Exact Zero Vacuum Energy in twisted $SU(N)$ Principal Chiral Field

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We present a finite set of equations for twisted PCF model. At the special twist in the root of unity we demonstrate that the vacuum energy is exactly zero at any size L. Also in $SU(2)$ case we numerically calculate the energy of the single particle state with zero rapidity, as a function of L.

INTRODUCTION

$SU(N)$ Principal Chiral Field on a cylinder of circumference L has the following classical action S:

$$S = \frac{1}{2g^2} \int_{-\infty}^{\infty} dt \int_0^L dx \ \mathrm{tr} \ \partial_\mu U \partial^\mu U^\dagger. \tag{1}$$

On the quantum level at $L \to \infty$ the theory with periodic boundary conditions has $N-1$ types of massive excitations which are in one to one correspondence with fundamental representations of $SU(N)$. The lowest mass $m = m_1$ is defined by the mechanism of dimensional transmutation $m = m_1 = \frac{\Lambda}{g} e^{-\frac{4\pi}{Ng^2}}$. All other particles appear as bound states and have masses $m_k = m \frac{\sin(\frac{\pi k}{N})}{\sin(\frac{\pi}{N})}$. The spectrum of the theory on $\mathbb{R}^2$ can be completely solved by Bethe Ansatz technique [1, 2]. In case of finite L and periodic boundary conditions the finite system of integral equations with solution in terms of Wronskian determinants was presented in [3].

In this paper we generalize the construction of [3] to the case of twisted boundary conditions $U(t, x+L) = e^{i\Phi} U(t,x) e^{-i\Phi}$. In the case of particular twist $e^{i\Phi} = \Omega = e^{i\frac{\pi}{N} \frac{1+(-1)^N}{2}} \mathrm{diag}[1, e^{\frac{2\pi i}{N}}, e^{\frac{4\pi i}{N}}, ..., e^{\frac{2\pi(N-1)i}{N}}]$ we calculate the energy of single particle state in the rest as a function of L numerically and show that the vacuum energy is exactly zero. This zero is quite unexpected because PCF is a pure bosonic theory without SUSY.

TWISTED TBA

Using the usual logic of the derivation of TBA we can write the vacuum energy as

$$E_0^d = -\lim_{R \to \infty} \frac{\log \mathrm{tr}(e^{-H(L)R})}{R} \tag{2}$$

and then go to the mirror model $\mathrm{Tr}(e^{-\tilde{H}(R)L}) = \mathrm{Tr}(e^{-H(L)R})$ with mirror Hamiltonian $\tilde{H}(R)$ and periodic boundary conditions. Twisted boundary conditions $U(t, x+L) = e^{i\Phi} U(t,x) e^{-i\Phi}$ in the original model can be formalized as an insertion of the defect operator in the mirror model [5]:

$$\mathrm{Tr}(e^{-\tilde{H}(R)L} e^{i\Phi}) \tag{3}$$

where twist $e^{i\Phi}$ commutes with the scattering matrix and acts on one particle states as $e^{i\Phi \otimes 1 - 1 \otimes i\Phi}$ where $e^{i\Phi} = \mathrm{diag}[e^{i\phi_1}, e^{i\phi_2}, ..., e^{i\phi_N}]$. This twist leads us to the following TBA equations ($a \in [1, ..., N-1]$):

$$\mu^{a,0} = mL \frac{\sin \frac{\pi a}{N}}{\sin \frac{\pi}{N}} \cosh \frac{2\pi}{N} u - \log \frac{1}{Y^{a,0}} +$$
$$+ \sum_{a'} \mathcal{K}^{(a',0),(a,0)} * \log(1 + Y^{a',0}) +$$
$$\sum_{s' \neq 0, a'} \mathcal{K}^{(a',s'),(a,0)} * \log(1 + \frac{1}{Y^{a',s'}}),$$
$$\mu^{a,s \neq 0} = -\log Y^{a,s} + \sum_{a'} \mathcal{K}^{(a',0),(a,s)} * \log(1 + Y^{a',0}) +$$
$$\sum_{s' \neq 0, a'} \mathcal{K}^{(a',s'),(a,s)} * \log(1 + \frac{1}{Y^{a',s'}}) \tag{4}$$

