2 into our definition of I_E because this term arises from the second order term in the expansion of $\exp(-S)$. We have to consider I_C^ν, I_D^ν and I_G^ν with $\nu = 1, 2, 3, 4$ that are given by all the possible combination obtained by expanding one of the elements in the first field $(V_1, \jmath_1, \bar{\jmath}_1)$ and one of the elements in the second field $(V_2, \jmath_2, \bar{\jmath}_2)$.

We would like to stress again that, unlike in the case of symmetric superspaces, the one loop dilation operator for semi-symmetric spaces does no longer act diagonally on the basis of fields we have selected. The diagonal terms of the dilation operator are obtained considering those pairs of fields from our basis that possess a non-vanishing tree level two-point function. We shall see that these diagonal terms are formally identical to $\mathbb{Z}_2$ case, at least after all cancelations are taken into account. The existence of off-diagonal entries in the dilation operator is quite crucial, however. As we have seen at the end of the previous section, for example, these terms conspire to make the one-loop anomalous dimension of the action vanish.

5.2 Detailed calculation of cases A-G

5.2.1 Contributions from case A

In this case the result is the same as for symmetric spaces. It is determined by the logarithmic contribution from I_A. Since the only logarithmic term arises from a contraction of the two dimension zero fields ϕ, without any derivative, the only expression that needs to be evaluated is

$$\int_{G/H} d\mu(g_0 H) \left\langle V^{(1)}(u,\bar{u}) \otimes V^{(1)}(v,\bar{v}) \right\rangle = \tag{53}$$
$$= R^{-2} \log \left| \frac{u-v}{\epsilon} \right|^2 \int_{G/H} d\mu(g_0 H) \left[\mathbf{Cas}_{\mathfrak{g}}^{\Lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\lambda} \right] V^{(0)} \otimes V^{(0)} .$$

The details of the calculations can be found in subsection 3.2.1 of [16]. Note that it is important here that we chose the parameters $p_\alpha = 1$. Beyond this conformal case, the results takes a more complicated form which cannot be written in terms of $\mathbf{Cas}_{\mathfrak{g}}^{\Lambda} - \mathbf{Cas}_{\mathfrak{h}}^{\lambda}$. Hence, while it was very easy to compute the corrections to the scaling behavior in symmetric spaces beyond the conformal case, see [16], a simple extension to non-conformal semi-symmetric spaces does not exist.

5.2.2 Contributions from case B

Let us now turn to the more interesting case B and analyze the following integral that is contained in it

$$\tilde{I}_B = - \int_{\mathbb{C}_\epsilon} d^2 z \left\langle \jmath^{(0)}_{\mathbf{m}}(u) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{m}}}(\bar{u}) \otimes \jmath^{(0)}_{\mathbf{n}}(v) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{n}}}(\bar{v})\, \Omega_4(z,\bar{z}) \right\rangle_0 . \tag{54}$$

Compared to the original I_B we just dropped the group theoretic factor that is associated with the zero modes. It may be reinstalled easily.

While an integral similar to $\tilde{I}_B$ also appears for symmetric spaces and was computed for these in [16], we now have to pay attention to the grading and the coefficients, in particular the nontrivial q_α, in the action. As a result, while a subset of terms turns out to reproduce those found for symmetric spaces, we shall also find new contributions that have no counterpart in the previous computation. To begin with, we can rewrite the quantity (54) in the form,

$$\tilde{I}_B = -\Pi \cdot \Bigg[\sum_{\rho,\sigma=1}^{r} \sum_{\bar{\rho},\bar{\sigma}=1}^{\bar{r}} \left\langle \jmath^{(0)}_{\mathbf{m}_\rho}(u) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{m}}_{\bar{\sigma}}}(\bar{u}) \otimes \jmath^{(0)}_{\mathbf{n}_\sigma}(v) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{n}}_{\bar{\sigma}}}(\bar{v}) \right\rangle_0 \otimes$$
$$\otimes \int_{\mathbb{C}_\epsilon} \frac{d^2 z}{\pi} \left\langle \partial^{m_\rho}\phi(u) \otimes \bar{\partial}^{\bar{m}_{\bar{\rho}}}\phi(\bar{u}) \otimes \partial^{n_\sigma}\phi(v) \otimes \bar{\partial}^{\bar{n}_{\bar{\sigma}}}\phi(\bar{v})\, \Omega_4(z,\bar{z}) \right\rangle_0 \Bigg] . \tag{55}$$

Here, $\jmath^{(0)}_{\mathbf{m}_\rho}$ denotes the tensor product (33) of currents with the ρ-th factor removed and we introduced a permutation Π that acts on a tensor power of $\mathfrak{m}$, see [16] for details. In evaluating $\tilde{I}_B$ we insert the representation (22) for the four vertex Ω_4.

Let us begin with a general statement and consider the integrand of eq. (55) with some definite choice of $\mathbb{Z}_4$ grading for the currents,

$$\mathcal{I} := \left\langle\, \partial^m\phi_\alpha(u)\otimes\bar\partial^{\bar m}\phi_\beta(\bar u)\otimes\partial^n\phi_\gamma(v)\otimes\bar\partial^{\bar n}\phi_\delta(\bar v)\,\left([\phi_\epsilon,\partial\phi_\varphi],[\phi_\rho,\bar\partial\phi_\chi]\right)(z,\bar z)\,\right\rangle_0 .$$

