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Rényi entropy and conformal defects.

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Abstract: We propose a field theoretic framework for calculating the dependence of Rényi entropies on the shape of the entangling surface in a conformal field theory. Our approach rests on regarding the corresponding twist operator as a conformal defect and in particular, we define the displacement operator which implements small local deformations of the entangling surface. We conjecture a simple relation between the coefficient defining the two-point function of the displacement operator and the conformal weight of the twist operator. With this approach, we are able to consolidate a number of apparently distinct conjectures found in the literature about the Rényi entropy. In particular, we examine a conjecture regarding the universal coefficient associated with a conical singularity in the entangling surface for CFTs in any number of spacetime dimensions. We also provide a general formula for the second order variation of the Rényi entropy arising from small deformations of a spherical entangling surface, extending Mezei's results for the entanglement entropy.

Keywords: Rényi entropy, twist operator, displacement operator, conformal defect.

Contents

1. Introduction

There has been a growing interest in entanglement and Rényi entropies as a probe of complex interacting quantum systems in a variety of areas ranging from condensed matter physics, *e.g.,* [1, 2, 3] to quantum gravity, *e.g.,* [4, 5, 6, 7]. While commonly regarded as a useful theoretical diagnostic, the past year has seen remarkable experimental advances where the Rényi entropy, as well as quantum purity and mutual information, of a system of delocalized interacting particles can be measured in the laboratory [8]. This experimental breakthrough strengthens the motivation to develop further theoretical insight into these entanglement measures, particularly in the framework of quantum field theory (QFT). In this paper, we focus our attention on Rényi entropies [9, 10] in the context of conformal field theories (CFTs).

Conformal symmetry obviously introduces additional constraints on the Rényi entropy beyond those in a general QFT, however, CFTs are still a very important class of QFTs since they describe physics at the quantum critical points and also shed light on the structure of gravity through the AdS/CFT duality.

In the case of holographic CFTs with a gravitational dual, the Ryu-Takayanagi prescription [11, 12] — and its generalizations [13, 14, 15, 16, 17] — provides an elegant and practical tool to evaluate the entanglement entropy across an arbitrary entangling surface. While the recent derivation [18] of this prescription presents a generalization to holographic Rényi entropies in principle, explicit holographic calculations of the Rényi entropy have been restricted to a spherical entangling surface [19, 20]. Similarly, efficient computational tools to study the Rényi entropies for more general CFTs are rare. Numerical techniques have been developed to evaluate the Rényi entropy in lattice models describing critical theories, *e.g.,* [21, 22, 23] but these are demanding and must be adapted for the specifics of a given model. Beyond these numerical studies, the existing literature considers primarily the Rényi entropy across a spherical entangling surface for a CFT living in flat space, *e.g.,* [19, 24, 25, 26, 27, 28].

In this paper, we build on conformal defects techniques to develop a field theoretic framework which allows for quantitative studies of the Rényi entropy. Conformal defects have a long story, both in two and higher dimensions - see e.g. [29, 30, 31]. In section 2, we begin from the basic definitions to draw a parallel between conformal defects and the twist operators, which enter the calculation of the Rényi entropy. This perspective demonstrates that the Rényi entropy readily lends itself to the application of defect CFT techniques. In particular, we define the so-called displacement operator for the twist operators, which then implements small deformations of the entangling surface. One of our key results will be a conjecture relating the coefficient defining the two-point function of the displacement operator to the conformal weight of the corresponding twist operator.

The displacement operator can be used for perturbative calculations of the Rényi entropy when small modifications are made in the geometry of the entangling surface. In particular, for a planar or spherical entangling surface, we show that the second order variations are fixed by the two-point function of the displacement operator. In section 3, we apply this approach to evaluate the second order variation of the Rényi entropy for a spherical region. In the limit that the Rényi index goes to one, we precisely recover Mezei's conjecture for variations of the entanglement entropy [32] — see also [33]. Further the displacement operator can also be used to examine the variation of the Rényi entropy for small but 'singular' deformations of the entangling surface. In section 4, we consider the case of a planar entangling surface which undergoes a singular deformation to create a small conical singularity. Our result for the change in the Rényi entropy matches previous conjectures with regards to cusp and cone geometries in the limit that the entangling surface is almost smooth [34, 35, 36, 37]. In section 5, we focus on the Rényi entropy across an arbitrary entangling surface in four spacetime dimensions. We are able to relate two coefficients in the universal contribution to the Rényi entropy to the conformal weight of the twist operator and to the coefficient in the two-point function of the displacement operator, respectively. Our relation between the latter two quantities then yields the equality of these coefficients, as was conjectured for all four-dimensional CFTS in [38]. Hence, interestingly, the relation which we conjecture between the coefficient of the two-point function of the displacement and the conformal weight of the twist operator underlies a number of existing conjectures in the literature about the Rényi entropy.

In section 6, we pose the question whether it may be possible to define the twist operator through the Operator Product Expansions generically available in the presence of defects, and we point out an intriguing universal feature of the fusion of the stress-tensor with this specific extended operator.

We conclude with a discussion of our results and possible future directions in section 7. We also have a number appendices where we present various technical calculations whose results are used in the main text. In Appendix A, we derive a set of Ward identities in the presence of a twist operator. In Appendix B, we list some useful formulas which describe variation of various geometrical objects around the flat space. We devote Appendix C to the example of a four dimensional free-scalar, where the displacement operator can be given a precise identity in terms of the elementary field.

Finally, let us add that while this paper was in the final stages of preparation, ref. [39] appeared. Although their discussion only considers the entanglement entropy, some of the results overlap with aspects of the present paper.

2. Twist operators as conformal defects

An central object for our discussion will be the twist operator which naturally arises in evaluating Rényi entropies in quantum field theory [26, 40]. Therefore, let us start by recalling the definition of our main player. We begin with a generic QFT in flat d-dimensional spacetime. On a given time slice, the QFT is in a global state described by the density matrix ρ — in fact, shortly we will restrict our attention to the vacuum state. We consider the density matrix ρ_A obtained when the state is restricted to a particular region A, *i.e.,* obtained by tracing over the degrees of freedom in the complementary region B of the time slice:

$$\rho_A = tr_B(\rho)\,. \tag{2.1}$$

The one-parameter family of Rényi entropies associated to the reduced density matrix ρ_A is defined as follows [9, 10]:

$$S_n = \frac{1}{1-n}\,\log \mathrm{Tr}(\rho_A^n). \tag{2.2}$$

The entanglement entropy, *i.e.,* the von Neumann entropy, is recovered with the limit:

$$\lim_{n\to 1} S_n = S_{\mathrm{EE}} = -\mathrm{Tr}(\rho_A \log \rho_A)\,. \tag{2.3}$$

Here we have implicitly considered the Rényi index n in (2.2) to be a real number. However, for specifically for integer n (with $n > 1$), a path integral construction, which is widely known as the replica trick, allows us to evaluate the Rényi entropies for a QFT. An analytic continuation is then required to make contact with the entanglement entropy but we will have nothing to add about the conditions under which the continuation is reliable.

The replica trick begins by evaluating the reduced density matrix ρ_A in terms of a (Euclidean) path integral on $\mathbb{R}^d$ but with independent boundary conditions fixed over the region A as it is approached from above and below in Euclidean time, *e.g.,* with $t_{\rm E} \to 0^{\pm}$. The expression $\mathrm{Tr}(\rho_A^n)$ is then evaluated by extending the above to a path integral on a n-sheeted geometry [40], where the consecutive sheets are sewn together on cuts running over A. Denoting the corresponding partition function as Z_n, we can write the Rényi entropy (2.2) as

$$S_n = \frac{1}{1-n}\ \log\frac{Z_n}{Z_1^n}\,, \tag{2.4}$$

where the denominator Z_1^n is introduced here to ensure the correct normalization, *i.e.,* $\mathrm{Tr}[\rho_A] = 1$. The partition function Z_n has an important symmetry. That is, even if in the above construction, we chose to glue the copies together along the codimension-one submanifold A on the $t_{\rm E} = 0$ slice, the precise location of the cut between different sheets is meaningless — see for instance section 3.1 of [41]. Hence the only source of breaking of translational invariance on each sheet is at the location of the entangling surface, *i.e.,* the boundary of A. Since the modification is local in this sense, it can be reinterpreted as the insertion of a *twist operator* τ_n. In defining τ_n, the above construction is replaced by a path integral over n copies of the underlying QFT on a single copy of the flat space geometry. The twist operator is then defined as a codimension-two surface operator in this n-fold replicated QFT, which extends over the entangling surface and whose expectation value yields

$$\langle\tau_n\rangle \equiv \frac{Z_n}{Z_1^n} = e^{(1-n)S_n}. \tag{2.5}$$

Hence eq. (2.5) implies that τ_n opens a branch cut over the region A which then connects consecutive copies of the QFT in the n-fold replicated theory. Note that here and in the following, we omit the A dependence of τ_n to alleviate the notation.

In proceeding, we restrict our attention to the case where the QFT of interest is a conformal field theory and the state is simply the flat space vacuum state. Now let us take a closer look at the residual symmetry group in the presence of the twist operator. In doing so, we restrict ourselves to a very symmetric situation where we choose A to be half of the space. That is, we choose τ_n to lie on a flat $(d–2)$-dimensional plane, which we denote as Σ. For concreteness, we parametrize $\mathbb{R}^d$ with coordinates $(x^1,\dots,x^d)$, and we locate the twist operator at $x^1 = 0 = x^2$. In the following, we will denote directions orthogonal to Σ with Latin indices from the beginning of the alphabet $(a,\, b, \dots)$ and parallel directions with Latin indices from the middle of the alphabet $(i,\ j,\dots)$, while $\mu = (i, a)$. Let us explicitly notice that since a spherical entangling surface can be obtained from the planar one by means of a conformal transformation, the following applies equally well to the spherical case.

Now, the stabilizer of a plane of dimension $(d-2)$ within the d-dimensional conformal algebra is the subalgebra $so(d-1,1) \times u(1)$. The first factor comprises the conformal transformations in $(d-2)$ dimensions, while the second consists of rotations in the transverse space. Let us choose the cut to lie along the half-plane in $\mathbb{R}^d$, *e.g.,* $x^1 < 0$ (and $x^2 = 0$), then a moment's thought is sufficient to realize that the gluing condition is preserved only if the same conformal transformation is applied to all the copies at the same time[1]. The rotations in the transverse plane, on the other hand, move the cut, which can be brought back to the original position through the symmetry of the partition function which we referred to above. This leads to a remark on the structure of the symmetry group. A rotation of an angle 2π has the net effect of shifting by one the labeling of the replicas: in a correlation function, an operator inserted in the i-th copy ends up in the $(i+1)$-th one. Therefore, the $u(1)$ algebra exponentiates in the n-fold cover of the group $O(2)$. Up to this subtlety, we see that the symmetry group preserved by the twist operator is the same as the one preserved by a flat conformally invariant extended operator, *i.e.,* a conformal defect.