with chemical potentials $\mu^{a,s}$:

$$\begin{cases} \mu_{a,s} = -is(\phi_{a+1} - \phi_a), & s > 0 \\ \mu_{a,s} = i|s|(\phi_{a+1} - \phi_a), & s < 0 \\ \mu_{a,s} = 0, & s = 0 \end{cases} \tag{5}$$

and all other ingredients are as in the untwisted case: kernels $\mathcal{K}$ are derivatives of logarithm of S-matrix which can be read off from [2] and Y-functions $Y^{a,0} = \frac{\rho^{a,0}}{\bar{\rho}^{a,0}}$, $Y^{a,s \neq 0} = \frac{\bar{\rho}^{a,s}}{\rho^{a,s}}$ are expressed through the hole and particle's densities $\bar{\rho}^{a,s}$, $\rho^{a,s}$. The convolution $*$ is defined in the usual way: $[f * g](u) = \int_{-\infty}^{\infty} dv f(u-v) g(v)$. In terms of Y-functions TBA (4) can be rewritten as a usual Y-system:

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a,s+1} Y_{a,s-1}} = \frac{1 + 1/Y_{a,s+1}}{1 + 1/Y_{a+1,s}} \frac{1 + 1/Y_{a,s-1}}{1 + 1/Y_{a-1,s}} \tag{6}$$

At large L all middle-node Y-functions are exponentially small $Y_{a,0} = const \times e^{-L p_a}$ and Y-system splits in two independent left and right wings. Here we have introduced the momentum of a's particle $p_a(\theta) = m \frac{\sin \frac{a\pi}{N}}{\sin \frac{\pi}{N}} \cosh \frac{2\pi}{N} \theta$

The asymptotic solution at $L \to \infty$ can be written as:

$$Y_{a,s} = \frac{\chi_{a,s+1}(e^{-i\Phi})\chi_{a,s-1}(e^{-i\Phi})}{\chi_{a+1,s}(e^{-i\Phi})\chi_{a-1,s}(e^{-i\Phi})}, \quad s > 0$$

$$Y_{a,0} = \chi_{a,1}(e^{i\Phi})\chi_{a,1}(e^{-i\Phi})e^{-Lp_a(u)}, \quad s = 0$$

$$Y_{a,s} = \frac{\chi_{a,|s|+1}(e^{i\Phi})\chi_{a,|s|-1}(e^{i\Phi})}{\chi_{a+1,|s|}(e^{i\Phi})\chi_{a-1,|s|}(e^{i\Phi})}, \quad s < 0 \quad (7)$$

where $\chi_{a,s}(g)$ is a character of g in the representation with rectangular Young tableau with a rows and s columns. The ansatz (7) obviously solves Y-system (6) up to $O(e^{-mL})$ terms. However Y-system (6) doesn't depend on twist and we have to check that solution (7) goes through TBA (4) and reproduces the right chemical potentials. Turn out that a special regularization is needed. Namely we should formally modify the twist as

$$e^{\pm i\Phi} \to \text{diag}\{e^{\pm i\phi_1}e^{\epsilon_1}, e^{\pm i\phi_2}e^{\epsilon_2}, ..., e^{\pm i\phi_N}e^{\epsilon_N}\} \quad (8)$$

where $0 > \epsilon_1 > \epsilon_2 > ... > \epsilon_N$, $\frac{\epsilon_i}{\epsilon_{i+1}}$ - fixed and $\epsilon_N \to 0$. Let's show how this prescription reproduces the chemical potentials in the case of $a \in [2, ..., N-2]$ and $s > 0$. The corresponding equation reads:

$$\mu_{a,s} = -\log(Y_{a,s}) - \sum_{s'=1}^{\infty} min(s,s')\log(1 + \frac{1}{Y_{a-1,s'}}) +$$

$$+ \sum_{s'=1}^{\infty}(2min(s,s') - \delta_{s,s'})\log(1 + \frac{1}{Y_{a,s'}}) -$$

$$- \sum_{s'=1}^{\infty} min(s,s')\log(1 + \frac{1}{Y_{a+1,s'}}) + O(e^{-mL}) \quad (9)$$

