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Toda 3-Point Functions From Topological Strings II

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Abstract

In [1] we proposed a formula for the 3-point structure constants of Toda field theory, derived using topological strings and the AGT-W correspondence from the partition functions of the non-Lagrangian T_N theories on S^4. In this article, we show how the semi-degeneration of one of the three primary fields on the Toda side corresponds to a particular Higgsing of the T_N theories and derive the well-known formula by Fateev and Litvinov.

Contents

1 Introduction

Obtaining the 3-point functions of the Toda CFT is a long standing problem in mathematical physics. Attacking this problem purely by using 2D CFT techniques is a notoriously difficult task and results exist only for particular specializations of the external momenta. Only the cases of degenerate or semi-degenerate primaries are known [2–4]. Our strategy in this paper, following [1], will be to employ string theory techniques, in particular topological stings and 5-brane web physics, that through the AGT-W correspondence will allow us tackle this problem using tools of a very different nature.

The AGT-W correspondence [5,6] is a relation between 4D $\mathcal{N}=2$ SU(N) quiver gauge theories and 2D $\mathbf{W}_N$ Toda CFT. The correlation functions of the 2D Toda CFT are obtained from the partition functions of the corresponding 4D $\mathcal{N}=2$ gauge theories as

$$\mathcal{Z}^{S^4} = \int [da] \left| \mathcal{Z}^{\text{4D}}_{\text{Nek}}(a, m, \epsilon_{1,2}) \right|^2 \propto \langle V_{\boldsymbol{\alpha}_1}(z_1) \cdots V_{\boldsymbol{\alpha}_n}(z_n) \rangle_{\text{Toda}} \,, \tag{1}$$

where the Omega deformation parameters are related to the Toda coupling constant[1] via $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$. The conformal blocks of the 2D CFTs are given by the appropriate instanton partition functions of Nekrasov [5,6], while the three point structure constants should be obtained by the S^4 partition functions of the T_N superconformal theories [7,8]. These partition functions were until recently [1,8,9] unknown, with the sole exception of the $\mathbf{W}_2$ case, *i.e.* the Liouville case, whose three point structure constants are given by the famous DOZZ formula [10,11].

[1] We also use the notation $\epsilon_+ = \epsilon_1 + \epsilon_2$. When we specialize $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$ in order to connect the topological string expressions to the Toda expressions, we have $\epsilon_+ = b + b^{-1} = Q$.

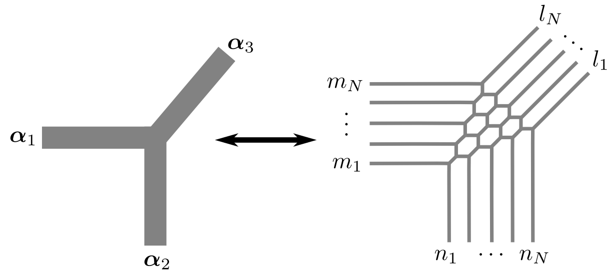

Figure 1: *This figure depicts the identification of the $\boldsymbol{\alpha}$ weights appearing on the Toda CFT side with the position of the flavor branes on the T_N side, here drawn for the case $N = 5$.*

A similar relation between 5D gauge theories and 2D q-CFT exists [8, 12–24], which relates the 5D Nekrasov partition functions on $S^4 \times S^1$ to correlation functions the of q-deformed Liouville/Toda field theory:

$$\mathcal{Z}^{S^4\times S^1} = \int [da] \Big| \mathcal{Z}^{\text{5D}}_{\text{Nek}}(a, m, \beta, \epsilon_{1,2}) \Big|^2 \propto \langle V_{\boldsymbol{\alpha}_1}(z_1) \cdots V_{\boldsymbol{\alpha}_n}(z_n) \rangle_{q\text{-Toda}} \,, \tag{2}$$

where $\beta = -\log q$ is the circumference of the S^1. The exponentiated Omega background parameters

$$\mathfrak{q} = e^{-\beta\epsilon_1} \,, \qquad \mathfrak{t} = e^{\beta\epsilon_2} \,, \tag{3}$$

are used in this case. The partition function on $\mathcal{Z}^{S^4\times S^1}$ is the 5D superconformal index, which as discussed in [25] can also be computed using topological string theory techniques

$$\mathcal{Z}^{S^4\times S^1} = \int [da] \, |\mathcal{Z}^{\text{5D}}_{\text{Nek}}(a)|^2 \propto \int [da] \, |\mathcal{Z}_{\text{top}}(a)|^2 \,. \tag{4}$$

In [8] we computed the partition functions of the 5D T_N theories on $S^4 \times S^1$ (see also [9]) and suggested that they should be interpreted as the 3-point structure constants of q-deformed Toda. We read them off from the toric-web diagrams of the T_N junctions of [26] by employing the refined topological vertex formalism of [27, 28]. In a subsequent paper [1], we showed how to take the 4D limit, corresponding to $\beta \to 0$ or equivalently to $q \to 1$, obtaining the partition function of 4D T_N theories on S^4

$$\mathcal{Z}^{S^4}_N = \text{const} \times \lim_{\beta\to 0} \beta^{-\frac{\chi_N}{\epsilon_1\epsilon_2}} \mathcal{Z}^{S^4\times S^1}_N \,, \tag{5}$$

where by "const" we mean a function of ϵ_1, ϵ_2 and β that is independent of the mass parameters of the theory. The degree of divergence was determined as proportional to the quadratic Casimir of $\text{SU}(N)^3$

$$\chi_N = -\sum_{1\le i<j\le N} \left[(m_i - m_j)^2 + (n_j - n_i)^2 + (l_i - l_j)^2 \right] = -N \sum_{i=1}^{3} \left(\boldsymbol{\alpha}_i - \mathcal{Q}, \boldsymbol{\alpha}_i - \mathcal{Q} \right) \,. \tag{6}$$

where $\mathcal{Q} := Q\rho = (b + b^{-1})\rho$ with the $\text{SU}(N)$ Weyl vector ρ defined in (A.8). After the first equality of (6), we have introduced the mass parameters m_i, n_i and l_i of the T_N theory, which, as shown in figure 1, are connected to the Toda theory parameters [1]

$$\begin{aligned} m_i &= (\boldsymbol{\alpha}_1 - \mathcal{Q}, h_i) = N \sum_{j=i}^{N-1} \alpha_1^j - \sum_{j=1}^{N-1} j\alpha_1^j - \frac{N+1-2i}{2} Q \,, \\ n_i &= -(\boldsymbol{\alpha}_2 - \mathcal{Q}, h_i) = -N \sum_{j=i}^{N-1} \alpha_2^j + \sum_{j=1}^{N-1} j\alpha_2^j + \frac{N+1-2i}{2} Q \,, \\ l_i &= -(\boldsymbol{\alpha}_3 - \mathcal{Q}, h_{N+1-i}) = -N \sum_{j=N+1-i}^{N-1} \alpha_3^j + \sum_{j=1}^{N-1} j\alpha_3^j - \frac{N+1-2i}{2} Q \,. \end{aligned} \tag{7}$$

It is important to note, that the mass parameters are not all independent, but obey

$$\sum_{i=1}^{N} m_i = \sum_{i=1}^{N} n_i = \sum_{i=1}^{N} l_i = 0 \,, \tag{8}$$

which is reflected in the fact that the sum of the weights h_i of the fundamental SU(N) representation is zero. Then the structure constants of three primary operators in the q-Toda theory is given by the T_N partition functions on $S^4 \times S^1$ as

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \text{const} \times \left[\prod_{j=1}^{3} Y_q(\boldsymbol{\alpha}_j) \right] (1-q)^{-\chi_N} \mathcal{Z}_N^{S^4 \times S^1} \,, \tag{9}$$

where we have used the following special functions that capture the non-trivial Weyl transformation properties of the structure constants:

$$Y_q(\boldsymbol{\alpha}) := \left[\frac{\left(1-q^b\right)^{2b^{-1}} \left(1-q^{b^{-1}}\right)^{2b}}{(1-q)^{2Q}} \right]^{-(\boldsymbol{\alpha},\rho)} \prod_{e>0} \Upsilon_q\left((\mathcal{Q} - \boldsymbol{\alpha}, e) \right) \,, \tag{10}$$

with the functions Υ_q defined in (A.24) and the product is taken over all positive roots e of SU(N). The partition function on $S^4 \times S^1$, or the superconformal index, for the T_N theory is given by an integral over the refined topological string amplitude[2]

$$\mathcal{Z}_N^{S^4 \times S^1} := \oint \prod_{l=1}^{N-2} \prod_{m=1}^{N-1-l} \left[\frac{d\tilde{A}_l^{(m)}}{2\pi i \tilde{A}_l^{(m)}} \left| M(\mathfrak{t}, \mathfrak{q}) \right|^2 \right] \left| \frac{\mathcal{Z}_N^{\text{top}}}{\mathcal{Z}_N^{\text{dec}}} \right|^2 \,, \tag{11}$$

after removing the decoupled degrees of freedom, referred to as "non-full spin content" in [8],

$$\begin{aligned} \left|\mathcal{Z}_N^{\text{dec}}\right|^2 &:= \prod_{1 \le i < j \le N} \left| \mathcal{M}(\tilde{M}_i \tilde{M}_j^{-1}) \mathcal{M}(\mathfrak{t}/\mathfrak{q}\, \tilde{N}_i \tilde{N}_j^{-1}) \mathcal{M}(\tilde{L}_i \tilde{L}_j^{-1}) \right|^2 \\ &= \text{const} \times \prod_{k=1}^{3} (1-q)^{N(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k - 2\mathcal{Q})} \left(\left(1-q^b\right)^{2b^{-1}} \left(1-q^{b^{-1}}\right)^{2b} \right)^{(\boldsymbol{\alpha}_k, \rho)} Y_q(\boldsymbol{\alpha}_k) \,. \end{aligned} \tag{12}$$

The partition function $\mathcal{Z}_N^{\text{dec}}$ captures additional degrees of freedom that are contained in the topological string calculation but then decouple from the 5D theory. Interestingly enough, as noted in [1], these

[2] Multiplied by an appropriate power of the refined MacMahon function $M(\mathfrak{t}, \mathfrak{q})$, see (A.30) for a definition.

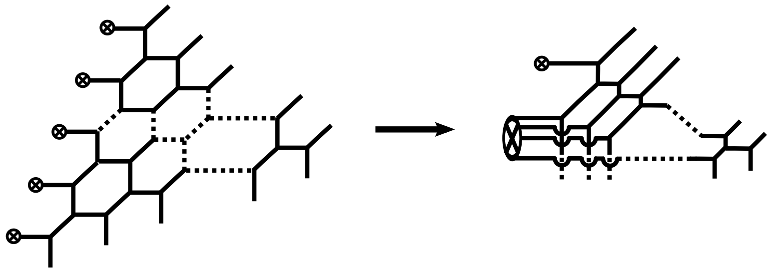

Figure 2: *The figure illustrates the desired Higgsing procedure for the general T_N diagram. We denote 7-branes by crossed circles. The left part of the figure shows the original T_N 5-brane web diagram, while the right one depicts the web diagram obtained by letting $N-1$ of the left 5-branes terminate on the same 7-brane.*

degrees of freedom are responsible for the Weyl covariance of the Toda structure constants. Here and elsewhere, we shall use the shorthand notation

$$|f(u_1,\dots,u_r;\mathfrak{t},\mathfrak{q})|^2 := f(u_1,\dots,u_r;\mathfrak{t},\mathfrak{q})f(u_1^{-1},\dots,u_r^{-1};\mathfrak{t}^{-1},\mathfrak{q}^{-1})\,, \tag{13}$$

while the functions $\mathcal{M}$ are defined in (A.19). Inserting (11) into (9), we find the nice expression

$$C_q(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) = \text{const}\times\oint\prod_{k=1}^{N-2}\prod_{l=1}^{N-1-l}\left[\frac{d\tilde{A}_k^{(l)}}{2\pi i\tilde{A}_k^{(l)}}\left|M(\mathfrak{t},\mathfrak{q})\right|^2\right]\left|\mathcal{Z}_N^{\text{top}}\right|^2\,. \tag{14}$$

The topological string amplitude is $\mathcal{Z}_N^{\text{top}}$ obtained from the T_N web-diagram by using the refined topological vertex formalism and reads

$$\mathcal{Z}_N^{\text{top}} = \mathcal{Z}_N^{\text{pert}}\mathcal{Z}_N^{\text{inst}}\,, \tag{15}$$

where the "perturbative" partition function[3] is

$$\mathcal{Z}_N^{\text{pert}} := \prod_{r=1}^{N-1}\prod_{1\le i<j\le N-r}\frac{\mathcal{M}\Big(\frac{\tilde{A}_i^{(r-1)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_{j+1}^{(r-1)}}\Big)}{\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r-1)}\tilde{A}_{j-1}^{(r)}}{\tilde{A}_{i-1}^{(r-1)}\tilde{A}_j^{(r)}}\Big)\mathcal{M}\Big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r-1)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r-1)}}\Big)}\prod_{1\le i<j\le N-r-1}\mathcal{M}\Big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{A}_i^{(r)}\tilde{A}_j^{(r)}}{\tilde{A}_{i-1}^{(r)}\tilde{A}_{j+1}^{(r)}}\Big), \tag{16}$$

and the "instanton" one is

$$\begin{aligned}\mathcal{Z}_N^{\text{inst}} := \sum_{\boldsymbol{\nu}}\prod_{r=1}^{N}\prod_{i=1}^{N-r}\left(\frac{\tilde{N}_r\tilde{L}_{N-r}}{\tilde{N}_{r+1}\tilde{L}_{N-r+1}}\right)^{\frac{|\nu_i^{(r)}|}{2}}\prod_{r=1}^{N}\prod_{i\le j=1}^{N-r}&\left[\frac{\mathsf{N}^{\beta}_{\nu_i^{(r-1)}\nu_j^{(r)}}\left(a_i^{(r-1)}+a_{j-1}^{(r)}-a_{i-1}^{(r-1)}-a_j^{(r)}-\epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_i^{(r-1)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r-1)}+a_j^{(r-1)}-a_{i-1}^{(r-1)}-a_{j+1}^{(r-1)}\right)}\right.\\ &\left.\times\frac{\mathsf{N}^{\beta}_{\nu_i^{(r)}\nu_{j+1}^{(r-1)}}\left(a_i^{(r)}+a_j^{(r-1)}-a_{i-1}^{(r)}-a_{j+1}^{(r-1)}-\epsilon_+/2\right)}{\mathsf{N}^{\beta}_{\nu_i^{(r)}\nu_j^{(r)}}\left(a_i^{(r)}+a_{j-1}^{(r)}-a_{i-1}^{(r)}-a_j^{(r)}-\epsilon_+\right)}\right],\end{aligned} \tag{17}$$

where the $a_i^{(j)}$ are defined via $\tilde{A}_i^{(j)} = e^{-\beta a_i^{(j)}}$, while the $\mathsf{N}^{\beta}_{\lambda\mu}$ are given in (A.36). The "interior" Coulomb moduli $\tilde{A}_j^{(i)} = e^{-\beta a_i^{(j)}}$ are independent, while the "border" ones are given by

$$\tilde{A}_i^{(0)} = \prod_{k=1}^{i}\tilde{M}_k\,,\qquad \tilde{A}_0^{(i)} = \prod_{k=1}^{i}\tilde{N}_k\,,\qquad \tilde{A}_i^{(N-i)} = \prod_{k=1}^{i}\tilde{L}_k\,, \tag{18}$$

[3]We put the words "perturbative" and "instanton" inside quotation marks because for the T_N there is not really a notion of instanton expansion, since there is no coupling constant.

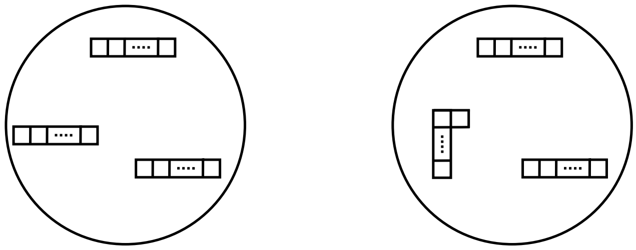

Figure 3: *On the left we depict the sphere with three full punctures that corresponds to the un-Higgsed T_N with $SU(N)^3$ global symmetry. On the right we show the sphere with two full punctures and one L-shaped $\{N-1,1\}$ puncture. This particular Higgsing of T_N leads to a theory with with $SU(N)\times SU(N)\times U(1)$ global symmetry. The partition function of this theory will lead to the Toda 3-point function with one semi-degenerate primary insertion.*

where $\tilde{M}_k := e^{-\beta m_k}$ and similarly for $\tilde{N}_k$ and $\tilde{L}_k$. See appendix A for more details on the parametrization of the T_N junction.

The formula (9) for the structure constants of three Toda primary fields, has the correct symmetry properties, the zeros that it should and, for $N=2$, gives the known answer for the Liouville CFT [1]. However, it is very implicit as one has first to perform $\frac{N(N-1)}{2}$ sums over the partitions $\nu_i^{(j)}$, followed by an $\frac{(N-1)(N-2)}{2}$ dimensional integral over the Coulomb moduli $\tilde{A}_i^{(j)}$(the number of faces in the left diagram in figure 2), and finally take the 4D or $q\to 1$ limit (5). In this article, we show how to derive from (9) the formula by Fateev and Litvinov [2–4] for the structure constants with one semi-degenerate primary (20). This provides a very strong check of the results in [1].

This article is organized as follows. We begin by presenting in section 2 the formula by Fateev and Litvinov that we wish to obtain as well as its generalization to the q-deformed Toda theory. We then argue in section 3 that the semi-degeneration of the primary field on the 2D CFT side is obtained on the 4D/5D side by Higgsing the T_N theory. Specifically, we Higgs the left part of the T_N web diagram, as illustrated in figure 2. The original, *i.e.* non-Higgsed, T_N partition function on $S^4\times S^1$ is given by a $\frac{(N-1)(N-2)}{2}$ dimensional integral over the Coulomb moduli, but the Higgsing "pinches" the integration contours and gets rid of all of them as we show in section 4 for the T_3 case and in appendix C for the T_4 case. The general T_N case is then presented in section 5. In deriving these results, we need to perform $\frac{N(N-1)}{2}$ sums over partitions, but the semi-degeneration trivializes $\frac{(N-1)(N-2)}{2}$ of them. Finally, we succeed in performing the $N-1$ leftover sums using the Kaneko-Macdonald-Warnaar sl(N) q-binomial identities, which we present in 5 and prove their applicability to our case in appendix B.

