

Five-loop quark mass and field anomalous dimensions for a general gauge group

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ABSTRACT: We present analytical five-loop results for the quark mass and quark field anomalous dimensions, for a general gauge group and in the $\overline{\text{MS}}$ scheme. We confirm the values known for the gauge group $\text{SU}(3)$ from an independent calculation, and find full agreement with results available from large- N_f studies.

KEYWORDS: Perturbative QCD, Renormalization Group

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Contents

1	Introduction	1
2	Setup	2
3	Notation	4
4	Quark mass renormalization	4
5	Quark field renormalization	6
6	Conclusions	8
A	Summary of large-N_f results	9

1 Introduction

High-precision determinations of Standard Model (SM) parameters are crucially important for precise predictions for the observables measured in collider physics experiments, such as currently performed at the LHC, or possibly at a future linear collider. A thorough comparison of these predictions with experimental results allows to scrutinize the details of the hugely successful SM, and might shed light on possible physics beyond the SM.

In order to pin down the theory’s fundamental parameters, such as coupling constants and masses, with sufficient precision, the knowledge of higher order perturbative corrections is required. This encompasses the evaluation of multi-loop Feynman diagrams and Feynman integrals, for which significant progress has been made in recent years, mainly with respect to the formulation of advanced algorithms that allow to treat the complexity level met in those higher-loop integrals.

We focus here on the strong interactions, which are embedded into the SM via Quantum Chromodynamics (QCD). The relevant parameters are then the (strong) gauge coupling and the quark masses, both of which run with the energy scale according to the renormalization group (RG) equations. In order to evolve e.g. the low-energy value of the coupling constant (measured with high precision from tau lepton decay) to high energies, the anomalous dimension of the gauge coupling (the so-called Beta function) is needed, as coefficient in the corresponding RG equation. Likewise, a high-order evaluation of the quark mass anomalous dimension gives access to precise values for e.g. charm and bottom quark masses, which are measured at low energies (typically a few GeV) but whose uncertainty at the high-energy scale of the Higgs mass $m_H = 125$ GeV is important in Higgs decay rates into such quark pairs. In particular, to match the precision of the known five-loop inclusive decay width of $H \rightarrow q\bar{q}$ [1], one should evolve the parameters (which are $\alpha_s(\mu)$ and the running quark mass $m_q(\mu)$ with μ being the renormalization scale) from low energies to $\mu = m_H$ at the same perturbative order, for full consistency and to avoid large logarithms $\ln(\mu^2/m_H^2)$.

Given this clear phenomenological motivation, we will present new results for the quark mass anomalous dimension here, applying a number of the above-mentioned algorithmic advances. While



Figure 1. 5-loop master integrals with 12 lines that contribute to eqs. (2.1)-(2.3). Each line denotes a massive propagator $1/(k^2 + m^2)$, and each dot stands for an extra power of the corresponding propagator.

this renormalization constant has been studied previously up to five loops in perturbation theory [2–7], we generalize it from the gauge group $SU(3)$ to a semi-simple Lie group. At the same time, we provide a truly independent check on the available $SU(3)$ result, since we utilize largely independent methods, as described below. We will also give results for a related quantity needed to renormalize the quark sector at five loops, namely the quark field anomalous dimension in Feynman gauge, again generalizing known $SU(3)$ results to a semi-simple Lie group.

The paper is organized as follows. We start by explaining our computational setup in section 2. Using some notation defined in section 3, we then present and discuss results in sections 4 and 5, before concluding in section 6. For convenience, an appendix reproduces large- N_f results from the literature, which we use for consistency checks.

2 Setup

Let us start by explaining our computational setup, which closely follows the one employed and tested in [8]. We base our highly automated setup on the diagram generator `qgraf` [9, 10] and various in-house `FORM` [11–13] codes. After generating the required fermionic 2-point functions, we apply projectors and perform the group algebra with `color` [14]. To extract the ultraviolet (UV) divergences, we then use the freedom to change the low-momentum behavior and make all propagators massive, which regulates the infrared at the cost of introducing a new counterterm for the unphysical regulator mass. Expanding to sufficient depth in the external momentum [15–17] and keeping all potentially UV divergent structures results in nullifying the external momentum. The coefficients of this expansion can then be mapped onto a family of fully massive vacuum integrals, which are labelled by 15 indices (corresponding to maximally 12 propagators plus 3 scalar products) at five loops [18]. As a next and fairly time-consuming step, we reduce those integrals to a small set of master integrals, powered by our own codes `crusher` [19] and `TIDE` [18], which are based on integration-by-parts (IBP) identities [20] and use a Laporta-type algorithm [21] for a systematic integral reduction.

At five loops, we end up with a set of 110 master integrals. These have been evaluated in an ε -expansion around $d = 4 - 2\varepsilon$ dimensions in previous works [18, 22], using an approach based on IBP reductions and difference equations [21] that has been realized in `C++` and uses `Fermat` [23] to perform the polynomial algebra that arises in solving systems of linear equations with large rational coefficients. The resulting high-precision numerical results for the coefficients of the ε -expansions finally allow us to utilize the integer-relation finding algorithm `PSLQ` [24] to discover the analytic content of some of these numbers, and to find relations between others. As a consequence, we are able to provide all our results given below in analytic form.

