Prepared for submission to JHEP
DESY-16-239, NORDITA-2016-123,WIS/07/16-DEC-DPPA

Harmony of Spinning Conformal Blocks

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Abstract: Conformal blocks for correlation functions of tensor operators play an increasingly important role for the conformal bootstrap programme. We develop a universal approach to such spinning blocks through the harmonic analysis of certain bundles over a coset of the conformal group. The resulting Casimir equations are given by a matrix version of the Calogero-Sutherland Hamiltonian that describes the scattering of interacting spinning particles in a 1-dimensional external potential. The approach is illustrated in several examples including fermionic seed blocks in 3D CFT where they take a very simple form.

Keywords: Conformal blocks, Harmonic analysis, Calogero-Sutherland models

Contents

1 Introduction

The conformal bootstrap programme, which was originally formulated in the [1–3], has raised hopes for a new non-perturbative construction of conformal field theories in any dimension, even of theories for which an action cannot be written down or a microscopic (UV) description is not known. The programme rests on a careful separation of kinematical and dynamical data in correlation functions, i.e. on the split into the kinematical conformal blocks and the dynamical coefficients of the operator product expansion. The latter are severely constrained by the so-called crossing symmetry equations, an infinite set of coupled equations for the operator product coefficients with kinematically determined coefficients. Over the last few years, numerical studies of these crossing symmetry equations have given access to critical exponents and operator product coefficients with enormous precision [4–8].

While initial work has focused on correlation functions involving one or two scalars, tensor fields are only now beginning to receive some attention in the bootstrap programme. The most important tensor field is clearly the stress tensor which, by definition, exists in any conformal field theory. If the conformal blocks for tensor fields were under good control one could explore the space of conformal field theories without assumptions on the scalar subsector. The study of such spinning conformal blocks was initiated in [9, 10]. A fairly generic approach was proposed in [11], based on the so-called shadow formalism of Ferrara et al. [12–15], see also [16, 17] for more recent work and further references. This leads to expressions in which conformal blocks are simply sewn together from 3-point functions. In the bootstrap programme, such formulas are difficult to work with, partly because they involve a the large number of integrations. On the other hand, recent work [18] clearly shows that explicit constructions of spinning blocks in higher dimensional conformal field theories in terms of known special functions are possible. The main motivation for our work is to pave the way for systematic extensions of such efficient formulas.

In order to achieve this, we generalize an interesting interpretation of conformal blocks as wave functions of an interacting 2-particle Schrödinger problem with Calogero-Sutherland potential that was recently uncovered in [19]. More precisely, it was shown that the Casimir equations for scalar conformal blocks [20] are equivalent to the eigenvalue equations for a hyperbolic Calogero-Sutherland Hamiltonian. The integrability of this Hamiltonian has been argued to provide a new avenue to scalar conformal blocks. Only very few Casimir equations for spinning blocks have been worked out in the literature, see however [18, 21]. Here we propose an independent approach that allows us to construct an appropriate Calogero-Sutherland model for any choice of external operators with spin. In comparison to the case of scalar blocks, the potentials become matrix valued and describe the motion of two interacting particles with spin in a 1-dimensional (spin-dependent) external potential. The associated Schrödinger problems are equivalent to the Casimir equations for spinning blocks.

Let us describe the main results and plan of this paper. Throughout the next two sections we shall set up a model for spinning conformal blocks in any dimension where the 4-point blocks are represented as sections in a certain vector bundle over the following double coset of the conformal group $G = \mathrm{SO}(1, d + 1)$

$$\mathcal{C} = \mathrm{SO}(1, d+1) /\!\!/ (\mathrm{SO}(1,1) \times \mathrm{SO}(d)) \ .$$

The denominator consists of dilations and rotations and we divide by both its right and its left action on the conformal group. As we shall argue in section 4, this coset space is 2-dimensional and parameterizes the conformally invariant cross ratios. Let us notice that, once we have divided by the right action, the left action of $\mathrm{SO}(1,1)\times\mathrm{SO}(d)$ in the quotient is stabilized by a subgroup $\mathrm{SO}(d-2) \subset \mathrm{SO}(d)$ of the rotation group.

Given four tensor fields that transform in representations with highest weight $\mu_i, i = 1, \dots, 4$ of the rotation group $\mathrm{SO}(d)$, the fibers of the relevant bundles over the double coset $\mathcal{C}$ are given by

$$T = (V_{\mu_1} \otimes V_{\mu_2} \otimes V_{\mu_3} \otimes V_{\mu_4})^{\mathrm{SO}(d-2)} \ .$$

Here, V_μ denotes the carrier space of the representation μ of the rotation group.[1] We consider the tensor product inside the brackets as a representation of the subgroup $\mathrm{SO}(d-2) \subset \mathrm{SO}(d)$ and select the subspace of $\mathrm{SO}(d-2)$ invariants. As we shall argue in section 2 and 3, elements of the resulting vector space should be considered as 4-point tensor structures. We stress that the global structure of the relevant vector bundles also depends on the choice of conformal weights. As described in section 3, the bundle can be realised as a space of equivariant functions over G which defined by their restriction to $\mathcal{C}$.

[1]Strikly speaking, the fibers also carry an action of dilations that is determined by the values of the external conformal weights. We will specify this later.

Once the model of conformal blocks is set up, we derive the relevant Casimir equations for spinning blocks from the Laplacian on the conformal group $\mathrm{SO}(1,d+1)$ in section 4. Following the logic of Hamiltonian reduction described in [22, 23], we argue that these equations can be brought into a matrix Schrödinger problem for two interacting particles with spin that are moving in a 1-dimensional external potential. In the case of non-trivial fermionic seed blocks in 3-dimensional conformal field theory, the relevant Hamiltonian is worked out explicitly, see section 5. It is associated with a 4-point correlation function of two scalars and two spin-1/2 fermions [2], i.e. two of the $\mathrm{Spin}(3) = \mathrm{SU}(2)$ representations μ are 1-dimensional while the other two are 2-dimensional. The fiber T of our bundle is 4-dimensional and the Hamiltonian has block-diagonal form $H = \mathrm{diag}\{-2\mathcal{M}_1, -2\mathcal{M}_2\}$ with the following entries

$$\mathcal{M}_1 = \begin{pmatrix} H_{CS}^{(a,b,1)} + \frac{5}{4} & 0 \\ 0 & H_{CS}^{(a,b,1)} + \frac{5}{4} \end{pmatrix} + \tag{1.1}$$

$$\begin{pmatrix} \frac{-1}{16}\left(\frac{1}{\cosh^2\frac{x}{2}} + \frac{1}{\cosh^2\frac{y}{2}} - \frac{2}{\sinh^2\frac{x-y}{4}} - \frac{2}{\sinh^2\frac{x+y}{4}}\right) & \frac{a+b}{4}\left(\frac{1}{\cosh^2\frac{x}{2}} - \frac{1}{\cosh^2\frac{y}{2}}\right) \\ \frac{a+b}{4}\left(\frac{1}{\cosh^2\frac{x}{2}} - \frac{1}{\cosh^2\frac{y}{2}}\right) & \frac{-1}{16}\left(\frac{1}{\cosh^2\frac{x}{2}} + \frac{1}{\cosh^2\frac{y}{2}} + \frac{2}{\cosh^2\frac{x-y}{4}} + \frac{2}{\cosh^2\frac{x+y}{4}}\right) \end{pmatrix}$$

$$\mathcal{M}_2 = \begin{pmatrix} H_{CS}^{(a,b,1)} + \frac{5}{4} & 0 \\ 0 & H_{CS}^{(a,b,1)} + \frac{5}{4} \end{pmatrix} + \tag{1.2}$$

$$\begin{pmatrix} \frac{1}{16}\left(\frac{1}{\sinh^2\frac{x}{2}} + \frac{1}{\sinh^2\frac{y}{2}} + \frac{2}{\sinh^2\frac{x+y}{4}} - \frac{2}{\cosh^2\frac{x-y}{4}}\right) & \frac{b-a}{4}\left(\frac{1}{\sinh^2\frac{x}{2}} - \frac{1}{\sinh^2\frac{y}{2}}\right) \\ \frac{b-a}{4}\left(\frac{1}{\sinh^2\frac{x}{2}} - \frac{1}{\sinh^2\frac{y}{2}}\right) & \frac{1}{16}\left(\frac{1}{\sinh^2\frac{x}{2}} + \frac{1}{\sinh^2\frac{y}{2}} + \frac{2}{\sinh^2\frac{x-y}{4}} - \frac{2}{\cosh^2\frac{x+y}{4}}\right) \end{pmatrix}$$

where $H_{CS}^{(a,b,1)}$ is a Calogero-Sutherland Hamiltonian of BC_2 type, see eq. (4.11). In the appendix A we map this Hamiltonian to the set of Casimir equations for 3D fermionic seed blocks that were worked out in [21]. Our matrix Hamiltonian describes the two spin-1/2 particles in a 1-dimensional external potential with an infinite wall at $x = 0, y = 0$. The interaction of the particles with the wall depends on the spin and it can induce spin flips, i.e. involves off-diagional terms, if the parameters $a \neq 0$ or $b \neq 0$. In addition, the particles possess a spin-dependent interaction. The latter is purely diagonal.

