# Effect of Aberration of Light in X-ray Free Electron Lasers

Gianluca Geloni, [a] Vitali Kocharyan, [b] Evgeni Saldin [b]

[a] *European XFEL GmbH, Hamburg, Germany*

[b] *Deutsches Elektronen-Synchrotron (DESY), Hamburg, Germany*

**Abstract**

We discuss the physics of a microbunched electron beam kicked by the dipole field of a corrector magnet by describing the kinematics of coherent undulator radiation after the kick. Particle tracking shows that the electron beam direction changes after the kick, while the orientation of the microbunching wavefront stays unvaried. Therefore, electrons motion and wavefront normal have different directions. Coherent radiation emission in a downstream undulator is expected to be dramatically suppressed as soon as the kick angle becomes larger than the divergence of the output radiation. In fact, according to conventional treatments, coherent radiation is emitted along the normal to the microbunching wavefront. Here we show that kinematics predicts a surprising effect. Namely, a description of coherent undulator radiation in the laboratory frame yields the radical notion that, due light aberration, strong coherent radiation is produced along the direction of the kick. We hold a recent FEL study made at the LCLS as a direct experimental evidence that coherent undulator radiation can be kicked by an angle of about five times the rms radiation divergence without suppression. We put forward our kinematical description of this experiment.

## 1 Introduction

This paper presents our explanation of the results obtained at the LCLS [1] in a recent experiment presented at the FEL2015 conference [2]. This experiment apparently demonstrated that after a microbunched electron beam is kicked on a large angle compared to the divergence of the FEL radiation pulse, the microbunching wavefront is readjusted along the new direction of motion of the kicked beam. There is a reason for arguing that this picture cannot be explained in the framework of classical mechanics. If this is correct, then we are inclined to say that there must be something wrong with the classical particle tracking model. However, we will see that there

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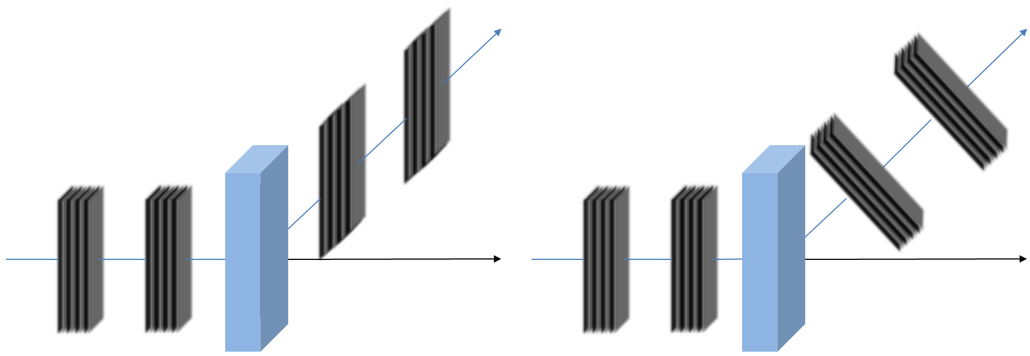

Fig. 1. Behavior of the microbunching wavefront when the direction of the electron beam is changed by a corrector magnet. The left plot qualitatively shows the result of numerical simulations based on particle tracking. The right plot qualitatively illustrates the apparent result of the experiment at the LCLS.

is nothing wrong with it. Here we try to convey the idea that the difference between the directions of the electron motion and of the normal to the microbunching wavefront after the kick, as predicted by particle tracking, is not in contradiction with the deflection (of the same angle as the electron beam) without suppression of the radiation in the downstream undulator, which occurs in the experiment.

Let us discuss the experimental results described in reference [2]. A well-known result of classical particle tracking states that after the electron beam is kicked there is a change in the trajectory of the electron beam, while the orientation of the microbunching wavefront remains as before, see Fig. 1 (left). In other words, the kick results in a difference between the directions of the electron motion and of the normal to the microbunching wavefront [3, 4]. In XFEL simulations it is generally accepted that coherent radiation from an undulator placed after the kicker is emitted along the normal to the microbunching wavefront. Therefore, when the angular kick exceeds the divergence of the output radiation, emission in the direction of the electron motion is suppressed [3].

The experiment [2] showed the surprising effect that radiation in the kicked direction is produced practically without suppression. This fact could be explained if, in contrast with the conventional understanding [3], the microbunching orientation is readjusted along the new direction of the electron beam, see Fig 1 (right). The experimental result is apparently at odds with the prediction from particle tracking. Therefore, the question arises about how to explain the coherent undulator radiation that occurs in nature and whether or not a phenomenon like deflection of coherent undulator radiation by a dipole magnet can be described in the framework of classical mechanics and electrodynamics.

In this paper we demonstrate that there is no 'wavefront readjusting' phenomenon. In the laboratory frame, due to the kick, electrons move with a uniform transverse velocity component along the direction of the microbunching wavefront, like in Fig. 1 (left), while the radiation beam is deflected. The effect of aberration of light [5] supplies the basis for an understanding of this phenomenon.

## 2 Experimental results and kinematical explanation

Let us summarize the experimental results described in reference [2]. The LCLS generates linearly polarized X-ray pulses from a planar undulator. A 3.2 m-long Delta undulator, which allows for a full control of the degree of polarization of the emitted radiation, was recently installed in place of the last LCLS undulator segment. Before going through the Delta undulator, the electron beam is microbunched in the preceding planar undulator segments. This enhances the radiation power by several orders of magnitude. Therefore, the Delta undulator is said to be operating in 'afterburner configuration'. Such configuration leads to the presence of linearly-polarized background radiation from the main undulator, which should be suppressed. In fact, when the efficiency of the regular afterburner mode of operation was tested, a maximum contrast ratio of about 2.5 was achieved. It has been recently proposed [6] that the background radiation component can be greatly reduced by a reverse undulator tapering configuration. By inverting the sign of the baseline undulator tapering the radiation emission is reduced, while microbunching can still develop. The efficiency of this mode of operation was tested and a contrast ratio of about 10 was reported in [2]. Under this conditions, practically, at the entrance of the Delta undulator there is only the micro-bunched beam. Reference [2] further reports a final improvement of the degree of polarization up to 100% by X-ray beam splitting at the photon energy of 0.7 keV. This was achieved by kicking the electron beam before entering the Delta undulator, in order to let electron beam and background radiation pass through the Delta undulator at different angles. The quadrupole at the end of the last planar undulator section includes a regular vertical corrector, which was used to control the magnitude of the kick. The maximal kick angle was about $3 \times 10^{-5}$ rad and was limited only by the 4 mm diameter of the beamline aperture at the distance of 80 m. At this maximum angle, the separation between the two radiation spots on the screen in the experimental hall was about 5 rms times the radiation spot size. The LCLS experiment apparently shows that the microbunching orientation is readjusted to produce coherent radiation in the kicked direction: the energy of the output radiation pulse with and without kick is practically the same.

However, classical particle tracking shows that while the electron beam

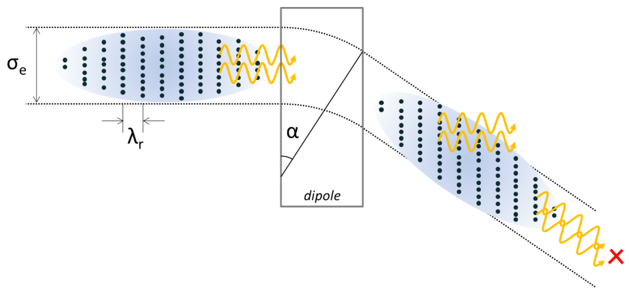


Fig. 2. Illustration of the problem, which arises according to classical particle tracking when a microbunched electron beam is deflected by a dipole magnet. After passing the dipole, the microbunching is preserved, but only along its original direction.

direction changes after the kick, the orientation of the microbunching wavefront stays unvaried. Therefore, the electron motion and the wavefront normal have different directions. Figure 2 illustrates the problem, which arises in classical treatments when a microbunched electron beam is deflected by a single dipole. The FEL process in a downstream undulator is expected to be dramatically suppressed as soon as the kick angle is larger than the divergence of the output coherent radiation.

