General ε-representation for scalar one-loop Feynman integrals

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Abstract

A systematic study of the scalar one-loop two-, three-, and four-point Feynman integrals is performed. We consider all cases of mass assignment and external invariants and derive closed expressions in arbitrary space-time dimension in terms of higher transcendental functions. The integrals play a role as building blocks in general higher-loop or multi-leg processes. We also perform numerical checks of the calculations using `AMBRE/MB` and `LoopTools/FF`.

Keywords: One-loop Feynman integrals, (generalized) hypergeometric functions.

1. Introduction

The physics at future colliders, like the LHC at high luminosities and the ILC [1–3], focuses on measuring the properties of the Higgs boson, of the top quark and vector bosons, as well as performing searches for signals beyond the Standard Model. These measurements will be performed at high precision, requiring higher order corrections. They necessitate detailed calculations for one-loop multi-leg processes and higher-loop calculations for selected scattering cross sections.

Scalar one-loop integrals in general space-time dimension $d = 4 + 2n − 2\varepsilon$, $n \in \mathbb{N}$, are important for several reasons. Within the general framework of higher-order corrections, higher terms in the ε-expansion for one-loop integrals form necessary building blocks. At the level of one-loop order, one may cure the inverse Gram determinant problem with Feynman integrals in higher dimensions than $d = 4$ [4–8].

The calculation of scalar one-loop integrals in $d = 4 − 2\varepsilon$ dimensions have been performed by many authors [9–15]. For the case of general dimension d,

we mention the work in [16–20]. Various packages are available for the numerical evaluation of massive one-loop integrals, such as `FF` [21], `LoopTools` [22], `XLOOP-GiNaC` [14], `AMBRE/MB` [23, 24] and others [25]. However, not all of these calculations and packages cover general dimension d with a general ε-expansion. In this paper, we study systematically the scalar one-loop 2-, 3-, and 4-point integrals, based on the method introduced in [19, 20]. We consider all cases of mass and external invariant assignment with propagator indices $\nu_k = 1$ and perform numerical checks using `LoopTools/FF` and `AMBRE/MB`.

2. The one-loop functions

In the following we present analytic results for scalar one-loop two-, three-, and four-point functions at general values of space-time dimension d.

2.1. Notations

Scalar one-loop n-point Feynman integrals are given by

$$J_n^{(d)} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{P_1 P_2 \ldots P_n}, \tag{1}$$

where $P_j = (k − p_j)^2 − m_j^2$, $p_{ij}^2 = (p_i − p_j)^2$ will denote the external momenta and m_j is the mass of the jth propagator.

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The following recurrence relation for $J_n^{(d)}$ in general space-time dimension

$$(d - n + 1)\, G_{n-1} J_n^{(d+2)} = \left[2\Delta_n + \sum_{k=1}^{n} (\partial_k \Delta_n)\, \mathbf{k}^-\right] J_n^{(d)} \quad (2)$$

has been given [20]. Here d is the space-time dimension, $\partial_k \equiv \partial/\partial m_k^2$, and $\mathbf{k}^-$ is an operator which shrinks P_k in the integrand of $J_n^{(d)}$. The kinematic variables in (2) are defined as follows

$$\Delta_n(\{p_1, m_1\}, \ldots \{p_n, m_n\}) = \begin{vmatrix} Y_{11} & Y_{12} & \ldots & Y_{1n} \\ Y_{12} & Y_{22} & \ldots & Y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \ldots & Y_{nn} \end{vmatrix},$$

with $Y_{ij} = -(p_i - p_j)^2 + m_i^2 + m_j^2,$ \quad (3)

and

$$G_{n-1}(p_1, \ldots, p_n) =$$
$$= -2^n \begin{vmatrix} p_1^2 & p_1 p_2 & \ldots & p_1 p_{n-1} \\ p_1 p_2 & p_2^2 & \ldots & p_2 p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1 p_{n-1} & p_2 p_{n-1} & \ldots & p_{n-1}^2 \end{vmatrix}. \quad (4)$$