The paper finally concludes with a list of open questions and further directions. Among them are the analysis of Casimir equations in dimension $d \geqslant 4$, the study of boundary and defect blocks as well as spinning blocks for non-BPS operators in superconformal field theories. Integrability and solutions of the Casimir equations are briefly commented on while details are left for future research.

2 Conformal blocks and Tensor Structures

In this section we shall review the basic model of spinning conformal blocks in the context of 4-point correlation functions on $\mathbb{R}^d$. We will work in Euclidian d-dimensional space so

[2] Strictly speaking, spin-1/2 fermions are in a representation of the universal covering group $\mathrm{Spin}(3) = \mathrm{SU}(2)$ of the rotation group $R = \mathrm{SO}(3)$. Throughout most of this text we shall not distinguish between $\mathrm{Spin}(d)$ and $\mathrm{SO}(d)$.

that the conformal group is $G = \mathrm{SO}(1, d+1)$. Primary fields of a conformal field theory sit in representations χ_π of G that are induced from a representation π of the subgroup $K = \mathrm{SO}(1,1) \times \mathrm{SO}(d) \subset G$. Here, the factor $D = \mathrm{SO}(1,1)$ is generated by dilations while $R = \mathrm{SO}(d)$ consists of all rotations r of the d-dimensional Euclidean plane. The choice of π encodes the conformal weight Δ and the highest weight μ of the rotation group $\mathrm{SO}(d)$. We shall use $\pi = \pi^\Delta_\mu$ to display the dependence on Δ and μ. From time to time we will also write $\pi = (\Delta, \mu)$.

It is well known that the correlation functions of two primary operators are uniquely fixed (up to normalization) by conformal symmetry to take the following form

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j^\dagger(x_2)\rangle = \frac{\delta_{ij} t_{ij}}{|x_{12}|^{2\Delta_i}} \tag{2.1}$$

where t is a unique tensor structure. As an example consider correlation function of two primary operators $\mathcal{O}^{\nu_1\cdots\nu_l}$ which transforms as symmetric traceless tensors under the action of the rotation group $R = \mathrm{SO}(d)$. It is customary to contract the indices of such fields with the indices of a lightlike vector ζ_ν, i.e. to introduce

$$\mathcal{O}(x;\zeta) \equiv \mathcal{O}^{\nu_1\cdots\nu_l}(x)\,\zeta_{\nu_1}\cdots\zeta_{\nu_l} \ .$$

The corresponding 2-point functions can be written as

$$\langle \mathcal{O}(x_1,\zeta_1)\,\mathcal{O}(x_2,\zeta_2)^\dagger\rangle = \frac{1}{|x_{12}|^{2\Delta}}\,(\zeta_{1,\nu} I^{\nu\eta}\zeta_{2,\eta})^l \tag{2.2}$$

where $I^{\nu\eta} = g^{\nu\eta} - 2x_{12}^\nu x_{12}^\eta/|x_{12}|^2$. Correlation function of three primary operators corresponding to representations $(\Delta_1,\mu_1), (\Delta_2,\mu_2)$ and (Δ_3,μ_3) can be written as a sum over conformally invariant tensor structures t^α

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\sum_{\alpha=1}^{N_3} \lambda_{123}^\alpha t_{123}^\alpha}{|x_{12}|^{\Delta_{12,3}}|x_{23}|^{\Delta_{23,1}}|x_{13}|^{\Delta_{13,2}}}, \tag{2.3}$$

where $\Delta_{12,3} = \Delta_1 + \Delta_2 - \Delta_3$ etc. and $N_3 = N_3(\mu_1,\mu_2,\mu_3)$ denotes the number of tensor structures t^α that can appear. Finally, λ_{123}^α are the structure constants that are not determined by conformal symmetry and carry dynamical information. Note that we have suppressed all tensor indices in eq. (2.3). In case two of the fields, let's say $\mathcal{O}_1$ and $\mathcal{O}_2$ are scalar and the field $\mathcal{O}_3$ is a symmetric traceless tensor of spin l, there is a unique tensor structure, i.e. $N_3 = 1$, and the correlator reads

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3,\zeta)\rangle = \frac{\lambda_{123}^\alpha Z^l}{|x_{12}|^{\Delta_{12}}|x_{23}|^{\Delta_{23}}|x_{13}|^{\Delta_{13}}}, \tag{2.4}$$

$$Z = \frac{|x_{23}||x_{13}|}{|x_{12}|}\left(\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}\right)\zeta_\mu \ . \tag{2.5}$$

In more general cases, the number N_3 of tensor structures can be computed in terms of the representation theory of the rotation group [24, 25]

$$N_3(\mu_1,\mu_2,\mu_3) = \sum_\mu N^\mu_{\mu_1,\mu_2} n_\mu(\mu_3) \ , \tag{2.6}$$

where the sum runs over irreducible representations μ of the rotation group and $N^{\mu}_{\mu_1,\mu_2}$ denotes the Clebsch-Gordon multiplicities for the decomposition of the tensor product of μ_1 and μ_2. The number n_μ denotes the number of SO($d-1$)$\subset$ SO(d) invariant linear maps from V_μ to $V_{\bar{3}}$ i.e.

$$n_\mu(\mu_3) = dim\left(Hom_{\mathrm{SO}(d-1)}(V_\mu, V_{\bar{3}})\right) \ . \tag{2.7}$$

Here V_μ and $V_{\bar{3}}$ ate the carrier spaces of the representations μ and $\bar{\mu}_3$, respectively. The subscript indicates that we consider only SO($d-1$) invariant maps. Let us note that the number N_3 of 3-point tensor structures t_{123} also counts the number of different tensor structures appearing in the operator product expansion of the first two fields $\mathcal{O}_1$ and $\mathcal{O}_2$ into the third $\mathcal{O}_3^\dagger$. From our description it is clear that we can construct the tensor structures in operator products as $t_{123} = \sum_\mu C_{12\mu} I_{\mu\bar{\mu}_3}$. Here, $C_{12\mu}$ is a SO(d) Clebsch-Gordon map from the tensor product $\mu_1 \otimes \mu_2$ into the SO(d) representations μ. The maps $I_{\mu\bar{\mu}_3}$, on the other hand, are SO($d-1$) intertwiners between the representations μ and $\bar{\mu}_3$ where both are restricted to representations of the subgroup SO($d-1$) $\subset$ SO(d).

Even though formula (2.6) seems to break the symmetry between $1, 2, 3$, the number it computes is actually completely symmetric. In fact, inserting eq. (2.7) into eq. (2.6) we obtain

$$N_3(\mu_1, \mu_2, \mu_3) = dim\left(Hom_{\mathrm{SO}(d-1)}(V_1 \otimes V_2, V_{\bar{3}})\right) = dim\left(V_1 \otimes V_2 \otimes V_3\right)^{\mathrm{SO}(d-1)} \ . \tag{2.8}$$

The relevance of the subgroup SO($d-1$) $\subset$ SO(d) is not too difficult to understand. Recall that we can use conformal transformations to move three points in $\mathbb{R}^d$ to the origin, the point $e_1 = (1, 0, \ldots, 0)$ and the point at infinity. Since all these points lie on a single line $\mathbb{R} \subset \mathbb{R}^d$, the configuration is left invariant by rotations of the transverse space $\mathbb{R}^{d-1}$.

After this preparation let us turn to the main object of our interest, namely the 4-point correlation function. Similarly to the case of 2 and 3-point correlation functions it can be decomposed into the sum over different tensor structures $t^I = t^I_{1234}$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \Omega_{(12)(34)}(x_i) \sum_{I=1}^{N_4} g^I(u,v) t^I_{1234}, \tag{2.9}$$

$$\Omega_{(12)(34)}(x_i) = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{14}}{x_{24}}\right)^{\Delta_2-\Delta_1} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_3-\Delta_4} \ .$$

The coefficients $g^I(u,v)$ depend on two anharmonic ratios $u = x_{12}^2 x_{34}^2 / x_{13}^2 x_{24}^2$ and $v = x_{14}^2 x_{23}^2 / x_{13}^2 x_{24}^2$ and N_4 is the number of different 4-point tensor structures,

$$N_4 \equiv N_4(\mu_1, \mu_2, \mu_3, \mu_4) \leqslant dim\left(V_1 \otimes V_2 \otimes V_3 \otimes V_4\right)^{\mathrm{SO}(d-2)} \ . \tag{2.10}$$

This formula is a direct extension of formula (2.8) for the number of 3-point structures. The main difference is that now we need to look for invariants with respect to the action of SO($d-2$) $\subset$ SO(d) rather than SO($d-1$). Once again, we can understand the relevance of this subgroup from the geometry of insertion points in $\mathbb{R}^d$. It is well known that conformal transformations allow to bring four such points into a 2-dimensional plane $\mathbb{R}^2 \subset \mathbb{R}^d$. The subgroup $\mathrm{SO}(d-2)$ is the symmetry group of the associated transverse space.