The symmetry algebra places constraints on observables, which in many cases have been worked out in the context of defect CFTs. In particular, this is the case for correlation functions of local operators [31]. It is especially interesting to consider the consequences of interactions among replicas, which distinguish the twist operator from a mere local modification of a CFT on $\mathbb{R}^d$: correlation functions of operators placed on different copies of the replicated geometry do not have a counterpart in the latter. However, these configurations do not escape the defect CFT framework and in particular can be handled with the classical tool available in any conformal field theory: the existence of Operator Product Expansions (OPEs), which converge inside correlation functions. In the presence of a defect, bulk excitations can be brought close to the extended operator, and be expressed as a sum over local defect operators. This corresponds to a new OPE channel, usually referred to as the defect OPE. If we denote with a hat a defect operator, the defect OPE of a bulk scalar of scaling dimension Δ takes the following form:

$$O(x^a, x^i) \sim b_{\widehat{O}_0}\, |x^a|^{\widehat{\Delta}_0 - \Delta} \widehat{O}_0(x^i) + \dots \tag{2.6}$$

The meaning of the label 0 given to the defect operator will become clear in a moment. Let us stress that operators in our formulae will always be thought of as inserted on a single copy and when present, sums over replicas will be written explicitly. Now consider a correlator of bulk primaries which belong to different factors of the n-fold replicated CFT. We can substitute to each of them the respective defect OPE, and since the latter converges inside correlation functions, the resulting sum over two-point functions of defect operators must reproduce the original correlator. In particular, we see that the expression on the right hand side of eq. (2.6) must retain the information about the copy in which the primary on the left hand side was inserted. This is possible thanks to the global structure of the symmetry group - that

[1]In the density matrix language, this is rephrased in the statement that the transformation $U\rho_A U^{-1}$ is a symmetry of Z_n only if applied to the n factors of ρ_A appearing in the trace (2.2).

is, the fact that rotations around the defect are combined non-trivially with the $\mathbb{Z}_n$ replica symmetry. The rotational symmetry around an extended operator is a global symmetry from the point of view of the theory on the defect. As a consequence, defect operators carry a $u(1)$ quantum number s. In our case, this transverse spin is rational: $s = k/n$, k being an integer. We see that the defect OPE of a bulk scalar contains in general terms of the form

$$O(x^a, x^i) \sim b_{\widehat{O}_s}\, |x^a|^{\widehat{\Delta}_s - \Delta} e^{is\phi} \widehat{O}_s(x^i) + \dots \tag{2.7}$$

where $\phi \in [0, 2\pi n)$ is the angle in a plane orthogonal to the defect and provides the information about the replica on which the bulk primary has been inserted. In appendix C, we shall see that OPEs of the form (2.7) arise in fact in free theory, and allow to decompose correlation functions of bulk primaries placed in arbitrary positions.

The breaking of translational invariance in the directions transverse to the entangling surface gives rise to an operator of transverse spin $s = 1$, which is always present on top of defects in local theories. Indeed, the Noether current which generates translations fails to be conserved only at the position of the defect, so that a new contact term should be present in the Ward identities of the stress-tensor. This defines the displacement operator D^a:

$$\sum_{\mathrm{m}=1}^{n} \partial_\mu T^{\mu a}_{(\mathrm{m})}(x^\nu) = \delta_\Sigma(x)\ D^a(x^i), \tag{2.8}$$

where the index m runs over the replicas[2] and δ_Σ denotes the delta function in the transverse space with support on the Σ. The sum over replicas appears because, as mentioned, symmetry transformations should be applied to all the sheets in the same way, resulting in a sum over insertions of the stress-tensor. Eq. (2.8) is written in a somewhat loose notation, which highlights the properties of the displacement. The right hand side should be intended as an additional contribution arising when the left hand side is inserted in a correlation function. We refer to appendix A for a derivation, and we content ourselves of making some remarks. It is important that the quantum numbers of D^a are fixed by the Ward identity: its scaling dimension is $\Delta = d - 1$ and it carries one unit of spin under rotations around the defect. Notice that its normalization is also fixed by (2.8). Therefore, its Zamolodchikov norm $C_D(n)$ is a property of the defect under consideration:

$$\langle D^a(x^i)\, D^b(0) \rangle_n = C_D(n)\ \frac{\delta^{ab}}{|x^i|^{2(d-1)}}\,. \tag{2.9}$$

Here and in the following the subscript n applied to expectation values implies the presence of the twist operator:

$$\langle O \rangle_n \equiv \frac{\langle \tau_n\, O \rangle}{\langle \tau_n \rangle}. \tag{2.10}$$

[2] We stress again that in general, our calculations will refer to bulk operators in a single copy of the replicated CFT. Hence $T^{\mu\nu}_{(\mathrm{m})}$ here denotes the stress tensor in the m'th copy of the CFT and the total stress tensor for the full theory would be given by $T^{\mu\nu}_{\mathrm{tot}} = \sum_{\mathrm{m}=1}^{n} T^{\mu\nu}_{(\mathrm{m})}$. However, in order to reduce clutter in expressions below, we will drop the subscript (m) but $T^{\mu\nu}$ still denotes the single-copy stress tensor. The total stress tensor will always be denoted as $T^{\mu\nu}_{\mathrm{tot}}$.

Let us finally mention that generic defects might have a more complicated structure of contact terms showing up in the divergence of the stress-tensor: more operators might be present, associated with derivatives of δ-functions appearing in eq. (2.8). They can be written down systematically [31] but we will not need this information here.

A consequence of the Ward identity (2.8) is that a small deformation $\delta x^a(x^i)$ of the defect, is obtained by integrating the displacement operator in the action. The first order variation under such a deformation can be written

$$\delta \langle X \rangle_n = - \int d^{d-2}x \, \delta x^a(x^i) \, \langle D^a(x^i) X \rangle_n \,. \tag{2.11}$$

where X is an arbitrary product of local operators. As already pointed out, a flat twist operator preserves a subgroup of the conformal transformations which includes dilatations. As an immediate consequence, scale invariance prevents defect operators from acquiring an expectation value. Hence, the first order variation of the partition function (2.5) vanishes for a flat (or spherical) entangling surface, or more precisely, it is non-universal. The second order variation is then related directly to C_D. Indeed, denoting the variation as $\epsilon \delta x^a$, we get

$$\frac{1}{\langle \tau_n \rangle} \frac{d^2}{d\epsilon^2} \langle \tau_n \rangle \bigg|_{\epsilon=0} = \int d^{d-2}x \int d^{d-2}x' \, \langle D^a(x) D^b(x') \rangle_n \, \delta x_a \delta x'_b. \tag{2.12}$$

The double integration is going to contain divergences. However, power-law divergences can be unambiguously tuned away, and finite or logarithmically divergent parts are universal well defined quantities, proportional to C_D.

The key result of this paper is the claim that for the specific case of twist operators, the value of C_D is entirely determined by the one-point function of the stress tensor in presence of the defect, also called the conformal dimension of the twist operator. The latter, dubbed h_n, is defined by the leading singularity of the one-point function $\langle T_{\mu\nu} \rangle_n \equiv \langle T_{\mu\nu} \, \tau_n \rangle / \langle \tau_n \rangle$. For a planar conformal defect in Euclidean flat geometry, this leading singularity is easily identified as it is completely fixed by symmetry

$$\langle T_{ij} \rangle = -\frac{h_n}{2\pi n} \frac{\delta_{ij}}{|x^a|^d} \,, \qquad \langle T_{aj} \rangle = 0 \,, \qquad \langle T_{ab} \rangle = \frac{h_n}{2\pi n} \frac{(d-1)\, \delta_{ab} - d \, n_a n_b}{|x^a|^d} \,. \tag{2.13}$$

Here n_a is a unit normalized vector normal to the entangling surface. The factor n in the denominator appears so that h_n is the coefficient in the one-point function for the total stress tensor (summed over all of the replicas), as defined in *e.g.*, [26]. We conjecture that, for a generic d-dimensional conformal field theory, $C_D(n)$ and h_n are related by the following equality

$$C_D(n) = d \, \Gamma\left(\tfrac{d+1}{2}\right) \left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1} h_n \tag{2.14}$$

One immediate consequence of this relation is $C_D(1) = 0$, as one could guess by the fact that the defect disappears for $n = 1$. If we analytically continue (2.14) to real n, we can consider the first order variation around $n = 1$:

$$\partial_n C_D|_{n=1} = d \, \Gamma\left(\tfrac{d+1}{2}\right) \left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1} \partial_n h_n|_{n=1} = \frac{2\pi^2}{d+1} C_T, \tag{2.15}$$

where we used the relation

$$\partial_n h|_{n=1} = 2\pi^{\frac{d}{2}+1}\frac{\Gamma(\frac{d}{2})}{\Gamma(d+2)}C_T, \tag{2.16}$$

first observed in [20] for holographic theories and then proven in [26] for general CFTs.

This conjecture is supported by performing a number of consistency checks, whose analysis highlights how the relation (2.14) constitutes a unified framework for the various attempts of considering geometrical perturbations to highly symmetric entangling surfaces. Let us list here the cases where we tested the validity of our conjecture either against known relations or by proving its equivalence with other well supported conjectures:

- In section 3, we find that formula (2.15) reproduces the variation of the entanglement entropy across a deformed sphere conjectured in [32] for arbitrary dimensions.[3]
- Equation (2.15) also allows one to compute the universal contribution to the Rényi entropy for an entangling surface with a (hyper)conical singularity of opening angle Ω. The leading coefficient in an expansion around the smooth entangling surface has been conjectured to be related the conformal weight h_n [37] — see also [34, 35, 36]. In section 4 we prove the equivalence of that conjecture and formula (2.15).
- In the four-dimensional case, for arbitrary n, equation (2.14) implies the equivalence of the coefficients $f_b(n)$ and $f_c(n)$ appearing in the universal part of the four-dimensional Rényi entropy, as discussed in [28, 38]. This is demonstrated in section 5 by relating f_b to C_D and f_c to h_n.

3. Rényi and entanglement entropy across a deformed sphere

In this section we study shape dependence of the Rényi entropy for a generic CFT in flat space. In particular, we calculate second order correction to the Rényi entropy induced by small perturbations of a perfect sphere. In the limit $n \to 1$ our findings agree with the holographic results previously found in [32].

Starting from (2.12), we note that the second order correction for a slightly deformed spherical defect is given by (2.12)

$$\delta S_n = \frac{1}{2(1-n)}\int_\Sigma\int_{\Sigma'} \langle D^a(x)D^b(x')\rangle\, \delta x_a\, \delta x'_b + \mathcal{O}(\delta x^3)\ , \tag{3.1}$$

where integrals run over a spherical entangling surface of radius R. We will restrict the deformation to the $t_{\rm E}=0$ time slice and denote $\delta\vec{x} = \epsilon f(x)\hat{r}$ where $\hat{r}$ is a unit vector in the radial direction. The relevant correlator (2.9) then beomes

$$\langle D^r(x)\, D^r(x')\rangle = \frac{C_D}{(x-x')^{2(d-1)}} = \frac{C_D}{(2R^2)^{d-1}}\,\frac{1}{(1-\cos\gamma)^{d-1}}\ , \tag{3.2}$$

[3] This conjecture was recently proven in [39].

with γ being the angle between $x, x' \in S^{d-2}$.

Let us now represent the two-point correlator (3.2) in the basis of spherical harmonics on S^N $(N \equiv d-2)$

$$Y_{\ell_N \ldots \ell_1}(\theta_N \ldots \theta_1) = \frac{1}{\sqrt{2\pi}}\, e^{i\ell_1\theta_1} \prod_{n=2}^{N} {}_n c_{\ell_n}^{\ell_{n-1}} (\sin\theta_n)^{\frac{2-n}{2}} P_{\ell_n + \frac{n-2}{2}}^{-\left(\ell_{n-1}+\frac{n-2}{2}\right)}(\cos\theta_n) \tag{3.3}$$

where $\ell_N \geq \ell_{N-1} \geq \ldots \geq |\ell_1|$ are integers and

$$ds_N^2 = d\theta_N^2 + \sin^2\theta_N ds_{N-1}^2 \ , \quad ds_1^2 = d\theta_1^2 \ , \tag{3.4}$$

$$\sqrt{g} = \sin^{N-1}\theta_N \sin^{N-2}\theta_{N-1} \cdots \sin\theta_2 \ , \tag{3.5}$$

$$P_\nu^{-\mu}(x) = \frac{1}{\Gamma(1+\mu)} \left(\frac{1-x}{1+x}\right)^{\mu/2} {}_2F_1\left(-\nu \ , \ \nu+1 \ ; \ 1+\mu \ ; \ \frac{1-x}{2}\right) \quad , \tag{3.6}$$

$${}_n c_L^l = \left[\frac{2L+n-1}{2} \frac{(L+l+n-2)!}{(L-l)!}\right]^{1/2} \quad . \tag{3.7}$$

For simplicity we assume that one of the points is sitting at the north pole, in which case only spherical harmonics with $\ell_{N-1} = \ell_{N-2} = \ldots = \ell_1 = 0$ contribute

$$Y_{\ell_N 0\ldots 0}(\theta_N) = \sqrt{\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}}\ {}_N c_{\ell_N}^0 (\sin\theta_N)^{\frac{2-N}{2}} P_{\ell_N + \frac{N-2}{2}}^{-\frac{N-2}{2}}(\cos\theta_N) \ . \tag{3.8}$$

Hence, by assumption $\gamma = \theta_N$ in (3.2), and the following identity holds

$$\begin{aligned} \langle D(x)D(x')\rangle &= \frac{C_D}{(2R^2)^{d-1}} \sum_{\ell_N} A_{\ell_N} Y_{\ell_N 0\ldots 0}(\gamma) \ , \\ A_{\ell_N} &= \sqrt{\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}}\ {}_N c_{\ell_N}^0 \int_{-1}^{1} dz\, \frac{(1-z^2)^{\frac{N-2}{4}}}{(1-z)^{N+1}} P_{\ell_N + \frac{N-2}{2}}^{-\frac{N-2}{2}}(z) \end{aligned} \tag{3.9}$$

where we introduced a new variable $z = \cos\gamma$.