In order to estimate the loss of radiation efficiency we make the assumption that the spatial profile of the bunching factor is close to that of the electron beam and has a Gaussian shape with standard deviation $\sigma_{\mathrm{b}}$. A bunched electron beam in an FEL amplifier can be considered as a sequence of periodically spaced oscillators. The radiation produced by these oscillators always interferes coherently at zero angle with respect to the undulator axis. In the limit for a small size of the electron beam the interference will be constructive within an angle of about $\Delta\theta_{\mathrm{c}} \simeq \sqrt{c/(\omega L_{\mathrm{g}})}$, where $L_{\mathrm{g}}$ is the FEL gain length. In the limit for a large size of the electron beam, the angle of coherence is about $\Delta\theta_{\mathrm{c}} \simeq c/(\omega\sigma_{\mathrm{b}})$ instead. The boundary between these two asymptotes is for sizes of about $\sigma_{\mathrm{dif}} \simeq \sqrt{cL_{\mathrm{g}}/\omega}$. It is worth noting that the condition $\sigma_{\mathrm{b}}^2 \gg \sigma_{\mathrm{dif}}^2$ is satisfied in our case of study at the LCLS. Thus, we can conclude that the angular distribution of the radiation power in the far zone has a Gaussian shape with standard deviation $\sigma_{\mathrm{c}} \simeq c/(\sqrt{2}\omega\sigma_{\mathrm{b}})$. After the electron beam is kicked, as already mentioned, in classical treatments we have a discrepancy between direction of the electron motion and wavefront normal. Then, the radiation intensity along the new direction of the electron beam can be approximated as $I \simeq I_0 \exp[-\theta^2/(2\sigma_{\mathrm{c}}^2)]$, where $I_0$ is the on-axis intensity without kick and $\theta$ is the kick angle. The exponential suppression factor is due to the microbunching wavefront tilt with respect to the direction of motion of the electrons.

Let us now discuss the absence of detuning effects in the case of radiation emitted along the direction of the kick. The effective undulator period is now given by $\lambda_w/\cos(\theta) \simeq (1 + \theta^2/2)\lambda_w$, where $\lambda_w$ is the actual undulator period. This induces a relative red shift in the resonance wavelength of about $\Delta\lambda/\lambda \simeq \theta^2/2$ which should be compared with the relative bandwidth of the resonance, the $\rho$ parameter, which is much larger. As a result, the red shift in the resonance wavelength due to the kick can be neglected in all situations of practical relevance. It is clear from the above that if a microbunched beam is at perfect resonance along the direction of motion without kick, then after the kick the same microbunched beam is at perfect resonance along the new direction of the electron beam motion.

Beam splitting at the LCLS was done by kicking the electron beam of an angle of about 5 standard deviations of the intensity distribution in the far zone. According to the estimations presented above, the intensity of the coherent radiation in the kicked direction should be suppressed by two orders of magnitude. In spite of this, the experiment showed that the radiation intensity in the kicked direction is practically the same as the intensity without kick at zero angle. This is in contradiction with the result in literature [3]. Our point is that reference [3] misses a non-trivial, but very important kinematical effect in the description of coherent undulator radiation emitted by a modulated electron beam with an amplitude front tilt. We offer an explanation based on special relativity [5].

The direction of a ray of light depends essentially on the velocity of the light source relative to the observer, a phenomenon commonly known as aberration. It is this phenomenon that was observed at the LCLS beam splitting experiment. Aberration of light is a shift of the direction of an incident beam of light due to the motion of the source relative to the observer. An elementary explanation of this effect is well-known [5]. This phenomenon is fully understandable in terms of transformation of velocities between different reference frames both in special relativity and in classical treatments [1]. The aberration of light can also be easily explained on the basis of the corpuscular theory of light. This is plausible if one keeps in mind that a light signal represents a certain amount of electromagnetic energy. Energy, like mass, is a quantity that is conserved, so that a light signal resembles, in many aspects, a material particle. Therefore, we should expect that group velocities of light signals obey the same addition theorem for particle velocities. A closer treatment based on wave theory of light confirms this expectation.

Figure 3 depicts a special case of vertical plane-wave emitter, formed by many elementary sources. All elementary sources flash simultaneously. A simple Huygens' construction shows that such emitter radiates a plane wavefront in the horizontal direction. However, the situation changes if the

[1] Note that the velocity of light plays no special role in classical theories. Doppler shift and aberration effects are caused by the finite velocity of the wave emitter and can be demonstrated with either light or sound waves.

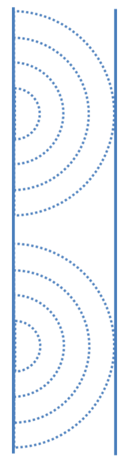

Fig. 3. Vertical plane-wave emitter. All elementary emitters flash simultaneously. A plane wavefront propagates in the horizontal direction.

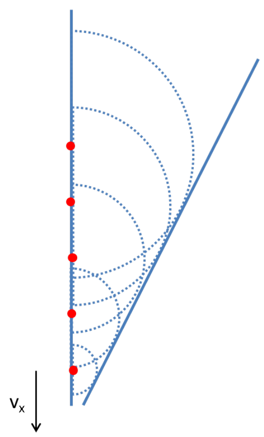


Fig. 4. Vertical plane-wave emitter. All elementary emitters flash simultaneously but they move with vertical speed $v_x$.

emitter moves at constant speed $v_x$ in the vertical direction, see Fig. 4. In that case, due to the motion of the emitter, the elementary sources along it produce a linear phase chirp. As a consequence of this phase chirp, the wavefront propagates at speed $c$ with an angle $\theta$ from the horizontal direction. In this way one obtains $\sin\theta = v_x/c$. This result can be explained purely phenomenologically on the basis of Einstein's theorem for the addition of velocities. If $v_x$ is small compared to $c$ the formula for the aberration of light is reduced to the simpler form $\theta = v_x/c$. Aberration of light in terms of a classical theory based on ether differs with respect to the prediction of special relativity by terms of second and higher orders in $v_x/c$. In other words, relativistic theory and classical theory agree only up to terms of first order in $v_x/c$ [5].

We conclude that if there is a microbunch front tilt is as in Fig. 1 (left), then due to aberration effects the radiation wavefront is tilted to be orthogonal to the kicked direction, and one cannot tell the difference with the case in Fig. 1 (right). Thus, aberration effects justify a completely novel standpoint: the front tilt of the mucrobunch has no definite objective meaning. The new experience gained from the accurate experiment at the LCLS, compel us to make a revision of kinematical concepts of beam physics. Only a closer analysis of the concept of front tilt, i.e. a discussion of the methods by which a measurement of the front tilt can actually be performed, opens up the possibility of an unambiguous description of physical phenomena in accordance with the principle of relativity.

Generalizing further we are led to consider the idea that also the concept of simultaneity between two events in different places is, to some extent, conventional, because we cannot give any experimental method by which simultaneity can be ascertained [2]. The concept of microbunch front tilt, at least in the ultrarelativistic case, seems to be a particular illustration of this more general statement, because the microbunch front tilt is just a plane of simultaneous events, and no study of mucrobunching phenomena can ever lead to a determination of the influence of the kick.