As in our analysis of 3-point structures, we obtain an alternative view on the tensor structures if we evaluate 4-point correlation functions by performing operator product expansion of two fields $\mathcal{O}_1$ and $\mathcal{O}_2$ into conformal primary fields $\mathcal{O} = \mathcal{O}_\pi$ and its descendants. The result reads as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_{\mathcal{O}_\pi}\sum_{\alpha,\beta} \lambda^\alpha_{12\pi}\lambda^\beta_{\bar{\pi}34} W^{\alpha\beta}_{1234,\pi}(x_1,x_2,x_3,x_4) \ . \tag{2.11}$$

The set of 3-point tensor structures α, β that appear in the two operator products depends on the intermediate operator $\mathcal{O}_\pi$ with $\pi = (\Delta, \mu)$. The individual block W may now be decomposed as

$$W^{\alpha\beta}_{1234,\pi}(x_1,x_2,x_3,x_4) = \Omega_{(12)(34)}(x_i)\, g^{\alpha\beta}_{\Delta,\mu}(u,v)\, t^{\alpha\beta}_{1234} \ , \tag{2.12}$$

where we introduced the 4-point tensor structures $t^{\alpha\beta}_{1234}$ that can be obtained from a composition of tensor structures $t^\alpha_{12\mu}$ and $t^\beta_{\bar{\mu}34}$ in the left and right operator product expansion, respectively. By the very construction it is clear that the number of such tensor structures is

$$N_4(\mu) = N_4(\mu_1,\mu_2,\mu_3,\mu_4;\mu) := N_3(\mu_1,\mu_2,\mu)N_3(\mu_3,\mu_4,\mu^\dagger) \leqslant N_4. \tag{2.13}$$

We can now perform the decompositions (2.11) and (2.12) on the coefficients $g^I(u,v)$ defined in eq. (2.9) to obtain the following expansion in terms of spinning conformal blocks $g^{I,\alpha\beta}_\pi(u,v)$,

$$g^I(u,v) = \sum_{\mathcal{O}_\pi}\sum_{\alpha,\beta} \lambda^\alpha_{12\pi}\lambda^\beta_{\bar{\pi}34} g^{I,\alpha\beta}_{\Delta,\mu}(u,v) \ . \tag{2.14}$$

The spinning conformal blocks $[g^{I,\alpha\beta}_{\Delta,\mu}(u,v)]$ with given $\pi = (\Delta,\mu)$ satisfy a set of second order differential equations of the form

$$\mathcal{C}^{(2)}[g^{I',\alpha'\beta'}_{\Delta,\mu}(u,v)] = C_{\Delta,\mu}\ [g^{I,\alpha\beta}_{\Delta,\mu}(u,v)] \ , \tag{2.15}$$

where $\mathcal{C}^{(2)}$ denotes the second order Casimir differential operator and $C_{\Delta,\mu}$ is the eigenvalue of the quadratic Casimir element of the conformal group in the representation χ_π that is induced from (Δ,μ). Such Casimir equations are well known for scalar blocks, see [20], and they were constructed for several examples involving fields with spin, see [18, 21]. Our main goal in this work is to develop a systematic approach to Casimir equations for spinning blocks.

3 Harmonic Analysis Approach to Conformal Blocks

In the previous section we described spinning conformal blocks as a set of functions $g^{\alpha\beta}_{\Delta,\mu}(u,v)$ of the two anharmonic rations one can build out of four points in $\mathbb{R}^d$. The main goal of the current section is to show that the same objects can also be realized as sections of a certain vector bundle over a 2-dimensional quotient of the conformal group G itself. While our discussion will remain a bit abstract, it mirrors the line of arguments we went through in the previous section. Many of the key elements will be illustrated in the next section when we discuss concrete examples.

In mathematical terms, 4-point conformal blocks are invariants in the tensor product of four continuous series representations $\chi_i, i = 1, \dots, 4$ of the conformal group G. In the principal continuous series, the conformal weights are of the form $\Delta = d/2 + ic$ with real parameter c. We shall adopt these values for now and only continue to real conformal weights at the very end once we derived the equations. In order to construct this space, we will first realize the tensor products $\chi_1 \otimes \chi_2$ and $\chi_3 \otimes \chi_4$ in a space of functions on G with certain equivariance properties under the left/right regular action of the subgroup $K \subset G$. According to theorem 9.2 of [26] the tensor product $\chi_{\pi_1} \otimes \chi_{\pi_2}$ can be realized as

$$\chi_{\pi_1} \otimes \chi_{\pi_2} \cong \Gamma^{(\pi_1,\pi_2)}_{K\backslash G} \quad \text{with} \tag{3.1}$$

$$\Gamma^{(\pi_1,\pi_2)}_{K\backslash G} = \left\{ f : g \to V_{\mu_1} \otimes V_{\mu_2} \,\middle|\, \begin{array}{ll} f(d(\lambda)g) = e^{\lambda(\Delta_2-\Delta_1)} f(g) & \text{for } \ d(\lambda) \in D \subset G \\ f(rg) = \mu_1(r) \otimes \mu_2(r) f(g) & \text{for } r \in R \subset G \end{array} \right\}.$$

Here, V_{μ_1} and V_{μ_2} denote the finite dimensional carrier spaces of our representations μ_1 and μ_2 of the rotation group and we wrote elements $d \in D$ as

$$d(\lambda) = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix} . \tag{3.2}$$

For a proof of this theorem see [26]. Elements of the space (3.1) are vector valued functions on the group that are covariantly constant along the orbits of the left K-action on G. Such functions are uniquely characterized by the values they assume on the space $K \setminus G$ of such orbits. This is why we shall often refer to Γ as a space of sections in a vector bundle over the quotient space $K \setminus G$. Similarly one can realise tensor product $\chi_{\pi_3} \otimes \chi_{\pi_4}$ on the right cosets G/K,

$$\chi_{\pi_3} \otimes \chi_{\pi_4} = \Gamma^{(\pi_3,\pi_4)}_{G/K} \quad \text{with} \tag{3.3}$$

$$\Gamma^{(\pi_3,\pi_4)}_{G/K} = \left\{ f : g \to V_{\mu_3} \otimes V_{\mu_4} \,\middle|\, \begin{array}{ll} f(gd(\lambda)^{-1}) = e^{\lambda(\Delta_3-\Delta_4)} f(g) & \text{for } \ d(\lambda) \in D \subset G \\ f(gr^{-1}) = \mu_3(r) \otimes \mu_4(r) f(g) & \text{for } r \in R \subset G \end{array} \right\}.$$

Let us note in passing that the spaces Γ we defined in eqs. (3.1) and (3.3) decompose into an infinite set of irreducible representations of the conformal group. The number of times a given representation $\chi_\pi = \chi_3$ appears in this decomposition is given by the formula (2.8) for the number of 3-point tensor structures.

Equipped with a good model for the tensor products of field multiplets we now want to realize G-invariants in the four-fold tensor product of representations. In order to keep the discussion as transparent as possible we shall first restrict to the case of four external scalars, i.e. we shall assume that $\pi_i = (\Delta_i, \mu_i)$ with $\mu_i = 0$. As before, we group these four fields into two pairs and apply the previous theorem to realize the products of representations $\chi_1 \otimes \chi_2$ and $\chi_3 \otimes \chi_4$ on the vector bundles (3.1) and (3.3), respectively. Since these bundles are defined over the left and right cosets $K \setminus G$ and G/K, respectively, they both carry an action of the conformal group G by right resp. left translations. More precisely, an element $g \in G$ acts on $K \setminus G \times G/H$ as $(g_1, g_2) \to (g_1 g^{-1}, g g_2)$. We can use this action to pass to the space of invariants,

$$\left(\bigotimes_{i=1}^{4} \chi_{\pi_i} \right)^G \cong \left(\Gamma_{K \backslash G}^{(\Delta_1,\Delta_2)} \otimes \Gamma_{G/K}^{(\Delta_3,\Delta_4)} \right)^G \cong \Gamma_{G/\!/K}^{(a,b)} \quad \text{with} \tag{3.4}$$

$$\Gamma_{G/\!/K}^{(a,b)} = \left\{ f : G \to \mathbb{C} \middle| f(d(\lambda)g) = e^{2a\lambda} f(g) \ , \ f(gd(\lambda)^{-1}) = e^{2b\lambda} f(g) \right\} \tag{3.5}$$

where $2a = \Delta_2 - \Delta_1$ and $2b = \Delta_3 - \Delta_4$. Since we have assumed that $\Delta_i = d/2 + ic_i$, the parameters a, b are purely imaginary before we continue to real Δ_i. We have now obtained a new model for the space of conformal blocks $g(u, v)$. Since we restricted to four external scalars, there is a single tensor structure only so that no indices $I, \alpha\beta$ appear. In our notations we indicate that we want to think of the space (3.4), as a space of sections in a line bundle over the double coset $G /\!/ K$. The latter appears since $(K \setminus G \times G/K)/G = K \setminus G/K \equiv G /\!/ K$ which follows from the obvious relation $(G \times G)/G = G$. As we will see in the next section, the double coset $G /\!/ K$ is two-dimensional and the two coordinates are related with the two anharmonic ratios u, v we used in the previous section. In complete analogy with the decomposition (2.14) we can decompose the space $\Gamma_{G/\!/K}^{(a,b)}$ of sections into a sum over intermediate channels,

$$\Gamma_{G/\!/K}^{(a,b)} = \bigoplus_{\Delta,\mu} \Gamma_{G/\!/K}^{(a,b),(\Delta,\mu)} \ . \tag{3.6}$$

Since we constructed Γ as a space of functions on G with certain equivariance properties, the Laplacian on the conformal group G descends to Γ and the decomposition (3.6) is the corresponding spectral decomposition. More precisely, the summands in the decomposition are eigenspaces of the Laplacian with eigenvalue $C_{\Delta,\mu}$ and certain boundary conditions.