As has already been mentioned elsewhere [8], our high-precision evaluation of all 5-loop master integrals has not yet produced results for the 12-line families, see figure 1. These do not contain divergences in four dimensions, and could therefore be avoided in evaluations of anomalous dimensions.

It turns out, however, that with our integral reduction criteria and lexicographic ordering prescription, we do get contributions from these integral classes since they are multiplied by prefactors with spurious poles as $d \rightarrow 4$. To fix the values of the 12-line master integrals that we need, we first note that in all our results, only three independent linear combinations appear:

$$\ell_0 = 689 I_{30527.1.1,5} - 3934 I_{30527.1.2,5} + 5464 I_{30527.1.3,5} + 1152 I_{30527.1.4,5} - 5228 I_{30527.6.3,5} , \quad (2.1)$$

$$\begin{aligned} \ell_1 = & 689 I_{30527.1.1,6} - 3934 I_{30527.1.2,6} + 5464 I_{30527.1.3,6} + 1152 I_{30527.1.4,6} - 5228 I_{30527.6.3,6} \\ & + 1968 I_{30527.1.1,5} + 3890 I_{30527.1.2,5} + 3844 I_{30527.1.3,5} - 6912 I_{30527.1.4,5} - 70 I_{30527.6.3,5} , \end{aligned} \quad (2.2)$$

$$\ell_2 = 11 I_{31740.1.1,5} - 72 I_{31740.1.3,5} . \quad (2.3)$$

Here, each integral $I_{\#,n}$ corresponds to the ε^n -coefficient of the respective fully massive master integral of figure 1, divided by the fifth power of the 1-loop tadpole J for normalization reasons¹. While it is conceivable that there exists a suitable basis transformation that eliminates these linear combinations altogether from the final results, we have not yet performed a systematic search of such transformations in our integral reduction tables, but opted for other criteria to fix the numerical values of the three linear combinations, with high precision, as we will explain now.

As a crude order-of-magnitude estimate, we have evaluated the set of 12-line integrals via Feynman parametric representations (see e.g. [25]) and subsequent (primary) sector decomposition, using the strategy explained in [26, 27] and as implemented in FIESTA [28] as well as own code (see [29]). Due to the large prefactors and cancellations in eqs. (2.1)-(2.3), a 6-digit evaluation results in the estimates

$$\ell_0 \approx -7.47(1) , \quad \ell_1 \approx -50.6(1) , \quad \ell_2 \approx -0.673(1) , \quad (2.4)$$

with 3-digit accuracy.

Turning now to high-precision evaluations, we first recall that the higher-order ε -poles of renormalization constants are completely determined by lower-order coefficients. Checking these, we obtain one consistency condition that allows to fix ℓ_0 . Second, we observe the occurrence of some rank-12 group invariants in another renormalization constant that we have evaluated using the same setup, namely for the ghost-gluon vertex [30]. Using eq. (2.4), to determine their coefficients, we find that they vanish at least to this low accuracy. Taking this zero for granted, we turn the argument around and require e.g. the structure $d_{FAA}^{444} N_f$ (where the group invariant is in the notation of [14]) to be absent from the final result; this gives us another condition, fixing ℓ_2 . Third, for fixing the remaining linear combination, we choose to compare our results in the SU(3) limit to the previously known 5-loop results. To be concrete, out of the many possible coefficients we choose the n_f term of γ_m as given in [7], giving us one constraint which fixes ℓ_1 . Along these lines, we obtain numerical values for all three linear combinations with 260 digits, the first 50 of which read

$$\ell_0 = -7.4750787021276651819913288152084850401974826928834 \dots , \quad (2.5)$$

$$\ell_1 = -50.56371484107199642853937259222326105092965639946 \dots , \quad (2.6)$$

$$\ell_2 = -0.67332086607447050046759024439428336720209195028580 \dots , \quad (2.7)$$

and which can be seen to be consistent with our low-precision estimates of eq. (2.4) that had been obtained by direct integration.

¹ $J = \int d^d k / (k^2 + m^2) \sim \Gamma(1 - d/2)$ has a simple pole as $d \rightarrow 4$; hence, finite 5-loop terms correspond to ε^5 .

