Spin decoherence in electron storage rings —more from a simple model

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Abstract

This is an addendum to the paper "Some models of spin coherence and decoherence in storage rings" by one of the authors [1] in which spin diffusion in simple electron storage rings is studied. In particular, we illustrate in a compact way, a key implication in the Epilogue of [1], namely that the exact formalism of [1] delivers a rate of depolarisation which can differ from that obtained by the conventional treatments of spin diffusion which rely on the use of the derivative $\partial\hat{n}/\partial\eta$ [2, 3, 4]. As a vehicle we consider a ring with a Siberian Snake and electron polarisation in the plane of the ring (Machine II in [1]). For this simple setup with its one-dimensional spin motion, we avoid having to deal directly with the Bloch equation [5, 6] for the polarisation density.

Our treatment, which is deliberately pedagogical, shows that the use of $\partial\hat{n}/\partial\eta$ provides a very good approximation to the rate of spin depolarisation in the model considered. But it then shows that the exact rate of depolarisation can be obtained by replacing $\partial\hat{n}/\partial\eta$ by another derivative as suggested in the Epilogue of [1], while giving a heuristic justification for the new derivative.

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1 Introduction[1]

The paper "Some models of spin coherence and decoherence in storage rings" by one of the authors [1] provides an introduction to the use of spin-polarisation transport equations of Fokker-Planck and Liouville type in electron storage rings. The work goes into great and careful detail and it uses two exactly solvable but simple model configurations called Machine I and Machine II as vehicles. These are perfectly aligned flat rings and only the effects of synchrotron radiation and of longitudinal oscillations on electron spins lying in the machine plane are considered. Moreover the orbit and spin dynamics are approximated by smoothed equations of motion with azimuth-independent parameters given by averages around the ring. Machine II contains a point–like Siberian Snake in addition [9]. As is often the case with "toy" models, the exact results for the spin distributions for Machines I and II can be expected to deliver useful insights and expose essential features. This is indeed the case and in this paper we use Machine II to:

- cover a topic mentioned, but not treated in detail in [1], namely the contrast between the exact calculations of the rate of depolarisation in [1] and the conventional approach to estimating depolarisation due to Derbenev, Kondratenko and Mane (DKM) [2, 3].
- in the process:
  - give, and comment on, the form of the derivative $\partial\hat{n}/\partial\eta$ in the DKM formula for this case,
  - show how to improve the DKM calculation in a heuristically obvious way, while confirming a claim in the Epilogue of [1] and thereby calibrating the DKM approach in this case.
- discuss approximations to $\partial\hat{n}/\partial\eta$ and misunderstandings in the literature.
- suggest further avenues of investigation.

For both model machines, polarisation build–up due to the Sokolov–Ternov effect [10, 4] is ignored. In any case the Sokolov–Ternov mechanism is ineffective in Machine II as explained in Section 3.2.

In Section 2 we outline the models. In Section 3 we present the exact solution for the rate of depolarisation of Machine II and the rate from the DKM theory. Then, after comparing the two results we show how to modify the DKM calculation to get agreement with the exact solution. Finally, in Section 4 we make further relevant comments and discuss other approximations and their utility. Machine II is the simplest non–trivial exactly solvable model that we are aware of.

[1] This paper was first prepared in 2000 following suggestions at that time that a Siberian Snake would help to preserve the polarisation in a putative high energy electron ring [7]. It was then put aside when the proponents sensibly dropped the idea. Now, after recent work [8], it is appropriate to make an updated version of this paper available.

2 The models

2.1 Machine I

According to (2.5) in [1], for Machine I the Langevin equations of motion for the orbit and the spin-expectation value (the "spin") are

$$\begin{pmatrix} \sigma'(s) \\ \eta'(s) \\ \psi'(s) \end{pmatrix} = \begin{pmatrix} 0 & -\kappa & 0 \\ \Omega_s^2/\kappa & -2\,\alpha_s/L & 0 \\ 0 & 2\pi\tilde{\nu}/L & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma(s) \\ \eta(s) \\ \psi(s) \end{pmatrix} + \sqrt{\omega} \cdot \begin{pmatrix} 0 \\ \zeta(s) \\ 0 \end{pmatrix}, \tag{2.1}$$

where

- s denotes the distance around the ring (the "azimuth"),
- L is the length of the ring,
- σ is the distance to the centre of the bunch,
- η is the fractional energy deviation,
- $\psi(s)$ is the angular position of the spin in the machine plane w.r.t. a horizontal direction precessing according to the T–BMT equation [11] at the rate $2\pi\{(g-2)/2\}\gamma_0/L$ on the design orbit, where $(g-2)/2$ is the gyromagnetic anomaly and where $\gamma_0 = E_0/(m_0 c_0^2)$ is the Lorentz factor at the design orbit energy E_0, whereby c_0 is the vacuum velocity of light and m_0 is the rest mass of the electron,
- α_s is the 1–turn synchrotron damping decrement,
- κ is the compaction factor,
- $\Omega_s = 2\pi \cdot Q_s/L$ where Q_s is the synchrotron tune for undamped motion.
- $\tilde{\nu} = \{(g-2)/2\}\gamma_0$,
- ω is s-independent and is the 1–turn averaged stochastic kick strength which is expressed in terms of an s-independent curvature $\bar{K}_x$

  $$\omega = \frac{55}{24\sqrt{3}} r_e \lambda\!\!\!^{-}_e \gamma_0^5 \bar{K}_x^3 := \frac{55}{24\sqrt{3}} r_e \lambda\!\!\!^{-}_e \gamma_0^5 \frac{1}{L}\int_0^L ds\, |K_x|^3,$$

  where r_e in the classical electron radius, $\lambda\!\!\!^{-}_e$ is the reduced Compton wavelength of the electron and where $\bar{K}_x^3$ is determined by the local curvature $K_x(s)$ of the design orbit in the horizontal plane of the original ring via the 1-turn average in the second equality, [2]
- ζ simulates the noise kicks due to the synchrotron radiation, via a Gaussian white noise process, i.e. with stochastic averages $<\zeta(s_1)\cdot\zeta(s_2)>= \delta(s_1 - s_2)$, $\quad <\zeta(s)>=0$, where δ denotes the Dirac delta function.

