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Spin Chains and Gustafson's Integrals

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Abstract. The Gustafson's integrals are the multidimensional generalization of the classical Mellin–Barnes integrals. We show that some of these integrals arise from relations between matrix elements in the Sklyanin's representation of Separated Variables in the spin chain models. We also present several new integrals.

1. Introduction

In his papers [1, 2] R.A. Gustafson evaluated integrals representing multidimensional generalization of the Mellin–Barnes integrals. The integrals associated with the classical $su(N)$ and $sp(N)$ Lie algebras take the form [1]

$$\left(\prod_{n=1}^{N}\int_{-i\infty}^{i\infty}\frac{dz_n}{2\pi i}\right)\frac{\prod_{k=1}^{N+1}\prod_{j=1}^{N}\Gamma(\alpha_k-z_j)\Gamma(\beta_k+z_j)}{\prod_{k<j}\Gamma(z_k-z_j)\Gamma(z_j-z_k)}=\frac{N!\prod_{k,j=1}^{N+1}\Gamma(\alpha_k+\beta_j)}{\Gamma\big(\sum_{k=1}^{N+1}(\alpha_k+\beta_k)\big)} \tag{1}$$

and

$$\left(\prod_{n=1}^{N}\int_{-i\infty}^{i\infty}\frac{dz_n}{2\pi i}\right)\frac{\prod_{k=1}^{2N+2}\prod_{j=1}^{N}\Gamma(\alpha_k\pm z_j)}{\prod_{k=1}^{N}\Gamma(\pm 2z_k)\prod_{k<j}\Gamma(z_k\pm z_j)\Gamma(-z_k\pm z_j)}=\frac{2^N N!\prod_{k<j}\Gamma(\alpha_k+\alpha_j)}{\Gamma\big(\sum_{k=1}^{2N+2}\alpha_k\big)}\,, \tag{2}$$

where $\Gamma(\alpha\pm\beta)\equiv\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)$ and the integration contours separate the series of poles of Γ functions, $\{\alpha_k+n_k\}$ and $\{-\beta_k-n_k\}$, $k=1,\dots,N$, $n_k\in\mathbb{Z}_+$, in the first integral, and $\{\alpha_k+n_k\}$ and $\{-\alpha_k-n_k\}$, $k=1,\dots,2N+2$ in the second one.

In this paper we show that Gustafson's integrals (1), (2) arise in a natural way in the integrable spin chain models. Namely, both integrals can be related to the matrix element of the shift operator T_γ (the operator of translations) in the Sklyanin's representation of Separated Variables (SoV) [3]. Moreover, we obtain a new identity which we were not able to derive from the Gustafson's integrals (1) and (2). It takes the form

$$\left(\prod_{n=1}^{N}\int\frac{dz_n}{2\pi i}\right)\frac{\prod_{j=1}^{N}\Big(\prod_{k=1}^{N+1}\Gamma(\alpha_k-z_j)\Big)\Big(\prod_{m=1}^{N}\Gamma(z_j\pm\beta_m)\Big)}{\prod_{k<j}\Gamma(z_k\pm z_j)\Gamma(z_j-z_k)}=\frac{N!\prod_{j=1}^{N}\prod_{k=1}^{N+1}\Gamma(\alpha_k\pm\beta_j)}{\prod_{j<k}^{N+1}\Gamma(\alpha_j+\alpha_k)}\,, \tag{3}$$

where it is supposed that the series of poles $\{\alpha_k+n_k\}$ and $\{\pm\beta_k-n_k\}$ are separated by the integration contours.

We calculate also the scalar products between the eigenfunctions of the different elements of monodromy matrix and show that evaluation of these scalar products in SoV representation gives rise to new integral identities.

The paper is organized as follows: Sect. 2 contains the basic facts about the spin chain models. In Sect. 3 we recall the construction of the SoV representation and provide the explicit expressions for the corresponding basis functions. Scalar products of certain eigenfunctions are calculated in sect. 4 and 5. We also show that the SoV representation for the matrix element of the translation operators gives rise to the Gustafson's integrals (1), (2). In sect. 6 we present several new integrals which follow from relations between the eigenfunctions of the monodromy matrix for closed spin chain. The final sect. 7 contains a short summary and outlook. Some technical details and elements of the diagrammatic technique are given in the Appendix A.

2. Spin chain models

One dimensional quantum mechanical lattice models with the dynamical variables being generators of some Lie algebra are usually called the spin chain magnets. We consider a model with the $SL(2,R)$ symmetry group. The dynamical variables are the generators of the $SL(2,R)$ group

$$S_+^{(k)}=z_k^2\partial_{z_k}+2s_kz_k,\qquad S_0^{(k)}=z_k\partial_{z_k}+s_k,\qquad S_-^{(k)}=-\partial_{z_k}\,, \tag{4}$$

where the index k enumerates the lattice sites, $k = 1, \ldots, N$ and the parameter s_k (spin) specifies the representation of the $SL(2,R)$ group in the k-th site. Henceforth we will consider the homogeneous spin chains, $s_1 = s_2 = \ldots = s_N = s$. The generators (4) act on the irreducible discrete series representation of the $SL(2,R)$ group, D_s^+, the spin s being a positive integer or half-integer number, which is realized on the space of functions holomorphic in the upper complex half-plane [4]. The Hilbert space of the model is given by the direct product of vector spaces of the representation D_s^+ in each site, $\mathcal{H}_N = \prod_{k=1}^N \otimes V_s$. Thus the space $\mathcal{H}_N$ is the space of functions of N complex variables holomorphic in each variable in the upper half-plane and equipped with the invariant scalar product [4], which takes the form

$$\big(f_1, f_2\big) = \prod_{k=1}^{N} \int \mathcal{D}z_k \Big(f_1(z_1, \ldots, z_N)\Big)^\dagger f_2(z_1, \ldots, z_N)\,. \tag{5}$$

Here the integration goes over the upper half-plane $y \geq 0$, $(z = x + iy)$ and the integration measure is defined as

$$\mathcal{D}z = \frac{2s-1}{\pi}(2y)^{2s-2} dx dy\,. \tag{6}$$

The scalar product (5) is invariant under the $SL(2,R)$ transformations

$$f(z_1, \ldots, z_N) \mapsto [T(g)f](z_1, \ldots, z_N) = \prod_{k=1}^{N} \frac{1}{(cz_k + d)^{2s}} f(z_1', \ldots, z_N')\,, \tag{7}$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)$ and $z_k' = (az_k + b)/(cz_k + d)$. The generators (4) are anti-hermitian w.r.t. this scalar product.

The Quantum Inverse Scattering Method (QISM) [5, 6, 7, 8] allows one to define a physically meaningful Hamiltonian as a function of the dynamical variables, $S_k^{(\alpha)}$, $k = 1, \ldots, N$ and provides effective tools for solving the corresponding spectral problem. The pivotal object for the QISM machinery (see for a review Refs. [9, 10]) is the monodromy matrix. It is given by a product of Lax operators [6]

$$L_k(u) = u + i \begin{pmatrix} S_0^{(k)} & S_-^{(k)} \\ S_+^{(k)} & -S_0^{(k)} \end{pmatrix}, \tag{8}$$

which are two by two matrices depending on the generators $S_\alpha^{(k)}$ and the spectral parameter $u \in \mathbb{C}$. For the *closed* [6] and *open* [11] spin chains the monodromy matrices are defined as follows:

$$T_N^{cl}(u) \;=\; L_1(u)L_2(u)\ldots L_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}, \tag{9}$$

$$\mathbb{T}_N^{op}(u) \;=\; T_N(-u)\sigma_2 T_N^t(u)\sigma_2 = \begin{pmatrix} \mathbb{A}_N(u) & \mathbb{B}_N(u) \\ \mathbb{C}_N(u) & \mathbb{D}_N(u) \end{pmatrix}. \tag{10}$$

Here σ_2 is the Pauli matrix and the T_N^t is the transposed matrix. The matrix elements $A_N(u), \ldots, D_N(u)$ $(\mathbb{A}_N(u), \ldots, \mathbb{D}_N(u))$ are the (differential) operators acting on the Hilbert space of the model. By construction they are polynomials in the spectral parameter u.

It is shown in the QISM that the entries of the monodromy matrix $T_N^{cl}(u)$ form commuting families,

$$[A_N(u), A_N(v)] = [B_N(u), B_N(v)] = [C_N(u), C_N(v)] = [D_N(u), D_N(v)] = 0\,. \tag{11}$$

For the open spin chains this property holds for the off-diagonal elements only,

$$[\mathbb{B}_N(u), \mathbb{B}_N(v)] = [\mathbb{C}_N(u), \mathbb{C}_N(v)] = 0. \tag{12}$$

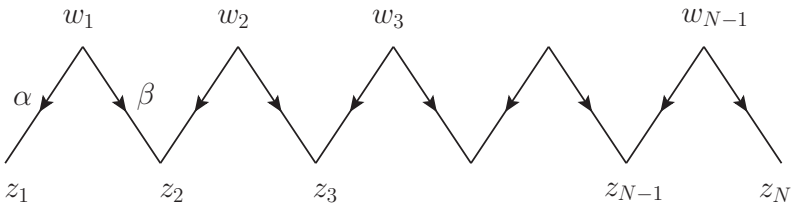

Figure 1. The diagrammatic representation of the operator $\Lambda_N(u)$, $\alpha(u) = s - iu$, $\beta(u) = s + iu$. The arrow from $\bar{w}$ to z with the index α stands for the propagator $D_\alpha(z, \bar{w})$, Eq. (16)

It follows from Eqs. (11), (12) that the eigenfunctions of the operators $A_N(u), B_N(u)$, etc. do not depend on the spectral parameter. At the same time the corresponding eigenvalues are polynomials in u. It turns out that an eigenfunction is completely determined by its eigenvalue. Therefore it is convenient to label eigenfunctions by the roots of the corresponding eigenvalue polynomials. For instance, if Ψ is the eigenfunction of $A_N(u)$ with the eigenvalue $a_N(u) = (u - x_1) \dots (u - x_N)$ we will denote it by $\Psi_{x_1,\dots,x_N}$,

$$A_N(u)\Psi_{x_1,\dots,x_N} = a_N(u)\Psi_{x_1,\dots,x_N} = (u - x_1) \dots (u - x_N)\Psi_{x_1,\dots x_N} \,. \tag{13}$$

The eigenfunctions of operators $B_N(u)$, $A_N(u)$ etc, provide the convenient bases (Sklyanin's representation of SoV [3]) for studies of the spin chain models. All these eigenfunctions admit an explicit representation in the form of multi-parametric integrals which we discuss in the next section.