We can now split the symbols $\phi_\alpha = t_a\phi^a_\alpha$ into products of fields and generators. Moving all generators to the left of the fields and outside the correlator, we obtain

$$\begin{aligned}\mathcal{I} &= (-1)^{|a|+|a||b|+|b|+|c||d|}(-1)^{|h|+|g||h|+|g|+|e||f|}t_a\otimes t_b\otimes t_c\otimes t_d\\ &\times\left\langle \partial^m\phi^a_\alpha(u)\bar\partial^{\bar m}\phi^b_\beta(\bar u)\partial^n\phi^c_\gamma(v)\bar\partial^{\bar n}\phi^d_\delta(\bar v)(\phi^e_\epsilon\partial\phi^f_\varphi\phi^g_\rho\bar\partial\phi^h_\chi)(z,\bar z)f_{ef}{}^i f_{ghi}\right\rangle_0 . \end{aligned}\tag{56}$$

We observe that the sign factors we picked up while moving the generators past fields cancel in all terms that are allowed by Wick contractions. After inserting the resulting expression for the integrand $\mathcal{I}$ back into the $\tilde{I}_B$ we can perform the integration with the help of the following integral formula that we derive in App. A

$$\begin{aligned}\int_{\mathbb{C}_\epsilon}\frac{d^2z}{\pi}&\frac{a!b!c!d!}{(z-u)^{a+1}(z-v)^{b+1}(\bar z-\bar u)^{c+1}(\bar z-\bar v)^{d+1}} =\\ &= 2\log\left|\frac{u-v}{\epsilon}\right|^2\times\frac{(-1)^{a+c}(a+b)!(c+d)!}{(u-v)^{a+b+1}(\bar u-\bar v)^{c+d+1}} + \text{non-log.}\end{aligned}\tag{57}$$

If we take all possible Wick contraction schemes into account and make repeated use of the properties (7) of our parameters we obtain

$$\begin{aligned}\tilde{I}_B = \Pi\cdot\Bigg[&\sum_{\rho,\sigma=1}^{r}\sum_{\bar\rho,\bar\sigma=1}^{\bar r}\sum_{\alpha+\beta+\gamma+\delta=0}\left\langle \jmath^{(0)}_{\mathbf{m}_\rho}(u)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{m}}_{\bar\sigma}}(\bar u)\,\otimes\,\jmath^{(0)}_{\mathbf{n}_\sigma}(v)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{n}}_{\bar\sigma}}(\bar v)\right\rangle_0\\ &\otimes\log\left|\frac{u-v}{\epsilon}\right|^2 f_{m_\rho\bar m_{\bar\rho}n_\sigma\bar n_{\bar\sigma}}(u,v)t_a\otimes t_b\otimes t_c\otimes t_d\\ \times\Bigg(-&\left(-\frac{4}{3}+p_{\alpha+\gamma}-\frac{1}{3}(q_\alpha-q_\beta+q_\gamma-q_\delta)+q_{\alpha+\gamma}\right)(-1)^{|c|+|d|+|c||d|}H^{abcd}_{\alpha\beta\gamma\delta}\\ &+\left(-\frac{2}{3}+\frac{p_{\alpha+\delta}}{2}-\frac{1}{6}(q_\alpha-q_\beta+q_\gamma-q_\delta)\right)(-1)^{|b|+|d|}A^{abcd}_{\alpha\beta\gamma\delta}\\ +&\left(-\frac{2}{3}+\frac{p_{\alpha+\beta}}{2}-\frac{1}{6}(q_\alpha-q_\beta+q_\gamma-q_\delta)\right)(-1)^{|c|+|d|}X^{abcd}_{\alpha\beta\gamma\delta}\Bigg)\Bigg] + \text{non-log.}\end{aligned}$$

where we introduced the following convenient shorthand that will appear frequently below,

$$f_{m\bar m n\bar n}(u,v) = \frac{(-1)^{n+m}(m+n-1)!(\bar m+\bar n-1)!}{R^6(u-v)^{m+n}(\bar u-\bar v)^{\bar m+\bar n}} .\tag{58}$$

In addition we have used the matrices H, A and X that were defined in eqs. (37)-(39). It follows from the Jacobi identity of the Lie superalgebra $\mathfrak{g}$ that these matrices obey,

$$-(-1)^{|c|+|d|+|c||d|}H^{abcd}_{\alpha\beta\gamma\delta} = (-1)^{|b|+|d|}A^{abcd}_{\alpha\beta\gamma\delta} + (-1)^{|c|+|d|}X^{abcd}_{\alpha\beta\gamma\delta} \tag{59}$$

so that we are able to bring our result for $\tilde{I}_B$ into the following form

$$\begin{aligned}\tilde{I}_B = \Pi \cdot \Bigg[\sum_{\rho,\sigma=1}^{r} \sum_{\bar{\rho},\bar{\sigma}=1}^{\bar{r}} \sum_{\alpha+\beta+\gamma+\delta=0} \Big\langle\, \jmath^{(0)}_{\mathbf{m}_\rho}(u) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{m}}_{\bar{\sigma}}}(\bar{u}) \,\otimes\, \jmath^{(0)}_{\mathbf{n}_\sigma}(v) \otimes \bar{\jmath}^{(0)}_{\bar{\mathbf{n}}_{\bar{\sigma}}}(\bar{v}) \,\Big\rangle_0 \\ \otimes \log\left|\frac{u-v}{\epsilon}\right|^2 f_{m_\rho \bar{m}_{\bar{\rho}} n_\sigma \bar{n}_{\bar{\sigma}}}(u,v) t_a \otimes t_b \otimes t_c \otimes t_d \\ \times \left(\left(2 - p_{\alpha+\gamma} + \frac{1}{2}(Q_{\alpha\gamma} - Q_{\beta\delta})\right) (-1)^{|c|+|d|+|c||d|} H^{abcd}_{\alpha\beta\gamma\delta} \right. \\ \left. + \frac{p_{\alpha+\delta}}{2}(-1)^{|b|+|d|}A^{abcd}_{\alpha\beta\gamma\delta} + \frac{p_{\alpha+\beta}}{2}(-1)^{|c|+|d|}X^{abcd}_{\alpha\beta\gamma\delta}\right) \Bigg] + \text{non-log.}\end{aligned} \tag{60}$$