Now let's exponentiate its right hand side:

$$e^{\text{r.h.s. of (9)}} =$$

$$= \lim_{p\to\infty}\left(\frac{\chi_{a-1,p+1}(e^{i\Phi})}{\chi_{a-1,p}(e^{i\Phi})}\frac{\chi_{a,p}(e^{i\Phi})^2}{\chi_{a,p+1}(e^{i\Phi})^2}\frac{\chi_{a+1,p+1}(e^{i\Phi})}{\chi_{a+1,p}(e^{i\Phi})}\right)^s$$

(10)

Using the above-mentioned epsilon-prescription and the first Weyl formula for characters it's easy to verify that:

$$\lim_{p\to\infty}\frac{\chi_{a,p+1}}{\chi_{a,p}} = e^{-i\phi_1}e^{-i\phi_2}...e^{-i\phi_a} \quad (11)$$

what gives us exponent of l.h.s. of (9):

$$e^{\text{r.h.s. of (9)}} = \left(\frac{e^{-i\phi_{a+1}}}{e^{-i\phi_a}}\right)^s = e^{\mu_{a,s}} \quad (12)$$

For other values of a and s the consideration is similar.

EXACT SOLUTION FOR VACUUM AT TWIST Ω

It is surprising that the asymptotic solution (7) turns out to be exact at the twist Ω. Indeed in case of large

L middle-node Y-functions $Y_{a,0}$ was exponentially small, leading to the decoupling of left and right wings. At the twist Ω all characters in fundamental representations are zero $\chi_{a,0}(\Omega) = 0$ and it leads to a vanishing middle-node Y-functions $Y_{a,0} = 0$. This formal solution is singular but it has a natural regularization with above-mentioned epsilon-prescription. In the case of $\epsilon_i \to 0$ the ansatz (7) solves Y-system (or TBA) up to the terms $o(\epsilon)$ and gives us solution at any L. Due to the vacuum energy formula:

$$E_\Omega^{exact}(L) =$$

$$= \lim_{\epsilon_i \to 0} -\frac{1}{N}\sum_{a=1}^{N-1}\int_{-\infty}^{\infty}d\theta p_a(\theta)\log(1 + o(\epsilon_i)) = 0 \quad (13)$$

we get exact zero for vacuum energy.

At the large L limit this zero energy can be seen directly from (3). Indeed the leading contribution is a sum over one-particle states and for the particle of the type a we get factor $\chi_{a,0}(e^{i\Phi})\chi_{a,0}(e^{-i\Phi})$ which comes from the sum over basis vectors of fundamental representation with a vertical boxes:

$$\sum_p \langle p|e^{i\Phi}|p\rangle = \chi_{a,0}(e^{i\Phi})\chi_{a,0}(e^{-i\Phi}) \quad (14)$$

what gives zero at twist Ω. The next correction is a Lüscher term corresponding to two-particle contribution and in the $SU(2)$ case it was calculated in [5]. Plugging the twist Ω in (3.43) of [5] it is easy to verify that the Lüscher term vanishes.

On the other hand, at small L, we have a weakly coupled theory and the one-loop Casimir energy was presented [?] in [6]:

$$E_{e^{i\Phi}}^{1-loop} = -\frac{1}{\pi L}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\left|tr(e^{in\Phi})\right|^2 - 1\right) \quad (15)$$

In the periodic case we have the standard Casimir energy

$$E_{periodic}^{1-loop} = -\frac{1}{\pi L}\sum_{n=1}^{\infty}\frac{1}{n^2}(N^2 - 1) = -\frac{\pi(N^2-1)}{6L},$$

corresponding to $N^2 - 1$ free bosons [3].

In the case of twist Ω we have:

$$E_\Omega^{1-loop} = -\frac{1}{\pi L}\left(\sum_{k=1}^{\infty}\frac{1}{(Nk)^2}(N)^2 - \sum_{n=1}^{\infty}\frac{1}{n^2}\right) = 0, \quad (16)$$

because $tr\,\Omega^n = 0$ for $n \neq Nk$

ABA

The Y-system (6) can be reformulated in the form of Hirota equations:

$$T_{a,s}^+T_{a,s}^- = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1} \quad (17)$$

and the original Y-functions are expressed through the set of $\{T_{a,s}(\theta)\}$ as $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$.

