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The anomalous magnetic moment of the muon: Theory update

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Abstract

We present a review on recent progress in perturbative calculations for the anomalous magnetic moment of the muon. We present recent calculations for leptonic contributions to $g-2$ and discuss the NNLO contributions to hadronic vacuum polarisation insertions.

Keywords: QED, $g-2$, hadronic contributions

1. Introduction

The anomalous magnetic moment of the muon $(g-2)_\mu$ has been both experimentally measured and theoretically calculated with astonishing precision. The difference between the experimental value [1, 2]

$$a_\mu^{\text{exp}} = 0.001\,165\,920\,80(54)(33)[63] \tag{1}$$

and the theory prediction [3]

$$a_\mu^{\text{theo}} = 0.001\,165\,918\,40(59) \tag{2}$$

has the size of about three standard deviations. On the theory side the contributions to a_μ^{theo} can be decomposed into three parts

$$a_\mu^{\text{theo}} = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{hadr}}\,, \tag{3}$$

where a_μ^{QED}, a_μ^{EW}, and a_μ^{hadr} denote the QED, electroweak, and hadronic contributions, respectively. The error on the theory prediction (2) is dominated by the uncertainty of the hadronic contributions [4, 5].

The electro-weak contributions have been calculated analytically in Refs. [6, 7, 8, 9, 10] and with the measurement of the mass of the Higgs boson all input parameters are now sufficiently well known which leads to the following contributions up to next-to-leading order

$$\begin{aligned}
a_\mu^{EW,(1)} &= (194.80 \pm 0.01) \times 10^{-11}, \\
a_{\mu,\text{bos}}^{EW,(2)} &= -(19.97 \pm 0.03) \times 10^{-11}, \\
a_{\mu,f\text{rest},H}^{EW,(2)} &= -(1.50 \pm 0.01) \times 10^{-11}, \\
a_\mu^{EW,(2)}(e,\mu,u,c,d,s) &= -(6.91 \pm 0.20 \pm 0.30) \times 10^{-11}, \\
a_\mu^{EW,(2)}(\tau,t,b) &= -(8.21 \pm 0.10) \times 10^{-11}, \\
a_{\mu,f\text{rest},noH}^{EW,(2)} &= -(4.64 \pm 0.10) \times 10^{-11}, \\
a_\mu^{EW,\geq 3\ell} &= (0 \pm 0.20) \times 10^{-11}\,.
\end{aligned}$$

The full result for the electro-weak corrections reads

$$a_\mu^{EW} = (153.6 \pm 1.0) \times 10^{-11}$$

with a conservative error estimate.

The QED contributions have been calculated up to five-loop order in [3, 11]. We want to stress that looking at the absolute size of the QED corrections (cf Tab. 1) one finds that the four-loop contribution is of the same size as the difference between theory and experiment. Therefore it is mandatory to verify the only existing calculation of these contributions by an independent one. First steps towards this are presented in Section 2. Corrections to a_μ^{QED} from vacuum polarization insertions have been calculated up to five-loop order and are discussed in Section 4. Even though the corrections contained in a_μ^{hadr} are of non-perturbative nature they still

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receive quantum corrections which can be addressed in perturbation theory and are discussed in Section 3.

2. Leptonic contributions at four-loop order

The pure QED contributions can be further decomposed as

$$a_\mu^{\text{QED}} = \sum_{n=1} \left(\frac{\alpha}{\pi}\right)^n A_\mu^{(n)} \tag{4}$$

$$A_\mu^{(n)} = A_1^{(n)} + A_2^{(n)}(M_e/M_\mu) + A_2^{(n)}(M_\mu/M_\tau) + A_3^{(n)}(M_e/M_\mu, M_\mu/M_\tau) \tag{5}$$

$$A_2^{(4)}(M_e/M_\mu) = n_l^3 A_2^{(43)} + n_l^2 A_2^{(42)a} + n_l^2 n_h A_2^{(42)b} + \cdots \tag{6}$$

where $A_1^{(n)}$ contains the universal contribution and n_l and n_h denote light electron and heavy muon loops, respectively.