It remains to extend the previous discussion to the case of spinning blocks, i.e. we need to drop the condition $\mu_i = 0$. Formula (3.4) possesses the following extension to cases with $\mu_i \neq 0$,

$$\left(\Gamma_{K \backslash G}^{(\Delta_1,\mu_1;\Delta_2,\mu_2)} \otimes \Gamma_{G/K}^{(\Delta_3,\mu_3;\Delta_4,\mu_4)} \right)^G \cong \Gamma_{G/\!/K}^{(a,\mu_1 \otimes \mu_2; b,\mu_3 \otimes \mu_4)} \ . \tag{3.7}$$

The labels a, b are determined by the conformal weights of the external fields as before. Extending our prescription (3.4), we specify vector bundle over $G /\!/ K$ that appear on the right hand side in the following way

$$\Gamma_{G/\!/K}^{(\mathcal{LR})} = \{ \, f : G \to V_{\mathcal{L}} \otimes V_{\mathcal{R}} \, | \, f(kg) = \mathcal{L}(k) f(g) \quad , \quad f(gk^{-1}) = \mathcal{R}(k) f(g) \ \} \ , \tag{3.8}$$

where the two representations $\mathcal{L} = (a, \mu_1 \otimes \mu_2)$, $\mathcal{R} = (b, \mu_3 \otimes \mu_4)$ act on $V_{\mathcal{L}} = V_1 \otimes V_2$ and $V_{\mathcal{R}} = V_3 \otimes V_4$, respectively, according to

$$\mathcal{L}(d(\lambda)r) = e^{2a\lambda} \mu_1(r) \otimes \mu_2(r) \quad , \quad \mathcal{R}(d(\lambda)r) = e^{2b\lambda} \mu_3 \otimes \mu_4(r) \ . \tag{3.9}$$

Our definition (3.8) selects a subspace among functions on the group that take values in the 4-fold tensor product $\bigotimes_{i=1}^4 V_{\mu_i}$ of the group K. The identification of this space as sections of a vector bundle over the coset space is a bit more tricky in $d > 3$ since the action of $K \times K$ on the conformal group G is not free beyond $d = 3$ dimensions. As we shall see explicitly in the next section, the stabilizer for the action of $K \times K$ on G is given by a subgroup $\mathrm{SO}(d-2) \subset \mathrm{SO}(d) \times \mathrm{SO}(d)$. If we now want to construct a function f in the space (3.8) by prescribing the values it takes on the double coset, we have to make sure that the covariance conditions with respect to the left and right action of K are compatible. This compatibility condition forces f to take values in the subspace

$$T = (V_{\mu_1} \otimes V_{\mu_2} \otimes V_{\mu_3} \otimes V_{\mu_4})^{\mathrm{SO}(d-2)} \ . \tag{3.10}$$

In conclusion, we can indeed think of the space (3.8) as a space of sections in a vector bundle over the double coset, as long as we remember to restrict the fibers to the space of $\mathrm{SO}(d-2)$ invariants in the tensor product of the spin representations. Note that the space T contains the space of 4-point tensor structures we introduced in the previous section.

As in eq. (3.6) we can decompose the space (3.7) into a sum of eigenspaces of the Laplacian of the conformal group,

$$\Gamma^{(\mathcal{LR})}_{G/\!/K} = \sum_{\Delta,\mu} \Gamma^{(\mathcal{LR}),(\Delta,\mu)}_{G/\!/K} \ . \tag{3.11}$$

We have now succeeded to model spinning conformal blocks through vector valued $K \times K$-covariant functions on G. The latter can also be thought of as sections in vector bundles over the double coset $G/\!/K$. Our next task is to analyse the action of the Laplacian on the spaces (3.11) and finally to compare the associated eigenvalue problem with the Casimir equations for conformal blocks.

4 Harmonic Analysis and Calogero-Sutherland Models

Our goal in this section is to describe an algorithm that allows us to write the action of the conformal Laplacian on the spaces (3.8) as a Hamiltonian for two interacting particles with spin that move on a 1-dimensional space. The latter will turn out to be of Calogero-Sutherland type. This extends a classical observation of Olshanetsky and Perelomov about a relation between certain harmonic analysis problems on groups and Calogero-Sutherland Hamiltonians [27–29] to the cases with spin, see also [22, 23]. In the context of conformal field theory, our findings generalize [19] to spinning blocks.

In order to achieve our goal we shall introduce a special set of coordinates on the conformal group that are based on a variant of the Cartan decomposition and suited for identification of double quotient $G//K$, see first subsection. We will then construct the Laplace-Beltrami operator on the conformal group in these coordinates. In a final step, we integrate over the $K \times K$ orbits to obtain second order differential operators on the 2-dimensional quotient space $G/\!/K$. The latter can be transformed into a Calogero-Sutherland type Hamiltonian.

4.1 Cartan decomposition of the conformal group

Let us begin by introducing a coordinate system on the conformal group $G = \mathrm{SO}(1, d+1)$ that is well adapted to the action of the $K \times K \subset G \times G$ on G. The action of $G \times G$ that we restrict to the subgroup $K \times K$ is the action on G by left and right regular translations. Our choice of $K = \mathrm{SO}(1,1) \times \mathrm{SO}(d)$ determines a so-called Cartan or KAK decomposition of G. In order to describe the details we note that Lie algebra $\mathfrak{k}$ of K contains all the elements of Lie algebra $\mathfrak{g}$ of conformal group G that are eigenvectors with eigenvalue $+1$ of automorphism Θ acting on $\xi \in \mathfrak{g}$ as $\Theta(\xi) = \theta\xi\theta,\ \theta = diag\ (-1,-1,1,\dots,1)$. The automorphism Θ determines a decomposition of the Lie algebra $\mathfrak{g}$ of the conformal group G as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the subgroup K and $\mathfrak{p}$ its orthogonal complement. The latter is the subspace on which Θ acts by multiplication with -1. What leads to $\mathbb{Z}_2$ grading on $\mathfrak{g}$

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \quad , \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \quad , \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \ .$$

Note that any *subalgebra* $\mathfrak{a} \subset \mathfrak{p}$ of $\mathfrak{g}$ that is contained in $\mathfrak{p}$ must be abelian. A maximally abelian subalgebra $\mathfrak{a}$ of this type is often referred to as the Cartan subalgebra of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. As we will see below, any two such Cartan subalgebras are conjugated to each other and the dimension of these subalgebras is $dim\ \mathfrak{a} = 2$. Exponentiating $\mathfrak{a}$ we get the abelian subgroup A and the Cartan decomposition reads as $G = KAK$.[3]

Now let us describe the Cartan decomposition explicitly. To this end, we shall work with the usual set of generators $M_{ij} = -M_{ji}$ of the conformal group $G = \mathrm{SO}(1, d{+}1)$ where i, j run through $i, j = 0, 1, 2, \dots d+1$. Here $i, j = 0$ correspond to the time-like direction while all other directions are space-like. Obviously, the Lie algebra $\mathfrak{k}$ of K is spanned by the generator $M_{0,1}$ of dilations along with the elements $M_{\mu\nu}$ for $\mu = 2, \dots, d+1$ that generate rotations. Our subspace $\mathfrak{p}$ in turn is spanned by $M_{0,\mu}$ and $M_{1,\mu}$. The choice of $\mathfrak{a}$ that we shall adopt is the one for which $\mathfrak{a}$ is spanned by $a_+ = M_{0,2}$ and $a_- = M_{1,3}$. These two generators commute with each other since they have no index in common. Clearly, the Cartan algebra cannot be extended beyond a_+, a_- since any other generator $\mathfrak{p}$ will necessarily have one index in common with the ones we have singled out as a_+ and a_-.

Through the Cartan decomposition we may write any element $g \in G$ of the conformal group as a product of the form

$$g = d(\lambda_1)\, r_1\, a(\tau_1, \tau_2)\, d(\lambda_2)\, r_2. \tag{4.1}$$

Here $d(\lambda_i) \in D = \mathrm{SO}(1,1)$ are considered as elements of the subgroup $D \subset G$. The group element $a(\tau_1, \tau_2)$ in turn is given by

$$a(\tau_1,\tau_2) = \begin{pmatrix} \cosh\frac{\tau_1}{2} & 0 & \sinh\frac{\tau_1}{2} & 0 \\ 0 & \cos\frac{\tau_2}{2} & 0 & -\sin\frac{\tau_2}{2} \\ \sinh\frac{\tau_1}{2} & 0 & \cosh\frac{\tau_1}{2} & 0 \\ 0 & \sin\frac{\tau_2}{2} & 0 & \cos\frac{\tau_2}{2} \end{pmatrix} . \tag{4.2}$$

[3] Let us stress that the decomposition $g = k_1 a k_2$ of an element $g \in G$, if it exists, is not unique for $d > 3$, see below.