As an illustration of this fact, we present a discussion on a non-trivial but important kinematical aspect of the kick influence associated with the so called 'smearing of microbunching' phenomenon presented in [3]. In section 2.4 the authors state: 'The finite angular spread of the electron beam results in a difference in time when each electron arrives at the same longitudinal position, and this spoils the phase coherence [...] This is the so called debunching effect. The time difference is enhanced by the error [kick] angle [...], which leads to a significant degradation of the bunching factor. Let us call this mechanism smearing of microbunching and distinguish from the normal debunching'. The question here is whether or not we can observe this effect, and what measuring device can serve to this end. One should describe an operative procedure for observing the debunching effect. Suppose that we measure the difference in energy between coherent undulator radiation emission as a function of undulator length in the cases with and without the microbunch front tilt. Reference [3] does not account for the effect of aberration. Due to the motion of the electron beam along the microbunching front, the electrons along the microbunching front will induce a linear phase chirp onto the radiated field. In accordance with this phase

[2] It is interesting to note that the discussion about the conventionality of simultaneity is usually carried out within the framework of special relativity. However, the thesis that the choice of standard synchrony is a convention has been argued by Poincare' in the framework of a classical theory at the end of 19th century [7]. Simultaneity is conventional in the same sense in which the gauge freedom that arises in field theory makes the choice between e.g. Coulomb and Lorentz gauges conventional.

chirp the electron beam will always propagate perpendicularly to the phase front of the radiation, and the smearing effect cannot exist at all.

Our fruitless attempts to find any influence of the microbunch front tilt on the emission of coherent undulator radiation, confirmed by the LCLS experiment, convinced us that this result has general validity and that the front tilt of the mucrobunching has no definite objective meaning. All physical phenomena will take the same course of development independently of the microbunch front tilt and one can never decide with an experiment which electron beam has a microbunch front tilt. All tilts become completely equivalent and a central requirement for a satisfactory physical theory of coherent emission is that all electron systems must be treated on the same footing.

In all treatments of vacuum-tube devices, one is continually concerned with the interaction of beams of electrons and electromagnetic fields, static or dynamic [3] . In this class of problems one is ultimately interested in finding current and charge density at various points in space from the equation of motion of the particles and from the electromagnetic fields. In obtaining the current and charge densities from the motion of the particles, one is essentially considering the electron beam as a fluid and is thus essentially applying hydrodynamical notions [8].

When treating hydrodynamical problems one can follow two equivalent approaches: Lagrangian or Eulerian. In the Lagrangian approach one follows the motion of the individual charges through space. The variables of interest are the coordinates of the individual particles. In the Eulerian approach, instead, one begins defining the charge density $\rho$ and the current density $\vec{j}$, i.e. particular field quantities, as a function of coordinates and time. One is no more interested in knowing the evolution of each single particle but only to observe charges passing through a given position, in order to obtain information about $\rho$ and $\vec{j}$. Once these two fields are determined, one can solve Maxwell's equations for the electric field, with $\rho$ and $\vec{j}$ as macroscopic electromagnetic sources. In this method, the charge and current density must be considered as field variables, exactly like the electromagnetic field.

The transformation from Lagrangian to Eulerian formulation is obtained by transforming the equations of motion. The acceleration of any particle that passes through a fixed point in space at a fixed time is related to the spatial and time derivatives of the velocity field by the well-known expression for the so-called particle: $d/dt = \partial/\partial t + (\vec{v} \cdot \vec{\nabla})$. The advantage of the Eulerian

[3] It is relevant to note that the principle of FELs operation is similar to that of conventional vacuum-tube devices: it is based on the interaction of electron beams with radiation in vacuum.

approach is that one deals with field variables that are functions of space and time and one has a set of differential equations for these field variables. This set of equations includes the equation of continuity, expressed as $\vec{\nabla} \cdot \vec{j} + \partial\rho/\partial t = 0$.

The problem of finding the coherent radiation in an undulator due to a fast, modulated electron beam moving with constant velocity $\vec{v}$ can be solved most readily by taking advantage of Fourier transformations. If the charge density of the electron fluid is transformed into a superposition of plane waves at different temporal frequencies, then the transformed equation of motion simply becomes $\omega = \vec{k} \cdot \vec{v}$, where $\vec{k}$ is the wave vector. Taking the gradient with respect to $\vec{k}$ we see that the group velocity is directed along the electron beam velocity as must be. When a modulated electron beam can be approximated with some accuracy by a monochromatic plane wave, its wave vector must be directed along the beam velocity, like in Fig. 1 (right) and equal in modulus to $k = \omega/v$. This analysis leads to what, at first sight, seems to be a paradoxical conclusion: in the space-frequency domain, the microbunch front tilt does not exist at all. This seems in contrast with the results obtained using a Lagrangian viewpoint, Fig. 1 (left). However, as we have just seen, in that case one needs to include aberration effects in the treatment, thus finding the same result.

## 3 Microscopic approach to the effect of aberration of light in XFELs

In literature it became tradition to derive the effect of aberration of light using Lorentz transformations. Such an approach usually forces the reader to believe that a description of FEL operation is impossible without a detailed knowledge of the special theory of relativity. We show below that the effect of aberration of light can be simply explained in the laboratory frame of reference and there is no need to use the laws of relativistic kinematics. When an electron can be treated as a point particle, such an approach always gives a reliable way to describe the processes of electron beam dynamics and radiation in given electromagnetic fields.

### *3.1 A derivation of a general formula for electron beam radiation*

As already discussed, we will focus on the description of an XFEL source in terms of coherent elementary sources, the electrons, which are arranged in a modulated electron beam. We will be interested in the case of an ultra relativistic electron beam going through a certain magnetic system. We will discuss the undulator case to illustrate our reasoning, but the considerations in this section, being fully general, apply to any other magnetic system (for instance a bending magnet) as well.

Coherent undulator radiation theory is naturally developed in the space-frequency domain, as one is usually interested into radiation properties at a given position in space at a certain frequency. In this article we introduce the relation between temporal and frequency domain via the following definition of Fourier transform pairs:

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} dt\ f(t) \exp(i\omega t) \leftrightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \bar{f}(\omega) \exp(-i\omega t)\ . \qquad (1)$$

Whatever the method used to present results, one needs to solve Maxwell's equations in unbounded space. We introduce a cartesian coordinate system, where a point in space is identified by a longitudinal coordinate $z$ and transverse position $\vec{r}$. Accounting for electromagnetic sources, i.e. in a region of space where current and charge densities are present, the following equation for the field in the space-frequency domain holds in all generality:

$$c^2\nabla^2\vec{\bar{E}} + \omega^2\vec{\bar{E}} = 4\pi c^2\vec{\nabla}\bar{\rho} - 4\pi i\omega\vec{\bar{j}}\ , \qquad (2)$$

where $\bar{\rho}(z,\vec{r},\omega)$ and $\vec{\bar{j}}(z,\vec{r},\omega)$ are the Fourier transforms of the charge density $\rho(z,\vec{r},t)$ and of the current density $\vec{j}(z,\vec{r},t)$. Eq. (2) is the well-known Helmholtz equation. Here $\vec{\bar{E}}$ indicates the Fourier transform of the electric field in the space-time domain.

A system of electromagnetic sources in the space-time can be conveniently described by $\rho(z,\vec{r},t)$ and $\vec{j}(z,\vec{r},t)$. Considering a single electron in uniform motion and using the Dirac delta distribution, we can write

$$\rho\,(z,\vec{r},t) = -e\delta\,(\vec{r} - \vec{r}_0(t))\,\delta(z - z_0(t)) \qquad (3)$$

$$\vec{j}\,(z,\vec{r},t) = \vec{v}\rho\,(z,\vec{r},t)\ \ , \qquad (4)$$

where $(z_0(t),\vec{r}_0(t)) = (z_0(t),x_0(t),y_0(t))$ and $\vec{v}$ are, respectively, position and velocity of the particle at a given time $t$ in a fixed reference frame. Here $(-e)$ is the electron charge.

It is useful to specialize our coordinate system to $(s,x',y')$ in order to have the velocity along one (let us say $s$) axis. Since $v_z \simeq v$ and the coordinate system $(s, x', y')$ is rotated by a small angle, the expression for the charge density may be written as

$$\rho\left(z, \vec{r}, t\right) = -\frac{e}{v}\delta\left(\vec{r} - \vec{r}_0(z)\right)\delta\left(\frac{s(z, \vec{r}_0(z))}{v} - t\right) \tag{5}$$

Fixing the initial conditions $t = 0$, $z = z_0$ and $\vec{r} = \vec{r}_0(z_0)$ we have $s_0 = s(\vec{r}_0(z_0), z_0)$.