3 Notation

Let us fix our notation here: we work with a semi-simple Lie algebra with hermitian generators T^a , whose real and antisymmetric structure constants f^{abc} are fixed by the commutation relation $[T^a, T^b] = if^{abc}T^c$. As usual, the quadratic Casimir operators of the fundamental (adjoint) representation (of dimensions N_F and N_A , respectively) are defined as $T^a T^a = C_F \mathbb{1}$ ($f^{acd}f^{bcd} = C_A \delta^{ab}$). Furthermore, traces are normalized as $\text{Tr}(T^a T^b) = T_F \delta^{ab}$, we denote the number of quark flavors with N_f , and find it convenient to define the following normalized combinations:

$$c_f = \frac{C_F}{C_A} \quad , \quad n_f = \frac{N_f T_F}{C_A} . \quad (3.1)$$

In our multi-loop diagrams, we will encounter traces of more than two group generators, giving rise to higher-order group invariants. It is useful to define traces over combinations of symmetric tensors [14], of which we need the following (writing $[F^a]_{bc} = -if^{abc}$ for the generators of the adjoint representation):

$$d_1 = \frac{[\text{sTr}(T^a T^b T^c T^d)]^2}{N_A T_F^2 C_A^2} , \quad d_2 = \frac{\text{sTr}(T^a T^b T^c T^d) \text{sTr}(F^a F^b F^c F^d)}{N_A T_F C_A^3} , \quad d_3 = \frac{[\text{sTr}(F^a F^b F^c F^d)]^2}{N_A C_A^4} . \quad (3.2)$$

Here, sTr stands for a fully symmetrized trace (such that $\text{sTr}(ABC) = \frac{1}{2}\text{Tr}(ABC + ACB)$ etc.). While dealing with the quark sector, it might seem more natural to normalize these traces with respect to the dimension of the fundamental representation; to this end, we note the relation $N_A T_F = N_F C_F$ which holds in general [14, 31]. For the gauge group $\text{SU}(N)$ (where $T_F = \frac{1}{2}$ and $C_A = N$), the normalized group invariants introduced above read [14]

$$n_f = \frac{N_f}{2N} , \quad c_f = \frac{N^2 - 1}{2N^2} , \quad d_1 = \frac{N^4 - 6N^2 + 18}{24N^4} , \quad d_2 = \frac{N^2 + 6}{24N^2} , \quad d_3 = \frac{N^2 + 36}{24N^2} . \quad (3.3)$$

The corresponding $\text{SU}(3)$ values, relevant for physical QCD, hence read

$$\text{SU}(3) : \quad n_f = \frac{N_f}{6} , \quad c_f = \frac{4}{9} , \quad d_1 = \frac{5}{216} , \quad d_2 = \frac{5}{72} , \quad d_3 = \frac{5}{24} . \quad (3.4)$$

4 Quark mass renormalization

The renormalization constant for the quark mass $m_{\text{bare}} = Z_m m_{\text{ren}}$, or equivalently its anomalous dimension $\gamma_m = -\partial_{\ln \mu^2} \ln Z_m$, has been known at two [2] and three loops [3, 4] for a long time. At four loops, γ_m is known for $\text{SU}(N)$ and QED [6] as well as for a general Lie group [5]. Presently, at five loops only the $\text{SU}(3)$ value is publicly available [7]. We present our corresponding result for a general Lie group below.

The structure of the quark mass anomalous dimension is

$$\partial_{\ln \mu^2} \ln m_q(\mu) \equiv \gamma_m(a) = -c_f a \left\{ 3 + \gamma_{m1} a + \gamma_{m2} a^2 + \gamma_{m3} a^3 + \gamma_{m4} a^4 + \dots \right\} , \quad (4.1)$$

$$a \equiv \frac{C_A g^2(\mu)}{16\pi^2} , \quad (4.2)$$

with $g(\mu)$ being the renormalized QCD gauge coupling constant that depends on the renormalization scale μ (we prefer to use the expansion parameter a which is nothing but a rescaled version of the renormalized strong coupling constant $\alpha_s = \frac{g^2(\mu)}{4\pi}$). We work in $d = 4 - 2\varepsilon$ dimensions and employ the

$\overline{\text{MS}}$ scheme. The coefficients γ_{mn} are polynomials in n_f and can be written in terms of our normalized group factors. Up to four loops, they read [5]

$$3^1 \gamma_{m1} = n_f \left[-10 \right] + \left[(9c_f + 97)/2 \right], \quad (4.3)$$

$$3^3 \gamma_{m2} = n_f^2 \left[-140 \right] + n_f \left[54(24\zeta_3 - 23)c_f - 4(139 + 324\zeta_3) \right] + \left[(6966c_f^2 - 3483c_f + 11413)/4 \right], \quad (4.4)$$

$$\begin{aligned} 3^4 \gamma_{m3} = & n_f^3 \left[-8(83 - 144\zeta_3) \right] + n_f^2 \left[48(19 - 270\zeta_3 + 162\zeta_4)c_f + 2(671 + 6480\zeta_3 - 3888\zeta_4) \right] \\ & + n_f \left[-216(35 - 207\zeta_3 + 180\zeta_5)c_f^2 - 3(8819 - 9936\zeta_3 + 7128\zeta_4 - 2160\zeta_5)c_f \right. \\ & \left. - (65459/2 + 72468\zeta_3 - 21384\zeta_4 - 32400\zeta_5) + 2592(2 - 15\zeta_3)d_1 \right] \\ & + \frac{9}{8} \left[-9(1261 + 2688\zeta_3)c_f^3 + 6(15349 + 3792\zeta_3)c_f^2 - 2(34045 + 5472\zeta_3 - 15840\zeta_5)c_f \right. \\ & \left. + (70055 + 11344\zeta_3 - 31680\zeta_5) - 1152(2 - 15\zeta_3)d_2 \right], \quad (4.5) \end{aligned}$$

where we have denoted values of the Riemann Zeta function as $\zeta_s = \zeta(s) = \sum_{n>0} n^{-s}$.