The above integral diverges at $z = 1$. This is not surprising given that the coefficients A_{ℓ_N} correspond to a spherical harmonic representation of a singular function (3.2). To regularize these coefficients let us modify the power of $(1-\cos\gamma)$ in (3.2) by introducing a new parameter α such that A_{ℓ_N} takes the form

$$A_{\ell_N} = \frac{\pi^{\frac{N}{4}}\ {}_N c_{\ell_N}^0}{2^{\frac{N+1}{2}}\, \Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)} \lim_{\alpha\to 0} \int_0^1 dy\, y^{\alpha - \frac{N+4}{2}}\ {}_2F_1\Big(-\ell_N - \frac{N}{2} + 1 \ , \ \ell_N + \frac{N}{2} \ ; \ \frac{N}{2} \ ; \ y\Big) \ . \tag{3.10}$$

where $y = (1-z)/2$ and we used (3.7) to express the associated Legendre polynomial in terms of the hypergeometric function. Now the integral can be readily evaluated assuming that α

is large enough[4]

$$A_{\ell_N} = \frac{\pi^{\frac{N}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N+1}{2}}\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)} \lim_{\alpha\to 0} \frac{{}_3F_2\Big(\alpha-\frac{N}{2}-1\ ,\ \ -\ell_N-\frac{N}{2}+1\ ,\ \ell_N+\frac{N}{2}\ ;\ \alpha-\frac{N}{2}\ ,\ \frac{N}{2}\ ;\ 1\Big)}{\alpha-\frac{N}{2}-1} \tag{3.11}$$

For odd N (odd d) the limit $\alpha \to 0$ is finite. However, it diverges for even N (even d). Therefore we analyze these cases separately.

3.1 Odd d

For odd d, we have

$$\begin{aligned} A_{\ell_N} &= -\frac{\pi^{\frac{N}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N-1}{2}}\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)} \frac{{}_3F_2\Big(-\frac{N}{2}-1\ ,\ \ -\ell_N-\frac{N}{2}+1\ ,\ \ell_N+\frac{N}{2}\ ;\ \ -\frac{N}{2}\ ,\ \frac{N}{2}\ ;\ 1\Big)}{N+2} \\ &= (-1)^{\frac{N-1}{2}} \frac{\pi^{\frac{N+4}{4}}\ {}_N c^0_{\ell_N}}{2^{\frac{N-3}{2}}\,N(N+2)\,\Gamma^{\frac{3}{2}}\left(\frac{N}{2}\right)\Gamma(N+1)} \prod_{k=1,\dots,N+2}(\ell_N+k-2) \end{aligned} \tag{3.12}$$

Using now addition theorem for spherical harmonics

$$Y_{\ell_N 0\dots 0}(\gamma) = \frac{1}{{}_N c^0_{\ell_N}} \sqrt{\frac{(4\pi)^{\frac{N}{2}}\Gamma\left(\frac{N}{2}\right)}{2}} \sum_{\ell_{N-1},\dots,\ell_1} Y^*_{\ell_N\dots\ell_1}(x) Y_{\ell_N\dots\ell_1}(x')\ , \tag{3.13}$$

we obtain from (3.9)

$$\begin{aligned} \langle D(x)D(x')\rangle \;=\; & C_D\,\frac{(-1)^{\frac{N-1}{2}}\,\pi^{\frac{N+2}{2}}}{2(2R^2)^{N+1}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)} \\ & \times \sum_{\ell_N,\dots,\ell_1} Y^*_{\ell_N\dots\ell_1}(x) Y_{\ell_N\dots\ell_1}(x') \prod_{k=1,\dots,N+2}(\ell_N+k-2)\ . \end{aligned} \tag{3.14}$$

Substituting this result into (3.1), yields

$$\delta S_n = \epsilon^2\,\frac{C_D}{(n-1)}\,\frac{(-1)^{\frac{d-1}{2}}\pi^{\frac{d}{2}}}{2^{d+1}\,\Gamma(d-1)\Gamma\left(\frac{d}{2}+1\right)} \sum_{\ell_N,\dots,\ell_1} |a_{\ell_N\dots\ell_1}|^2 \prod_{k=1,\dots,d}(\ell_N+k-2) + \mathcal{O}(\epsilon^3)\ , \tag{3.15}$$

where $a_{\ell_N\dots\ell_1}$ are the coefficients of $f(x)$ in a spherical harmonics representation. This result agrees with [32] for any odd d provided that (2.15) holds.

3.2 Even d

For even d the limit $\alpha \to 0$ in (3.11) is singular due to logarithmic divergence. To extract the numerical coefficient of this divergence we expand the integrand in (3.9) around $z = 1$ and keep only logarithmically divergent term

$$\frac{(1-z^2)^{\frac{N-2}{4}}}{(1-z)^{N+1}}\,P^{-\frac{N-2}{2}}_{\ell_N+\frac{N-2}{2}}(z) = \frac{(-1)^{\frac{N}{2}}\prod_{k=1,\dots,N+2}(\ell_N+k-2)}{2^{\frac{N+2}{2}}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)}\,\frac{1}{z-1} + \dots\ , \tag{3.16}$$

[4] As usual, small values of α are treated by analytic continuation.

where ellipsis encode terms which do not contribute to logarithmic divergence.

Hence,

$$A_{\ell_N} = (-1)^{\frac{N+2}{2}} \sqrt{\frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}}\ {}_N c^0_{\ell_N}\ \frac{\prod_{k=1,\dots,N+2}(\ell_N + k - 2)}{2^{\frac{N}{2}}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)} \log(R/\delta) + \dots\,, \tag{3.17}$$

with $\delta = R \cdot \delta\gamma$ being the UV cut off. Using now (3.13), we obtain

$$\begin{aligned}\langle D(x)D(x')\rangle &= \log(R/\delta)\, C_D\, \frac{(-1)^{\frac{N+2}{2}}\, \pi^{\frac{N}{2}}}{(2R^2)^{N+1}\Gamma(N+1)\Gamma\left(\frac{N}{2}+2\right)} \\ &\quad \times \sum_{\ell_N,\dots,\ell_1} Y^*_{\ell_N\dots\ell_1}(x) Y_{\ell_N\dots\ell_1}(x') \prod_{k=1,\dots,N+2} (\ell_N + k - 2) + \dots\ .\end{aligned} \tag{3.18}$$

Substituting this result into (3.1), yields

$$\begin{aligned}\delta S_n &= \epsilon^2\, \frac{C_D}{(n-1)}\, \frac{(-\pi)^{\frac{d-2}{2}}}{2^d\,\Gamma(d-1)\Gamma\left(\frac{d}{2}+1\right)} \log(R/\delta) \\ &\quad \times \sum_{\ell_N,\dots,\ell_1} |a_{\ell_N\dots\ell_1}|^2 \prod_{k=1,\dots,d} (\ell_N + k - 2) + \dots\ ,\end{aligned} \tag{3.19}$$

where $a_{\ell_N\dots\ell_1}$ are coefficients of $f(x)$ in a spherical harmonics representation. Combined with (2.15) this result is in full agreement with [32].

4. The cone conjecture

In this section we analyze the conjecture proposed in [34, 35] for three-dimensional entanglement entropy and then extended to Rényi entropy [36] in arbitrary dimensions [37]. In order to introduce the claim of the conjecture, we consider a deformation of a flat entangling surface which consists in creating a conical singularity. The three- and four-dimensional cases are shown in figure 1 of ref. [37]. The universal contribution to the Rényi (and consequently entanglement) entropy is affected by this modification. In particular, if the twist operator is smooth, the universal contribution would be logarithmically divergent in even dimensions and constant (i.e. regulator independent) in odd dimensions. When a conical singularity is present an additional logarithm emerges and the universal contribution to the Rényi entropy takes the form

$$S_n^{\text{univ}}(V) = \begin{cases} (-1)^{\frac{d-1}{2}}\ a_n^{(d)}(\Omega)\ \log\frac{R}{\delta} & d \text{ odd} \\ (-1)^{\frac{d-2}{2}}\ a_n^{(d)}(\Omega)\ \log^2\frac{R}{\delta} & d \text{ even}\end{cases} \tag{4.1}$$

Here Ω is the opening angle of the cone, varying in the interval $[0, \frac{\pi}{2}]$ and approaching $\frac{\pi}{2}$ in the limit of smooth surface [5]. The function $a_n^{(d)}$ is the universal contribution to the Rényi

[5]The angle Ω actually varies over the full range $[0, \pi]$, but, since the Rényi entropy evaluated for a pure state is equal for the region A or for its complement $\bar{A}$, the function $a_n^{(d)}$ is symmetric for reflections with respect to $\Omega = \frac{\pi}{2}$, i.e. $a_n^{(d)}(\Omega) = a_n^{(d)}(\pi - \Omega)$ and we can consistently focus on the interval $[0, \frac{\pi}{2}]$

entropy and depends on the angle Ω only. R and δ are the IR and UV regulators respectively. The former can be thought of as a length scale characterizing the entangling region V (the one enclosed by the twist operator), whereas the latter can be taken to be a short-distance cut-off originating from the infinite number of short-distance correlations in proximity of the twist-operator. The cusp conjecture, in the most general formulation of [37], states that, for an arbitrary conformal field theory, the leading contribution to $a_n^{(d)}$ for $\Omega \to \frac{\pi}{2}$ is controlled by the constant h_n introduced in (2.13). Explicitly

$$a_n^{(d)}(\Omega) \overset{\Omega\to\pi/2}{\sim} 4\,\sigma_n^{(d)}(\Omega-\tfrac{\pi}{2})^2 \qquad \sigma_n^{(d)} = \frac{h_n}{n(n-1)}\,\frac{(d-1)(d-2)\,\pi^{\frac{d-4}{2}}\,\Gamma\left[\frac{d-1}{2}\right]^2}{16\,\Gamma[d/2]^3} \times \begin{cases} \pi & d \text{ odd}\,, \\ 1 & d \text{ even}\,. \end{cases} \tag{4.2}$$

Restricting to the case $n = 1$ and using (2.16) one finds the following relation between the small angle contribution to the entanglement entropy and the central charge C_T of a CFT

$$\sigma_1^{(d)} \equiv \sigma^{(d)} = C_T \frac{\pi^{d-1}(d-1)(d-2)\Gamma[\frac{d-1}{2}]^2}{8\,\Gamma[d/2]^2\,\Gamma[d+2]} \times \begin{cases} \pi & d \text{ odd}\,, \\ 1 & d \text{ even}\,. \end{cases} \tag{4.3}$$

In the following we will apply the theoretical framework introduced in section (2) to this particular deformation and find a connection between σ_n and C_D. This will allow to prove the equivalence of the cusp conjecture and equation (2.14)

4.1 Displacement operator for a conical deformation

One of the appealing features of the displacement operator is that equation (2.12) is valid for any kind of deformation of the defect, regardless of its smoothness. It is then clear that the response (4.1) of the Rényi entropy to a conical singularity in the limit $\Omega \to \frac{\pi}{2}$ can be related to the two-point function of the displacement operator (2.12) integrated over a planar defect with the appropriate profile. In particular combining (2.5) and (4.2) we obtain

$$\frac{1}{2}\Sigma^{(d)} \equiv \frac{1}{2}\,\frac{1}{\langle\tau_n\rangle}\frac{d^2}{d\epsilon^2}\,\langle\tau_n\rangle\bigg|_{\epsilon=0} = 4(n-1)\,\sigma_n^{(d)} \times \begin{cases} (-1)^{\frac{d+1}{2}} \;\log\frac{R}{\delta} & d \text{ odd} \\ (-1)^{\frac{d}{2}} \;\log^2\frac{R}{\delta} & d \text{ even} \end{cases} \tag{4.4}$$

where the first equality is just the definition of $\Sigma^{(d)}$. In the following we will compute $\Sigma^{(d)}$ in terms of C_D using (2.12). Then, exploiting the conjectured relation (2.14) we will reproduce the cusp conjecture (4.2).