2 The formula of Fateev and Litvinov

In this section we introduce the formula by Fateev and Litvinov [2–4] for the Toda 3-point structure constants with one semi-degenerate primary that we want to re-derive using our formula (9). The coordinate dependence of the 3-point functions of three primary fields $V_{\boldsymbol{\alpha}_i}$ is fixed by conformal symmetry up to an overall coefficient $C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$ called the 3-point structure constants

$$\langle V_{\boldsymbol{\alpha}_1}(z_1,\bar{z}_1)V_{\boldsymbol{\alpha}_2}(z_2,\bar{z}_2)V_{\boldsymbol{\alpha}_3}(z_3,\bar{z}_3)\rangle = \frac{C(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}\,, \tag{19}$$

where $z_{ij}:=z_i-z_j$ and Δ_i is the conformal dimension of the primary $V_{\boldsymbol{\alpha}_i}$. Using 2D CFT techniques it has so far only been possible to derive the formula for the 3-point structure constants of the $\mathbf{W}_N$ Toda CFT in some special cases, see [2–4] for the state of the art. The formula for the structure constants in the *semi-degenerate case*, in which one of the three weights becomes proportional to the first ω_1 or to the last ω_{N-1} fundamental SU(N) weight was derived in [2]. Specifically, one sets[4] $\boldsymbol{\alpha}_1 = N\varkappa\omega_1$ or $\boldsymbol{\alpha}_1 = N\varkappa\omega_{N-1}$ and obtains

$$C(N\varkappa\omega_{N-1}, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^{3}\boldsymbol{\alpha}_i,\rho\right)}{b}} \times \\ \times \frac{\Upsilon'(0)^{N-1}\Upsilon(N\varkappa)\prod_{e>0}\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^{N}\Upsilon(\varkappa+(\boldsymbol{\alpha}_2-\mathcal{Q},h_i)+(\boldsymbol{\alpha}_3-\mathcal{Q},h_j))}, \tag{20}$$

where the functions γ and Υ are to be found in appendix A.3. For our conventions concerning SU(N), see appendix A.2. In [1], we argued that the q-deformation of (20) should be given by

$$C_q(N\varkappa\omega_{N-1}, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \left(\frac{\left(1-q^b\right)^2\left(1-q^{b^{-1}}\right)^{2b^2}}{(1-q)^{2(1+b^2)}}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^{3}\boldsymbol{\alpha}_i,\rho\right)}{b}} \\ \times \frac{\Upsilon_q'(0)^{N-1}\Upsilon_q(N\varkappa)\prod_{e>0}\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^{N}\Upsilon_q(\varkappa+(\boldsymbol{\alpha}_2-\mathcal{Q},h_i)+(\boldsymbol{\alpha}_3-\mathcal{Q},h_j))}. \tag{21}$$

We want to emphasize that the q-deformed version of Toda field theory does not have a known Lagrangian description. Thus, everything is defined algebraically in analogy to the usual case via a deformation of the $\mathbf{W}_N$ algebra, see [29] and references therein. Since no Lagrangian description is known for the q-deformed Toda field theory, we can compute everything, up to overall factors containing the cosmological constant. Therefore, we defined the 5D correlation functions (21) up to the $\pi\mu\gamma(b^2)$ term, since they together form the $b \to b^{-1}$ invariant combination. Explicitly, we have for the q-deformed 3-point structure constants

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) \xrightarrow{q\to 1} \left(\pi\mu\gamma(b^2)\right)^{-\frac{\left(2\mathcal{Q}-\sum_{i=1}^{3}\boldsymbol{\alpha}_i,\rho\right)}{b}} C(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3), \tag{22}$$

so that it is clear how one can put back the appropriate $\pi\mu\gamma(b^2)$ factors for a given correlation function.

3 Higgsing the T_N theories

In this section we argue that a particular way of Higgsing the T_N theories, as depicted in figure 2, corresponds to the degeneration that we are in interested on the Toda side. We do this by using the physics of (p, q) 5-brane webs, considering their symmetries and counting the dimension of their moduli spaces, both Higgs and Coulomb. In the next sections we will use the intuition we acquired here to explicitly substitute the values dictated by the web diagram, (32) and (28), in the formula (9) in order to explicitly obtain the formula (20) by Fateev and Litvinov. The physics of the (p, q) 5-brane webs that we will need in this section is well known and extensively studied, in the context that we need, in [9,26,30,31]. We give a short review of their relevant results.

A very useful way of realizing 4D $\mathcal{N} = 2$ quiver gauge theories in string theory is by using type IIA string theory and the Hanany-Witten construction [32] of D4 branes suspended between NS5 branes [33]. This configuration can be lifted to M-theory, where both the D4 and the NS5 branes become a single M5 brane with non-trivial topology, physically realizing the Seiberg-Witten curve in which all the low energy data are encoded [33]. Similarly, 5D $\mathcal{N} = 1$ gauge theories can be realized using type IIB string theory with D5 branes suspended between NS5 branes forming (p, q) 5-brane webs [34, 35]. A large class of $\mathcal{N} = 2$ SCFTs, called class $\mathcal{S}$, can be reformulated (from the realization in [33] with a single M5 brane

[4] We use a slightly different convention than [2]. One has to change $\varkappa \to \frac{\varkappa}{N}$ to obtain the same expressions.

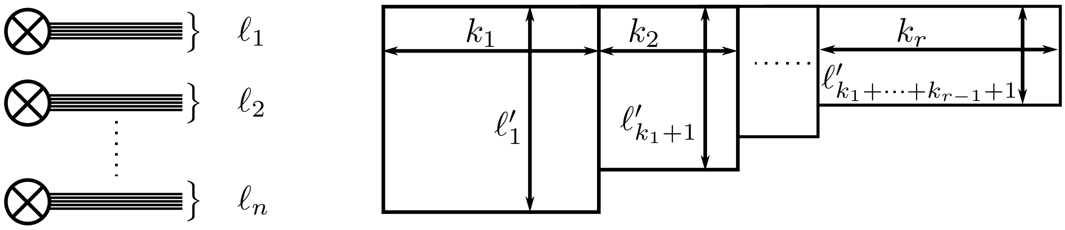

Figure 4: *On the left part of this figure, we see N 5-branes ending on n 7-branes in bunches of $\ell_1, \ldots, \ell_n$ 5-branes each. On the right side of the figure, we depict the Young diagram $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$ that gives the flavor symmetry of the corresponding puncture. Having n bunches of 5-branes, each ending of a 7-brane leads to a puncture in the Gaiotto curve with flavor symmetry $S(U(k_1) \times \cdots \times U(k_r))$, where the widths k_i of the boxes are equal to the numbers of stacks with the same number of branes per stack.*

with non-trivial topology) as a compactification of N M5 branes on a sphere [36]. This point of view is very useful since intersections of these N M5 branes with other M5 branes can be thought of as insertions of defect operators on the world volume of the M5 branes and thus punctures on the sphere. The name *simple puncture* is used for defects that are obtained from the intersection of the original N M5 branes with a single M5 brane (originating from D4's ending on an NS5 in the Hanany-Witten construction), while *full or maximal punctures* stem from defects corresponding to intersections with N semi-infinite M5 branes (external flavor semi-infinite D4's in [33]).

More general punctures, naturally labeled by Young diagrams consisting of N boxes, are also possible [36,37]. In the (p, q) 5-brane web language, they can be described when additional 7-branes are introduced [26]. Semi-infinite (p, q) 5-branes are equivalent to (p, q) 5-branes ending on (p, q) 7-branes [38]. Consider N 5-branes and let them end on n 7-branes, as shown on the left of figure 4. The j^{th} 7-brane carries ℓ_j 5-branes. We define the numbers ℓ'_j as a permutation of the ℓ_j such that they are ordered

$$\ell'_1 \geq \ell'_2 \geq \cdots \geq \ell'_n \,, \tag{23}$$

and arrange them as the columns of a Young diagram[5] $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$, see the right hand side of figure 4. As we started with N 5-branes, the ℓ'_js must obey the condition $\sum_{j=1}^{n} \ell'_j = N$. The integers k_a are defined recursively

$$k_a = \{\# \, \ell'_j : \ell'_j = \ell'_{k_1+\cdots k_{a-1}+1}\} \,, \tag{24}$$

and are equal to the number of columns of equal height. Since the diagonal U(1) of the whole set of the N 5-branes is not realized on the low energy theory [38], the flavor symmetry of the corresponding puncture in the Gaiotto curve is $\mathrm{S}(\mathrm{U}(k_1) \times \cdots \times \mathrm{U}(k_r))$ [36].

Having this toolkit at hand, the authors of [26] where able to show that the Coulomb branch of the T_N theories, corresponding to normalizable deformations of the web which do not change its shape at infinity, has dimension equal to the number of faces in the T_N web diagram, see the left part of figure 2, and has dimension $\frac{(N-1)(N-2)}{2}$, as it should [37]. Moreover, they where able to count the dimension of the Higgs branch of the T_N theories (that was known to be $\frac{3N^2-N-2}{2}$ [37]) by terminating all the external semi-infinite 5-branes on 7-branes and counting the independent degrees of freedom for moving them around on the web-plane. Finally, the global symmetry $\mathrm{SU}(N)^3$ of the T_N theories is realized on the 7-branes.

Beginning with the T_N 5-brane webs which correspond to the sphere with three full punctures (labeled by the Young diagrams $\{1^N\}$) and grouping the N parallel 5 branes of the punctures into smaller bunches

[5] In this article, we draw the Young diagrams in the English notation. By $\{c_1, \ldots, c_r\}$ we mean a Young diagram with r columns for which the j-th column has c_j boxes, $j = 1, \ldots, r$. Furthermore, we use the notation $\{a^b\}$ for the partition $\{a, \ldots, a\}$ with b columns.

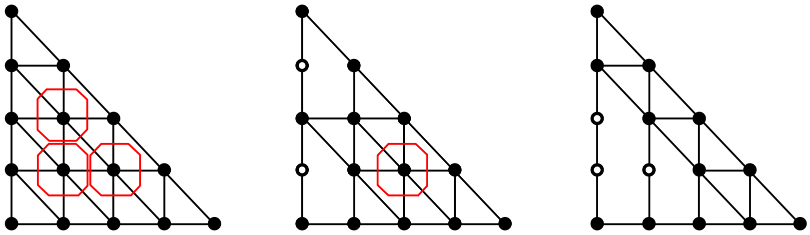

Figure 5: *In this figure we present the dot diagrams of T_4 with three different Higgsings. On the left we have the un-Higgsed dot diagram with three full punctures, $SU(4)^3$ global symmetry and three Coulomb moduli. In the middle, the four D5 branes end on two D7 branes with two D5 branes on each, which corresponds to the Young diagram $\{2,2\}$. This theory has apparent global symmetry $SU(4)^2 \times SU(2)$ and one closed polygon corresponding to one leftover Coulomb modulus. Finally, on the right we have the fully-Higgsed theory with three D5 branes on the first D7 brane and one D5 brane on the second D7. This theory has no Coulomb moduli left.*

(labeled by the Young diagrams $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$), the authors of [26] obtained 5-brane configurations which realize 5D theories with $E_{6,7,8}$ flavor symmetry. These theories have Coulomb and Higgs branches of smaller dimension than the original T_N and can be counted using a generalization of the s-rule [39–41] by using the so called dot diagrams[6]. These theories where further studied in [9, 30, 31]. For us, the important result from [26] is that the dimension of the Higgs moduli space of a puncture corresponding to the Young diagram depicted in figure 4 is

$$\dim_{\mathbb{H}} \mathcal{M}_H^p = \sum_{j=1}^{n} (j-1)\,\ell_j\,, \tag{25}$$

and that the Coulomb branch is the number of closed dual polygons in the dot diagram.

We need to decide which puncture (Young diagram $\{\ell'_1, \ell'_2, \ldots, \ell'_n\}$) corresponds to the Fateev-Litvinov semi-degenerate primary operator. This puncture should have only $U(1)$ symmetry (for $N > 2$). Thus, it can be obtained by grouping the N5-branes in two bunches of unequal number of 5-branes, $N-1$ and 1 respectively, forming the L-shaped Young diagram $\{N-1, 1\}$ shown in figure 3. For $N = 2$, the puncture has an SU(2) flavor symmetry, while for $N \geq 3$ the flavor symmetry gets reduced to $U(1)$, as required for the semi-degenerate field. The Higgs moduli space of this configuration has $\dim_{\mathbb{H}} \mathcal{M}_H^{\text{semi-deg}} = 1$ which is consistent with the fact that we have only one parameter $\varkappa$ in the CFT side. Finally, the dot diagrams tell us that the dimension of the Coulomb branch in this case is zero, which, as we will see later, is consistent with what one gets by just substituting (29) in (9).

Now, let us discuss what happens with the Kähler moduli that parametrize the T_N partition functions as we bring together $N-1$ parallel horizontal external D5 branes on a single D7 brane. These we will then translate in the language of mass parameters m_i, n_i, l_i ($i = 1, \ldots, N$) and Coulomb moduli a_r ($r = 1, \ldots, {}^{(N-1)(N-2)}/_2$) using the dictionary of appendix A.1 and in particular equation (A.4) and, finally, to the Toda weights $\boldsymbol{\alpha}_{1,2,3}$ using (7). We follow closely the discussion in [30]. For simplicity, we begin with two parallel D5 branes that originally end on different D7 branes. This process is depicted in figure 6. First we need to shrink u_2 of $U_2 = e^{-\beta u_2}$ to zero while still having two 7-branes. In the process of sending the u_1 of $U_1 = e^{-\beta u_1}$ to zero, one of the two D7 branes will meet a D5 brane and the two parallel D5 branes will fractionate on the D7 branes. After moving the cut piece to infinity it effectively decouple from the rest of the web.

[6] The dot diagrams are the dual graphs of the web diagrams with the additional information about the 7-branes encoded in white and black dots.

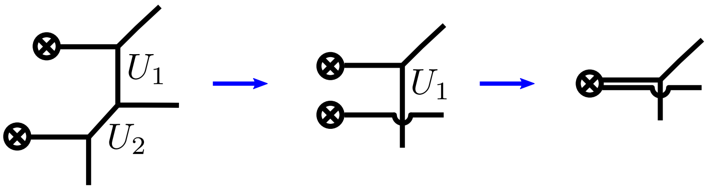

Figure 6: *This figure shows the way two 5-branes are brought on the same 7-brane [9].*

For the unrefined topological strings, *i.e.* for $\epsilon_2 = -\epsilon_1$, shrinking the length of a 5-brane that is parametrized[7] by $U = e^{-\beta u}$ corresponds to setting $U = 1$. This is not true any more in the case of the refined topological string where zero size will correspond either to $U = \sqrt{\mathfrak{t}/\mathfrak{q}}$ or $U = \sqrt{\mathfrak{q}/\mathfrak{t}}$ [42–45]. It turns out that both choices are equivalent as is extensively discussed in [30]. In this paper we wish to consider only the parameter space that corresponds to Toda with $Q = \epsilon_1 + \epsilon_2 > 0$, *i.e.* $\mathfrak{t}/\mathfrak{q} > 1$, and thus we have to pick $U = \sqrt{\mathfrak{t}/\mathfrak{q}}$.

For the T_3 case the situation is exactly the same as the simple example depicted in figure 6. The following two Kähler parameters

$$Q^{(1)}_{m;1} = \mathsf{A}^{-1}\tilde{M}_1\tilde{N}_1 \quad \text{and} \quad Q^{(1)}_{l;1} = \mathsf{A}\tilde{M}_2^{-1}\tilde{N}_1^{-1} \tag{26}$$

are the ones we have to shrink, where $\mathbf{A} \equiv \tilde{A}^{(1)}_1$ is the Coulomb modulus of T_3. See appendix A.1 for notations and figure 9 for the web diagram of T_3. Thus, we have to set

$$Q^{(1)}_{m;1} = Q^{(1)}_{l;1} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\,. \tag{27}$$

In general for T_N as depicted in figure 13 we must tune

$$Q^{(j)}_{m;i} = Q^{(j)}_{l;i} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}} \qquad \text{with} \qquad i = 1,\ldots,N-2, \quad j = 1,\ldots,N-1-i\,. \tag{28}$$

Going back to the Toda side, we wish to semi-degenerate the weight $\boldsymbol{\alpha}_1$, *i.e.* set it to

$$\boldsymbol{\alpha}_1 = N\varkappa\omega_{N-1} \qquad \Longleftrightarrow \qquad m_i = \begin{cases} \varkappa - \frac{N+1-2i}{2}Q & i = 1,\ldots,N-1\,, \\ -(N-1)\varkappa + \frac{N-1}{2}Q & i = N\,, \end{cases} \tag{29}$$

where the implications from (7) of the semi-degeneration on the mass parameters are written on the right. For the T_3 case that implies for the exponentiated mass parameters that

$$\tilde{M}_1 = \tilde{K}\frac{\mathfrak{t}}{\mathfrak{q}} = e^{-\beta(\varkappa - Q)} \qquad \text{and} \qquad \tilde{M}_2 = \tilde{K} \tag{30}$$

which is consistent with (26) and (27) when the Coulomb moduli is tuned to the value

$$\mathsf{A} = \tilde{K}\tilde{N}_1\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\,. \tag{31}$$

This is compatible with the statement that after Higgsing, the T_3 the dimension of the Coulomb branch is zero, and also with the fact that we will discuss in next section, the contour integral gets pinched once one substitutes (29) in (9). In the general T_N case, Higgsing forces the Coulomb parameters to become

$$\tilde{A}^{(j)}_i = \tilde{K}^i \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{i(N-i-j)}{2}} \prod_{k=1}^{j} \tilde{N}_k\,, \tag{32}$$

[7]The parameter u in the exponent is the length of the 5 brane segment.

where $i, j = 1, \ldots, N-2$, $i+j \leq N-1$ and $\tilde{K} = e^{-\beta\varkappa}$. This implies that the Kähler parameters obey (28).

A remark on the physicality condition for the $\mathbf{W}_N$ Toda weights $\boldsymbol{\alpha}$ is in order. Denoting by $\Delta(\boldsymbol{\alpha})$ the conformal dimension of the primary field $V_{\boldsymbol{\alpha}}$, the formula for the two point functions

$$\langle V_{\boldsymbol{\alpha}'}(z', \bar{z}')V_{\boldsymbol{\alpha}}(z, \bar{z})\rangle = \frac{(2\pi)^{N-1}\delta(\boldsymbol{\alpha} + \boldsymbol{\alpha}' - 2\mathcal{Q}) + \text{Weyl-reflections}}{|z - z'|^{4\Delta(\boldsymbol{\alpha})}}\,, \tag{33}$$

tells us that requiring that $V_{\boldsymbol{\alpha}'}$ be the conjugate field to $V_{\boldsymbol{\alpha}}$ leads to the following reality condition[8]

$$\Re(\boldsymbol{\alpha}) = \mathcal{Q} \qquad \Longleftrightarrow \qquad m_i, n_i, l_i \in i\mathbb{R}\,. \tag{34}$$

On the Toda side, the physicality condition for the Toda weights (34) implies through the dictionary (7) that the mass parameters are purely imaginary. On the (p, q) 5-brane web diagram side, distances are measured by the real part of the mass parameters, see equations (2.7-2.12) of [17] for a review of the conventions. When the 5-branes are on top of each other, *i.e.* when their distance is zero[9], T_N has $\mathrm{SU}(N)^3$ symmetry [26] and we can have physical Toda states. Since $\mathcal{Q} = Q\sum_{i=1}^{N-1}\omega_i$ and since semi-degeneration requires that $\boldsymbol{\alpha} = N\varkappa\omega_{N-1}$, we see that semi-degeneration/Higgsing is incompatible with the physicality condition (34). This is compatible with CFT intuition [46].

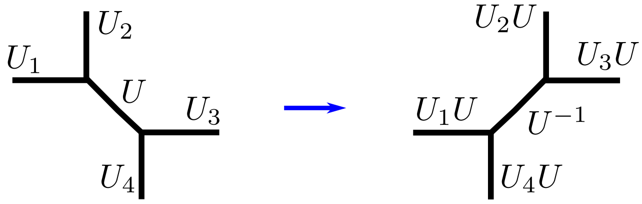

Figure 7: *The figure illustrates the change of the Käher parameters upon flopping.*

We wish to conclude this section by stressing that the formulas we are dealing with have different domains with different convergent expansions depending on the values of the masses, just like in (A.19). In the topological string language they correspond to different geometries that are related to each other by flopping. For each Kähler parameter U, we distinguish between the region $|U| > 1$ and the one with $|U| < 1$; to each is associated a different (p, q) 5-brane web diagram. Going from one region to the other involves "flopping" which transforms the Kähler parameters as depicted in figure 7. See [47] for a recent discussion of the topic. In the next section, we explain how the contour in (9) is to be chosen and we argue that the contour is dictated by the choice of the flopping frame.

4 The Higgsed T_3 theory and the semi-degenerate $\mathbf{W}_3$ 3-point functions

In this section, we illustrate the relationships between Higgsing and semi-degeneration with the simplest example, namely T_3. For this, we show how semi-degeneration of the masses m_i "pinches" the contour integral, so that the result is given by only one residue. Furthermore, we discuss the relationship between the potential residues from the contour integral and the different Higgsed geometries.

[8]See section 4 and 11 of [46] for a detailed discussion of the physicality condition in the Liouville case.
[9]In the refined topological vertex, the Seiberg-Witten curve is replaced by its quantum version in which zero distance is understood as integer multiples of ϵ_+.