In order to read off the contributions to the dilation operator we still have to reexpress the function $f(u,v)$ through currents. This is possible and in the process we actually absorb all the sign factors that appear in front of the matrices H, A and X. As an example, let us consider the terms involving H and observe that

$$\begin{aligned}\big\langle\, \partial^m \phi_\alpha(u) \otimes \bar{\partial}^{\bar{m}} \phi_\beta(\bar{u}) \otimes \partial^n \phi_\gamma(v) \otimes \bar{\partial}^{\bar{n}} \phi_\delta(\bar{v}) \,\big\rangle_0 = \\ = (-1)^{|a||b|+|a|+|b|} f_{m\bar{m}n\bar{n}}(u,v)\ t_a \otimes t_b \otimes t^{a'} \otimes t^{b'} \frac{\delta_{c,a'}\delta_{d,b'}}{p_\alpha p_\beta}\ .\end{aligned} \tag{61}$$

once this is inserted into our expression for $\tilde{I}_B$, we can read off the corresponding contribution to the dilation operator from the coefficient of the tree level correlation function. For details we refer to section 3.2.1 of [16]. A similar analysis may be performed for the terms involving A and H. In these cases, the relevant correlation functions are

$$\big\langle\, \partial^m \phi_\alpha(u) \otimes \bar{\partial}^{\bar{m}} \phi_\beta(\bar{u}) \otimes \bar{\partial}^{\bar{n}} \phi_\gamma(\bar{v}) \otimes \partial^n \phi_\delta(v) \,\big\rangle_0$$

and

$$\big\langle\, \partial^m \phi_\alpha(u) \otimes \partial^n \phi_\gamma(v) \otimes \bar{\partial}^{\bar{m}} \phi_\beta(\bar{u}) \otimes \bar{\partial}^{\bar{n}} \phi_\delta(\bar{v}) \,\big\rangle_0\ .$$

We leave all details to the reader. The general conclusion is that the factor $f(u,v)$ along with the grading signs in front of H, A and X are absorbed. Similar comments apply at many places below even if we refrain from stressing this again.

5.2.3 Contributions from case C

Next we need to compute the part of the one-loop correction that arises from the expansion of the currents. In our discussion we shall work with the component fields $\phi_i := (\phi, t_i)$. When written in terms of these component fields, the subleading part of the current becomes

$$\jmath^{(1)} = \frac{1}{2} f^{ij}{}_a \phi_i \partial \phi_j t^a \ .$$

Here i, j and a run over a basis of the quotient space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. In this case we can divide the discussion in diagonal and off-diagonal contributions. We call diagonal those contributions that do not cancel at tree level, i.e. come from correlation functions of two fields that have complementary grading of the (anti)holomorphic tails.

Diagonal contributions

In this case only one type of terms appears which looks as follows

$$\begin{aligned} &\langle \partial^m \jmath^{(1)}(u) \otimes \partial^n \jmath^{(1)}(v) \rangle = \\ &= -\frac{(-1)^{m+|a|}}{4R^4} \ln \left| \frac{u-v}{\varepsilon} \right|^2 \frac{(m+n+1)!}{(u-v)^{m+n+2}} f^{ij}{}_a f^{kl}{}_b \eta_{ik} \eta_{jl} t^a \otimes t^b + \mathcal{O}(\varepsilon). \end{aligned} \tag{62}$$

Similar contributions arise from the anti-holomorphic currents. If we take into account that the Killing form of G vanishes by assumption and then compare with eq. (5.8) in [8] we can identify the combination of the structure constants that appears in the previous expression with the Ricci tensor of the coset space, i.e.

$$f^{ij}{}_a f^{kl}{}_b \eta_{ik} \eta_{jl} = 4 R_{ab}(G/H) \ . \tag{63}$$

Below we shall see that a similar term involving the Ricci tensor also arises from case E. The latter actually cancels the contributions from case C.

Off-diagonal contributions

We take an example. From case C we have two possible contributions, that contribute equally, namely

$$X = \frac{1}{4} \langle \partial^m \phi_3(u) \otimes \left[\phi_2 \, , \, \bar{\partial}^{\bar{m}} \phi_1 \right](u, \bar{u}) \otimes \left[\phi_2 \, , \, \partial^n \phi_1 \right](v, \bar{v}) \otimes \bar{\partial}^{\bar{n}} \phi_3(\bar{v}) \rangle$$

and

$$\tilde{X} = \frac{1}{4} \langle \left[\phi_2 \, , \, \partial^m \phi_1 \right](u, \bar{u}) \otimes \bar{\partial}^{\bar{m}} \phi_3(\bar{u}) \otimes \partial^n \phi_3(v) \otimes \left[\phi_2 \, , \, \bar{\partial}^{\bar{n}} \phi_1 \right](v, \bar{v}) \rangle \ .$$

In both expressions we have moved all the derivatives into the commutator, using the fact that terms in which derivatives act on the non-derivative field cannot give rise to logarithms. The sum of the previous two quantities is easily seen to give

$$\begin{aligned} X + \tilde{X} = -\frac{1}{2}\log\left|\frac{u-v}{\epsilon}\right|^2 t_{3i}\otimes t_{3j}\otimes t_{3k}\otimes t_{3l}\left(\left[t_1^l\,,\,t_1^j\right],\left[t_1^k\,,\,t_1^i\right]\right)\times \\ \times f_{m\bar{m}n\bar{n}}(u,v) + \text{non-log.} \end{aligned} \tag{64}$$