There is one small subtlety that is associated with elements r_1 and r_2 of the rotation group. Let us note that the two generators a_+ and a_- of our subgroup $A \subset G$ are left invariant by all generators of the form $M_{\mu\nu} \in \mathfrak{k}$ with $\mu, \nu = 4, \ldots, d+1$. These generate a subgroup $B = \mathrm{SO}(d-2) \subset \mathrm{SO}(d) \subset K$. Consequently, the decomposition (4.1) is not unique as we can move factors $b \in B$ between r_1 and r_2. We can fix this freedom by choosing r_2 to be a representative of a point on the coset space $K/B = \mathrm{SO}(d)/\,\mathrm{SO}(d-2)$. Once this choice is adopted, the KAK decomposition becomes unique up to discrete identifications. One may verify that the dimensions indeed match

$$dimG = dimK + dimA + dimK - dimB \ .$$

To complete our description of coordinates on the conformal group it remains to parametrize the elements r_i of the rotation group. The detailed choice does not matter since these coordinates will be integrated over later.

In the remainder of this work we shall assume that $d \leqslant 3$ so that the group B is trivial. Extending our calculations beyond $d = 3$ is the subject of a future paper [30].

Example: Throughout this section we shall illustrate all our statements and constructions at the example of the 2-dimensional conformal group SO(1, 3). In this case we shall parametrize the elements $r_1 = r_1(\phi_1)$ and $r_2 = r_2(\phi_2)$ such that

$$k_i(\lambda_i, \phi_i) = d(\lambda_i) r(\phi_i) = \begin{pmatrix} \cosh\lambda_i & \sinh\lambda_i & 0 & 0 \\ \sinh\lambda_i & \cosh\lambda_i & 0 & 0 \\ 0 & 0 & \cos\phi_i & -\sin\phi_i \\ 0 & 0 & \sin\phi_i & \cos\phi_i \end{pmatrix} .$$

Thereby we have now parametrized an arbitrary element of the conformal group SO(1, 3) with the help of the product formula (4.1) through the six coordinates $\lambda_i, \phi_i, \tau_i$ for $i = 1, 2$. These coordinates possess the following ranges: $\tau_1 \in (-\infty, \infty)$, $\tau_2 \in [0, 4\pi)$, $\lambda_i \in (-\infty, \infty)$, $\phi_i \in [0, 2\pi)$.

4.2 The Laplacian on the Cartan subgroup

Our next task is to construct the Laplacian on the conformal group in the coordinate system we have introduced in the previous subsection. This is straightforward. The Laplace-Beltrami operator on any Riemannian manifold may be computed from the metric g through

$$\Delta_{\mathrm{LB}} = \sum_{\alpha,\beta} |\det(g_{\alpha\beta})|^{-\frac{1}{2}} \partial_\alpha g^{\alpha\beta} |\det(g_{\alpha\beta})|^{\frac{1}{2}} \partial_\beta \ . \tag{4.3}$$

On a group manifold the metric $g_{\alpha\beta}$ is obtained with the help of the Killing form as

$$g_{\alpha\beta}(x) = -2 \ \mathrm{tr} \ h^{-1}\partial_\alpha h \ h^{-1}\partial_\beta h, \quad h \in G \ . \tag{4.4}$$

By construction, the Laplace-Beltrami operator Δ_{LB} commutes with the $G \times G$ action on the group G by left and right regular transformations. Since it is a second order differential operator, it can be written as a quadratic expression in the left or right invariant vector fields on G in which the vector fields are contracted with the Killing form, i.e. the Laplace-Beltrami operator coincides with the action of the quadratic Casimir element on functions.

In the setup we described in the previous section, the Laplace-Beltrami operator acts on functions f on the conformal group that take values in the vector spaces $V_{\mathcal{L}} \otimes V_{\mathcal{R}}$. Since the bundle over the group G is trivial, the Laplace operator acts simply component-wise. We will not distinguish in notation between the Laplacian on the group itself and on trivial vector bundles.

Using the metric on G we can also construct the invariant Haar measure $d\mu_G$ on G. Its density is given by $\sqrt{\det g_{\alpha,\beta}}$. The Haar measure can then be used to introduce a scalar product for (vector-valued) functions on G. The associated space of square integrable functions will be denoted as usual by $L^2_G = L^2(G, V_{\mathcal{L}} \otimes V_{\mathcal{R}}; d\mu_G)$. The Laplace-Beltrami operator is a densely defined on this space and it is Hermitian with respect to the scalar product.

Example: Using the coordinates on SO(1,3) that we introduced at the end of the previous subsection, the metric takes the form

$$\begin{aligned} g_{\alpha\beta}dx^\alpha dx^\beta &= 4(d^2\phi_l + d^2\phi_r - d^2\lambda_l - d^2\lambda_r) - d^2\tau_1 + d^2\tau_2 \\ &\quad - 8\sinh\frac{\tau_1}{2}\sin\frac{\tau_2}{2}(d\lambda_l d\phi_r + d\lambda_r d\phi_l) + 8\cosh\frac{\tau_1}{2}\cos\frac{\tau_2}{2}(d\phi_l d\phi_r - d\lambda_l d\lambda_r)\ . \end{aligned} \tag{4.5}$$

It is easy to work out the Haar measure on the conformal group from the determinant of the metric,

$$d\mu_G = 8(\cosh\tau_1 - \cos\tau_2)d\lambda_l d\phi_l d\tau_1 d\tau_2 d\lambda_r d\phi_r\ .$$

We leave it as an exercise to construct the associated Laplace-Beltrami operator.

In the context of d-dimensional conformal blocks we are now instructed to restrict the Laplace-Beltrami operator to the space (3.8) and to study the spectrum and eigenfunctions of this restriction. The elements of the space (3.8) are $K \times K$ covariant functions on the group G and hence they are uniquely characterized by their dependence on the two coordinates τ_1 and τ_2. We can equip functions in the Cartan subgroup A with a measure[4]

$$m(\tau_1,\tau_2)d\tau_1 d\tau_2 := d\mu_A(\tau_1,\tau_2) = \frac{1}{Z}\int_{K\times K}^{\text{reg}} d\mu_G$$

with $Z = \mathit{Vol}(\mathrm{SO}(d))^2$. Note that $K = D \times R$ contains the non-compact factor D that makes the integration over K divergent. We can regularize this divergence e.g. through the prescription

$$\int_{\mathbb{R}}^{\text{reg}} d\lambda = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} d\lambda\ .$$

Having fixed a measure on A we can now take a function $f_A \in L^2_A = L^2(A, V_{\mathcal{L}} \otimes V_{\mathcal{R}}; d\mu_A)$ on the Cartan subgroup A with values in the linear space $V_{\mathcal{L}} \otimes V_{\mathcal{R}}$. Such a function may be

[4] Recall that we assumed $d \leqslant 3$. For $d \geqslant 4$, the integration is over fibers of the $K \times K$ action on G.

extended uniquely to a $V_{\mathcal{L}} \otimes V_{\mathcal{R}}$-valued covariantly constant function on G. The latter is square integrable provided we agree to regularize the integration over λ_l and λ_r as outlined above. In other words, there is an isomorphism of Hilbert spaces

$$L^2(\Gamma^{(\mathcal{LR})}; d\mu_G) \cong L^2_A = L^2(A, V_{\mathcal{L}} \otimes V_{\mathcal{R}}; d\mu_A) \ . \tag{4.6}$$

This isomorphism induces a correspondence between $K \times K$ invariant Hermitian differential operators $\mathcal{D}$ acting on L^2_G and Hermitian differential operators $\mathcal{D}^A$ on the Cartan subgroup A such that

$$\int_A d\mu_A \langle f_A, \mathcal{D}^A g_A \rangle = \frac{1}{Z} \int_G^{\text{reg}} d\mu_G \langle f(k_l a k_r), \mathcal{D}\, g(k_l a k_r) \rangle, \tag{4.7}$$

$$\begin{aligned} \text{where} \quad f(k_l a k_r) &= [\mathcal{L}(k_l) \otimes \mathcal{R}(k_r^{-1})] f_A(\tau_1, \tau_2) = \\ &= e^{2b\lambda_l + 2a\lambda_r} [(\mu_1 \otimes \mu_2)(r_l) \otimes (\mu_3 \otimes \mu_4)(r_r^{-1})] f_A(\tau_1, \tau_2) \ . \end{aligned}$$

Here, f and g are two covariantly constant functions on G, i.e. two elements of the space (3.8). The symbols f_A and g_A denote their restriction to the Cartan subgroup $A \subset G$. Elements k_l, k_r are parametrized as $k_l = d(\lambda_l) r_l$, $k_r = d(\lambda_r) r_r$. In addition we used $\langle \cdot, \cdot \rangle$ for the scalar product on the finite dimensional linear space $V_{\mathcal{L}} \otimes V_{\mathcal{R}}$.

