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Counting the number of master integrals for sunrise diagrams via the Mellin–Barnes representation

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Abstract: A number of irreducible master integrals for L-loop sunrise and bubble Feynman diagrams with generic values of masses and external momenta are explicitly evaluated via the Mellin–Barnes representation.

Contents

1 Introduction

The sunrise or watermelon diagram (see Fig. 1) is one of the simplest Feynman diagrams which have been studied by the physics community as well as by mathematicians over the past fifty years [2–14]. This diagram has a few different representations. Within dimensional regularization [1] in momentum space, it is defined as

$$J(\vec{M}_j^2; \vec{\alpha_j}; p^2) = \int \prod_{j=1}^{L} \frac{d^n k_j}{[k_j^2 - M_j^2]^{\alpha_j}} \times \frac{1}{[(p-k_1-\cdots-k_L)^2 - M_{L+1}^2]^{\alpha_{L+1}}} \,, \tag{1.1}$$

where α_j are positive integers, M_j^2 and p^2 are some (in general, complex) parameters, and n is a (in general, non-integer) parameter of dimensional regularization. The parametric representation of this diagram has the following form:[1]

$$J \sim \Gamma\left(\alpha - \frac{n}{2}L\right) \int_0^\infty \prod_{i=1}^{L+1} dx_i \frac{x_i^{\alpha_i - 1}}{\Gamma(\alpha_i)} \delta\left(1 - \sum_i x_i\right) \frac{F^{\frac{n}{2}L - \alpha}}{U(x)^{\frac{n}{2}(L+1) - \alpha}} \,, \tag{1.2}$$

[1] It could also be rewritten in projective space [2, 3].

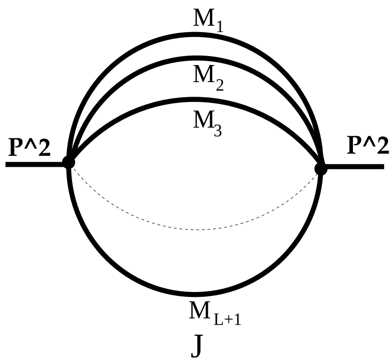

Figure 1. L-loop sunrise diagram.

where F and U are Symanzik polynomials [15] and $\alpha = \sum_{j=1}^{L+1} \alpha_j$. Using the coordinate representation of the Feynman propagator and integrating over the angle, it is easy to get [4] a one-fold integral representation of this diagram (for details, see Refs. [10, 11] and Appendix B). Applying the algorithm of Ref. [16], a Mellin–Barnes integral representation [17] for the sunrise diagram can be deduced [6] (see Eq. (4.1)).

The aim of the present paper is to extend the approach described in Refs. [18, 19] to the multivariable case with reducible monodromy. As an illustration, the number of irreducible master integrals of the L-loop sunrise diagram is evaluated. In Appendix A, the generalized sunrise diagram is considered. The application of our analysis to integrals including products of Bessel functions is discussed in Appendix B.

2 Mellin–Barnes integral versus Horn hypergeometric function

Let us consider the function Φ defined through a K-fold Mellin–Barnes integral,

$$\Phi(A, B; \{C_k\}; \{z_j\}) = \int \prod_{j=1}^{K} dt_j \Gamma(-t_j) \Gamma\left(C_j - t_j\right) z_j^{t_j} \frac{\Gamma\left(A+t\right)}{\Gamma\left(B-t\right)} , \tag{2.1}$$

where $t = \sum_{j=1}^{K} t_j$ and $\{z_a\} = (z_1, z_2, \cdots, z_K)$. This function depends on $K+2$ discrete parameters A, B, and $\{C_j\} = (C_1, \cdots, C_K)$ and K variables $z_1, \cdots, z_K$. Let us briefly recall some basic steps of the differential-reduction procedure [20] applied to the Mellin–Barnes integral. The differential contiguous relations for the function Φ follow directly from the Mellin–Barnes representation and have the following form:

$$\Phi(A,B;\{\},C_j+1,\{\};\{z_k\}) = (C_j-\theta_j)\Phi(A,B;\{\},C_j,\{\};\{z_k\}) \,, \tag{2.2a}$$

$$\Phi(A+1,B;\{C_k\};\{z_k\}) = (A+\sum_{j=1}^{K}\theta_j)\Phi(A,B;\{C_k\};\{z_k\}) \,, \tag{2.2b}$$

$$\Phi(A,B-1;\{C_k\};\{z_k\}) = (B-1-\sum_{j=1}^{K}\theta_j)\Phi(A;B;\{C_k\};\{z_k\}) \,, \tag{2.2c}$$

where

$$\theta_j = z_j \frac{d}{dz_j} \,, \quad j=1,\cdots,K.$$

We denote the set of differential operators on the r.h.s. of Eq. (2.2) as $\mathbf{B}^{+}_{C_j,A,B}$. A linear system of partial differential equations (PDEs) for the function Φ can be derived in two steps. In the first step, we define the polynomials P and Q as

$$\frac{P_j}{Q_j} = \frac{\phi(t_j+1)}{\phi(t_j)} \,.$$

In the second step, we set up the corresponding system of PDEs,

$$L_j : \left(Q_j|_{t_j\to\theta_j} \frac{1}{z_j}\Phi = P_j|_{t_j\to\theta_j}\,\Phi \right) \,,$$

where, for simplicity, we have introduced the short-hand notation: $\Phi \equiv \Phi(A,B;\{C_k\};\{z_k\})$. For the function under consideration, we have

$$\frac{P_j^{\Phi}}{Q_j^{\Phi}} = -\frac{(A+t)\,(1-B+t)}{(1-C_j+t_j)(1+t_j)} \Rightarrow \tag{2.3a}$$

$$L_j^{\Phi} : (\theta_j - C_j)\,\theta_j\Phi = -z_j \left(\sum_{j=1}^{K}\theta_j + A\right)\left(\sum_{j=1}^{K}\theta_j + (1-B)\right)\Phi \,, \quad j=1,\cdots,K. \tag{2.3b}$$

To get the full system of PDEs for the function Φ, a prolongation procedure should be applied [21, 22], which consists in applying new derivatives to the system of PDEs, so that the system of PDEs in Eq. (2.3b) can be written in a Pfaffian form:[2]

$$d\vec{\phi} = \Omega\vec{\phi} \,, \tag{2.4}$$

where the matrix Ω only depends on the values of the parameters and the singular locus of the system of PDEs, and the vector function $\vec{\phi}$ is defined as $(\Phi, \theta_i\Phi, \theta_{ij}\Phi, \cdots, \theta_{j_1,j_2,\cdots,j_m}\Phi)$. The rank r of the matrix Ω at the point z_0 (in our case, $z_0 = \vec{z}_0 = 0$) in Eq. (2.4) is equal to the number of independent solutions of the full system of PDEs.

According to the algorithm described in Ref. [20], the differential operators b_j^- inverse to the operators defined by Eq. (2.2) can be constructed so that

$$b_j^- \mathbf{B}_j^+ \Phi(A,B;\{C_j\};\{z_j\}) = \Phi(A,B;\{C_j\};\{z_j\}) \,, \quad j=1,\cdots,K \,. \tag{2.5}$$

[2]The condition of complete integrability is valid.