In the space-frequency domain the electromagnetic sources transform to:

$$\bar{\rho}(\vec{r}, z, \omega) = -\frac{e}{v}\delta\left(\vec{r} - \vec{r}_0(z)\right)\exp\left[\frac{i\omega s(z)}{v}\right] \tag{6}$$

and

$$\vec{\bar{j}}(\vec{r}, z, \omega) = \vec{v}\bar{\rho}(\vec{r}, z, \omega) \tag{7}$$

Let us consider the problem of determining the Fourier transformed charge density when the electron motion is only in two dimensions. Consider an electron which moves with constant velocity $\vec{v} = (v_x, 0, v_z)$. We know where the electron was at $t = 0$: this is the point $z = 0$, $x = x_0$, $y = y_0$, which is fixed by the initial conditions above. Using the Dirac delta function, we can write $\rho(x, y, z, t) = -e\delta(x - x_0 - v_x t)\delta(y - y_0)\delta(z - v_z t)$. In the space-frequency domain the charge density in the plane $z = 0$ amounts to $\bar{\rho} = (-e/v_z)\delta(x - x_0)\delta(y - y_0)\exp(i\omega v_x x_0/v^2)$. What is surprising and needs to be understood is the phase factor $\exp(i\omega v_x x_0/v^2)$. At $z = 0$, when the electron moves only along the $z$ direction, i.e. when $v_x = 0$, we simply have $\bar{\rho} = (-e/v_z)\delta(x - x_0)\delta(y - y_0)$ and the phase factor is absent. However, when $v_x \neq 0$ the function $s(x, z)$ depends on two independent variables $x$ and $z$. In other words, the initial condition for the function $s$ can be written as $s_0(x_0, 0) = v_x x_0/v$. This can be seen by transforming the reference system $(z, x, y)$ into the reference system $(s, x', y')$, with $s$ parallel to the particle velocity. Recalling the Eq. (6) we then obtain the phase factor $\exp(i\omega v_x x_0/v^2)$.

Moving forward with our process of generalization we consider what happens if, instead of a single electron, we have many electrons that move in synchrony. Suppose we have a plane full of electrons. We let the plane of charges be the $x - y$ plane. All electrons move with the same velocity $\vec{v} = (v_x, 0, v_z)$ and all have the same arrival time $t = 0$ at $z = 0$. In order to get the total electron density in the space-frequency domain, we must add the contribution of all the charges in the plane. At $z = 0$, this results in a charge distribution with a linear phase chirp $\bar{\rho} = (-e/v_z)\sigma_0\exp(i\omega v_x x_0/v^2)$. Here $\sigma_0$ is the number of electrons per unit surface. We can now see why the electron motion along the front of simultaneity gives an extra phase chirp, and we may now claim to understand the effect of aberration. Indeed, we demonstrated mathematically that in the space-frequency domain the microbunch front tilt does not exist at all.

We point out that the temporal Fourier transform that we performed opens up the possibility of an unambiguous description of aberration phenomena without the need for the theory of relativity. From the standpoint of electrodynamic theory, after this temporal Fourier transformation is taken, the electron density is a field variable like the electromagnetic field. One is no more interested in knowing the velocity of each particle and the plane of simultaneity, but only to observe charge passing through a given point at every instant in time. From the standpoint of electron fluid theory, the microbunch front tilt phenomena does not exist at all. A demonstration of the effect of aberration on coherent undulator radiation is now just a technical problem.

Let us consider a single electron. Suppose we are interested in the radiation generated by an electron and observed far away from it. In this case it is possible to find a relatively simple expression for the electric field. We indicate the electron velocity in units of $c$ with $\vec{\beta}(t)$, the Lorentz factor with $\gamma$, the electron trajectory in three dimensions with $\vec{R}(t)$ and the observation position with $\vec{R}_0 \equiv (z_0, \vec{r}_0)$. Finally, we introduce the unit vector

$$\vec{n} = \frac{\vec{R}_0 - \vec{R}(t)}{|\vec{R}_0 - \vec{R}(t)|} \tag{8}$$

pointing from the retarded position of the electron to the observer. In the far zone, by definition, the unit vector $\vec{n}$ is nearly constant in time. If the position of the observer is far away enough from the charge, one can make the expansion

$$\left|\vec{R}_0 - \vec{R}(t)\right| = R_0 - \vec{n} \cdot \vec{R}(t) \ . \tag{9}$$

We then obtain the following approximate expression for the the radiation field in the frequency domain [4]:

[4] A different constant of proportionality in Eq. (10) compared to textbooks (see e.g. [9]) is to be ascribed to the use of different units and definition of the Fourier transform.

$$\vec{\bar{E}}(\vec{R}_0,\omega) = -\frac{i\omega e}{cR_0}\exp\left[\frac{i\omega}{c}\vec{n}\cdot\vec{R}_0\right]\int\limits_{-\infty}^{\infty} dt\,\vec{n}\times\left[\vec{n}\times\vec{\beta}(t)\right]\exp\left[i\omega\left(t-\frac{\vec{n}\cdot\vec{R}(t)}{c}\right)\right] . \tag{10}$$

Using the complex notation, in this and in the following sections we assume, in agreement with Eq. (1), that the temporal dependence of fields with a certain frequency is of the form:

$$\vec{E} \sim \vec{\bar{E}}(z,\vec{r},\omega)\exp(-i\omega t) . \tag{11}$$

With this choice for the temporal dependence we can describe a plane wave traveling along the positive $z$-axis with

$$\vec{E} = \vec{E}_0\exp\left(\frac{i\omega}{c}z - i\omega t\right) . \tag{12}$$

In the following we will always assume that the ultra-relativistic approximation is satisfied, which is the case for XFEL setups. As a consequence, the paraxial approximation applies too. The paraxial approximation implies a slowly varying envelope of the field with respect to the wavelength. It is therefore convenient to introduce the slowly varying envelope of the transverse field components as

$$\vec{\widetilde{E}}(z,\vec{r},\omega) = \vec{\bar{E}}(z,\vec{r},\omega)\exp\left(-i\omega z/c\right) . \tag{13}$$

Introducing angles $\theta_x = x_0/z_0$ and $\theta_y = y_0/z_0$, the transverse components of the envelope of the field in Eq. (10) in the far zone and in paraxial approximation can be written as

$$\vec{\widetilde{E}}(z_0,\vec{r}_0,\omega) = -\frac{i\omega e}{c^2 z_0}\int\limits_{-\infty}^{\infty} dz'\exp\left[i\Phi_T\right]\left[\left(\frac{v_x(z')}{c}-\theta_x\right)\vec{e_x}+\left(\frac{v_y(z')}{c}-\theta_y\right)\vec{e_y}\right] , \tag{14}$$

where the total phase $\Phi_T$ is

$$\Phi_T = \omega\left[\frac{s(z')}{v}-\frac{z'}{c}\right] \\ +\frac{\omega}{2c}\left[z_0(\theta_x^2+\theta_y^2)-2\theta_x x(z')-2\theta_y y(z')+z(\theta_x^2+\theta_y^2)\right] . \tag{15}$$

Here $v_x(z')$ and $v_y(z')$ are the horizontal and the vertical components of the transverse velocity of the electron, $x(z')$ and $y(z')$ specify the transverse position of the electron as a function of the longitudinal position, $\vec{e}_x$ and $\vec{e}_y$ are unit vectors along the transverse coordinate axis. Finally, $s(z')$ is longitudinal coordinate along the trajectory. The electron is moving with velocity $\vec{v}$, whose magnitude is constant and equal to $v = ds/dt$.