[2] In [1] ω was written in Gaussian units and in terms of a product of factors C_1 and C_2. Here we work in SI units and express the product in terms of more convenient factors.

More details can be found in [1] but the content of (2.1) is clear: the motion of σ and η is that of a damped harmonic oscillator subject to the noise $\sqrt{\omega}\,\zeta$ in the variable η. Moreover spin is a "passenger", and w.r.t. the rotating reference direction it precesses at the rate $(2\pi\tilde{\nu}/L)\eta$. Then the noise acting on η feeds through onto the spin to cause the spin diffusion which we wish to study. Equation (2.1) is the cleanest way to package that message.

In terms of the parameters

$$a \equiv -\kappa \ , \quad b \equiv \Omega_s^2/\kappa = (4\pi^2 Q_s^2)/(\kappa L^2) \ , \quad c \equiv -2\alpha_s/L \ , \tag{2.2}$$

the asymptotic ($s \to +\infty$) r.m.s widths of the distributions of σ and η are given by

$$\sigma_\sigma^2 \ \equiv \ \frac{\omega a}{2bc} > 0 \ , \qquad \sigma_\eta^2 \equiv -\frac{\omega}{2c} = \frac{\omega L}{4\alpha_s} = -\frac{b}{a}\sigma_\sigma^2 > 0 \ . \tag{2.3}$$

We also need $d \equiv 2\pi\tilde{\nu}/L$. For Machine I the 1-turn periodic solution to the T-BMT equation on the design orbit, $\hat{n}_0$, is perpendicular to the machine plane and $\tilde{\nu}$ is the *design-orbit spin tune*, ν_0, namely the number of spin precessions around $\hat{n}_0$ per turn around the ring for a particle on the design orbit.

For the HERA electron ring [12] running at about 27 GeV the values of the parameters are

$$\begin{aligned} Q_s &\approx 6.0 \cdot 10^{-2}, & \alpha_s &\approx 3.2 \cdot 10^{-3}, & \kappa &\approx 6.9 \cdot 10^{-4}, \\ \omega &\approx 2.0 \cdot 10^{-12}\,\mathrm{m}^{-1}, & L &\approx 6.3 \cdot 10^{3}\,\mathrm{m}, & d &\approx 6.2 \cdot 10^{-2}\,\mathrm{m}^{-1}, \end{aligned}$$

and we adopt these parameters to illustrate our arguments. Then

$$a \approx -6.9 \cdot 10^{-4}, \quad b \approx 5.2 \cdot 10^{-6}\,\mathrm{m}^{-2}, \quad ab \approx -3.6 \cdot 10^{-9}\,\mathrm{m}^{-2}, \quad c \ \approx \ -1.0 \cdot 10^{-6}\,\mathrm{m}^{-1},$$

while

$$\sigma_\eta^2 \approx 1.0 \cdot 10^{-6} \ , \qquad \sigma_\sigma^2 \approx 1.3 \cdot 10^{-4}\,\mathrm{m}^2 \ , \tag{2.4}$$

For Machine I, the corresponding σ_ψ^2 depends on the initial distribution of ψ and an interesting point there is that the distribution of ψ reaches a stationary form after a few damping times. Details can be found in [1, 13, 14]. This is in contrast to earlier calculations which suggest that the ψ distribution spreads out with a width proportional to $\sqrt{s}$ [15].

However, when viewed from the fixed machine coordinates the spin distribution is not stationary. On the contrary it is rotating continuously in the machine plane at the rate $2\pi\nu_0/L$ and cannot be periodic if $\nu_0 \neq$ integer. Such a distribution cannot be handled using DKM methods for reasons to be explained in the Commentary.

2.2 Machine II

For Machine II a single point–like Siberian Snake is included at $s = 0$. This rotates a spin by the angle π around the radial direction independently of σ and η, so that it is a "Type 2" snake [16]. In Machine I, $\hat{n}_0$, is perpendicular to the machine plane. But for Machine II $\hat{n}_0$ is in the horizontal plane and is given by (3.3) in [1] [3]:

$$\hat{n}_0(s) \ \equiv \ \cos\Big(g_6(s)\Big)\hat{e}_1 + \sin\Big(g_6(s)\Big)\hat{e}_2 \ , \tag{2.5}$$

[3]For clarity we omit the subscript II used in [1].

where $\hat{e}_1$ and $\hat{e}_2$ are unit vectors transverse and parallel to the design orbit respectively and

$$g_6(s) \equiv d\Big(s - L/2 - L\mathcal{G}(s/L)\Big) , \tag{2.6}$$

where the "stairway" function $\mathcal{G}$ is defined by:

$$\mathcal{G}(s/L) \equiv N \qquad \text{if } N < s/L < (N+1) , \tag{2.7}$$

for which N is an integer. In the range $0 < s < L$ one has

$$g_6(s) = d\Big(s - L/2\Big) . \tag{2.8}$$

For Machine II it is this horizontal $\hat{n}_0(s)$ which is taken as the reference direction so that for Machine II ψ is the angle between a spin and $\hat{n}_0(s)$. The conclusions of this paper are the same for a snake with a longitudinal rotation axis ("Type 1"), provided, for example, in the real world, by a solenoid. In that case the spin motion on the design orbit is illustrated in Figure 8 in [16]. The reader can easily make a corresponding sketch for our Type 2 snake.