Let us first make a trivial example to illustrate our situation. Let g be a meromorphic function in a domain $D \subset \mathbb{C}$ that has only simple poles at the points a, b and p_i, meaning that it can be written as

$$g(z) = \frac{f(z)}{(z-a)(z-b)\prod_i (z-p_i)}, \tag{35}$$

where f is a holomorphic function in D. Let $\mathcal{C}$ be a closed contour in D that encircles a as well as the p_i but not b. We write $a = p + \delta$ and $b = p - \delta$ and take the limit $\delta \to 0$, thus letting the two points a and b collide on the contour $\mathcal{C}$ on both sides, as depicted in figure 8. If we now compute the contour integral of

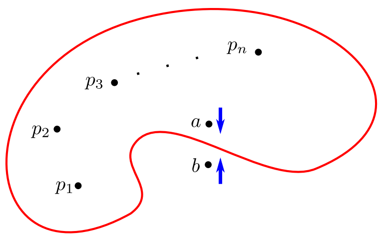

Figure 8: *The figure shows an example of contour pinching. As the poles at a and at b collide, the contour integral diverges, which is why we regulate it by multiplying with $a - b$. In the limit $a \to b$, the integral is given by a single residue.*

g around $\mathcal{C}$ and multiply it by $a - b$, we obtain

$$\begin{aligned} (a-b)\oint_{\mathcal{C}} \frac{dz}{2\pi i} g(z) &= \frac{f(a)}{\prod_i (a-p_i)} + \sum_i \frac{(a-b)f(p_i)}{(p_i-a)(p_i-b)\prod_{j\neq i}(p_i-p_j)} \\ &\overset{\delta\to 0}{\longrightarrow} \frac{f(p)}{\prod_i (p-p_i)} = \lim_{a\to b}\left[(a-b)\mathrm{Res}(g(z),a)\right] . \end{aligned} \tag{36}$$

Thus, in the limit $a \to b$, the contour gets pinched at the point $a = b = p$ and the integral is given by a single residue. This is essentially the contour integral version of the identity $\lim_{\varepsilon\to 0} \frac{\varepsilon}{(x+i\varepsilon)(x-i\varepsilon)} = \pi\delta(x)$. This example can also be easily generalized to the case in which g has not only simple poles, but we will not need it.

We now want to explain how this simple example applies to our integral formulas for the correlation functions of T_3. In the T_3 case, formula (14) for the structure constants reads

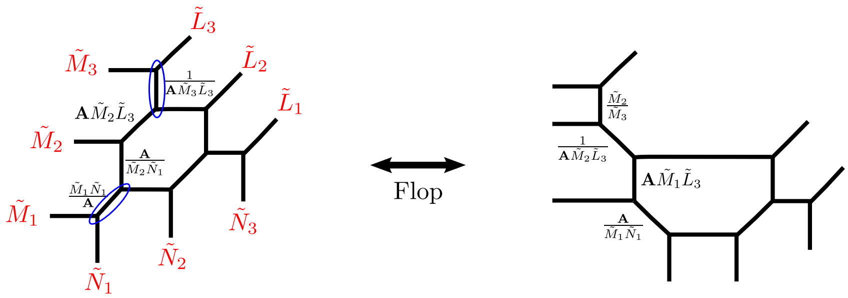

Figure 9: *This figure shows the two flopping frames for T_3 for which it is possible to choose an integration contour for $\boldsymbol{A}$ that gets pinched in the semi-degenerate limit. The geometry on the left corresponds to the domain (38), while the one on the right corresponds to (39). One can obtain the right geometry from the left one by applying two flopping moves, see figure 7, to the encircled segments.*

$$C_q(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = \text{const} \times \oint \frac{d\mathbf{A}}{2\pi i \mathbf{A}} \, |M(\mathfrak{t}, \mathfrak{q})|^2 \, |\mathcal{Z}_3^{\text{top}}|^2 \,, \tag{37}$$

where $A_1^{(1)} \equiv \mathbf{A} = e^{-\beta \mathbf{a}}$. We first need to determine the contour. As we discussed in the previous section, for each flopping frame, the Käher parameters are subject to specific constraints. In our case, we have two relevant flopping frames as shown in figure 9. The first one is valid for

$$\left| \frac{\tilde{M}_1 \tilde{N}_1}{\mathbf{A}} \right|^2 > 1 \,, \qquad \left| \frac{\mathbf{A}}{\tilde{M}_2 \tilde{N}_1} \right|^2 > 1 \,, \qquad \left| \mathbf{A} \tilde{M}_2 \tilde{L}_3 \right|^2 > 1 \,, \tag{38}$$

while the second one requires

$$\left| \frac{\mathbf{A}}{\tilde{M}_1 \tilde{N}_1} \right|^2 > 1 \,, \qquad \left| \mathbf{A} \tilde{M}_1 \tilde{L}_3 \right|^2 > 1 \,, \qquad \left| \frac{1}{\mathbf{A} \tilde{M}_2 \tilde{L}_3} \right|^2 > 1 \,. \tag{39}$$

They are related to each other by two flopping moves as illustrated in figure 7. For a given flopping frame, the contour integral in $\mathbf{A}$ has to be chosen such that the inequalities (38) or (39) are satisfied. As we will show in the rest of this section, picking different flopping frames, *i.e.* picking different contours, leads to having different poles contribute to the integral, which however leaves the final answer invariant up to a sign.

Let us now turn our attention to the poles in the contour integral. From (15), we get for $N = 3$ the following expression for the topological string amplitude

$$\begin{aligned}
|\mathcal{Z}_3^{\text{top}}|^2 = |\mathcal{Z}_3^{\text{pert}}|^2 |\mathcal{Z}_3^{\text{inst}}|^2 &= \frac{\left| \left[\prod_{1 \le i < j \le 3} \mathcal{M}\left(\frac{\tilde{M}_i}{\tilde{M}_j}\right) \right] \mathcal{M}\left(\mathbf{A}^2 \frac{\tilde{L}_3}{\tilde{N}_1}\right) \mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}} \mathbf{A}^2 \frac{\tilde{L}_3}{\tilde{N}_1}\right) \right|^2}{\left| \left[\prod_{k=1}^3 \mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}} \mathbf{A} \tilde{M}_k \tilde{L}_3\right) \mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}} \frac{\mathbf{A}}{\tilde{M}_k \tilde{N}_1}\right) \right] \mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}} \frac{\mathbf{A} \tilde{N}_2}{\tilde{L}_1}\right) \mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}} \frac{\mathbf{A} \tilde{N}_3}{\tilde{L}_2}\right) \right|^2} \\
&\times \left| \sum_{\boldsymbol{\nu}} \left(\frac{\tilde{N}_1 \tilde{L}_2}{\tilde{N}_2 \tilde{L}_3} \right)^{\frac{|\nu_1^{(1)}| + |\nu_2^{(1)}|}{2}} \left(\frac{\tilde{N}_2 \tilde{L}_1}{\tilde{N}_3 \tilde{L}_2} \right)^{\frac{|\nu_1^{(2)}|}{2}} \frac{\prod_{k=1}^3 \left[\mathsf{N}^\beta_{\nu_1^{(1)} \emptyset}(\mathbf{a} - m_k - n_1 - Q/2) \, \mathsf{N}^\beta_{\emptyset \nu_2^{(1)}}(\mathbf{a} + m_k + l_3 - Q/2) \right]}{\mathsf{N}^\beta_{\nu_1^{(1)} \nu_1^{(1)}}(0) \, \mathsf{N}^\beta_{\nu_2^{(1)} \nu_2^{(1)}}(0) \, \mathsf{N}^\beta_{\nu_1^{(2)} \nu_1^{(2)}}(0)} \right. \\
&\times \left. \frac{\mathsf{N}^\beta_{\nu_1^{(1)} \nu_1^{(2)}}(\mathbf{a} + n_2 - l_1 - Q/2) \, \mathsf{N}^\beta_{\nu_1^{(2)} \nu_2^{(1)}}(\mathbf{a} + n_3 - l_2 - Q/2)}{\mathsf{N}^\beta_{\nu_1^{(1)} \nu_2^{(1)}}(2\mathbf{a} - n_1 + l_3) \, \mathsf{N}^\beta_{\nu_2^{(1)} \nu_1^{(1)}}(-2\mathbf{a} + n_1 - l_1)} \right|^2 .
\end{aligned} \tag{40}$$

Since we wish to evaluate the contour integral (37) in the semi-degenerate limit $\boldsymbol{\alpha}_1 = 3\varkappa \omega_2$, we introduce a regulator δ and parametrize the three masses labeling the positions of the branes on the left as

$$m_1 = \varkappa + \delta - Q \,, \qquad m_2 = \varkappa - \delta \,, \qquad m_3 = -2\varkappa + Q \,, \tag{41}$$

which implies that the exponentiated masses $\tilde{M}_i = e^{-\beta m_i}$ are

$$\tilde{M}_1 = \tilde{K} e^{-\beta \delta} \frac{\mathfrak{t}}{\mathfrak{q}} \,, \qquad \tilde{M}_2 = \tilde{K} e^{\beta \delta} \,, \qquad \tilde{M}_3 = \tilde{K}^{-2} \frac{\mathfrak{q}}{\mathfrak{t}} \,, \tag{42}$$

with $\tilde{K} = e^{-\beta \varkappa}$. The semi-degenerate limit then corresponds to $\delta \to 0$. For these values of the masses, the numerator of $|\mathcal{Z}_3^{\text{top}}|^2$ in (40) goes to zero, just like the term $a - b$ in (36), since

$$|\mathcal{M}(\tilde{M}_1 \tilde{M}_2^{-1})|^2 = (1 - e^{-2\beta\delta}) \times \text{reg.} \approx \delta \times \text{reg.} \,, \tag{43}$$

where "reg" are terms that don't vanish for $\delta \to 0$. Let us now analyze the poles in the integrand of (40) and determine which ones will contribute in the semi-degenerate limit. We make the assumption that only poles from the "perturbative" part, *i.e.* the first line of (40), are relevant for this computation, which will be justified by the final result. We need to find poles that lie on different sides of the contour and that collide when the regulator is removed. The relevant poles in the integrand come from the zeroes of the functions $|\mathcal{M}(u)|^2$ in the first line of (40). Since, in order to obtain the Toda theory from topological strings we wish to have $b > 0$, so that $|\mathfrak{q}| < 1$ and $|\mathfrak{t}| > 1$, we get from (A.19) the expression

$$|\mathcal{M}(u;\mathfrak{t},\mathfrak{q})|^2 = \mathcal{M}(u;\mathfrak{t},\mathfrak{q})\mathcal{M}(u^{-1};\mathfrak{t}^{-1},\mathfrak{q}^{-1}) = \prod_{i,j=1}^{\infty}(1-u\mathfrak{t}^{-i}\mathfrak{q}^{j})(1-u^{-1}\mathfrak{t}^{1-i}\mathfrak{q}^{j-1})\,. \tag{44}$$

Thus, the zeroes of $|\mathcal{M}(u)|^2$ are to be found on the points

$$u = \mathfrak{t}^{-m}\mathfrak{q}^{n}\,, \qquad u = \mathfrak{t}^{m+1}\mathfrak{q}^{-n-1}\,, \tag{45}$$

for $m, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We see that there are two classes of poles of $|\mathcal{Z}^{\text{top}}|^2$, namely those that condense around zero in the $\mathbf{A}$ complex plane and those that condense around infinity.

We now pick our contour according to the two different choices of flopping frames in figure 9. On one hand, if we choose the first geometry, the contour has to lie in the domain (38), which in the semi-degenerate limit implies

$$\left|\tilde{K}\tilde{N}_1 e^{\beta\delta}\right| < |\mathbf{A}| < \left|\tilde{K}\tilde{N}_1 e^{-\beta\delta}\frac{\mathfrak{t}}{\mathfrak{q}}\right|\,, \qquad |\mathbf{A}| > \left|\tilde{K}^{-1}\tilde{L}_3^{-1}e^{-\beta\delta}\right|\,. \tag{46}$$

On the other hand, the other geometry forces the contour to be in the domain (39), which means that

$$\left|\tilde{K}^{-1}\tilde{L}_3^{-1}e^{\beta\delta}\frac{\mathfrak{q}}{\mathfrak{t}}\right| < |\mathbf{A}| < \left|\tilde{K}^{-1}\tilde{L}_3^{-1}e^{-\beta\delta}\right|\,, \qquad |\mathbf{A}| > \left|\tilde{K}\tilde{N}_1 e^{\beta\delta}\frac{\mathfrak{t}}{\mathfrak{q}}\right|\,. \tag{47}$$

When we then take the limit $\delta \to 0$, some poles from the exterior of the contour integral with coincide with some from the interior, leading to a divergence that will cancel the zero of (44), just like in the simple example of equation (36). We easily see that the relevant terms in the denominator of the first line of (40) are

$$\left|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_1^{-1}\tilde{N}_1^{-1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_2^{-1}\tilde{N}_1^{-1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_1\tilde{L}_3\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathbf{A}\tilde{M}_2\tilde{L}_3\right)\right|^2\,. \tag{48}$$

The other zeroes in the denominator will not pinch the integral once the regulator δ is set to zero and can be ignored, just like the p_i terms in (36). Numbering the functions $\mathcal{M}$ as 1 to 4 in (48) from left to right, using (45) and the parametrization (41), we know that we have first order poles in the integrand if

$$\begin{aligned}
&(1) && \mathbf{A} = \tilde{K}\tilde{N}_1 e^{-\beta\delta}\mathfrak{t}^{-m+\frac{1}{2}}\mathfrak{q}^{n-\frac{1}{2}}\,, && (\bar{1}) && \mathbf{A} = \tilde{K}\tilde{N}_1 e^{-\beta\delta}\mathfrak{t}^{m+\frac{3}{2}}\mathfrak{q}^{-n-\frac{3}{2}}\,,\\
&(2) && \mathbf{A} = \tilde{K}\tilde{N}_1 e^{\beta\delta}\mathfrak{t}^{-m-\frac{1}{2}}\mathfrak{q}^{n+\frac{1}{2}}\,, && (\bar{2}) && \mathbf{A} = \tilde{K}\tilde{N}_1 e^{\beta\delta}\mathfrak{t}^{m+\frac{1}{2}}\mathfrak{q}^{-n-\frac{1}{2}}\,,\\
&(3) && \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1}e^{\beta\delta}\mathfrak{t}^{-m-\frac{3}{2}}\mathfrak{q}^{n+\frac{3}{2}}\,, && (\bar{3}) && \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1}e^{\beta\delta}\mathfrak{t}^{m-\frac{1}{2}}\mathfrak{q}^{-n+\frac{1}{2}}\,,\\
&(4) && \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1}e^{-\beta\delta}\mathfrak{t}^{-m-\frac{1}{2}}\mathfrak{q}^{n+\frac{1}{2}}\,, && (\bar{4}) && \mathbf{A} = \tilde{K}^{-1}\tilde{L}_3^{-1}e^{-\beta\delta}\mathfrak{t}^{m+\frac{1}{2}}\mathfrak{q}^{-n-\frac{1}{2}}\,,
\end{aligned} \tag{49}$$

for $m, n \in \mathbb{N}_0$. We have labeled with a $\bar{\cdot}$ those sets of poles that coalesce around $\mathbf{A} = \infty$. The choice of the domain (46) or (47) does not fully constrain the form of the contour. We argue that it is natural to choose the contour such that for generic values of the parameters it lies between the sets of poles in (49) that condense around zero and those that condense around infinity. For the first choice of the domain (46), we show a possible contour in figure 10. The contour has to also pass between two poles coming the lines $\bar{3}$ and $\bar{4}$, which will not lead to pinching in the limit $\delta \to 0$. We see that, due to set of poles 1

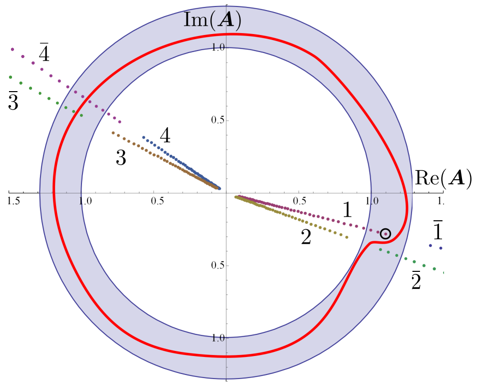

Figure 10: *The figure shows the integration contour for the choice of allowed domain* (46)*. The allowed domain is the shaded region, shown here for* $|\tilde{K}| = |\tilde{N}_1| = |\tilde{L}_3| = 1$*. As the variable* δ *is sent to zero, the contour gets pinched between two zeroes and the contributions are given by a residues on one of the poles of type* 1 *whose position is indicated by a black circle. The set of poles are labeled according to* (49)*.*

colliding with the set of poles $\bar{2}$ for $m = n = 0$, the integral gets pinched as $\delta \to 0$ and that the result is given by the residue at

$$\mathbf{A} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1 e^{-\beta\delta} \,. \tag{50}$$

We see that for the choice of contour in figure 10, the fact that for $\delta \to 0$ we get an overlap between a pole from 3 and a pole from $\bar{4}$ is of no consequence since they both lie of the same side of the contour. The geometry corresponding to the residue at (50) is depicted in figure 11. Let us now compute the residue of

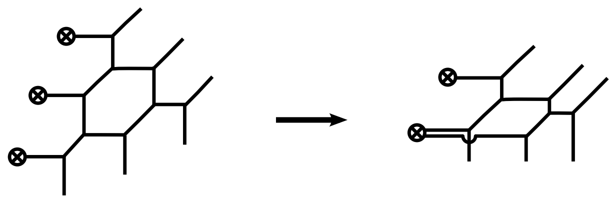

Figure 11: *The figure shows the Higgsed geometry corresponding to the residue* (50)*. For this residue, the Kähler parameters take the values* (27)*.*

$|\mathcal{Z}_3^{\text{top}}|^2$ at (50) directly. We can use the fact that for a function f that has no pole at $B\mathfrak{t}^k\mathfrak{q}^l$, we have

$$\text{Res}\left(\frac{f(\mathbf{A})}{\mathbf{A}|\mathcal{M}(\mathbf{A}B^{-1})|^2}, \mathbf{A} = B\mathfrak{t}^k\mathfrak{q}^l\right) = \frac{\mathfrak{g}_{-k,l}}{|M(\mathfrak{t},\mathfrak{q})|^2} f(B\mathfrak{t}^k\mathfrak{q}^l) \,. \tag{51}$$

Here $|M(\mathfrak{t},\mathfrak{q})|^2$ is the norm squared of the refined MacMahon function defined in (A.30) and the function $\mathfrak{g}_{kl}$ is defined as

$$\mathfrak{g}_{kl}(\mathfrak{t},\mathfrak{q}) := \lim_{u\to 1}\frac{|\mathcal{M}(u)|^2}{|\mathcal{M}(u\mathfrak{t}^{-k}\mathfrak{q}^l)|^2} = \prod_{i=1}^{k}\frac{(\mathfrak{t}^{-i}\mathfrak{q}^{l+1};\mathfrak{q})_\infty}{(\mathfrak{t}^{i}\mathfrak{q}^{-l};\mathfrak{q})_\infty}\prod_{j=1}^{l}\frac{(\mathfrak{t}^{-1}\mathfrak{q}^{j};\mathfrak{t}^{-1})_\infty}{(\mathfrak{q}^{-j};\mathfrak{t}^{-1})_\infty} \,, \tag{52}$$

where we have used the shift properties (A.23) of the $\mathcal{M}$ functions and the last equality is only valid for $k, l \in \mathbb{N}_0$. The above expression can be continued for negative k and l with $\mathfrak{g}_{kl} = -\mathfrak{g}_{-k-1,-l-1}$. In particular $\mathsf{g}_{-n,0} = \mathsf{g}_{0,-n} = 0$ for $n \geq 1$. Therefore, choosing the contour to lie in the first domain (46), letting $\delta \to 0$ and using (51)

$$\lim_{\delta\to 0} \oint \frac{d\mathbf{A}}{2\pi i \mathbf{A}} |M(\mathfrak{t},\mathfrak{q})|^2 |\mathcal{Z}_3^{\rm top}|^2 = |M(\mathfrak{t},\mathfrak{q})|^2 \,\mathrm{Res}\left(|\mathcal{Z}_3^{\rm top}|^2, \mathbf{A} = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1 e^{-\beta\delta}\right) \tag{53}$$

$$= \frac{\left|\mathcal{M}(\tilde{K}^{-3})\right|^2}{\left|\prod_{k=1}^3 \mathcal{M}\left(\frac{\tilde{N}_k \tilde{L}_{4-k}}{\tilde{K}}\right)\right|^2} \left|Z_3^{\rm inst}\right|^2_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1} .$$

One can observe that due to (A.38), the sum over $\nu_1^{(1)}$ in $\left|\mathcal{Z}_3^{\rm inst}\right|^2_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1}$ drops out and we obtain the result

$$\left(\mathcal{Z}_3^{\rm inst}\right)\Big|_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1} = \sum_{\nu_1,\nu_2} \left(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{N}_3\tilde{L}_2}\right)^{\frac{|\nu_1|}{2}} \left(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{N}_2\tilde{L}_3}\right)^{\frac{|\nu_2|}{2}}$$
$$\times \frac{\mathsf{N}^\beta_{\nu_1\emptyset}(n_3+l_1-\varkappa)\mathsf{N}^\beta_{\nu_2\nu_1}(n_2+l_2-\varkappa)\mathsf{N}^\beta_{\emptyset\nu_2}(n_1+l_3-\varkappa)}{\mathsf{N}^\beta_{\nu_1\nu_1}(0)\mathsf{N}^\beta_{\nu_2\nu_2}(0)}, \tag{54}$$

where we denoted $\nu_1^{(2)} \equiv \nu_1$, $\nu_2^{(1)} \equiv \nu_2$.