Consider a planar defect, parametrized by parallel coordinates x^i with $i = 3, ..., d$, and its deformation into a configuration with a conical singularity in the origin. The two coordinates for the orthogonal directions are x^a with $a = 1, 2$. To deform the plane into a cone we introduce spherical coordinates $\{r, \theta^1, ..., \theta^{d-3}\}$ in the directions parallel to the entangling surface and we consider a variation δx^a in the direction 2, proportional to the radius r

$$\delta x^a = \delta^a_2\, r. \tag{4.5}$$

Plugging this expression into (2.12), combined with (2.9) and using the symmetries of the problem to perform the angular integrations we are left with

$$\Sigma^{(d)} = C_D\,\Omega_{d-3}\Omega_{d-4}\int dr_1\,dr_2\int_0^{2\pi} d\theta_{12}\frac{r_1^{d-2}r_2^{d-2}\sin^{d-4}\theta_{12}}{(r_1^2+r_2^2-2r_1r_2\cos\theta_{12})^{d-1}}. \tag{4.6}$$

where θ_{12} is the angle described by the position of the two displacement operators in the plane defined by them and the origin. Further $\Omega_m = 2\pi^{\frac{m+1}{2}}/\Gamma(\frac{m+1}{2})$ is the volume of a unit m-sphere. The integration over θ_{12} yields

$$\begin{aligned}\Sigma^{(d)} = C_D&\frac{2^{d-3}\Gamma\left(\frac{d-3}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(d-1)}\\ &\times\int dr_1\,dr_2\left[\left|r_1^2-r_2^2\right|^{-d-1}r_1^{d-2}r_2^{d-2}\Big((d-2)r_1^4+2d\,r_1^2r_2^2+(d-2)r_2^4\Big)\right]\end{aligned} \tag{4.7}$$

One has to be particularly careful in the integration over r_1 and r_2 since we expect a singularity along the line $r_1 = r_2$. Therefore it is useful to note the symmetry of the integral under the exchange $r_1 \leftrightarrow r_2$ and restrict the integration contour to the region $r_1 > r_2$. We then regulate the divergences for $r_1 \to r_2$ and for $r_1, r_2 \to 0$ with a UV cut-off δ, and the divergence for $r_1, r_2 \to \infty$ with an IR cut-off R. Introducing the variables $x = r_1 + r_2$ and $y = r_1 - r_2$ the integral takes the form

$$\begin{aligned}\Sigma^{(d)} = C_D&\frac{2^{3-2d}\pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)}\int_\delta^R\frac{dx}{x}\\ &\times\int_\delta^x dy\left[\left(x^2-y^2\right)^{d-2}(xy)^{-d-1}\Big((d-1)x^4+2(d-3)x^2y^2+(d-1)y^4\Big)\right]\end{aligned} \tag{4.8}$$

An additional change of variables $w = z^2$ yields

$$\begin{aligned}\Sigma^{(d)} = C_D&\frac{2^{2-2d}\pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)}\int_\delta^R\frac{dx}{x}\\ &\times\int_{\left(\frac{\delta}{x}\right)^{\frac{1}{2}}}^1 dw\left[(1-w)^{d-2}\,w^{-1-\frac{d}{2}}\Big((d-1)w^2+2(d-3)w+d-1\Big)\right]\end{aligned} \tag{4.9}$$

Since the treatment of this integral differs substantially in even and odd dimensions, it is convenient to analyze the two cases separately.

Even dimension

It is useful to note that, for integer d, the binomial $(1-w)^{d-2}$ can be converted in a finite sum over powers of w. Furthermore, if d is even also the exponent of $w^{1-d/2}$ is an integer, which implies that the integral over w contains a first logarithmic divergence for small w. We focus on that contribution and we perform the first integration, which yields

$$\Sigma^{(d)}_{\text{even}} = C_D\frac{(-1)^{\frac{d}{2}}2^{5-2d}\pi^{d-2}\Gamma(d)}{d(d-1)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)^3}\int_\delta^R\frac{dx}{x}\log\frac{x}{\delta}+\ldots \tag{4.10}$$

where the missing terms are power-like divergences. The last integration can be trivially carried out and the final result is

$$\Sigma^{(d)}_{\text{even}} = C_D \frac{(-1)^{\frac{d}{2}} 2^{-d} \pi^{d-\frac{5}{2}} d\, \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right) \Gamma\left(\frac{d}{2}+1\right)^2} \log^2 \frac{R}{\delta} + ... \tag{4.11}$$

Comparing this result with eqs. (4.3) and (4.4), we find perfect agreement when using (2.14) for C_D.

Odd dimension

For odd dimension it is still true that the binomial $(1-w)^{d-2}$ can be expanded as a finite sum, but $1-\frac{d}{2}$ is not an integer anymore and we are left with an integral of the form

$$\begin{aligned} \Sigma^{(d)}_{\text{odd}} = C_D \frac{2^{2-2d} \pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \int_\delta^R \frac{dx}{x} \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^k \\ \times \int_{\left(\frac{\delta}{x}\right)^{\frac{1}{2}}}^1 dw \left((d-1) w^{k-1-\frac{d}{2}} + 2(d-3) w^{k-\frac{d}{2}} + (d-1) w^{k+1-\frac{d}{2}}\right) \end{aligned} \tag{4.12}$$

For odd d, all the exponents in the last bracket are semi-integers, and the integration over w only leads to power-like divergences. The only logarithmic term comes from the integration over x, combined with the finite part of the integration over w, i.e.

$$\begin{aligned} \Sigma^{(d)}_{\text{odd}} = C_D \frac{2^{2-2d} \pi^{d-2}}{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}\right)} \log \frac{R}{\delta} \\ \times \sum_{k=0}^{d-2} \binom{d-2}{k} (-1)^k \left(\frac{d-1}{k-\frac{d}{2}} + 2\frac{d-3}{k+1-\frac{d}{2}} + \frac{d-1}{k+2-\frac{d}{2}}\right) + ... \end{aligned} \tag{4.13}$$

Performing the finite sums we get

$$\Sigma^{(d)}_{\text{odd}} = C_D \frac{(-1)^{\frac{d+1}{2}} 2^{-d} \pi^{d-\frac{3}{2}} d\, \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}-1\right) \Gamma\left(\frac{d}{2}+1\right)^2} \log \frac{R}{\delta} + ... \tag{4.14}$$

Again, substituting for C_D using (2.14) produces precise agreement with eqs. (4.3) and (4.4).

4.2 Wilson lines in supersymmetric theories and entanglement in $d=3$.

The relation between the expectation value of the stress tensor and the two-point function of the displacement operator has been explored, in fact, at least in one other example of a defect CFT, i.e. for Wilson lines [42]. In that context, C_D is better known as the Bremsstrahlung function. Indeed, a sudden acceleration of a charged source creates a cusp in the Wilson line that describes its trajectory, and it can be shown that the Zamolodchikov norm of the displacement operator measures the energy emitted in the process [43]. The precise relation between the two quantities is

$$C_D^{WL} = 12\, B \tag{4.15}$$

where B is the Bremsstrahlung function. The authors of [42] observed that the ratio between B and h is theory dependent. However, a restricted form of universality is valid within a certain class of conformal gauge theories, whose Bremsstrahlung function is related to the one-point function of the stress tensor through a coefficient that only depends on the dimension of spacetime. This class includes theories with $\mathcal{N} = 4$ [42] and four-dimensional $\mathcal{N} = 2$ [44] supersymmetry. In particular, in three dimensions, the general formula conjectured in [42] yields

$$C_D^{WL} = 24\, h^{WL} \tag{4.16}$$

where h^{WL} is the constant entering the one-point function of the stress-tensor in the presence of a Wilson line.

Now the three-dimensional case is especially interesting for us, because twist operators become one-dimensional line operators in three dimensions. Furthermore, if we consider holographic CFTs, the calculation of the Wilson line [45, 46] and the Ryu-Takayanagi prescription [11, 12] for holographic entanglement entropy both reduce to evaluating the area of extremal surfaces anchored on the AdS boundary. The only difference in the two calculations is the overall factor multiplying the extremal area in evaluating the final physical quantity, but this constant factor will cancel out in the ratio between C_D and h. Hence for theories which possess a holographic dual and belong to the class for which (4.16) is valid, *e.g.,* ABJM theory [47], the relation between $\partial_n C_D|_{n=1}$ and $\partial_n h_n|_{n=1}$ has to coincide with (4.16) — at strong coupling. Hence it is a nontrivial check that, indeed, formula (2.14) reduces to (4.16) for $d = 3$. Let us make two additional remarks: This agreement is better than required, *i.e.,* $C_D(n) = 24\, h_n$ for all n whereas our argument only indicated a match in the $n \to 1$ limit. Notice, furthermore, that both eqs. (4.16) and (2.14) are independent of the coupling. Hence this special relation between the CFT data for the two separate physical observables, *i.e.,* Wilson lines and Rényi entropies, which are apparently unrelated, not only agree at strong coupling but also at any coupling.

5. Entanglement entropy and anomalies in 4d Defect CFTs.

In any even number of dimensions, the universal contribution to the Rényi entropy (2.2) depends only on the shape of the spatial region A through local geometric quantities. In four dimensions, in particular, when the theory is conformal, Weyl invariance fixes the universal contribution up to three functions of n. If we denote by ℓ a characteristic length scale of A, then the Renyi entropy takes the form[6]

$$S_n = \left(-\frac{f_a(n)}{2\pi} \int_{\partial A} R_\Sigma - \frac{f_b(n)}{2\pi} \int_{\partial A} \tilde{K}^a_{ij} \tilde{K}^a_{ij} + \frac{f_c(n)}{2\pi} \int_{\partial A} \gamma^{ij} \gamma^{kl} C_{ikjl} \right) \log(\mu \ell) + c \ , \tag{5.1}$$

[6]In what follows we suppress the well-known 'area law' $\sim (\mu\ell)^2$. Its coefficient is scheme dependent and thus non-universal. In particular, it vanishes within dimensional regularization scheme which we employ throughout this paper.

where γ^{ij} is the inverse of the induced metric on the entangling surface, μ is an arbitrary mass scale typically chosen to be of order of the inverse cut off, ℓ is a characteristic length scale of $\Sigma = \partial A$, c is a non-universal constant and $\tilde{K}^a_{ij}$ is a traceless part of the extrinsic curvature of ∂A

$$\tilde{K}^a_{ij} = K^a_{ij} - \frac{K^a}{2}\gamma_{ij} \ , \tag{5.2}$$

where $K^a = \gamma^{kl}K^a_{kl}$. Now, two of the functions appearing in (5.1) are conjecturally [48] equal to each other:

$$f_b(n) = f_c(n). \tag{5.3}$$

This has been proven for $n = 1$, but remains an open question in general. On the other hand, from our defect CFT point of view, the expression (5.1) has the form of a conformal anomaly, which simply arises because the presence of a defect in the vacuum provides additional ways to violate Weyl invariance. Since the a and c coefficients of anomalies in a generic even dimensional CFT appear in correlation functions of the stress-tensor, one might wonder if the same happens in a defect CFT. In this section we show that this is indeed the case, in the sense that f_b and f_c are just other names for C_D and h respectively:

$$f_c(n) = \frac{3\pi}{2}\frac{h_n}{n-1} \, , \qquad f_b(n) = \frac{\pi^2}{16}\frac{C_D(n)}{n-1} \ . \tag{5.4}$$