Owing to the presence of the snake, ν_0 is not $\tilde{\nu}$ but 1/2 [1, 16]. Thus the condition *spin–orbit resonance* occurs around $Q_s = 1/2$. Near resonance, spin diffusion effects can be particularly strong. [4]

3 The polarisation evolution for Machine II

3.1 Using exact distribution functions

We now look again at the exact evaluation of the distribution functions for Process 3 of Machine II [1, Section 3.5]. For this process the orbital phase-space distribution is in its stationary state and the spins are initially all parallel to $\hat{n}_0$. Then by (3.58) and (3.62) in [1], and after transients have died away in the first few damping times the polarisation of the whole beam evolves like

$$||\vec{P}^{w_3}_{tot}(s)|| \propto \exp\Big(-\frac{g_{14}(s)}{2}\Big) \cdot \exp\Big(-\frac{s\ g_{15}}{2}\Big) \tag{3.1}$$

where the function g_{14} is 1–turn periodic and g_{15}, which is positive, is given by

$$g_{15} \equiv \frac{2\ d^2\ g_{11}\ \sigma_\eta^2}{a\ b\ L\ \lambda}\Big(2\lambda \sinh(cL/2) - c \sin(\lambda L)\Big) , \tag{3.2}$$

with $\lambda \equiv \sqrt{-ab - c^2/4}$ and $g_{11} \equiv 1/\{\cosh(cL/2) + \cos(\lambda L)\}$.

Whereas $\lambda_0 \equiv \sqrt{-ab} = \Omega_s$ is proportional to the synchrotron tune in the absence of damping, $\lambda \equiv \sqrt{-ab - c^2/4}$ is the corresponding tune in the presence of damping. Since $c^2 << -ab$, $\lambda \approx \lambda_0$.

At large times the polarisation vanishes:

$$||\vec{P}^{w_3}_{tot}(+\infty)|| = 0 . \tag{3.3}$$

[4] The resonance condition should be more precisely expressed in terms of the *amplitude-dependent spin tune* [17, 18]. But for typical electron/positron rings the amplitude-dependent spin tune differs only insignificantly from ν_0.

The depolarisation rate w.r.t. distance is:

$$\frac{1}{\tau_{spin}} \equiv \frac{g_{15}}{2} . \tag{3.4}$$

For the HERA electron ring parameters listed above one gets

$$\tau_{spin} \approx 7.6 \cdot 10^7 \text{ m} , \tag{3.5}$$

which corresponds to about 12000 turns, i.e. about 250 milliseconds.

For the HERA parameters, and when transients have died away after about 1000 turns ($s \approx 6.3 \cdot 10^6$ m), the variation of $g_{14}(s)$ over a turn causes a variation of $g_{14}(s) + sg_{15}$ in the exponent of (3.1) of about 15 percent. If Q_s were close to 1/2 so that λL were close to π, g_{15} would, because of its factor g_{11}, become very large and τ_{spin} would be very small. This is exactly what one expects when sitting close to a spin-orbit resonance [17].

Of course this model *only* includes the effects on spin of smoothed synchrotron motion and radiation in the main body of the ring. No account is taken here of the detailed dependence of spin motion on the orbital variables in a real snake and there is no horizontal or vertical betatron motion.

It can be seen from [1] that the asymptotic depolarisation rate is $g_{15}/2$ for any initial distribution of spins in the machine plane.

For comparison with the results of the next section we use the relations $\sigma_\eta^2 = -\omega/2c$ and $\lambda_0 = \sqrt{-ab}$ to obtain

$$\tau_{spin}^{-1} = \frac{d^2}{\lambda_0^2} \cdot \frac{\omega}{2\, c\, \lambda\, L} \cdot \frac{1}{\{\cosh(cL/2) + \cos(\lambda L)\}} \cdot \Big(2\lambda \sinh(cL/2) - c \sin(\lambda L)\Big) . \tag{3.6}$$

3.2 Using the Derbenev–Kondratenko–Mane approach

The conventional way to calculate the rate of depolarisation is to use the spin diffusion term in the Derbenev–Kondratenko–Mane formula [2, 3] for the equilibrium polarisation. In this approach it is assumed that at orbital equilibrium, the combined effect of the depolarisation and the S–T mechanism is to cause the polarisation at a point in phase space to be aligned along the vector $\hat{n}$ of the *invariant spin field* [17, 18]. This is a special solution to the T–BMT equation along the trajectory $(\sigma(s), \eta(s))$ satisfying the periodicity condition $\hat{n}(\sigma, \eta; s) = \hat{n}(\sigma, \eta; s + L)$. In general $\hat{n}$ is a function of all six phase-space coordinates but in our models only σ and η come into play. In the absence of radiation, and spin-orbit equilibrium, the polarisation at a point in phase space, which we call the *local polarisation* is indeed aligned along $\hat{n}$ [19]. Then, since the characteristic times for the action of synchrotron radiation, namely the damping time and the polarisation and depolarisation times, are very large compared to the characteristic times for the orbital and spin-precession dynamics, the above assumption about the direction of the local polarisation for electrons with radiation is reasonable since the orbital and spin dynamics still dominate on short time scales, at least away from spin-orbit resonances. We return to this in Section 3.4. The DKM approach also assumes that the value of the local polarisation and its rate of change are independent of the position in phase space. These two assumptions are motivated by recognition that the stochastic photon emission and damping cause electrons to continually diffuse through phase space and thereby, in the end, have effectively interchangeable histories.