We also calculated the contour integral for the second choice of the flopping frame, *i.e.* the domain (46). We find that for that choice of the contour, the result is given by the residue of $\left|\mathcal{Z}_3^{\rm top}\right|^2$ at

$$\mathbf{A} = \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\tilde{K}^{-1}\tilde{L}_3^{-1} e^{-\beta\delta}, \tag{55}$$

which, together with (41) implies for $\delta \to 0$ the Higgsed geometry shown in figure 12. Computing the

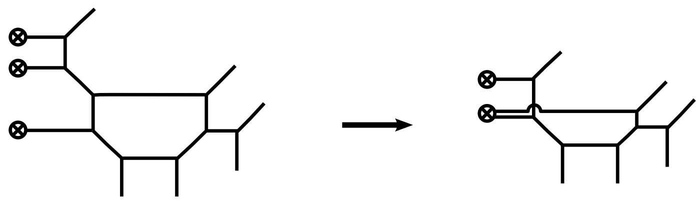

Figure 12: *The figure shows the Higgsed geometry corresponding to the residue* (55).

residue, we find that the "perturbative" contribution, *i.e.* the prefactor of $\left|\mathcal{Z}_3^{\rm inst}\right|^2$ in (54), is the same as before. Furthermore, we find after relabeling $\nu_2^{(1)} \leftrightarrow \nu_1^{(1)}$ and using (A.39) that the "instanton" contribution in (54) is unchanged, *i.e.*

$$\left(\mathcal{Z}_3^{\rm inst}\right)\Big|_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1} = \left(\mathcal{Z}_3^{\rm inst}\right)\Big|_{\mathbf{A}=\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\tilde{K}^{-1}\tilde{L}_3^{-1}} . \tag{56}$$

In order to complete the computation, we need to calculate the sum in (54) over the two remaining partitions. For this purpose, we shall use the following identity that we shall state in full generality in section 5 and prove in appendix B:

$$\sum_{\nu_1,\nu_2}\left(V_1\sqrt{U_1U_2}\right)^{|\nu_1|}\left(V_2\sqrt{U_2U_3}\right)^{|\nu_2|}\frac{\mathsf{N}^{\beta}_{\nu_1\emptyset}\left(u_1-Q/2\right)\mathsf{N}^{\beta}_{\nu_2\nu_1}\left(u_2-Q/2\right)\mathsf{N}^{\beta}_{\emptyset\nu_2}\left(u_3-Q/2\right)}{\mathsf{N}^{\beta}_{\nu_1\nu_1}(0)\,\mathsf{N}^{\beta}_{\nu_2\nu_2}(0)} \tag{57}$$
$$=\frac{\mathcal{M}(U_1V_1)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_1U_2)\mathcal{M}(U_2V_2)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_2U_3)\mathcal{M}(U_1V_1U_2V_2)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_1U_2V_2U_3)}{\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_1)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_2)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_1V_1U_2)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_1U_2V_2)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_2V_2U_3)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_1V_1U_2V_2U_3)},$$

where $U_i := e^{-\beta u_i}$. Upon making the following substitutions in (57)

$$U_k=\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{\tilde{N}_{4-k}\tilde{L}_k}{\tilde{K}},\qquad V_1=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{K}}{\tilde{N}_3\tilde{L}_2},\qquad V_2=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{K}}{\tilde{N}_2\tilde{L}_3}, \tag{58}$$

where $k=1,2,3$, we arrive at

$$\left(\mathcal{Z}_3^{\text{inst}}\right)\big|_{\mathbf{A}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{K}\tilde{N}_1}=\frac{\mathcal{M}(\frac{\tilde{L}_1}{\tilde{L}_2})\mathcal{M}(\frac{\tilde{L}_2}{\tilde{L}_3})\mathcal{M}(\frac{\tilde{L}_1}{\tilde{L}_3})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_1}{\tilde{N}_2})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_2}{\tilde{N}_3})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_1}{\tilde{N}_3})}{\mathcal{M}(\frac{\tilde{N}_1\tilde{L}_1}{\tilde{K}})\mathcal{M}(\frac{\tilde{N}_1\tilde{L}_2}{\tilde{K}})\mathcal{M}(\frac{\tilde{N}_2\tilde{L}_1}{\tilde{K}})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_2\tilde{L}_3})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_3\tilde{L}_2})\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_3\tilde{L}_3})}. \tag{59}$$

Inserting the above into (53), we arrive at

$$\begin{aligned}\lim_{\delta\to 0}\oint\frac{d\mathbf{A}}{2\pi i\mathbf{A}}\left|M(\mathfrak{t},\mathfrak{q})\right|^2\left|\mathcal{Z}_3^{\text{top}}\right|^2 &= \frac{\left|\mathcal{M}(\tilde{K}^{-3})\prod_{1\le i<j\le 3}\mathcal{M}\left(\tilde{N}_j/\tilde{N}_i\right)\mathcal{M}\left(\tilde{L}_i/\tilde{L}_j\right)\right|^2}{\left|\prod_{i,j=1}^3\mathcal{M}(\tilde{N}_i\tilde{L}_j\tilde{K}^{-1})\right|^2}\\ &= \frac{(1-q)^{\tilde{\varphi}_3}}{\Lambda^2}\frac{\Upsilon_q(3\varkappa)\prod_{1\le i<j\le 3}\Upsilon_q(n_i-n_j)\Upsilon_q(l_{4-i}-l_{4-j})}{\prod_{i,j=1}^3\Upsilon_q(\varkappa-n_i-l_{4-j})},\end{aligned} \tag{60}$$

where we have used (A.24), (A.25) and defined the exponent

$$\begin{aligned}\tilde{\varphi}_3 &= \left(\frac{Q}{2}-3\varkappa\right)^2+\sum_{1\le i<j\le 3}\left[\left(\frac{Q}{2}+n_j-n_i\right)^2+\left(\frac{Q}{2}+l_{4-j}-l_{4-i}\right)^2\right]-\sum_{i,j=1}^3\left(\frac{Q}{2}+n_i+l_{4-j}-\varkappa\right)^2\\ &= 2Q\left(3\varkappa+\sum_{i=1}^3 i(n_i+l_{4-i})\right)-\frac{Q^2}{2}=-2Q\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)-\frac{Q^2}{2},\end{aligned} \tag{61}$$

where in the last line we have used our SU(3) conventions, see appendix A.2 and equation (7). Now we employ (A.31) and rearrange the prefactors of (60) to obtain the q-deformed $\mathbf{W}_3$ Fateev-Litvinov structure constants (21) in the form conjectured by [1]:

$$\begin{aligned}&C_q(3\varkappa\omega_2,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)=\\ &=\left(\beta\left|M(\mathfrak{t},\mathfrak{q})\right|^2\right)^2\left(\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}\right)^{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}\lim_{\delta\to 0}\oint\frac{d\mathbf{A}}{2\pi i\mathbf{A}}\left|M(\mathfrak{t},\mathfrak{q})\right|^2\left|\mathcal{Z}_3^{\text{top}}\right|^2\\ &=\left(\frac{\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}}{(1-q)^{2Q}}\right)^{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}\frac{\Upsilon_q'(0)^2\Upsilon_q(3\varkappa)\prod_{e>0}\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^3\Upsilon_q(\varkappa+(\boldsymbol{\alpha}_2-\mathcal{Q},\,h_i)+(\boldsymbol{\alpha}_3-\mathcal{Q},\,h_j))}.\end{aligned} \tag{62}$$

Taking the 4D limit $q\to 1$ then leads to the usual formula (20), up to a prefactor depending on b and the cosmological constant μ.

5 The general T_N case

Having computed the structure constants for the T_3 case in the previous section, we now want to turn our attention to the general case. Starting with T_4, we have to consider multiple integrals and we relegate the investigation of the subtleties that are associated to them to appendix C.

We begin by considering the integration contour. From (A.4), we take the parametrization of the relevant Kähler parameters for the T_N geometry depicted on the left of figure 2

$$Q^{(j)}_{l;i} = \frac{\tilde{A}^{(j)}_i \tilde{A}^{(j-1)}_i}{\tilde{A}^{(j)}_{i-1} \tilde{A}^{(j-1)}_{i+1}}\,, \qquad Q^{(j)}_{m;i} = \frac{\tilde{A}^{(j-1)}_i \tilde{A}^{(j)}_{i-1}}{\tilde{A}^{(j)}_i \tilde{A}^{(j-1)}_{i-1}}\,, \tag{63}$$

where the "boundary" $\tilde{A}$'s are to be expressed via the masses through equation (A.1). We choose to perform the contour integral in the flopping frame of figure 2, which implies that the $\tilde{A}^{(l)}_k$ that we integrate over have to be in the domain in which

$$\left|Q^{(j)}_{l;i}\right| > 1\,, \qquad \left|Q^{(j)}_{m;i}\right| > 1\,, \tag{64}$$

where $i, j \leq 1$ and $i + j \leq N$. As before, we are interested in the semi-degenerate limit. This, we parametrize the masses as follows

$$\tilde{M}_i = \tilde{K} d_i \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{N+1-2i}{2}} \qquad \text{for } i = 1, \ldots, N-1, \qquad \tilde{M}_N = \tilde{K}^{1-N}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{N-1}{2}}\,, \tag{65}$$

where the $d_i = e^{-\beta\delta_i}$ are regulators satisfying $\prod_{i=1}^{N-1} d_i = 1$ and $\tilde{K} = e^{-\beta\varkappa}$. The numerator of $|\mathcal{Z}^{\text{top}}|^2$ has a zero of order $\frac{(N-2)(N-1)}{2}$ in the limit $\delta_i \to 0$ since

$$\prod_{1\leq i<j\leq N} \left|\mathcal{M}\Big(\frac{\tilde{M}_i}{\tilde{M}_j}\Big)\right|^2 = \text{reg} \times \prod_{1\leq i<j\leq N-1} \left|\mathcal{M}\Big(\Big(\frac{\mathfrak{t}}{\mathfrak{q}}\Big)^{j-i}\frac{d_i}{d_j}\Big)\right|^2\,, \tag{66}$$

and $\left|\mathcal{M}\left((\mathfrak{t}/\mathfrak{q})^n\right)\right|^2 = 0$ for $n \geq 0$. These zeroes can all be canceled by divergences coming from the pinching of the $\frac{(N-2)(N-1)}{2}$ integrals if we choose the contour carefully, see for instance figure 14 for an example in the T_4 case.Thus the final answer is obtained by taking the residues in the integration variables $\tilde{A}^{(l)}_k$ at

$$\tilde{A}^{(j)}_i = \tilde{K}^i \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{i(N-i-j)}{2}} \prod_{k=1}^{j} \tilde{N}_k\,. \tag{67}$$

Computing the residues, we obtain the result

$$\begin{aligned}
&\lim_{\delta_a\to 0} \oint \prod_{l=1}^{N-2}\prod_{m=1}^{N-1-l}\left[\frac{d\tilde{A}^{(m)}_l}{2\pi i \tilde{A}^{(m)}_l}|M(\mathfrak{t},\mathfrak{q})|^2\right]|\mathcal{Z}^{\text{top}}_N|^2 = \\
&= \frac{\left|\mathcal{M}(\tilde{K}^{-N})\right|^2}{\left|\prod_{k=1}^{N}\mathcal{M}\big(\tilde{N}_k\tilde{L}_{N+1-k}\tilde{K}^{-1}\big)\right|^2} \times \Bigg| \sum_{\nu_1,\ldots,\nu_{N-1}} \left[\prod_{i=1}^{N-1}\left(\frac{\tilde{N}_{N-i}\tilde{L}_i}{\tilde{N}_{N-i+1}\tilde{L}_{i+1}}\right)^{\frac{|\nu_i|}{2}}\right] \\
&\times \frac{\mathsf{N}^{\beta}_{\nu_1\emptyset}(n_N + l_1 - \varkappa)\left[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\nu_{i+1}\nu_i}(n_{N-i} + l_{i+1} - \varkappa)\right]\mathsf{N}^{\beta}_{\emptyset\nu_{N-1}}(n_1 + l_N - \varkappa)}{\prod_{i=1}^{N-1}\mathsf{N}^{\beta}_{\nu_i\nu_i}(0)}\Bigg|^2\,.
\end{aligned} \tag{68}$$

Here ν_i for $i = 1, \dots, N-1$ denote the partitions corresponding to the $N-1$ brane junctions not affected by Higgsing at the given pole. For our choice of flopping frame, see figure 2, these partitions are readily identified as $\nu_i := \nu_i^{(N-i)}$, $i = 1, \dots, N-1$, see figure 4 of [1] for the notation.

The remaining sums in (68) will be now performed by using the summation identity (B.18) proven in appendix B.2, which we reproduce here for convenience

Theorem

$$\sum_{\nu_1,\dots,\nu_{N-1}} \left[\prod_{i=1}^{N-1} \frac{\left(V_i\sqrt{U_iU_{i+1}}\right)^{|\nu_i|}}{\mathsf{N}^{\beta}_{\nu_i\nu_i}(0)}\right] \mathsf{N}^{\beta}_{\nu_1\emptyset}(u_1 - \epsilon_+/2) \left[\prod_{i=1}^{N-2} \mathsf{N}^{\beta}_{\nu_{i+1}\nu_i}(u_{i+1} - \epsilon_+/2)\right] \mathsf{N}^{\beta}_{\emptyset\nu_{N-1}}(u_N - \epsilon_+/2) =$$
$$= \prod_{i=1}^{N-1}\prod_{j=1}^{N-i} \frac{\mathcal{M}\big(\prod_{s=j}^{i+j-1} U_sV_s\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{U_{i+j}}{U_j}\cdot\prod_{s=j}^{i+j-1} U_sV_s\big)}{\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_{i+j}\prod_{s=j}^{i+j-1} U_sV_s\big)\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{1}{U_j}\prod_{s=j}^{i+j-1} U_sV_s\big)}. \quad (69)$$

Setting the parameters here to be equal to

$$U_i = \sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{\tilde{N}_{N-i+1}\tilde{L}_i}{\tilde{K}}, \qquad V_j = \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{K}}{\tilde{N}_{N-j+1}\tilde{L}_{j+1}}, \quad (70)$$

for $i = 1, \cdots N$ and $j = 1, \cdots N-1$, one straightforwardly obtains:

$$\sum_{\nu_1,\dots,\nu_{N-1}} \left[\prod_{i=1}^{N-1}\left(\frac{\tilde{N}_{N-i}\tilde{L}_i}{\tilde{N}_{N-i+1}\tilde{L}_{i+1}}\right)^{\frac{|\nu_i|}{2}}\right]$$
$$\times \frac{\mathsf{N}^{\beta}_{\nu_1\emptyset}(n_N + l_1 - \varkappa)\left[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\nu_{i+1}\nu_i}(n_{N-i} + l_{i+1} - \varkappa)\right]\mathsf{N}^{\beta}_{\emptyset\nu_{N-1}}(n_1 + l_N - \varkappa)}{\prod_{i=1}^{N-1}\mathsf{N}^{\beta}_{\nu_i\nu_i}(0)}$$
$$= \prod_{1\le i<j\le N} \frac{\mathcal{M}\big(\frac{\tilde{L}_i}{\tilde{L}_j}\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{N}_{N-j+1}}{\tilde{N}_{N-i+1}}\big)}{\mathcal{M}\big(\frac{\tilde{N}_{N-j+1}\tilde{L}_i}{\tilde{K}}\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\tilde{K}}{\tilde{N}_{N-i+1}\tilde{L}_j}\big)}. \quad (71)$$

Substituting (71) in (68) and expressing everything in term of the Υ_q functions through formula (A.24) one obtains

$$\lim_{\delta_a\to 0}\oint\prod_{l=1}^{N-2}\prod_{m=1}^{N-1-l}\left[\frac{d\tilde{A}_l^{(m)}}{2\pi i\tilde{A}_l^{(m)}}|M(\mathfrak{t},\mathfrak{q})|^2\right]|\mathcal{Z}_N^{\text{top}}|^2 =$$
$$= \frac{(1-q)^{\tilde{\varphi}_N}}{\Lambda^{N-1}}\frac{\Upsilon_q(N\varkappa)\prod_{1\le i<j\le N}[\Upsilon_q(n_i - n_j)\Upsilon_q(l_{N+1-i} - l_{N+1-j})]}{\prod_{i,j=1}^{N}\Upsilon_q(\varkappa - n_i - l_{N+1-j})} \quad (72)$$

where the exponent

$$\tilde{\varphi}_N = \left(\frac{Q}{2} - N\varkappa\right)^2 + \sum_{1\le i<j\le N}\left[\left(\frac{Q}{2} + n_j - n_i\right)^2 + \left(\frac{Q}{2} + l_{N+1-j} - l_{N+1-i}\right)^2\right]$$
$$- \sum_{i,j=1}^{N}\left(\frac{Q}{2} + n_i + l_{N+1-j} - \varkappa\right)^2 \quad (73)$$

after a little algebra simplifies into

$$\tilde{\varphi}_N = 2Q\left(\frac{N(N-1)}{2}\varkappa + \sum_{i=1}^{N} i(n_i + l_{N+1-i})\right) - \frac{N-1}{4}Q^2 = -2Q\left(2\mathcal{Q} - \sum_{i=1}^{3}\boldsymbol{\alpha}_i, \rho\right)\frac{N-1}{4}Q^2. \quad (74)$$

Now we will employ our SU(N) conventions, see appendix A.2, equation (7) as well as equations (A.25), (A.31) and rearrange the prefactors to obtain the the q-deformed Fateev-Litvinov three-point function in the form conjectured by [1]:

$$\begin{aligned}C_q(N\varkappa\omega_{N-1},\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3) &= \left(\beta\,|M(\mathfrak{t},\mathfrak{q})|^2\right)^{N-1}\left(\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}\right)^{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}\\ &\quad\times\lim_{\delta_a\to 0}\oint\prod_{l=1}^{N-2}\prod_{m=1}^{N-1-l}\left[\frac{d\tilde{A}_l^{(m)}}{2\pi i\tilde{A}_l^{(m)}}|M(\mathfrak{t},\mathfrak{q})|^2\right]|\mathcal{Z}_N^{\text{top}}|^2 \qquad (75)\\ &= \left(\frac{\left(1-q^b\right)^{2b^{-1}}\left(1-q^{b^{-1}}\right)^{2b}}{(1-q)^{2Q}}\right)^{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}\frac{\Upsilon_q'(0)^{N-1}\Upsilon_q(N\varkappa)\prod_{e>0}\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon_q((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^N\Upsilon_q(\varkappa+(\boldsymbol{\alpha}_2-\mathcal{Q},\,h_i)+(\boldsymbol{\alpha}_3-\mathcal{Q},\,h_j))}\,.\end{aligned}$$

Taking the 4D limit $q\to 1$ here yields the Fateev-Litvinov formula (20) for the semi-degenerate three-point function of the $\mathbf{W}_N$ Toda field theory, up to a factor depending on b and the cosmological constant μ.