The results of this analysis can be summarized through the following expression for the off-diagonal contributions to $\tilde{I}_C$,

$$\begin{aligned} \tilde{I}_C^{\text{off}} = \Pi\cdot\Bigg[\sum_{\rho,\sigma=1}^{r}\sum_{\bar\rho,\bar\sigma=1}^{\bar r}\sum_{\alpha+\beta+\gamma+\delta=0}\Big\langle \jmath^{(0)}_{\mathbf{m}_\rho}(u)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{m}}_{\bar\sigma}}(\bar u)\otimes\jmath^{(0)}_{\mathbf{n}_\sigma}(v)\otimes\bar\jmath^{(0)}_{\bar{\mathbf{n}}_{\bar\sigma}}(\bar v)\Big\rangle_0 \\ \otimes\frac{p_{\alpha+\gamma}}{2}\log\left|\frac{u-v}{\epsilon}\right|^2 f_{m_\rho\bar m_{\bar\rho}n_\sigma\bar n_{\bar\sigma}}(u,v) \\ t_a\otimes t_b\otimes t_c\otimes t_d(-1)^{|c|+|d|+|c||d|}H^{abcd}_{\alpha\beta\gamma\delta}\Bigg] \end{aligned} \tag{65}$$

where we introduced $p_{\alpha+\gamma}$ to implement the fact that these terms are off-diagonal.

5.2.4 Contributions from case D

In the calculation we shall make use of the following integral formula

$$\begin{aligned} \int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\frac{1}{(z-x)^{a+1}(\bar z-\bar x)^{b+1}(\bar z-\bar y)^{c+1}} = \\ = \delta_{a,0}\frac{(-1)^{b+1}}{a!}\binom{b+c}{c}\frac{\ln\left|\frac{x-y}{\varepsilon}\right|^2}{(\bar x-\bar y)^{b+c+1}} + \mathcal{O}(\varepsilon) \end{aligned} \tag{66}$$

This formula is derived in App. A. Let us point out that logarithmic singularities only exist when $a = 0$. This implies that most terms that appear in the evaluation of case D do not contain any logarithmic divergencies. Having made this observation, let us discuss the contributions from I_D. Once again we shall distinguish between diagonal and off-diagonal contributions.

Diagonal contribution

Recall that $\Omega_3 = \Omega_3^a$ and that we can add a total derivative in order to bring Ω_3^a into the simple form $\Omega_3'^a$. This is the form we shall use. We address the two terms of $\Omega_3'^a$ separately. Given our introductory comment, there are few terms that contribute. The following integral is an example

$$
\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \langle \left[\phi_3\,,\, \partial^{m+1}\phi_2\right](u,\bar u) \otimes \partial^{n+1}\phi_3(v) (\partial\phi_1, [\phi_2, \bar\partial\phi_1])(z,\bar z)\rangle = \\
&\frac{(-1)^{m+1}}{R^4} \ln\left|\frac{u-v}{\varepsilon}\right|^2 \frac{(m+n+1)!}{(u-v)^{m+n+2}} \left[t_{2i}\,,\, t_{3j}\right] \otimes \left[t_2^i\,,\, t_1^j\right] + \text{non-log.}
\end{aligned}
\tag{67}
$$

This term is actually canceled by the following contact term

$$
\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \langle \left[\phi_2\,,\, \partial^{m+1}\phi_3\right](u,\bar u) \otimes \partial^{n+1}\phi_3(v) (\partial\phi_1, [\phi_2, \bar\partial\phi_1])(z,\bar z)\rangle = \\
&-\frac{1}{R^4}\partial_u^n \partial_v^m \int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \ln\left|\frac{u-z}{\varepsilon}\right|^2 \frac{1}{u-z}\delta^2(v-z) \left[t_{2i}\,,\, t_{3j}\right] \otimes \left[t_2^i\,,\, t_1^j\right] = \\
&-\frac{(-1)^{m+1}}{R^4} \ln\left|\frac{u-v}{\varepsilon}\right|^2 \frac{(m+n+1)!}{(u-v)^{m+n+2}} \left[t_{2i}\,,\, t_{3j}\right] \otimes \left[t_2^i\,,\, t_1^j\right] + \text{non-log.}
\end{aligned}
\tag{68}
$$

A similar cancelation occurs if we consider terms in which the current at v is expanded and the second contribution of $\Omega_3'^a$ employed in the contractions. Following this reasoning one may see that all diagonal contributions for case D cancel each other.

Off-diagonal contribution

In this case we shall work with the generic expression (18) for Ω_3. First we consider the expansion of one of the holomorphic currents

$$
\begin{aligned}
\int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \langle (\left[\phi_\epsilon\,,\, \partial^{m+1}\phi_\varphi\right] \otimes \bar\partial^{\bar m+1}\phi_\beta)(u,\bar u) \otimes (\partial^{n+1}\phi_\gamma \otimes \bar\partial^{\bar n+1}\phi_\delta)(v,\bar v) \\
\times (\partial\phi_\rho, [\phi_\chi, \bar\partial\phi_\iota])(z,\bar z)\rangle\ .
\end{aligned}
\tag{69}
$$

from this we get

$$
-\frac{1}{4} Q_{\beta\delta} \log\left|\frac{u-v}{\epsilon}\right|^2 f_{m\bar m n\bar n}(u,v)\, t_a \otimes t_b \otimes t_c \otimes t_d (-1)^{|c|+|d|+|c||d|} H^{abcd}_{\alpha\beta\gamma\delta}\ , \tag{70}
$$

and similarly for the expansion of the other holomorphic current. If we consider the expansion of the anti-holomorphic current we obtain instead

$$
+\frac{1}{2} Q_{\alpha\gamma} \log\left|\frac{u-v}{\epsilon}\right|^2 f_{m\bar m n\bar n}(u,v)\, t_a \otimes t_b \otimes t_c \otimes t_d (-1)^{|c|+|d|+|c||d|} H^{abcd}_{\alpha\beta\gamma\delta}\ . \tag{71}
$$