As will be discussed later, the analytic results for scalar one-loop n-point integrals will be presented as a function of the ratio of the above determinants. Therefore, it is worth to introduce the following index variables

$$\lambda_{ij\cdots n} = \Delta_n(\{p_1, m_1\}, \ldots \{p_n, m_n\}), \quad (5)$$
$$g_{ij\cdots n} = G_{n-1}(p_1, \ldots, p_n), \quad (6)$$
$$r_{ij\ldots n} = -\frac{\lambda_{ij\ldots n}}{g_{ij\ldots n}}. \quad (7)$$

The solution of (2) has been presented in [20]:

$$J_n^{(d)} = b_n(\varepsilon) -$$ \quad (8)
$$- \sum_{k=1}^{n} \left(\frac{\partial_k \Delta_n}{2\Delta_n}\right) \sum_{r=0}^{\infty} \left(\frac{d-n+1}{2}\right)_r \left(\frac{G_{n-1}}{\Delta_n}\right)^r \mathbf{k}^- J_n^{(d+2r)}.$$

The boundary term $b_n(\varepsilon)$ is determined by the asymptotic behavior $J_n^{(d)}$ at $d \to \infty$.

2.2. Two-point functions

We first consider the simplest case, the scalar one-loop two-point functions for $p_{ij}^2 \neq 0$ and $r_{ij} \neq 0$. The recurrence relation for $J_2^{(d)}$ takes the form

$$J_2^{(d)} = b_2(\varepsilon) -$$

$$- \sum_{k=1}^{2} \left(\frac{\partial_k \lambda_{ij}}{2\lambda_{ij}}\right) \sum_{r=0}^{\infty} \left(\frac{d-1}{2}\right)_r \left(-\frac{1}{r_{ij}}\right)^r \mathbf{k}^- J_2^{(d+2r)} \quad (9)$$

and $k = 1, 2$ label the internal masses. The operator $\mathbf{k}^- J_2^{(d+2r)}$ reduces $J_2^{(d)}$ to scalar one-loop one-point functions in $d + 2r$ space-time dimensions. By using the formula for the scalar one-loop one-point functions in [20] one obtains

$$\mathbf{k}^- J_2^{(d+2r)} = (-1)^{r+1} \frac{\Gamma(\frac{d}{2})\Gamma(1 - \frac{d}{2})}{\Gamma(\frac{d}{2} + r)} (m_k^2)^{\frac{d-2}{2} + r}. \quad (10)$$

Inserting this result into (9), the following representation is obtained

$$\frac{2\lambda_{ij} J_2^{(d)}}{\Gamma\left(1 - \frac{d}{2}\right)} = b_2(\varepsilon) + \frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(\frac{m_j^2}{r_{ij}}\right)^r$$
$$+ \frac{\partial_j \lambda_{ij}}{(m_i^2)^{1-\frac{d}{2}}} \sum_{r=0}^{\infty} \frac{\left(\frac{d-1}{2}\right)_r}{\left(\frac{d}{2}\right)_r} \left(\frac{m_i^2}{r_{ij}}\right)^r. \quad (11)$$

Here $b_2(\varepsilon)$ is determined by the asymptotic behavior of $J_2^{(d)}$, $d \to \infty$. The infinite series is given in terms of a Gauss hypergeometric function [26], yielding

$$\frac{2\lambda_{ij}\, J_2^{(d)}}{\Gamma\left(1 - \frac{d}{2}\right)} = -\frac{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} r_{ij}^{\frac{d-2}{2}} \left[\frac{\partial_i \lambda_{ij}}{\sqrt{1 - \frac{m_j^2}{r_{ij}}}} + \frac{\partial_j \lambda_{ij}}{\sqrt{1 - \frac{m_i^2}{r_{ij}}}}\right]$$
$$+ \left\{\frac{\partial_i \lambda_{ij}}{(m_j^2)^{1-\frac{d}{2}}}\, {}_2F_1\!\left[\begin{matrix} 1, \frac{d-1}{2}\,; \\ \frac{d}{2}\,; \end{matrix}\ \frac{m_j^2}{r_{ij}}\right] + (i \leftrightarrow j)\right\}, \quad (12)$$