We can now apply the prescription (4.7) to the Laplacian $\mathcal{D} = \Delta_{\text{LB}}$. In order to bring the reduced Laplacian Δ^A_{LB} into the form of a Calogero-Sutherland Hamiltonian on a space with measure $d\tau_1 d\tau_2$, it remains to remove the non-trivial factor $m(\tau_1, \tau_2)$ in the measure on the Cartan subgroup by an appropriate gauge transformation. This is achieved by rescaling the functions $f_A \in L^2_A$ such that

$$\psi_A(\tau_1, \tau_2) = \sqrt{m(\tau_1, \tau_2)}\, f_A(\tau_1, \tau_2) \ .$$

On the 2-particle wave functions $\psi_A(\tau_1, \tau_2)$ the reduced Laplacian indeed takes the form of a Calogero-Sutherland type Hamiltonian,

$$H_{(\mathcal{L},\mathcal{R})} \;=\; \sqrt{m(\tau_1, \tau_2)}\, \Delta^A \frac{1}{\sqrt{m(\tau_1, \tau_2)}} =: -\frac{d^2}{d\tau_1^2} + \frac{d^2}{d\tau_2^2} + V_{(\mathcal{L},\mathcal{R})}(\tau_1, \tau_2) \ . \tag{4.8}$$

After performing the gauge transformation that trivialized the measure, we can read off the matrix valued potential $V_{(\mathcal{L},\mathcal{R})}$. It depends on the choice of the representations $\mathcal{L}, \mathcal{R}$ and acts on the space $V_{\mathcal{L}} \otimes V_{\mathcal{R}}$. Our construction guarantees that the Hamiltonian $H_{(\mathcal{L},\mathcal{R})}$ is Hermitian with respect to the measure $d\tau_1 d\tau_2$ as it descends from the Hermitian Laplace-Beltrami operator on the conformal group G. In conclusion, we have now described an algorithm that associates a family of matrix valued potentials $V_{(\mathcal{L},\mathcal{R})} = V_{(a,\mu_1\otimes\mu_2;b,\mu_3\otimes\mu_4)}$ to any spinning conformal block. In order to make the kinetic term of the model look more standard, we will often use the coordinates $\tau_1 = x + y$ and $\tau_2 = i(x - y)$.

Example: Returning to our example of $G = \text{SO}(1,3)$ we want to determine the action of the Laplace-Beltrami operator on scalar blocks. In the case of scalars with parameters a, b, the covariantly constant functions on G read

$$f(x) = e^{2b\lambda_l + 2a\lambda_r} f_A(\tau_1, \tau_2) \ .$$

Our reduction formula (4.7) for the Laplacian becomes

$$\int d\mu_A \bar{f}_A(\tau_1,\tau_2)\Delta^A g_A(\tau_1,\tau_2) =$$
$$= \int d\tau_1 d\tau_2 \,(\cosh\tau_1 - \cos\tau_2) e^{-2b\lambda_l - 2a\lambda_r} \bar{f}_A(\tau_1,\tau_2)\Delta_{\text{LB}}\left(e^{2b\lambda_l+2a\lambda_r} g_A(\tau_1,\tau_2)\right) \,. \tag{4.9}$$

Here, $\bar{f}_A$ is the complex conjugate and we have used that a and b are purely imaginary. The measure $d\mu_A$ on A is given by $d\mu_A = m d\tau_1 d\tau_2$ with a non-trivial density function $m(\tau_1,\tau_2) = \cosh\tau_1 - \cos\tau_2$. If we perform the transformation (4.8) with the square root $m = (\cosh\tau_1 - \cos\tau_2)^{\frac{1}{2}}$ of the measure factor we obtain the famous Calogero-Sutherland Hamiltonian of BC_2 type

$$H = \frac{1}{2} H^{(a,b,0)}_{C.S} + \frac{1}{4} \tag{4.10}$$

where

$$H^{(a,b,\epsilon)}_{CS} = -\partial_x^2 - \partial_y^2 + V^{(a,b,\epsilon)}_{C.S.}, \qquad \epsilon = d-2 \tag{4.11}$$

$$V^{(a,b,\epsilon)}_{C.S.} = V^{(a,b)}_{PT}(x) + V^{(a,b)}_{PT}(y) + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x-y}{2}} + \frac{\epsilon(\epsilon-2)}{8\sinh^2\frac{x+y}{2}}, \tag{4.12}$$

$$V^{(a,b)}_{PT}(x) = \frac{(a+b)^2 - \frac{1}{4}}{\sinh^2 x} - \frac{ab}{\sinh^2\frac{x}{2}} \,. \tag{4.13}$$

Here we have written the Calogero-Sutherland Hamiltonian for arbitrary values of the coupling $\epsilon = d-2$. It appears when we evaluate the Laplace-Beltrami operator on the line bundles (3.4) associated with scalar representations of the conformal group, see also next section. In the case of d-dimensional scalar blocks there is an additional constant $(d^2-2d+2)/8$ which evaluates to $1/4$ for $d=2$. According to [19] the resulting Hamiltonians can be transformed into the usual Casimir operator [20] for scalar 4-point blocks in 2-dimensional conformal field theory, provided the coordinates $x_1 = x$ and $x_2 = y$ on the Cartan subgroup A are related to the usual variables $z_1 = z$ and $z_2 = \bar{z}$ through

$$z_i = -\sinh^{-2}\frac{x_i}{2} \,. \tag{4.14}$$

Note that this relation is independent of the dimension d.

5 Example: Seed conformal blocks in 3D

It has been argued [10, 31] that all conformal blocks in 3-dimensional conformal field theory may be obtained from two seed blocks by application of derivatives. These seed blocks include the usual scalar blocks along with one type of spinning blocks in which two of the four external fields transform in a 2-dimensional representation of the rotation group or rather its universal covering group Spin(3) = SU(2). Our goal is to construct the Casimir equations for these seed blocks from the Laplace-Beltrami operator on the 3-dimensional conformal group SO(1, 4). Following the procedure we have outlined above, we shall end up with two Calogero-Sutherland Hamiltonians. For scalar blocks, the result agrees with [19]. In the case of spinning blocks, on the other hand, we obtain a new formulation of the Casimir equations that were originally written in [21]. A verification that the two sets of Casimir equations are equivalent may be found in Appendix A.

5.1 3D scalar blocks

For scalar blocks the construction of the potential V proceeds exactly as in our 2-dimensional example in the previous section. In order to build the Laplacian on the conformal group, we parametrize the two elements $r_i \in \mathrm{SO}(3)$ in the KAK decomposition (4.1) through three angles,

$$r_i = \begin{pmatrix} \cos\phi_i & -\sin\phi_i & 0 \\ \sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_i & -\sin\theta_i \\ 0 & \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} \cos\psi_i & -\sin\psi_i & 0 \\ \sin\psi_i & \cos\psi_i & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{5.1}$$

The angles parametrizing r_i take the values $\phi_i, \psi_i \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$. The remaining variables τ_i and λ_i run through the same domain as in our 2-dimensional example.

It is straightforward to compute the metric and to construct the associated Laplacian. In the case at hand, the Haar measure is given by

$$d\mu_G = 128(\cosh\tau_1 - \cos\tau_2)\sin\theta_1\sin\theta_2\sinh\frac{\tau_1}{2}\sin\frac{\tau_2}{2}\prod_{i=1}^{2} d\phi_i d\theta_i d\psi_i d\tau_i d\lambda_i . \tag{5.2}$$

If this measure is used to integrate out the angular variables ϕ_i, ψ_i and θ_i, see eq. (4.7), and the Laplacian is gauge transformed with the square root of the function

$$m = (\cosh\tau_1 - \cos\tau_2)\sinh\frac{\tau_1}{2}\sin\frac{\tau_2}{2} \tag{5.3}$$

as described in eq. (4.8), we obtain

$$H = \frac{1}{2}H_{C.S}^{(a,b,1)} + \frac{d^2 - 2d + 2}{8}\Big|_{d=3} . \tag{5.4}$$

The result is in complete agreement with the Casimir equation for scalar 4-point functions constructed in [20] as was shown in [19].

5.2 3D fermionic seed block

The fermionic seed block analysed in [21] involves two spin-1/2 fermions at x_1 and x_4 and two scalar fields that are inserted at x_2 and x_3. Consequently, it corresponds to $\mu_1 \otimes \mu_2 = \frac{1}{2} \otimes 0 = \frac{1}{2}$ and $\mu_3 \otimes \mu_4 = \frac{1}{2}$. In other words, our representations $\mathcal{L}$ and $\mathcal{R}$ map the elements $k = k_l$ and $k = k$ in the universal covering SU(2) of the rotation group to the following 2×2 matrices

$$\mathcal{L}(k) = \begin{pmatrix} \cos\frac{\theta}{2}e^{i\frac{\phi+\psi}{2}} & i\sin\frac{\theta}{2}e^{i\frac{\phi-\psi}{2}} \\ i\sin\frac{\theta}{2}e^{-i\frac{\phi-\psi}{2}} & \cos\frac{\theta}{2}e^{-i\frac{\phi+\psi}{2}} \end{pmatrix} = \mathcal{R}(k) . \tag{5.5}$$