6 Conclusions and Outlook

In this article, we used formula (9), taken from [1], for the Toda structure constants of three generic primary operators to rederive the known formula for the structure constants involving one semi-degenerate primary, which was originally obtained by Fateev and Litvinov in [2]. We showed in section 3 how the degeneration of the primary fields on the Toda side corresponds to Higgsing on the (p,q) 5-brane web diagram side. After committing to the choice of the flopping frame which then dictates the form of the contour, we demonstrated that in the semi-degenerate limit, the contour integral expressing the Toda structure constants is given by a single residue. Then, using q-binomial identities (69), we proved that the sums over partitions still present in the residues can be computed exactly and that the final result (75) agrees with the expression of Fateev and Litvinov after one takes the $q\to 1$ limit and reintroduces (22) the terms depending on the cosmological constant μ. Thus, we have obtained a very non-trivial check of the general formula for the Toda structure constants given in [1].

We would of course want to obtain further checks of (9). A natural next step would involve rederiving (3.11) of [4], which is a generalization of (20) for the T_3 case. Specifically, if one generalizes the semi-degenerate condition $\boldsymbol{\alpha}_1 = N\kappa\omega_2$ to $\boldsymbol{\alpha}_1 = N\varkappa\omega_2 - mb\omega_1$, where m is a positive integer, then one obtains the expression

$$C(N\varkappa\omega_2 - mb\omega_1,\boldsymbol{\alpha}_2\boldsymbol{\alpha}_3) = \Xi(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3|N\varkappa)_m\mathfrak{J}_m, \qquad (76)$$

where Ξ is given by a generalization of (20) as

$$\begin{aligned}\Xi(\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3|N\varkappa)_m :=&(\pi\mu)^{2m}\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\left(2\mathcal{Q}-\sum_{i=1}^3\boldsymbol{\alpha}_i,\rho\right)}{b}}\times\\ &\times\frac{\Upsilon'(-mb)^2\Upsilon(N\varkappa)\prod_{e>0}\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_2,e))\Upsilon((\mathcal{Q}-\boldsymbol{\alpha}_3,e))}{\prod_{i,j=1}^3\Upsilon(\varkappa-mb\delta_{ij}+\frac{mb}{3}+(\boldsymbol{\alpha}_2-\mathcal{Q},h_i)+(\boldsymbol{\alpha}_3-\mathcal{Q},h_j))}\,.\end{aligned} \qquad (77)$$

and $\mathfrak{J}_m$ is given by an $4m^5$ dimensional Coulomb integral, see appendix B of [4]. It would be quite interesting to obtain this formula using our formula (9).

Furthermore, another interesting direction involves investigating the semi-classical regime. In that limit, the combinatorial functions $\mathsf{N}^\beta_{\lambda\mu}$ factorize in a product

$$\mathsf{N}^\beta_{\lambda\mu}(m;b,b^{-1}) \overset{b\to\infty}{\longrightarrow} \mathsf{N}^\beta_{\lambda\emptyset}(m;b,b^{-1})\mathsf{N}^\beta_{\emptyset\mu}(m;b,b^{-1})\,. \qquad (78)$$

Thus, the sums over partitions disentangle, allowing one to use generalizations of the q-binomial identities to compute them. This direction is especially interesting because computing the correlation function in the semi-classical limit could help derive the still unknown Lagrangian of the q-deformed Toda theory. One should begin by looking for the Lagrangian of the q-deformed Liouville theory, returning to the work of [18, 19]. It could be that the 2D space has to be made non-commutative [48–50].

The ultimate goal is of course to compute the contour integral in (14) exactly for generic values of the parameters, which requires finding a closed form expression for the "instanton" sum of (17). In order to accomplish this, a suitable generalization of the q-binomial identities that we used in this article to perform the sums in (9) must be found. A more modest goal with which one can practice involves computing the sums for the cases with $E_{6,7,8}$ flavor symmetry studied in [9,26,30,31], which are obtained from the general T_N by a less severe Higgsing than the one we perform here.

In [1] we gave a formula for the 3-point functions of three Toda primary fields. Knowledge of all the 3-point correlation functions of primary fields is in itself not enough to obtain all generic 3-point functions, thus solving the Toda CFT. In order to achieve that, we need to also compute the correlation functions of descendants, which for the general $\mathbf{W}_N$ Toda CFTs is not that simple. It is however rather straightforward to see from the topological strings point of view what are the steps that lead to them. Specifically, in the computation of the topological partition function (15), we have to let the partitions on the edge of the diagram be arbitrary instead of empty, see figure 4 of [1]. This would lead to the general Ding-Iohara algebra interwiners. The Ding-Iohara algebra [51] in the free boson representation with N free bosons is known to factorize [52] as

$$\mathcal{A} = \mathbf{W}_N \otimes \mathbf{H} \tag{79}$$

where $\mathbf{H}$ is the Heisenberg algebra. This algebra is precisely the one needed to describe the descendent operators in the context of the AGT-W relation, see [53, 54]. In particular, it would be quite easy to obtain the 3-point function of two primaries and one descendant and such 3-point functions will already lead to many higher point functions thanks to the conformal bootstrap and addressing this problem is work in progress [55]. Moreover, it would be important to better understand the q-deformed AGT-W correspondence and its relation to topological strings by more actively studying the Ding-Iohara algebra as in [56].

The degeneration we study in this paper, and in general Higgsing, can also be understood on the 4D/5D side using co-dimension two half-BPS surface defects [57] as in [42, 58–60] and [61, 62]. More concretely, this can be done by studying the generalization of the 2D/4D story of [58–63], which is a generalization of the AGT correspondence including surface operators, to the 3D/5D relation that was initiated by [18] and further studied by [19–21]. See [64] for the latest advancements on the subject.

We finish by observing that the Higgsed geometry corresponding to the degeneration, see the right side of figure 2, is related to the strip geometry, see figure 23 in [8], by the Hanany-Witten effect. We refer to [65] for a nice discussion on the subject. The invariance of the topological string amplitude under the Hanany-Witten transition is non-trivial and it would be important to see how one can relate formula (21) for the q-deformed structure constants to the topological string amplitude for the strip, see equation (4.66) of [8].

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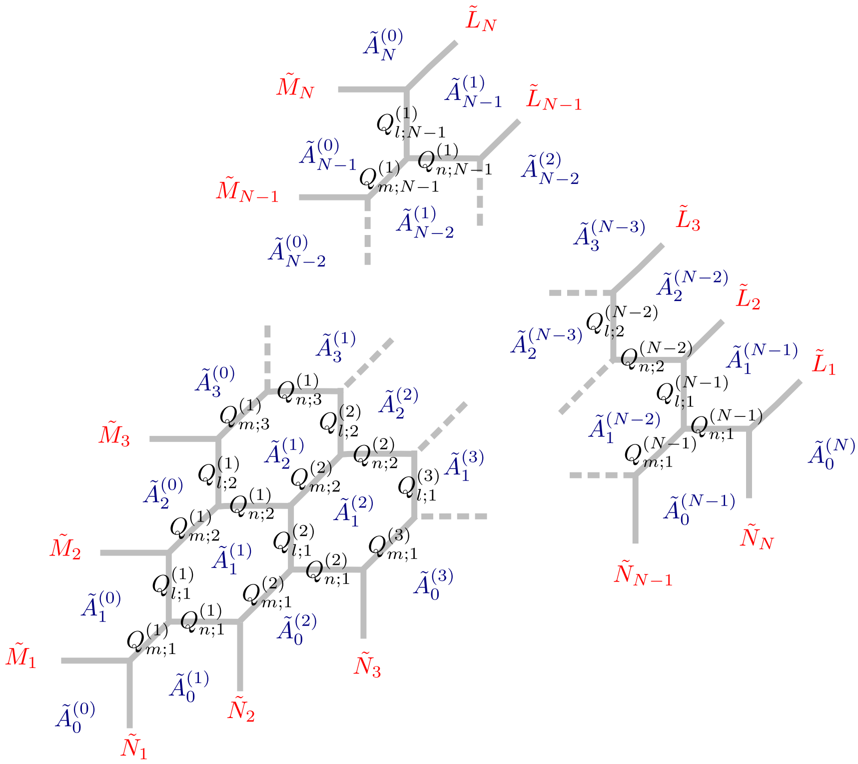

Figure 13: *Parametrization for T_N. We denote the Kähler moduli parameters corresponding to the horizontal lines as $Q_{n;i}^{(j)}$, to the vertical lines as $Q_{l;i}^{(j)}$, and to tilted lines as $Q_{m;i}^{(j)}$. We denote the breathing modes as $\tilde{A}_i^{(j)}$. The index j labels the strips in which the diagram can be decomposed.*

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A Notations, conventions and special functions

In this appendix, we summarize our conventions and the main properties of the special functions that we use the most.

A.1 Parametrization of the T_N junction

We gather in this appendix all necessary formulas for the parametrizations of the Kähler moduli of the T_N. First, the "interior" Coulomb moduli $\tilde{A}_j^{(i)} = e^{-\beta a_i^{(j)}}$ are independent, while the "border" ones are given by

$$\tilde{A}_i^{(0)} = \prod_{k=1}^{i} \tilde{M}_k, \qquad \tilde{A}_0^{(j)} = \prod_{k=1}^{j} \tilde{N}_k, \qquad \tilde{A}_i^{(N-i)} = \prod_{k=1}^{i} \tilde{L}_k. \tag{A.1}$$

The parameters labeling the positions of the flavors branes obey the relations

$$\prod_{k=1}^{N}\tilde{M}_k=\prod_{k=1}^{N}\tilde{N}_k=\prod_{k=1}^{N}\tilde{L}_k=1\Longleftrightarrow\sum_{k=1}^{N}m_k=\sum_{k=1}^{N}n_k=\sum_{k=1}^{N}l_k=0. \tag{A.2}$$

Therefore, $\tilde{A}_0^{(0)}=\tilde{A}_N^{(0)}=\tilde{A}_0^{(N)}=1$ and we can invert relation (A.1) as

$$\tilde{M}_i=\frac{\tilde{A}_i^{(0)}}{\tilde{A}_{i-1}^{(0)}},\qquad \tilde{N}_i=\frac{\tilde{A}_0^{(i)}}{\tilde{A}_0^{(i-1)}},\qquad \tilde{L}_i=\frac{\tilde{A}_i^{(N-i)}}{\tilde{A}_{i-1}^{(N-i+1)}}. \tag{A.3}$$

All placements are illustrated in figure 13. The Kähler parameters associated to the edges of the T_N junction are related to the $\tilde{A}_i^{(j)}$ as follows

$$Q_{n;i}^{(j)}=\frac{\tilde{A}_i^{(j)}\tilde{A}_{i-1}^{(j)}}{\tilde{A}_i^{(j-1)}\tilde{A}_{i-1}^{(j+1)}},\qquad Q_{l;i}^{(j)}=\frac{\tilde{A}_i^{(j)}\tilde{A}_i^{(j-1)}}{\tilde{A}_{i-1}^{(j)}\tilde{A}_{i+1}^{(j-1)}},\qquad Q_{m;i}^{(j)}=\frac{\tilde{A}_i^{(j-1)}\tilde{A}_{i-1}^{(j)}}{\tilde{A}_i^{(j)}\tilde{A}_{i-1}^{(j-1)}}. \tag{A.4}$$

For each inner hexagon of (13), the following two constraints are satisfied

$$Q_{l;i}^{(j)}Q_{m;i+1}^{(j)}=Q_{m;i}^{(j+1)}Q_{l;i}^{(j+1)},\qquad Q_{n;i}^{(j)}Q_{m;i}^{(j+1)}=Q_{m;i+1}^{(j)}Q_{n;i+1}^{(j)}. \tag{A.5}$$

A.2 Conventions and notations for SU(N)

For the convenience of the reader we summarize here our SU(N) conventions. The weights of the fundamental representation of SU(N) are h_i with $\sum_{i=1}^N h_i=0$. We remind that the scalar product is defined via $(h_i,h_j)=\delta_{ij}-\frac{1}{N}$. The simple roots are

$$e_k:=h_k-h_{k+1}\,,\qquad k=1,\dots,N-1\,, \tag{A.6}$$

and the positive roots $e>0$ are contained in the set

$$\Delta^+:=\{h_i-h_j\}_{i<j=1}^N=\{e_i\}_{i=1}^{N-1}\cup\{e_i+e_{i+1}\}_{i=1}^{N-2}\cup\cdots\cup\{e_1+\cdots+e_{N-1}\}\,. \tag{A.7}$$

The Weyl vector ρ for SU(N) is given by

$$\rho:=\frac{1}{2}\sum_{e>0}e=\frac{1}{2}\sum_{i<j=1}^{N}(h_i-h_j)=\sum_{i=1}^{N}\frac{N+1-2i}{2}h_i=\omega_1+\cdots+\omega_{N-1}, \tag{A.8}$$

and it obeys $(\rho,e_i)=1$ for all i. The $N-1$ fundamental weights ω_i of SU(N) are given by

$$\omega_i=\sum_{k=1}^{i}h_k\,,\qquad i=1,\dots,N-1 \tag{A.9}$$

and the corresponding finite dimensional representations are the i-fold antisymmetric tensor product of the fundamental representation. They obey the scalar products $(e_i,\omega_j)=\delta_{ij}$, *i.e.* they are a dual basis. Furthermore, we find the following scalar products useful

$$(\rho,h_j)=\frac{N+1}{2}-j,\qquad(\rho,\omega_i)=\frac{i(N-i)}{2},\qquad(h_j,\omega_i)=\begin{cases}1-\frac{i}{N} & j\leq i\\ -\frac{i}{N} & j>i\end{cases}\,, \tag{A.10}$$

as well as

$$(\omega_i,\omega_j)=\frac{\min(i,j)\,(N-\max(i,j))}{N},\qquad(\rho,\rho)=\frac{N(N^2-1)}{12}\,. \tag{A.11}$$

The Weyl group of $\mathrm{SU}(N)$ is isomorphic to S_N and is generated by the $N-1$ Weyl reflections associated to the simple roots. If $\boldsymbol{\alpha}$ is a weight, we define the Weyl reflections with respect to the simple root e_i

$$\mathsf{w}_i \cdot \boldsymbol{\alpha} := \boldsymbol{\alpha} - 2\frac{(e_i, \boldsymbol{\alpha})}{(e_i, e_i)} e_i = \boldsymbol{\alpha} - (e_i, \boldsymbol{\alpha})\, e_i\,. \tag{A.12}$$

Furthermore, we define the affine Weyl reflections with respect to e_i as follows

$$\mathsf{w}_i \circ \boldsymbol{\alpha} := \mathcal{Q} + \mathsf{w}_i \cdot (\boldsymbol{\alpha} - \mathcal{Q}) = \mathsf{w}_i \cdot \boldsymbol{\alpha} + Q e_i = \boldsymbol{\alpha} - (\boldsymbol{\alpha} - \mathcal{Q}, e_i)\, e_i\,, \tag{A.13}$$

where $\mathcal{Q} := Q\rho = (b + b^{-1})\rho$.

A.3 Special functions

In this section we gather the definitions and properties of all special functions used in the main text. First we begin by defining the shifted factorials[10]

$$(x;q)_p := \prod_{i=1}^{p} (1 - x q^{i-1}) \tag{A.14}$$

for positive p, which is continued to negative p according to

$$(x;q)_p = \frac{1}{(xq^p; q)_{-p}}\,. \tag{A.15}$$

In particular for $p \to \infty$, and for arbitrary number of q's, we have (we require for convergence that $|q_i| < 1$ for all i)

$$(x; q_1, \dots, q_r)_\infty := \prod_{i_1=0,\dots,i_r=0}^{\infty} (1 - x q_1^{i_1} \cdots q_r^{i_r})\,. \tag{A.16}$$

We can extend the definition of the shifted factorial for all values of q_i by imposing the relations

$$(x; q_1, \dots, q_i^{-1}, \dots, q_r)_\infty = \frac{1}{(xq_i; q_1, \dots, q_r)_\infty}\,. \tag{A.17}$$

Furthermore, they obey the following shifting properties

$$(q_j x; q_1, \dots, q_r)_\infty = \frac{(x; q_1, \dots, q_r)_\infty}{(x; q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_r)_\infty}\,. \tag{A.18}$$

We then define the function $\mathcal{M}(u; \mathfrak{t}, \mathfrak{q})$ as

$$\mathcal{M}(u; \mathfrak{t}, \mathfrak{q}) := (u\mathfrak{q}; \mathfrak{t}, \mathfrak{q})_\infty^{-1} = \begin{cases} \prod_{i,j=1}^{\infty} (1 - u\mathfrak{t}^{i-1}\mathfrak{q}^{j})^{-1} & \text{for } |\mathfrak{t}| < 1, |\mathfrak{q}| < 1 \\ \prod_{i,j=1}^{\infty} (1 - u\mathfrak{t}^{i-1}\mathfrak{q}^{1-j}) & \text{for } |\mathfrak{t}| < 1, |\mathfrak{q}| > 1 \\ \prod_{i,j=1}^{\infty} (1 - u\mathfrak{t}^{-i}\mathfrak{q}^{j}) & \text{for } |\mathfrak{t}| > 1, |\mathfrak{q}| < 1 \\ \prod_{i,j=1}^{\infty} (1 - u\mathfrak{t}^{-i}\mathfrak{q}^{1-j})^{-1} & \text{for } |\mathfrak{t}| > 1, |\mathfrak{q}| > 1 \end{cases}\,, \tag{A.19}$$

converging for all u. This function can be written as a plethystic exponential

$$\mathcal{M}(u; \mathfrak{t}, \mathfrak{q}) = \exp\left[\sum_{m=1}^{\infty} \frac{u^m}{m} \frac{\mathfrak{q}^m}{(1 - \mathfrak{t}^m)(1 - \mathfrak{q}^m)}\right]\,, \tag{A.20}$$

[10] A good source for the properties of the shifted factorials is [66].

which converges for all $\mathfrak{t}$ and all $\mathfrak{q}$ provided that $|u| < \mathfrak{q}^{-1+\theta(|\mathfrak{q}|-1)}\mathfrak{t}^{\theta(|\mathfrak{t}|-1)}$. Here and elsewhere $\theta(x) = 1$ if $x > 0$ and is zero otherwise. The following identity is obvious from the definition

$$\mathcal{M}(u;\mathfrak{q},\mathfrak{t}) = \mathcal{M}(u\mathfrak{t}/\mathfrak{q};\mathfrak{t},\mathfrak{q})\,. \tag{A.21}$$

From the analytic properties of the shifted factorials (A.17), we read the identities

$$\mathcal{M}(u;\mathfrak{t}^{-1},\mathfrak{q}) = \frac{1}{\mathcal{M}(u\mathfrak{t};\mathfrak{t},\mathfrak{q})}, \qquad \mathcal{M}(u;\mathfrak{t},\mathfrak{q}^{-1}) = \frac{1}{\mathcal{M}(u\mathfrak{q}^{-1};\mathfrak{t},\mathfrak{q})}\,, \tag{A.22}$$

while from (A.18) we take the following shifting identities

$$\mathcal{M}(u\mathfrak{t};\mathfrak{t},\mathfrak{q}) = (u\mathfrak{q};\mathfrak{q})_\infty \mathcal{M}(u;\mathfrak{t},\mathfrak{q}), \qquad \mathcal{M}(u\mathfrak{q};\mathfrak{t},\mathfrak{q}) = (u\mathfrak{q};\mathfrak{t})_\infty \mathcal{M}(u;\mathfrak{t},\mathfrak{q})\,. \tag{A.23}$$

We define the q-deformed Υ function as

$$\begin{aligned}\Upsilon_q(x|\epsilon_1,\epsilon_2) =&(1-q)^{-\frac{1}{\epsilon_1\epsilon_2}(x-\frac{\epsilon_+}{2})^2}\prod_{n_1,n_2=0}^{\infty}\frac{(1-q^{x+n_1\epsilon_1+n_2\epsilon_2})(1-q^{\epsilon_+-x+n_1\epsilon_1+n_2\epsilon_2})}{(1-q^{\epsilon_+/2+n_1\epsilon_1+n_2\epsilon_2})}\\ =&(1-q)^{-\frac{1}{\epsilon_1\epsilon_2}(x-\frac{\epsilon_+}{2})^2}\left|\frac{\mathcal{M}(q^{-x};\mathfrak{t},\mathfrak{q})}{\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}};\mathfrak{t},\mathfrak{q})}\right|^2\,,\end{aligned} \tag{A.24}$$

where we have used the definition (13) for the norm squared. From time to time we will use the short-hand notation

$$\Lambda := |\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}};\mathfrak{t},\mathfrak{q})|^2\,. \tag{A.25}$$

If follows from the definition (A.24) that $\Upsilon_q(\epsilon_+/2|\epsilon_1,\epsilon_2) = 1$, that $\Upsilon_q(x|\epsilon_1,\epsilon_2) = \Upsilon_q(\epsilon_+ - x|\epsilon_1,\epsilon_2)$ and that $\Upsilon_q(x|\epsilon_1,\epsilon_2) = \Upsilon_q(x|\epsilon_2,\epsilon_1)$. Furthermore, from the shifting identities for $\mathcal{M}$, we can easily prove that

$$\Upsilon_q(x+\epsilon_1|\epsilon_1,\epsilon_2) = \left(\frac{1-q}{1-q^{\epsilon_2}}\right)^{1-2\epsilon_2^{-1}x}\gamma_{q^{\epsilon_2}}(x\epsilon_2^{-1})\Upsilon_q(x|\epsilon_1,\epsilon_2)\,, \tag{A.26}$$

together with a similar equation for the shift with ϵ_2. Here, we have used the definition of the q-deformed Γ and γ functions

$$\Gamma_q(x) := (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty}, \qquad \gamma_q(x) := \frac{\Gamma_q(x)}{\Gamma_q(1-x)} = (1-q)^{1-2x}\frac{(q^{1-x};q)_\infty}{(q^x;q)_\infty}\,, \tag{A.27}$$

valid for $|q| < 1$. They obey $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$, implying $\gamma_q(x+1) = \frac{(1-q^x)(1-q^{-x})}{(1-q)^2}\gamma_q(x)$. Because of the normalization of $\Upsilon_q(x|\epsilon_1,\epsilon_2)$ and since the factors of the right hand side of (A.26) have a well defined limit for $q \to 1$, we find by comparing functional identities that

$$\Upsilon_q(x+\epsilon_1|\epsilon_1,\epsilon_2) \xrightarrow{q\to 1} \Upsilon(x|\epsilon_1,\epsilon_2) := \frac{\Gamma_2\big(\frac{\epsilon_+}{2}|\epsilon_1,\epsilon_2\big)^2}{\Gamma_2\big(x|\epsilon_1,\epsilon_2\big)\Gamma_2\big(\epsilon_+ - x|\epsilon_1,\epsilon_2\big)}\,. \tag{A.28}$$

where Γ_2 is the Barnes Double Gamma function. In particular, the usual function $\Upsilon(x)$ introduced in [11] is equal to $\Upsilon(x|b,b^{-1})$. We shall often just write $\Upsilon_q(x)$ instead of $\Upsilon_q(x|\epsilon_1,\epsilon_2)$ and indicate in the text whether the ϵ_i parameters are arbitrary or whether $b = \epsilon_1 = \epsilon_2^{-1}$.