provided that $\mathrm{Re}\big((d-1)/2\big) > 0$ and $\left|m_{i,j}^2/r_{ij}\right| < 1$. If one applies Eqs. (1.3.13), (1.8.10), (1.3.3.5) of [26], one obtains

$$\frac{g_{ij}\, J_2^{(d)}}{\Gamma\left(2 - \frac{d}{2}\right)} = -\frac{\partial_i \lambda_{ij}}{(m_j^2)^{\frac{4-d}{2}}}\, {}_2F_1\!\left[\begin{matrix} 1, \frac{4-d}{2}\,; \\ \frac{3}{2}\,; \end{matrix}\ 1 - \frac{r_{ij}}{m_j^2}\right] \quad (13)$$
$$- \frac{\partial_j \lambda_{ij}}{(m_i^2)^{\frac{4-d}{2}}}\, {}_2F_1\!\left[\begin{matrix} 1, \frac{4-d}{2}\,; \\ \frac{3}{2}\,; \end{matrix}\ 1 - \frac{r_{ij}}{m_i^2}\right],$$

provided that $\left|1 - \frac{r_{ij}}{m_{i,j}^2}\right| < 1$. Eqs. (12) and (13) reproduce (53) and (59) in [20].

It is important to note that the arguments of the hypergeometric functions in (12) and (13) may have a different behavior. In general, it is not possible to write a single expression for $J_n^{(d)}$; one rather has to refer to the corresponding analytic continuations, cf. e.g. [26]. We will treat all the special cases such as $r_{ij} = \{0, m_i^2, m_j^2\}$,

$g_{ij} = 0$, $m_i^2 = m_j^2 = 0$, etc., in [27]. In the present paper, we consider $r_{ij} = 0$ as an example. One has

$$_2F_1 \left[\begin{array}{c} 1, \frac{4-d}{2} \; ; \\ \frac{3}{2} \; ; \end{array} 1 \right] = \frac{1}{2} \int_0^1 dt \, (1-t)^{\frac{d-3}{2}-1} = \frac{1}{d-3}, \quad (14)$$

provided that $\mathrm{Re}((d-3)/2) > 0$. From (13), one gets

$$\frac{(d-3) \, J_2^{(d)}}{\Gamma\left(2 - \frac{d}{2}\right)} = \left\{ \frac{\partial_j \lambda_{ij}}{4 p_{ij}^2} \left(m_i^2\right)^{\frac{d}{2}-2} + (i \leftrightarrow j) \right\}. \quad (15)$$

2.3. Three-point functions

In a similar manner we can write a complete formula for $J_3^{(d)}$ as

$$\begin{aligned} J_3^{(d)} = {} & \sqrt{8} \, \pi \Gamma\left(2 - \frac{d}{2}\right) \frac{\sqrt{-g_{ijk}}}{\lambda_{ijk}} \left(r_{ijk}\right)^{\frac{d-2}{2}} \Theta(\tau_{ijk}) \\ & + C_{ijk}^0 + C_{ikj}^0 + C_{kji}^0. \end{aligned} \quad (16)$$

The first term in (16) derives from the boundary condition solving (8) also defining the function $\Theta(\tau_{ijk})$, see [27]. Depending on the problem it can be chosen in different ways. The functions C_{ijk}^0 read