Closing this section we note that the operators, B_N and C_N, A_N and D_N, $\mathbb{B}_N$ and $\mathbb{C}_N$ are related to each other by an inversion [15], so that is is sufficient to consider the operators B_N, A_N and $\mathbb{B}_N$ only.

3. Sklyanin's representation of Separated Variables

The eigenfunctions of the operators $B_N(u)$, $\mathbb{B}_N(u)$ and $A_N(u)$ were constructed in Refs. [12, 13, 14], respectively. In this section we present the explicit expressions for these eigenfunctions and discuss their properties.

3.1. B_N-system

In order to present the result in a compact form we define an auxiliary operator $\Lambda_N(u)$. This operator, where $u \in \mathbb{C}$ is a spectral parameter, maps a function of $N - 1$ variables to a function of N variables according the following rule

$$[\Lambda_N(u)f](z_1, \dots, z_N) = \left(\prod_{k=1}^{N-1} \int \mathcal{D}w_k \right) \Lambda_N^{(u)}(z_1, \dots, z_N | w_1, \dots w_{N-1})\, f(w_1, \dots, w_{N-1}) \,, \tag{14}$$

where

$$\Lambda_N^{(u)}(z_1, \dots, z_N | w_1, \dots w_{N-1}) = \prod_{j=1}^{N-1} D_{s-iu}(z_j, \bar{w}_j)\, D_{s+iu}(z_{j+1}, \bar{w}_j) \tag{15}$$

and the function D_α (propagator) is defined as follows:

$$D_\alpha(z, \bar{w}) = \left(\frac{i}{z - \bar{w}} \right)^\alpha \,. \tag{16}$$

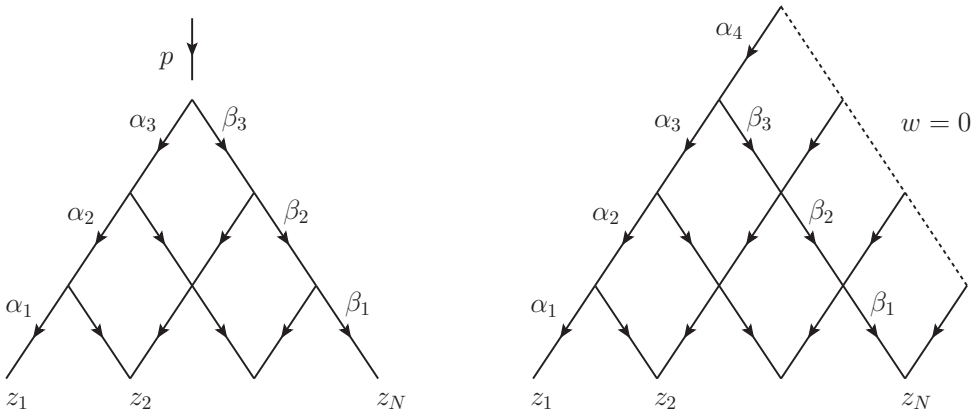

Figure 2. The diagrammatic representation of the eigenfunction $\Psi_B^{(N)}(p,\vec{x}|\vec{z})$, left diagram, and $\Psi_A^{(N)}(\vec{x}|\vec{z})$, right diagram, for $N=4$. The indices are defined as follows: $\alpha_k = s - ix_k$, $\beta_k = s + ix_k$. In a given layer the parallel arrows have the same indices. The arrows attached to the dashed line start from the point $w = 0$.

Note, that under conjugation the propagator transforms as follows, $(D_\alpha(z,\bar{w}))^\dagger = D_{\alpha^*}(w,\bar{z})$.

The operator Λ_N has the following properties: first, $\Lambda_N(u)$ is annihilated by the operator $B_N(u)$,

$$B_N(u)\Lambda_N(u) = 0 \tag{17}$$

and, second, it satisfies the following exchange relation [13]

$$\Lambda_N(u_1)\Lambda_{N-1}(u_2) = \Lambda_N(u_2)\Lambda_{N-1}(u_1)\,. \tag{18}$$

The eigenfunctions of the operator B_N are obtained by a consecutive application of the operators Λ_k to the exponential function

$$\Psi_B^{(N)}(p,\vec{x}|\vec{z}) = b_N(p)\Lambda_N(x_1)\Lambda_{N-1}(x_2)\dots\Lambda_2(x_{N-1})\,e^{ipw}\,. \tag{19}$$

The separated variables $\vec{x} = \{x_1,\dots,x_{N-1}\}$ are real numbers, $x_k \in \mathbb{R}$, and $p \geq 0$. The normalization factor is convenient to choose as follows:

$$b_N(p) = p^{Ns-\frac{1}{2}}(\Gamma(2s))^{-N^2/2}. \tag{20}$$

It follows from Eqs. (17), (18) that $\Psi_B^{(N)}(p,\vec{x}|\vec{z})$, Eq. (19), is the eigenfunction of the operator $B_N(u)$,

$$B_N(u)\Psi_B^{(N)}(p,\vec{x}|\vec{z}) = p(u-x_1)\dots(u-x_{N-1})\,\Psi_B^{(N)}(p,\vec{x}|\vec{z})\,. \tag{21}$$

It is symmetric under permutation of the separated variables $\{x_1,\dots,x_{N-1}\}$. Since the operator $B_N(u)$ is self-adjoint for real u, $(B_N(u))^\dagger = B_N(u)$, the eigenfunctions are mutually orthogonal [12]

$$\left(\Psi_B^{(N)}(p',\vec{x}'),\Psi_B^{(N)}(p,\vec{x})\right) = (2\pi)^{N-1}\;\delta(p-p')\left(\sum_S \delta(x-Sx')\right)\frac{\prod_{j\neq k}\Gamma(i(x_k-x_j))}{\prod_{k=1}^{N-1}\left[\Gamma(\alpha_{x_k})\Gamma(\beta_{x_k})\right]^N}\,, \tag{22}$$

where $\alpha_x = s - ix$, $\beta_x = s + ix$,

$$\left(\sum_S \delta(x-Sx')\right) \equiv \sum_{S\in\mathcal{S}_{N-1}} \delta(x_1 - x'_{s_1})\dots\delta(x_{N-1} - x'_{s_{N-1}}) \tag{23}$$

and the sum goes over all permutations.

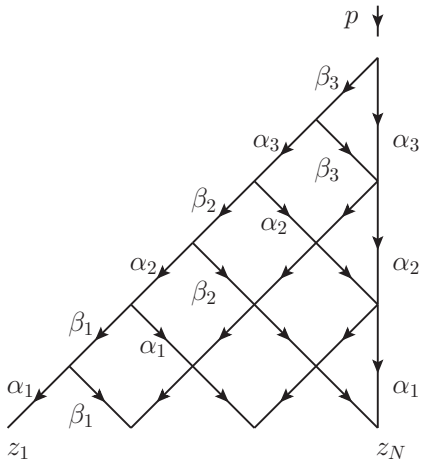

Figure 3. The diagrammatic representation of the eigenfunction $\Psi^{(N)}_{\mathbb{B}}(p,\vec{x}|\vec{z})$, $N=4$. The notations are the same as in Fig. 2.

3.2. A_N-system

The eigenfunctions of the operator $A_N(u)$ is constructed in the similar way [14]:

$$\Psi^{(N)}_{A}(\vec{x}|\vec{z}) = a_N(\vec{x})\,\widehat{\Lambda}_N(x_1)\widehat{\Lambda}_{N-1}(x_2)\dots\widehat{\Lambda}_1(x_N)\,, \tag{24}$$

where

$$a_N = (\Gamma(2s))^{-N^2/2} \tag{25}$$

and the operator $\widehat{\Lambda}_N$ is defined as follows

$$[\widehat{\Lambda}_N(x)f](z_1,\dots,z_N) = D_{s+ix}(z_N,0)\,[\Lambda_N(x)f](z_1,\dots,z_N)\,. \tag{26}$$

The operator satisfies equations similar to Eqs. (17) and (18):

$$A_N(u)\widehat{\Lambda}_N(u) = 0\,, \qquad\qquad \widehat{\Lambda}_N(u_1)\widehat{\Lambda}_{N-1}(u_2) = \widehat{\Lambda}_N(u_2)\widehat{\Lambda}_{N-1}(u_1)\,. \tag{27}$$

which ensure that

$$A_N(u)\Psi^{(N)}_{A}(\vec{x}|\vec{z}) = (u-x_1)\dots(u-x_N)\Psi^{(N)}_{A}(\vec{x}|\vec{z}). \tag{28}$$

The function $\Psi^{(N)}_{A}(\vec{x}|\vec{z})$ is symmetric under permutation of the spectral parameters x_k, and for the scalar product one gets [14]

$$\left(\Psi^{(N)}_{A}(\vec{x}'),\Psi^{(N)}_{A}(\vec{x})\right) = (2\pi)^N\left(\sum_S \delta\left(x-Sx'\right)\right)\frac{\prod_{j\neq k}\Gamma(i(x_k-x_j))}{\prod_{k=1}^N\left[\Gamma(\alpha_{x_k})\Gamma(\beta_{x_k})\right]^N}\,. \tag{29}$$

3.3. $\mathbb{B}_N$ system

The operator $\mathbb{B}_N(u)$ vanishes at $u=-i/2$, $\mathbb{B}_N(-i/2)=0$, so it is convenient to redefine the operator to get rid of this zero [13],

$$\widehat{\mathbb{B}}_N(u) = \mathbb{B}_N(u)/(2u+i). \tag{30}$$

The new operator is a polynomial of degree $2N-2$ in the spectral parameter u which is symmetric under $u \to -u$, $\widehat{\mathbb{B}}_N(u) = \widehat{\mathbb{B}}_N(-u)$.