The operators b_a^- are defined[3] modulo the full system of PDEs. The differential reduction has the form of a product of several operators b_i^- and B_j^+. In symbolic form, this can be written as

$$\Phi(\vec{I}+\{A,B,\{C_j\}\};\{z_j\}) = \left[Q_0 + \sum_{j=1}^{K} Q_j \theta_j + \sum_{\substack{i,j=1 \\ i<j}}^{K} Q_{ij}\theta_{ij} + \cdots \right] \Phi(A,B,\{C_j\};\{z_j\}) \quad (2.6)$$

where $\vec{I}$ is a set of integers, $\{A,B,\{C_j\}\}$ is a set of parameters, and Q_i are some rational functions of $\{z_i\}$ and $A,B,\{C_j\}$.

The fundamental system of solutions of Eq. (2.3b) is expressible in terms of the Lauricella [21] function $F_C^{(K)}$ of K variables and include the following functions (see, for example, Eq. (13) in Ref. [21] or Eq. (19) in Ref. [23]):

$$\begin{aligned}
& F_C^{(K)}(A,1-B;\{1-C_j\};\{-z_j\}) \, , \\
& (-z_j)^{C_j} \times F_C(A+C_j, 1-B+C_j; 1-C_1,\cdots,1-C_{j-1},1+C_j,1-C_k;\{-z_j\}) \, , \\
& \quad j=1,\cdots,K \, , \\
& (-z_{j_1})^{C_{j_1}}(-z_{j_2})^{C_{j_2}} \times F_C^{(K)}(A+C_{j_1}+C_{j_2},1-B+C_{j_1}+C_{j_2};\{1-C_k\},1+C_{j_1},1+C_{j_2};\{-z_j\}) \, , \\
& \quad j_1,j_2=1,\cdots,K \, , \\
& \cdots \\
& \prod_{j_a=1}^{K}(-z_{j_a})^{C_{j_a}} \times F_C^{(K)}(A+\sum_{j=1}^{K}C_j, 1-B+\sum_{j=1}^{K}C_j;\{1+C_j\};\{-z_j\}) \, ,
\end{aligned} \quad (2.7)$$

where the Lauricella function $F_C^{(K)}$ is defined as

$$F_C^{(K)}(a,b;\{c_j\};\{z_j\}) = \sum_{j_1,\cdots,j_k=0}^{\infty} (a)_{j_1+\cdots+j_K}(b)_{j_1+\cdots+j_K} \prod_{p=1}^{K} \frac{z_p^{j_p}}{j_p!(c_p)_{j_p}} \, , \quad (2.8)$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$ being the Pochhammer symbol. Any particular solution of the system defined by Eq. (2.3b) about the points $z_i = 0$ for a generic set of parameters is a linear combination of solutions defined by Eq. (2.7) with undetermined coefficients. To fix these coefficients, it is necessary to evaluate the Mellin–Barnes integral as a power series solution [24]. In particular, under the condition that the monodromy is irreducible (see

[3] Due to the differential relation

$$\theta_p \Phi(A,B;\{C_j\};\{z_j\}) = -z_p \Phi(1+A,B-1;C_p-1,\{C_j\};\{z_j\}) \equiv -z_p B_A^+ B_B^+ b_C^- \Phi(A,B;\{C_j\};\{z_j\}),$$

not all operators b_c^- are independent.

Eq. (2.10b)), the following representation is valid (see Ref. [16] for details):[4]

$$
\begin{aligned}
&\Phi(A,B,\{C_j\};\{z_j\}) = \frac{\Gamma(A)}{\Gamma(B)}\prod_{j=1}^{K}\Gamma(C_j)\times F_C^{(K)}(A,1-B;\{1-C_j\};\{-z_j\}) \\
&+\sum_{j=1}^{K}(-z_j)^{C_j}\frac{\Gamma(A+C_j)\Gamma(-C_j)}{\Gamma(B-C_j)}\prod_{\substack{a=1\\a\neq j}}^{K}\Gamma(C_a) \\
&\quad\times F_C\left(A+C_j,1-B+C_j;1-C_1,\cdots,1-C_{j-1},1+C_j,1-C_k;\{-z_j\}\right) \\
&+\sum_{j_1,j_2=1}^{K}(-z_{j_1})^{C_{j_1}}\Gamma(-C_{j_1})\times(-z_{j_2})^{C_{j_2}}\Gamma(-C_{j_2})\times\frac{\Gamma(A+C_{j_1}+C_{j_2})}{\Gamma(B-C_{j_1}-C_{j_2})} \\
&\quad\times\prod_{\substack{a=1\\a\neq j_1,j_2}}^{K}\Gamma(C_a)\times F_C^{(K)}\left(A+C_{j_1}+C_{j_2},1-B+C_{j_1}+C_{j_2};\{1-C_k\},1+C_{j_1},1+C_{j_2};\{-z_j\}\right) \\
&+\cdots \\
&+\sum_{b=1}^{K}\frac{\Gamma(C_{j_b})}{(-z_{j_b})^{C_{j_b}}\Gamma(-C_{j_b})}\times\frac{\Gamma\left(A+\sum_{j=1}^{K}C_j-C_b\right)}{\Gamma\left(B-\sum_{j=1}^{K}C_j-C_b\right)}\times\prod_{a=1}^{K}(-z_{j_a})^{C_{j_a}}\Gamma(-C_{j_a}) \\
&\quad\times F_C^{(K)}(A+\sum_{j=1}^{K}C_j-C_b,1-B+\sum_{j=1}^{K}C_j-C_n;1-C_b,\{1+C_j\};\{-z_j\}) \\
&+\prod_{j_a=1}^{K}(-z_{j_a})^{C_{j_a}}\Gamma(-C_{j_a})\frac{\Gamma\left(A+\sum_{j=1}^{K}C_j\right)}{\Gamma\left(B-\sum_{j=1}^{K}C_j\right)} \\
&\quad\times F_C^{(K)}(A+\sum_{j=1}^{K}C_j,1-B+\sum_{j=1}^{K}C_j;\{1+C_j\};\{-z_j\})\,.
\end{aligned}
\tag{2.9}
$$

We wish to point out that, for the sunrise diagram, the last term in Eq. (2.9) is proportional to $1/\Gamma(0)$ and so equal to zero.[5]

The holonomic[6] rank[7] of the system of PDEs in Eq. (2.3b) for a generic set of parameters $\{A,B,C_j\}$ is equal to 2^K [21] (see also Refs. [27–30]). However, if the parameters $A,B,\{C_i\}$ satisfy certain linear relations, then additional differential operators are generated, so that Puiseux-type solutions appear. The main questions are how to find such linear relations between the parameters and how to define a minimal set of the additional PDEs. Our approach [18, 19] to these problems is based on studying the inverse differential operators: the exceptional case of parameters, where the dimension of the solution space is reduced, corresponds to the condition that the denominators of the functions Q_i entering Eq. (2.6)

[4]The interrelations between Mellin–Barnes integrals and multiple residues were discussed in Ref. [25].

[5]Another example of such a cancellation was presented in Ref. [26].

[6]We adopt the following definition of holonomic function [20]: a function is called holonomic if it satisfies a system of linear differential equations with polynomial coefficients whose solutions form a finite-dimensional vector space.

[7]The dimension of the space of solutions of a system of PDEs near some generic point is called holonomic rank.

are equal to zero for arbitrary values of z_i [31]. The same recipe works in its application to Mellin–Barnes integrals [24], which can be treated as a particular case of the Gelfand–Kapranov–Zelevinsky (GKZ) hypergeometric system [32]. However, the inverse differential operators have a very complicated structure (see, for example, Ref. [30]), which gives rise to technical problems in the analysis of the number of independent PDEs.