At five loops, from app. A, we have LO and NLO large- N_f terms to all orders, coinciding with the leading terms above, and predicting the first two terms of the 5-loop contributions as

$$6^5 \gamma_{m4} = \gamma_{m44} \left[4n_f \right]^4 + \gamma_{m43} \left[4n_f \right]^3 + \gamma_{m42} \left[4n_f \right]^2 + \gamma_{m41} \left[4n_f \right] + \gamma_{m40}, \quad (4.6)$$

$$\gamma_{m44} = -6(65 + 80\zeta_3 - 144\zeta_4), \quad (4.7)$$

$$\gamma_{m43} = 3(4483 + 4752\zeta_3 - 12960\zeta_4 + 6912\zeta_5)c_f + (18667/2 + 32208\zeta_3 + 29376\zeta_4 - 55296\zeta_5). \quad (4.8)$$

We have evaluated the remaining coefficients, mapping all diagrams onto fully massive vacuum tadpoles; IBP-reducing them to master integrals; using high-precision numerical evaluations thereof plus some additional consistency conditions to fix linear combinations of 12-line master integrals as explained in section 2; and finally employing PSLQ at 200 digits for discovery, and at 250 digits for confirmation, we confirm the two large- N_f expressions in eqs. (4.7) and (4.8), and obtain the three missing coefficients of eq. (4.6) in analytic form, containing only Zeta values²:

$$\begin{aligned} \gamma_{m42} = & \left\{ c_f^2, c_f, d_1, 1 \right\} \cdot \left\{ 9(45253 - 230496\zeta_3 + 48384\zeta_3^2 + 70416\zeta_4 + 144000\zeta_5 - 86400\zeta_6), \right. \\ & 375373 + 323784\zeta_3 - 1130112\zeta_3^2 + 905904\zeta_4 - 672192\zeta_5 + 129600\zeta_6, \\ & -864(431 - 1371\zeta_3 + 432\zeta_4 + 420\zeta_5), \\ & \left. 4(13709 + 394749\zeta_3 + 173664\zeta_3^2 - 379242\zeta_4 - 119232\zeta_5 + 162000\zeta_6) \right\}, \quad (4.9) \end{aligned}$$

$$\begin{aligned} \gamma_{m41} = & \left\{ c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1 \right\} \cdot \left\{ -54(48797 - 247968\zeta_3 + 24192\zeta_4 + 444000\zeta_5 - 241920\zeta_7), \right. \\ & -18(406861 + 216156\zeta_3 - 190080\zeta_3^2 + 254880\zeta_4 - 606960\zeta_5 - 475200\zeta_6 + 362880\zeta_7), \\ & -62208(11 + 154\zeta_3 - 370\zeta_5), \\ & 753557 + 15593904\zeta_3 - 3535488\zeta_3^2 - 6271344\zeta_4 - 17596224\zeta_5 + 1425600\zeta_6 + 1088640\zeta_7, \\ & 1728(3173 - 6270\zeta_3 + 1584\zeta_3^2 + 2970\zeta_4 - 13380\zeta_5), \\ & 1728(380 - 5595\zeta_3 - 1584\zeta_3^2 - 162\zeta_4 + 1320\zeta_5), \\ & \left. -2(4994047 + 11517108\zeta_3 - 57024\zeta_3^2 - 5931900\zeta_4 - 15037272\zeta_5 + 4989600\zeta_6 + 3810240\zeta_7) \right\}, \quad (4.10) \end{aligned}$$

²To make the group structure more visible, we resort to a vector notation here and below, where a dot between two curly brackets denotes a scalar product as e.g. in $\{c_f, 1\} \cdot \{a, b\} = c_f a + b$.

$$\begin{aligned}
\gamma_{m40} = & \left\{ c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1 \right\} \cdot \left\{ 972(50995 + 6784\zeta_3 + 16640\zeta_5), \right. \\
& -54(2565029 + 1880640\zeta_3 - 266112\zeta_4 - 1420800\zeta_5), \\
& 108(2625197 + 1740528\zeta_3 - 125136\zeta_4 - 2379360\zeta_5 - 665280\zeta_7), \\
& 373248(141 + 80\zeta_3 - 530\zeta_5), \\
& -8(25256617 + 16408008\zeta_3 + 627264\zeta_3^2 - 812592\zeta_4 - 40411440\zeta_5 + 3920400\zeta_6 - 5987520\zeta_7), \\
& -6912(9598 + 453\zeta_3 + 4356\zeta_3^2 + 1485\zeta_4 - 26100\zeta_5 - 1386\zeta_7), \\
& 5184(537 + 2494\zeta_3 + 5808\zeta_3^2 + 396\zeta_4 - 7820\zeta_5 - 1848\zeta_7), \\
& \left. 4(22663417 + 10054464\zeta_3 + 1254528\zeta_3^2 - 1695276\zeta_4 - 41734440\zeta_5 + 7840800\zeta_6 + 5987520\zeta_7) \right\}. \tag{4.11}
\end{aligned}$$