The eigenfunctions have the form (19) with a different type of the "layer" operator $\Lambda_N \mapsto \widetilde{\Lambda}_N$

$$\Psi^{(N)}_{\mathbb{B}}(p,\vec{x}|\vec{z}) = c_N(p)\widetilde{\Lambda}_N(x_1)\widetilde{\Lambda}_{N-1}(x_2)\dots\widetilde{\Lambda}_2(x_{N-1})\, e^{ipw}\,, \tag{31}$$

where

$$c_N(p) = p^{Ns-\frac{1}{2}}\left(\Gamma(2s)\right)^{-N(N-1/2)}. \tag{32}$$

The operator $\widetilde{\Lambda}_N(x)$ is defined as follows [13]

$$[\widetilde{\Lambda}_N(x)f](z_1,\dots,z_N) = \left(\prod_{k=1}^{N-1}\int \mathcal{D}w_k\right)\widetilde{\Lambda}^{(x)}_N(z_1,\dots,z_N|w_1,\dots w_{N-1})\, f(w_1,\dots,w_{N-1})\,, \tag{33}$$

and the kernel $\widetilde{\Lambda}^{(x)}_N$ has the form

$$\begin{aligned}\widetilde{\Lambda}^{(x)}_N(z_1,\dots,z_N|w_1,\dots w_{N-1}) = D_{s-ix}(z_N,\bar{w}_{N-1})\left(\prod_{k=1}^{N-1}\int \mathcal{D}\xi_k\right)\prod_{i=1}^{N-1} D_{s-ix}(z_i,\bar{\xi}_i)D_{s+ix}(z_{i+1},\bar{\xi}_i)\\ \times\left(\prod_{k=1}^{N-2} D_{s+ix}(\xi_k,\bar{w}_k)D_{s-ix}(\xi_{k+1},\bar{w}_k)\right)D_{s+ix}(\xi_{N-2},\bar{w}_{N-1}).\end{aligned} \tag{34}$$

Again, this operator satisfies two equations,

$$\widehat{\mathbb{B}}_N(x)\widetilde{\Lambda}^{(x)}_N = 0\,, \qquad\qquad \widetilde{\Lambda}_N(x_1)\widetilde{\Lambda}_{N-1}(x_2) = \widetilde{\Lambda}_N(x_2)\widetilde{\Lambda}_{N-1}(x_1)\,. \tag{35}$$

The scalar product of two eigenfunctions takes the form [13]

$$\begin{aligned}\left(\Psi^{(N)}_{\mathbb{B}}(p',\vec{x}'),\Psi^{(N)}_{\mathbb{B}}(p,\vec{x})\right) = (2\pi)^{N-1}\,\delta(p-p')\left(\sum_{\mathcal{S}}\delta(x'-\mathcal{S}x)\right)\\ \times\prod_{n=1}^{N-1}\Gamma(2ix_n)\Gamma(-2ix_n)\,\frac{\prod_{j<k}\Gamma(i(x_k\pm x_j))\Gamma(-i(x_k\pm x_j))}{\prod_{k=1}^{N-1}\left[\Gamma(\alpha_{x_k})\Gamma(\beta_{ix_k})\right]^{2N}}\,.\end{aligned} \tag{36}$$

It appears quite helpful to use a diagrammatic representation for all objects – Λ-operator, eigenfunctions, etc, – considered above. They can be represented in the form of Feynman diagrams. The examples are shown in Figs. 2 and 3. In these figures the arrow line with the index α stands for the propagator, D_α, Eq. (16), and the integration over all vertices with the measure (6) is implied. Identities like (18) are equivalent to the statement of the equality of the corresponding diagrams and can be proved with the help of few diagrammatical rules given in Appendix A.

The operators $B_N(u), A_N(u), \mathbb{B}_N(u)$ are symmetric operators for real u. Provided that they can be extended to self-adjoint operators their eigenfunctions will form a complete system in the Hilbert space. The direct proof of completeness is also possible and will be given elsewhere. In particular, the completeness of the B_N and A_N systems is equivalent to the completeness of the SoV representation for the Toda spin chain which was proved by K. Kozlowski [18]. In what follows we take for granted that each of these systems provide a basis in the Hilbert space $\mathcal{H}_N$.

Finally, we need two more identities for the eigenfunctions. Namely,

$$A_N(x_1)\Psi^{(N)}_B(p,x_1,\dots,x_{N-1}|\vec{z}) = (u+is)^N\Psi^{(N)}_B(p,x_1+i,\dots,x_{N-1}|\vec{z})\,, \tag{37a}$$

$$B_N(x_1)\Psi^{(N)}_A(x_1,\dots,x_N|\vec{z}) = -i(u+is)^N\Psi^{(N)}_A(x_1,\dots,x_N|\vec{z})\,, \tag{37b}$$

i.e. the operators A_N and B_N act as shift operators on the separated variables. Since the eigenfunctions are symmetric in x_k the similar equations hold also for all others x_k. Equations (37) can be derived from the fundamental commutation relations (FCR) for the operators A_N, B_N, see e.g. Refs. [10, 3] or by the "gauge rotation" trick for the Lax operators, [16, 12].

4. Matrix elements

In this section we discuss calculation of the scalar product of the eigenfunctions $\Psi_B^{(N)}(p,\vec{u})$ and $\Psi_A^{(N)}(\vec{x})$ and the matrix element of the shift operator, $T_\gamma = \exp\{-\gamma S_-\}$, where $S_- = \sum_{k=1}^N S_-^{(k)}$ is the shifts generator

$$T_\gamma f(z_1,\dots,z_N) = f(z_1+\gamma,\dots,z_N+\gamma)\,, \qquad \gamma\in\mathbb{R}\,. \tag{38}$$

We introduce the following notation

$$S_N^{BA}(p,\vec{u};\vec{x}) = \Big(\Psi_B^{(N)}(p,\vec{u}),\Psi_A^{(N)}(\vec{x})\Big)\,, \qquad T_\gamma(\vec{x},\vec{x}') = \Big(\Psi_A^{(N)}(\vec{x}'),T_\gamma\Psi_A^{(N)}(\vec{x})\Big)\,. \tag{39}$$

The matrix elements (39) have been calculated in Ref. [14]. It was done by going over to the Feynman diagram representation for the quantities in question and subsequent evaluation of these diagrams. The calculation is rather straightforward and will not be repeated here, see Ref. [14] for details. Another examples of the diagrammatic technique can be found in Refs. [12, 13, 17] and in Sect. 5 of the present work †. Here we give some arguments explaining why these matrix elements can be calculated in the closed form.

It can be shown that both these matrix elements satisfy difference equations. In order to derive a difference equation for the scalar product $S_N^{BA}(p,\vec{u};\vec{x})$ we consider the matrix element of the operator $A_N(u_1)$ between the eigenstates $\Psi_B^{(N)}(p,\vec{u})$ and $\Psi_A^{(N)}(\vec{x})$. Since the function $\Psi_A^{(N)}(\vec{x}|\vec{z})$ is the eigenfunction of the operator $A_N(u_1)$ and $A_N(u_1)$ acts as shift operator on $\Psi_B^{(N)}(p,\vec{u})$ we get

$$\Big(\Psi_B^{(N)}(p,\vec{u}),A_N(u_1)\Psi_A^{(N)}(\vec{x})\Big) = \prod_{k=1}^N(u_1-x_k)\Big(\Psi_B^{(N)}(p,\vec{u}),\Psi_A^{(N)}(\vec{x})\Big) = \prod_{k=1}^N(u_1-x_k)S_N^{BA}(p,\vec{u};\vec{x}) \tag{40}$$

and

$$\begin{aligned}\Big(\Psi_B^{(N)}(p,\vec{u}),A_N(u_1)\Psi_A^{(N)}(\vec{x})\Big) &= \Big(A_N(u_1)\,\Psi_B^{(N)}(p,\vec{u}),\Psi_A^{(N)}(\vec{x})\Big)\\ &= (u+is)^N\Big(\Psi_B^{(N)}(p,\vec{u}+i\vec{e}_1),\Psi_A^{(N)}(\vec{x})\Big) = (u+is)^N S_N^{BA}(p,\vec{u}+i\vec{e}_1;\vec{x}),\end{aligned} \tag{41}$$

where $\vec{u}+i\vec{e}_1 = \{u_1+i,u_2,\dots,u_{N-1}\}$. Thus we get a recurrence relation on the function $S_N^{BA}(p,\vec{u}+i\vec{e}_1;\vec{x})$ in the variable u_1, which takes the form

$$(u+is)^N S_N^{BA}(p,\vec{u}+i\vec{e}_1;\vec{x}) = \prod_{k=1}^N(u_1-x_k)S_N^{BA}(p,\vec{u};\vec{x})\,. \tag{42}$$

The solution of the difference equation (up to multiplication by a periodic function of u_1) has the form $\prod_{k=1}^N\Gamma(i(x_k-u_1))/\Gamma^N(s-iu_1)$. Next, proceeding in the same way and considering the matrix

† For an application of this technique to the Toda spin chain see Ref. [19].