Fortunately, there is a simpler way [24, 33–35]) to find the conditions of reducibility and to define the dimension of the (ir)reducible subspace of solutions. In our case, the system of PDEs is irreducible if (for details, see Section 3)

$$F_C : \left\{ a \notin \mathbb{Z}, \quad b \notin \mathbb{Z}, \quad a - \sum_{s_1} c_p \notin \mathbb{Z}, \quad b - \sum_{s_2} c_p \notin \mathbb{Z} \right\}, \tag{2.10a}$$

$$\Phi : \left\{ A \notin \mathbb{Z}, \quad B \notin \mathbb{Z}, \quad A + \sum_{S_1} C_p \notin \mathbb{Z}, \quad B - \sum_{S_2} C_p \notin \mathbb{Z} \right\}, \tag{2.10b}$$

where s_i and S_j are any subsets of $\{c_i\}$ and $\{C_j\}$, respectively. Eq. (2.10a) corresponds to the set of exceptional values of the parameters of the hypergeometric function F_C,[8] and Eq. (2.10b) defines the exceptional set of parameters for the function Φ. The conditions defined by Eq. (2.10b) are invariant with respect to a linear change of variables $\{t_j\} \to \{A_1 \pm \sum_{A \in \{1,\cdots,p\}} t_A\}$ in the Mellin–Barnes integral in Eq. (2.1).

In a similar manner, we can consider the function Ψ defined as a K-fold Mellin–Barnes integral,

$$\Psi(A, D; \{C_k\}; z_1, \cdots, z_K) = \int \prod_{j=1}^{K} dt_j \Gamma(-t_j) \Gamma\left(C_j - t_j\right) z_j^{t_j} \Gamma\left(A + t\right) \Gamma\left(D + t\right) \ . \tag{2.11}$$

In this case, we have

$$\frac{P_j^\Psi}{Q_j^\Psi} = \frac{(A+t)\,(D+t)}{(1-C_j+t_j)(1+t_j)} \Rightarrow \tag{2.12a}$$

$$L_j^\Psi : (\theta_j - C_j)\,\theta_j \Psi = z_j \left(\sum_{j=1}^{K} \theta_j + A \right) \left(\sum_{j=1}^{K} \theta_j + D \right) \Psi \ , \quad j = 1, \cdots, K. \tag{2.12b}$$

The system of PDEs for the function Ψ is irreducible if

$$\Psi : \left\{ A \notin \mathbb{Z}, \quad D \notin \mathbb{Z}, \quad A + \sum_{S_1} C_p \notin \mathbb{Z}, \quad D + \sum_{S_2} C_p \notin \mathbb{Z} \right\}, \tag{2.13}$$

where S_1 and S_2 are any subsets of $\{C_j\}$. The solution of Eq. (2.12b) for a generic set of parameters about the points $z_i = 0$ can be written as linear combination of Lauricella

[8] This set of parameters complies with the condition that the monodromy group of the Lauricella function F_C is reducible [27]. For recent results on the evaluation of the monodromy of the GKZ system, see Ref. [33].

functions $F_C^{(K)}$ of K variables. For completeness, we present it here:

$$
\begin{aligned}
&\Psi(A,D,\{C_j\};\{z_j\}) = \Gamma(A)\Gamma(D)\prod_{j=1}^{K}\Gamma(C_j)\times F_C^{(K)}(A,D;\{1-C_j\};\{z_j\}) \\
&+\sum_{j=1}^{K}(z_j)^{C_j}\Gamma(A+C_j)\Gamma(D+C_j)\Gamma(-C_j)\prod_{\substack{a=1\\a\neq j}}^{K}\Gamma(C_a) \\
&\quad\times F_C^{(K)}(A+C_j,D+C_j;1-C_1,\cdots,1-C_{j-1},1+C_j,1-C_K;\{z_j\}) \\
&+\sum_{j_1,j_2=1}^{K}(z_{j_1})^{C_{j_1}}\Gamma(-C_{j_1})\times(z_{j_2})^{C_{j_2}}\Gamma(-C_{j_2})\times\Gamma(A+C_{j_1}+C_{j_2})\times\Gamma(D+C_{j_1}+C_{j_2}) \\
&\quad\times\prod_{\substack{a=1\\a\neq j_1,j_2}}^{K}\Gamma(C_a)\times F_C^{(K)}(A+C_{j_1}+C_{j_2},D+C_{j_1}+C_{j_2};\{1-C_k\},1+C_{j_1},1+C_{j_2};\{z_j\}) \\
&+\cdots \\
&+\sum_{b=1}^{K}\frac{\Gamma(C_{j_b})}{(z_{j_b})^{C_{j_b}}\Gamma(-C_{j_b})}\times\Gamma\left(A+\sum_{j=1}^{K}C_j-C_b\right)\times\Gamma\left(D+\sum_{j=1}^{K}C_j-C_b\right) \\
&\quad\times\prod_{a=1}^{K}(z_{j_a})^{C_{j_a}}\Gamma(-C_{j_a})\times F_C^{(K)}(A+\sum_{j=1}^{K}C_j-C_b,D+\sum_{j=1}^{K}C_j-C_n;1-C_b,\{1+C_j\};\{z_j\}) \\
&+\prod_{j_a=1}^{K}(z_{j_a})^{C_{j_a}}\Gamma(-C_{j_a})\Gamma\left(A+\sum_{j=1}^{K}C_j\right)\Gamma\left(D+\sum_{j=1}^{K}C_j\right) \\
&\quad\times F_C^{(K)}(A+\sum_{j=1}^{K}C_j,D+\sum_{j=1}^{K}C_j;\{1+C_j\};\{z_j\})\,.
\end{aligned}
\tag{2.14}
$$

3 What happens if the monodromy is reducible?

Here, we present a short algebraic deviation of the condition in Eq. (2.10) for the system of PDEs related to the Mellin–Barnes integral in Eq. (2.1) to be irreducible and count the dimension of the invariant subspace of the differential contiguous operators defined by Eqs. (2.2b) and (2.2c). We follow an idea presented in Ref. [34] (see also Ref. [35]).

Let Φ be the set of solutions for the system of linear differential operators L_j^Φ defined by Eq. (2.3b) as

$$
L_j^\Phi(A,B,\vec{C}):\left[(\theta_j-C_j)\,\theta_j+z_j\left(\sum_{j=1}^{K}\theta_j+A\right)\left(\sum_{j=1}^{K}\theta_j+(1-B)\right)\right]\,,\quad j=1,\cdots,K.
$$

Let $S(A,B,\vec{C})$ denote the local solution space of the operators $L_j^\Phi(A,B,\vec{C})$ about some point z_0. The contiguous differential operators B^+_{A,B,C_j} defined by Eq. (2.2) map the solution space $S(A,B,\vec{C})$ to the solution space $S(A\pm I_1,B\pm I_2,\vec{C}\pm\vec{I})$, where $\{I_a\}$ are a set of integers. If the monodromy is reducible, then there is a monodromy-invariant subspace (invariant under the action of the monodromy) in the space of solutions. In this case, the contiguous differential operators B^+_{A,B,C_j} have a nontrivial kernel, and it is necessary to evaluate their dimension.