$$\begin{aligned} \frac{C_{ijk}^0}{\Gamma\left(2 - \frac{d}{2}\right)} = {} & -\frac{\sqrt{\pi} \, \Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{r_{ij}^{\frac{d}{2}-1}}{2\lambda_{ij}}\right) \left(\frac{\partial_k \lambda_{ijk}}{2\lambda_{ijk}}\right) \\ & \times \left\{ \frac{\partial_i \lambda_{ij}}{\sqrt{1 - \frac{m_j^2}{r_{ij}}}} + \frac{\partial_j \lambda_{ij}}{\sqrt{1 - \frac{m_i^2}{r_{ij}}}} \right\} {}_2F_1 \left[\begin{array}{c} 1, \frac{d-2}{2} \; ; \\ \frac{d-1}{2} \; ; \end{array} \frac{r_{ij}}{r_{ijk}} \right] \\ & + \frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\partial_k \lambda_{ijk}}{2\lambda_{ijk}}\right) \quad (17) \\ & \times \left\{ \frac{\partial_j \lambda_{ij}}{\sqrt{1 - \frac{m_i^2}{r_{ij}}}} \frac{\left(m_i^2\right)^{\frac{d}{2}-1}}{2\lambda_{ij}} F_1\left(\frac{d-2}{2}; 1; \frac{1}{2}; \frac{d}{2}; \frac{m_i^2}{r_{ijk}}, \frac{m_i^2}{r_{ij}}\right) \right. \\ & \left. + (i \leftrightarrow j) \right\}. \end{aligned}$$

Analogously for C_{ikj}^0 and C_{kji}^0. Eq. (17) reproduces Eq. (74) in [20]. Eq. (17) is valid provided that $\left|\frac{m_i^2}{r_{ij}}\right| < 1$, $\left|\frac{r_{ij}}{r_{ijk}}\right| < 1$ and $\mathrm{Re}\left(\frac{d-2}{2}\right) > 0$. The latter condition is always met when $d > 2$. The kinematic variables r_{ijk}, r_{ij} and m_i, etc., usually do not satisfy the former conditions. Therefore, if the absolute value of the arguments of $_2F_1$ and the Appell functions F_1 in (16) are larger than one, one has to perform analytic continuations [26, 28].

The Appell function in (17) obeys a simple integral representation [26]

$$F_1\left(\frac{d-2}{2}; 1; \frac{1}{2}; \frac{d}{2}; x, y\right) = \frac{d-2}{2} \int_0^1 \frac{du \, u^{\frac{d-4}{2}}}{(1-xu)\sqrt{1-yu}}, \quad (18)$$

provided that $|x|$, $|y| < 1$ and $\mathrm{Re}(\frac{d}{2} - 1) > 0$.

All the special cases for $J_3^{(d)}$ will be listed and calculated in detail in Ref [27]. In the present note we consider the massless example. We relabel the external momenta as $p_{ij} = p_i$, $p_{jk} = p_j$ and $p_{ki} = p_k$. One then confirms that $g_{ijk} = 2 \, \lambda(p_i^2, p_j^2, p_k^2)$, with λ being the Källen function, $\lambda_{ijk} = -2 p_i^2 p_j^2 p_k^2$. By taking the derivatives of λ_{ijk} with respect to m_i^2 in (3) and setting $m_i^2 \to 0$, we obtain

$$\begin{aligned} \partial_i \lambda_{ijk} &= 2 p_j^2 (p_i^2 + p_k^2 - p_j^2), \\ \partial_j \lambda_{ijk} &= 2 p_k^2 (p_i^2 + p_j^2 - p_k^2), \quad (19) \\ \partial_k \lambda_{ijk} &= 2 p_i^2 (p_j^2 + p_k^2 - p_i^2). \end{aligned}$$

The analytic solution for $J_3^{(d)}$ is given by (16) by replacing

$$\begin{aligned} \frac{C_{ijk}^0}{\Gamma\left(2 - \frac{d}{2}\right)} = {} & \frac{\sqrt{\pi} \, \Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(-\frac{p_i^2}{4}\right)^{\frac{d}{2}-2} \quad (20) \\ & \times \left(\frac{1}{p_j^2 + p_k^2 - p_i^2}\right) {}_2F_1\left[\begin{array}{c} 1, \frac{1}{2} \; ; \\ \frac{d-1}{2} \; ; \end{array} \frac{\lambda(p_i^2, p_j^2, p_k^2)}{(p_j^2 + p_k^2 - p_i^2)^2} \right]. \end{aligned}$$

For $p_i^2 > 0$ one applies $p_i^2 + i\epsilon$ in (20) and reproduces the results given in [29], using another method.