The relation between f_c and h_n was recently found in the context of entanglement entropy [49], but both equalities turn out to be true in a generic defect CFT. In the case of the replica defect, they also establish the equivalence of the conjecture (5.3) with the four-dimensional version of eq. (2.14). As a first step towards eq. (5.4), we notice that by dimensional analysis (or direct calculation), we have

$$\mu\frac{\partial}{\partial\mu}S_n - \ell\frac{\partial}{\partial\ell}S_n = 0 \quad \Leftrightarrow \quad \mu\frac{\partial}{\partial\mu}S_n = S_n^{\text{univ}} \ , \tag{5.5}$$

where S_n^{univ} denotes the universal Renyi entropy

$$S_n^{\text{univ}} = \left(-\frac{f_a(n)}{2\pi}\int_\Sigma R_\Sigma - \frac{f_b(n)}{2\pi}\int_\Sigma \tilde{K}^a_{ij}\tilde{K}^a_{ij} + \frac{f_c(n)}{2\pi}\int_\Sigma \gamma^{ij}\gamma^{kl}C_{ikjl} \right) \ . \tag{5.6}$$

Varying both sides of (5.5) with respect to the metric and using (2.4), yields

$$\frac{1}{1-n}\,\mu\frac{\partial}{\partial\mu}\sum_{\text{m}}\left(\langle T^{\mu\nu}_{(\text{m})}(x)\rangle_n - \langle T^{\mu\nu}(x)\rangle_1\right) = \frac{-2}{\sqrt{g(x)}}\frac{\delta S_n^{\text{univ}}}{\delta g_{\mu\nu}(x)} \ , \tag{5.7}$$

$$\frac{1}{1-n}\,\mu\frac{\partial}{\partial\mu}\Big(\sum_{\text{l,m}}\langle T^{\mu\nu}_{(\text{l})}(x)T^{\alpha\beta}_{(\text{m})}(y)\rangle_n - n\langle T^{\mu\nu}(x)T^{\alpha\beta}(y)\rangle_1\Big) = \frac{4}{\sqrt{g(y)}}\frac{\delta}{\delta g_{\alpha\beta}(y)}\frac{1}{\sqrt{g(x)}}\frac{\delta S_n^{\text{univ}}}{\delta g_{\mu\nu}(x)} \ , \tag{5.8}$$

where indices m and n run over the replicas. In the next subsection, we build on eq. (5.7) to prove that f_c appears in the one-point function of the stress-tensor, while eq. (5.8) will be needed in subsection 5.2 to match f_b with the two-point function of the displacement.

5.1 f_c and the expectation value of the stress-tensor

Substituting $d = 4$ into eq. (2.13), nontrivial terms in the one-point function of the stress tensor become[7]

$$\begin{aligned}\langle T^{ij}_{\rm tot}\rangle_n &= -\frac{h_n}{2\pi}\frac{\delta^{ij}}{r^4} + \dots\ ,\\ \langle T^{ab}_{\rm tot}\rangle_n &= \frac{h_n}{2\pi}\frac{3\,\delta^{ac}\,r^2 - 4\,x^a x^c}{r^6} + \dots\ ,\end{aligned} \tag{5.9}$$

where the indices a, c and i, j denote the two transverse directions and two parallel directions to the entangling surface and $r^2 = \delta_{ac}\,x^a x^c$. Note that h_n in the above expression is a constant, *i.e.,* we are in the regime when the surface and the background are flat and thus all curvatures can be ignored. While eq. (2.13) was written for a planar twist operator, this expression also coincides with the singularity for general entangling surfaces if x is sufficiently close to Σ. In particular, the same constant appears for the conformal weight h_n independent of the geometry of the entangling surface.

Of course, (5.9) is independent of μ, and thus one might think that we reached a contradiction with (5.7). However, this conclusion is too fast. The right hand side of (5.7) vanishes unless $r = 0$, but $r = 0$ corresponds to a singular point of (5.9). This singularity should be carefully defined as distribution. As we will see, this results in a dependence on a mass scale μ.

In what follows we use dimensional regularization and expand all the results around $d = 4$. In particular, we start from the analog of (5.9) with dimension of the entangling surface being fixed (two in our case), while the transverse space to the entangling surface is assumed to have dimension $d - 2$ (rather than two in 4D). Hence, the analog of (5.9) reads

$$\begin{aligned}\langle T^{ij}_{\rm tot}\rangle_n &= -\frac{h_n}{2\pi}\frac{\delta^{ij}}{r^d}\ ,\\ \langle T^{ab}_{\rm tot}\rangle_n &= \frac{h_n}{2\pi}\frac{1}{d-3}\frac{3\delta^{ac}\,r^2 - d\,x^a x^c}{r^{d+2}} = \frac{h_n}{2\pi(d-2)(d-3)}\left(\delta^{ac}\partial_\perp^2 - \partial^a\partial^c\right)\frac{1}{r^{d-2}}\ ,\end{aligned} \tag{5.10}$$

where $\partial_\perp^2 = \delta^{ac}\partial_a\partial_c$ is Laplace operator in the transverse space.

Using now the standard Fourier integral

$$\int \frac{d^{d-2}k}{(2\pi)^{d-2}}\frac{e^{ik\cdot x}}{(k^2)^\alpha} = \frac{\Gamma(d/2-\alpha-1)}{(4\pi)^{(d-2)/2}\Gamma(\alpha)}\left(\frac{4}{x^2}\right)^{d/2-\alpha-1}\ , \tag{5.11}$$

we deduce

$$\frac{1}{(r^2)^{d/2-\alpha-1}} = \frac{4^\alpha\pi^{(d-2)/2}\Gamma(\alpha)}{\Gamma(d/2-\alpha-1)}(-\partial_\perp^2)^{-\alpha}\delta_\Sigma\ , \tag{5.12}$$

where equality holds between the distributions and we recall that δ_Σ denotes the delta fundtion in the transverse space with support on Σ. Substituting now $\alpha = -1 + \epsilon$ and $\alpha = 0 + \epsilon$ with

[7] For convenience in this section, we work with the total energy-momentum tensor of the replicated CFT: $T^{\mu\nu}_{\rm tot} = \sum_{\rm m=1}^n T^{\mu\nu}_{\rm (m)}$.

$\epsilon \ll 1$, and replacing $(-\partial_\perp^2)^\epsilon \to \mu^{2\epsilon}$ yields

$$\begin{aligned}\frac{1}{r^{d-2}} &= -\Omega_{d-3}\left(\frac{1}{2\epsilon}+\log(\mu\, r)+\dots\right)\delta_\Sigma\ ,\\ \frac{1}{r^d} &= -\frac{\Omega_{d-1}}{4\pi}\left(\frac{1}{2\epsilon}+\log(\mu\, r)+\dots\right)\partial_\perp^2\delta_\Sigma\ .\end{aligned}\tag{5.13}$$

where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ and ellipsis correspond to a finite μ-independent constant as $\epsilon \to 0$. Consequently, r^{-d} and $r^{-(d-2)}$ although defined by analytic continuation in d are singular when $d = 4$. Hence, to define (5.10) as a sensible distribution one has to subtract the singular part,

$$\begin{aligned}\mathcal{R}\frac{1}{r^{d-2}} &= -\Omega_{d-3}\left(\log(\mu\, r)+a\right)\delta_\Sigma\ ,\\ \mathcal{R}\frac{1}{r^d} &= -\frac{\Omega_{d-1}}{4\pi}\left(\log(\mu\, r)+a\right)\partial_\perp^2\delta_\Sigma\ ,\end{aligned}\tag{5.14}$$

with a an arbitrary constant (which may be absorbed into μ). Note that such subtraction modifies (5.10) in the limit of coincident points only. Furthermore, the details of this subtraction are not important as long as the result is used in (5.7)

$$\begin{aligned}-\frac{h_n}{4(n-1)}\delta^{ij}\partial_\perp^2\delta_\Sigma &= -2\frac{\delta S_n^{\rm univ}}{\delta g_{ij}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}\ ,\\ \frac{h_n}{2(n-1)}\left(\delta^{ac}\partial_\perp^2-\partial^a\partial^c\right)\delta_\Sigma &= -2\frac{\delta S_n^{\rm univ}}{\delta g_{ab}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}\ .\end{aligned}\tag{5.15}$$

Next we use (5.6) to evaluate the variation on the right hand side. We start from noting that the term proportional to $f_a(n)$ is topological, and therefore its variation vanishes. Hence, in general we have to vary $f_b(n)$ and $f_c(n)$ terms only. Now in four dimensions the following relations hold

$$\begin{aligned}C^\lambda{}_{\mu\sigma\nu} &= R^\lambda{}_{\mu\sigma\nu} - \left(g^\lambda_{[\sigma}R_{\nu]\mu} - g_{\mu[\sigma}R^\lambda_{\nu]}\right) + \frac{1}{3}R\, g^\lambda_{[\sigma}g_{\nu]\mu}\ ,\\ \gamma^{ij}\gamma^{kl}C_{ikjl} &= \gamma^{ij}\gamma^{kl}R_{ikjl} - \gamma^{ij}R_{ij} + \frac{1}{3}R\\ &= \frac{1}{3}\left(\gamma^{ij}\gamma^{kl}R_{ikjl} - \gamma_{ij}g^\perp_{\mu\nu}R^{i\mu j\nu} + g^\perp_{\mu\nu}g^\perp_{\alpha\beta}R^{\mu\alpha\nu\beta}\right)\ ,\end{aligned}\tag{5.16}$$

where $g^\perp_{\mu\nu} = n^a_\mu n^c_\nu \delta_{ac}$ is the metric in the transverse space to Σ, *i.e.*, $g_{\mu\nu} = \gamma_{\mu\nu} + g^\perp_{\mu\nu}$. One can use the Gauss-Codazzi relation

$$\gamma^{ij}\gamma^{kl}R_{ikjl} = R_\Sigma + K^a_{ij}K^{ij}_a - K^aK_a\ ,\tag{5.17}$$

where R_Σ is the intrinsic curvature of the entangling surface, to write

$$\gamma^{ij}\gamma^{kl}C_{ikjl} = \frac{1}{3}\left(R_\Sigma + K^a_{ij}K^{ij}_a - K^aK_a - \gamma_{ij}g^\perp_{\mu\nu}R^{i\mu j\nu} + g^\perp_{\mu\nu}g^\perp_{\alpha\beta}R^{\mu\alpha\nu\beta}\right)\tag{5.18}$$

Now recall that (5.10) is valid in the limit when all curvatures (extrinsic, intrinsic and background) are negligibly small. Hence, we expand the relevant curvature components around the flat space, $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$,

$$
\begin{aligned}
\delta R^{ia}{}_{ia} &= \frac{1}{2}\left(2\,\partial^a\partial^i h_{ai} - \partial_\perp^2 h^i{}_i - \partial^i\partial_i h^a{}_a\right) + \mathcal{O}(h^2)\ , \\
\delta R^{ab}{}_{ab} &= \partial^a\partial^b h_{ab} - \partial_\perp^2 h^a{}_a + \mathcal{O}(h^2)\ ,
\end{aligned}
\tag{5.19}
$$

where we used the results listed in section (B) and summation over the repeated indices is assumed.

Next we use again that the integral of intrinsic curvature over a two-dimensional manifold is a topological invariant, and therefore its variation vanishes. As a result, we obtain[8]

$$
\begin{aligned}
-2\frac{\delta S_n^{\rm univ}}{\delta g_{ij}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= -\frac{f_c(n)}{6\pi}\delta^{ij}\partial_\perp^2\delta_\Sigma\ , \\
-2\frac{\delta S_n^{\rm univ}}{\delta g_{ab}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= -\frac{f_c(n)}{3\pi}\left(\partial^a\partial^b - \delta^{ab}\partial_\perp^2\right)\delta_\Sigma\ ,
\end{aligned}
\tag{5.20}
$$

where we have used the following identities

$$
\frac{\delta g_{\alpha\beta}(y)}{\delta g_{\mu\nu}(x)} = \delta^\mu_{(\alpha}\delta^\nu_{\beta)}\delta(x-y)\ , \qquad \int_\Sigma \delta(x-y) = \delta_\Sigma(x_a)\ \text{for}\ y\in\Sigma\ . \tag{5.21}
$$

Combining (5.15) and (5.20), yields

$$
f_c(n) = \frac{3\pi}{2}\frac{h_n}{n-1}\ . \tag{5.22}
$$

In full agreement with the existing results for free fields [38]. As we mentioned, this result was also found with a complementary argument in [49].