According to the DKM approach, with a stable phase-space distribution, the local polarisation settles to the equilibrium value

$$P_{\rm dkm} \;=\; -\frac{8}{5\sqrt{3}} \frac{\oint ds \left\langle |K(s)|^3 \hat{b}\cdot(\hat{n} - \frac{\partial\hat{n}}{\partial\eta})\right\rangle_s}{\oint ds \left\langle |K(s)|^3 (1 - \frac{2}{9}(\hat{n}\cdot\hat{s})^2 + \frac{11}{18}\left(\frac{\partial\hat{n}}{\partial\eta}\right)^2)\right\rangle_s} \tag{3.7}$$

where $\hat{s}$ = direction of the particle motion, $\hat{b} = (\hat{s}\times\dot{\hat{s}})/|\dot{\hat{s}}|$, $K(s)$ is the curvature (in the horizontal or vertical planes) and $<\;>_s$ denotes an average over phase space at azimuth s. The unit vector $\hat{b}$ is the magnetic field direction if the electric field vanishes and the motion is perpendicular to the magnetic field. The polarisation of the beam as a whole is

$$\vec{P}_{\rm dkm}(s) \;=\; P_{\rm dkm}\, \langle\hat{n}\rangle_s \tag{3.8}$$

In the DKM formula, the depolarisation rate w.r.t. time is

$$\tau^{-1}_{\rm dep} \;=\; \frac{5\sqrt{3}}{8} r_{\rm e} \lambda\!\!\!^{-}_e c_0 \gamma^5 \frac{1}{L} \oint ds \left\langle |K(s)|^3 \frac{11}{18}\left(\frac{\partial\hat{n}}{\partial\eta}\right)^2 \right\rangle_s \tag{3.9}$$

where, by the very nature of the DKM approach, it is assumed that all transients have died away. By (3.8) this is also the depolarisation rate of the whole beam. For a ring with constant curvature $\bar{K}_x$ (3.9) becomes

$$\tau^{-1}_{\rm dep} = \frac{55}{48\sqrt{3}} r_{\rm e} \lambda\!\!\!^{-}_e c_0 \gamma^5 \frac{\bar{K}_x^3}{L} \oint ds \left\langle \left(\frac{\partial\hat{n}}{\partial\eta}\right)^2 \right\rangle_s$$

so that the depolarisation rate w.r.t. distance is

$$(c_0\;\tau_{\rm dep})^{-1} \;=\; \frac{\omega}{2L}\int_0^L ds \left\langle \left(\frac{\partial\hat{n}}{\partial\eta}\right)^2 \right\rangle_s \;. \tag{3.10}$$

To evaluate this we need to know $\hat{n}$ at each azimuth and at each point in phase space. From (B.1) in Appendix B in [1], in the arc the T–BMT equation reads as

$$\frac{\partial\hat{n}}{\partial s} \;=\; -a\;\eta\;\frac{\partial\hat{n}}{\partial\sigma} - b\;\sigma\;\frac{\partial\hat{n}}{\partial\eta} + d(1+\eta)\hat{e}_3\times\hat{n}\;. \tag{3.11}$$

This is in fact just the partial differential form of the T–BMT equation

$$\frac{\partial\hat{n}}{\partial s} + \frac{d\sigma}{ds}\frac{\partial\hat{n}}{\partial\sigma} + \frac{d\eta}{ds}\frac{\partial\hat{n}}{\partial\eta} \;=\; d(1+\eta)\hat{e}_3\times\hat{n} \tag{3.12}$$

for which the derivatives $d\sigma/ds$ and $d\eta/ds$ have been obtained from (2.1) by switching off the radiation.

By choosing the ansatz [5]

$$\hat{n} \;\equiv\; \cos(f)\hat{e}_1 + \sin(f)\hat{e}_2 \tag{3.13}$$

[5] For clarity we still omit the subscript II used in [1].

(3.11) becomes

$$\frac{\partial f}{\partial s} = -a\ \eta\ \frac{\partial f}{\partial \sigma} - b\ \sigma\ \frac{\partial f}{\partial \eta} + d\ \eta + d\ . \tag{3.14}$$

Then by taking the snake into account and enforcing 1–turn periodicity we obtain

$$f(\sigma, \eta; s) = g_6(s) + \sigma g_{19}(s) + \eta g_{20}(s)\ , \tag{3.15}$$

with

$$\begin{aligned} g_{19}(s) &= \frac{d\ b}{\lambda_0^2} \frac{1}{1+\cos(\lambda_0 L)} \Bigg[\cos\Big(\lambda_0 [s - L - L\mathcal{G}(s/L)] \Big) \\ &\quad + \cos\Big(\lambda_0 [s - L\mathcal{G}(s/L)] \Big) - \cos(\lambda_0 L) - 1 \Bigg]\ , \\ g_{20}(s) &= \frac{d}{\lambda_0} \frac{1}{1+\cos(\lambda_0 L)} \Bigg[\sin\Big(\lambda_0 [s - L - L\mathcal{G}(s/L)] \Big) + \sin\Big(\lambda_0 [s - L\mathcal{G}(s/L)] \Big) \Bigg] \end{aligned} \tag{3.16}$$

where both are independent of σ and η.

The resulting $\hat{n}$ is a solution of the T–BMT equation along the trajectory $(\sigma(s), \eta(s))$ and like $\hat{n}_0$ it lies in the machine plane. As required, it is 1–turn periodic in s for all σ and η and reduces to $\hat{n}_0$ at $\sigma = 0, \eta = 0$. If $\hat{n}$ had a component perpendicular to the machine plane it would, at simplest, be 2–turn periodic.

One also sees that a singularity in $\hat{n}$ occurs if the fractional part of the orbital tune $Q_s = (\lambda_0 \cdot L)/(2\pi)$ equals 1/2, i.e. if one is at a spin-orbit resonance defined in terms of ν_0 (see Footnote 3). In that case $\hat{n}(\sigma, \eta; s)$ is not unique and a different formulation is needed.

Obviously, $(\partial \hat{n}/\partial \eta)^2 = g_{20}^2$ which is independent of σ and η. Then to obtain the rate of depolarisation within the DKM framework we just need to evaluate

$$(c_0\ \tau_{\text{dep}})^{-1} \quad = \quad \frac{\omega}{2L} \int_0^L g_{20}^2(s')\ ds'. \tag{3.17}$$

In the range $0 < s < L$ we have

$$g_{20}(s) = \frac{d}{\lambda_0} \frac{1}{1+\cos(\lambda_0 L)} \Bigg[\sin\Big(\lambda_0 (s - L) \Big) + \sin(\lambda_0 s) \Bigg] \tag{3.18}$$

so that we find

$$(c_0\ \tau_{\text{dep}})^{-1} \quad = \quad \frac{d^2}{{\lambda_0}^2} \cdot \frac{\omega}{2\ \lambda_0\ L} \cdot \frac{1}{\{1+\cos(\lambda_0 L)\}} \cdot \Big(\lambda_0 L - \sin(\lambda_0 L) \Big)\ . \tag{3.19}$$

This scales like γ_0^7 at fixed λ_0. Thus the depolarisation due to noisy damped synchrotron motion becomes very strong at high energy.