In Ref. [13] a first step towards an independent calculation of the electronic contributions $A_2^{(4)}(M_e/M_\mu)$ to the anomalous moment of the muon has been made. Contributions with at least two closed electron loops have been calculated. The results are accurate up to terms M_e/M_μ and are shown in the following

$$A_2^{(43)} = \frac{1}{54}L_{\mu e}^3 - \frac{25}{108}L_{\mu e}^2 + \left(\frac{317}{324} + \frac{\pi^2}{27}\right)L_{\mu e} - \frac{2\zeta_3}{9} - \frac{25\pi^2}{162} - \frac{8609}{5832} \approx 7.196\,66\,, \tag{7}$$

$$A_2^{(42)a} = L_{\mu e}^2\left[\pi^2\left(\frac{5}{36} - \frac{a_1}{6}\right) + \frac{\zeta_3}{4} - \frac{13}{24}\right] + L_{\mu e}\left[-\frac{a_1^4}{9} + \pi^2\left(-\frac{2a_1^2}{9} + \frac{5a_1}{3} - \frac{79}{54}\right) - \frac{8a_4}{3} - 3\zeta_3 + \frac{11\pi^4}{216} + \frac{23}{6}\right] - \frac{2a_1^5}{45} + \frac{5a_1^4}{9} + \pi^2\left(-\frac{4a_1^3}{27} + \frac{10a_1^2}{9} - \frac{235a_1}{54} - \frac{\zeta_3}{8} + \frac{595}{162}\right) + \pi^4\left(-\frac{31a_1}{540} - \frac{403}{3240}\right) + \frac{40a_4}{3} + \frac{16a_5}{3} - \frac{37\zeta_5}{6} + \frac{11167\zeta_3}{1152} - \frac{6833}{864} \approx -3.624\,27\,,$$

$$A_2^{(42)b} = \left(\frac{119}{108} - \frac{\pi^2}{9}\right)L_{\mu e}^2 + \left(\frac{\pi^2}{27} - \frac{61}{162}\right)L_{\mu e} - \frac{4\pi^4}{45} + \frac{13\pi^2}{27} + \frac{7627}{1944} \approx 0.494\,05\,,$$

loop order	with α^{-1}(Rb)[$\times 10^{-11}$]	with $\alpha^{-1}(a_e)$[$\times 10^{-11}$]
1	116 140 973.318 (77)	116 140 973.212 (30)
2	413 217.6291 (90)	413 217.6284 (89)
3	30 141.902 48 (41)	30 141.902 39 (40)
4	381.008 (19)	381.008 (19)
5	5.0938 (70)	5.0938 (70)
a_μ(QED)	116 584 718.951 (80)	116 584 718.845 (37)

Table 1: QED corrections by loop order using values for α obtain from the anomalous magnetic moment of the electron [11] and the second best determination of α from Ref. [12]. The errors are indicated originate from α, the parametric uncertainty of the mass ratio M_e/M_μ and the error from the numerical integration.

group	$10^2 \cdot A_{2,\mu}^{(4)}(M_\mu/M_\tau)$	
	Ref. [14]	Ref. [3]
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	−0.6292808(6)	−0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	−2.316756(5)	−2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)	0.612661(5)	0.6106(31)
IV(c)	−1.83010(1)	−1.823(11)

Table 2: Results from Ref. [14] in comparison with Ref. [3]. We refer to Ref. [3] for the definition of the diagram classes.

with $L_{\mu e} = \ln(M_\mu^2/M_e^2)$, $\zeta_n = \sum_{k=1} 1/k^n$, $a_1 = \ln 2$ and $a_n = \text{Li}_n(1/2)$, $n \geq 4$. Excellent agreement with the results in the literature has been found.

The contributions from τ-leptons to the anomalous magnetic moment of the muon can very efficiently be calculated by performing an asymptotic expansion in the mass ratio $z = M_\mu/M_\tau \approx 6 \cdot 10^{-2}$ leading to a power series

$$A_2^{(4)}(M_\mu/M_\tau) = \sum_{n=1}^{\infty} C_{4,n} z^{2n}\,. \tag{8}$$

After performing the expansion on the diagram level one is left with at most the calculation of four-loop vacuum diagrams, which have been extensively studied in the literature. To obtain a good numerical accuracy it is sufficient to consider the first three terms in the expansion

$$10^2 A_2^{(4)}(M_\mu/M_\tau) \approx 4.21670 + 0.03257 + 0.00015 = 4.24941(2)(53)\,, \quad (9)$$

where the errors indicated originate from the truncation of the series in z and the parametric uncertainty of the mass ratio. We compare the obtained results for the various diagram classes defined in [3] in Tab. 2. For all classes excellent agreement has been found.