Let us now discuss some checks on this new result. The authors of [7] have published the full 5-loop result for the case of SU(3). To compare, we recall the definition of our expansion parameter in eq. (4.2) and put all group invariants to their SU(3) values as given in eq. (3.4); the $\gamma_{m1..4}$ as listed above then coincide with the expressions given in [7]. Furthermore, the same group has very recently generalized their work to a general Lie group as well [32]. We have cross-checked their preliminary result with our γ_{m4} as given above, and found full agreement. Since both five-loop results have been obtained with completely different methods (with the exception of also relying on **qgraf** for diagram generation, in [32] the 5-loop renormalization constants are mapped onto massless 4-loop two-point functions [33–35], and integral reduction is done via $1/d$ expansions [36, 37]), this agreement constitutes an extremely strong check.

5 Quark field renormalization

For completeness, let us also present our new five-loop result for the quark field anomalous dimension $\gamma_2 = -\partial_{\ln \mu^2} \ln Z_2$, where the renormalization constant Z_2 relates bare and renormalized quark fields as $\psi_{\text{bare}} = \sqrt{Z_2} \psi_{\text{ren}}$. As opposed to the quark mass, this quantity is not physical and hence gauge dependent. Lower-loop results can be found for SU(N) and covariant gauge in [38], and for a general Lie group with ξ^0, ξ^1 terms (which corresponds to a NLO expansion around Feynman gauge) in [39]. At five loops, γ_2 is known for SU(3) in Feynman gauge [7], and we will below once more present the generalization to a general Lie group.

Up to four loops, we obtain (ξ being the covariant gauge parameter, such that the values $\xi = 0/1$ correspond to Feynman/Landau gauge)

$$\gamma_2 = -c_f a \left\{ (1 - \xi) + \gamma_{21} a + \gamma_{22} a^2 + \gamma_{23} a^3 + \gamma_{24} a^4 + \dots \right\}, \tag{5.1}$$

$$2^2 \gamma_{21} = n_f \left[-8 \right] + \left[-6c_f + 34 - 10\xi + \xi^2 \right], \tag{5.2}$$

$$\begin{aligned}
2^5 3^2 \gamma_{22} = & n_f^2 \left[640 \right] + n_f \left[8(108c_f - 1301 + 153\xi) \right] \\
& + \left[432c_f^2 - 72(143 - 48\zeta_3)c_f + 2(10559 - 1080\zeta_3) - 9\xi(371 + 48\zeta_3) + 27\xi^2(23 + 4\zeta_3) - 90\xi^3 \right], \tag{5.3}
\end{aligned}$$

$$\begin{aligned}
2^4 3^5 \gamma_{23} = & n_f^3 \left[13440 \right] + n_f^2 \left[6912(19 - 18\zeta_3)c_f + 16(6835 + 9072\zeta_3) + 64\xi(269 - 324\zeta_3) \right] \\
& + n_f \left[5184(19 - 48\zeta_3)c_f^2 + (-108(2407 - 1584\zeta_3 - 1296\zeta_4 - 5760\zeta_5)) \right. \\
& + 324\xi(767 - 528\zeta_3 - 144\zeta_4)c_f + 497664d_1 - (1365691 + 154224\zeta_3 + 97200\zeta_4 + 311040\zeta_5) \\
& \left. + \xi(48865 + 152928\zeta_3 + 29160\zeta_4) - 54\xi^2(109 + 84\zeta_3 - 18\zeta_4) \right]
\end{aligned}$$

$$\begin{aligned}
& + \left[-486(1027 + 3200\zeta_3 - 5120\zeta_5)c_f^3 + 324(5131 + 10176\zeta_3 - 17280\zeta_5)c_f^2 \right. \\
& + (-108(23777 + 7704\zeta_3 + 2376\zeta_4 - 28440\zeta_5) - 1944\xi(6 - 7\zeta_3 + 10\zeta_5))c_f \\
& + 486(16(-33 + 95\zeta_3 - 85\zeta_5) - 8\xi(1 + 48\zeta_3 - 70\zeta_5) - 8\xi^2(7\zeta_3 + 5\zeta_5) + 20\xi^3(2\zeta_3 - \zeta_5) \\
& - \xi^4(7\zeta_3 - 5\zeta_5))d_2 + (10059589/4 - 241218\zeta_3 + 168156\zeta_4 - 604260\zeta_5) \\
& - \xi(2127929/8 + 164106\zeta_3 - 21141\zeta_4 - 107730\zeta_5) + 27\xi^2(13883 + 9108\zeta_3 - 1548\zeta_4 \\
& \left. - 1920\zeta_5)/8 - 81\xi^3(263 + 65\zeta_3 - 9\zeta_4 + 20\zeta_5)/2 + 81\xi^4(57 + \zeta_3 + 10\zeta_5)/4 \right]. \quad (5.4)
\end{aligned}$$