For Machine II, the vector $\hat{n}$ is in the machine plane so that $\hat{b} \cdot \hat{n}$ is zero and the S–T mechanism is inoperative [4]. However, in this case the vector $\partial \hat{n}/\partial \eta$ acquires a vertical component. This leads to the phenomenon of *kinetic polarisation* embodied in the term linear in $\partial \hat{n}/\partial \eta$ in the numerator in (3.7) [20, 16]. We will not pursue that here. The formalisms in [2, 3] leading to (3.7) and (3.9) are semiclassical and the concept of depolarisation does not immediately appear in those papers, Nevertheless, the expression for τ_{dep}^{-1} in (3.9) can be obtained from classical notions as in [21, 22]. This fact will be useful in later discussions.

3.3 Comparison

Since $c\ L \approx -6.3 \cdot 10^{-3}$, and $\lambda \approx \lambda_0$ it is immediately clear that the result for $(c_0\ \tau_{\rm dep})^{-1}$ in (3.19) is very close to τ_{spin}^{-1} in (3.6) and in fact the relative difference between the two rates is $\approx 4 \cdot 10^{-6}$.

Thus the DKM estimate, that used is all analytical calculations of the depolarisation rate, is very accurate and perfectly adequate for Machine II with the parameters used here. In fact it is expected to be very accurate in general. But it is not exact. This is easy to understand: the DKM result is based on the impressive insight that the equilibrium polarisation should be parallel to $\hat{n}$ at each point in phase space but as we will see in the next section the polarisation is not exactly parallel to $\hat{n}$.

If we set Q_s close to 1/2 so that we are close to spin orbit resonance [6] the DKM estimate deviates significantly from the exact result. For example for $Q_s = 0.4998$ the relative difference is about 65%. We expect this to be the case in general for realistic rings too. But the depolarisation rate would then be so large that the results would be of no practical interest.

3.4 A modified DKM calculation

Given the (admittedly small) differences between the two values for the depolarisation rate we are motivated to explain them by exploiting our exact expressions in [1] for the asymptotic spin distributions and thereby examine the assumptions in Section 3.2 underlying the DKM approach.

Thus, as shown by (2.120) in [1], the unit vector $\vec{P}^w_{dir}(\sigma, \eta; s)$ describing the polarisation direction at each point in phase space does not(!) obey the T–BMT equation. In particular, damping and diffusion effects must be included. Then at orbital equilibrium and after transforming (3.33) in [1] into the machine frame we find

$$\frac{\partial \vec{P}^w_{dir}}{\partial s} = -a\ \eta\ \frac{\partial \vec{P}^w_{dir}}{\partial \sigma} - b\ \sigma\ \frac{\partial \vec{P}^w_{dir}}{\partial \eta} + \vec{\Omega}_{II} \times \vec{P}^w_{dir} + c\ \eta\ \frac{\partial \vec{P}^w_{dir}}{\partial \eta}\ . \tag{3.20}$$

It was shown in [1] that after transients have died away in the first few damping times, $\vec{P}^w_{dir}(\sigma, \eta; s)$ becomes 1–turn periodic is s although the local polarisation itself then decreases smoothly to zero. From now on we will denote the periodic $\vec{P}^w_{dir}(\sigma, \eta; s)$ by the unit vector $\hat{p}(\sigma, \eta; s)$ and we will now show how to obtain it in a way paralleling the construction of $\hat{n}(\sigma, \eta; s)$. According to (3.20), in the arc $\hat{p}$ fulfills

$$\frac{\partial \hat{p}}{\partial s} = -a\ \eta\ \frac{\partial \hat{p}}{\partial \sigma} - b\ \sigma\ \frac{\partial \hat{p}}{\partial \eta} + c\ \eta\ \frac{\partial \hat{p}}{\partial \eta} + d(1+\eta)\hat{e}_3 \times \hat{p}\ . \tag{3.21}$$

Then, writing

$$\hat{p}\ \equiv\ \cos(\tilde{f})\hat{e}_1 + \sin(\tilde{f})\hat{e}_2\ . \tag{3.22}$$

we have, in contrast to (3.14),

$$\frac{\partial \tilde{f}}{\partial s} = -a\ \eta\ \frac{\partial \tilde{f}}{\partial \sigma} - b\ \sigma\ \frac{\partial \tilde{f}}{\partial \eta} + c\ \eta\ \frac{\partial \tilde{f}}{\partial \eta} + d\ \eta + d\ . \tag{3.23}$$

[6] Of course one never tries to run a storage ring in that way — the r.f. cavity voltage would be enormous and the smoothed equations of motion used in the model would no longer be reasonable.