We will also need to evaluate the derivative of $\Upsilon_q(x)$ at $x = 0$. Since the zero of $\Upsilon_q(x)$ at $x = 0$ is due to the factor $(1-q^x)$ in the numerator of (A.24), we find that the only piece of the derivative that survives is

$$\Upsilon_q'(0) = \frac{\beta}{1-q}\Upsilon_q(b)\,. \tag{A.29}$$

From this formula we can then obtain an identity useful for the calculations of the main text. Let us define the norm squared of the refined McMahon function following [25]:

$$|M(\mathfrak{t},\mathfrak{q})|^2 := \lim_{u\to 1} \frac{|\mathcal{M}(u;\mathfrak{t},\mathfrak{q})|^2}{1-u^{-1}} = |\mathcal{M}(\mathfrak{q}^{-1};\mathfrak{t},\mathfrak{q})|^2 = (1-q)^{\frac{(\epsilon_1-\epsilon_2)^2}{4\epsilon_1\epsilon_2}} \Lambda\Upsilon_q(\epsilon_1)\,. \tag{A.30}$$

Then, from (A.25) and (A.29) we get for $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$

$$|M(\mathfrak{t},\mathfrak{q})|^2 = \frac{1}{\beta}(1-q)^{\left(\frac{Q}{2}\right)^2} \Lambda\Upsilon_q'(0)\,. \tag{A.31}$$

A.4 Combinatorial functions

We shall use in the following

$$|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i, \qquad ||\lambda||^2 := \sum_{i=1}^{\ell(\lambda)} \lambda_i^2, \qquad n(\lambda) := \sum_{i=1}^{\ell(\lambda)} (\lambda_i - 1) = \frac{||\lambda^t||^2 - |\lambda|}{2}\,, \tag{A.32}$$

where $\ell(\lambda)$ is the number of rows of the partition λ. We also define the relative arm-length $a_\mu(s)$, arm-colength $a'_\mu(s)$, leg-length $l_\mu(s)$ and leg-colength $l'_\mu(s)$ of a given box s of the partition λ with respect to another partition μ as:

$$a_\mu(s) := \mu_i - j\,, \qquad a'_\mu(s) := j-1\,, \qquad l_\mu(s) := \mu_j^t - i\,, \quad l'_\mu(s) := i-1\,. \tag{A.33}$$

It is of course also possible to have $\lambda = \mu$. The $(\mathfrak{q},\mathfrak{t})$-deformed factorial of A depending on a partition λ is then given as a following product over its boxes:

$$(A;\mathfrak{q},\mathfrak{t})_\lambda := \prod_{i=1}^{\ell(\lambda)} (A\mathfrak{t}^{1-i};\mathfrak{q})_{\lambda_i} = \prod_{s\in\lambda} (1 - A\mathfrak{q}^{a'(s)}\mathfrak{t}^{-l'(s)})\,. \tag{A.34}$$

The next piece of notation that we need are the $(\mathfrak{q},\mathfrak{t})$-deformations of the hook product of a Young diagram λ. There are two inequivalent ways for this number to be deformed to a two-variable polynomial, namely:

$$h_\lambda(\mathfrak{q},\mathfrak{t}) := \prod_{s\in\lambda} (1 - \mathfrak{q}^{a(s)}\mathfrak{t}^{l(s)+1})\,, \qquad h'_\lambda(\mathfrak{q},\mathfrak{t}) := \prod_{s\in\lambda} (1 - \mathfrak{q}^{a(s)+1}\mathfrak{t}^{l(s)})\,. \tag{A.35}$$

Our last definition is that of the 5D uplift of Nekrasov functions, which we write as

$$\begin{aligned} \mathsf{N}^\beta_{\lambda\mu}(m;\epsilon_1,\epsilon_2) := &\prod_{(i,j)\in\lambda} 2\sinh\frac{\beta}{2}\left[m + \epsilon_1(\lambda_i - j + 1) + \epsilon_2(i - \mu_j^t)\right] \\ &\times \prod_{(i,j)\in\mu} 2\sinh\frac{\beta}{2}\left[m + \epsilon_1(j - \mu_i) + \epsilon_2(\lambda_j^t - i + 1)\right] \\ &= \prod_{s\in\lambda} 2\sinh\frac{\beta}{2}\left[m + \epsilon_1\left(a_\lambda(s)+1\right) - \epsilon_2 l_\mu(s)\right] \prod_{s\in\mu} 2\sinh\frac{\beta}{2}\left[m - \epsilon_1 a_\mu(s) + \epsilon_2\left(l_\lambda(s)+1\right)\right] \end{aligned} \tag{A.36}$$

where the products are taken over boxes of partitions λ and μ, respectively. By pulling some factors out of the products, the definition can also be rewritten as

$$\begin{aligned} \mathsf{N}^\beta_{\lambda\mu}(m;\epsilon_1,\epsilon_2) := &\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{1}{U}\right)^{\frac{|\lambda|+|\mu|}{2}} \mathfrak{t}^{\frac{||\lambda^t||^2 - ||\mu^t||^2}{4}} \mathfrak{q}^{\frac{||\mu||^2 - ||\lambda||^2}{4}} \prod_{(i,j)\in\lambda}\left(1 - U\mathfrak{t}^{\mu_j^t - i}\mathfrak{q}^{\lambda_i - j + 1}\right) \\ &\times \prod_{(i,j)\in\mu}\left(1 - U\mathfrak{t}^{-\lambda_j^t + i - 1}\mathfrak{q}^{-\mu_i + j}\right), \end{aligned} \tag{A.37}$$

where $U := e^{-\beta m}$. For particular values of the parameter m, the introduced functions behave like Kronecker$-\delta$ functions, namely

$$\mathsf{N}^{\beta}_{\lambda\emptyset}(-\epsilon_+) = \mathsf{N}^{\beta}_{\emptyset\lambda}(0) = \delta_{\lambda\emptyset}, \tag{A.38}$$

where $\epsilon_+ = \epsilon_1 + \epsilon_2$. Furthermore, they obey the exchange identities

$$\begin{aligned} \mathsf{N}^{\beta}_{\lambda\mu}(m; -\epsilon_2, -\epsilon_1) &= \mathsf{N}^{\beta}_{\mu^t\lambda^t}(m - \epsilon_+; \epsilon_1, \epsilon_2), \\ \mathsf{N}^{\beta}_{\lambda\mu}(-m; \epsilon_1, \epsilon_2) &= (-1)^{|\lambda|+|\mu|}\mathsf{N}^{\beta}_{\mu\lambda}(m - \epsilon_+; \epsilon_1, \epsilon_2), \\ \mathsf{N}^{\beta}_{\lambda\mu}(m; \epsilon_2, \epsilon_1) &= \mathsf{N}^{\beta}_{\lambda^t\mu^t}(m; \epsilon_1, \epsilon_2). \end{aligned} \tag{A.39}$$

Finally, there are two relations involving the functions we just defined, namely

$$\frac{1}{h_\lambda(\mathfrak{q},\mathfrak{t})h'_\lambda(\mathfrak{q},\mathfrak{t})} = \frac{(-1)^{|\lambda|}\mathfrak{t}^{-\frac{||\lambda^t||^2}{2}}\mathfrak{q}^{-\frac{||\lambda||^2}{2}}}{\mathsf{N}^{\beta}_{\lambda\lambda}(0)} \tag{A.40}$$

as well as

$$(U)_\lambda \equiv (U;\mathfrak{q},\mathfrak{t})_\lambda = \left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U\right)^{\frac{|\lambda|}{2}} \mathfrak{t}^{-\frac{||\lambda^t||^2}{4}}\mathfrak{q}^{\frac{||\lambda||^2}{4}}\mathsf{N}^{\beta}_{\lambda\emptyset}(m-\epsilon_+), \tag{A.41}$$

where $U := e^{-\beta m}$.

B The sl(N) Kaneko-Macdonald-Warnaar hypergeometric functions

This appendix contains the derivation of the summation formula (69) used in the main text. It exploits a binomial identity for the Kaneko-Macdonald-Warnaar extension of basic hypergeometric functions [67] which generalizes the Kaneko-Macdonald sl(2) identity of [68–70].

B.1 The sl(N) KMW hypergeometric functions and their $\mathfrak{q}$-binomial identity

The *Macdonald polynomials* $P_\lambda(\mathsf{x};\mathfrak{q},\mathfrak{t})$, which are referred to in the case of infinite alphabet x as the Macdonald symmetric functions, are labeled by a number partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$ and form an especially convenient basis in the ring of symmetric functions of $\mathsf{x} = (\mathsf{x}_1, \mathsf{x}_2, \dots)$ over the field $\mathbb{F} = \mathbb{Q}(\mathfrak{q},\mathfrak{t})$ of rational functions in two variables $\mathfrak{q}$ and $\mathfrak{t}$ [71].

Having many nice properties, the Macdonald polynomials have been applied in various areas of contemporary mathematics. One of them is the theory of sl(N) *Kaneko-Macdonald-Warnaar analogues of basic hypergeometric functions.* These functions of type $(r+1, r)$ are defined as

$$\begin{aligned} &{}_{r+1}\Phi_r\left[\begin{matrix} A_1, \dots, A_{r+1} \\ B_1, \dots, B_r \end{matrix}; \mathfrak{q}, \mathfrak{t}; \mathsf{x}^{(1)}, \dots, \mathsf{x}^{(N-1)}\right] := \\ &\sum_{\lambda^{(1)},\dots,\lambda^{(N-1)}} \frac{(A_1,\dots,A_{r+1};\mathfrak{q},\mathfrak{t})_{\lambda^{(N-1)}}}{(\mathfrak{q}\mathfrak{t}^{k_{N-1}-1}, B_1,\dots,B_r;\mathfrak{q},\mathfrak{t})_{\lambda^{(N-1)}}} \prod_{s=1}^{N-1}\left[\mathfrak{t}^{n(\lambda^{(s)})}\frac{(\mathfrak{q}\mathfrak{t}^{k_s-1};\mathfrak{q},\mathfrak{t})_{\lambda^{(s)}}}{h'_{\lambda^{(s)}}(\mathfrak{q},\mathfrak{t})}P_{\lambda^{(s)}}(\mathsf{x}^{(s)};\mathfrak{q},\mathfrak{t})\right] \\ &\times \prod_{s=1}^{N-2}\prod_{i=1}^{k_s}\prod_{j=1}^{k_{s+1}} \frac{(\mathfrak{q}\mathfrak{t}^{j-i-1+k_s-k_{s+1}};\mathfrak{q})_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}}{(\mathfrak{q}\mathfrak{t}^{j-i+k_s-k_{s+1}};\mathfrak{q})_{\lambda_i^{(s)}-\lambda_j^{(s+1)}}}, \end{aligned} \tag{B.1}$$

where the summations are performed over partitions $\lambda^{(s)}$, $1 \le s \le N-1$ whose lengths $k_s := \ell(\lambda^{(s)})$ satisfy $0 \equiv k_0 < k_1 < k_2 < \cdots < k_{N-1}$ and whose entries are ordered according to $\lambda_i^{(s)} \ge \lambda_{i-k_s+k_{s+1}}^{(s+1)}$ for $1 \le s \le N-1$. We have used here the definitions (A.14), (A.32), (A.34), (A.35).

In the following, it will be enough to restrict ourselves to a so-called *principal specialization* of a Macdonald polynomial, for which the string of arguments x is set to $\tilde{\mathsf{x}} := z(1, t, \ldots, t^{k-1})$ with $k = \ell(\lambda)$:

$$P_\lambda(\tilde{\mathsf{x}}; \mathsf{q}, \mathsf{t}) = z^{|\lambda|} \mathsf{t}^{n(\lambda)} \frac{(\mathsf{t}^k; \mathsf{q}, \mathsf{t})_\lambda}{h_\lambda(\mathsf{q}, \mathsf{t})}. \tag{B.2}$$

The corresponding specialization of the sl(N) *q-binomial theorem* is then written as:

Theorem [See [67], Cor. 3.1]

$$\begin{aligned} {}_1\Phi_0\left[\begin{matrix} A \\ - \end{matrix}; \mathsf{q}, \mathsf{t}; \tilde{\mathsf{x}}^{(1)}, \ldots, \tilde{\mathsf{x}}^{(N-1)}\right] &= \prod_{s=1}^{N-1} \prod_{i=1}^{k_s - k_{s-1}} \frac{(A z_s \cdots z_{N-1} \mathsf{t}^{i+s+k_{s-1}+\cdots+k_{N-2}-N}; \mathsf{q})_\infty}{(z_s \cdots z_{N-1} \mathsf{t}^{i+s+k_{s-1}+\cdots+k_{N-2}-N}; \mathsf{q})_\infty} \\ &\times \prod_{1\le s\le r\le N-2} \prod_{i=1}^{k_s-k_{s-1}} \frac{(\mathsf{q} z_s \cdots z_r \mathsf{t}^{i+s-r+k_{s-1}+\cdots+k_r-k_{r+1}-2}; \mathsf{q})_\infty}{(z_s \cdots z_r \mathsf{t}^{i+s-r+k_{s-1}+\cdots+k_{r-1}-1}; \mathsf{q})_\infty}, \end{aligned} \tag{B.3}$$

where $\tilde{\mathsf{x}}^{(s)} := z_s(1, \mathsf{t}, \ldots, \mathsf{t}^{k_s-1})$ for $1 \le s \le N-1$.

B.2 The summation formula

It will be convenient for the subsequent argument to rewrite the above formula (B.3) in the topological string conventions. This turns out to be possible due to the identities (A.19), (A.40), (A.41) and the following lemma:

Lemma

$$\prod_{i=1}^{k_1}\prod_{j=1}^{k_2} \frac{(A\mathsf{t}^{j-i})_{\lambda_{1,i}-\lambda_{2,j}}}{(A\mathsf{t}^{j-i+1})_{\lambda_{1,i}-\lambda_{2,j}}} = \mathsf{t}^{\frac{k_1|\lambda_2|-k_2|\lambda_1|}{2}} \frac{\mathsf{N}^\beta_{\lambda_2\lambda_1}(-a)}{\mathsf{N}^\beta_{\lambda_2\emptyset}(-a-k_1\epsilon_2)\, \mathsf{N}^\beta_{\emptyset\lambda_1}(-a+k_2\epsilon_2)}, \tag{B.4}$$

where $\ell(\lambda_1) = k_1$, $\ell(\lambda_2) = k_2$ and $A := e^{-\beta a}$.

Proof: Let us first notice that by using definition A.37 as well as exchange identities A.39, the right-hand side of the above formula can be written as a following product:

$$\begin{aligned} &\mathsf{t}^{\frac{k_1|\lambda_2|-k_2|\lambda_1|}{2}} \frac{\mathsf{N}^\beta_{\lambda_2\lambda_1}(-a)}{\mathsf{N}^\beta_{\lambda_2\emptyset}(-a-k_1\epsilon_2)\, \mathsf{N}^\beta_{\emptyset\lambda_1}(-a+k_2\epsilon_2)} \\ &\qquad = \prod_{(i,j)\in\lambda_1} \frac{1 - A\frac{\mathsf{t}}{\mathsf{q}}\mathsf{t}^{\lambda^t_{2,j}-i}\mathsf{q}^{\lambda_{1,i}-j+1}}{1 - A\frac{\mathsf{t}}{\mathsf{q}}\mathsf{t}^{k_2-i}\mathsf{q}^{\lambda_{1,i}-j+1}} \prod_{(i,j)\in\lambda_2} \frac{1 - A\frac{\mathsf{t}}{\mathsf{q}}\mathsf{t}^{-\lambda^t_{1,j}+i-1}\mathsf{q}^{-\lambda_{2,i}+j}}{1 - A\frac{\mathsf{t}}{\mathsf{q}}\mathsf{t}^{-k_1+i-1}\mathsf{q}^{-\lambda_{2,i}+j}}. \end{aligned} \tag{B.5}$$

In proving the lemma, we will deal with formal power series in variables t and q, so that we will not be concerned with issues of convergence of the intermediate expressions, requiring only that $\mathsf{t}, \mathsf{q} \ne 1$. We also extend the entries of partitions λ_1 and λ_2, such that

$$\lambda_{1,i} := 0,\ i > k_1, \qquad \lambda_{2,i} := 0,\ i > k_2\,. \tag{B.6}$$

So, let us start with the following obvious identity:

$$\sum_{i,j=1}^{\infty} \mathsf{t}^{j-i}\left(1 - \mathsf{q}^{\lambda_{1,i}-\lambda_{2,j}}\right) = \left(\sum_{i=1}^{k_1}\sum_{j=1}^{k_2} + \sum_{i=k_1+1}^{\infty}\sum_{j=1}^{k_2} + \sum_{i=1}^{k_1}\sum_{j=k_2+1}^{\infty}\right) \mathsf{t}^{j-i}\left(1 - \mathsf{q}^{\lambda_{1,i}-\lambda_{2,j}}\right). \tag{B.7}$$