5.2 f_b and the two-point function of the displacement operator.

We now turn to the second one of the equalities (5.4). Since we look for C_D, we are interested in evaluating[9]

$$
\langle D^a(x)D^c(y)\rangle_n\delta_\Sigma(x)\,\delta_\Sigma(y) = -\langle\nabla_\mu T^{\mu a}_{tot}(x)\nabla_\nu T^{\nu c}_{tot}(y)\rangle_n \quad \text{for} \quad x\neq y\in\Sigma\ . \tag{5.23}
$$

Of course, for x, y disjoint from Σ this correlator vanishes identically. However, as we will see, it does not vanish when x and y hit the entangling surface Σ. This is why we explicitly included δ_Σ in the definition of $\langle D^a(x)D^c(y)\rangle_n$. The latter (surface) contact term has a simple interpretation in terms of the standard Ward identity – it represents a local deformation of Σ with $D^a(x)$ being the displacement operator which translates the surface at point x. In particular, we are interested in the leading order singularity of $\langle D^a(x)D^c(y)\rangle$ when x

[8] Note that the third term in (5.19) is a total derivative.

[9] See the Ward identity (A.5).

approaches y. In this limit curvature corrections are subleading, *i.e.,* both the entangling surface and the background can be regarded as flat. From (5.8), we have

$$\frac{1}{1-n}\,\mu\frac{\partial}{\partial\mu}\langle\partial_\mu T^{\mu a}_{\text{tot}}(x)\;\partial_\nu T^{\nu c}_{\text{tot}}(y)\rangle_n\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} = 4\,\frac{\partial^2}{\partial y^\nu\,\partial x^\mu}\,\frac{\delta^2 S_n^{\text{univ}}}{\delta g_{\nu c}(y)\delta g_{\mu a}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} - 2\,\delta^{\alpha\beta}\,\delta(x-y)\,\frac{\delta S_n^{\text{univ}}}{\delta g_{\mu\nu}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}}\,. \tag{5.24}$$

The results of appendix B yield[10]

$$\begin{aligned}\delta^2 R^{ia}{}_{ia} =&\frac{1}{4}\left(2\partial_a h_{i\mu}\partial^i h^{a\mu}+\partial_a h_{i\mu}\partial^a h^{i\mu}-2\partial_a h_{i\mu}\partial^\mu h^{ai}+\partial_i h_{a\mu}\partial^i h^{a\mu}\right.\\&\left.-2\partial_i h_{a\mu}\partial^\mu h^{ai}+\partial_\mu h_{ai}\partial^\mu h^{ai}\right)\\&-\frac{1}{4}\left(4\partial^a h_{a\mu}\partial_i h^{i\mu}-2\partial_a h^{a\mu}\partial_\mu h^i_i-2\partial_\mu h^a_a\partial_i h^{i\mu}+\partial_\mu h^a_a\partial^\mu h^i_i\right)\\&+\frac{1}{2}h^{ab}\left(\partial_a\partial_b h^i_i-2\partial_i\partial_b h^i_a+\partial_i\partial^i h_{ab}\right)\\&+\frac{1}{2}h^{ij}\left(\partial^a\partial_a h_{ij}-2\partial_a\partial_j h^i_a+\partial_i\partial_j h^a_a\right)\\&+h^{ai}\left(\partial_a\partial_i h^j_j-\partial_a\partial_j h^j_i-\partial_i\partial_j h^j_a+\partial^j\partial_j h_{ai}\right).\end{aligned} \tag{5.25}$$

Similarly,

$$\begin{aligned}\delta^2 R^{ab}{}_{ab} =&\frac{1}{2}\left(\partial_a h_{b\mu}\partial^b h^{a\mu}+\partial_a h_{b\mu}\partial^a h^{b\mu}-\partial_a h_{b\mu}\partial^\mu h^{ab}-\partial_b h_{a\mu}\partial^\mu h^{ab}+\frac{1}{2}\partial_\mu h_{ab}\partial^\mu h^{ab}\right)\\&-\partial^a h_{a\mu}\partial_b h^{b\mu}+\partial_a h^{a\mu}\partial_\mu h^b_b-\frac{1}{4}\partial_\mu h^a_a\partial^\mu h^b_b\\&+h^{ab}\left(\partial_a\partial_b h^c_c-2\partial_b\partial_c h^c_a+\partial_c\partial^c h_{ab}\right)\\&-2h^{ai}\left(\partial_a\partial_b h^b_i-\partial_b\partial^b h_{ai}-\partial_a\partial_i h^b_b+\partial_i\partial_b h^b_a\right).\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}\delta^2\left(K^a_{ij}K^{ij}_a\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \frac{1}{2}\partial_i h_{aj}\partial^i h^{aj}+\frac{1}{2}\partial_j h_{ai}\partial^i h^{aj}-\partial_a h_{ij}\partial^i h^{aj}+\frac{1}{4}\partial_a h_{ij}\partial^a h^{ij}\;,\\ \delta^2\left(K^a K_a\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \partial_i h^{ai}\partial_j h^j_a-\partial_i h^{ai}\partial^a h^j_j+\frac{1}{4}\partial_a h^i_i\,\partial^a h^j_j\;.\end{aligned} \tag{5.27}$$

These expansions together with (5.19) are sufficient to evaluate the variation on the right hand side of (5.24). There is, however, a significant simplification if we notice that the general term of this variation contains: two delta functions, δ_Σ, which restrict the final answer to the entangling surface, one delta function intrinsic to the entangling surface and 4 derivatives, ∂_a and ∂_i, which act on these delta functions. Among all such terms only those with four

[10] As before summation over the repeated indices is assumed.

derivatives parallel to the entangling surface will contribute to the leading singularity of $\langle D^a(x)D^b(y)\rangle_n$ as x approaches y. Hence, the relevant part of the variations are

$$\begin{aligned}
\delta^2 R^{ia}{}_{ia} &= \frac{1}{2}\left(\partial_i h_{aj}\partial^i h^{aj} - \partial_i h_{aj}\partial^j h^{ai}\right) + h^{ai}\left(-\partial_i\partial_j h^j_a + \partial^j\partial_j h_{ai}\right) + \dots \\
\delta^2\left(K^a_{ij}K^{ij}_a\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \frac{1}{2}\left(\partial_i h_{aj}\partial^i h^{aj} + \partial_j h_{ai}\partial^i h^{aj}\right) + \dots\ , \\
\delta^2\left(K^a K_a\right)\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} &= \partial_i h^{ai}\partial_j h^j_a + \dots
\end{aligned} \tag{5.28}$$

where ellipsis encode terms which do not contribute to the leading singularity of $\langle D^a(x)D^b(y)\rangle_n$ as x approaches y.

Now it follows from (5.18) that the term proportional to $f_c(n)$ in (5.6) does not contribute to the leading singularity of $\langle D^a(x)D^b(y)\rangle_n$ while $f_b(n)$ gives

$$4\,\frac{\partial^2}{\partial y^\nu\,\partial x^\mu}\,\frac{\delta^2 S_n^{\rm univ}}{\delta g_{\nu c}(y)\delta g_{\mu a}(x)}\Big|_{g_{\mu\nu}=\delta_{\mu\nu}} = -\frac{f_b(n)}{2\pi}\,\delta^{ac}\,(\partial_i\partial^i)^2\delta_\parallel(x-y)\,\delta_\Sigma(x)\,\delta_\Sigma(y)\ , \tag{5.29}$$

where $\delta_\parallel(x-y)$ is the delta function intrinsic to Σ. Substituting into (5.24) and using (5.23), yields[11]

$$\frac{1}{1-n}\mu\frac{\partial}{\partial\mu}\langle D^a(x)D^b(y)\rangle_n = \frac{f_b(n)}{2\pi}\,\delta^{ab}\,(\partial_i\partial^i)^2\delta_\parallel(x-y)\quad \text{for}\quad x,y\in\Sigma\ . \tag{5.30}$$

Now let us recall that the leading singularity of $\langle D^a(x)D^b(y)\rangle_n$ is entirely fixed based on translation invariance along the flat entangling plane and scaling dimension of $\partial_\mu T^{\mu\,a}$, *i.e.*, up to a constant C_D, we have

$$\langle D^a(x)D^b(y)\rangle_n = C_D\frac{\delta^{ab}}{|x-y|^6} = C_D\,\frac{\delta^{ab}}{(x-y)^{2(d-1)}}\quad \text{for}\quad x,y\in\Sigma\ . \tag{5.31}$$

In particular, we should use the analog of (5.12) to interpret this correlator in the limit $x\to y$.[12] The final answer takes the form

$$\frac{1}{r^6} = -\frac{\pi}{32}\left(\frac{1}{2\epsilon} + \log(\mu\, r) + \dots\right)(\partial_i\partial^i)^2\,\delta_\parallel(r)\ . \tag{5.32}$$

Combining altogether, yields

$$\frac{C_D}{n-1} = \frac{16}{\pi^2}\,f_b(n)\ . \tag{5.33}$$

In full agreement with (2.15) since $f_b(1) = c = \pi^4\,C_T/40$.

[11] It follows from (5.20) that the first variation of $S_n^{\rm univ}$ does not have the same singularity structure as $\langle D^a D^b\rangle$, and therefore it does not contribute.

[12] The analog is obtained by replacing δ_Σ and $\partial^2_\perp$ with $\delta_\parallel$ and $\partial^i\partial_i$ respectively.

6. Twist operators and the (defect) CFT data.

In the most general sense, a conformal field theory is defined by a set of data, whose knowledge is sufficient to compute all the observables in the theory. A minimal definition of the CFT data includes the spectrum of scaling dimensions of local operators and the OPE coefficients which regulate their fusion. Knowledge of such a set of numbers is sufficient to compute correlation functions with any number of points. However, one might argue that a more complete definition of the CFT data should include those associated to non-local probes, *e.g.,* defects: certainly, they are part of the observables of a theory. A possible objection is that the set of defects that can be inserted in a higher dimensional conformal field theory may be very large, even nearly as large as the set of lower dimensional conformal field theories. The exception is of course given by boundaries and interfaces in two dimensions, where a beautiful and simple picture has been uncovered [50, 51]. In higher dimensions, it is perhaps better to think of a theory with a defect as a separate problem, more similar in spirit to the question of which new fixed points can be obtained by coupling two CFTs together. The CFT data that describe a defect CFT are then again associated to correlation functions of local operators in this system, and therefore to the spectrum of primaries and their fusion rules. As we mentioned in section 2, the main news in the defect CFT setup are given by the spectrum of defect operators, and by the existence of a defect OPE, again regulated by a set of OPE coefficients.

It is then natural to ask what is the set of CFT data which characterizes the twist operator. This question is not only a simple curiosity. The definition of the replica defect is through a boundary condition in the path-integral. This is often sufficient, but a new definition in terms of CFT data would apply to any conformal field theory, irrespectively of the availability of a path-integral description.[13] Again, some care is needed in setting up this question. The large majority of the OPE coefficients appearing in formulae such as (2.7) will depend on the theory in which the twist operator is inserted. However, if an unambiguous characterization exists, it should be possible to single out some universal pattern, unique to this defect and independent of the CFT. In fact, in the present paper, we highlighted the presence of two interesting features. First, the CFT data associated to a flat twist operator always includes a spectrum of defect primaries with rational spin under rotations around the defect. Second, the coefficient of the two-point function of the displacement and the one of the expectation value of the stress-tensor are conjecturally related by eq. (2.14). Both these facts are theory independent, but both of them are not unique to the twist operator. Codimension-two defects supporting operators with non-integer transverse spin can be easily constructed — see for instance [52] — while the relation (2.14) is shared by Wilson lines in a class of three-dimensional supersymmetric gauge theories, as we recalled in section 4.