Let us consider the equation $B^+_A \Phi = 0$, where B^+_A is defined by Eq. (2.2b). Then, the system of equations defined by Eq. (2.3b) reduces to

$$L^{\Phi}_j(A,B,\vec{C})\Phi \equiv (\theta_j - C_j)\,\theta_j \Phi \equiv 0 \,, \quad j = 1, \cdots, K. \tag{3.1}$$

If all $C_j \notin \mathbb{Z}$, which is true for the considered Feynman diagrams, then the solution Φ_0 of Eq. (3.1) has the following form:

$$\Phi_0 = c_0 + \sum_{i=1}^{K} c_i z_i^{C_i} + \sum_{\substack{i,j=1 \\ i<j}}^{K} c_{i,j} z_i^{C_i} z_j^{C_j} + \cdots + \text{Const.} \times \prod_{i=1}^{K} z_i^{C_i} \,. \tag{3.2}$$

Applying the operator B^+_A to the function Φ_0, we get

$$\begin{aligned} B^+_A \Phi_0 \equiv 0 = {} & A c_0 + \sum_{i=1}^{K} c_i (A + C_i) z_i^{C_i} + \sum_{\substack{i,j=1 \\ i<j}}^{K} c_{i,j} (A + C_i + C_j) z_i^{C_i} z_j^{C_j} + \cdots \\ & + \left(\prod_{i=1}^{K} z_i^{C_i} \right) \times \sum_{a=1}^{K} \frac{\tilde{c}_a}{z_a^{C_a}} \left(A + \sum_{j=1}^{K} C_j - C_a \right) + \text{Const.} \times \prod_{i=1}^{K} z_i^{C_i} \left(A + \sum_{j=1}^{K} C_j \right) , \end{aligned} \tag{3.3}$$

where $c_i, c_{i,j}, \ldots, \tilde{c}_i$ are some constants. As follows from Eq. (3.3), $B^+_A \Phi = 0$ if and only if $A + \sum_S C_p = 0$, where S are any subsets of $\{C_j\}$. In this way, under the conditions that

- $A \notin \mathbb{Z}$,
- $C_a \notin \mathbb{Z} \,, \quad \forall a = 1, \cdots, K$,
- $A + \sum_{j=1}^{K} C_j - C_a = 0 \,, \quad \forall a = 1, \cdots, K$,

there is an invariant subspace of dimension K for the operator B^+_A.

A similar consideration can also be made for the operator B^+_B defined by Eq. (2.2c). In particular, it is easy to show that, under the conditions that

- $B \notin \mathbb{Z}$,
- $C_a \notin \mathbb{Z} \,, \quad \forall a = 1, \cdots, K$,
- $B - \sum_{j=1}^{K} C_j = 0$,

there is a one-dimensional invariant subspace. Collecting the previous results, we get the following lemma:
Lemma Φ*:*
Under the conditions that

- A and $B \notin \mathbb{Z}$,

- $C_a \notin \mathbb{Z}\,, \quad \forall a = 1, \cdots, K,$
- $B - \sum_{j=1}^{K} C_j \in \mathbb{Z}\,,$
- $A + \sum_{j=1}^{K} C_j - C_a \in \mathbb{Z}\,, \quad \forall a = 1, \cdots, K\,,$

there is a $(K+1)$-dimensional invariant subspace (of Puiseux-type solutions) in the space of solutions of the linear PDEs defined by Eq. (2.3b), and the dimension of the space of nontrivial solutions is equal to

$$N_\Phi = 2^K - (K+1)\,. \tag{3.4}$$

A similar consideration can also be made for the function Ψ in Eq. (2.11).

In the application to the Lauricella function $F_C^{(K)}(a, b; \{c_j\}; \vec{z})$ of K variables, our analysis is equivalent to the following lemma:
Lemma F:
Under the conditions that

- a and $b \notin \mathbb{Z}\,,$
- $c_j \notin \mathbb{Z}\,, \quad \forall \;\; j = 1, \cdots, K,$
- $b + \sum_{j=1}^{K} c_j \in \mathbb{Z}\,,$
- $a + \sum_{j=1}^{K} c_j - c_k \in \mathbb{Z}\,, \quad \forall \;\; k = 1, \cdots, K\,,$

there is a $(K+1)$-dimensional invariant subspace (of Puiseux-type solutions) so that the dimension of the space of nontrivial solutions is equal to

$$N_F = 2^K - (K+1)\,, \tag{3.5}$$

and the differential reduction applied to the hypergeometric function $F_C^{(K)}(a, b; \{c_j\}; \vec{z})$ of K variables has the following form:

$$F_C^{(K)}(\vec{I} + \{a, b, c_j\}; \vec{z}) = \sum_{j=0}^{K-2} \vec{Q}_j \vec{\theta}^j F_C^{(K)}(a, b; \{c_j\}; \vec{z}) + \sum_{j=1}^{K+1} P_j(\vec{z})\,, \tag{3.6}$$

where $\vec{I}$ is a set of integers, $\vec{Q}_j$ are some rational functions, $P_j(\vec{z})$ denote the Puiseux-type solutions, and $\vec{\theta}^p$ means $\theta_{\substack{i_1, i_2, \cdots, i_p \\ i_1 < i_2 < \cdots < i_p}}$.

The number of symmetric derivatives $\theta_{i_1, i_2, \cdots, i_m}$, where $i_1 < i_2 < \cdots < i_m$, is equal to $\frac{K!}{m!(K-m)!}$, and the number of terms entering the reduction procedure of Eq. (3.6) is equal to $\sum_{j=0}^{K-2} \frac{K!}{j!(K-j)!} = 2^K - (K+1)$, which coincides with Eq. (3.5). Consequently, the highest differential operators, namely, one operator of order K, $\theta_{i_1, i_2, \cdots, i_K}$, and K operators of order $K-1$, $\theta_{i_1, i_2, \cdots, i_{K-1}}$, are expressible in terms of low-dimensional differential operators.
Remark A: As follows from Eq. (3.3), for a special set of parameters, the $K+1$ Puiseux-type solutions have the following form: K solutions are of the type $1/z_a^{C_a} \times \left[\prod_{j=1}^{K} z_j^{C_j}\right]$, and one solution is $\left[\prod_{j=1}^{K} z_j^{C_j}\right]$.

Remark B: For sunrise and bubble diagrams, each term of the hypergeometric representation in Eqs. (2.9) and (2.14) satisfies the conditions of Lemma F, so that the dimension of the space of nontrivial solutions of each term is defined by Eq. (3.5) and the differential reduction of each term is described by Eq. (3.6).

4 L-loop sunrise diagram

The Mellin–Barnes representation for the Feynman diagram defined by Eq. (1.1) follows from the algorithm presented in Ref. [16] (see also Ref. [6]) and has the following form:

$$J^{(L)}(M_1^2,\cdots,M_{L+1}^2;\alpha_1,\cdots,\alpha_{L+1};p^2) = (p^2)^{\frac{n}{2}L-\alpha}[i^{1-n}\pi^{n/2}]^L \\ \times \int \left\{ \prod_{j=1}^{L+1} dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2}-\alpha_j-t_j\right)}{\Gamma(\alpha_j)} \left(-\frac{M_j^2}{p^2}\right)^{t_j} \right\} \frac{\Gamma\left(\alpha-\frac{n}{2}L+t\right)}{\Gamma\left(\frac{n}{2}(L+1)-\alpha-t\right)} , \tag{4.1}$$

where

$$\alpha = \sum_{j=1}^{L+1} \alpha_j \; , \quad t = \sum_{j=1}^{L+1} t_j \; ,$$

α_j, L are positive integers, M_j^2 and p^2 are some (in general, complex) parameters, and d is a parameter (in general, non-integer) of dimensional regularization [1].