Eq. (14) can be used to characterize the far field from an electron moving on any trajectory. When the single-electron fields are specified at a certain position $z_1$, the fields at any other position $z_2$ can be found by propagating forward or backward in free-space according to the paraxial law

$$\widetilde{E}\left(\vec{\eta},\vec{l},z_2,\vec{r}_2,\omega\right) = \frac{i\omega}{2\pi c(z_2-z_1)} \int d\vec{r}_1\ \widetilde{E}\left(\vec{\eta},\vec{l},z_1,\vec{r}_1,\omega\right) \exp\left[\frac{i\omega\left|\vec{r}_2-\vec{r}_1\right|^2}{2c(z_2-z_1)}\right]. \tag{16}$$

In particular, one may decide to backpropagate the field even at positions well inside the magnetic structure under study. In this case, the field distribution is obviously virtual in nature, because it is not actually there, but it fully characterizes the radiation field from a single electron with given offset and deflection. Within the paraxial approximation, single-electron fields are fully characterized when they are known on a transverse plane at one arbitrary position $z$. Because of this, all positions $z$ are actually equivalent. As we will see there can be, however, a privileged position $z = z_s$ where the electric field assumes a particularly simple form: at this position, in many cases of practical interest including undulator and bending magnet radiation, the field wavefront from a single electron is simply 'plane' [5]. With this expression we mean, with some abuse of language that the phase curvature at the plane $z = z_s$ is zero. Without loss of generality one can set $z_s = 0$ for simplicity and call this the source position. Then, the relation between the field from a single electron at the source $\widetilde{E}\left(\vec{\eta},\vec{l},0,\vec{r},\omega\right)$ and the field in the far zone, $\widetilde{E}\left(\vec{\eta},\vec{l},z_0,\vec{\theta},\omega\right)$, follows once more from Eq. (16):

$$\widetilde{E}\left(\vec{\eta},\vec{l},0,\vec{r},\omega\right) = \frac{iz_0\omega}{2\pi c} \int d\vec{\theta}\ \widetilde{E}\left(\vec{\eta},\vec{l},z_0,\vec{\theta},\omega\right) \exp\left(-\frac{i\theta^2 z_0\omega}{2c}\right) \exp\left(\frac{i\omega\vec{r}\cdot\vec{\theta}}{c}\right) \tag{17}$$

[5] In an undulator this position is just in the middle of the setup.

$$\widetilde{E}\left(\vec{\eta},\vec{l},z_0,\vec{\theta},\omega\right) = \frac{i\omega}{2\pi c z_0}\exp\left(\frac{i\theta^2 z_0\omega}{2c}\right)\int d\vec{r}\,\widetilde{E}\left(\vec{\eta},\vec{l},0,\vec{r},\omega\right)\exp\left(-\frac{i\omega\vec{r}\cdot\vec{\theta}}{c}\right) \tag{18}$$

We assume that a plane wave traveling along the positive $z$-axis can be expressed as in Eq. (12). Then, the negative sign in the exponential factor $\exp(-i\omega z/c)$ in Eq. (13) determines the sign of the exponential in Eq. (16) and consequently the sign of the exponential that appears in the integrand in Eq. (18), which is the solution of the propagation problem in the far zone.

There are practical situations when offset and deflection of an electron lead to the same offset and deflection of the radiation beam from that electron. This is the case for an undulator setup without electron beam focusing. In such situation the expression for the distribution of coherent undulator radiation can be greatly simplified. The presence of an electron offset $\vec{l}$ shifts the single-electron field source, while a deflection $\vec{\eta}$ tilts the source. Therefore

$$\widetilde{E}\left(\vec{l},\vec{\eta},0,\vec{r},\omega\right) = \widetilde{E}_0\left(\vec{r}-\vec{l}\right)\exp\left[i\omega\vec{\eta}\cdot\vec{r}/c\right] \tag{19}$$

where we set $\widetilde{E}_0\left(\vec{r}\right) \equiv \widetilde{E}\left(0,0,0,\vec{r},\omega\right)$.

### 3.2 Undulator elementary source

We consider a planar undulator, so that the transverse velocity of an electron can be written as

$$\vec{v}_\perp(z) = -\frac{cK}{\gamma}\sin\left(k_w z\right)\vec{e}_x\ , \tag{20}$$

where $k_w = 2\pi/\lambda_w$ with $\lambda_w$ the undulator period and $K$ the undulator parameter

$$K = \frac{\lambda_w e H_w}{2\pi m_\mathrm{e} c^2}\ , \tag{21}$$

$m_\mathrm{e}$ being the electron mass and $H_w$ being the maximum of the magnetic field produced by the undulator on the $z$ axis.

We will assume, for simplicity, that the resonance condition with the fundamental harmonic is satisfied. In this way, our treatment leads to an analytical description of undulator radiation at the source position, i.e. in the middle of the undulator, at $z = 0$. The resonance condition with the fundamental harmonic is given by

$$\frac{\omega}{2\gamma^2 c}\left(1+\frac{K^2}{2}\right)=\frac{2\pi}{\lambda_w} \ . \tag{22}$$

A well-known expression for the angular distribution of the first harmonic field in the far-zone (see Appendix A for a detailed derivation) can be obtained from Eq. (14). Such expression is axis-symmetric, and can therefore be presented as a function of a single observation angle $\theta$, where

$$\theta^2 = \theta_x^2 + \theta_y^2 \ , \tag{23}$$

$\theta_x$ and $\theta_y$ being angles measured from the undulator $z$-axis in the horizontal and in the vertical direction. One obtains the following distribution for the slowly varying envelope of the electric field:

$$\widetilde{E}(z_0,\theta) = -\frac{K\omega e L}{2c^2 z_0 \gamma} A_{JJ} \exp\left[i\frac{\omega z_0}{2c}\theta^2\right] \mathrm{sinc}\left[\frac{\omega L \theta^2}{4c}\right] \ , \tag{24}$$

where the field is polarized in the horizontal direction. Here $L = \lambda_w N_w$ is the undulator length and $N_w$ the number of undulator periods. Finally, $A_{JJ}$ is defined as

$$A_{JJ} = J_o\left(\frac{K^2}{4+2K^2}\right) - J_1\left(\frac{K^2}{4+2K^2}\right) \ , \tag{25}$$

$J_n$ being the n-th order Bessel function of the first kind. Eq.(24) describes a field with spherical wavefront centered in the middle of the undulator. Eq. (17) can now be used to calculate the field distribution at the virtual source yielding

$$\widetilde{E}(0,r) = i\frac{K\omega e}{2c^2\gamma} A_{JJ}\left[\pi - 2\mathrm{Si}\left(\frac{\omega r^2}{Lc}\right)\right] \ , \tag{26}$$

where $\mathrm{Si}(z) = \int_0^z dt \sin(t)/t$ indicates the sin integral function and $r = |\vec{r}|$ is the distance from the $z$ axis on the virtual-source plane. Note that $\widetilde{E}(0,r)$ is axis-symmetric. Eq. (26) describes a virtual field with a plane wavefront. Let us compare this virtual field with a laser-beam waist. In laser physics, the waist is located in the center of the optical cavity. In analogy with this, in our case the virtual source is located in the center of the undulator. Both in laser physics and in our situation the waist has a plane wavefront and the transverse dimension of the waist is much longer than the wavelength. Note that the phase of the wavefront in Eq. (26) is shifted by $-\pi/2$ with respect to the spherical wavefront in the far zone. Such phase shift is analogous to the Guoy phase shift in laser physics.