2.4. Four-point functions

By applying the same procedure, one can write a compact formula for $J_4^{(d)}$ as follows

$$\begin{aligned} J_4^{(d)} = {} & 8\pi^{\frac{3}{2}} \frac{\Gamma\left(\frac{d}{2}\right) \Gamma\left(2 - \frac{d}{2}\right)}{(d-2)\Gamma\left(\frac{d-3}{2}\right)} \frac{\sqrt{-g_{ijkl}}}{\lambda_{ijkl}} r_{ijkl}^{\frac{d-3}{2}} \Theta(\tau_{ijkl}) \\ & + D_{ijkl}^0 + D_{lijk}^0 + D_{klij}^0 + D_{jkli}^0, \end{aligned} \quad (21)$$

with

$$\begin{aligned} \frac{D_{ijkl}^0}{\Gamma\left(2 - \frac{d}{2}\right)} = {} & -\sqrt{8}\pi \frac{\sqrt{-g_{ijk}}}{\lambda_{ijk}} \left(r_{ijk}\right)^{\frac{d-2}{2}} \\ & \times \left(\frac{\partial_l \lambda_{ijkl}}{2\lambda_{ijkl}}\right) {}_2F_1\left[\begin{array}{c} \frac{d-3}{2}, 1 \; ; \\ \frac{d}{2} - 1 \; ; \end{array} \frac{r_{ijk}}{r_{ijkl}} \right] \Theta(\tau_{ijk}) \\ & + \left\{ \frac{\sqrt{\pi} \, \Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{\partial_l \lambda_{ijkl}}{2\lambda_{ijkl}}\right) \left(\frac{\partial_k \lambda_{ijk}}{2\lambda_{ijk}}\right) \right. \end{aligned}$$

$$\times \left(\frac{r_{ij}^{\frac{d}{2}-1}}{2\lambda_{ij}}\right)\left(\frac{\partial_i \lambda_{ij}}{\sqrt{1-\frac{m_j^2}{r_{ij}}}}\right)\left(\frac{1}{\sqrt{1-\frac{r_{ij}}{r_{ijk}}}}\right) \qquad (22)$$

$$\times F_1\left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{r_{ij}}{r_{ijkl}}, \frac{r_{ij}}{r_{ijk}}\right)$$

$$-\frac{\Gamma\left(\frac{d}{2}-1\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{\partial_l \lambda_{ijkl}}{2\lambda_{ijkl}}\right)\left(\frac{\partial_k \lambda_{ijk}}{2\lambda_{ijk}}\right)\left(\frac{\partial_j \lambda_{ij}}{2\lambda_{ij}}\right)$$

$$\times \left(\frac{r_{ijk}}{r_{ijk}-m_i^2}\right)\left(\frac{r_{ij}}{r_{ij}-m_i^2}\right)(m_i^2)^{\frac{d}{2}-1} \times$$

$$F_S\left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; \frac{m_i^2}{r_{ijkl}}, \frac{m_i^2}{m_i^2 - r_{ijk}}, \frac{m_i^2}{m_i^2 - r_{ij}}\right)$$

$$+(i \leftrightarrow j) \Bigg\}$$

$$+\left\{(i, j, k) \leftrightarrow (k, i, j)\right\} + \left\{(i, j, k) \leftrightarrow (j, k, i)\right\}.$$

Eq. (22) reproduces Eq. (99) in [20]. The terms D_{lijk}^0, D_{klij}^0, D_{jkli}^0 are obtained from D_{ijkl}^0 by circular permutation of the indices i, j, k, l, and the first first term in (21) results form the boundary condition, cf. [27] for details. The representation for $J_4^{(d)}$ in (21) with D_{ijkl}^0, and equivalently for D_{lijk}^0, D_{klij}^0, D_{jkli}^0 in (22), is valid under the conditions that $\text{Re}\left(\frac{d-3}{2}\right) > 0$ and that the absolute values of arguments of the hypergeometric functions are smaller than one. If the absolute value of the arguments are larger than one, one has to perform analytic continuations of the hypergeometric and Appell F_1 functions, cf. [26, 28]. Further, the Saran function F_S may be expressed by a Mellin-Barnes representation in this case.