Let us introduce the variables $z_j = -\frac{M_j^2}{p^2}$ $(j = 1,2,\cdots,L+1)$ and define the functions Φ_J as

$$\Phi_J = \prod_{k=1}^{L+1} \frac{\Gamma(\alpha_k)}{[i^{1-n}\pi^{n/2}]^L (p^2)^{\frac{n}{2}L-\alpha}} \times J^{(L)}(M_1^2,\cdots,M_{L+1}^2;\alpha_1,\cdots,\alpha_{L+1};p^2) \; . \tag{4.2}$$

After this redefinition, the results of Sections 2 and 3 are directly applicable to the analysis of the sunrise diagram. In particular, in the application to the L-loop sunrise diagram, we have

$$A = \alpha - \frac{n}{2}L \; , \quad B = \frac{n}{2}(L+1) - \alpha \; , \quad C_j = \frac{n}{2} - \alpha_j \; , \quad j = 1,\cdots,L+1 \; . \tag{4.3}$$

As follows from Eq. (3.3), to find the dimension of the invariant subspace, it is necessary to find all solutions of the following system of algebraic equations:

$$\frac{n}{2}L - \sum_{S_1} \frac{n}{2} = 0 \quad (\mathrm{mod}\ \mathbb{Z}) \; , \tag{4.4a}$$

$$\frac{n}{2}(L+1) - \sum_{S_2} \frac{n}{2} = 0 \quad (\mathrm{mod}\ \mathbb{Z}) \; , \tag{4.4b}$$

where S_1 and S_2 are any subsets of $1,\ldots,L+1$, $\mathbb{Z} = \{\ldots,-1,0,1,\ldots\}$, and n is non-integer. The subset S_1 in Eq. (4.4a) can be constructed in $L+1$ different ways (it includes all possible combinations of L out of the $L+1$ massive lines), and there is only one solution of Eq. (4.4b) (the subset S_2 includes all the lines). In this way, among the 2^{L+1} solutions of the system of PDEs related to the sunrise diagram, there are $L+1+1$ Puiseux-type

solutions.[9] Then, the following theorem is valid:
The number N_J of irreducible master integrals of the L-loop sunrise diagram with generic values of masses and momenta is equal to

$$\boxed{N_J = 2^{L+1} - L - 2} \, . \tag{4.5}$$

For example, for $L = 1, 2, 3, 4, 5, 6$, we have $N_J = 1, 4, 11, 26, 57, 120$, respectively.

The result of the differential reduction can be written in a more familiar form via propagators with dots: the term without derivatives on the r.h.s. of Eq. (2.6) corresponds to the diagram itself, whereas terms with derivative(s) $\theta_i \Phi$ correspond to diagrams with dot(s). In the application to the sunrise diagram, we have

$$J^{(L)}(\{M_i^2\}; \vec{I} + \vec{\alpha}; p^2) = \sum_{j=0}^{L-1} \vec{Q}_j \vec{\partial}^j J^{(L)}(\{M_i^2\}; \vec{\alpha}; p^2) + \sum_{j=1}^{L+1} P_j(\vec{M}) \, , \tag{4.6}$$

where we have introduced the short-hand notation

$$J^{(L)}(\{M_i^2\}; \vec{\alpha}; p^2) \equiv J^{(L)}(M_1^2, \cdots, M_{L+1}^2; \alpha_1, \cdots, \alpha_{L+1}; p^2),$$

$\vec{Q}_j$ are some rational functions, $P_j(\vec{z})$ denote the Puiseux-type solutions, and $\vec{\partial}^j$ means symmetric derivatives with respect to the masses M_j^2, i.e.

$$\vec{\partial}^j \equiv \left(M_{i_1}^2 \frac{\partial}{\partial M_{i_1}^2} \right) \left(M_{i_2}^2 \frac{\partial}{\partial M_{i_2}^2} \right) \cdots \left(M_{i_j}^2 \frac{\partial}{\partial M_{i_j}^2} \right) \, , \tag{4.7}$$

and $i_1 < i_2 < \cdots < i_j$. Indeed, the number of symmetric derivatives $\vec{\partial}^j$ in the considered case is equal to $\frac{(L+1)!}{j!(L+1-j)!}$, so that

$$2^{L+1} - (L+2) = \sum_{j=0}^{L-1} \frac{(L+1)!}{j!(L+1-j)!} \, . \tag{4.8}$$

In other words, (i) an L-loop sunrise diagram with two or more derivatives on one of its lines is reducible to a linear combination of sunrise diagrams with no more than one derivative on any of its lines, and (ii) the number of lines with one dot is less or equal to $L-1$.[10] The latter statement can also be understood as follows: instead of excluding higher-order derivatives, the function (one element) and its first derivatives ($L+1$ elements) can be excluded in favor of higher derivatives, and the reduction procedure has the following form:

[9] As was pointed out in Ref. [19], a Puiseux-type solution corresponds to a product of one-loop bubble diagrams.

[10] Let us recall that there are $L+1$ massive lines in this case.

$$J^{(L)}(\{M_i\},\vec{1};p^2) = \sum_{j=2}^{L+1} \vec{\tilde{Q}}_j \vec{\partial}^j J^{(L)}(\{M_i\},\{1\};p^2) + \sum_{j=1}^{L+1} \tilde{P}_j(\vec{M}) \; , \tag{4.9a}$$

$$M_a^2 \frac{\partial}{\partial M_a^2} J^{(L)}(\{M_i\},\vec{1};p^2) = \sum_{j=2}^{L+1} \vec{\hat{Q}}_j \vec{\partial}^j J^{(L)}(\{M_i\},\{1\};p^2) + \sum_{j=1}^{L+1} \hat{P}_j(\vec{M}) \; , \quad \forall a = 1, \cdots, L+1, \tag{4.9b}$$

$$J^{(L)}(\{M_i\},\vec{I}+\{\alpha\};p^2) = \sum_{j=2}^{L+1} \vec{Q}_j \vec{\partial}^j J^{(L)}(\{M_i\},\{1\};p^2) + \sum_{j=1}^{L+1} P_j(\vec{M}) \; , \tag{4.9c}$$

(4.9d)

where $\vec{Q}_j, \vec{\tilde{Q}}, \vec{\hat{Q}}$ are some rational functions, $P_j(\vec{z}), \tilde{P}_j(\vec{M}), \hat{P}_j(\vec{M})$ denote the Puiseux-type solutions, and $\vec{\partial}^j$ is defined by Eq. (4.7). As an illustration of these relations, let us consider some special cases.

- At the two-loop level ($L = 2$), only first derivatives with respect to masses enter the reduction procedure, in agreement with Refs. [36–38].
- At the three-loop level ($L = 3$), according to Eq. (4.6), the second symmetric derivatives with respect to masses $\frac{\partial^2}{\partial M_i^2 \partial M_j^2} J^{(3)}(M_i^2;\vec{1};p^2)$ where $i < j$ and $i, j = 1, 2, 3, 4$, are generated $(1, 4, 6$ terms) or, according to Eq. (4.9), the basis can be constructed from the second, third, and fourth symmetric derivatives $(6, 4, 1$ terms), $\frac{\partial^2}{\partial M_i^2 \partial M_j^2} J^{(3)}(M_i^2;\vec{1};p^2)$, $\frac{\partial^3}{\partial M_i^2 \partial M_j^2 \partial M_k^2} J^{(3)}(M_i^2;\vec{1};p^2)$, and $\frac{\partial^4}{\partial M_1^2 \partial M_2^2 \partial M_3^2 \partial M_4^2} J^{(3)}(M_i^2;\vec{1};p^2)$, where $i < j < k$ and $i, j, k = 1, 2, 3, 4$.
- At the four-loop level ($L = 4$), according to Eq. (4.6), the third symmetric derivatives with respect to masses $\frac{\partial^3}{\partial M_i^2 \partial M_j^2 \partial M_k^2} J^{(4)}(M_i^2;\vec{1};p^2)$, where $i < j < k$ and $i, j, k = 1, 2, 3, 4, 5$, are generated $(1, 5, 10, 10$ terms) or, according to Eq. (4.9), the basis includes the second, third, fourth, and fifth symmetric derivatives $(10, 10, 5, 1$ terms), $\frac{\partial^2}{\partial M_i^2 \partial M_j^2} J^{(4)}(M_i^2;\vec{1};p^2)$, $\frac{\partial^3}{\partial M_i^2 \partial M_j^2 \partial M_k^2} J^{(4)}(M_i^2;\vec{1};p^2)$, $\frac{\partial^4}{\partial M_i^2 \partial M_j^2 \partial M_k^2 \partial M_l^2} J^{(4)}(M_i^2;\vec{1};p^2)$, and $\frac{\partial^5}{\partial M_1^2 \partial M_2^2 \partial M_3^2 \partial M_4^2 \partial M_5^2} J^{(4)}(M_i^2;\vec{1};p^2)$, where $i < j < k < l$ and $i, j, k, l = 1, 2, 3, 4, 5$.