Summing all the contributions we find

$$\tilde{I}_D^{\rm off} = \frac{1}{2}\Pi \cdot \Bigg[\sum_{\rho,\sigma=1}^{r} \sum_{\bar\rho,\bar\sigma=1}^{\bar r} \sum_{\alpha+\beta+\gamma+\delta=0} \Big\langle \jmath^{(0)}_{\mathbf{m}_\rho}(u) \otimes \bar\jmath^{(0)}_{\bar{\mathbf{m}}_{\bar\sigma}}(\bar u) \otimes \jmath^{(0)}_{\mathbf{n}_\sigma}(v) \otimes \bar\jmath^{(0)}_{\bar{\mathbf{n}}_{\bar\sigma}}(\bar v) \Big\rangle_0$$
$$(Q_{\alpha\gamma} - Q_{\beta\delta}) \log \left| \frac{u-v}{\epsilon} \right|^2 f_{m_\rho \bar m_{\bar\rho} n_\sigma \bar n_{\bar\sigma}}(u,v)$$
$$t_a \otimes t_b \otimes t_c \otimes t_d (-1)^{|c|+|d|+|c||d|} H^{abcd}_{\alpha\beta\gamma\delta} \Bigg] . \quad (72)$$

This contribution is antisymmetric in fields of grading 1 and 3 and it is identical to the antisymmetric contribution from case B.

5.2.5 Contributions from case E

Let us now address the final case with two insertions of the vertex Ω_3. Recall that in a theory with vanishing one-loop beta-function, the three vertex is purely antisymmetric, i.e. $\Omega_3 = \Omega_3^a$. To ease the calculation we added a total derivative. Throughout this section we shall use the three vertex $\Omega_3'^a$, as we did before.

In the calculation we have two types of terms. The first type consists of contributions which involve two contractions between the two Ω_3 and two more contractions between the vertices and two currents. These only contribute to the diagonal part. On the other hand, we can also have terms in which there is a single contraction between the two vertices and four contractions with the tails of currents. We shall argue that the first type cancels the contributions from case C while the second type contributes new terms to the dilation operator.

In the case of two contractions we get only terms that contribute to the diagonal part of the anomalous dimension.

$$\frac{1}{8}\int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 w}{\pi} \langle \partial^{m+1}\phi_2(u) \otimes \partial^{n+1}\phi_2(v) (\partial\phi_1, [\phi_2, \bar\partial\phi_1])(z,\bar z) \times$$
$$\times (\partial\phi_3, [\phi_2, \bar\partial\phi_3])(w,\bar w) \rangle_0 = \quad (73)$$
$$= -\frac{(-1)^m}{8R^4} \ln \left| \frac{u-v}{\varepsilon} \right|^2 \frac{(m+n+1)!}{(u-v)^{m+n+2}} f^{ij}{}_a f^{kl}{}_b \eta_{ik}\eta_{jl} t_2^a \otimes t_2^b + \text{non-log.}$$

Here we have inserted the first term of $\Omega_3'^a$ at $(z,\bar z)$ and the second term at $(w,\bar w)$. The opposite choice turns out to give exactly the same result so that we get a numerical prefactor -1/4 in place of -1/8 after summing both contributions. We see that the result cancels against the contribution from case C for a and b labeling basis elements of $\mathfrak{g}_2$, i.e. when $|a| = |b| = 0$. In evaluating the relevant integrals, we have used the formula

$$\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\frac{1}{(z-x)(w-y)(z-w)^2(\bar z-\bar w)^2}=$$

$$=\ -\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\frac{1}{(z-x)}\frac{1}{(z-y)^2}\frac{1}{(\bar z-\bar y)}=\frac{\ln\left|\frac{u-v}{\varepsilon}\right|^2}{(x-y)^2(\bar x-\bar y)^2}$$

and derivatives thereof. This can be derived using formulas in Appendix A. We can perform a similar analysis in case the currents from the tails possess odd grade rather than even. The result is

$$\frac{1}{8}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\big\langle\partial^{m+1}\phi_1(u)\otimes\partial^{n+1}\phi_3(v)(\partial\phi_1,[\phi_2,\bar\partial\phi_1])(z,\bar z)\times$$
$$\times(\partial\phi_3,[\phi_2,\bar\partial\phi_3])(w,\bar w)\big\rangle_0= \tag{74}$$
$$=\frac{(-1)^m}{8R^4}\ln\left|\frac{u-v}{\varepsilon}\right|^2\frac{(m+n+1)!}{(u-v)^{m+n+2}}f^{ij}{}_af^{kl}{}_b\eta_{ik}\eta_{jl}t_1^a\otimes t_3^b+\text{non-log.}$$

In the process we have used the integral formula (89) from Appendix A. Once again, the result cancels the contribution from case C.