3. Hadronic vacuum polarization contributions at NNLO

Contributions from the hadronic vacuum polarization are calculated by integrating the measured R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{\text{pt}}} \quad (10)$$

with $\sigma_{\text{pt}} = 4\pi\alpha^2/(3s)$ over a kernel function $K(s)$

$$a_\mu^{\text{had}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2} \mathrm{d}s \frac{R(s)K(s)}{s} \quad (11)$$

where $K(s)$ is at leading order given by

$$K(s) = \int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2 + (1-x)s/m_\mu^2} \quad (12)$$

The kernel function receives higher-order corrections from perturbation theory. The next-to-leading order correction has been calculated in [15]. At next-to-next-to-leading order corrections have recently been calculated in [16] by performing an asymptotic expansion in an artificial heavy photon mass. The diagram classes contributing are displayed in Fig. 1. They lead to the individual contributions

$$\begin{aligned} a_\mu^{(3a)} &= 0.80 \times 10^{-10}\,, \\ a_\mu^{(3b)} &= -0.41 \times 10^{-10}\,, \\ a_\mu^{(3b,\text{lbl})} &= 0.91 \times 10^{-10}\,, \\ a_\mu^{(3c)} &= -0.06 \times 10^{-10}\,, \\ a_\mu^{(3d)} &= 0.0005 \times 10^{-10}\,, \end{aligned} \quad (13)$$

and finally to the sum

$$a_\mu^{\text{had,NNLO}} = 1.24 \pm 0.01 \times 10^{-10}. \quad (14)$$

Including this contribution in the theory prediction reduces the discrepancy between theory and experiment by 0.2 standard deviations.

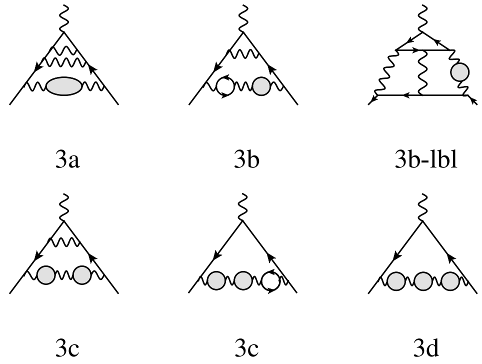

Figure 1: Diagrams contributing to the hadronic vacuum polarization at next-to-next-to-leading order.

4. Leptonic vacuum polarization contributions at five-loop order

Similar to the hadronic vacuum polarization insertions leptonic ones can be calculated by integrating over the vacuum polarization function $\Pi(q^2)$

$$a_\mu^{\text{lep-vacpol}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x(1-x)\frac{1}{1+\Pi(s_x)}\,,\ s_x = -\frac{x^2}{1-x}m_\mu^2\,. \quad (15)$$

This analysis has been done at four loops in [17]. At five loops the method has first been implemented using only the leading term in the high-energy expansion as approximation for $\Pi(q^2)$[18]. The analysis showed unexpected deviations from the results in [3] and was improved in [19] where a Padé approximation was used for the vacuum polarization. For the construction of the Padé approximation of the vacuum polarization function at four loops all available information in the low- and high-energy and in threshold region has been used. In this follow-up analysis the discrepancies were resolved and we compare all three results in Tab. 3. As can be seen there is good agreement between the new analysis and the numerical results for all diagram classes.

5. Conclusions

We reviewed recent progress in perturbative calculations for the anomalous magnetic moment of the muon. Much progress has been made to further improve the theory prediction. Higher-order corrections to the hadronic vacuum polarization contribution reduced the difference between experiment and theory by about 0.2 standard deviations.

	Ref. [19]	Ref. [18]	Refs. [20, 21, 22, 23]
I(a)	20.142 813	20.183 2	20.142 93(23)
I(b)	27.690 061	27.718 8	27.690 38(30)
I(c)	4.742 149	4.817 59	4.742 12(14)
I(d+e)	6.241 470	6.117 77	6.243 32(101)(70)
I(e)	-1.211 249	-1.331 41	-1.208 41(70)
I(f+g+h)	$4.446\,8^{+6}_{-4}$	4.391 31	4.446 68(9)(23)(59)
I(i)	$0.074\,6^{+8}_{-19}$	0.252 37	0.0 87 1(59)
I(j)	$-1.246\,9^{+4}_{-3}$	-1.214 29	-1.247 26(12)

Table 3: Comparison of the results from Ref. [19], Ref. [18] and Refs. [20, 21, 22, 23]. We refer to Ref. [3] for the definition of the diagram classes.

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