The case of integer n requires an extra analysis [39].

4.1 L-loop sunrise diagram with R massive lines

Let us consider the L-loop sunrise diagram in which only R lines ($R \leq L$) have different masses,

$$J_R(\{M_i^2\};\{\alpha_j\},\{\beta_k\};p^2) = \int \prod_{j=1}^{R} \frac{d^n(k_1 \cdots k_L)}{[k_j^2 - M_k^2]^{\alpha_j} [k_{R+1}^2]^{\beta_1} \cdots [(p - k_1 - \cdots - k_L)^2]^{\beta_{L+1-R}}} \; . \tag{4.10}$$

The Mellin–Barnes representation of this diagram is

$$
\begin{aligned}
&J_R(\{M_R^2\};\{\alpha_j\},\{\beta_k\};p^2) = (p^2)^{\frac{n}{2}L-\alpha-\beta}[i^{1-n}\pi^{n/2}]^L \\
&\times\left\{\prod_{k=1}^{L+1-R}\frac{\Gamma\left(\frac{n}{2}-\beta_k\right)}{\Gamma(\beta_k)}\right\}\frac{\Gamma\left(\beta-\frac{n}{2}(L-R)\right)}{\Gamma\left(\frac{n}{2}(L-R+1)-\beta\right)} \\
&\times\int\left\{\prod_{j=1}^{R}dt_j\frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2}-\alpha_j-t_j\right)}{\Gamma(\alpha_j)}\left(-\frac{M_j^2}{p^2}\right)^{t_j}\right\}\frac{\Gamma\left(\alpha+\beta-\frac{n}{2}L+t\right)}{\Gamma\left(\frac{n}{2}(L+1)-\alpha-\beta-t\right)} \ , \qquad (4.11)
\end{aligned}
$$

where

$$\alpha=\sum_{j=1}^{R}\alpha_j \ , \quad \beta=\sum_{j=1}^{L-R+1}\beta_j \ , \quad t=\sum_{j=1}^{R}t_j \ .$$

In this case, we have:

$$A=\alpha+\beta-\frac{n}{2}L \ , \quad B=\frac{n}{2}(L+1)-\alpha-\beta \ , \quad C_j=\frac{n}{2}-\alpha_j \ , \quad j=1,\cdots,R.$$

To find the dimension of the invariant subspace, we use the algorithm described in Section 3. In this case, it is necessary to find all solutions (all subsets) of the following system of algebraic equations:

$$\frac{n}{2}L-\sum_{S_1}\frac{n}{2}=0 \quad (\mathrm{mod} \ \ \mathbb{Z}) \ , \qquad (4.12a)$$

$$\frac{n}{2}(L+1)-\sum_{S_2}\frac{n}{2}=0 \quad (\mathrm{mod} \ \ \mathbb{Z}) \ , \qquad (4.12b)$$

where S_1 and S_2 are any subsets of $1,\ldots,R$, and n is non-integer. There is only one solution for the subset S_1 (if $R=L$), and there is no solution for Eq. (4.12b). In this way, among the 2^R solutions, there is only one Puiseux-type solution (if $R=L$). Then we have the following theorem:

The number $N_{J,R}$ of irreducible master integrals of the L-loop sunrise diagram with R massive lines $(R\leq L)$ is equal to

$$\boxed{N_{L,R}=2^R-\delta_{0,L-R}} \ . \qquad (4.13)$$

As follows from this relation, the sunrise diagrams with R massive and two or more massless lines are irreducible, and their holonomic ranks near $\vec{z}=0$ coincide with the holonomic ranks of the hypergeometric functions $F_C^{(R)}$ with irreducible monodromies.

There is a one-dimensional invariant subspace (Puiseux-type solution) if the sunrise diagram has one massless line.[11] If at least one of the values of β_j is non-integer, which happens if a massless line is dressed by other massless lines, then the number of irreducible master integrals is equal to 2^R.

Example:

$$N_{1,1}=1 \ ; \quad N_{2,2}=3 \ , \quad N_{2,1}=2 \ ; \quad N_{3,3}=7 \ , \quad N_{3,2}=4 \ , \quad N_{3,1}=2 \ .$$

[11] At the two-loop level, this was shown explicitly by Tarasov in Ref. [36] (see also the discussion in Ref. [37]).

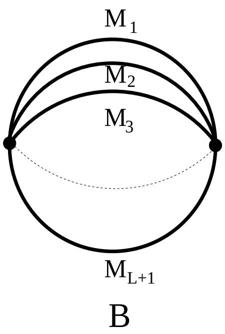

Figure 2. L-loop bubble diagram.

5 L-loop bubble diagram

In a similar manner, let us consider the L-loop bubble diagram (see Fig. 2) defined as

$$B(M_1^2,\cdots,M_{L+1}^2;\alpha_1,\cdots,\alpha_{L+1}) = \int \frac{d^n(k_1\cdots k_L)}{[k_1^2-M_1^2]^{\alpha_1}\cdots[k_L^2-M_L^2]^{\alpha_L}[(k_1-\cdots-k_L)^2-M_{L+1}^2]^{\alpha_{L+1}}} \ , \tag{5.1}$$

where α_j are integers. Its Mellin–Barnes representation is

$$B(M_1^2,\cdots,M_{L+1}^2;\alpha_1,\cdots,\alpha_{L+1}) = (-M_{L+1}^2)^{\frac{n}{2}L-\alpha} \times \frac{[i^{1-n}\pi^{n/2}]^L}{\Gamma\left(\frac{n}{2}\right)\Gamma(\alpha_{L+1})}$$
$$\times \int \left\{ \prod_{j=1}^{L} dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2}-\alpha_j-t_j\right)}{\Gamma(\alpha_j)} \left(\frac{M_j^2}{M_{L+1}^2}\right)^{t_j} \right\} \Gamma\left(\alpha+\alpha_{L+1}-\frac{n}{2}L+t\right)\Gamma\left(\alpha-\frac{n}{2}(L-1)+t\right) \ , \tag{5.2}$$

where

$$\alpha = \sum_{j=1}^{L}\alpha_j \ , \quad t = \sum_{j=1}^{L} t_j \ .$$

Let us introduce the variables $z_j = \frac{M_j^2}{M_{L+1}}$, $j = 1,\cdots,K$, and define the functions Ψ_B as

$$\Psi_B = \prod_{k=1}^{L+1}\Gamma(\alpha_k) \times \frac{\Gamma\left(\frac{n}{2}\right)}{[i^{1-n}\pi^{n/2}]^L(-M_{L+1}^2)^{\frac{n}{2}L-\alpha}} \times B(M_1^2,\cdots,M_{L+1}^2;\alpha_1,\cdots,\alpha_{L+1}) \ .$$