Let us then turn to the second type of terms in which we have four contractions between the vertices and the tail of currents. As above, we illustrate the computations with a concrete example,

$$\frac{1}{8}\int_{\mathbb{C}_\epsilon}\frac{\mathrm{d}^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2z}{\pi}\langle\partial^m\phi_1(u)\otimes\bar\partial^{\bar m}\phi_2(\bar u)\otimes\partial^n\phi_3(v)\otimes\bar\partial^{\bar n}\phi_2(\bar v)\ \times$$
$$\Big(\big(\partial\phi_1,\big[\phi_2\,,\,\bar\partial\phi_1\big]\big)(z,\bar z)\big(\partial\phi_3,\big[\phi_2\,,\,\bar\partial\phi_3\big]\big)(w,\bar w)\rangle_0= \tag{75}$$
$$=\frac{1}{4}\ln\left|\frac{u-v}{\varepsilon}\right|^2f_{m\bar m n\bar n}(u,v)\times$$
$$t_{1i}\otimes t_{2j}\otimes t_{3k}\otimes t_{2l}\Big[\Big(\Big[t_2^l\,,\,t_1^k\Big]\,,\Big[t_3^i\,,\,t_2^j\Big]\Big)-\Big(\Big[t_2^j\,,\,t_1^k\Big]\,,\Big[t_3^i\,,\,t_2^l\Big]\Big)\Big]+\text{non-log.}$$

If we insert the second non-trivial term in $\Omega_3'^a$ we obtain an identical contribution so that we just have to multiply the right hand side of the previous formula by a factor of two.

There is another qualitatively somewhat different example we want to consider in which the only contraction between the two vertices is a contraction between non-derivative fields. This happens e.g. in

$$\frac{1}{8}\int_{\mathbb{C}_\epsilon}\frac{\mathrm{d}^2 z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{\mathrm{d}^2 z}{\pi}\langle\partial^m\phi_1(u)\otimes\bar{\partial}^{\bar{m}}\phi_1(\bar{u})\otimes\partial^n\phi_3(v)\otimes\bar{\partial}^{\bar{n}}\phi_3(\bar{v})\ \times$$

$$\Big(\big(\partial\phi_1,\big[\phi_2\,,\,\bar{\partial}\phi_1\big]\big)(z,\bar{z})\,\big(\partial\phi_3,\big[\phi_2\,,\,\bar{\partial}\phi_3\big]\big)(w,\bar{w})\rangle_0 = \qquad (76)$$

$$= -\frac{1}{4}\ln\left|\frac{u-v}{\varepsilon}\right|^2 f_{m\bar{m}n\bar{n}}(u,v)\ \times$$

$$t_{1i}\otimes t_{1j}\otimes t_{3k}\otimes t_{3l}\left(\left[t_1^l\,,\,t_1^k\right],\left[t_3^i\,,\,t_3^j\right]\right) + \text{non-log.}$$

This and similar terms can be evaluated using the integral formula (88). Off diagonal terms are evaluated similarly and all the relevant integrals can be found in appendix A. The final result reads

$$\tilde{I}_E = \Pi\cdot\Bigg[\sum_{\rho,\sigma=1}^{r}\sum_{\bar{\rho},\bar{\sigma}=1}^{\bar{r}}\sum_{\alpha+\beta+\gamma+\delta=0}\Big\langle\, \jmath^{(0)}_{\mathbf{m}_\rho}(u)\otimes\bar{\jmath}^{(0)}_{\bar{\mathbf{m}}_{\bar{\sigma}}}(\bar{u})\ \otimes\ \jmath^{(0)}_{\mathbf{n}_\sigma}(v)\otimes\bar{\jmath}^{(0)}_{\bar{\mathbf{n}}_{\bar{\sigma}}}(\bar{v})\,\Big\rangle_0$$

$$\otimes\ \frac{(-1)^{|a|+|c|}}{2}\log\left|\frac{u-v}{\epsilon}\right|^2 f_{m_\rho\bar{m}_{\bar{\rho}}n_\sigma\bar{n}_{\bar{\sigma}}}(u,v)\,t_a\otimes t_b\otimes t_c\otimes t_d$$

$$\times\Big(-Q_{\alpha\gamma}Q_{\beta\delta}(-1)^{|c|+|d|+|c||d|}H^{abcd}_{\alpha\beta\gamma\delta} + Q_{\alpha\delta}Q_{\beta\gamma}(-1)^{|b|+|d|}A^{abcd}_{\alpha\beta\gamma\delta}$$

$$+Q_{\alpha\beta}Q_{\gamma\delta}(-1)^{|c|+|d|}X^{abcd}_{\alpha\beta\gamma\delta}\Big)\Bigg]\ . \qquad (77)$$

5.2.6 Contributions from case F and G

These two cases turn out to contribute only off-diagonal terms. Contributions from case G are symmetric in the exchange of the grading 1 and 3, those from case F are antisymmetric. Note that these contributions appear only if all the fields in the tail of the two fields have complementary grading except for one. This means that from these terms we read the off-diagonal entries of the dilation operator that, at one loop, change the grading of one of the fields in the tail at the time. In case G the evaluation of the new terms is fairly straightforward since no integrals need to be performed. The calculation of F is very similar to the one for case D. Without presenting any more details we state the answer for two vertex operators that differ in the grading of one holomorphic field,

$$I_F + I_G = -\Pi \cdot \int_{G/H} d\mu(g_0 H) \left[\sum_{\rho,\sigma=1}^{r} \sum_{\alpha,\beta=1,2,3} V^{(1)}(t_c) \otimes V^{(0)} \right.$$
$$\otimes \jmath^{(0)}_{\mathbf{m}_\rho} \otimes t_a \otimes \jmath^{(0)}_{\bar{\mathbf{m}}} \otimes \jmath^{(0)}_{\mathbf{n}_\sigma} \otimes t_b \otimes \jmath^{(0)}_{\bar{\mathbf{n}}}$$
$$\times (1 + (-1)^{|c|} Q_{\alpha\beta})$$
$$\left. \frac{(-1)^{m_\rho}}{4R^4} \ln \left| \frac{u-v}{\varepsilon} \right|^2 \frac{(m_\rho + n_\sigma + 1)!}{(u-v)^{m_\rho + n_\sigma + 2}} f^{cab} \right] . \quad (78)$$

A similar results applies for the anti-holomorphic case. This concludes our derivation of the one-loop dilation operator.