We will continue to parametrize the left elements $k_l \in \mathrm{SU}(2)$ by angles ϕ_l, ψ_l and θ_l and use ϕ_r, ψ_r and θ_r for $k_r \in \mathrm{SU}(2)$. Note that the action of the right transformations involves $\mathcal{R}(k_r^{-1})$, i.e. it contains an additional inversion. The equivariance law in eq. (4.7) allows to construct the four components u_i of a function $f = e^{2b\lambda_l + 2a\lambda_r}(u_1, u_2, u_3, u_4)^T$ from a set of functions $u_i^A = u_i^A(\tau_1, \tau_2)$ on the Cartan subgroup of the KAK decomposition

$$u_1 = e^{\frac{i}{2}(\phi_l - \psi_l - \phi_r - \psi_r)}(e^{i\psi_l} \cos\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_1^A - ie^{i(\psi_l + \phi_r)} \cos\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_2^A$$
$$+ i \sin\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_3^A + e^{i\phi_r} \sin\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_4^A) \tag{5.6}$$
$$u_2 = e^{\frac{i}{2}(\phi_l - \psi_l - \phi_r + \psi_r)}(-ie^{i\psi_l} \cos\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_1^A + e^{i(\psi_l + \phi_r)} \cos\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_2^A$$
$$+ \sin\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_3^A + ie^{i\phi_r} \sin\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_4^A) \tag{5.7}$$
$$u_3 = e^{\frac{i}{2}(-\phi_l - \psi_l - \phi_r - \psi_r)}(ie^{i\psi_l} \sin\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_1^A + e^{i(\psi_l + \phi_r)} \sin\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_2^A$$
$$+ \cos\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_3^A - ie^{i\phi_r} \cos\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_4^A) \tag{5.8}$$
$$u_4 = e^{\frac{i}{2}(-\phi_l - \psi_l - \phi_r + \psi_r)}(e^{i\psi_l} \sin\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_1^A + ie^{i(\psi_l + \phi_r)} \sin\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_2^A$$
$$- i \cos\frac{\theta_l}{2} \sin\frac{\theta_r}{2} u_3^A + e^{i\phi_r} \cos\frac{\theta_l}{2} \cos\frac{\theta_r}{2} u_4^A) \tag{5.9}$$

It is now straightforward to work out an expression for the reduction of the Laplace-Beltrami operator to the Cartan subgroup by inserting the previous list of formulas for the components of two functions f and g into the general prescription (4.7) and performing the integral over the various angle variables. After our gauge transformation with the function m given in eq. (5.3), the Laplacian takes a block diagonal form, $H = diag(H_1, H_2)$ with two 2×2 matrices of Calogero-Sutherland like matrix Hamiltonians H_1 and H_2. An additional constant matrix valued gauge transformation of the form

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} H_1 \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{1}{4}\mathcal{M}_1 \tag{5.10}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} H_2 \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = -\frac{1}{4}\mathcal{M}_2 \tag{5.11}$$

maps these Hamiltonians to the expressions for $\mathcal{M}_1$, $\mathcal{M}_2$ we quoted at the end of the Introduction. In the Appendix A we demonstrate that this Hamiltonian is equivalent to the Casimir equations derived in [21].

6 Discussion, Outlook and Conclusions

In this work we build a model of spinning conformal blocks through sections of a vector bundle over a double-coset of the conformal group to derive Casimir equations from the Laplace-Beltrami differential operator on $\mathrm{SO}(1, d+1)$. We argued that the resulting eigenvalue equation takes the form of a Calogero-Sutherland Schrödinger problem for two interacting particles with spin that move in a 1-dimensional external potential. This potential depends on the choice of tensor structures and conformal weights of the external fields and on the dimension d of the space. It was worked out in a few examples, including the case of 3-dimensional fermionic seed blocks for which the Casimir equation had originally been derived in [21]. The algorithm we described extends to higher dimensions $d \geqslant 4$ with only one significant change, namely that the KAK decomposition is no longer unique. In order to fix the issue, one can restrict one of the factors K to the coset space K/B where $B = SO(d-2)$. At the same time, the fibers of the relevant vector bundles must be projected to the subspace of $\mathrm{SO}(d-2)$ invariants. We will describe this in more detail in a forthcoming paper [30] on Casimir equations for 4-dimensional seed blocks, see [18].

There are a number of other extensions that seem worth pursuing. To begin with, it would be interesting to work out the Calogero-Sutherland Hamiltonians for blocks of scalar and tensor fields in supersymmetric theories. Most of the existing work on Casimir equations in such theories focuses on correlation functions of BPS operators. If all four external operators are BPS, the Casimir equations resemble those for scalar blocks in bosonic theories [32, 33, 36] and hence they can be cast into a Calogero-Sutherland like form. Things become more interesting when we admit non-BPS operators. There are only a few cases in which the Casimir equations for such setups have been worked out, see e.g. [32] and [37] for 2-dimensional theories with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry, respectively.

Other interesting extensions concern correlation functions of local operators in the presence of boundaries and defects. All these scenarios can be cast into the framework we outlined above. The main difference is that the left and right subgroups $K_l = K$ and $K_r = K$ that we divided by above must be chosen according to the geometry of the configuration. In particular, they are usually not equal to each other any longer. If we want to describe conformal blocks for two bulk fields in the presence of a boundary, for example, we have to consider the coset $K_l \setminus G/K$ where $K_l = \mathrm{SO}(1,d)$ is the $d-1$ dimensional conformal group and $K_r = K$ is the same as before. We plan to work out a number of such examples and to compare with known Casimir equations whenever they are available, see e.g. [38–40].

For technical reasons we worked with the principle series representations of conformal weight $\Delta = d/2 + ic$ and performed an analytic continuation to the real weights of local fields only in the very last step. On the other hand, there could exist direct applications to a broader class of operators. In [41] one of authors introduced a new class of nonlocal light-ray operators that realize the principle series representation of $\mathrm{sl}(2|4)$ and then calculated their correlation function in BFKL regime [42, 43]. It would be very tempting to extend the bootstrap programme to such type of operators.

What we have explored here so far is a very universal new approach to conformal blocks that may be applied to a wide variety of setups, including boundaries, defects and supersymmetric theories. As we have also seen in the example of the 3D seed blocks, it casts the Casimir equations into a new and often simpler looking form. But the main interest of our approach is that it embeds the theory of conformal blocks into the rich world of (super-)integrable quantum systems. In the case that is relevant for conformal blocks of scalar fields, super-integrability is firmly established, see [19] and references therein, though it still remains to be exploited [44, 45]. The analysis presented above suggests that the connection between blocks and integrability goes much deeper and, in particular, also includes blocks with external tensor fields. Let us explain this in a bit more detail. Harmonic analysis on a Lie group is usually not an integrable problem. In fact, the number of independent commuting (differential) operators is given by the rank r of the group and hence is much smaller than the number $dimG$ of coordinates. In performing the reduction to coset geometries, however, we reduce the number of coordinates while keeping the same number of commuting operators unless they start to become dependent. The conformal group possesses $r = [d + 2/2]$ independent Casimir elements. So, when we reduce to our double coset, these outnumber the coordinates and hence the quantum mechanical system becomes integrable at least before we add spin degrees of freedom. The first case in which there are infinitely many spinning conformal seed blocks appears in $d = 4$ dimensions. At this dimension, the number r of Casimir invariants jumps from $r = 2$ for $d < 4$ to $r = 3$, i.e. there is one more Casimir invariant than there are cross ratios or coordinates on the double coset. It seems likely that the additional Casimir invariant makes the corresponding spinning quantum mechanical systems integrable. For the spinning A_n Calogero-Sutherland Hamiltonians which are associated to bundles over adjoint coset spaces G/G, super-integrability (or degenerate integrability) has recently been proven in [46]. It remains to extend such an analysis to BC_n root systems and thereby to spinning conformal blocks.

Super-integrability is a powerful feature. As is well known from the Runge-Lenz vector of the hydrogen atom, the spectrum generating symmetries of super-integrable systems can make them algebraically solvable. In the case of conformal blocks, all the known recurrence relations [47] are direct consequences of super-integrability [44, 45]. We believe that the remarkable formulas for 4-dimensional seed blocks that were found in [18] can be understood through the super-integrability of the associated Calogero-Sutherland systems. If this was true, it would pave the way for extensions, e.g. to other dimensions. We plan to return to these questions in future research.

Acknowledgments

We thank Lazlo Feher, Denis Karateev, Zohar Komargodski, Madalena Lemos, Pedro Liendo, Vincent Pasquier, Joao Penedones and Didina Serban for comments and discussions. E.Sobko thanks the Department of Mathematics, King's college London, and in particular the DESY Theory Group for their hospitality during the earlier stages of this research. The work of E. Sobko was supported by the grant "Exact Results in Gauge and String Theories" from the Knut and Alice Wallenberg foundation and the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA Grant Agreement No 317089 (GATIS). The work of M. Isachenkov was supported by an Israel Science Foundation center for excellence grant, by the I-CORE program of the Planning and Budgeting Committee and the Israel Science Foundation (grant number 1937/12), by the Minerva foundation with funding from the Federal German Ministry for Education and Research, by a Henri Gutwirth award from the Henri Gutwirth Fund for the Promotion of Research, by the ISF within the ISF-UGC joint research program framework (grant no. 1200/14) and by the ERC STG grant 335182.