We have presented the full gauge parameter dependence above. Note that this fills a gap in the literature and constitutes new information at four loops: in [39] only the terms linear in ξ have been evaluated for a general Lie group, while with full gauge dependence only the SU(N) result is available [38]. However, due to the degeneracies $2d_1 = 6c_f^2 - 5c_f + 13/12$ and $2d_2 = 7/12 - c_f$ in the SU(N) limit, one cannot uniquely extract the Lie group structure from the latter reference. Needless to say that our result for γ_{23} given in eq. (5.4) reproduces the ξ^0 and ξ^1 terms given in [39], and in the SU(N) limit reduces to the respective expressions of [38], for all powers of ξ .

Expanding the all-order large- N_f Landau gauge result of eq. (A.3) in the coupling a_f allows to confirm eqs. (5.2)-(5.4) to NLO in n_f , and to predict the first two terms of γ_{24} in that gauge as

$$24^3 \gamma_{24}|_{\xi=1} = \frac{83 - 144\zeta_3}{72} [16n_f]^4 + \gamma_{243}^{\xi=1} [16n_f]^3 + \dots, \quad (5.5)$$

$$\gamma_{243}^{\xi=1} = \left\{ c_f, 1 \right\} \cdot \left\{ -659/18 + 312\zeta_3 - 216\zeta_4, -1783/36 - 248\zeta_3 + 216\zeta_4 \right\}. \quad (5.6)$$

At five loops, along the same steps as explained in section 4, we have obtained the new Feynman gauge result³

$$24^3 \gamma_{24} = \frac{83 - 144\zeta_3}{72} [16n_f]^4 + \gamma_{243} [16n_f]^3 + \gamma_{242} [16n_f]^2 + \gamma_{241} [16n_f] + \gamma_{240} + \mathcal{O}(\xi), \quad (5.7)$$

where the coefficients again contain only Zeta values up to weight 7,

$$\gamma_{243} = \left\{ c_f, 1 \right\} \cdot \left\{ -659/18 + 312\zeta_3 - 216\zeta_4, -3443/48 - 255\zeta_3 + 252\zeta_4 \right\}, \quad (5.8)$$

$$\begin{aligned}
\gamma_{242} = & \left\{ c_f^2, c_f, d_1, 1 \right\} \cdot \left\{ -2(2497 - 1200\zeta_3 + 3456\zeta_4 - 8640\zeta_5), \right. \\
& 477433/12 - 45636\zeta_3 + 4608\zeta_3^2 + 11448\zeta_4 - 65088\zeta_5 + 28800\zeta_6, -384(115 - 33\zeta_3 - 90\zeta_5), \\
& \left. 3015955/72 + 69509\zeta_3 - 2304\zeta_3^2 - 12861\zeta_4 + 16662\zeta_5 - 14400\zeta_6 - 11907\zeta_7 \right\}, \quad (5.9)
\end{aligned}$$

$$\begin{aligned}
\gamma_{241} = & \left\{ c_f^3, c_f^2, c_f d_1, c_f, d_1, d_2, 1 \right\} \cdot \left\{ 24(29209 + 89984\zeta_3 + 12288\zeta_3^2 - 28800\zeta_4 - 187520\zeta_5 + 76800\zeta_6), \right. \\
& -4(296177 + 517020\zeta_3 + 26784\zeta_3^2 - 469908\zeta_4 - 4104720\zeta_5 + 1069200\zeta_6 + 3011904\zeta_7), \\
& -2304(748 + 4536\zeta_3 - 1368\zeta_3^2 - 6780\zeta_5 + 3255\zeta_7), \\
& 8(115334 - 37764\zeta_3 - 123012\zeta_3^2 - 49923\zeta_4 - 1124556\zeta_5 + 133650\zeta_6 + 1519308\zeta_7), \\
& 192(16732 + 39912\zeta_3 - 10944\zeta_3^2 - 72960\zeta_5 + 36771\zeta_7), \\
& \left. 96(6158 - 13952\zeta_3 - 372\zeta_3^2 + 2880\zeta_4 - 39475\zeta_5 - 3900\zeta_6 + 45696\zeta_7), \right\} \quad (5.10)
\end{aligned}$$

³The restriction to $\xi = 0$ is for practical reasons only. To evaluate the ξ -dependent coefficients, one would need to enlarge the integral reduction tables as produced by `crusher` and `TIDE` to integrals with higher propagator powers (or dots), roughly one more dot per power of the gauge parameter. Since the present calculation is at the limit of what the computing resources available to us are able to handle, we defer this to future work.