6 Conclusion

In this paper we calculated the one-loop dilation operator on the space of all vertex operators in NLSMs on semi-symmetric coset superspaces, at least for a special choice of the background parameters that appears in pure spinor and hybrid formulations and makes the one-loop beta-function vanish. While the diagonal terms are identical to the ones found for conformal NLSMs on symmetric spaces [16], there exist interesting new off-diagonal contributions to the dilation operators. These involve a new XXZ-like interaction term between left and right moving currents in the vertex operator. Of course, it would be very interesting to diagonalize the dilation operator for general states, not just for the small subsector of fields we considered in section 4.2.

In the case of symmetric spaces it was very easy to extend the analysis beyond the conformal case. In fact, [16] discusses a simple additional term that must be added to the formula for anomalous dimensions in order to be applicable for symmetric spaces G/H with numerator group G other than $G = PSU(N|N), OSP(2N+2|2N)$ or $D(2,1;\alpha)$ when the corresponding sigma model ceases to be conformal. The extension of our analysis for semi-symmetric spaces to couplings with non-vanishing beta-function, however, would certainly require significantly more work and not lead to a simple result.

Let us stress once again that the calculations we have performed apply to all NLSMs that appear in the hybrid formulation of AdS backgrounds [7,8]. This includes e.g. the coset $PSU(1,1|2)/U(1) \times U(1)$ whose bosonic base is $AdS_2 \times S^2$ [8]. It would be interesting to list all vertex operators for this case, to compute the full one-loop partition function and to extend the required input from harmonic analysis to higher dimensional AdS backgrounds for which the stability subgroup H is non-compact.

A key motivation for this work arises from the challenge to find a dual world-sheet description of strongly curved AdS backgrounds. In order to test any future proposal for such a 2-dimensional dual model, one will have to compare at least parts of the spectra. We have recently outlined the contours of such investigations at the example of the supersphere sigma model [18]. The target space $S^{3|2}$ that was used in the analysis is a symmetric superspace. In this case, a dual Gross-Neveu model description had been proposed by Candu and Saleur. We showed how recent all-loop results in deformed WZNW models could be used to test this duality by matching the spectrum of the free sigma model and its one-loop corrections. The results of the present work, which can be straightforwardly applied to the sigma model e.g. on $AdS_2 \times S^2$, provide plenty of data that could be matched with a potential dual formulation. Unlike the supersphere model, for which no string embedding is known, the $AdS_2 \times S^2$ model can be made into a true string theory. This brings in additional techniques which could complement the analytic tools developed here and help to resolve some of the issues that were left open in our previous investigation of supersphere models, such as e.g. the role of strongly RG relevant high gradient operators.

In this work we have not discussed the relation between various different formulations of superstring theory and we focused on the models that appear in the hybrid formalism. It would certainly be interesting to repeat our analysis within the (κ-gauge fixed) Green-Schwarz formulation.

A Derivation of integral formulas

First of all we recall a useful formula that was derived in [16]

$$\int_{\mathbb{C}_\epsilon} \frac{d^2 z}{\pi} \frac{1}{(z-x)(z-y)(\bar{z}-\bar{x})(\bar{z}-\bar{y})} = \frac{2\log\left|\frac{x-y}{\epsilon}\right|^2}{|x-y|^2} + \mathcal{O}(\epsilon) \ . \tag{79}$$

By taking now the appropriate number of derivatives in x, y, $\bar{x}$ or $\bar{y}$ on both sides of the above equation, we recover eq. (57).

Another useful integral is

$$\begin{aligned}\int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \frac{1}{(z-x)(\bar{z}-\bar{x})(\bar{z}-\bar{y})} &= \frac{1}{\bar{x}-\bar{y}} \int_{\mathbb{C}_\varepsilon} \frac{\mathrm{d}^2 z}{\pi} \bar{\partial}_z \frac{\ln\left|\frac{z-x}{z-y}\right|^2}{z-x} \\ = \frac{-1}{\bar{x}-\bar{y}} \oint_{\partial\mathbb{C}_\varepsilon} \frac{\mathrm{d}z}{2\pi i} \frac{\ln\left|\frac{z-x}{z-y}\right|^2}{z-x} &= -\frac{\ln\left|\frac{x-y}{\varepsilon}\right|^2}{(\bar{x}-\bar{y})} + \mathcal{O}(\varepsilon)\end{aligned} \tag{80}$$

Note the differences with the previous case with four factors in the denominator. In particular, only taking derivatives with respect to barred variables retains the logarithmic factor. This explains the delta factor in the above formula. The difference by a factor $\frac{1}{2}$ is due to the difference in the number of poles.