The integral representation of F_S reads [30]

$$F_S(\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_1, \gamma_1; x, y, z) = \qquad (23)$$

$$= \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_1)\Gamma(\gamma_1 - \alpha_1)} \int_0^1 dt \frac{t^{\gamma_1 - \alpha_1 - 1}(1-t)^{\alpha_1 - 1}}{(1 - x + tx)^{\beta_1}}$$

$$\times F_1(\alpha_2, \beta_2, \beta_3; \gamma_1 - \alpha_1; ty, tz),$$

provided that $|x|, |y|, |z| < 1$ and $\text{Re}(\gamma_1 - \alpha_1 - \alpha_2) > 0$.

All the special cases are treated in Ref [27] in detail. Again we consider the massless case as an example. We perform the analytic continuation of the result for $J_4^{(d)}$ in (21, 22) in this case. Furthermore, taking $m_i^2 \to 0$, one notices that

$$F_S\left(\frac{d-3}{2}, 1, 1; 1, 1, \frac{1}{2}; \frac{d}{2}, \frac{d}{2}, \frac{d}{2}; 0, 0, 0\right) = 1, \qquad (24)$$

provided that $\text{Re}\left(\frac{d-3}{2}\right) > 0$. On the other hand, $(m_i^2)^{\frac{d}{2}-1} \to 0$ whenever $d > 2$. Therefore the terms

related to F_S in (22) are vanishing in the massless case. As a result, the term D_{ijkl}^0 can be written as

$$\frac{D_{ijkl}^0}{\Gamma\left(2-\frac{d}{2}\right)} = -\sqrt{8}\pi \frac{\sqrt{-g_{ijk}}}{\lambda_{ijk}}\left(r_{ijk}\right)^{\frac{d-2}{2}}$$

$$\times \left(\frac{\partial_l \lambda_{ijkl}}{2\lambda_{ijkl}}\right){}_2F_1\left[\begin{matrix} \frac{d-3}{2}, 1; \\ \frac{d}{2}-1; \end{matrix} \frac{r_{ijk}}{r_{ijkl}}\right]\Theta(\tau_{ijk})$$

$$+\frac{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}-1\right)}{2\,\Gamma\left(\frac{d-1}{2}\right)}\left(\frac{\partial_l \lambda_{ijkl}}{2\lambda_{ijkl}}\right)\left(\frac{\partial_k \lambda_{ijk}}{2\lambda_{ijk}}\right)$$

$$\times \left(\frac{r_{ij}^{\frac{d}{2}-2}}{\sqrt{1-\frac{r_{ij}}{r_{ijk}}}}\right)F_1\left(\frac{d-3}{2}; 1, \frac{1}{2}; \frac{d-1}{2}; \frac{r_{ij}}{r_{ijkl}}, \frac{r_{ij}}{r_{ijk}}\right)$$

$$+\left\{(i, j, k) \leftrightarrow (j, k, i)\right\} + \left\{(i, j, k) \leftrightarrow (k, i, j)\right\}. \qquad (25)$$

The ε-expansion of all the above expressions can be performed using the packages Sigma, EvaluateMultiSums and Harmonic Sums [31]. Compact expressions for the numerics of 5-point and higher-point functions are given in [8, 32].

3. Numerical checks

The analytic results have been implemented as Mathematica v8.0 package ONELOOP234.m. In Tables 1–5 we compare ONELOOP234.m with AMBRE/MB v1.2 and LoopTools/FF v.2.10. The results show a very good agreement in all cases.