In this case, we have

$$A = \alpha+\alpha_{L+1}-\frac{n}{2}L \ , \quad D = \alpha-\frac{n}{2}(L-1) \ , \quad C_j = \frac{n}{2}-\alpha_j \ , \quad j = 1,\cdots,L \ .$$

As follows from Eq. (3.3), the dimension of the invariant subspace it defined by the number of solutions of the following system of algebraic equations:

$$\frac{n}{2}L - \sum_{S_1} \frac{n}{2} = 0 \quad (\mathrm{mod} \quad \mathbb{Z}) \ , \tag{5.3a}$$

$$\frac{n}{2}(L-1) - \sum_{S_2} \frac{n}{2} = 0 \quad (\mathrm{mod} \quad \mathbb{Z}) \ , \tag{5.3b}$$

where S_1 and S_2 are any subsets of $1, \ldots, L$, and n is non-integer. There is only one solution for the subset S_1, and there are L solutions for Eq. (5.3b). In this case, there are $L+1$ Puiseux-type solutions, and the following theorem is valid:

The number N_B of irreducible master integrals for the L-loop bubble diagram with generic values of masses is equal to

$$\boxed{N_B = 2^L - L - 1} \ . \tag{5.4}$$

For example, for $L = 1, 2, 3, 4, 5, 6$, there are $N_B = 0, 1, 4, 11, 26, 57$ nontrivial master integrals. Let us explain the zero result for $L = 1$. The one-loop tadpole A_0 is treated as a constant in the framework of the differential algebra (see the discussion in Refs. [18, 19]).

Remark 1: The result in Eq. (5.4) can be derived from Eq. (4.5) by considering bubble diagrams as sunrise diagrams with external momenta put equal to zero, which effectively reduces the number of independent variables by one.

Remark 2: In contrast to the sunrise diagram, the reduction for the bubble diagrams with $L \geq 3$ does not have a completely symmetric structure with respect to mass derivatives. At the three-loop level, this statement was confirmed in Ref. [40]. In the framework of integration-by-parts (IBP) relations [41], there is a so-called {`dim`} relation (see the discussion in Ref. [42]) connecting the diagram in Eq. (5.1) without derivatives to a linear combination of diagrams with a first derivative,

$$\left\{ \sum_{j=1}^{L+1} M_j^2 \frac{\partial}{\partial M_j^2} - \left(\frac{n}{2}L - \sum_{j=1}^{L+1} \alpha_j \right) \right\} B(M_1^2, \cdots, M_{L+1}^2; \alpha_1, \cdots, \alpha_{L+1}) = 0 \ , \tag{5.5}$$

so that

$$B(\{M_{L+1}^2\}; \vec{1}) = \frac{1}{\left[\frac{n}{2}L - (L+1)\right]} \left(\sum_{j=1}^{L+1} B(\{M_i^2\}; \vec{1} + \vec{e_j}) \right) \ , \tag{5.6}$$

where $\vec{e_j}$ is the unit vector with unity in the j-th place and we have introduced the shorthand notation $B(\{M_{L+1}^2\}; \vec{1}) \equiv B(M_1^2, \cdots, M_{L+1}^2; 1, \cdots, 1)$.

Remark 3: Let us find out which diagrams form the symmetric set of master integrals for the bubble diagrams with $L \geq 4$. It is easy to see that the lists of diagrams are different for even and odd numbers of loops. If L is even, then the symmetric set of master integrals is defined by the following expression:

$$\sum_{j=0}^{L-2} \frac{(L+1)!}{j!(L+2-j)!} \frac{(1+(-1)^j)}{2} \vec{\partial}^j B(M_1^2, \cdots, M_{L+1}^2; 1, \cdots, 1) \ , \tag{5.7}$$

where $\vec{\partial}^j$ is defined by Eq. (4.7). For odd values of L, a similar expression is valid, namely

$$\sum_{j=0}^{L-2} \frac{(L+1)!}{j!(L+2-j)!} \frac{(1-(-1)^j)}{2} \vec{\partial}^j B(M_1^2, \cdots, M_{L+1}^2; 1, \cdots, 1) \,. \tag{5.8}$$

As an illustration of these relations, let us consider some special cases.

- At the four-loop level ($L = 4$), there are 11 irreducible master integrals, and the symmetric set of master integrals includes the diagram $B(\{M_i^2\};\{1\})$ (1 term) and its second symmetric derivatives with respect to masses, $\frac{\partial^2}{\partial M_i^2 \partial M_j^2} B(\{M_i^2\};\{1\})$, where $i < j$ and $i, j = 1, \cdots, 5$ (10 terms).
- At the five-loop level ($L = 5$), there are 26 irreducible master integrals, and the symmetric set of master integrals include their first derivatives with respect to masses, $\frac{\partial}{\partial M_i^2} B(\{M_i^2\};\{1\})$, where $j = 1, \cdots, 6$ (6 terms), and their third symmetric derivatives with respect to masses, $\frac{\partial^3}{\partial M_i^2 \partial M_j^2 \partial M_k^2} B(\{M_i^2\};\{1\})$, where $i < j < k$ and $i, j, k = 1, \cdots, 6$ (20 terms).
- At the six-loop level ($L = 6$), there are 57 irreducible master-integrals, and the symmetric set of master integrals include the diagram $B(\{M_i^2\};\{1\})$ (1 term), its second and fourth symmetric derivatives with respect to masses, $\frac{\partial^2}{\partial M_i^2 \partial M_j^2} B(\{M_i^2\};\{1\})$ and $\frac{\partial^4}{\partial M_i^2 \partial M_j^2 \partial M_k^2 \partial M_r^2} B(\{M_i^2\};\{1\})$, where $i < j < k < r$ and $i, j, k, r = 1, \cdots, 7$ (21 and 35 terms, respectively).

5.1 *L*-loop bubble diagram with *R* massive lines

Let us consider the L-loop bubble diagram for the case where only R lines ($R \leq L$) have different masses,

$$B_R(M_1^2, \cdots, M_R^2; \{\alpha_j\}, \{\beta_k\}) = \int \frac{d^n(k_1 \cdots k_L)}{[k_1^2 - M_1^2]^{\alpha_1} \cdots [k_R^2 - M_R^2]^{\alpha_R} [k_{R+1}^2]^{\beta_1} \cdots [(k_1 - \cdots - k_L)^2]^{\beta_{L+1-R}}} \,. \tag{5.9}$$

The Mellin–Barnes representation of this diagram is

$$\begin{aligned} B_R(M_1^2, \cdots, M_R^2; \{\alpha_j\}, \{\beta_k\}) = (-M_R^2)^{\frac{n}{2}L-\alpha-\beta} \frac{[i^{1-n}\pi^{n/2}]^L}{\Gamma\left(\frac{n}{2}\right)\Gamma(\alpha_{L+1})} \times \left\{ \prod_{k=1}^{L+1-R} \frac{\Gamma\left(\frac{n}{2}-\beta_k\right)}{\Gamma(\beta_k)} \right\} \\ \times \frac{\Gamma\left(\beta - \frac{n}{2}(L-R)\right)}{\Gamma\left(\frac{n}{2}(L-R+1)-\beta\right)} \times \int \left\{ \prod_{j=1}^{R-1} dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2}-\alpha_j-t_j\right)}{\Gamma(\alpha_j)} \left(\frac{M_j^2}{M_R}\right)^{t_j} \right\} \\ \times \Gamma\left(\alpha+\beta-\frac{n}{2}(L-1)+t\right) \Gamma\left(\alpha+\beta+\alpha_R-\frac{n}{2}L+t\right) \,, \end{aligned} \tag{5.10}$$

where

$$\alpha = \sum_{j=1}^{R-1} \alpha_j \,, \quad \beta = \sum_{j=1}^{L-R+1} \beta_j \,, \quad t = \sum_{j=1}^{R-1} t_j \,.$$