Eq. (24) and Eq. (26) can be generalized to the case of a particle with a given offset $\vec{l}$ and deflection angle $\vec{\eta}$ with respect to the longitudinal axis, assuming that the magnetic field in the undulator is independent of the transverse coordinate of the particle. Although this can be done using Eq. (14) directly, it is sometimes possible to save time by getting the answer with some trick. For example, in the undulator case one takes advantage of the following geometrical considerations, which are in agreement with rigorous mathematical derivation. First, we consider the effect of an offset $\vec{l}$ on the transverse plane, with respect to the longitudinal axis $z$. Since the magnetic field experienced by the particle does not change, the far-zone field is simply shifted by a quantity $\vec{l}$. Eq. (24), can be immediately generalized by systematic substitution of the transverse coordinate of observation, $\vec{r}_0$ with $\vec{r}_0 - \vec{l}$. This means that $\vec{\theta} = \vec{r}_0/z_0$ must be substituted by $\vec{\theta} - \vec{l}/z_0$, thus yielding

$$\widetilde{E}\left(z_0, \vec{l}, \vec{\theta}\right) = -\frac{K\omega e L}{2c^2 z_0 \gamma} A_{JJ} \exp\left[i\frac{\omega z_0}{2c}\left|\vec{\theta} - \frac{\vec{l}}{z_0}\right|^2\right] \mathrm{sinc}\left[\frac{\omega L\left|\vec{\theta} - \left(\vec{l}/z_0\right)\right|^2}{4c}\right] . \quad (27)$$

Let us now discuss the effect of a deflection angle $\vec{\eta}$. Since the magnetic field experienced by the electron is assumed to be independent of its transverse coordinate, the trajectory followed is still sinusoidal, but the effective undulator period is now given by $\lambda_w/\cos(\eta) \simeq (1 + \eta^2/2)\lambda_w$. This induces a relative red shift in the resonant wavelength $\Delta\lambda/\lambda \sim \eta^2/2$. In practical cases of interest we may estimate $\eta \sim 1/\gamma$. Then, $\Delta\lambda/\lambda \sim 1/\gamma^2$ should be compared with the relative bandwidth of the resonance, that is $\Delta\lambda/\lambda \sim 1/N_w$, $N_w$ being the number of undulator periods. For example, if $\gamma > 10^3$, the red shift due to the deflection angle can be neglected in all situations of practical relevance. As a result, the introduction of a deflection angle only amounts to a rigid rotation of the entire system. Performing such rotation we should account for the fact that the phase factor in Eq. (27) is indicative of a spherical wavefront propagating outwards from position $z = 0$ and remains thus invariant under rotations. The argument in the sinc($\cdot$) function in Eq. (27), instead, is modified because the rotation maps the point $(z_0, 0, 0)$ into the point $(z_0, -\eta_x z_0, -\eta_y z_0)$. As a result, after rotation, Eq. (27) transforms to

$$\widetilde{E}\left(z_0,\vec{\eta},\vec{l},\vec{\theta}\right) = -\frac{K\omega e L A_{JJ}}{2c^2 z_0\gamma}\exp\left[i\frac{\omega z_0}{2c}\left|\vec{\theta}-\frac{\vec{l}}{z_0}\right|^2\right]\exp\left(\frac{i\omega}{c}\vec{\eta}\cdot\vec{l}\right)$$
$$\times\mathrm{sinc}\left[\frac{\omega L\left|\vec{\theta}-\left(\vec{l}/z_0\right)-\vec{\eta}\right|^2}{4c}\right] \tag{28}$$

Here we note the important fact that the exponent $\exp(i\omega\vec{\eta}\cdot\vec{l})$ of Eq. (28) is the exponent $\exp(i\omega s(z)/v)$ at $z=0$ of Eq.(15). We already know the meaning of this phase factor and why it answers the questions which we posed in our discussion about the effect of aberration of light in the last section. Finally, in the far-zone case, we can always work in the limit for $l/z_0 \ll 1$, that allows one to neglect the term $\vec{l}/z_0$ in the argument of the sinc($\cdot$) function, as well as the quadratic term in $\omega l^2/(2cz_0)$ in the phase. Thus Eq. (28) can be further simplified, giving the generalization of Eq. (24) in its final form:

$$\widetilde{E}\left(z_0,\vec{\eta},\vec{l},\vec{\theta}\right) = -\frac{K\omega e L A_{JJ}}{2c^2 z_0\gamma}\exp\left[i\frac{\omega}{c}\left(\frac{z_0\theta^2}{2}+(\vec{\eta}-\vec{\theta})\cdot\vec{l}\right)\right]$$
$$\times\mathrm{sinc}\left[\frac{\omega L\left|\vec{\theta}-\vec{\eta}\right|^2}{4c}\right]. \tag{29}$$

The expression for the field at virtual source, Eq. (26), should be modified accordingly. Namely, one has to plug Eq. (29) into Eq. (17), which gives

$$\widetilde{E}\left(0,\vec{\eta},\vec{l},\vec{r}\right) = -\frac{iK\omega^2 e L A_{JJ}}{4\pi c^3\gamma}\int d\vec{\theta}\exp\left[i\frac{\omega}{c}\left[\vec{\theta}\cdot\left(\vec{r}-\vec{l}\right)+\vec{\eta}\cdot\vec{l}\right]\right]$$
$$\times\mathrm{sinc}\left[\frac{\omega L\left|\vec{\theta}-\vec{\eta}\right|^2}{4c}\right] \tag{30}$$

yielding

$$\widetilde{E}\left(0,\vec{\eta},\vec{l},\vec{r}\right) = i\frac{K\omega e}{2c^2\gamma}A_{JJ}\exp\left[i\frac{\omega}{c}\vec{\eta}\cdot\vec{r}\right]\left[\pi-2\mathrm{Si}\left(\frac{\omega\left|\vec{r}-\vec{l}\right|^2}{Lc}\right)\right] \tag{31}$$

as final result. The meaning of Eq. (31) is that offset and deflection of the single electron motion with respect to the longitudinal axis of the system result in a transverse shift and a tilting of the waist plane. The combination $(\vec{r} - \vec{l}\,)$ in Eq. (31) describes the shift, while the phase factor represents the tilting of the waist plane.

To sum up, the diffraction size of the undulator radiation beam is about $\sqrt{\lambdabar L} \gg \lambdabar$. This means that the radiation from an ultra-relativistic electron can be interpreted as generated from a virtual source, which produces a laser-like beam. In principle, such virtual source can be positioned everywhere down the beam, but there is a particular position where it is similar, in many aspects, to the waist of a laser beam. In the case of an undulator this location is the center of the insertion device, where virtual source exhibits a plane wavefront. For a particle moving on-axis, the field amplitude distribution at the virtual source is axially symmetric, Eq. (26). When the particle offset is different from zero, the laser-like beam is shifted. When the particle also has a deflection, the laser-like beam is tilted, but the wavefront remains plane.

### *3.3 Modulated electron beam with amplitude front-tilt in an undulator*

Coherent radiation from a kicked modulated electron beam is a coherent collection of laser-like beams with different offsets and the same deflection. Let us consider an electron beam with a given gradient profile of the charge density. To be specific, we write down an explicit expression for the case of a Gaussian transverse distribution:

$$-ev_z n_0(l_x, l_y) = -\frac{I_0 a_m}{2\pi\sigma^2} \exp\left(-\frac{l_x^2 + l_y^2}{2\sigma^2}\right) , \qquad (32)$$

where $I_0$ is the total beam current and $a_m$ is the modulation strength.

A subject of particular interest is the angular distribution of the radiation intensity , which will be denote $I$. All we need to do in order to find the intensity $I$ is to integrate Eq. (29) over the electron offset distribution in Eq. (32). Upon introduction of normalized quantities $\vec{\hat{\theta}} = \vec{\theta}\sqrt{\omega L/c}$, $\vec{\hat{\eta}} = \vec{\eta}\sqrt{\omega L/c}$ and of the Fresnel number $N = \omega\sigma^2/(cL)$ one obtains

$$I(\vec{\hat{\theta}}, \vec{\hat{\eta}}) = \mathrm{const} \times \exp\left[-N \mid \vec{\hat{\theta}} - \vec{\hat{\eta}} \mid^2\right] \mathrm{sinc}^2\left[\frac{1}{4} \mid \vec{\hat{\theta}} - \vec{\hat{\eta}} \mid^2\right] . \qquad (33)$$

This expression shows that the radiation intensity in the kicked direction $\vec{\hat{\eta}}$ is the same as the intensity without kick, i.e. at zero angle $\vec{\eta} = 0$. The foregoing microscopic analysis in the laboratory frame leads to the same conclusion as before: there is no kick influence on the radiation efficiency.