Although insufficient to formulate a definition, eq. (2.14) is nevertheless remarkable, and one might wonder whether it is possible to understand it from the abstract perspective that we are adopting here. In fact, something special does happen in the defect OPE of the stress

[13] We thank Davide Gaiotto for a discussion on this point.

tensor, when this relation is fulfilled, as we now show. The appearance of the displacement operator in the defect OPE of the stress tensor is constrained by Lorentz and scale invariance to take the following form:

$$T^{ij}(x) \sim \cdots + \alpha \frac{x_b D^b\, \delta^{ij}}{r^2} + \beta\, x_b\, \partial_i \partial_j D^b + \gamma\, x_b\, \delta^{ij}\, \partial_k \partial^k D^b + \dots \tag{6.1a}$$

$$T^{bi}(x) \sim \cdots + \delta\, \frac{x^b x_c\, \partial^i D^c}{r^2} + \epsilon\, \partial^i D^b + \dots \tag{6.1b}$$

$$T^{bc}(x) \sim \cdots + \zeta\, \frac{x^b x^c x_a D^a}{r^4} + \eta\, \frac{\delta^{bc} x_a D^a}{r^2} + \lambda \frac{D^b x^c + D^c x^b}{r^2} + \dots \tag{6.1c}$$

where r denotes the transverse distance from the defect, as usual. In the first and third lines, the first ellipsis alludes to the identity and to operators which might be lighter than the displacement, and in every case, the second ellipsis indicates less-singular contributions, including higher descendants of the displacement itself. Conformal invariance and energy conservation place constraints on the coefficients in eq. (6.1), and in fact it can be shown that only two of them are independent. More interestingly, all the coefficients are in fact fixed in terms of the conformal weight h and the coefficient C_D. A proof will appear in [31], but it is not difficult to understand how this may come about. All the coefficients are determined by the two-point function of the displacement operator with the stress-tensor $\langle T^{\mu\nu}(x) D^a(y) \rangle$. Now imagine integrating the displacement along the defect: this is equivalent to an infinitesimal translation in the direction labeled by a. We see that the integrated two-point function is proportional to the derivative $\partial_a \langle T^{\mu\nu} \rangle$, whence the relation to h. On the other hand, if we contract the same two-point function with a derivative, we obtain the two-point function of the displacement via eq. (2.8).

Let us now consider the most singular contributions in every component in (6.1). In Euclidean signature, all terms in the the OPE of T^{bi} and T^{bc} have the same degree of singularity. We can still define the most singular terms in Lorentzian signature, by considering a spacelike defect — this is especially natural when talking about entanglement entropy. Now as the insertion approaches the null cone, the individual x^a may remain finite while r approaches zero. In this circumstance, the most singular terms are those multiplied by α, δ and ζ. It turns out that the three constants are all proportional to the same linear combination of C_D and h_n. Remarkably, this linear combination is precisely the one that vanishes when eq. (2.14) holds, i.e.

$$\alpha = \delta = \zeta = 0 \iff C_D(n) = d\, \Gamma\left(\tfrac{d+1}{2}\right) \left(\tfrac{2}{\sqrt{\pi}}\right)^{d-1} h_n \tag{6.2}$$

This observation is appealing, even if its meaning remains somewhat obscure. One may speculate that the twist operator is a "mild" defect, in some sense. It is obtained through a modification of the geometry, rather than the addition of local degrees of freedom, and now we see that the OPE of the stress-tensor is less singular than for a generic defect. However, this idea should not be taken too literally. The identity appears in the same defect OPE, with a more severe singularity. Moreover, lighter defect operators with respect to the displacement might exist — in fact, they do in a free scalar theory, as discussed in Appendix

C. Some of them may also appear in the defect OPE of the stress tensor. Whatever the right interpretation may be, it is worth emphasizing that it would have been probably difficult to recognize the special character of the relation (2.14), without adopting the defect CFT perspective.

7. Discussion

Twist operators were originally defined in examining Rényi entropies in two-dimensional CFTs [40] and they are easily understood in this context since they are local primary operators. As discussed in 2, twist operators are formally defined for general QFTs through the replica trick, as in eq. (2.5). In higher dimensions then, they become nonlocal surface operators and their properties are less well understood. In the present paper, we have begun to explore twist operators for CFTs in higher dimensions from the perspective of conformal defects. This approach naturally introduces a number of tools that are unfamiliar in typical discussions of Rényi entropies. In particular, our discussion has focused on the displacement operator D^a, which appears with the new contact term in the Ward identity (2.8).

A key role of the displacement operator is to implement small local deformations of the entangling surface, as in eq. (2.11). As shown in eq. (2.12), the expectation value of the twist operator itself only varies at second order for such deformations of a planar (or spherical) entangling surfaces and is determined by the two-point function (2.9) of the displacement operator. This behaviour was previously seen in holographic studies of the so-called entanglement density [53] and more recently in [39]. These results would relate to the special case of the $n \to 1$ limit in eq. (2.12). We might also like to note that the connection with Wilson lines in holographic conformal gauge theories discussed in section 4.2 would also relate these entanglement variations to the wavy-line behaviour of Wilson lines [54].

Our main result is the claim that for a general CFT in any number of dimensions, the coefficient defining the two-point function of the displacement operator is simply related to the conformal weight of the twist operator, as given in eq. (2.14). This simple relation seems to tie together a variety of conjectures about the Rényi entropy that were made at various points in the literature, as summarized at the end of section 2. While these connections were already considered in [37] — see also discussion in [39] — eq. (2.14) appears to provide the root source with a relation between two pieces of CFT data characterizing the twist operators. It would be interesting to explore other implications which this relation has for Rényi entropies in other geometries and other dimensions. For example, it could provide a relation (for arbitrary n) between different coefficients appearing in the universal contribution to the Rényi entropy in $d = 6$ or higher even dimensions, along the lines of our four-dimensional discussion in section 5. We might also add that the new proof in [39] in fact assures that our conjectured relation must hold in the limit $n \to 1$, *i.e.,* eq. (2.15) now stands as a established fact.

Recalling that the twist operator is a local primary in two-dimensional CFTs, we might ask how the displacement operator appears in this context. Here, the natural object is the first descendant, *i.e.,* derivative, of the twist operator which would be analogous to the combination of the displacement and twist operators together. This matches the appropriate contact term in the two-dimensional version of the Ward identity (2.8). Here we refer to an analogy (rather than a precise match) keeping in mind that as a local operator, the two-dimensional twist operator can be moved but not deformed. Still one might make sense of the two-point correlator (2.9) by considering a "spherical" entangling surface. In two dimensions, the (zero-dimensional) sphere would correspond to two points whose separation defines the diameter of the sphere. Hence eq. (2.9) would be given by taking derivatives of the correlator of two twist operators and hence one finds that the corresponding C_D is indeed proportional to the conformal weight h_n.

Our discussion has highlighted h_n and C_D as two pieces of CFT data which characterize twist operators. With this perspective of regarding the twist operator as a conformal defect, we began in section 6 to consider the question of what are the defining characteristics of the twist operator? Certainly the relation (2.14) is one important feature and as we noted there, it has an interesting impact on the defect OPE with the stress tensor. However, as described in section 2, this property is also shared by Wilson line operators in certain superconformal gauge theories. Another important property discussed in section 2 is that the spectrum of defect operators can contain operators with fractional spins k/n. Certainly, our analysis of the free scalar theory in appendix C explicitly reveals the presence of such operators. But again twist operators are not unique in this regard. Another interesting point that arises in our discussion is that the twist operators are naturally defined for integer n but in discussing h_n and C_D, as well as the Rényi entropy, one continues the results to real n almost immediately. Here derivatives of correlators with respect to the Rényi entropy index are naturally defined in terms of the modular Hamiltonian [26, 55]. This seems to point to a unique characteristic of twist operators in higher dimensions. In any event, better understanding the definition of the twist operator as a conformal defect remains an open question. Undoubtedly it is a question whose answer will produce a better understanding of the entanglement properties of CFTs, and perhaps QFTs more generally.

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A. Ward identities in the presence of twist operator

This appendix is devoted to the Ward identities obeyed by the stress tensor in the presence of a twist operator. We shall focus on the displacement operator and opt for a streamlined derivation. We refer to [31] for a more detailed account. Let us consider a q-point correlator of the scalar fields on an arbitrary replicated manifold $\mathcal{M}_n$

$$\Gamma(x_1, x_2, \dots x_q, \Sigma, g_{\mu\nu}) \equiv \langle \phi(x_1)\phi(x_2)\cdots\phi(x_q)\rangle_n \equiv \langle \phi(x_1)\phi(x_2)\cdots\phi(x_q)\Phi_\Sigma\rangle \ , \tag{A.1}$$

where n is the replica parameter and Φ_Σ is the twist operator associated with Σ. By definition $\Gamma(x_1, x_2, \dots x_q, \Sigma, g_{\mu\nu})$ transforms as a scalar under diffeomorphisms of the manifold. This means it will be unchanged if we simultaneously make the following infinitesimal replacements

$$\begin{aligned} \delta x_i^\mu &= \xi^\mu|_{x_i} \quad \text{for} \quad i = 1, \dots q \ , \\ \delta x^\mu &= (\xi_\alpha \cdot n_c^\alpha)\, n_c^\mu \quad \text{for} \quad x^\mu \in \Sigma \ , \\ \delta g_{\mu\nu} &= -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \ , \end{aligned} \tag{A.2}$$

where n_c^μ are two normal vectors to Σ. Thus to leading order in ξ^μ we have

$$\begin{aligned} 0 = &\sum_{i=1}^{q} \xi^\mu|_{x_i} \langle \phi(x_1)\cdots\partial_\mu\phi(x_i)\cdots\phi(x_q)\rangle_n + \langle \phi(x_1)\phi(x_2)\cdots\phi(x_q) \int_\Sigma \xi^\alpha \cdot D_\alpha \rangle_n \\ &+ \langle \phi(x_1)\phi(x_2)\cdots\phi(x_q) \int_{\mathcal{M}_n} T^{\mu\nu}\nabla_\mu \xi_\nu \rangle_n \ , \end{aligned} \tag{A.3}$$

where $D_\alpha(y) = n_\alpha^c \cdot D_c(y)$ is a local operator which implements displacement of the surface operator, Φ_Σ, at $y^\mu \in \Sigma$ (analog of ∂_μ for a scalar operator $\phi(x)$). Now recall that ξ^μ is arbitrary, but must be the same vector on all the sheets in the replicated geometry. With this in mind, we arrive at the following Ward identity

$$\begin{aligned} 0 = &\sum_{i=1}^{q} \delta(x - x_i)\langle \phi(x_1)\cdots\partial_\nu\phi(x_i)\cdots\phi(x_q)\rangle_n + \delta_\Sigma(x)\, \langle \phi(x_1)\phi(x_2)\cdots\phi(x_q) D_\nu(x)\rangle_n \\ &- \sum_{m=1}^{n} \langle \phi(x_1)\phi(x_2)\cdots\phi(x_q)\nabla_\mu T^\mu_\nu(x_m)\rangle_n \ , \end{aligned} \tag{A.4}$$

where x_m si a point on the m-th replica. The Ward identity defines the displacement operator by specifying its matrix elements: the only additional input is the one of locality of the theory and of the defect, which guarantees that the displacement is a local operator.

Next we assume that there are no scalar field insertions and consider a special case when ξ^μ is peaked around the two given disjoint points x and y, but otherwise is arbitrary. Then expanding to linear order in ξ^μ around x and y and using the above Ward identity, results in (from the cross term $\xi^\mu(x)\xi^\nu(y)$)[14]

$$\delta_\Sigma(x)\, \delta_\Sigma(y)\, \langle D^a(x) D^c(y)\rangle_n + \langle \nabla_\mu T^{\mu a}_{\text{tot}}(x) \nabla_\nu T^{\nu c}_{\text{tot}}(y)\rangle_n = 0 \quad \text{for} \quad x \neq y \ . \tag{A.5}$$

[14]Note that there are two cross terms of the form $\delta_\Sigma(y)\langle \nabla_\mu T^{\mu a}_{\text{tot}}(x) D^c(y)\rangle_n$. They vanish identically since only one stress tensor hits the defect, whereas the correlator $\langle T^{\mu a}_{\text{tot}}(x) D^c(y)\rangle_n$ is conserved.