So there is an extra term depending on the damping rate. Taking into account the action of the snake, the 1–turn periodic solution for $\tilde{f}$ is

$$\tilde{f}(\sigma,\eta;s) = g_6(s) + \sigma\tilde{g}_{19}(s) + \eta\tilde{g}_{20}(s) \tag{3.24}$$

where in the range $0 < s < L$

$$\tilde{g}_{19}(s) = -\frac{d}{2\lambda\ a}\left[i\ g_1(s)\ g_{11}\ \exp(-c\ L/2) + i\ g_1(s-L)\ g_{11}\ \exp(c\ L/2) - 2\ \lambda\right], \tag{3.25}$$

$$\tilde{g}_{20}(s) = -\frac{d}{2\ \lambda}\left[i\ g_2(s)\ g_{11}\ \exp(-c\ L/2) + i\ g_2(s-L)\ \ g_{11}\ \exp(c\ L/2)\right]. \tag{3.26}$$

with

$$\begin{aligned} g_1(s) &= i\ \exp(c\ s/2)\ [c\ \sin(\lambda\ s) - 2\ \lambda\ \cos(\lambda\ s)]\ , \\ g_2(s) &= 2i\ \sin(\lambda\ s)\ \exp(c\ s/2)\ , \end{aligned} \tag{3.27}$$

so that (3.22) and (3.24) are equivalent to (3.72) in [1]. Thus $\hat{p}(\sigma,\eta;s) \neq \hat{n}(\sigma,\eta;s)$ owing to the dependence of $\hat{p}$ on c but for $c \to 0$ we see that $\tilde{g}_{19}(s) \to g_{19}(s)$, $\tilde{g}_{20}(s) \to g_{20}(s)$ and $\hat{p} \to \hat{n}$ as one would expect. Moreover, the expression for τ_{spin}^{-1} in (3.6) then reduces to the expression for $(c_0\ \tau_{\rm dep})^{-1}$ in (3.16). The function $\tilde{g}_{20}(s)$ can be written in the form

$$\tilde{g}_{20}(s) = \frac{d}{\lambda}\ g_{11}\left[\ \sin(\lambda\ s)\ \exp(c\ (s-L)/2) +\ \sin(\lambda\ (s-L))\ \exp(c\ s/2)\right] \tag{3.28}$$

and apart from the exponential factors containing c, it is reminiscent of g_{20} in (3.18).

We now turn to the local polarisation $||\vec{P}^{w_3}_{loc}(\sigma,\eta;s)||$. In the DKM picture it is assumed that at orbital equilibrium $||\vec{P}^{w_3}_{loc}(\sigma,\eta;s)||$ and its rate of change are independent of (σ,η). As seen in (3.71) in [1], this is indeed the case in our model after transients have died away. This emerges naturally – there has been no need to appeal to heuristic arguments. In particular, at orbital equilibrium the long-term s dependence of $||\vec{P}^{w_3}_{tot}(s)||$ reflects the s dependence of $||\vec{P}^{w_3}_{loc}(\sigma,\eta;s)||$. However the direction of the local polarisation is $\hat{p}$, not $\hat{n}$. Processes 4 and 5 in [1] reflect on this too.

As we are seeing, instead of relying on intuition, for Machine II one can find the exact result for the asymptotic depolarisation rate by examining the development of the distribution functions. In the general case, such an analytical treatment is not available. But to stay close to the philosophy of using distribution functions one can examine the properties of solutions of the Bloch equation for the polarisation density [1, 5, 6] and then, as suggested in the Epilogue in [1], $(\partial\hat{n}/\partial\eta)^2$ should be replaced by $(\partial\hat{p}/\partial\eta)^2$ in the expression for $(c_0\ \tau_{\rm dep})^{-1}$ of the DKM picture. It is then no surprise to find that

$$(c_0\ \tilde{\tau}_{\rm dep})^{-1} \quad = \quad \frac{\omega}{2L}\int_0^L \tilde{g}_{20}^2(s')\ ds'. \tag{3.29}$$

gives precisely the τ_{spin}^{-1} in (3.6)!

This is also exactly what one expects given the physical picture in [21, 22] used to arrive at the use of $(\partial\hat{n}/\partial\eta)^2$ if the polarisation were parallel to $\hat{n}$. There, a simple geometrical construction was used to estimate the rate of change of the average of spin projections along the local polarisation direction, which was taken to be $\hat{n}$. For a stored non–radiating beam in a stationary spin–orbit state, i.e. a non–radiating beam for which the phase-space distribution and the polarisation distribution repeat from turn to turn after a chosen starting s, the polarisation is indeed parallel to $\hat{n}$ at each point in phase space [17]. But in the presence of radiation the direction of the polarisation is $\hat{p}$ and if $\hat{p}$ had been used in [21, 22] instead, then $(\partial\hat{p}/\partial\eta)^2$ would have been needed as we have already discovered by appealing to the Epilogue in [1]. Thus the exact agreement between (3.29) and (3.6) supports the construction used in [21, 22] once $\hat{n}$ has been replaced by $\hat{p}$. The necessity of using $\hat{p}$ instead of $\hat{n}$ emphasises that the DKM approach is essentially perturbative beginning with $\hat{n}$.

Of course, the "improvement" embodied in (3.29) is so small that it is of no practical significance. Nevertheless our discussion demonstrates how our toy model can be used to expose details of spin dynamics which would not otherwise be accessible.

As seen in [1], at the phase-space point where $\sigma = \sigma_\sigma$ and $\eta = \sigma_\eta$ and for Q_s well away from $1/2$, the angle between $\hat{n}$ and $\hat{p}$ at the snake is given approximately by

$$\frac{d \cdot c \cdot \sigma_\eta}{2 \cdot \lambda_0^2} \cdot \frac{\lambda_0 \cdot L - \sin(\lambda_0 \cdot L)}{1 + \cos(\lambda_0 \cdot L)} \ .$$

For HERA parameters this is about 0.04 milliradians whereas the angle between $\hat{n}$ and $\hat{n}_0$ is of the order of 200 milliradians. The angle between $\hat{n}$ and $\hat{p}$ is proportional to c so that, as one might expect, the angle becomes larger as stronger damping causes the equation of motion for $\hat{p}$ to deviate more from the T-BMT equation. Moreover, by comparing the denominators in g_{11} and (3.16) we see that $\hat{p}$ differs strongly from $\hat{n}$ near $Q_s = 1/2$, thereby confirming our suspicion in Section 3.2 that near spin-orbit resonances, care is needed with the argument based on time scales. This, then, is the origin of the failure of the near-resonance DKM estimate for the depolarisation rate mentioned in Section 3.3. Note that as seen in [1] the characteristic time for the decay of transients remains the damping time, even close to the resonance. In [23] it is suggested that very close to spin-orbit resonances, extra terms containing delta functions should be added to the DKM value (3.9) for the rate of depolarisation as a result of so-called uncorrelated resonance crossing. These would be due to fluctuations in the rate of spin precession around $\hat{n}_0$ caused by energy fluctuations. See [8]. However, for Machine II spins precess only around the vertical in the arc, not around $\hat{n}_0$ which is horizontal in the arc. So this effect will not be seen here.