Using (79) and (80) and their derivatives we can calculate a series of double integrals:

$$\begin{gathered}\int_{\mathbb{C}_\varepsilon}\frac{d^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{u-z}\frac{1}{v-w}\frac{1}{(\bar u-\bar z)^2}\frac{1}{(\bar v-\bar w)^2}\frac{1}{(z-w)^2}=\\ =-\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar u-\bar v)^2}\end{gathered}\tag{81}$$

That can be derived by starting with the w-integral:

$$\begin{aligned}&\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{v-w}\frac{1}{(\bar v-\bar w)^2}\frac{1}{(z-w)^2}=\\ &=-\bar\partial_v\partial_z\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{w-v}\frac{1}{\bar w-\bar v}\frac{1}{w-z}=\\ &=\bar\partial_v\partial_z\frac{\ln|\frac{v-z}{\varepsilon}|^2}{v-z}+\mathcal{O}(\varepsilon)=\frac{1}{(v-z)^2}\frac{1}{\bar v-\bar z}+\mathcal{O}(\varepsilon)\end{aligned}\tag{82}$$

Plugging this in the double integral:

$$\begin{gathered}\int_{\mathbb{C}_\varepsilon}\frac{d^2z}{\pi}\frac{1}{u-z}\frac{1}{(v-z)^2}\frac{1}{(\bar u-\bar z)^2}\frac{1}{\bar v-\bar z}=\\ =-\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar u-\bar v)^2}\end{gathered}\tag{83}$$

Similarly one can calculate:

$$\begin{gathered}\int_{\mathbb{C}_\varepsilon}\frac{d^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{u-w}\frac{1}{v-z}\frac{1}{(\bar u-\bar z)^2}\frac{1}{(\bar v-\bar w)^2}\frac{1}{(z-w)^2}=\\ =+\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar u-\bar v)^2}\end{gathered}\tag{84}$$

$$\begin{gathered}\int_{\mathbb{C}_\varepsilon}\frac{d^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{(u-z)^2}\frac{1}{(v-w)^2}\frac{1}{(\bar u-\bar z)^2}\frac{1}{\bar v-\bar w}\frac{1}{z-w}=\\ =-\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar u-\bar v)^2}\end{gathered}\tag{85}$$

$$\begin{gathered}\int_{\mathbb{C}_\varepsilon}\frac{d^2z}{\pi}\int_{\mathbb{C}_\varepsilon}\frac{d^2w}{\pi}\frac{1}{(v-z)^2}\frac{1}{(u-w)^2}\frac{1}{(\bar u-\bar z)^2}\frac{1}{\bar v-\bar w}\frac{1}{z-w}=\\ =+\frac{2\ln|\frac{u-v}{\varepsilon}|^2}{(u-v)^2(\bar u-\bar v)^2}\end{gathered}\tag{86}$$

In the main text we need also some double integrals containing logarithms. For this reason we need to calculate:

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{(u-w)^2} \frac{1}{(\bar{u}-\bar{w})^2} \ln\left|\frac{z-w}{\varepsilon}\right|^2 = \\
&\qquad = \partial_u \partial_{\bar{u}} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \partial_{\bar{w}} \left(\frac{\ln\left(\frac{\bar{u}-\bar{w}}{\bar{u}-\bar{z}}\right) \ln\left|\frac{z-w}{\varepsilon}\right|^2 + \mathrm{Li}_2\left(\frac{\bar{w}-\bar{z}}{\bar{u}-\bar{z}}\right)}{w-u} \right) = \\
&\qquad = \partial_u \partial_{\bar{u}} \left(\frac{\pi^2}{6} + \ln\left(\frac{\varepsilon}{\bar{u}-\bar{z}}\right) \ln\left|\frac{u-z}{\varepsilon}\right|^2 \right) = \\
&\qquad = -\frac{1}{u-z} \frac{1}{\bar{u}-\bar{z}}
\end{aligned} \tag{87}$$

Where we have used the fact that $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$. This result can be used to calculate:

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2z}{\pi} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{(u-w)^2} \frac{1}{(v-z)^2} \frac{1}{(\bar{u}-\bar{w})^2} \frac{1}{(\bar{v}-\bar{z})^2} \ln\left|\frac{z-w}{\varepsilon}\right|^2 = \\
&\qquad\qquad = -\frac{2\ln\left|\frac{u-v}{\varepsilon}\right|^2}{(u-v)^2(\bar{u}-\bar{v})^2}
\end{aligned} \tag{88}$$

Other useful logarithmic integrals are:

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{(u-w)^2} \frac{1}{(\bar{z}-\bar{w})^2} \ln\left|\frac{z-w}{\varepsilon}\right|^2 = \\
&\qquad = -\partial_u \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \partial_{\bar{w}} \left(\frac{1+\ln\left|\frac{z-w}{\varepsilon}\right|^2}{(w-u)(\bar{w}-\bar{z})} \right) = \\
&\qquad = -\partial_u \left(\frac{1+\ln\left|\frac{z-u}{\varepsilon}\right|^2}{(\bar{u}-\bar{z})} \right) = \\
&\qquad = -\frac{1}{u-z} \frac{1}{\bar{u}-\bar{z}}
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
&\int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \frac{1}{(u-w)^2} \frac{1}{(\bar{v}-\bar{w})^2} \ln\left|\frac{z-w}{\varepsilon}\right|^2 = \\
&\qquad = \partial_u \partial_{\bar{v}} \int_{\mathbb{C}_\varepsilon} \frac{d^2w}{\pi} \partial_{\bar{w}} \left(\frac{\ln\left(\frac{\bar{v}-\bar{w}}{\bar{v}-\bar{z}}\right) \ln\left|\frac{z-w}{\varepsilon}\right|^2 + \mathrm{Li}_2\left(\frac{\bar{w}-\bar{z}}{\bar{v}-\bar{z}}\right)}{w-u} \right) = \\
&\qquad = \partial_u \partial_{\bar{u}} \left(\mathrm{Li}_2\left(\frac{\bar{u}-\bar{z}}{\bar{v}-\bar{z}}\right) + \ln\left(\frac{\varepsilon}{\bar{u}-\bar{z}}\right) \ln\left|\frac{u-z}{\varepsilon}\right|^2 \right) = \\
&\qquad = \frac{(\bar{z}-\bar{u})}{(\bar{u}-\bar{v})(u-z)(\bar{v}-\bar{z})}
\end{aligned} \tag{90}$$

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