element of the operator $B_N(x_1)$ one can fix the x_1-dependence of $S_N^{BA}(p,\vec{u};\vec{x})$. Taking into account that $S_N^{BA}(p,\vec{u};\vec{x})$ is symmetric in $\{x\}$ and $\{u\}$ one gets

$$S_N^{BA}(p,\vec{u};\vec{x}) = \frac{1}{\sqrt{p}}p^{-iX}\prod_{k=1}^{N}\frac{1}{\Gamma(s-ix_k)}\prod_{j=1}^{N-1}\frac{\Gamma(i(u_j-x_k))}{\Gamma(s-ix_k)\Gamma(s+iu_j)}\times\varphi(\vec{x},\vec{u})\,, \tag{43}$$

where we put $X=\sum_{k=1}^{N}x_k$ and $\varphi(\vec{x},\vec{u})$ is a periodic function in each variable. The p-dependence follows from two relations

$$iS_0\,\Psi_A^{(N)}(\vec{x}|\vec{z}) = -\Big(\sum_k x_k\Big)\Psi_A^{(N)}(\vec{x}|\vec{z}) \quad\text{and}\quad iS_0\Psi_B^{(N)}(p,\vec{u})|\vec{z}) = i\Big(p\partial_p - 1/2\Big)\Psi_B^{(N)}(p,\vec{u})|\vec{z}). \tag{44}$$

In order to fix the periodic function $\varphi(\vec{x},\vec{u})$ one can either analyse analytic properties of the function S_N^{BA} or calculate it directly with the help of diagrammatic technique. The diagrammatic approach appears to be more effective in all cases considered in this work. For the matrix element (43) it gives $\varphi(\vec{x},\vec{u})=1$. Nevertheless the very possibility to obtain a matrix element by solving difference equations usually indicates that the corresponding Feynman diagram can be calculated in a closed form, though, sometimes, it is not so straightforward calculation as for S_N^{BA}, [14]. The corresponding examples are considered in the next section.

One has also to take care about singularities in (43) arising when $u_j\to x_k$. All Γ-functions in the numerator of (43) come from the integration of the propagator's chains (see. Ref. [14]),

$$\int DwD_{s+iu}(z,\bar{w})D_{s-ix}(w,\bar{\zeta}) = \frac{\Gamma(2s)}{\Gamma(\alpha_x)\Gamma(\beta_u)}\int_0^\infty\frac{dp}{p}\frac{e^{ip(z-\bar{\zeta})}}{p^{i(x-u)}} = \frac{\Gamma(2s)\Gamma(i(u-x))}{\Gamma(\beta_u)\Gamma(\alpha_x)}D_{i(u-x)}(z,\bar{\zeta}). \tag{45}$$

For $x=u$ the momentum integral diverges at the lower limit. To make sense of this integral for $x=u$ one can introduce the regulator, $i(u-x)\to i(u-x)+\epsilon$. Technically, in order to not destroy the balance of indices that makes possible calculation of diagrams in a closed form it is preferable to ascribe a small positive imaginary part to the variables x_k,u_k, and replace $u_k\to\bar{u}_k=u_k^*$ in (43). Thus we assume that $\mathrm{Im}\,x_k>0$, and $\mathrm{Im}\,u_k>0$ and write (43) in the form

$$S_N^{BA}(p,\vec{u};\vec{x}) = \frac{1}{\sqrt{p}}p^{-iX}\prod_{k=1}^{N}\frac{1}{\Gamma(s-ix_k)}\prod_{j=1}^{N-1}\frac{\Gamma(i(\bar{u}_j-x_k))}{\Gamma(s-ix_k)\Gamma(s+i\bar{u}_j)}\,. \tag{46}$$

The matrix element $T_\gamma(\vec{x},\vec{x}')$ was calculated in [14] with the help of diagrammatic technique. Here we only briefly discuss derivation of the recurrence relation. Making use of the commutation relation

$$[S_-,A_N(u)] = B_N(u) \tag{47}$$

which is a consequence of the FCR [10], and taking into account that $(A_N(x))^\dagger = A_N(\bar{x})$ one derives

$$\begin{aligned}\prod_{k=1}^{N}(x_1-\bar{x}'_k)T_\gamma(\vec{x},\vec{x}') &= \Big(A_N(\bar{x}_1)\Psi_A^{(N)}(\vec{x}'),T_\gamma\Psi_A^{(N)}(\vec{x})\Big) = \Big(\Psi_A^{(N)}(\vec{x}'),A_N(x_1)T_\gamma\Psi_A^{(N)}(\vec{x})\Big)\\ &= \Big(\Psi_A^{(N)}(\vec{x}'),T_\gamma\Big(A_N(x_1)+\gamma B_N(x_1)\Big)\Psi_A^{(N)}(\vec{x})\Big) = -i\gamma(x_1+is)^N T_\gamma(\vec{x}+i\vec{e}_1,\vec{x}')\,.\end{aligned} \tag{48}$$

The direct calculation results in the following expression for T_γ

$$T_\gamma(\vec{x},\vec{x}') = (\gamma+i0)^{i(X-\bar{X}')}e^{\frac{\pi}{2}(X-\bar{X}')}\prod_{k,j}\frac{\Gamma(i(\bar{x}'_j-x_k))}{\Gamma(s-ix_j)\Gamma(s+i\bar{x}'_k)}\,, \tag{49}$$

which as can be easily checked satisfies the above recurrence relation. Note also that the operator $e^{-\gamma S_-}$ is a well–defined operator on $\mathcal{H}_N$ provided that $\mathrm{Im}(\gamma)\geq 0$. Indeed, if $f\in\mathcal{H}_N$, then $\varphi=e^{-\gamma S_-}f$, $(\varphi(z_1,\ldots,z_N)=f(z_1+\gamma,\ldots,z_N+\gamma))$ also belongs to $\mathcal{H}_N$.

4.1. First Gustafson's integral

Expanding the eigenfunctions $\Psi_A^{(N)}$ over $\Psi_B^{(N)}$ (B-system) one gets the following integral representation for the matrix element $T_\gamma(\vec{x},\vec{x}')$,

$$T_\gamma(\vec{x},\vec{x'}) = \frac{1}{(N-1)!}\int_0^\infty dp\, e^{i\gamma p}\int_{-\infty}^{\infty}\prod_{k=1}^{N-1}\frac{du_k}{2\pi}\,\mu_N(\vec{u})\,S_N^{BA}(p,\vec{u},\vec{x})\left(S_N^{BA}(p,\vec{u},\vec{x}')\right)^\dagger, \tag{50}$$

where the measure is defined as follows, see Eq. (22)

$$\mu_N(\vec{u}) = \frac{\prod_{k=1}^{N-1}\left[\Gamma(s+iu_k)\Gamma(s-iu_k)\right]^N}{\prod_{j\neq k}\Gamma(i(u_k-u_j))}\,. \tag{51}$$

Calculating the momentum integral and canceling common factors on both sides of Eq. (50) one gets the identity

$$\frac{1}{(N-1)!}\left(\prod_{n=1}^{N-1}\int\frac{du_n}{2\pi}\right)\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}\Gamma(i(\bar{x}'_k-u_j))\Gamma(i(u_j-x_k))}{\prod_{k<j}\Gamma(i(u_k-u_j))\Gamma(i(u_j-u_k))} = \frac{\prod_{k,j=1}^{N}\Gamma(i(\bar{x}'_k-x_j))}{\Gamma(i\sum_{k=1}^{N-1}(\bar{x}'_k-x_k))}\,, \tag{52}$$

which is nothing else as the first Gustafson's integral (1).

Starting from the addition theorem for the shift operator, $T_{\gamma_1+\gamma_2}=T_{\gamma_1}T_{\gamma_2}$, one derives an integral identity for the matrix elements (49). It takes the form

$$\frac{1}{N!}\prod_{n=1}^{N}\int\frac{du_n}{(2\pi)}\,\zeta^{iU}\,\frac{\prod_{k,j=1}^{N}\Gamma(i(\bar{x}'_k-u_j))\Gamma(i(u_j-x_k))}{\prod_{k<j}\Gamma(i(u_k-u_j))\Gamma(i(u_j-u_k))} = \frac{\zeta^{iX'}}{(1+\zeta)^{i(\bar{X}'-X)}}\prod_{k,j=1}^{N}\Gamma(i(\bar{x}'_k-x_j))\,. \tag{53}$$

Note, that (53) can be obtained from (52) by sending parameters, x_N and x'_N to infinity. Further, dividing both sides of (53) by ζ and integrating from zero to infinity one reproduces the integral (9.2) in Ref. [1].

5. Mixed scalar products

In order to derive the second Gustafson's integral we consider the scalar products between the functions of the $\mathbb{B}$ system (*open chain*) and the B and A systems (*closed chain*). At first step we derive the recurrence relations for the matrix elements

$$\mathbb{S}_N^A(p,\vec{u}|\vec{x}) = \left(\Psi_{\mathbb{B}}^{(N)}(p,\vec{u}),\Psi_A^{(N)}(\vec{x})\right), \qquad \delta(p-q)\mathbb{S}_N^B(\vec{u}|\vec{x}) = \left(\Psi_{\mathbb{B}}^{(N)}(p,\vec{u}),\Psi_B^{(N)}(q,\vec{x})\right). \tag{54}$$

Analysis is almost the same for both products so we consider only $\mathbb{S}_N^A(p,\vec{u}|\vec{x})$. We start by noticing that

$$\mathbb{B}_N(u) = B_N(-u)A_N(u) - A_N(-u)B_N(u) = (-1)^{N-1}(2u+i)\Big\{iS_-u^{2N-1}+\dots\Big\} \tag{55}$$

and

$$\mathbb{B}_N(u)\Psi_{\mathbb{B}}^{(N)}(p,\vec{u}|\vec{z}) = (-1)^{N-1}(2u+i)p\prod_{k=1}^{N-1}(u^2-u_k^2)\Psi_{\mathbb{B}}^{(N)}(p,\vec{u}|\vec{z}). \tag{56}$$