4 Commentary

Some further remarks are now in order —some obvious, some less so.

(1) It is clear from the discussion above that $\hat{n}_0$ and $\hat{n}$ are two different quantities which only coincide on the closed design orbit. Indeed, as we have just mentioned, the angle between them is of the order of 200 milliradians for our parameters. However, they have often been confused in the literature. For example $\partial\hat{n}/\partial\eta$ was originally written as $\gamma(\partial\hat{n}/\partial\gamma)$ [21, 2] and that led some to calculate $\gamma_0(\partial\hat{n}_0/\partial\gamma_0)$. See, for example [16]. For general problems where all six phase-space coordinates must be included this can give completely misleading results. In particular there would be no resonant increase of the depolarisation rate when

an orbital tune were close to ν_0. Nevertheless, if, as in our models, horizontal and vertical betatron motion are being neglected, it can happen that $(\gamma_0 \partial\hat{n}_0/\partial\gamma_0)^2$ provides a useful initial approximation to $(\partial\hat{n}/\partial\eta)^2$. That is the case with Machine II as we now show.

Using (2.5) and (2.8) and the definition of d and $\tilde{\nu}$

$$(\gamma_0 \frac{\partial \hat{n}_0}{\partial \gamma_0})^2 = (\gamma_0 \frac{\partial g_6}{\partial \gamma_0})^2 = d^2\ (s - L/2)^2 = \{\tilde{\nu}\ (2\pi\ s/L - \pi)\}^2\ . \tag{4.1}$$

This is to be compared with

$$(\frac{\partial \hat{n}}{\partial \eta})^2 = g_{20}^2 = \frac{d^2}{{\lambda_0}^2} \cdot \frac{2}{\{1 + \cos(\lambda_0 L)\}} \cdot \sin^2\lambda_0 (s - L/2)\ . \tag{4.2}$$

The expression $\{\tilde{\nu}\ (2\pi\ s/L - \pi)\}^2$ in (4.1) is used in [20] and its origin is clear from the absence of the resonance factor $2/\{1 + \cos(\lambda_0 L)\}$ which takes the value ≈ 1.036 instead of 1 for our value: $Q_s = 6.0 \cdot 10^{-2}$. By replacing $(\partial\hat{n}/\partial\eta)^2$ in (3.19) by $(\gamma_0\partial\hat{n}_0/\partial\gamma_0)^2$ and using the result

$$\frac{1}{L} \int_0^L (\gamma_0 \frac{\partial g_6}{\partial \gamma_0})^2\ ds' = \frac{d^2\ L^2}{12} = \frac{(\pi\nu)^2}{3}. \tag{4.3}$$

one obtains a value about 3% lower than from (3.19) so that in this case $(\gamma_0\partial\hat{n}_0/\partial\gamma_0)^2$ provides an adequate approximation. But of course it would become a bad approximation for a large Q_s. Note that the approximation works for Machine II because the snake ensures that the spin tune, ν_0, is far from Q_s. In other situations one should never rely on this approximation. The use of $(\gamma_0\partial\hat{n}_0/\partial\gamma_0)^2$ delivers the correct result for Machine II at $Q_s = 0$, i.e. when the energy is the same from turn to turn.

(2) The vector $\hat{n}$ is a s–periodic solution to the partial differential equation (3.11). Whereas $\hat{n}_0$ can be obtained as the unit real eigenvector of the 1–turn spin map on the closed orbit [16], $\hat{n}$ is *not* the eigenvector of the 1–turn spin map beginning at some σ, η and s unless Q_s is an integer. At each chosen fixed s, each of the three components of $\hat{n}$ lies on a component-specific closed curve in the (σ, η) plane corresponding to the closed ellipse in the (σ, η) plane mapped out by a non–radiating particle of fixed amplitude. But if a particular (σ, η) pair are chosen at some initial s and the corresponding $\hat{n}(\sigma, \eta; s)$ is transported according to (3.11) for one turn, this $\hat{n}$ does not in general return to its starting value so that it is *not* a “closed spin solution”.

(3) Some authors still use the symbol $\hat{n}$ when they actually mean $\hat{n}_0$! The tendency to create confusion seems to be deep rooted.

(4) The vector $\partial\hat{n}/\partial\eta$ (which is still often written as $\gamma\partial\hat{n}/\partial\gamma$) is sometimes called the “spin chromaticity”. We prefer the terms “spin–orbit coupling function” or “spin-field derivative” so that “spin chromaticity” can be reserved for the rate of change of a amplitude-dependent spin tune w.r.t. a fractional energy change. In any case in the full theory, the DKM formula (3.7) must be modified to include (usually) relatively small terms involving derivatives of $\hat{n}$ w.r.t. the two transverse canonical momenta [17, 24] and for such terms the name “chromaticity” is clearly unsuitable.

(5) If $|\sigma g_{19} + \eta g_{20}| << 1$, i.e. if the angle between $\hat{n}$ and $\hat{n}_0$ is small, then $\hat{n}$ in (3.13) and (3.15) can be approximated by

$$\hat{n} = \hat{n}_0 + \Big(\sigma g_{19}(s) + \eta g_{20}(s)\Big)\, \hat{m} \tag{4.4}$$

where

$$\hat{m} = -\sin\Big(g_6(s)\Big)\hat{e}_1 + \cos\Big(g_6(s)\Big)\hat{e}_2 \tag{4.5}$$

is a unit vector in the machine plane perpendicular to $\hat{n}$. This is the "SLIM approximation" [25] whereby $\hat{n}$ deviates from $\hat{n}_0$ by a function linear in the phase-space coordinates [4]. In this approximation we again find $(\partial\hat{n}/\partial\eta)^2 = g_{20}^2$ so that for Machine II the SLIM approximation actually delivers the correct result for $(\partial\hat{n}/\partial\eta)^2$. Of course, a direct application of the eigenvectors and matrices of the SLIM formalism delivers (4.4) too.