Let us review what we have found. In macroscopic electrodynamics one is always considering electrons that constitute the source of electromagnetic field as a fluid. In a hydrodynamical analysis of such electron fluid one may choose two approaches, one known as Lagrangian, the other as Eulerian. In the left part of Fig. 1 the evolution of the electron beam is described in the Lagrangian formulation. This approach consists in 'labeling' the particles, following them one by one, and calculating their contribution as they traverse various regions of the system. The variables of interest are the coordinates of the individual particles. Downstream of the kicker one discusses about the microbunch front tilt, that is a plane of simultaneous events, and about the motion of the particles along this plane. In the right part of Fig. 1 the electron beam evolution is described, instead, according to the Eulerian formulation. In this approach, one starts by defining certain field quantities as a functions of spatial coordinates and time. The evolution of these field quantities is described by partial derivatives, and the tilt phenomenon does not exist at all. This is an example of different physical viewpoints and different mathematical formulations that are all equivalent to one another. It must be clearly understood that in the right part of the wave equation in macroscopic electrodynamics, sources must be field variables, exactly like the electromagnetic fields are. In the Lagrangian formulation, upstream of the kicker there is no particle motion along the plane of simultaneity and there is no need for translation to the Eulerian formulation either. Downstream of the kicker, only the Eulerian formulation can be used in the electrodynamical part of the problem, and in particular for calculating coherent undulator radiation emission.

## 4 Conclusions

We showed that the microbunch front tilt in an XFEL is related to a conventional choice. Also the concept of simultaneity between two events in different places includes a certain degree of conventionality, and the case of the microbunch front tilt in an ultrarelativistic electron beam seems a particular case of this.

The microbunch front tilt is conventional in the same sense in which the gauge freedom that arises in field theory makes the choice between Coulomb and Lorentz gauges conventional: a study of mucrobunching phenomena can never lead to a determination of the influence of the microbunch front tilt. This viewpoint is in contradiction with results in literature that miss a non-trivial but important kinematical aspect in the description of coherent undulator radiation of a modulated electron beam with an amplitude front tilt. In fact, according to conventional treatments, coherent radiation is emitted along the normal to the microbunching wavefront. We explained this point phenomenologically on the basis of the effect of aberration of light in the framework of the theory of special relativity. However, such an approach can lead the reader to believe that a description of aberration effects in undulator radiation is impossible without the use of the special theory of relativity. In contrast to this, we showed that the effect of aberration of light can be simply explained in the laboratory frame of reference, and there is no need to use the laws of relativistic kinematics.

The problem of electromagnetic wave amplification in an undulator refers to a class of self-consistent problems. It can be separated into two parts. First, the solution of the dynamical problem, i.e. finding the motion of particles under the action of a given electromagnetic fields. Second, the solution of the electrodynamic problem, i.e. finding the electromagnetic fields generated by a given distribution of charges and currents. To close the problem, the field equations and the equations of motion should be solved simultaneously. In this away, computers allow one to perform simulations of the XFEL process. However, these simulations are not a 'first principle' or 'full physics' description of the XFEL process. They are undoubtedly very important scientific tools, but they follow specific models. The correctness of their outcome is related to the correctness of these models, meaning that the validity of simulations should always be cross-checked with analytical results. By considering the case in which a microbunch front tilt is present, it can be seen that there are difficulties with the conventional coupling of fields and particles in XFEL codes. The performance of an XFEL can be reduced if any of a number of undulator parameters deviates from its optimal value. Analysis of the effect of trajectory errors on the XFEL amplification process showed that they become more and more important at shorter wavelengths. Previous numerical studies of this critical aspect in the design of a XFEL sources are, however, based on a kinematically incorrect model. Therefore, the tolerances predicted are more stringent than they need be. This can be considered one of the reason for the exceptional progress in Angstrom XFEL developments over the last decade.

## Appendix. Undulator radiation in resonance approximation. Far zone.

In this appendix we present a simple derivation of the frequency representation of the radiated field produced by an electron in an undulator. For the electron transverse velocity we assume

$$v_x(z) = -c\theta_s \sin(k_w z) = -\frac{c\theta_s}{2i}\left[\exp(ik_w z) - \exp(-ik_w z)\right] . \qquad (34)$$

Here $k_w = 2\pi/\lambda_w$, and $\lambda_w$ is the undulator period. Moreover, $\theta_s = K/\gamma$, where $K$ is the deflection parameter defined as

$$K = \frac{e\lambda_w H_w}{2\pi m_e c^2} , \qquad (35)$$

$m_e$ being the electron mass at rest and $H_w$ being the maximal magnetic field of the undulator on axis.

We write the undulator length as $L = N_w\lambda_w$, where $N_w$ is the number of undulator periods. With the help of Eq. (14) we obtain an expression, valid in the far zone at frequency $\omega$ and observation angle $\vec{\theta}$:

$$\vec{\bar{E}} = \frac{i\omega e}{c^2 z_0}\int_{-L/2}^{L/2} dz' \exp\left[i\Phi_T\right]\exp\left[i\frac{\omega\theta^2 z_0}{2c}\right]\left[\frac{K}{\gamma}\sin\left(k_w z'\right)\vec{e}_x + \vec{\theta}\right] . \qquad (36)$$

Here

$$\Phi_T = \left(\frac{\omega}{2c\bar{\gamma}_z^2} + \frac{\omega\theta^2}{2c}\right)z' - \frac{K\theta_x}{\gamma}\frac{\omega}{k_w c}\cos(k_w z') - \frac{K^2}{8\gamma^2}\frac{\omega}{k_w c}\sin(2k_w z') , \qquad (37)$$

where the average longitudinal Lorentz factor $\bar{\gamma}_z$ is defined as

$$\bar{\gamma}_z = \frac{\gamma}{\sqrt{1+K^2/2}} . \qquad (38)$$

The choice of the integration limits in Eq. (36) implies that the reference system has its origin in the center of the undulator.

Usually, it does not make sense to calculate the intensity distribution from Eq. (36) alone, without extra-terms (both interfering and not) from the other parts of the electron trajectory. This means that one should have complete information about the electron trajectory and calculate extra-terms to be added to Eq. (36) in order to have the total field from a given setup. Yet, we can find *particular situations* for which the contribution from Eq. (36) is dominant with respect to others. In this case Eq. (36), alone, has independent physical meaning.

One of these situations is when the resonance approximation is valid. This approximation does not replace the paraxial one, based on $\gamma^2 \gg 1$, but it is used together with it. It takes advantage of another parameter that is usually large, i.e. the number of undulator periods $N_w \gg 1$. In this case, the integral in $dz'$ in Eq. (36) exhibits simplifications, independently of the frequency of interest due to the long integration range with respect to the scale of the undulator period.

In all generality, the field in Eq. (36) can be written as

$$\vec{\widetilde{E}} = \exp\left[i\frac{\omega\theta^2 z_0}{2c}\right]\frac{i\omega e}{c^2 z_0} \times \int_{-L/2}^{L/2} dz' \left\{\frac{K}{2i\gamma}\left[\exp\left(2ik_w z'\right) - 1\right]\vec{e}_x + \vec{\theta}\exp\left(ik_w z'\right)\right\} \times \exp\left[i\left(C + \frac{\omega\theta^2}{2c}\right)z' - \frac{K\theta_x}{\gamma}\frac{\omega}{k_w c}\cos(k_w z') - \frac{K^2}{8\gamma^2}\frac{\omega}{k_w c}\sin(2k_w z')\right] . \tag{39}$$

Here $\omega = \omega_r + \Delta\omega$, $C = k_w \Delta\omega/\omega_r$ and

$$\omega_r = 2k_w c\bar{\gamma}_z^2 , \tag{40}$$

is the fundamental resonance frequency.