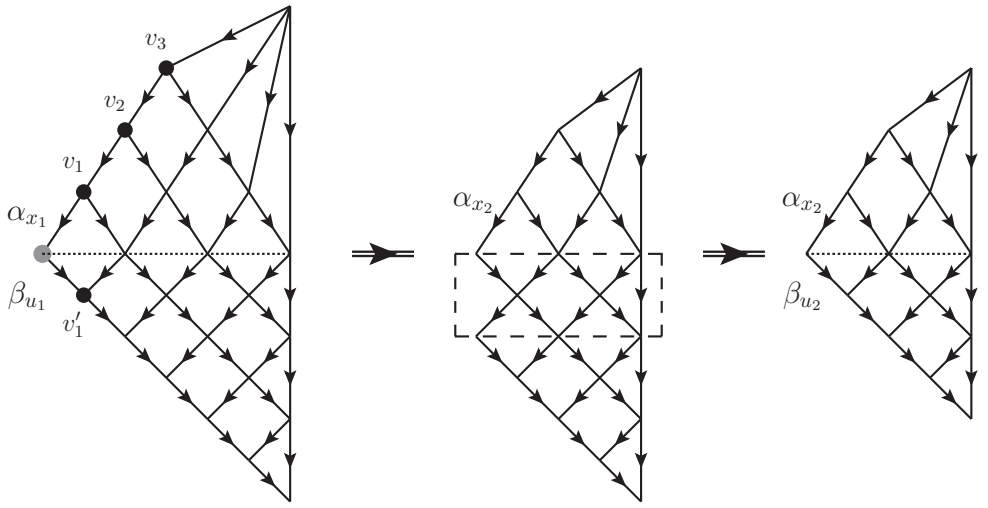

Figure 4. The diagrammatic calculation of the scalar product $\mathbb{S}^A_N(p,\vec{u}|\vec{x})$, Eq. (54). The leftmost diagram corresponds to the scalar product $\mathbb{S}^A_{N=4}(p,\{u_1,u_2,u_3\}|\{x_1,x_2,x_3,x_4\})$ and the rightmost – to $\mathbb{S}^A_{N=3}(p,\{u_2,u_3\}|\{x_2,x_3,x_4\})$. The upper part of the diagrams, above the dotted line, corresponds to the eigenfunction A_N (closed chain), and the lower part – to the eigenfunction $\mathbb{B}_N$ (open chain). The indices are not shown explicitly. They can be easily read of the diagrams in Figs. 2 and 3 (notice that $\alpha_{x_1}=s-ix_1$, and $\beta_{u_1}=s+iu_1=(\alpha_{u_1})^*$).

Then, considering the matrix element $\left(\Psi^{(N)}_{\mathbb{B}}(p,\vec{u}),\mathbb{B}_N(x_1)\,\Psi^{(N)}_A(\vec{x})\right)$ and taking into account that the operator $A_N(x_1)$ annihilates the eigenfunction $\Psi^{(N)}_A(\vec{x})$ while $B_N(x_1)$ shifts the separated variables as given by Eq. (37b) one derives after some algebra

$$p\prod_{k=1}^{N-1}(x_1^2-u_k^2)\,\mathbb{S}^A_N(p,\vec{u}|\vec{x})=i(x_1+is)^N\prod_{k=2}^{N}(x_1+x_k)\,\mathbb{S}^A_N(p,\vec{u}|\vec{x}+i\vec{e}_1)\,. \tag{57}$$

Solving this recurrence relation and taking into account that the function $\mathbb{S}^A_N(p,\vec{u}|\vec{x})$ is a symmetric function of the separated variables $x_1,\ldots,x_N$ we get

$$\mathbb{S}^A_N(p,\vec{u}|\vec{x})=\frac{1}{\sqrt{p}}p^{-iX}\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}\Gamma(\pm iu_j-ix_k)}{\prod_{k=1}^{N}\Gamma^N(s-ix_k)\prod_{k<j}\Gamma(-i(x_k+x_j))}\times\Phi_N(u)\,. \tag{58}$$

Of course, there is always the possibility to multiply this expression by a periodic function in x. Let for a moment assume that Eq. (58) correctly reproduces the x-dependence ‡ of the function $\mathbb{S}^A_N$. Then one can see that the Γ-functions in the second product in the denominator have the "nonstandard" arguments and can hardly be obtained from the integration of the propagator chains. It means that the diagram which represents the matrix element $\mathbb{S}^A_N(p,\vec{u}|\vec{x})$, see Fig. 4, cannot be calculated with the help of identities given in the Appendix A only. One can check that the approach based on the successive use of the chain rule and the permutation identities does not work already for $N=3$.

In order to calculate $\mathbb{S}^A_N(p,\vec{u}|\vec{x})$ we make use of the fact that the x-dependence of this function is known. Thus we have to determine the function $\Phi_N(u)$ in Eq. (58) only. Therefore it is sufficient to calculate the corresponding Feynman diagram for some specific value of variables $\{x_k\}$. For a natural

‡ For $N=1,2$ the function $\mathbb{S}^A_N$ can be easily calculated by the diagrammatic technique that gives some insight into the structure of the answer for general N.

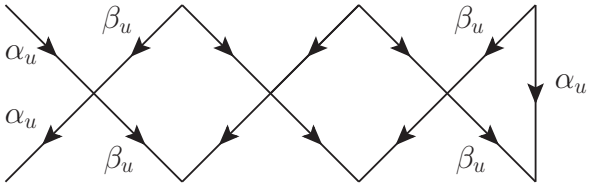

Figure 5. The diagrammatic representation of the operator $\mathbb{F}_N(u)$ (N=4) The indices are defined as follows $\alpha_u = s - iu$, $\beta_u = s + iu$.

choice, $x_1 = u_1$, the r.h.s. in Eq. (58) becomes singular. However, the integration over a "free" vertex in the diagram for $\mathbb{S}^A_N(p, \vec{u}|\vec{x})$ (the leftmost gray vertex in Fig. 4 with only two propagators attached) produces the factor (see Eq. (45))

$$a(s + iu_1, s - ix_1) = \frac{\Gamma(2s)\Gamma(i(u_1 - x_1))}{\Gamma(s + iu_1)\Gamma(s - ix_1)}\,, \tag{59}$$

which is also singular at $x_1 \to u_1$. Canceling the singular factor $\Gamma(i(u_1 - x_1))$ on both sides one can safely put $x_1 = u_1$. Since at $x_1 \to u_1$ the propagator arising due to the integration, $D_{i(u_1-x_1)}(v_1', \bar{v}_1) \to 1$, the line connecting vertices v_1 and v_1' disappears. The resulting diagram can be simplified as follows:

(i) one integrates over the vertex v_1' and move the resulting line to the right with the help of the permutation relations given in Appendix A. Then one consequently applies the same procedure to the vertices v_1, v_2 and so on. Each integration produces the factor $a(\alpha, \beta)$ and the successive application of the permutation relations results in a rearrangement of indices. Namely, the integration over the vertices $v_1', v_1, v_2, \ldots$ gives the factors $a(\alpha_{u_1}, \alpha_{u_1})$, $a(\beta_{u_1}, \alpha_{x_2})$, $a(\beta_{u_1}, \alpha_{x_3}), \ldots$, respectively. After these steps the upper part of the resulting diagram (the middle diagram in Fig. 4) corresponds to the diagram for the eigenfunction $\Psi^{(N-1)}_A(x_2, \ldots, x_N)$.

(ii) The lower part of the middle diagram in Fig. 4 has the form

$$\widetilde{\Lambda}^\dagger_2(u_{N-1} \ldots \widetilde{\Lambda}^\dagger_{N-1}(u_2)\, \mathbb{F}_{N-1}(u_1)\,, \tag{60}$$

where the operator $\mathbb{F}_N$ (depicted by the dashed lines in Fig. 4) has the diagrammatic representation shown in Fig. 5. The operators $\widetilde{\Lambda}^\dagger_N$ and $\mathbb{F}_N$ obey the following exchange relation

$$\widetilde{\Lambda}^\dagger_N(v)\, \mathbb{F}_N(u) = a(\alpha_u, \beta_v) a(\alpha_u, \alpha_v)\, \mathbb{F}_{N-1}(u) \widetilde{\Lambda}^\dagger_N(v)\,. \tag{61}$$

The proof of this relation is straightforward and illustrated in Fig. 6. Using this relation one gets for the product (60)

$$\widetilde{\Lambda}^\dagger_2(u_{N-1}) \ldots \widetilde{\Lambda}^\dagger_{N-1}(u_2)\, \mathbb{F}_{N-1}(u_1) = \prod_{k=2}^{N-1} a(\alpha_{u_1}, \beta_{u_k}) a(\alpha_{u_1}, \alpha_{u_k}) \mathbb{F}_1(u_1) \widetilde{\Lambda}^\dagger_2(u_{N-1} \ldots \widetilde{\Lambda}^\dagger_{N-1}(u_2)\,. \tag{62}$$

Finally, taking into account that $\mathbb{F}_1(u)(z, \bar{w}) = D_{s-iu}(z, \bar{w})$ and performing Fourier transform one finds that the lower part of the diagram corresponds to the diagram for the eigenfunction $\Psi^{(N-1)}_{\mathbb{B}}(p, u_2, \ldots, u_{N-1})$.