In the presence of the twist $e^{i\boldsymbol{\Phi}}$ the general solution of Hirota equation (up to a gauge degrees of freedom) has the following generating functional for T-functions $\lim_{L\to\infty} T^R$:

$$\hat{W}^R = \frac{1}{(1 - e^{-i\phi_N} X_{(N)}^R(\theta)e^{i\partial_\theta})} \cdot$$

$$\cdot \frac{1}{(1 - e^{-i\phi_{N-1}} X_{(N-1)}^R(\theta)e^{i\partial_\theta})} \cdot \ldots \cdot \frac{1}{(1 - e^{-i\phi_1} X_{(1)}^R(\theta)e^{i\partial_\theta})}$$

$$= \sum_{s=0}^\infty \frac{T_{1,s}^R(\theta + \frac{i}{2}(s-1))}{\varphi(\theta - i\frac{N}{4})} e^{is\partial_\theta}, \qquad (18)$$

where X_k^R and $\varphi(\theta)$ are defined as in [3], in terms of some polynomial functions Q_k^R encoding the excitations defining a given state. The exponent R stands for a specific choice of gauge where $\lim_{L\to\infty} T_{a,s}^R = 0$ is zero if $s < 0$ with $0 < a < N$, whereas for other values of (a,s) the limit is a polynomial.

Similarly, there exists a gauge denoted T^L where $\lim_{L\to\infty} T_{a,s}^L$ vanishes if $s > 0$ with $0 < a < N$. $T_{1,-s}^L$ has a generating series which differs from the generating series (18) of $T_{1,s}^R$ by the substitution $\phi_i \to -\phi_i$

Canceling poles at $T_{1,1}^R$ at $w - \frac{i}{2}(\frac{N}{2} - k + 1)$ we get the twisted auxiliary Bethe equations:

$$-e^{i(\phi_{k-1}-\phi_k)} = \frac{Q_{k-2}^R(w - \frac{i}{2})Q_{k-1}^R(w + i)Q_k^R(w - \frac{i}{2})}{Q_{k-2}^R(w + \frac{i}{2})Q_{k-1}^R(w - i)Q_k^R(w + \frac{i}{2})} \qquad (19)$$

and similar for the left wing.

Asymptotic form of middle-node Y-functions reads as:

$$Y_{a,0}(\theta) \sim e^{-mLp_a} \frac{T_{a,1}T_{a,-1}^L}{T_{a+1,0}T_{a-1,0}} \frac{\varphi^{[-\frac{N}{2}-a+1]}\varphi^{[-\frac{N}{2}-a+1]}}{\varphi^{[-\frac{N}{2}+a-1]}\varphi^{[-\frac{N}{2}+a+1]}} \cdot$$

$$\cdot \frac{1}{\Pi_a \left((S^{[-\frac{N}{2}]})^2 \chi_{CDD}^{[-\frac{N}{2}]} \right)} \qquad (20)$$

and it leads to the massive Bethe equation $Y_{1,0}(\theta_j + i\frac{N}{4}) = 0$ as in the untwisted case[?]:

$$-1 = \frac{e^{-imL\sinh\frac{2\pi}{N}\theta_j}}{\chi_{CDD}S^2(\theta_j)} \frac{Q_{N-1}^R(\theta_j - \frac{i}{2})Q_{N-1}^L(\theta_j - \frac{i}{2})}{Q_{N-1}^R(\theta_j + \frac{i}{2})Q_{N-1}^L(\theta_j + \frac{i}{2})} \qquad (21)$$

FINITE L

Up to some gauge degrees of freedom, the general solution of Hirota equation can be represented through the following Wronskian determinant:

$$T_{a,s}^{(R)}(\theta) = i^{\frac{N(N-1)}{2}} \text{Det}(c_{j,k})_{1 \leqslant j,k \leqslant N} \qquad (22)$$

where $c_{j,k} = \begin{cases} e^{i\phi_j(s/2-k)}\overline{q}_j^{[s+a+1+\frac{N}{2}-2k]} & \text{if } k \leqslant a \\ e^{i\phi_j(-s/2-k)}q_j^{[-s+a+1+\frac{N}{2}-2k]} & \text{if } k > a, \end{cases}$