A Comparing with 3D fermionic blocks from [21]

In this section we rewrite Casimir equations for the fermionic seed blocks that were derived in [21] as a matrix valued Calogero-Sutherland like eigenvalue equation that may be compared with the expressions we obtained by our reduction from the Laplacian on the conformal group. We start by reproducing the equations (A.10) from [21]

$$\left[\begin{pmatrix} \mathcal{L}_D^+ & \mathcal{L}_A^+ \\ \mathcal{L}_A^+ & \mathcal{L}_D^+ \end{pmatrix} + \begin{pmatrix} 0 & -\frac{8r(a+b)}{1+r^2-2r\eta} \\ 0 & -\frac{8r(\eta-2r+r^2\eta)(a+b)}{(1+r^2-2r\eta)2} \end{pmatrix}\right]\begin{pmatrix} \tilde{g}^1_{\Delta,l} \\ \tilde{g}^2_{\Delta,l} \end{pmatrix} = C_{\Delta,l}\begin{pmatrix} \tilde{g}^1_{\Delta,l} \\ \tilde{g}^2_{\Delta,l} \end{pmatrix} \tag{A.1}$$

$$\left[\begin{pmatrix} \mathcal{L}_D^- & \mathcal{L}_A^- \\ \mathcal{L}_A^- & \mathcal{L}_D^- \end{pmatrix} + \begin{pmatrix} -\frac{8r(\eta+2r+r^2\eta)b}{(1+r^2+2r\eta)^2} & \frac{8ra}{1+r^2+2r\eta} \\ \frac{8rb}{1+r^2+2r\eta} & -\frac{8r(\eta+2r+r^2\eta)a}{(1+r^2+2r\eta)^2} \end{pmatrix}\right]\begin{pmatrix} \tilde{g}^3_{\Delta,l} \\ \tilde{g}^4_{\Delta,l} \end{pmatrix} = C_{\Delta,l}\begin{pmatrix} \tilde{g}^3_{\Delta,l} \\ \tilde{g}^4_{\Delta,l} \end{pmatrix} \tag{A.2}$$

where

$$\begin{aligned} \mathcal{L}_D^\pm &= r^2\partial_r^2 + (\eta^2-1)\partial_\eta^2 + \\ &+ \left(\frac{-8r^2\eta(1-r^2)(a+b)}{(1+r^2-2r\eta)(1+r^2+2r\eta)} - \frac{r(1+3r^2)}{1-r^2} - \frac{r(1-r^2)(1+r^2\mp 2r\eta)}{(1+r^2+2r\eta)(1+r^2-2r\eta)}\right)\partial_r \\ &+ \left(\frac{-8(\eta^2-1)(r^3+r)(a+b)}{(1+r^2+2\eta r)(1+r^2-2\eta r)} + \frac{(3\eta(1+r^2)\pm 2r(4\eta^2-1))(1+r^2\mp 2r\eta)}{(1+r^2+2\eta r)(1+r^2-2\eta r)}\right)\partial_\eta \\ &+ \left(\frac{3}{4} - \frac{16abr(\eta+2r+r^2\eta)}{(1+r^2+2r\eta)^2}\right) \end{aligned} \tag{A.3}$$

$$\mathcal{L}_A^\pm = \frac{2r^2}{1-r^2}\partial_r \pm \partial_\eta \tag{A.4}$$

and $\Delta_{12} = -2a$, $\Delta_{43} = -2b$. To begin rewriting these expressions, we perform the following change of variables

$$r = e^{\frac{x+y}{2}}, \tag{A.5}$$

$$\eta = -\cosh\frac{x-y}{2} \tag{A.6}$$

After this change of variables the system of equations (A.1-A.2) continues to possess the matrix form

$$\begin{aligned} \tilde{\mathcal{M}}_1\begin{pmatrix} \tilde{g}^1_{\Delta,l} \\ \tilde{g}^2_{\Delta,l} \end{pmatrix} &= C_{\Delta,l}\begin{pmatrix} \tilde{g}^1_{\Delta,l} \\ \tilde{g}^2_{\Delta,l} \end{pmatrix}, \\ \tilde{\mathcal{M}}_2\begin{pmatrix} \tilde{g}^3_{\Delta,l} \\ \tilde{g}^4_{\Delta,l} \end{pmatrix} &= C_{\Delta,l}\begin{pmatrix} \tilde{g}^3_{\Delta,l} \\ /\tilde{g}^4_{\Delta,l} \end{pmatrix} \end{aligned} \tag{A.7}$$

Explicit formulas for the matrices $\tilde{\mathcal{M}}_i$ of differential operators in x and y are easily worked out. Once they are derived, we perform the following transformations

$$\mathcal{M}_1 = -\frac{1}{2}\begin{pmatrix} \chi_1(x,y) & \chi_2(x,y) \\ -\chi_1(x,y) & \chi_2(x,y) \end{pmatrix}^{-1} \tilde{\mathcal{M}}_1 \begin{pmatrix} \chi_1(x,y) & \chi_2(x,y) \\ -\chi_1(x,y) & \chi_2(x,y) \end{pmatrix} \tag{A.8}$$

$$\mathcal{M}_2 = -\frac{1}{2}\begin{pmatrix} \chi_3(x,y) & \chi_4(x,y) \\ -\chi_3(x,y) & \chi_4(x,y) \end{pmatrix}^{-1} \tilde{\mathcal{M}}_2 \begin{pmatrix} \chi_3(x,y) & \chi_4(x,y) \\ -\chi_3(x,y) & \chi_4(x,y) \end{pmatrix} \tag{A.9}$$

where

$$\begin{aligned}
\chi_1(x,y) &= \frac{\cosh\frac{x}{2}^{-a-b}\sinh\frac{x}{2}^{-\frac{1}{2}+a+b}\cosh\frac{y}{2}^{-a-b}\sinh\frac{y}{2}^{-\frac{1}{2}+a+b}}{(\cosh\frac{y}{2}-\cosh\frac{x}{2})^{\frac{3}{2}}(\cosh\frac{y}{2}+\cosh\frac{x}{2})^{\frac{1}{2}}}, \\
\chi_2(x,y) &= \frac{\cosh\frac{x}{2}^{-a-b}\sinh\frac{x}{2}^{-\frac{1}{2}+a+b}\cosh\frac{y}{2}^{-a-b}\sinh\frac{y}{2}^{-\frac{1}{2}+a+b}}{(\cosh\frac{y}{2}-\cosh\frac{x}{2})^{\frac{1}{2}}(\cosh\frac{y}{2}+\cosh\frac{x}{2})^{\frac{3}{2}}}, \\
\chi_3(x,y) &= \frac{\cosh\frac{x}{2}^{-\frac{1}{2}-a-b}\sinh\frac{x}{2}^{a+b}\cosh\frac{y}{2}^{-\frac{1}{2}-a-b}\sinh\frac{y}{2}^{a+b}}{(\sinh\frac{x}{2}-\sinh\frac{y}{2})^{\frac{1}{2}}(\sinh\frac{x}{2}+\sinh\frac{y}{2})^{\frac{3}{2}}}, \\
\chi_4(x,y) &= \frac{\cosh\frac{x}{2}^{-\frac{1}{2}-a-b}\sinh\frac{x}{2}^{a+b}\cosh\frac{y}{2}^{-\frac{1}{2}-a-b}\sinh\frac{y}{2}^{a+b}}{(\sinh\frac{x}{2}-\sinh\frac{y}{2})^{\frac{3}{2}}(\sinh\frac{x}{2}+\sinh\frac{y}{2})^{\frac{1}{2}}}.
\end{aligned} \tag{A.10}$$

After this transformation, the system A.1-A.2 now reads

$$\begin{aligned}
\mathcal{M}_1 \begin{pmatrix} g^1_{\Delta,l} \\ g^2_{\Delta,l} \end{pmatrix} &= -\frac{C_{\Delta,l}}{2}\begin{pmatrix} g^1_{\Delta,l} \\ g^2_{\Delta,l} \end{pmatrix}, \\
\mathcal{M}_2 \begin{pmatrix} g^3_{\Delta,l} \\ g^4_{\Delta,l} \end{pmatrix} &= -\frac{C_{\Delta,l}}{2}\begin{pmatrix} g^3_{\Delta,l} \\ g^4_{\Delta,l} \end{pmatrix}
\end{aligned} \tag{A.11}$$

where the operators $\mathcal{M}_1,\ \mathcal{M}_2$ are the same as in the Introduction and the new functions $\{g^i_{\Delta,l}\}$ are related to conformal blocks $\{\tilde{g}^i_{\Delta,l}\}$ as

$$\begin{pmatrix} g^1_{\Delta,l} \\ g^2_{\Delta,l} \end{pmatrix} = \begin{pmatrix} \chi_1(x,y) & \chi_2(x,y) \\ -\chi_1(x,y) & \chi_2(x,y) \end{pmatrix}^{-1}\begin{pmatrix} \tilde{g}^1_{\Delta,l} \\ \tilde{g}^2_{\Delta,l} \end{pmatrix} \tag{A.12}$$

$$\begin{pmatrix} g^3_{\Delta,l} \\ g^4_{\Delta,l} \end{pmatrix} = \begin{pmatrix} \chi_3(x,y) & \chi_4(x,y) \\ -\chi_3(x,y) & \chi_4(x,y) \end{pmatrix}^{-1}\begin{pmatrix} \tilde{g}^3_{\Delta,l} \\ \tilde{g}^4_{\Delta,l} \end{pmatrix} \tag{A.13}$$

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