B. Small variations of the metric

Consider a small perturbation of the flat metric $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, then

Christoffel symbols

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\delta^{\lambda\rho}\left(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}\right) \tag{B.1}$$

Riemann tensor

$$\begin{aligned} R_{\rho\mu\sigma\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\sigma g_{\rho\nu} + \partial_\rho\partial_\nu g_{\mu\sigma} - \partial_\mu\partial_\nu g_{\rho\sigma} - \partial_\rho\partial_\sigma g_{\mu\nu}\right) + g_{\lambda\beta}\left(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\beta}_{\rho\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\beta}_{\rho\sigma}\right) \\ \delta R_{\rho\mu\sigma\nu} &= \frac{1}{2}\left(\partial_\mu\partial_\sigma h_{\rho\nu} + \partial_\rho\partial_\nu h_{\mu\sigma} - \partial_\mu\partial_\nu h_{\rho\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu}\right) \ , \\ \delta^2 R_{\rho\mu\sigma\nu} &= \delta_{\lambda\beta}\left(\delta\Gamma^{\lambda}_{\mu\sigma}\delta\Gamma^{\beta}_{\rho\nu} - \delta\Gamma^{\lambda}_{\mu\nu}\delta\Gamma^{\beta}_{\rho\sigma}\right) \end{aligned} \tag{B.2}$$

Surface forming normal vectors

$$\begin{aligned} g^{\mu\nu}\, n^a_\mu n^c_\nu = \delta^{ac} \quad &\Rightarrow \quad \delta n^a_\mu n^{c\mu} + n^{a\mu}\delta n^c_\mu = n^{a\mu} n^{c\nu}\delta g_{\mu\nu} \ , \\ n^a_\mu t^\mu_i = 0 \quad &\Rightarrow \quad \delta n^a_\mu t^\mu_i = 0 \ , \end{aligned} \tag{B.3}$$

where $t^\mu_i = \partial x^\mu/\partial y^i$ are tangent vectors to the entangling surface. Thus,

$$\delta n^a_\mu = A^a{}_c n^c_\mu \ , \tag{B.4}$$

with

$$A^1{}_1 = \frac{1}{2}n^{1\mu}n^{1\nu}\delta g_{\mu\nu} \ , \quad A^2{}_2 = \frac{1}{2}n^{2\mu}n^{2\nu}\delta g_{\mu\nu} \ , \quad A^1{}_2 + A^2{}_1 = n^{1\mu}n^{2\nu}\delta g_{\mu\nu} \ . \tag{B.5}$$

Extrinsic curvatures

$$\delta K^a_{ij} = \delta\left(\nabla_i n^a_j\right) = -\delta\Gamma^\mu_{ij} n^a_\mu + \nabla_i \delta n^a_j = -\frac{1}{2}n^{a\mu}\left(\nabla_i\delta g_{\mu j} + \nabla_j \delta g_{\mu i} - \nabla_\mu \delta g_{ij}\right) + A^a{}_c K^c_{ij} \ . \tag{B.6}$$

Transverse metric

$$g^\perp_{\mu\nu} = n^a_\mu n^c_\nu \delta_{ac} \quad \Rightarrow \quad \delta g^\perp_{\mu\nu} = A^a{}_b n^b_\mu n^c_\nu \delta_{ac} + n^a_\mu A^c{}_b n^b_\nu \delta_{ac} \ , \tag{B.7}$$

or equivalently

$$\delta g^\perp_{\mu\nu} = (n^{1\alpha}n^{1\beta}\delta g_{\alpha\beta})n^1_\mu n^1_\nu + (n^{2\alpha}n^{2\beta}\delta g_{\alpha\beta})n^2_\mu n^2_\nu + (n^{1\alpha}n^{2\beta}\delta g_{\alpha\beta})(n^2_\mu n^1_\nu + n^1_\mu n^2_\nu) \ . \tag{B.8}$$

C. Displacement operator for the free scalar

In this appendix we consider the theory of a free scalar in four dimensions, and we explore the defect OPE of the low lying bulk primaries. In doing so, we give a concrete identity to the displacement operator in a double sheeted geometry, in terms of Fourier modes of the fundamental field. Given the Lagrangian of a four dimensional free massless boson

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 \tag{C.1}$$

the propagator in presence of a conical singularity placed in $r = 0$ can be derived:

$$\langle \phi(x)\phi(x')\rangle_n = \frac{\sinh(\frac{\eta}{n})}{8\pi^2 n r r' \sinh\eta \left(\cosh(\frac{\eta}{n}) - \cos(\frac{\theta}{n})\right)}, \tag{C.2}$$

where

$$\cosh\eta = \frac{r^2 + r'^2 + y^2}{2rr'}. \tag{C.3}$$

In the following, we use alternatively polar coordinates around the defect with $x = (r, \theta, y^1, y^2)$, $x' = (r', 0, 0, 0)$ or complex coordinates $x = (z, \bar{z}, y^i)$, $x' = (z', \bar{z}', y'^i)$ with $z = re^{i\theta}$. Assuming integer values of n and expanding (C.2) in the defect OPE limit, i.e. for $r \to 0$ and $r' \to 0$, one finds

$$\langle \phi(r,\theta,y^i)\phi(r',0,y^i)\rangle_n = \frac{1}{4n\pi^2}\left(\frac{1}{y^2} + 2\sum_{k=1}^{n-1} \frac{r^{\frac{k}{n}} r'^{\frac{k}{n}}}{(y^2)^{1+\frac{k}{n}}} \cos\frac{k\theta}{n} - \frac{r^2 + r'^2 - 2rr'\cos\theta}{y^4} + \ldots\right) \tag{C.4}$$

where the dots indicate higher powers of r and r'. This result can be precisely reproduced by the following OPE expansion for the field ϕ [15]

$$\phi(z,\bar{z}) = \phi(0) + \frac{1}{2\pi\sqrt{n}} \sum_{k\in\mathrm{N}} \left(z^{\frac{k}{n}} O_{\frac{k}{n}} + \bar{z}^{\frac{k}{n}} \bar{O}_{\frac{k}{n}}\right) + \ldots \tag{C.5}$$

where the dots indicate contributions from the descendants and the operators $O_{\frac{k}{n}}$ are defect primaries with transverse spin $s = \frac{k}{n}$ and scaling dimension $\Delta = s + 1$. This twist one[16] defect spectrum can be easily understood through the requirement that every conformal family appearing on the r.h.s. of (C.5) is annihilated by the Laplace operator. Indeed, the latter reduces to the two-dimensional $\partial_z \partial_{\bar{z}}$ differential operator once we disregard descendants, and the holomoprhicity property of the contribution of defect primaries to the OPE quickly follows. On the other hand the possible values of the spin are fixed by the symmetry preserved by the defect, i.e. a n-fold cover of $SO(2)$. The normalization of the operators is fixed by

$$\langle O_{\frac{k}{n}} \bar{O}_{\frac{k}{n}} \rangle_n = \frac{1}{(y^2)^{1+\frac{k}{n}}} \tag{C.6}$$

Let us make one more comment on the nature of the defect spectrum. The twist operator is responsible for the presence of a tower of primaries with non integer transverse spin. While

[15] The two contributions proportional to r^2 and r'^2 in (C.4) originate from the descendant $\partial_i \phi$.

[16] We're calling twist the difference between the scaling dimension and the charge under a transverse rotation. However, let us stress that the latter is a global symmetry from the point of view of the defect theory

these Fourier modes do not possess a local expression in terms of the elementary field, this is not so for the defect operators with integer spin. Their contribution to the defect OPE is modified by the defect, but we can still identify them with derivatives of ϕ in directions orthogonal to the defect[17]. In particular, it will be important in a moment that a defect operator $O_1 = \partial_z \phi$ exists.

We expect to find evidence of the presence of the displacement operator in the defect OPE expansion of the scalar operator ϕ^2. Therefore we consider the connected correlator

$$\langle \phi(x)^2 \phi(x')^2 \rangle_n - \langle \phi(x)^2 \rangle_n \langle \phi(x')^2 \rangle_n = 2 \langle \phi(x)\phi(x') \rangle_n{}^2 \tag{C.7}$$

in the defect OPE limit and we extract the contribution given by operators of dimension 3 (spin 1), which reads

$$\langle \phi(x)^2 \phi(x')^2 \rangle_n \big|_{\text{spin } 1} \sim \frac{r r' \cos\theta}{4n^2\pi^4 y^6}(n+1) \tag{C.8}$$

The defect OPE of ϕ^2 in free theory can be obtained by studying the fusion of two ϕ OPEs. That is, we can pick a basis of operators $O_{\left(\frac{k}{n},\frac{k'}{n}\right)} = O_{\frac{k}{n}} O_{\frac{k'}{n}}$. In particular at dimension 3 one has several possible contributions coming from the combination of all the possible spins summing to 1. Among them the operator $O_{(0,1)} = 4\pi^2 n \phi \partial_z \phi$ occupies a special role, since it is there also for $n = 1$. This degeneracy makes it impossible to single out the displacement operator in general. There is however a very simple case, i.e. the one of a double sheeted surface. When $n = 2$ there are only two operators of dimension 3 and spin 1. Those are $O_{\left(\frac{1}{2},\frac{1}{2}\right)}$ and $O_{(0,1)}$. Since the presence of the latter is unrelated to the existence of the twist defect, it is natural to identify the former with the displacement. Luckily, the Ward identities give an easy way of checking whether this expectation is correct. Let us first subtract the contribution of the operators $O_{(0,1)}$ and $\bar{O}_{(0,1)}$ and then focus on the case $n = 2$. In order to single out the contribution of $O_{(0,1)}$ we need to compute its coupling with ϕ^2

$$\langle \phi^2(z,\bar{z},y^i) O_{(0,1)}(z',\bar{z}',0) \rangle_{n,c} \tag{C.9}$$

and extract the relevant constant in the OPE expansion of ϕ^2

$$\phi^2(z,\bar{z}) \sim \beta z O_{(0,1)} + \bar{\beta}\bar{z}\bar{O}_{(0,1)} \tag{C.10}$$

After doing that, by simple Wick contraction and with due care in subtracting the power-like divergences we obtain $\beta = \bar{\beta} = \frac{1}{2\pi^2 n}$. Subtracting this contribution from (C.8) we get

$$\langle \phi(x)^2 \phi(x')^2 \rangle_n \big|_{\text{spin } 1} - \langle \phi(x)^2 \phi(x')^2 \rangle_n \big|_{O_{(0,1)},\bar{O}_{(0,1)}} = \frac{r r' \cos\theta}{4n^2\pi^4 y^6}(n-1) \tag{C.11}$$

Restricting to the case $n = 2$ we argue that

$$\langle \phi(x)^2 \phi(x')^2 \rangle_2 \big|_{\text{Disp}} = \frac{r r' \cos\theta}{16\pi^4 y^6} \tag{C.12}$$

[17]This is somewhat loose: a defect primary will in general be a combination of derivatives orthogonal and parallel to the defect. The one exception is $\partial_a \phi$, for which no mixing happens.

which, given the relevant term in the defect OPE of ϕ^2

$$\phi^2(x^a) \sim \alpha x^a D_a \,, \tag{C.13}$$

implies

$$\alpha^2 C_D = \frac{1}{16\pi^4} \tag{C.14}$$

We recall that the normalization of the displacement operator is fixed by the Ward identity

$$\int d^2y \, \langle \phi^2(x^a,0) D^b(y^i) \rangle_n = \partial^b \, \langle \phi^2(x^a,0) \rangle_n \tag{C.15}$$

which yields

$$\alpha C_D = -\frac{1}{12\pi^3}\left(\frac{1}{n^2} - 1\right) \tag{C.16}$$

We stress that this equality has to hold for any value of n, whereas the identification (C.14) can be univocally made only for $n = 2$. In the latter case we can combine (C.16) and (C.14) to give

$$C_D(n=2) = \frac{1}{16\pi^2} \tag{C.17}$$

Comparing this result with the value of h_n for a free scalar in four dimensions

$$h_n = \frac{n^4 - 1}{720\pi \, n^3}, \tag{C.18}$$

and using the relation (2.14) for $d = 4$

$$C_D(n) = \frac{24 h_n}{\pi} \tag{C.19}$$

we find perfect agreement with the expected value of C_D for $n = 2$. This result unambiguously identifies the operator $O_{(\frac{1}{2},\frac{1}{2})}$ as the displacement operator - up to a normalization - for a double sheeted geometry.

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