in terms of $2N$ complex functions $\{q_j \bar{q}_j\}$, where we use the notation $f^{[n]}(\theta) \equiv f(\theta + n\frac{i}{2})$. It turns out [3] that q_j (resp $\bar{q}_j$) is analytic on the lower (resp upper) half plane, and that q_j and $\bar{q}_j$ are complex conjugated, so that one has $\overline{T_{a,s}(\theta)} = T_{N-a,s}(\bar{\theta})$. In addition $q_j - \bar{q}_j$ decreases at large θ as $e^{-L\cosh(2\pi/N\theta)}$, hence the parameterisation

$$P_j + i\mathcal{C} * f_j = \begin{cases} \bar{q}_j & \text{if Im}(\theta) > 0 \\ q_j & \text{if Im}(\theta) < 0 \end{cases} \qquad (23)$$

where $\mathcal{C} \equiv \frac{1}{2i\pi\theta}$ is the Cauchy Kernel, $f_j \equiv i(\bar{q}_j - q_j)$ is a real jump density and the P_j's are polynomial whose $L \to \infty$ limit is related (through Wronskian determinants) to the polynomials Q_j^R. In the periodic case, it was shown [3] how to write equations for the densities f_j for symmetric states, i.e. for states such that $T_{a,s}^R = T_{a,-s}^L$.

For the single particle state with zero rapidity, which we denote Θ_0 and which defines the mass gap in the periodic case, this symmetry $T_{a,s}^R = T_{a,-s}^L$ is actually broken by the introduction of the twist. One can however see that for several states such as the vacuum and this state Θ_0, the introduction of the twist preserves a slightly different symmetry:

$$T_{a,-s}^{(L)}(\theta) = (-1)^{\mathcal{N}} T_{N-a,s}(-\theta) \qquad (24)$$

where $\mathcal{N}$ denotes the number of Bethe roots (for instance it is zero for vacuum). One can easily follow the lines of [3] to write equations for the densities of such state. In what follows we will do this for both the vacuum and the single-particle state Θ_0, in the $N = 2$ case.

For Vacuum we have $Q_j^R = 1 = Q_j^L$ (for all j), which corresponds to $P_j = 1$ in (23). By contrast, the considered single-particle state Θ_0 has $Q_1^R = Q_1^L = 1$ and $Q_2^R(\theta) = Q_2^L(\theta) = \theta$, i.e. $P_1 = 1$ and $P_2(\theta) = \theta + c$ where $\lim_{mL\to\infty} c = -\frac{1}{2\tan\phi}$. Gauge freedom allows to set $q_1 = 1$, and then one can notice that $T_{1,-1} = -i(\bar{q}_2 - q_2) = -f_2$, hence the general expression (22) reduces to

$$T_{1,s}^{(R)} = ie^{-i(s+1)\phi} \left((\theta^{[s+1]} + c)^{\mathcal{N}} + i\mathcal{C}^{[s+1]} * T_{1,-1}^{(R)} \right)$$

$$- ie^{i(s+1)\phi} \left((\theta^{[-s-1]} + c)^{\mathcal{N}} + i\mathcal{C}^{[-s-1]} * T_{1,-1}^{(R)} \right), \quad (25)$$

where the equation is valid for $|\text{Im}(\theta)| < s + 1$, we denote $\phi \equiv \phi_1 = -\phi_2$ and the single particle state Θ_0 corresponds to $\mathcal{N} = 1$ while the vacuum corresponds to $\mathcal{N} = 0$.

If $\mathcal{N} = 1$, the expression (25) involves a constant c which is fixed at finite size by the condition $T_{1,1}^R\big|_{\theta=0} = 0$. This condition comes from the requirement that $T_{1,1}^R(\theta)$ and $T_{1,1}^L(\theta) = -T_{1,1}^R(-\theta)$ are gauge-equivalent, and if the gauge transformation is regular this means that the zeroes of $T_{1,1}^R$ are symmetric under $\theta \to -\theta$. Hence the zero which lies at origin in the $L \to \infty$ limit stays exactly at origin at finite L.