$$\begin{aligned}
& -34919359/9 - 753797\zeta_3 + 548148\zeta_3^2 - 135063\zeta_4 + 1759474\zeta_5 + 265350\zeta_6 - 2647806\zeta_7 \Big\}, \\
\gamma_{240} = & \left\{ c_f^4, c_f^3, c_f^2, c_f d_2, c_f, d_2, d_3, 1 \right\} \cdot \left\{ 1728(4977 + 128000\zeta_3 + 19968\zeta_3^2 + 180800\zeta_5 - 381024\zeta_7), \right. \\
& -96(835739 + 8494144\zeta_3 + 1182336\zeta_3^2 - 316800\zeta_4 + 3983360\zeta_5 + 844800\zeta_6 - 17852688\zeta_7), \\
& 192(825361 + 5472068\zeta_3 + 651816\zeta_3^2 - 335808\zeta_4 - 1140420\zeta_5 + 950400\zeta_6 - 8056377\zeta_7), \\
& 4608(10 + 53226\zeta_3 - 15264\zeta_3^2 + 2145\zeta_5 - 45885\zeta_7), -16(84040774/9 \\
& + 33396648\zeta_3 + 2804616\zeta_3^2 - 838782\zeta_4 - 18160944\zeta_5 + 6252300\zeta_6 - 41015331\zeta_7), \\
& -384(43066 + 628802\zeta_3 - 160998\zeta_3^2 + 36540\zeta_4 - 201125\zeta_5 - 53475\zeta_6 - 403263\zeta_7), \\
& -72(20566 - 218812\zeta_3 - 79080\zeta_3^2 - 13212\zeta_4 + 760220\zeta_5 + 20100\zeta_6 - 660667\zeta_7), 804023630/9 \\
& \left. + 101490400\zeta_3 + 3143352\zeta_3^2 + 7356024\zeta_4 - 86186276\zeta_5 + 18372900\zeta_6 - 115799439\zeta_7 \right\}. \quad (5.11)
\end{aligned}$$

As a speculation, comparing the Landau gauge prediction eq. (5.5) with the Feynman gauge result eq. (5.8), the full result for γ_{243} could be simply adding $\delta\gamma_{243} = \xi(3197/144 + 7\zeta_3 - 36\zeta_4)$; more generally however, consistency only requires that $\delta\gamma_{243} = c_f u_1(\xi) + u_2(\xi)$ with $u_1(0) = 0 = u_1(1)$ as well as $u_2(0) = 0$ and $u_2(1) = 3197/144 + 7\zeta_3 - 36\zeta_4$, the simplest choice being the one speculated above.

As a check, replacing the group invariants with the values of eq. (3.4) in our Feynman gauge result for γ_{24} , we find perfect agreement with the known SU(3) result of [7] (see also eq. (46) of [40]).

6 Conclusions

We have provided new results for two fundamental renormalization coefficients, at five loops and for a semi-simple Lie group. In particular, our expression for the gauge-invariant quark mass anomalous dimension γ_m coincides in various limits (large N_f as well as SU(3)) with previously known results, and coincides exactly with recent results of another group [32]. We have also provided the Feynman gauge result for the quark field anomalous dimension γ_2 , which again could be checked against known expressions in the abovementioned limits. From these two quantities, one can reconstruct the two renormalization constants Z_m and Z_2 of the quark sector, an electronic version of which is available by downloading the source of this article from <http://arXiv.org/abs/1612.05512>.

As had already been observed in [5], looking at e.g. the 4-loop result for the quark mass anomalous dimension eq. (4.5), all Zeta terms (and also the higher group invariants d_n) vanish at $\{c_f = 1, n_f = \frac{1}{2}, d_1 = d_2\}$, which corresponds to $\mathcal{N} = 1$ supersymmetry. The same had happened for the 4-loop Beta function (generalizing the last condition to $d_1 = d_2 = d_3$). For these parameters values, from section 4 we have

$$\gamma_m = -a \left\{ 3 + 16a + \frac{310}{3}a^2 + \frac{2228}{3}a^3 + \left(\frac{671075}{108} - 194d_1 + \left(\frac{1483}{2} - 2028d_1 \right) \zeta_3 - 20(55 + 354d_1)\zeta_5 \right) a^4 \right\},$$

where we observe that the cancellation pattern does not hold through five loops – although the structure becomes much simpler, due to cancellation of all terms containing $\{\zeta_3^2, \zeta_4, \zeta_6, \zeta_7\}$.

To conclude the renormalization program at five loops, one needs to determine three more renormalization constants, which can be chosen to be those of the gluon and ghost fields, Z_3 and Z_3^c , respectively, and of the ghost-gluon vertex Z_1^{ccg} . From these, due to gauge invariance, one can then construct the renormalization constant for the gauge coupling (aka the Beta function, whose five-loop coefficient is so far known for SU(3) only [41]) as well as those for the remaining vertices. While the same methods that we have used here are sufficient to calculate those missing coefficients (which

indeed already led to the NLO terms at large N_f [8]), due to the complexity of the determination of Z_3 we leave the evaluation of the full coefficients for future work.