4. Conclusions

A systematic study of the scalar one-loop two-, three- and four-point integrals in arbitrary space-time dimensions is presented. For considering all cases of mass- and external invariant assignments, and for the systematic derivation of proper ε expansions we refer to [27]. We performed numerical sample checks with LoopTools/FF and AMBRE/MB, finding perfect agreement. Packages both in Mathematica and Fortran

Table 1: Comparison of the ε^0 term of $J_2^{(4-2\varepsilon)}$ with LoopTools/FF.

(p^2, m_1^2, m_2^2)	This work/LoopTools/FF
$(1000, 25, 36)$	$-4.6291944000901360 - 2.9439271197312254\,i$
	$-4.6291944000901353 - 2.9439271197312253\,i$
$(-1000, 25, 36)$	$-5.1755714223812398 + 0.0\,i$
	$-5.1755714223812399 + 0.0\,i$

providing the corresponding expressions will be made available in the near future.

Table 2: Comparison of the ε^0 term of $J_3^{(4-2\varepsilon)}$ with `LoopTools/FF`, with $m_1^2 = 16$, $m_2^2 = 25$, $m_3^2 = 36$.

$(p_1^2, p_2^2, p_3^2) \times 10^2$	This work/`LoopTools/FF` $(\times 10^{-2})$
$(1, 5, 2)$	$1.62458585289425548 - 0.81633722940756023\, i$
	$1.62458585289425653 - 0.81633722940756131\, i$
$(-1, -5, -2)$	$-0.622465213238832772 + 0.0\, i$
	$-0.622465213238832665 + 0.0\, i$

Table 3: Comparison of the ε^0 term of $J_4^{(4-2\varepsilon)}$ with `LoopTools/FF` in the massless case.

$(p_1^2, p_2^2, p_3^2, p_4^2, s, t)$	This work/`LoopTools/FF`
$(1, 5, 1, 7, 15, 1)$	$0.22052449908760818 + 0.0\, i$
	$0.22052449908760813 + 0.0\, i$
$(-1, -5, -1, -7, -25, -1)$	$0.18243214608579919 + 0.0\, i$
	$0.18243214608579872 + 0.0\, i$

Table 4: Comparison of $J_2^{(12-2\varepsilon)}(-100; 100, 400) \times 10^8$ with `AMBRE/MB`. The Monte Carlo error of `AMBRE/MB` is $O(10^{-12})$.

This work	$+3.42724867724867725\varepsilon^{-1} - 12.8050782926747736\varepsilon^0$ $+29.2335688024947187\varepsilon^1 - 49.6610273102650704\varepsilon^2$ $+70.9545648069212449\varepsilon^3 - 89.4570353152169962\varepsilon^4$
`AMBRE`	$+3.427248677248677\varepsilon^{-1} - 12.805078292674778\varepsilon^0$ $+29.23356880249472\varepsilon^1 - 49.66102731026506\varepsilon^2$ $+70.95456480692126\varepsilon^3 - 89.457035315217\varepsilon^4$

Table 5: Comparison of $J_3^{(12-2\varepsilon)}(-100, -500, -200; 16, 25, 36)$ with `AMBRE`. The numbers in brackets show the Monte Carlo error of `AMBRE/MB`.

This work	$+86945.7178571428571\, \varepsilon^{-1} - 252266.637358344924\, \varepsilon^0$ $+499059.587505861549\, \varepsilon^1 - 741026.396679848443\, \varepsilon^2$ $+968401.630700711348\, \varepsilon^3 - 1.12777896348311805 \cdot 10^6\, \varepsilon^4$
`AMBRE`	$+86945.7\, \varepsilon^{-1} - 252265.\, \varepsilon^0\ (\pm 7.) + 499054.\, \varepsilon\ (\pm 46.)$ $-740919.\, \varepsilon^2\ (\pm 249.) + 967850.\, \varepsilon^3\ (\pm 1256.)$ $-1.12699 \cdot 10^6\, \varepsilon^4 (\pm 6398.)$

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