If we denote by $\hat{T}_s$ the function on the r.h.s. of (25), the equation (25) now reads: $|\text{Im}(\theta)| < s + 1 \Rightarrow \hat{T}_s(\theta) = T_{1,s}^{(R)}(\theta)$. Then we can follow the lines of [7] to find integral equations for the function $T_{1,-1}^{(R)}$ entering equations (25) are obtained following[7]. First, one obtains

$$\hat{T}_0(\theta) = \begin{cases} T_{2,0}^{(R)}(\theta - \frac{i}{2}) & \text{if } \text{Im}(\theta) > \frac{1}{2} \\ T_{1,0}^{(R)}(\theta) & \text{if } \text{Im}(\theta) \in]-\frac{1}{2}, \frac{1}{2}[\\ T_{0,0}^{(R)}(\theta + \frac{i}{2}) & \text{if } \text{Im}(\theta) < -\frac{1}{2}. \end{cases} \quad (26)$$

Next, one gets (for $|\text{Im}(\theta)| < 1/2$)

$$Y_{1,0} = e^{-mL\cosh(\pi\theta)} \frac{(-1)^{\mathcal{N}+\sigma} \hat{T}_1\{\hat{T}_1\}}{\left(|\hat{T}_0^{[+2]}\{\hat{T}_0^{[+2]}\}|^2\right)^{*s}} \quad (27)$$

where we use the reality of $\hat{T}$ and we denote $f^{*s} = \exp(\log f * (1/(2\cosh(\pi\theta))))$.

The parameter $\sigma \in \{0, 1\}$ is an a priori remaining freedom, from the fact that the equation for $Y_{1,0}^+ Y_{1,0}^-$ only fixes $Y_{1,0}$ up to a sign. This sign can also be interpreted as an ambiguity in the choice of the branch of the log in $f^{*s} = exp(\log(f) * s)$. In the periodic case ($\phi = 0$), this sign is $\sigma = 0$ and it is fixed by requiring $Y_{1,0}$ to be a positive function. In our numeric resolution, we assumed that $\sigma = 0$ holds also at arbitrary twist.

Finally, since we have on the real axis $Y_{1,0} = \frac{\hat{T}_1 T_{1,-1}^{(R)}}{|\hat{T}_0^{[+1+0]}|^2}$, we get

$$T_{1,-1}^{(R)} = e^{-mL\cosh(\pi\theta)} \frac{(-1)^{\mathcal{N}+\sigma}|\hat{T}_0^{[+1+0]}|^2\{\hat{T}_1\}}{\left(|\hat{T}_0^{[+2]}\{\hat{T}_0^{[+2]}\}|^2\right)^{*s}}. \quad (28)$$

This equation is the twisted analog of equation (43) of [7] for the two states we consider, and it allows to numerically find their energy $E = \mathcal{N} - \frac{m}{2}\int\cosh(\pi\theta)\log(1 + Y_{10})\mathrm{d}\theta$ for arbitrary twist.

The numeric resolution relies on the fact that equations (28, 25) give a relation of the form $T_{1,-1}^R = F(T_{1,-1}^R)$, where F is a contraction mapping whose fixed points are found iteratively.

NUMERICAL RESULTS

At large L, numerical results can be compared to large-L expressions of the energy (Lüscher corrections). These large L expressions are obtained from (27) where the T-functions are replaced with their large-L expression, which can be obtained for instance by setting $T_{1,-1}^R = 0$ in (25). On figures Fig.1 and Fig.2 these large L energies are plotted in gray, while numeric energies of the vacuum and the state Θ_0 are plotted in black. We see that the energy of the states are smooth functions of the twist, which converge, in the $\phi \to 0$ limit, to the periodic model's vacuum energy and mass gap, which were already produced in the literature [8].