Taking the last two sums of the right-hand side, shifting their summation indices and using convention (B.6), one gets:

$$\left(\sum_{i=k_1+1}^{\infty}\sum_{j=1}^{k_2}+\sum_{i=1}^{k_1}\sum_{j=k_2+1}^{\infty}\right)\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)=\sum_{i=1}^{\infty}\sum_{j=1}^{k_2}\mathfrak{t}^{j-i-k_1}\left(1-\mathfrak{q}^{-\lambda_{2,j}}\right)+\sum_{i=1}^{k_1}\sum_{j=1}^{\infty}\mathfrak{t}^{j-i+k_2}\left(1-\mathfrak{q}^{\lambda_{1,i}}\right)$$
$$=\frac{1}{\mathfrak{t}^{-1}-1}\left(-\sum_{j=1}^{k_2}\mathfrak{t}^{j-i-k_1}\left(1-\mathfrak{q}^{-\lambda_{2,j}}\right)+\sum_{i=1}^{k_1}\mathfrak{t}^{-i+k_2}\left(1-\mathfrak{q}^{\lambda_{1,i}}\right)\right),\tag{B.8}$$

where in the last step we used the sum of an infinite geometric progression. Substituting this back and multiplying the whole expression by $\mathfrak{t}^{-1}-1$, we obtain:

$$(\mathfrak{t}^{-1}-1)\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)=(\mathfrak{t}^{-1}-1)\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)$$
$$-\sum_{j=1}^{k_2}\mathfrak{t}^{j-1-k_1}\left(1-\mathfrak{q}^{-\lambda_{2,j}}\right)+\sum_{i=1}^{k_1}\mathfrak{t}^{-i+k_2}\left(1-\mathfrak{q}^{\lambda_{1,i}}\right).\tag{B.9}$$

Now we will use the following identity which the reader can find for instance in [27]:

$$-(\mathfrak{t}^{-1}-1)\sum_{i=1}^{\infty}\mathfrak{q}^{\lambda_{1,i}}\mathfrak{t}^{1-i}=(\mathfrak{q}^{-1}-1)\sum_{i=1}^{\infty}\mathfrak{t}^{-\lambda^t_{1,i}}\mathfrak{q}^{i}.\tag{B.10}$$

Multiplying it by $\sum_{j=1}^{\infty}\mathfrak{t}^{j-1}\mathfrak{q}^{-\lambda_{2,j}}$ and subtracting from the result the same with λ_1, λ_2 set to zero, we find:

$$(\mathfrak{t}^{-1}-1)\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-i}\left(1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}\right)=(\mathfrak{q}^{-1}-1)\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-1}\mathfrak{q}^{i}\left(\mathfrak{t}^{-\lambda^t_{1,i}}\mathfrak{q}^{-\lambda_{2,j}}-1\right).\tag{B.11}$$

Substituting this back as a left-hand side of (B.9) and dividing everything by $\mathfrak{q}^{-1}-1$, we obtain the following:

$$\sum_{i,j=1}^{\infty}\mathfrak{t}^{j-1}\mathfrak{q}^{i}\left(\mathfrak{t}^{-\lambda^t_{1,i}}\mathfrak{q}^{-\lambda_{2,j}}-1\right)=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\mathfrak{q}\left(\mathfrak{t}^{j-i-1}-\mathfrak{t}^{j-i}\right)\frac{1-\mathfrak{q}^{\lambda_{1,i}-\lambda_{2,j}}}{1-\mathfrak{q}}$$
$$+\sum_{j=1}^{k_2}\mathfrak{q}^{1-\lambda_{2,j}}\mathfrak{t}^{j-1-k_1}\frac{1-\mathfrak{q}^{-\lambda_{2,j}}}{1-\mathfrak{q}}+\sum_{i=1}^{k_1}\mathfrak{q}\mathfrak{t}^{-i+k_2}\frac{1-\mathfrak{q}^{\lambda_{1,i}}}{1-\mathfrak{q}},\tag{B.12}$$

where one can now use the formula for finite geometric progression to get rid of the fractions in the right-hand side:

$$\sum_{i,j=1}^{\infty}\left(\mathfrak{t}^{j-1-\lambda^t_{1,i}}\mathfrak{q}^{i-\lambda_{2,j}}-\mathfrak{t}^{j-1}\mathfrak{q}^{i}\right)=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{l=1}^{\lambda_{1,i}-\lambda_{2,j}}\left(\mathfrak{t}^{j-i-1}-\mathfrak{t}^{j-i}\right)\mathfrak{q}^{l}$$
$$+\sum_{j=1}^{k_2}\sum_{i=1}^{\lambda_{2,j}}\mathfrak{t}^{j-1-k_1}\mathfrak{q}^{i-\lambda_{2,j}}+\sum_{i=1}^{k_1}\sum_{j=1}^{\lambda_{1,i}}\mathfrak{t}^{-i+k_2}\mathfrak{q}^{j}.\tag{B.13}$$

For the left-hand side one now should employ an identity from [72] (our $\mathfrak{t}$ and $\mathfrak{q}$ are interchanged with respect to the formula there):

$$\sum_{i,j=1}^{\infty}\left(\mathfrak{t}^{j-1-\lambda^t_{1,i}}\mathfrak{q}^{i-\lambda_{2,j}}-\mathfrak{t}^{j-1}\mathfrak{q}^{i}\right)=\sum_{s\in\lambda_1}\mathfrak{t}^{l_{\lambda_2}(s)}\mathfrak{q}^{a_{\lambda_1}(s)+1}+\sum_{s\in\lambda_2}\mathfrak{t}^{-l_{\lambda_1}(s)-1}\mathfrak{q}^{-a_{\lambda_2}(s)}$$
$$\equiv\sum_{(i,j)\in\lambda_1}\mathfrak{t}^{\lambda^t_{2,j}-i}\mathfrak{q}^{\lambda_{1,i}-j+1}+\sum_{(i,j)\in\lambda_2}\mathfrak{t}^{i-\lambda^t_{1,j}-1}\mathfrak{q}^{j-\lambda_{2,i}}\tag{B.14}$$

Interchanging the indices in the second summand of the right-hand side of (B.13), changing the summation order in the third summand and moving them to the left, one finally obtains:

$$\sum_{(i,j)\in\lambda_1}\left(\mathfrak{t}^{\lambda^t_{2,j}-i}-\mathfrak{t}^{k_2-i}\right)\mathfrak{q}^{\lambda_{1,i}-j+1}+\sum_{(i,j)\in\lambda_2}\left(\mathfrak{t}^{-\lambda^t_{1,j}+i-1}-\mathfrak{t}^{-k_1+i-1}\right)\mathfrak{q}^{-\lambda_{2,i}+j}$$
$$=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\sum_{l=1}^{\lambda_{1,i}-\lambda_{2,j}}\left(\mathfrak{t}^{j-i-1}-\mathfrak{t}^{j-i}\right)\mathfrak{q}^{l}.\tag{B.15}$$

Substituting here $\mathfrak{t},\mathfrak{q}\longrightarrow\mathfrak{t}^r,\mathfrak{q}^r$, multiplying by $\left(A\frac{\mathfrak{t}}{\mathfrak{q}}\right)^r/r$ and using the series expansion of a logarithm $\ln(1-x)=-\sum_{r=1}^{\infty}\frac{x^r}{r}$, we get

$$\sum_{(i,j)\in\lambda_1}\ln\left(\frac{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{\lambda^t_{2,j}-i}\mathfrak{q}^{\lambda_{1,i}-j+1}}{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{k_2-i}\mathfrak{q}^{\lambda_{1,i}-j+1}}\right)+\sum_{(i,j)\in\lambda_2}\ln\left(\frac{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{-\lambda^t_{1,j}+i-1}\mathfrak{q}^{-\lambda_{2,i}+j}}{1-A\frac{\mathfrak{t}}{\mathfrak{q}}\mathfrak{t}^{-k_1+i-1}\mathfrak{q}^{-\lambda_{2,i}+j}}\right)$$
$$=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\ln\left(\prod_{l=1}^{\lambda_{1,i}-\lambda_{2,j}}\frac{1-A\mathfrak{t}^{j-i}\mathfrak{q}^{l-1}}{1-A\mathfrak{t}^{j-i+1}\mathfrak{q}^{l-1}}\right).\tag{B.16}$$

Exponentiation concludes the proof.

Having proven the above lemma, it is straightforward to show that (B.3) is equivalent to:

$$\sum_{\lambda^{(1)},\ldots,\lambda^{(N-1)}}\left[\prod_{i=1}^{N-2}\left(\frac{z_i}{\mathfrak{t}}\mathfrak{t}^{\frac{k_{i-1}}{2}+k_i-\frac{k_{i+1}}{2}}\right)^{|\lambda^{(i)}|}\right]\cdot\left(\sqrt{A\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{z_{N-1}}{\mathfrak{t}}\mathfrak{t}^{\frac{k_{N-2}+k_{N-1}}{2}}\right)^{|\lambda^{(N-1)}|}$$
$$\times\left[\prod_{i=1}^{N-1}\frac{\mathsf{N}^{\beta}_{\lambda^{(i)}\lambda^{(i-1)}}\left((k_{i-1}-k_i)\epsilon_2-\epsilon_+\right)}{\mathsf{N}^{\beta}_{\lambda^{(i)}\lambda^{(i)}}(0)}\right]\cdot\mathsf{N}^{\beta}_{\emptyset\lambda^{(N-1)}}(-a)\tag{B.17}$$
$$=\prod_{1\leq i\leq j\leq N-2}\frac{\mathcal{M}\big(\mathfrak{t}^{i-(j+1)+k_i-k_{j+1}}\cdot\prod_{s=i}^{j}(z_s\mathfrak{t}^{k_s})\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\cdot\mathfrak{t}^{(i-1)-j+k_{i-1}-k_j}\cdot\prod_{s=i}^{j}(z_s\mathfrak{t}^{k_s})\big)}{\mathcal{M}\big(\mathfrak{t}\cdot\mathfrak{t}^{(i-1)-(j+1)+k_{i-1}-k_{j+1}}\cdot\prod_{s=i}^{j}(z_s\mathfrak{t}^{k_s})\big)\mathcal{M}\big(\frac{1}{\mathfrak{q}}\cdot\mathfrak{t}^{i-j+k_i-k_j}\cdot\prod_{s=i}^{j}(z_s\mathfrak{t}^{k_s})\big)}$$
$$\times\prod_{i=1}^{N-1}\frac{\mathcal{M}\big(\frac{A}{\mathfrak{q}}\cdot\mathfrak{t}^{i-(N-1)+k_i-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathfrak{t}^{k_s})\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\cdot\mathfrak{t}^{(i-1)-(N-1)+k_{i-1}-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathfrak{t}^{k_s})\big)}{\mathcal{M}\big(\frac{A\mathfrak{t}}{\mathfrak{q}}\cdot\mathfrak{t}^{(i-1)-(N-1)+k_{i-1}-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathfrak{t}^{k_s})\big)\mathcal{M}\big(\frac{1}{\mathfrak{q}}\cdot\mathfrak{t}^{i-(N-1)+k_i-k_{N-1}}\cdot\prod_{s=i}^{N-1}(z_s\mathfrak{t}^{k_s})\big)}.$$

Now we are in position to prove the required summation formula:

Theorem

$$\sum_{\lambda^{(1)},\dots,\lambda^{(N-1)}}\left[\prod_{i=1}^{N-1}\frac{\left(V_i\sqrt{U_iU_{i+1}}\right)^{|\lambda^{(i)}|}}{\mathsf{N}^{\beta}_{\lambda^{(i)}\lambda^{(i)}}(0)}\right]\mathsf{N}^{\beta}_{\lambda^{(1)}\emptyset}(d_1-\epsilon_+/2)$$
$$\times\left[\prod_{i=1}^{N-2}\mathsf{N}^{\beta}_{\lambda^{(i+1)}\lambda^{(i)}}(u_{i+1}-\epsilon_+/2)\right]\mathsf{N}^{\beta}_{\emptyset\lambda^{(N-1)}}(u_N-\epsilon_+/2) \tag{B.18}$$
$$=\prod_{i=1}^{N-1}\prod_{j=1}^{N-i}\frac{\mathcal{M}\big(\prod_{s=j}^{i+j-1}(V_sU_s)\big)\mathcal{M}\big(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{U_{i+j}}{U_j}\cdot\prod_{s=j}^{i+j-1}(V_sU_s)\big)}{\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_{i+j}\cdot\prod_{s=j}^{i+j-1}(V_sU_s)\big)\mathcal{M}\big(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{1}{U_j}\cdot\prod_{s=j}^{i+j-1}(V_sU_s)\big)},$$

with N site parameters $U_i=e^{-\beta d_i}$ and $N-1$ link parameters V_j. One can visualize the right-hand side of this formula by noticing that the arguments of numerator are exactly all the simply-connected combinations of even number of site and link parameters (multiplied by $\frac{\mathfrak{t}}{\mathfrak{q}}$ when starting with a link parameter), whereas the arguments of denominator are represented by all simply-connected combinations of odd number of site and link parameters (multiplied by $\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}$, single site parameters are excluded).

Proof: We use a so-called *specialization technique* [71]. Let us make the following specialization of U_i ($k_0\equiv 0$):

$$U_i=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\,\mathfrak{t}^{k_i-k_{i-1}},\qquad i=1,\dots,N-1 \tag{B.19}$$

and reparametrize the remaining variables as

$$V_j=\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\,\frac{z_j}{\mathfrak{t}}\,\mathfrak{t}^{k_{j-1}+k_j-k_{j+1}},\qquad j=1,\dots,N-2 \tag{B.20}$$

as well as

$$U_N=\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\,\frac{1}{A},\qquad V_{N-1}=\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\,A\,\frac{z_{N-1}}{\mathfrak{t}}\,\mathfrak{t}^{k_{N-2}}. \tag{B.21}$$

One can readily check that the formula (B.18) then reproduces the established sl(N) $\mathfrak{q}$-binomial identity (B.17). Upon the continuation (see, e.g. page 41 of [8]) from discrete to continuous values of parameters (from compact to non-compact functions), this specialized identity is promoted to the summation formula (B.18) with arbitrary U_i, concluding our short argument.

Finally, let us remark that the summation formula (B.18) for $N=2$

$$\sum_{\lambda^{(1)}}\left(V_1\sqrt{U_1U_2}\right)^{|\lambda^{(1)}|}\frac{\mathsf{N}^{\beta}_{\lambda^{(1)}\emptyset}(d_1-\epsilon_+/2)\,\mathsf{N}^{\beta}_{\emptyset\lambda^{(1)}}(d_2-\epsilon_+/2)}{\mathsf{N}^{\beta}_{\lambda^{(1)}\lambda^{(1)}}(0)}=\frac{\mathcal{M}(U_1V_1)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_1U_2)}{\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_1)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_1V_1U_2)} \tag{B.22}$$

reproduces the non-trivial part of (5.3) of [7], whereas, taken for $N=3$

$$\sum_{\lambda^{(1)},\lambda^{(2)}}\left(V_1\sqrt{U_1U_2}\right)^{|\lambda^{(1)}|}\left(V_2\sqrt{U_2U_3}\right)^{|\lambda^{(2)}|}\frac{\mathsf{N}^{\beta}_{\lambda^{(1)}\emptyset}(d_1-\epsilon_+/2)\,\mathsf{N}^{\beta}_{\lambda^{(2)}\lambda^{(1)}}(d_2-\epsilon_+/2)\,\mathsf{N}^{\beta}_{\emptyset\lambda^{(2)}}(d_3-\epsilon_+/2)}{\mathsf{N}^{\beta}_{\lambda^{(1)}\lambda^{(1)}}(0)\,\mathsf{N}^{\beta}_{\lambda^{(2)}\lambda^{(2)}}(0)}$$
$$=\frac{\mathcal{M}(U_1V_1)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_1U_2)\mathcal{M}(U_2V_2)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_2U_3)\mathcal{M}(U_1V_1U_2V_2)\mathcal{M}(\frac{\mathfrak{t}}{\mathfrak{q}}V_1U_2V_2U_3)}{\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_1)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_2)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_1V_1U_2)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}V_1U_2V_2)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_2V_2U_3)\mathcal{M}(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}U_1V_1U_2V_2U_3)}, \tag{B.23}$$

it is equivalent to the formula (6.7) conjectured in [9].

C Higgsing and iterated integrals for the T_4 case

We saw in section 4 how for T_3 the semi-degeneration of the mass parameters m_i pinches the integral contour so that the structure constants are given by a single residue. The purpose of this section is to perform the same computation in the T_4 case, so as to illustrate the complexities that arise when we are confronted with iterated contour integrals. For simplicity of notation, we set $\mathbf{A}_1 \equiv A_1^{(1)}$, $\mathbf{A}_2 \equiv A_2^{(1)}$ and $\mathbf{A}_3 \equiv A_1^{(2)}$. From (16), we read the "perturbative" part of the the topological string partition function

$$\begin{aligned}
\left|\mathcal{Z}_4^{\text{pert}}\right|^2 = \prod_{i<j=1}^{4} \left|\mathcal{M}(\tilde{M}_i\tilde{M}_j^{-1})\right|^2 & \frac{\left|\mathcal{M}\left(\frac{\mathbf{A}_1^2}{\tilde{N}_1\mathbf{A}_2}\right)\mathcal{M}\left(\frac{\tilde{N}_1\mathbf{A}_2}{\mathbf{A}_1^2}\right)\mathcal{M}\left(\frac{\tilde{L}_4\mathbf{A}_2^2}{\mathbf{A}_1}\right)\mathcal{M}\left(\frac{\mathbf{A}_1}{\tilde{L}_4\mathbf{A}_2^2}\right)\right|^2}{\prod_{j=1}^4\left|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_1}{\tilde{M}_j\tilde{N}_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\tilde{L}_4\tilde{M}_j\mathbf{A}_2\right)\right|^2} \\
\times & \frac{\left|\mathcal{M}\left(\frac{\mathbf{A}_1\mathbf{A}_2\tilde{L}_4}{\tilde{N}_1}\right)\mathcal{M}\left(\frac{\tilde{N}_1}{\mathbf{A}_1\mathbf{A}_2\tilde{L}_4}\right)\mathcal{M}\left(\frac{\mathbf{A}_3^2}{\tilde{L}_1\tilde{L}_2\tilde{N}_1\tilde{N}_2}\right)\mathcal{M}\left(\frac{\tilde{L}_1\tilde{L}_2\tilde{N}_1\tilde{N}_2}{\mathbf{A}_3^2}\right)\right|^2}{\left|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{N}_4\mathbf{A}_3}{\tilde{L}_2}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{N}_3\mathbf{A}_3}{\tilde{L}_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_3\mathbf{A}_2}{\tilde{L}_1\tilde{L}_2\mathbf{A}_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_1\mathbf{A}_3}{\tilde{N}_1\tilde{N}_2\mathbf{A}_2}\right)\prod_{j=1}^4\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{M}_j\mathbf{A}_1}{\mathbf{A}_2}\right)\right|^2} \\
\times & \frac{1}{\left|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{N}_2\mathbf{A}_1}{\mathbf{A}_3}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_1\mathbf{A}_3}{\tilde{L}_1\tilde{L}_2\tilde{N}_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_2}{\tilde{L}_3\mathbf{A}_3}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{L}_4\mathbf{A}_3\mathbf{A}_2}{\tilde{N}_1\tilde{N}_2}\right)\right|^2}\,.
\end{aligned} \tag{C.1}$$