Using the Anger-Jacobi expansion:

$$\exp\left[ia\sin(\psi)\right] = \sum_{p=-\infty}^{\infty} J_p(a)\exp\left[ip\psi\right] , \tag{41}$$

where $J_p(\cdot)$ indicates the Bessel function of the first kind of order $p$, to write the integral in Eq. (39) in a different way:

$$\vec{\widetilde{E}} = \exp\left[i\frac{\omega\theta^2 z_0}{2c}\right]\frac{i\omega e}{c^2 z_0}\sum_{m,n=-\infty}^{\infty} J_m(u)J_n(v)\exp\left[\frac{i\pi n}{2}\right]$$

$$\times \int_{-L/2}^{L/2} dz' \exp\left[i\left(C + \frac{\omega\theta^2}{2c}\right)z'\right]$$
$$\times \left\{\frac{K}{2i\gamma}\left[\exp\left(2ik_w z'\right) - 1\right]\vec{e}_x + \vec{\theta}\exp\left(ik_w z'\right)\right\}\exp\left[i(n+2m)k_w z'\right] , \quad (42)$$

where [6]

$$u = -\frac{K^2\omega}{8\gamma^2 k_w c} \quad \text{and} \quad v = -\frac{K\theta_x\omega}{\gamma k_w c} . \quad (43)$$

Up to now we just re-wrote Eq. (36) in a different way. Eq. (36) and Eq. (42) are equivalent. The definition of the detuning parameter $C$ is suited to investigate frequencies around the fundamental harmonic but no approximation is taken besides the paraxial approximation.

Whenever

$$C + \frac{\omega\theta^2}{2c} \ll k_w , \quad (44)$$

the first phase term in $z'$ under the integral sign in Eq. (42) is varying slowly on the scale of the undulator period $\lambda_w$. As a result, simplifications arise when $N_w \gg 1$, because fast oscillating terms in powers of $\exp[ik_w z']$ effectively average to zero. When these simplifications are taken, resonance approximation is applied, in the sense that one exploits the large parameter $N_w \gg 1$. This is possible under condition (44). Note that (44) restricts the range of frequencies for positive values of $C$ independently of the observation angle $\theta$, but for any value $C < 0$ (i.e. for wavelengths longer than $\lambdabar_r = c/\omega_r$) there is always some range of $\theta$ such that Eq. (44) can be applied. Altogether, application of the resonance approximation is possible for frequencies around $\omega_r$ and lower than $\omega_r$. Once any frequency is fixed, (44) poses constraints on the observation region where the resonance approximation applies. Similar reasonings can be done for frequencies around higher harmonics with a more convenient definition of the detuning parameter $C$.

Within the resonance approximation we further select frequencies such that

$$\frac{|\Delta\omega|}{\omega_r} \ll 1 , \quad \text{i.e. } |C| \ll k_w . \quad (45)$$

[6] Here the parameter $v$ should not be confused with the velocity.

Note that this condition on frequencies automatically selects observation angles of interest $\theta^2 \ll 1/\gamma_z^2$. In fact, if one considers observation angles outside the range $\theta^2 \ll 1/\gamma_z^2$, condition (44) is not fulfilled, and the integrand in Eq. (42) exhibits fast oscillations on the integration scale $L$. As a result, one obtains zero transverse field, $\vec{\widetilde{E}} = 0$, with accuracy $1/N_w$. Under the constraint imposed by (45), independently of the value of $K$ and for observation angles of interest $\theta^2 \ll 1/\gamma_z^2$, we have

$$|v| = \frac{K|\theta_x|}{\gamma}\frac{\omega}{k_w c} = \left(1+\frac{\Delta\omega}{\omega_r}\right)\frac{2\sqrt{2}K}{\sqrt{2+K^2}}\bar{\gamma}_z|\theta_x| \lesssim \bar{\gamma}_z|\theta_x| \ll 1 \; . \tag{46}$$

This means that, independently of $K$, $|v| \ll 1$ and we may expand $J_n(v)$ in Eq. (42) according to $J_n(v) \simeq [2^{-n}/\Gamma(1+n)]\, v^n$, $\Gamma(\cdot)$ being the Euler gamma function

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} \exp[-t] \; . \tag{47}$$

Similar reasonings can be done for frequencies around higher harmonics with a different definition of the detuning parameter $C$. However, around odd harmonics, the before-mentioned expansion, together with the application of the resonance approximation for $N_w \gg 1$ (fast oscillating terms in powers of $\exp[ik_w z']$ effectively average to zero), yields extra-simplifications.

Here we are dealing specifically with the first harmonic. Therefore, these extra-simplifications apply. We neglect both the term in $\cos(k_w z')$ in the phase of Eq. (39) and the term in $\vec{\theta}$ in Eq. (39). First, non-negligible terms in the expansion of $J_n(v)$ are those for small values of $n$, since $J_n(v) \sim v^n$, with $|v| \ll 1$. The value $n = 0$ gives a non-negligible contribution $J_0(v) \sim 1$. Then, since the integration in $dz'$ is performed over a large number of undulator periods $N_w \gg 1$, all terms of the expansion in Eq. (42) but those for $m = -1$ and $m = 0$ average to zero due to resonance approximation. Note that surviving contributions are proportional to $K/\gamma$, and can be traced back to the term in $\vec{e}_x$ only, while the term in $\vec{\theta}$ in Eq. (42) averages to zero for $n = 0$. Values $n = \pm 1$ already give negligible contributions. In fact, $J_{\pm 1}(v) \sim v$. Then, the term in $\vec{e}_x$ in Eq. (42) is $v$ times the term with $n = 0$ and is immediately negligible, regardless of the values of $m$. The term in $\vec{\theta}$ would survive averaging when $n = 1$, $m = -1$ and when $n = -1$, $m = 0$. However, it scales as $\vec{\theta}v$. Now, using condition (45) we see that, for observation angles of interest $\theta^2 \ll 1/\gamma_z^2$, $|\vec{\theta}|\,|v| \sim (\sqrt{2}\,K / \sqrt{2+K^2}\,)\,\bar{\gamma}_z\theta^2 \ll K/\gamma$. Therefore, the term in $\vec{\theta}$ is negligible with respect to the term in $\vec{e}_x$ for $n = 0$, that scales as $K/\gamma$. All terms corresponding to larger values of $|n|$ are negligible.

Summing up, all terms of the expansion in Eq. (41) but those for $n = 0$ and $m = -1$ or $m = 0$ give negligible contribution. After definition of

$$A_{JJ} = J_0\left(\frac{\omega K^2}{8k_w c\gamma^2}\right) - J_1\left(\frac{\omega K^2}{8k_w c\gamma^2}\right) , \tag{48}$$

that can be calculated at $\omega = \omega_r$ since $|C| \ll k_w$, we have

$$\vec{\widetilde{E}} = -\frac{K\omega e}{2c^2 z_0\gamma} A_{JJ} \exp\left[i\frac{\omega\theta^2 z_0}{2c}\right] \int_{-L/2}^{L/2} dz' \exp\left[i\left(C + \frac{\omega\theta^2}{2c}\right)z'\right] \vec{e}_x , \tag{49}$$

yielding the well-known free-space field distribution:

$$\vec{\widetilde{E}}(z_0, \vec{\theta}) = -\frac{K\omega eL}{2c^2 z_0\gamma} A_{JJ} \exp\left[i\frac{\omega\theta^2 z_0}{2c}\right] \mathrm{sinc}\left[\frac{L}{2}\left(C + \frac{\omega\theta^2}{2c}\right)\right] \vec{e}_x , \tag{50}$$

where $\mathrm{sinc}(\cdot) \equiv \sin(\cdot)/(\cdot)$. Therefore, the field is horizontally polarized and azimuthal symmetric.


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