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A Summary of large- N_f results

Some coefficients of QCD anomalous dimensions are known to all loop orders, from a large N_f expansion. Taking n_f and c_f as above and defining

$$a_f \equiv \frac{N_f T_F g^2(\mu)}{12\pi^2} = \frac{4n_f a}{3}, \quad \eta(\varepsilon) \equiv \frac{(2\varepsilon - 3)\Gamma(4 - 2\varepsilon)}{16\Gamma^2(2 - \varepsilon)\Gamma(3 - \varepsilon)\Gamma(\varepsilon)}, \quad (\text{A.1})$$

the all-order leading- N_f [44] and next-to-leading- N_f [45, 46] expressions that we need here read

$$\gamma_m = \frac{4c_f}{n_f} \left\{ \eta(a_f) + \frac{\eta_3(a_f)}{8n_f} + \mathcal{O}\left(\frac{1}{n_f^2}\right) \right\}, \quad (\text{A.2})$$

$$\gamma_2|_{\xi=1} = -\frac{2a_f c_f}{n_f} \left\{ \eta(a_f) + \frac{1}{n_f} \frac{\eta_4(a_f)}{4a_f} + \mathcal{O}\left(\frac{1}{n_f^2}\right) \right\}, \quad (\text{A.3})$$

where the fact that the asymptotic expansions have been performed in Landau gauge only does not affect the physical and gauge invariant quark mass anomalous dimension γ_m . To define the coefficient functions η_3 and η_4 , it is convenient to define the linear combinations

$$\Psi(\varepsilon) = \psi_0(1 - 2\varepsilon) + \psi_0(1 + \varepsilon) - \psi_0(1 - \varepsilon) - \psi_0(1), \quad (\text{A.4})$$

$$\Phi(\varepsilon) = \psi_1(1 - 2\varepsilon) - \psi_1(1 + \varepsilon) - \psi_1(1 - \varepsilon) + \psi_1(1), \quad (\text{A.5})$$

$$\Theta(\varepsilon) = \psi_1(1 - \varepsilon) - \psi_1(1), \quad (\text{A.6})$$

where $\psi_n(x) = \partial_x^{n+1} \ln \Gamma(x)$ is the PolyGamma function. Then [45, 46]

$$\begin{aligned} \eta_3(\varepsilon) \equiv & \left(-\frac{11}{4} + \sum_{n>0} \frac{f_n \varepsilon^n}{n} \right) 8\varepsilon \partial_\varepsilon \eta(\varepsilon) - \frac{16\eta^2(\varepsilon)}{(3-2\varepsilon)(1-\varepsilon)} \left\{ 3(2-\varepsilon)^2(1-\varepsilon)^2 \Theta(\varepsilon) - (5+5\varepsilon-11\varepsilon^2+4\varepsilon^3), \right. \\ & (88-372\varepsilon+551\varepsilon^2-380\varepsilon^3+160\varepsilon^4-64\varepsilon^5+16\varepsilon^6) - 4\varepsilon(3-2\varepsilon)(1-2\varepsilon)(2-\varepsilon)(1-\varepsilon)^2 (\Psi^2(\varepsilon) + \Phi(\varepsilon)) \\ & \left. + 2(1-\varepsilon)(24-144\varepsilon+249\varepsilon^2-146\varepsilon^3+12\varepsilon^4+8\varepsilon^5)\Psi(\varepsilon) \right\} \cdot \left\{ \frac{2}{2-\varepsilon} c_f, \frac{1}{4\varepsilon(3-2\varepsilon)(1-2\varepsilon)} \right\}, \quad (\text{A.7}) \end{aligned}$$

$$\eta_4(\varepsilon) = \frac{\varepsilon \eta_3(\varepsilon)}{2} + \left(-\frac{11}{4} + \sum_{n>0} \frac{f_n \varepsilon^n}{n} \right) 4\varepsilon \eta(\varepsilon) + \frac{2\eta^2(\varepsilon)}{3-2\varepsilon} \left\{ -8(1-4\varepsilon+2\varepsilon^2), \frac{(2-5\varepsilon+2\varepsilon^2)^2}{1-\varepsilon} \right\} \cdot \{c_f, 1\}, \quad (\text{A.8})$$

$$\text{with } \sum_{j>0} f_j \varepsilon^j \equiv -\eta(\varepsilon) \left\{ 4(1+\varepsilon)(1-2\varepsilon)c_f + \frac{4\varepsilon^4-14\varepsilon^3+32\varepsilon^2-43\varepsilon+20}{(1-\varepsilon)(3-2\varepsilon)} \right\}. \quad (\text{A.9})$$

The coefficients a_f , $\eta(\varepsilon)$ and f_j are the same as we had defined in [8].

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