In addition, the "instanton" part (17) takes for $N = 4$ the form

$$\begin{aligned}
\mathcal{Z}_4^{\text{inst}} = & \sum_{\boldsymbol{\nu}}\left(\frac{\tilde{L}_3\tilde{N}_1}{\tilde{L}_4\tilde{N}_2}\right)^{\frac{1}{2}|\nu_1^{(1)}|+\frac{1}{2}|\nu_2^{(1)}|+\frac{1}{2}|\nu_3^{(1)}|}\left(\frac{\tilde{L}_2\tilde{N}_2}{\tilde{L}_3\tilde{N}_3}\right)^{\frac{1}{2}|\nu_1^{(2)}|+\frac{1}{2}|\nu_2^{(2)}|}\left(\frac{\tilde{L}_1\tilde{N}_3}{\tilde{L}_2\tilde{N}_4}\right)^{\frac{1}{2}|\nu_1^{(3)}|} \\
& \times\frac{\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_2^{(2)}}\left(-\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3-l_1-l_2-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_2^{(2)}\nu_3^{(1)}}\left(\mathbf{a}_2-\mathbf{a}_3-l_3-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\emptyset\nu_2^{(1)}}\left(\mathbf{a}_1-\mathbf{a}_2+m_1-\frac{Q}{2}\right)}{\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_1^{(1)}}(0)\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_1^{(2)}}(0)\mathsf{N}^{\beta}_{\nu_1^{(3)}\nu_1^{(3)}}(0)\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_2^{(1)}}(0)} \\
& \times\frac{\mathsf{N}^{\beta}_{\emptyset\nu_3^{(1)}}\left(\mathbf{a}_2+l_4+m_1-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\emptyset\nu_2^{(1)}}\left(\mathbf{a}_1-\mathbf{a}_2+m_2-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\emptyset\nu_3^{(1)}}\left(\mathbf{a}_2+l_4+m_2-\frac{Q}{2}\right)}{\mathsf{N}^{\beta}_{\nu_2^{(2)}\nu_2^{(2)}}(0)\mathsf{N}^{\beta}_{\nu_3^{(1)}\nu_3^{(1)}}(0)\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_3^{(1)}}(-\mathbf{a}_1+2\mathbf{a}_2+l_4)} \\
& \times\frac{\mathsf{N}^{\beta}_{\nu_2^{(1)}\emptyset}\left(-\mathbf{a}_1+\mathbf{a}_2-m_3-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\emptyset\nu_3^{(1)}}\left(\mathbf{a}_2+l_4+m_3-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_2^{(1)}\emptyset}\left(-\mathbf{a}_1+\mathbf{a}_2-m_4-\frac{Q}{2}\right)}{\mathsf{N}^{\beta}_{\nu_2^{(1)}\nu_3^{(1)}}(-\mathbf{a}_1+2\mathbf{a}_2-Q+l_4)\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_2^{(1)}}(2\mathbf{a}_1-\mathbf{a}_2-n_1)\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_2^{(1)}}(2\mathbf{a}_1-\mathbf{a}_2-Q-n_1)} \\
& \times\frac{\mathsf{N}^{\beta}_{\nu_3^{(1)}\emptyset}\left(-\mathbf{a}_2-l_4-m_4-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_2^{(2)}}\left(\mathbf{a}_1+\mathbf{a}_3-l_1-l_2-n_1-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_1^{(1)}\emptyset}\left(\mathbf{a}_1-m_2-n_1-\frac{Q}{2}\right)}{\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_3^{(1)}}(\mathbf{a}_1+\mathbf{a}_2+l_4-n_1)\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_3^{(1)}}(\mathbf{a}_1+\mathbf{a}_2-Q+l_4-n_1)} \\
& \times\frac{\mathsf{N}^{\beta}_{\nu_1^{(1)}\emptyset}\left(\mathbf{a}_1-m_3-n_1-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_1^{(1)}\emptyset}\left(\mathbf{a}_1-m_4-n_1-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\emptyset\nu_1^{(1)}}\left(-\mathbf{a}_1+m_1+n_1-\frac{Q}{2}\right)}{\mathsf{N}^{\beta}_{\nu_1^{(2)},\nu_2^{(2)}}(2\mathbf{a}_3-l_1-l_2-n_1-n_2)\mathsf{N}^{\beta}_{\nu_1^{(2)},\nu_2^{(2)}}(2\mathbf{a}_3-Q-l_1-l_2-n_1-n_2)} \\
& \times\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_2^{(1)}}\left(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-n_1-n_2-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_1^{(2)}\nu_3^{(1)}}\left(\mathbf{a}_2+\mathbf{a}_3+l_4-n_1-n_2-\frac{Q}{2}\right) \\
& \times\mathsf{N}^{\beta}_{\nu_1^{(1)}\nu_1^{(2)}}\left(\mathbf{a}_1-\mathbf{a}_3+n_2-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_1^{(3)}\nu_2^{(2)}}\left(\mathbf{a}_3-l_2+n_4-\frac{Q}{2}\right)\mathsf{N}^{\beta}_{\nu_1^{(2)},\nu_1^{(3)}}\left(\mathbf{a}_3-l_1+n_3-\frac{Q}{2}\right)
\end{aligned} \tag{C.2}$$

Let us try to perform the contour integrals over the Coulomb moduli $\mathbf{A}_i$'s. We fix the contours over the $\mathbf{A}_i$'s as explained in section 4 by choosing the flopping frame shown in the left part of figure 2. Using the

T_4 parametrization of (A.4), we find the expressions for the Kähler parameters $Q^{(j)}_{m;i}$ and $Q^{(j)}_{l;i}$. The fact that the lengths in the picture have to be positive, implies the following domain

$$\left|\frac{\tilde{M}_1\tilde{N}_1}{\mathbf{A}_1}\right| > 1\,, \quad \left|\frac{\mathbf{A}_1}{\tilde{M}_2\tilde{N}_1}\right| > 1\,, \quad \left|\frac{\mathbf{A}_1\tilde{M}_2}{\mathbf{A}_2}\right| > 1\,, \quad \left|\frac{\mathbf{A}_2}{\mathbf{A}_1\tilde{M}_3}\right| > 1\,, \quad \left|\mathbf{A}_2\tilde{M}_3\tilde{L}_4\right| > 1\,,$$
$$\left|\frac{\mathbf{A}_1\tilde{N}_2}{\mathbf{A}_3}\right| > 1\,, \quad \left|\frac{\mathbf{A}_1\mathbf{A}_3}{\mathbf{A}_2\tilde{N}_1\tilde{N}_2}\right| > 1\,, \quad \left|\frac{\mathbf{A}_2\mathbf{A}_3}{\mathbf{A}_1\tilde{L}_1\tilde{L}_2}\right| > 1\,, \quad \left|\frac{\mathbf{A}_2}{\mathbf{A}_3\tilde{L}_3}\right| > 1\,, \quad \left|\frac{\mathbf{A}_3\tilde{N}_3}{\tilde{L}_1}\right| > 1\,, \quad \left|\frac{\mathbf{A}_3\tilde{N}_4}{\tilde{L}_2}\right| > 1\,. \tag{C.3}$$

The mass parameters for the 5-branes on the left side of the T_4 junction are parametrized as follows

$$\tilde{M}_1 = \tilde{K}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{3}{2}} d_1\,, \qquad \tilde{M}_2 = \tilde{K}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{1}{2}} d_2\,, \qquad \tilde{M}_3 = \tilde{K}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{-\frac{1}{2}} d_3\,, \qquad \tilde{M}_4 = \tilde{K}^{-3}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{-\frac{3}{2}}\,, \tag{C.4}$$

with $\prod_{i=1}^3 d_i = 1$. We set $d_i = e^{-\beta\delta_i}$ with $\sum_{i=1}^3 \delta_i = 0$. We will compute the integrals in the order $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ and are interested in the result in the limit $\delta_a \to 0$. Thus, in the calculation of the contour integrals, we will only keep the residues that will diverge when the regulators δ_i are finally all set to zero. Their divergences will be canceled in the limit by the zeroes coming from the $\left|\mathcal{M}\big(\tilde{M}_i\tilde{M}_j^{-1}\big)\right|^2$ in the numerator.

Let us now consider the contour integral over $\mathbf{A}_1$. The possible contributing poles come from the following terms in the denominator of (C.1)

$$\left|\prod_{j=1}^{3}\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_1}{\tilde{M}_j\tilde{N}_1}\right)\prod_{k=1}^{3}\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\tilde{M}_k\mathbf{A}_1}{\mathbf{A}_2}\right)\right|^2. \tag{C.5}$$

We number the terms with $j = 1,2,3$ as 1 to 3 and those with $k = 1,2,3$ as 4 to 6 and we need to investigate which of them might pinch the integral contour. Plugging (C.4) into (C.3), we find that the allowed domain for the contour of $\mathbf{A}_1$ is

$$\left|\tilde{K}\tilde{N}_1 d_2\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{1}{2}}\right| < |\mathbf{A}_1| < \left|\tilde{K}\tilde{N}_1 d_1\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{3}{2}}\right|\,, \qquad \left|\mathbf{A}_2\tilde{K}^{-1}d_2^{-1}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{1}{2}}\right| < |\mathbf{A}_1| < \left|\mathbf{A}_2\tilde{K}^{-1}d_3^{-1}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{1}{2}}\right|\,. \tag{C.6}$$

The situation for imaginary δ_a is depicted in figure 14. We see that for $|\tilde{K}| > 1$ and imaginary masses n_i and l_i, the two domains in (C.6) do not overlap and after some contemplation we find that the contour for $\mathbf{A}_1$ can be chosen in such a way that in the limit $\delta_a \to 0$ only one residue contributes, namely the one for

$$\mathbf{A}_1 = \tilde{K}\tilde{N}_1 d_1 \frac{\mathfrak{t}}{\mathfrak{q}}\,. \tag{C.7}$$

Thus, we can compute the integral over $\mathbf{A}_1$ just as in the T_3 case and, after some simplifications, obtain

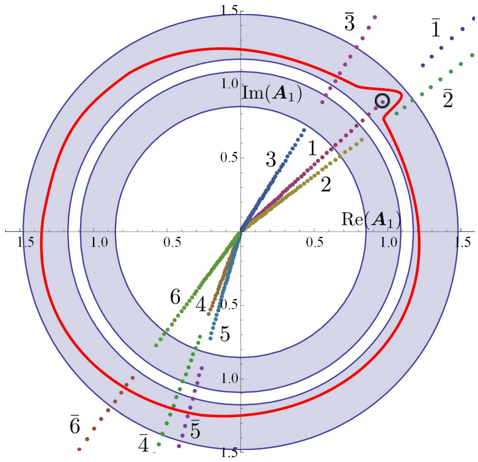

Figure 14: *The figure presents our choice of the integration contour for $\boldsymbol{A}_1$. The allowed domains for $\boldsymbol{A}_1$ taken from* (C.6) *are shaded in blue. As the regulators δ_a are taken to zero, the integral is given by just one residue whose position is indicated by a small circle.*

the integral expression

$$
\begin{aligned}
&\lim_{\delta_a\to 0}\oint\prod_{k=1}^{3}\left[|M(\mathfrak{t},\mathfrak{q})|^2\frac{d\mathbf{A}_k}{2\pi i\mathbf{A}_k}\right]\left|\mathcal{Z}_4^{\text{top}}\right|^2=\lim_{\delta_a\to 0}\oint\prod_{k=2}^{3}\left[|M(\mathfrak{t},\mathfrak{q})|^2\frac{d\mathbf{A}_k}{2\pi i\mathbf{A}_k}\right]\text{Res}\left(\left|\mathcal{Z}_4^{\text{top}}\right|^2,\mathbf{A}_1=\tilde{K}\tilde{N}_1d_1\frac{\mathfrak{t}}{\mathfrak{q}}\right)\\
&=\lim_{\delta_a\to 0}\oint\prod_{k=2}^{3}\left[|M(\mathfrak{t},\mathfrak{q})|^2\frac{d\mathbf{A}_k}{2\pi i\mathbf{A}_k}\right]\frac{\left|\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{d_2}{d_3}\right)\mathcal{M}\left(\frac{\mathfrak{t}^2}{\mathfrak{q}^2}\tilde{K}^4d_2\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\tilde{K}^4d_3\right)\right|^2}{\left|\mathcal{M}\left(\frac{\mathbf{A}_2d_2}{\tilde{K}^2\tilde{N}_1}\right)\mathcal{M}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{\mathbf{A}_2d_3}{\tilde{K}^2\tilde{N}_1}\right)\mathcal{M}\left(\mathbf{A}_2\tilde{K}\tilde{L}_4d_3\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\mathbf{A}_2\tilde{K}d_2\tilde{L}_4\right)\right|^2}\\
&=\frac{\left|\mathcal{M}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{\mathbf{A}_2}{\tilde{K}^2\tilde{N}_1d_1^2}\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\mathbf{A}_2\tilde{K}\tilde{L}_4d_1\right)\mathcal{M}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{\mathbf{A}_2^2\tilde{L}_4}{\tilde{K}\tilde{N}_1d_1}\right)\mathcal{M}\left(\frac{\mathbf{A}_2^2\tilde{L}_4}{\tilde{K}\tilde{N}_1d_1}\right)\right|^2}{\left|\mathcal{M}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{\mathbf{A}_2\tilde{L}_4}{\tilde{K}^3}\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\mathbf{A}_2\tilde{K}^2}{\tilde{N}_1d_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_2}{\mathbf{A}_3\tilde{L}_3}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{\mathbf{A}_2\tilde{N}_2}{\mathbf{A}_3\tilde{K}d_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{\mathbf{A}_2\mathbf{A}_3}{\tilde{K}\tilde{L}_1\tilde{L}_2\tilde{N}_1d_1}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_2\mathbf{A}_3\tilde{L}_4}{\tilde{N}_1\tilde{N}_2}\right)\right|^2}\\
&=\frac{\left|\mathcal{M}\left(\frac{\mathbf{A}_3^2}{\tilde{L}_1\tilde{L}_2\tilde{N}_1\tilde{N}_2}\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\frac{\mathbf{A}_3^2}{\tilde{L}_1\tilde{L}_2\tilde{N}_1\tilde{N}_2}\right)\right|^2}{\left|\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_3\tilde{N}_4}{\tilde{L}_2}\right)\mathcal{M}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{\mathbf{A}_3\tilde{N}_3}{\tilde{L}_1}\right)\mathcal{M}\left(\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{3}{2}}\frac{\mathbf{A}_3\tilde{K}d_1}{\tilde{L}_1\tilde{L}_2}\right)\mathcal{M}\left(\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{3}{2}}\frac{\tilde{K}\tilde{N}_1\tilde{N}_2d_1}{\mathbf{A}_3}\right)\right|^2}\left|\mathcal{Z}_4^{\text{inst}}\right|^2\Big|_{\mathbf{A}_1=\tilde{K}\tilde{N}_1d_1\frac{\mathfrak{t}}{\mathfrak{q}}}
\end{aligned}
\tag{C.8}
$$

where we have used (51).

We must now perform the integration over $\mathbf{A}_2$. We find that the relevant terms in the denominator of the integrand in (C.8) are

$$\left|\mathcal{M}\left(\frac{\mathbf{A}_2d_2}{\tilde{K}^2\tilde{N}_1}\right)\mathcal{M}\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{\mathbf{A}_2d_3}{\tilde{K}^2\tilde{N}_1}\right)\mathcal{M}\left(\mathbf{A}_2\tilde{K}d_3\tilde{L}_4\right)\mathcal{M}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\mathbf{A}_2\tilde{K}d_2\tilde{L}_4\right)\right|^2. \tag{C.9}$$

From the above, we read that there are two poles that are *potentially* relevant for the semi-degenerate limit, namely those for

$$\mathbf{A}_2=\tilde{K}^2\tilde{N}_1d_3^{-1}\frac{\mathfrak{t}}{\mathfrak{q}},\qquad \mathbf{A}_2=\tilde{K}^{-1}\tilde{L}_4^{-1}d_3^{-1}. \tag{C.10}$$

These are the two residues that *could* contribute due to pinching. We need now to set the exact integral contour for $\mathbf{A}_2$ to see which one of them actually contributes. Plugging (C.3) and (C.4) in (C.7), we find

$$\left|\tilde{K}^2\tilde{N}_1 d_2^{-1}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{1}{2}}\right| < |\mathbf{A}_2| < \left|\tilde{K}^2\tilde{N}_1 d_3^{-1}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{3}{2}}\right|, \qquad |\mathbf{A}_2| > \left|\tilde{K}^{-1}\tilde{L}_4^{-1}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{1}{2}}\right|. \tag{C.11}$$

The above constraints teach us that the contour can be chosen in such a way as to have the residue at $\mathbf{A}_2 = \tilde{K}^2\tilde{N}_1 d_3^{-1}\frac{\mathfrak{t}}{\mathfrak{q}}$, but not in such a way as to have $\mathbf{A}_2 = \tilde{K}^{-1}\tilde{L}_4^{-1}d_3^{-1}$. This latter point is due to the last constraint in (C.11) which implies that both poles that collide $\mathbf{A}_2 = \tilde{K}^{-1}\tilde{L}_4^{-1}$ when the regulators are removed have to lie on the same side of the contour. The argument is the same as the one used in the T_3 case to exclude (55) for the first flopping frame of figure 9. Taking all this into consideration, we can compute the integral over $\mathbf{A}_2$ in (C.8).

Finally, we have to compute the integral over $\mathbf{A}_3$. Arguments similar to the ones used for $\mathbf{A}_2$ tell us that the contour should be chosen such as to have a pinching when the regulators are removed and we get the residue at

$$\mathbf{A}_3 = \tilde{K}\tilde{N}_1\tilde{N}_2 d_1\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}. \tag{C.12}$$

Performing the same kind of computation that led to (C.8), we obtain the integral in the semi-degenerate limit

$$\lim_{\delta_a\to 0}\oint\prod_{k=1}^{3}\left[|M(\mathfrak{t},\mathfrak{q})|^2\frac{d\mathbf{A}_k}{2\pi i\mathbf{A}_k}\right]\left|\mathcal{Z}_4^{\text{top}}\right|^2 =$$

$$= \left|\frac{\mathcal{M}\left(\tilde{K}^{-4}\right)}{\prod_{i=1}^{4}\mathcal{M}\left(\frac{\tilde{L}_i\tilde{N}_{5-i}}{\tilde{K}}\right)}\right|^2 \left|\mathcal{Z}_4^{\text{inst}}\right|^2_{\left|\tilde{A}_i^{(j)}\to\tilde{K}^i\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{i(4-i-j)}{2}}\prod_{k=1}^{j}\tilde{N}_k\right.}. \tag{C.13}$$

Computing the "instanton" contribution to residues, we find that inserting the values of he Coulomb moduli, namely (C.7), the left part of (C.10) as well as (C.12) into (C.2) immediately gets rid of the sums over $\nu_1^{(1)}$, $\nu_1^{(2)}$ and $\nu_2^{(1)}$ due to (A.38). Thus, we obtain the "instanton" contribution to the contour integral in the semi-degenerate limit

$$\left(\mathcal{Z}_4^{\text{inst}}\right)\Big|_{\mathbf{A}_1=\tilde{K}\tilde{N}_1 d_1\frac{\mathfrak{t}}{\mathfrak{q}},\mathbf{A}_2=\tilde{K}^2\tilde{N}_1\frac{\mathfrak{t}}{\mathfrak{q}},\mathbf{A}_3=\tilde{K}\tilde{N}_1\tilde{N}_2\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}} = \sum_{\boldsymbol{\nu}}\left(\frac{\tilde{L}_3\tilde{N}_1}{\tilde{L}_4\tilde{N}_2}\right)^{\frac{|\nu_3^{(1)}|}{2}}\left(\frac{\tilde{L}_2\tilde{N}_2}{\tilde{L}_3\tilde{N}_3}\right)^{\frac{|\nu_2^{(2)}|}{2}}\left(\frac{\tilde{L}_1\tilde{N}_3}{\tilde{L}_2\tilde{N}_4}\right)^{\frac{|\nu_1^{(3)}|}{2}}$$

$$\times\frac{\mathsf{N}^{\beta}_{\nu_3^{(1)}\nu_2^{(2)}}(l_3+n_2-\varkappa)\mathsf{N}^{\beta}_{\nu_2^{(2)}\nu_1^{(3)}}(l_2+n_3-\varkappa)\mathsf{N}^{\beta}_{\emptyset\nu_3^{(1)}}(l_4+n_1-\varkappa)\mathsf{N}^{\beta}_{\nu_1^{(3)}\emptyset}(l_1+n_4-\varkappa)}{\mathsf{N}^{\beta}_{\nu_1^{(3)}\nu_1^{(3)}}(0)\mathsf{N}^{\beta}_{\nu_2^{(2)}\nu_2^{(2)}}(0)\mathsf{N}^{\beta}_{\nu_3^{(1)}\nu_3^{(1)}}(0)}. \tag{C.14}$$

We can now plug the summation formula (69) in (C.14) and inserting the result in (C.13) we get the final result

$$\lim_{\delta_a\to 0}\oint\prod_{k=1}^{3}\left[|M(\mathfrak{t},\mathfrak{q})|^2\frac{d\mathbf{A}_k}{2\pi i\mathbf{A}_k}\right]\left|\mathcal{Z}_4^{\text{top}}\right|^2 = \frac{\left|\mathcal{M}(\tilde{K}^{-4})\prod_{i<j=1}^{4}\mathcal{M}\left(\tilde{N}_j/\tilde{N}_i\right)\mathcal{M}\left(\tilde{L}_i/\tilde{L}_j\right)\right|^2}{\left|\prod_{i,j=1}^{4}\mathcal{M}(\tilde{N}_i\tilde{L}_j\tilde{K}^{-1})\right|^2}. \tag{C.15}$$

Thus, we obtain our general formula (68), specialized for $N = 4$.

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