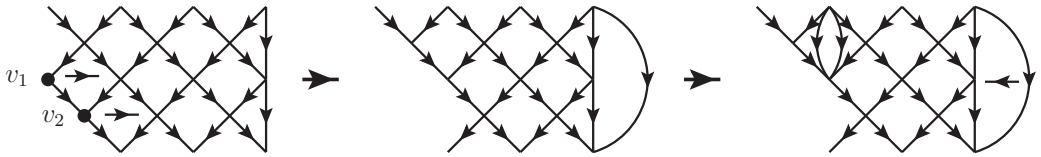

Figure 6. The diagrammatic proof of the exchange relation (61). The right diagram corresponds to the product $\widetilde{\Lambda}^{\dagger}_N(v)\,\mathbb{F}_N(u)$. On the first step one integrates over the vertex v_1 and moves the resulting line to the right with the help of the permutation relations ii and iii in Appendix A. It ends up as the arched line in the middle diagram. On the next step one repeats the same procedure for the vertex v_2. On the third step we insert the lines with the indices $\pm i(u-v)$ as shown in the rightmost diagram and move them to the left and right interchanging on the way the indices of the two upper layers. On the final step one moves the arched line to the left. The resulting diagram corresponds to the product $\mathbb{F}_{N-1}(u)\widetilde{\Lambda}^{\dagger}_N(v)$.

Thus one expresses the N-sites scalar product $\mathbb{S}^A_N$ for the special choice of parameters, $x_1 = u_1$, via the $N-1$ sites scalar product. Collecting all factors and taking into account the normalization coefficients (25), (32) we obtain:

$$\frac{\mathbb{S}^A_N(p,\vec{u}|\vec{x})}{\Gamma(i(u_1-x_i))}\Big|_{u_1=x_1} = \mathbb{S}^A_{N-1}(p,\{u_2,\dots,u_{N-1}\}|\{x_2,\dots,u_N\})\frac{\Gamma(2s)a(\alpha_{u_1},\alpha_{u_1})}{\Gamma(\alpha_{u_1})\Gamma(\beta_{u_1})}\prod_{k=2}^{N}a(\beta_{u_1},\alpha_{x_k})$$
$$\times\prod_{k=2}^{N-1}a(\alpha_{u_1},\beta_{u_k})a(\alpha_{u_1},\alpha_{u_k})\times p^{-iu_1}\frac{\Gamma(2s)}{\Gamma(\alpha_{u_1})}\left(\Gamma(2s)\right)^{-3N+2}. \tag{63}$$

Substituting $\mathbb{S}^A_N$ and $\mathbb{S}^A_{N-1}$ in the form (58) we get for $\Phi_N(u)$

$$\Phi_N(u_1,\dots,u_{N-1}) = \frac{1}{\Gamma^N(\alpha_{u_1})\Gamma^N(\beta_{u_1})}\prod_{k=2}^{N-1}\frac{1}{\Gamma(\alpha_{u_k})\Gamma(\beta_{u_k})}\Phi_{N-1}(u_2,\dots,u_{N-1}). \tag{64}$$

Taking into account the boundary condition $\Phi_1 = 1$ we get

$$\Phi_N(u_1,\dots,u_{N-1}) = \left(\prod_{k=1}^{N-1}\Gamma(\alpha_{u_k})\Gamma(\beta_{u_k})\right)^{-N} \tag{65}$$

and, hence

$$\mathbb{S}^A_N(p,\vec{u}|\vec{x}) = \frac{1}{\sqrt{p}}p^{-iX}\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}\Gamma(\pm iu_j - ix_k)}{\left(\prod_{k=1}^{N-1}\Gamma(\alpha_{u_k})\Gamma(\beta_{u_k})\prod_{k=1}^{N}\Gamma(\alpha_{x_k})\right)^N\prod_{k<j}\Gamma(-i(x_k+x_j))}. \tag{66}$$

The calculation of the scalar product $\mathbb{S}^B_N(\vec{u}|\vec{x})$ goes exactly along the same lines so we quote the final result only

$$\mathbb{S}^B_N(\vec{u}|\vec{x}) = \frac{\prod_{k=1}^{N-1}\prod_{j=1}^{N-1}\Gamma(\pm iu_j - ix_k)}{\left(\prod_{k=1}^{N-1}\Gamma(\alpha_{u_k})\Gamma(\beta_{u_k})\Gamma(\alpha_{x_k})\right)^N\prod_{k<j}\Gamma(-i(x_k+x_j))}. \tag{67}$$

In deriving Eq. (66) we have neglected the possibility to multiply the solution (58) of the recurrence equation (57) by a periodic function of x. In order to see that Eq. (66) gives the right

answer one can proceed a bit differently, finally arriving to the same result. Namely, it can be shown by a straightforward application of the integration rules to the diagram for the scalar product $\mathbb{S}^A_N(p,\vec{u}|\vec{x})$, (the leftmost diagram in Fig. 4) that it can be represented in the form $\mathbb{F}(u_1,\vec{x})\times S(u_2,\ldots,u_{N-1}|\vec{x})$, where $\mathbb{F}(u_1,\vec{x})$ is given by a product of Γ-functions. Since the function $\mathbb{S}^A_N(p,\vec{u}|\vec{x})$ is a symmetric function of $\vec{u}$ variables one concludes that $\mathbb{S}^A_N(p,\vec{u}|\vec{x})=\prod_{k=1}^{N-1}\mathbb{F}(u_k,\vec{x})\times\Psi_N(\vec{x})$. Finally, in order to determine $\Psi_N(\vec{x})$ one applies the same "$u_1\to x_1$" trick described above.

5.1. Second Gustafson's integral

The second Gustafson's integral is also related to the matrix element $T_\gamma(\vec{x},\vec{x}')$, Eq. (39). Only this time we use expansion over the eigenfunctions $\Psi^{(N)}_{\mathbb{B}}(p,\vec{u})$. One gets

$$T_\gamma(\vec{x},\vec{x'})=\frac{1}{(N-1)!}\int_0^\infty dp\,e^{i\gamma p}\left(\prod_{k=1}^{N-1}\int_{-\infty}^{\infty}\frac{du_k}{4\pi}\right)\widetilde{\mu}_N(\vec{u})\,\mathbb{S}^A_N(p,\vec{u},\vec{x})\left(\mathbb{S}^A_N(p,\vec{u},\vec{x}')\right)^\dagger, \tag{68}$$

where the measure is defined as follows, see Eq. (36),

$$\widetilde{\mu}_N(\vec{u})=\frac{\prod_{k=1}^{N-1}\left[\Gamma(s+iu_k)\Gamma(s-iu_k)\right]^{2N}}{\prod_{j<k}\Gamma(i(u_k\pm u_j))\Gamma(-i(u_k\pm u_j))}\,. \tag{69}$$

Substituting $\mathbb{S}^A_N(p,\vec{u},\vec{x})$, Eq. (66), in (68) and integrating over p one gets,

$$\frac{1}{(N-1)!}\left(\prod_{n=1}^{N-1}\int\frac{du_n}{4\pi}\right)\frac{\prod_{k=1}^{N}\prod_{j=1}^{N-1}\Gamma(i(x'_k\pm u_j))\Gamma(-i(x_k\pm u_j))}{\prod_{k=1}^{N-1}\Gamma(2iu_k)\Gamma(-2iu_k)\prod_{k<j}^{N-1}\Gamma(i(u_k\pm u_j))\Gamma(-i(u_k\pm u_j))}=$$
$$=\Gamma^{-1}(i(X'-X))\prod_{k,j=1}^{N}\Gamma(i(x'_k-x_j))\prod_{k<j}^{N}\Gamma(i(x'_k+x'_j))\Gamma(-i(x_k+x_j))\,. \tag{70}$$

This integral, after redefinition $\alpha_k=ix'_k$, $\alpha_{N+k}=-ix_k$ and $N-1\to N$, coincides with the integral (2).

Writing down the scalar product $\mathbb{S}^A_N(p,\vec{x},\vec{x}')$ in the $\Psi^{(N)}_B(q,\vec{u})$ basis one gets the following identity

$$\mathbb{S}^A_N(p,\vec{x},\vec{x}')=\frac{1}{(N-1)!}\left(\prod_{k=1}^{N-1}\int\frac{du_k}{2\pi}\right)\mu_N(\vec{u})\,\mathbb{S}^B_N(\vec{x},\vec{u})\,S^{BA}_N(p,\vec{u},\vec{x}')\,, \tag{71}$$

where the measure $\mu_N(\vec{u})$ is defined by Eq. (51). Substituting explicit expressions for the scalar products, Eqs. (46), (66), (67), one obtains after some redefinition the integral (3).

For $N=1$ the integral (3) is a special case of the integral (1). Contrary, for $N>1$ the integral (1) follows from the integrals (2) and (3). Indeed, let us send $\alpha_{2N+2}\to\infty$ in (2) and compare the asymptotic of both sides. It give rises to the following identity

$$\left(\prod_{n=1}^{N}\int_{-i\infty}^{i\infty}\frac{dz_n}{2\pi i}\right)\frac{\prod_{k=1}^{2N+1}\prod_{j=1}^{N}\Gamma(\alpha_k\pm z_j)}{\prod_{k=1}^{N}\Gamma(\pm 2z_k)\prod_{k<j}\Gamma(z_k\pm z_j)\Gamma(-z_k\pm z_j)}=2^N N!\prod_{k<j}\Gamma(\alpha_k+\alpha_j)\,. \tag{72}$$

Then multiplying both sides of (3) by

$$\frac{\prod_{j=1}^{N+1}\prod_{m=1}^{N}\Gamma(\gamma_j\pm\beta_m)}{\prod_{k=1}^{N}\Gamma(\pm 2\beta_k)\prod_{k<j}\Gamma(\beta_k\pm\beta_j)\Gamma(-\beta_k\pm\beta_j)} \tag{73}$$

and carrying out β-integrals with the help of (2) and (72) one obtains the first Gustafson's integral (1).