(6) The reason that the DKM approach cannot be used on the spin distribution of Machine I is as follows. The DKM formalism assumes that transients have died away and that the polarisation is locally parallel to $\hat{n}$. But in Machine I the magnetic field is perpendicular to the machine plane everywhere so that $\hat{n}_0$ and $\hat{n}$ are also perpendicular to the machine plane. Then since the spin distribution of Machine I is set up in the machine plane the spins are perpendicular to $\hat{n}$ and the distribution precesses at the rate ν_0 around $\hat{n}_0$. In Machine II there is a magnetic field in the machine plane at the snake and $\hat{n}_0$ and $\hat{n}$ are in the machine plane together with the spins themselves. In this case the polarisation can settle down to be almost parallel to $\hat{n}$.

(7) We have seen how the fields $\hat{n}(\sigma, \eta; s)$ and $\hat{p}(\sigma, \eta; s)$ for Machine II are found by enforcing 1-turn periodicity while solving the partial differential equations (PDE) (3.11) and (3.21) respectively. For real rings, the most reliable method for finding $\hat{n}$ is stroboscopic averaging [26] for a point in phase space. That involves integrating the T-BMT equation for spins along a particle trajectory through that point in phase space and averaging the spins. The 1-turn periodicity emerges automatically from the algorithm. For (3.11) the trajectory is, of course, defined by the relations $d\sigma/ds = a\eta$ and $d\eta/ds = b\sigma$. But these just define the characteristic curves of the PDE, familiar from the method of characteristics for solving linear PDE's [27]. Stroboscopic averaging for $\hat{n}$ relies on the fact that the T-BMT equation (equivalently (3.11)) is linear. The equation of motion for $\hat{p}$ in (3.21) and for real rings is also linear. Thus, since stroboscopic averaging works well for $\hat{n}$, it is tempting to use it for $\hat{p}$. In that case one would again integrate along a characteristic curve. However, the defining equations for a characteristic curve would now include damping. Then care would be needed. In any case stroboscopic averaging is only useful if the average is normalisable.

(8) In principle the asymptotic direction of the local polarisation at points in phase space and its dependence on the distance to spin-orbit resonance could be discovered with a Monte-Carlo spin-orbit tracking simulation [28, 8]. This would require a large number of particles and corresponding computing power in order to sufficiently populate a sufficient number of points in phase space. In addition a precise numerical determination of $\hat{n}$ at each point in phase space would be needed for comparison and that would require stroboscopic averaging [26] or the SODOM algorithm [29]. Away from spin-orbit resonances the

required number of particles could perhaps be reduced by just looking (say) turn-by-turn at the projection of a spin on the plane perpendicular to $\hat{n}$ for each particle and then finding the average projection. This would represent the deviation of the direction of the local polarisation from $\hat{n}$, averaged over phase space.

(9) Although it would be satisfying to calculate the rate of depolarisation for real electron/positron machines by integrating the Bloch equation for the polarisation density, that would be cumbersome in practice. In any case the DKM approach, as well as being elegant, provides an extremely good approximation perfectly suited to the operating regimes of storage rings in operation up to now and it provides the only practical analytical approach for real rings with arbitrary discrete magnet structures. Computer codes which evaluate the DKM formula numerically for real rings are listed in [4]. Unfortunately, beyond the first order approximation of SLIM, they all require large amounts of computing time.

Nevertheless, the Bloch equation for the polarisation density, augmented by a Baier-Katkov-Strakhovenko expression for the influence of the Sokolov-Ternov effect [5, 6, 30] remains the key to a general description of polarisation dynamics in electron/positron rings. Note that as explained in [5, 6] and hinted at in the Epilogue in [1], with this combination it will be possible to arrive at the DKM estimates from first principles. A first approach, based on a semiclassical calculation, can be found in [23]. This programme will involve approximations but will enable analytical exploration of the limitations of the DKM formula without recourse to heuristics. See [8] for Monte-Carlo simulations of the depolarisation process at very high energy and for a study of whether the extra terms for the depolarisation rate mentioned in Section 3.4 are needed.

Conclusion

Analytical calculations for the polarisation in electron/positron storage rings are usually based on the DKM formalism. This, in turn, is based on some implicit reasonable considerations of the various time scales involved in spin-orbit motion. See, for example, Figure 1 in [16]. Nevertheless there are open questions about the applicability of the DKM formula at very high energy [8].

It therefore seems desirable to return to first principles and obtain the rate of depolarisation from a study of the solutions of the Bloch equation for the polarisation density [5, 6]. To obtain the equilibrium polarisation, the Sokolov-Ternov effect must be included via the Baier-Katkov-Strakhovenko formalism.

This paper has used a simple, non-trivial model of spin diffusion to show that the DKM expression for the depolarisation rate is only a good approximation in this case. Although the Bloch equation for the polarisation density was not used directly, its use was implicit and our results support the claim in the Epilogue of [1] that the derivative $\partial\hat{n}/\partial\eta$ should be replaced by $\partial\hat{p}/\partial\eta$. Nevertheless, further work will be needed to properly establish the range of applicability of the basic DKM formalism and to address the open questions mentioned in [8].

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We take this opportunity to mention some typing errors on page 7 of the original.
Line 10: replace p_σ by σ_{p_σ}. Line 11: replace σ_{p_σ} by σ_ψ. Line 24: replace $2(2\pi\nu\sigma_{p_\sigma}/Q_s)$ with $\sqrt{2}(2\pi\nu\sigma_{p_\sigma}/Q_s)$. Line 31: replace σ_{p_σ} by σ_ψ.
Corresponding corrections should also be made in [13].

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