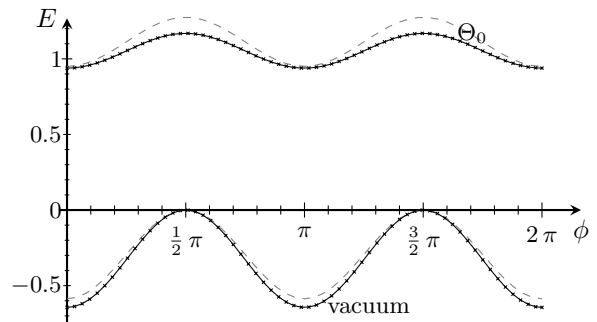FIG. 1. Energies of vacuum and Θ_0 as functions of twist at $L = 1$. Dashed gray lines are the large L expression of energy as a function of twist. In all numerical calculations we put $m = 1$.
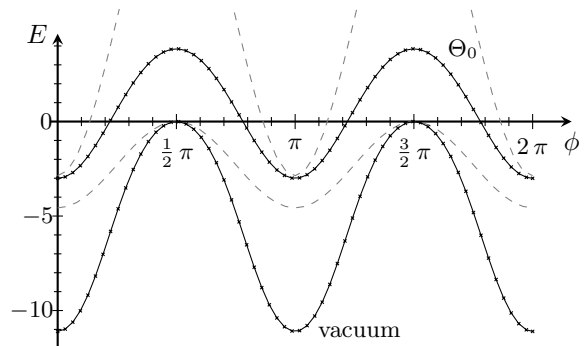

FIG. 2. Energies of vacuum and Θ_0 as functions of twist at $L = 1/10$. Deviation from the dashed gray lines shows the importance of finite-size effects.

DISCUSSION

One of the main results of this paper is the observation that the vacuum energy is exactly zero at the special twist Ω. Such behaviour is highly surprising in the absence of supersymmetry which could produce a vanishing vacuum energy by the mechanism of cancellation between fermionic and bosonic degrees of freedom[?]. In the language of integrability this vacuum state looks very similar to the vacuum state in deformed AdS/CFT [5, 10, 11] which has exactly zero energy at the trivial twist, but in $\mathcal{N} = 4$ SYM it is just a consequence of unbroken supersymmetry. It would certainly be fascinating to understand the underlying mechanism or symmetry which is responsible for vanishing of vacuum energy in our case.

In the case of vacuum state the vanishing middle-node Y-functions $Y_{a,0}$ can be interpreted as the vanishing of densities $\rho^{a,0}$ of massive particles what makes it similar to the vacuum at $L = \infty$. It is most likely related to the idea of adiabatic continuity proposed in [4]. Using resurgence techniques the authors of [4] also proposed that the mass gap has confined form and scales as $\sim L$ at small L. As we see from Fig.3 the state Θ_0 scales as in the weak
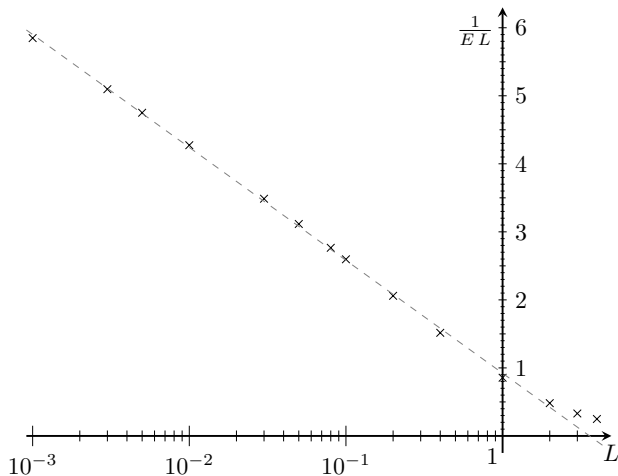
FIG. 3. Energy of the state Θ_0 at twist $\phi = \pi/2$. Crosses are numeric results and the dashed gray line is a linear fit. In spite of the limited numeric precision, it is manifest that at small L, $\frac{1}{EL}$ scales linearly with $\log L$.

coupling regime and is proportional to $\frac{1}{L \log L}$. One can expect that in the presence of the twist, the mass gap is not the difference of energies of the state Θ_0 and vacuum. Probably another state has lower energy than the state Θ_0, and that either this state does not correspond to a physical solution of Y-system in the periodic case, or that it is the same solution as vacuum in this limit. This would be similar to what happens for $SU(2)$ Heisenberg spin chains where the introduction of a twist lifts the degeneracy of some states belonging to the same multiplet, including states in the multiplet of vacuum. At the level of the Y-system (or equivalent finite system of integral equations), the search of the corresponding state is not a trivial task and should be addressed in a future paper.

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