6. $N \times (N-1)$ scalar products

In this section we calculate another set of scalar products that gives us a possibility to obtain several new Γ-integrals. Namely, we consider the scalar products of the eigenfunctions $\Psi_B^{(N)}(p,\vec{x}|\vec{z})$, $\Psi_A^{(N)}(\vec{x}|\vec{z})$ with functions

$$\Psi_B^{(N-1)}(p,x_1,\dots x_{N-2}|\vec{z}_{N-1})\times M_\nu(z_N),\qquad \Psi_A^{(N-1)}(p,x_1,\dots x_{N-1}|\vec{z}_{N-1})\times M_\nu(z_N), \tag{74}$$

where $M_\nu(z)$ is a power function (see also Appendix A)

$$M_\nu(z)=(\Gamma(2s))^{-1/2}\Gamma(s+i\nu)D_{s+i\nu}(z,0)=(\Gamma(2s))^{-1/2}\Gamma(s+i\nu)e^{i\pi/2(s+i\nu)}z^{-s-i\nu}. \tag{75}$$

All four scalar products can be calculated by the diagrammatic technique. The calculation goes along the following lines: one starts with the diagram G_N for the N-point scalar product of two functions. Then using the chain rule and permutation identities given in Appendix A one transforms G_N into the form $G_N=F_N\times G_{N-1}$. Here F_N is a factor depending on the spectral parameters and given by a product of Γ functions and G_{N-1} is the diagram for the $N-1$ point scalar product. So one immediately gets that $G_N=F_NF_{N-1}\dots F_3\times G_2$. In all cases, the starting point of the recursion, the diagram $G_{N=2}$, can be easily evaluated.

We obtained the following expressions for the scalar products:

$$\begin{aligned}\Big(\Psi_B^{(N-1)}(p,\vec{u})\times M_\nu,\Psi_A^{(N)}(\vec{x})\Big)&=\frac{p^{-i(\nu+X)}}{\sqrt{p}}\frac{\Gamma(\alpha_\nu)}{\Gamma(\beta_\nu)}\prod_{k=1}^{N}\frac{\Gamma(i(x_k+\nu))}{\Gamma(\alpha_{x_k})\Gamma(\beta_{x_k})}\\&\quad\times\prod_{k=1}^{N-2}\frac{\Gamma(\beta_{u_k})}{\Gamma(i(u_k+\nu))}\prod_{k=1}^{N}\prod_{j=1}^{N-2}\frac{\Gamma(i(u_j-x_k))}{\Gamma(\alpha_{x_k})\Gamma(\beta_{u_j})},\\\Big(\Psi_A^{(N-1)}(\vec{u})\times M_\nu,\psi_A^{(N)}(\vec{x})\Big)&=2\pi\delta(\nu+X-U)\frac{\Gamma(\alpha_\nu)}{\Gamma(\beta_\nu)}\frac{\prod_{k=1}^{N-1}\Gamma(\beta_{u_k})}{\prod_{k=1}^{N}\Gamma(\beta_{x_k})}\prod_{k=1}^{N}\prod_{j=1}^{N-1}\frac{\Gamma(i(u_j-x_k))}{\Gamma(\alpha_{x_k})\Gamma(\beta_{u_j})}.\end{aligned} \tag{76}$$

and

$$\begin{aligned}\Big(\Psi_B^{(N-1)}(q,\vec{u})\times M_\nu,\Psi_B^{(N)}(p,\vec{x})\Big)&=\theta(p-q)p^{iU+\frac{1}{2}}q^{-iX-\frac{1}{2}}(p-q)^{i(X-U-\nu)-1}\\&\quad\times\prod_{k=1}^{N-1}\frac{1}{|\Gamma(\alpha_{x_k})|^2}\prod_{j=1}^{N-2}\frac{\Gamma\big(i(u_j-x_k)\big)}{\Gamma(\beta_{u_j})\Gamma(\alpha_{x_k})},\\\Big(\Psi_A^{(N-1)}(\vec{x})\times M_\nu,\Psi_B^{(N)}(p,\vec{u})\Big)&=\frac{p^{i(-\nu+X)}}{\sqrt{p}}\prod_{k=1}^{N-1}\frac{1}{\Gamma(\beta_{u_k})}\frac{\Gamma(i(u_k-\nu))}{\Gamma(i(x_k-\nu))}\prod_{k,j=1}^{N-1}\frac{\Gamma(i(x_k-u_j))}{\Gamma(\beta_{x_k})\Gamma(\alpha_{u_j})}.\end{aligned} \tag{77}$$

Here and below $X=\sum_k x_k$, $Y=\sum_k y_k$ and $U=\sum_k u_k$. It is tacitly assumed that the "$+0$" prescription is used for all Γ functions in the numerators, i.e. $\Gamma(i(u-x))\mapsto\Gamma(i(u-x)+\epsilon)$.

The scalar product of two functions $\Psi_1(\vec{y})$, $\Psi_2(\vec{x})$ can be written, schematically, in the form

$$\big(\Psi_1(\vec{y}),\Psi_2(\vec{x})\big)=\int d\mu(\vec{u})\big(\Psi_1(\vec{y}),\Psi_3(\vec{u})\big)\big(\Psi_3(\vec{u}),\Psi_2(\vec{x})\big), \tag{78}$$

where $\Psi_3(\vec{u})$ is the complete system associated with $\Psi_B^{(N)}$, $\Psi_A^{(N)}$ or with the functions (74) and $\mu(\vec{u})$ is the corresponding measure. The r.h.s. of Eq. (78) has the form of the multidimensional Mellin-Barnes integral. We list below some of the integrals arising in this way. The most interesting ones are those for which the number of external parameters minus the number of integrations is maximal.

Presenting results we change the integration variables $iu_k \to u_k$ and do the same for the external parameters. Since the prescription for a bypass of the poles is fixed and integrals are convergent, the corresponding integral identities hold for the complex parameters as well. We will also sometimes shift $N \to N+1, N+2$.

- The first integral arise from (78) for the choice:
  $\left\{\Psi_1, \Psi_2, \Psi_3\right\} = \left\{\Psi_A^{(N-1)}(\vec{y}) \times M_\nu, \psi_A^{(N)}(\vec{x}), \Psi_B^{(N-1)}(p, \vec{u}) \times M_\nu\right\}$. It takes the form

$$\frac{1}{N!}\int \prod_{k=1}^{N} \frac{du_k}{2\pi i} \frac{\prod_{j=1}^{N}\prod_{k=1}^{N+1}\Gamma(y_k-u_j)\prod_{k=1}^{N+2}\Gamma(u_j+x_k)}{\prod_{k=1}^{N}\Gamma(\nu+u_k)\prod_{i\neq j}\Gamma(u_i-u_j)} = \frac{\prod_{k=1}^{N+2}\prod_{j=1}^{N+1}\Gamma(y_i+x_k)}{\prod_{k=1}^{N+2}\Gamma(\nu-x_k)}\,, \tag{79}$$

  where $\nu = Y + X = \sum_{k=1}^{N+1} y_k + \sum_{k=1}^{N+2} x_k$ and $\mathrm{Re} x_k > 0$, $\mathrm{Re} y_k > 0$ and coincides with the integral (3.2) in Ref. [2].

- Considering $\left\{\Psi_B^{(N-1)}(p, \vec{u}) \times M_\nu, \Psi_A^{(N)}(\vec{x}), \Psi_A^{(N-1)}(\vec{u}) \times M_\nu\right\}$ we get

$$\frac{2\pi i}{N!}\int \prod_{k=1}^{N} \frac{du_k}{2\pi i}\, \delta\left(\sum_{k=1}^{N} u_k\right) \frac{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\Gamma(y_k-u_j)\prod_{k=1}^{N+1}\Gamma(x_k+u_j)}{\prod_{k\neq j}\Gamma(u_k-u_j)} = \\ = \frac{\prod_{k=1}^{N+1}\Gamma\left(X-x_k\right)}{\prod_{k=1}^{N-1}\Gamma\left(X+y_k\right)} \prod_{k=1}^{N+1}\prod_{j=1}^{N-1}\Gamma(y_j+x_k)\,, \tag{80}$$

  where $X = \sum_{j=1}^{N+1} x_j$ and $\mathrm{Re} x_k > 0$, $\mathrm{Re} y_j > 0$ and integration goes along the imaginary axis. For $N = 2$ this identity is equivalent to the Wilson - de Branges integral [20, 21].

- The triple $\left\{\Psi_A^{(N-1)}(\vec{y}) \times M_\nu,\ \Psi_B^{(N)}(p, \vec{x}),\ \Psi_B^{(N-1)}(q, \vec{u}) \times M_\nu\right\}$ gives rise to

$$\frac{1}{N!}\int \prod_{k=1}^{N} \frac{du_k}{2\pi i} \frac{\Gamma(\nu-X-U)}{\Gamma(\nu+Y-U)} \frac{\prod_{k=1}^{N+1}\prod_{j=1}^{N}\Gamma(y_k-u_j)\Gamma(x_k+u_j)}{\prod_{k\neq j}\Gamma(u_k-u_j)} \\ = \prod_{k=1}^{N+1} \frac{\Gamma(\nu-x_k)}{\Gamma(\nu+y_k)} \frac{\prod_{k,j=1}^{N+1}\Gamma(y_j+x_k)}{\Gamma(X+Y)}\,, \tag{81}$$

  where $\mathrm{Re}\, x_k > 0$, $\mathrm{Re}\, y_k > 0$, and $\mathrm{Re}\, \nu > \mathrm{Re} X$. For $N = 1$ it is equivalent to the second Barnes Lemma, while for general N it is a modification of the first Gustafson's integral (1).

- For $\left\{\Psi_B^{(N)}(p, \vec{y}) \times M_\nu,\ \Psi_A^{(N)}(\vec{x}),\ \Psi_A^{(N-1)}(\vec{u}) \times M_\nu\right\}$ one gets

$$\frac{1}{N!}\int \prod_{k=1}^{N} \frac{du_k}{2\pi i} \frac{\Gamma(s-X-U)}{\Gamma(s+X+U)} \prod_{k=1}^{N} \frac{\Gamma(X+U-y_k)}{\Gamma(X+U-u_k)} \frac{\prod_{j=1}^{N}\prod_{k=1}^{N}\Gamma(y_k-u_j)\prod_{k=1}^{N+1}\Gamma(x_k+u_j)}{\prod_{k\neq j}\Gamma(u_k-u_j)} = \\ = \prod_{k=1}^{N+1} \frac{\Gamma(s-x_k)}{\Gamma(s+x_k)} \prod_{k=1}^{N} \frac{\Gamma(s-y_k)}{\Gamma(s+y_k)} \prod_{k=1}^{N+1}\prod_{j=1}^{N}\Gamma(y_j+x_k), \tag{82}$$

  where $\mathrm{Re}\, x_k > 0$, $\mathrm{Re}\, y_k > 0$, $\mathrm{Re}\, X > \mathrm{Re}\, y_k$ and $\mathrm{Re}\, s > \mathrm{Re} X$. Again, for $N = 1$ it reduces to the second Barnes Lemma.

- The last integral comes from $\left\{\Psi_B^{(N)}(p,\vec{y}),\, \Psi_A^{(N)}(\vec{x}),\, \Psi_B^{(N-1)}(p,\vec{u})\times M_\nu\right\}$

$$\frac{1}{(N-1)!}\int\frac{d\nu}{2\pi i}\int\prod_{k=1}^{N-1}\frac{du_k}{2\pi i}\frac{\Gamma(s-X-\nu)}{\Gamma(s+X+\nu)}\frac{\prod_{k=1}^{N+1}\Gamma(\nu+X-x_k)}{\prod_{k=1}^{N-1}\Gamma(\nu+X+u_k)}\frac{\Gamma(Y-\nu)\Gamma(\nu+U+X-Y)}{\Gamma(X+U)}$$
$$\times\frac{\prod_{j=1}^{N-1}\prod_{k=1}^{N}\Gamma(y_k-u_j)\prod_{k=1}^{N+1}\Gamma(x_k+u_j)}{\prod_{k\neq j}\Gamma(u_k-u_j)}=\prod_{k=1}^{N+1}\frac{\Gamma(s-x_k)}{\Gamma(s+x_k)}\prod_{k=1}^{N}\frac{\Gamma(s-y_k)}{\Gamma(s+y_k)}\prod_{k=1}^{N+1}\prod_{j=1}^{N}\Gamma(y_j+x_k)\,. \tag{83}$$

Here $\mathrm{Re}x_k>0$, $\mathrm{Re}y_k>0$, $\mathrm{Re}(X-Y)>0$ and $\mathrm{Re}(s-X)>0$. For $N=1$ it is equivalent to the second Barnes Lemma.

7. Summary

The eigenfunctions of the matrix elements of the monodromy matrix (both for the closed and open spin chains) provide convenient bases for the study of the spectral problem for the corresponding spin magnets. Remarkably enough these eigenfunctions can be constructed explicitly as multi-parametric integrals and represented by Feynman diagrams of a certain type. The scalar products between the different eigenfunctions can be calculated with the help of the diagrammatic technique and, as a rule, are given by a product of the Γ-functions with arguments depending on the parameters labeling the eigenfunctions (separated variables). In the SoV representations the scalar product or matrix elements take the form of multidimensional Mellin–Barnes integrals. Studying different scalar products we succeeded to reproduce all relevant integrals in Ref. [1, 2] except two ones – the integrals (9.6) in Ref. [1] and (5.4) in Ref. [2] – and derived several new integrals which we were not able to derive from the Gustafson's integrals.

In this work we have considered only the homogeneous spin chains. However, the eigenfunctions can be constructed in a similar way and for general case of the inhomogeneous spin chains with impurities [12]. The Gustafson's integrals (1), (2) are not sensitive to all these modifications. At the same time we expect that inclusion of additional parameters (spins and impurities) into consideration could have modify integrals given in sect. 6.

We hope also that this approach can be extended to the noncompact $SL(2,\mathbb{C})$ spin magnets [22] and allows one to obtain a corresponding extension of Gustafson's integrals. Some insights into the possible structure of such integrals can be derived from the works [23, 24, 25].

Acknowledgments

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Appendices

Appendix A. Diagram technique

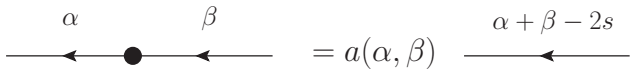
In this Appendix we present the basic elements of the diagram technique which was used throughout the paper. The propagator

$$D_\alpha(z,\bar{w})=\left(\frac{i}{z-\bar{w}}\right)^\alpha=\frac{1}{\Gamma(\alpha)}\int_0^\infty dp\, e^{ip\,(z-\bar{w})}\,p^{\alpha-1} \tag{A.1}$$

is shown by the arrow directed from $\bar{w}$ to z with the index α attached to it. Under complex conjugation it behaves as follows: $(D_\alpha(z,\bar{w}))^* = D_{\alpha^*}(w,\bar{z})$.

There are several useful identities involving propagators:

(i) Chain rule: the integral of two propagators is the propagator again:

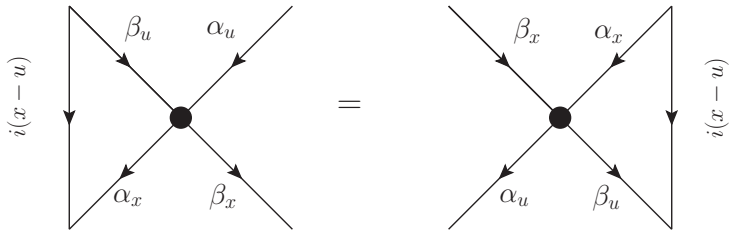


$$\int \mathcal{D}w\, D_\alpha(z,\bar{w}) D_\beta(w,\bar{\zeta}) = a(\alpha,\beta) D_{\alpha+\beta-2s}(z,\bar{\zeta})\,, \tag{A.2}$$

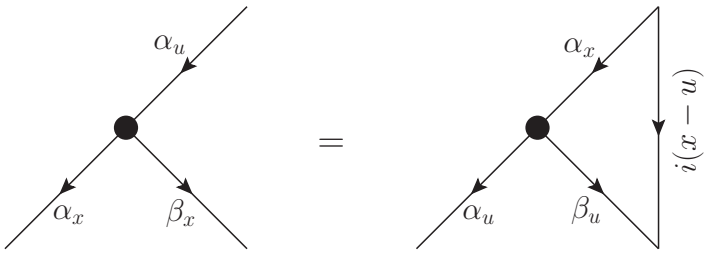
where

$$a(\alpha,\beta) = \frac{\Gamma(2s)\Gamma(\alpha+\beta-2s)}{\Gamma(\alpha)\Gamma(\beta)}\,. \tag{A.3}$$

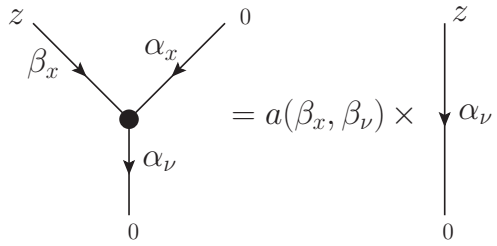
(ii) Permutation relation:



(iii) Reduced permutation relation:



(iv) The propagator identity:



All these identities can be easily checked by going over to the momentum representation.

There are two standard bases in the (one-particle) Hilbert space $\mathcal{H}$.

- The plane waves: $E_p(z) = p^{s-1/2}e^{ipz}/\Gamma^{1/2}(2s)$, $p > 0$:

$$(E_{p'}, E_p) = \int \mathcal{D}z\, E_p(z)\overline{E_{p'}(z)} = \delta(p - p')\,. \tag{A.4}$$

- powers:

$$M_\nu(z) = (\Gamma(2s))^{-1/2}\Gamma(s + i\nu)D_{s+i\nu}(z, 0) = (\Gamma(2s))^{-1/2}\Gamma(s + i\nu)e^{i\pi/2(s+i\nu)}z^{-s-i\nu}, \tag{A.5}$$

where $\nu \in \mathbb{R}$,

$$(M_{\nu'}, M_\nu) = \int \mathcal{D}z\, M_\nu(z)\overline{M_{\nu'}(z)} = 2\pi\delta(\nu - \nu')\,. \tag{A.6}$$

For the transition matrix element one gets

$$\Big(M_\nu, E_p\Big) = p^{-i\nu-1/2}\,. \tag{A.7}$$

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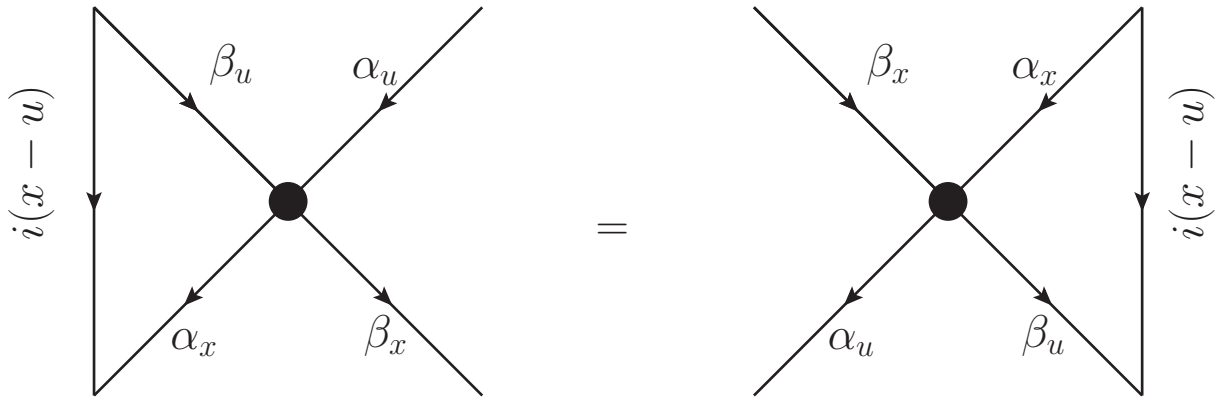
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