In terms of the notations of Section 2, we have

$$A = \alpha + \beta - \frac{n}{2}(L-1)\ , \quad D = \alpha + \beta + \alpha_R - \frac{n}{2}L\ , \quad C_j = \frac{n}{2} - \alpha_j\ , \quad j = 1, \cdots, R-1.$$

Applying again the algorithm described in Section 3, we get

$$\frac{n}{2}(L-1) - \sum_{S_1} \frac{n}{2} = 0 \pmod{\mathbb{Z}}\ , \tag{5.11a}$$

$$\frac{n}{2}L - \sum_{S_2} \frac{n}{2} = 0 \pmod{\mathbb{Z}}\ , \tag{5.11b}$$

where S_1 and S_2 are any subsets of $1, \ldots, R-1$, and n is non-integer. There is only one solution for the subset S_1 (if $R = L$), and there is no solution for Eq. (5.11b). In this way, among the 2^{R-1} solutions, there is only one Puiseux-type solution (if $R = L$). Then we have the following theorem:

The number $B_{L,R}$ of irreducible master integrals of the L-loop bubble diagram with R massive lines ($R \leq L$) is equal to

$$\boxed{B_{L,R} = 2^{R-1} - \delta_{0,L-R}}\ . \tag{5.12}$$

Example:

$$B_{2,2} = 1\ , \quad B_{3,3} = 3\ , \quad B_{3,2} = 2\ , \quad B_{4,4} = 7\ , \quad B_{4,3} = 4\ , \quad B_{4,2} = 2\ .$$

6 Independent verification

Fortunately, there are a few others ways to cross-check our key results, Eqs. (4.5), (4.13), (5.4), and (5.12). One is based on the reduction of sunrise and bubble diagrams to some bases by using IBP relations [41]. However, there is no guarantee that the freely available programs will perform the complete reduction (see the discussions in Refs. [14, 43, 44]).

Another way of cross-checking our result is to apply the Lee-Pomenransky approach [45] (see also the discussion in Ref. [46]). This algorithm is based on counting the critical points of the sum of the Symanzik polynomials $F + U$ defined by Eq. (1.2). As was pointed out by the authors of Ref. [45], there are situations where their program does not reproduce the correct number of master integrals. Here, we present a pedagogical example in which differential reduction allows us to predict and construct an algebraic relation between two master integrals [43], which is not predictable by the algorithm described in Ref. [45] or by the program `LiteRed` [47].[12] Let us consider the two-loop sunrise diagram with arbitrary kinematics. The sum of the corresponding Symanzik polynomials has the following form:

$$G \equiv F + U = z_1 z_2 + z_1 z_3 + z_2 z_3 + z_1 z_2 z_3 p^2 - (z_1 z_2 + z_1 z_3 + z_2 z_3)\left(z_1 M_1^2 + z_2 M_2^2 + z_3 M_3^2\right)\ .$$

In this case, there are eight critical points, defined by the conditions $G = 0$ and $\partial_{z_i} G = 0$, $i = 1, 2, 3$. Four of them are trivial, $G(\{q_1, q_2, q_3, q_4\}) = 0$, where $q_1 = (1/M_1^2, 0, 0)$,

[12] For the considered example, the diagram `J011`, the program `Azurite` [48] also produces two irreducible master integrals. We thank the authors of Ref. [48] for confirming and verifying this result.

$q_2 = (0, 1/M_2^2, 0)$, $q_3 = (0, 0, 1/M_3^2)$, and $q_4 = (0, 0, 0)$. The remaining four points are algebraically independent for a generic set of masses and momenta, so that there are four independent master integrals, in agreement with the result of Ref. [36] . The number and the values of critical points do not depend on the values of α_i (power of propagators) and the dimension of space-time n, and the product of one-loop bubble diagrams does not enter the counting of master integrals.

Let us consider as another particular case the two-loop on-shell diagram, which is denoted as `J011` in Ref. [43], with $M_3 = 0$ and all other masses on-mass shell, $M_1^2 = M_2^2 = p^2 = 1$. In this case, there are six non-degenerate[13] critical points,

$$
\begin{aligned}
&q_1 = (0,0,0), \quad q_2 = (1,0,0), \quad q_3 = (0,1,0), \\
&q_4 = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right), \quad q_5 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right), \quad q_6 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right),
\end{aligned}
$$

and

$$
G(q_1) = G(q_2) = G(q_3) = 0; \quad G(q_4) = G(q_5) = G(q_6) = -\frac{4}{27}.
$$

The index[14] of points $q_1 = q_2 = q_3$ is equal to 2, and the index of points $q_4 = q_5 = q_6$ is equal to 1. According to the criteria suggested in Ref. [45], there are two independent critical points, q_4 and q_6, and there are two master integrals which are not reducible to products of one-loop bubble diagrams. However, as was shown in Refs. [43, 49], there is only one non-trivial master integral in this case, the second one only being a product of Γ functions.[15]

The difference between the differential-reduction technique and the counting of the critical points is related to the treatment of the values of the propagator powers and the space-time dimension. In the framework of the differential-reduction technique, one can consider the parameters α_i (powers of propagators) as well as the dimension of space-time n as non-integer, and, as a consequence, the number of additional differential equations for the Mellin–Barnes integrals should be different in this case. In the Lee-Pomeransky approach, the dimension of space-time and the powers of the propagators do not enter the analysis. Nevertheless, if α_i are integer, then the results of our evaluations and those of the application of `Mint` should be equal. Roman Lee kindly agreed to cross-check our results by the help of his packages [45, 47] and got full agreement with our expressions through $L = 5$ for the sunrise diagrams and through $L = 6$ for the bubble diagrams.

7 Conclusions

The method of counting master integrals described in Refs. [18, 19] was applied to the multivariable case with reducible monodromy. In contrast to our previous considerations

[13]The corresponding Hessian matrix is non-singular at these points.

[14] The index of a critical point is the dimension of the negative eigenspace of the corresponding Hessian matrix at this point.

[15]Roman Lee made the following comment on this: since the algebraic relation between the master integrals established in Refs. [14, 43] does not follow from IBP relations related to the sunrise diagram [49], our results do not contradict the algorithm described in Ref. [45].

[43, 50], the case where some additional differential equations are generated simultaneously was analysed in the present paper. Our technique is based on the methods developed for the analysis of the monodromy of GKZ hypergeometric functions [24, 33–35].

For completeness, we recall the three basic steps of our algorithm: (i) get a system of linear PDEs for a given Feynman diagram via its Mellin-Barnes representation, which does not exploit IBP relations [41]; (ii) evaluate the holonomic rank (including zero and Puiseux-type solutions) of the system of PDEs by the help of the prolongation procedure; (iii) evaluate the dimension of the invariant subspace of the differential contiguous operators.

As a demonstration of the validity of this technique applied to Feynman diagrams, the numbers of irreducible master integrals for the L-loop sunrise and bubble diagrams with generic sets of masses and momenta were evaluated (see Eqs. (4.5), (4.13), (5.4), and (5.12)). In the considered cases, the non-integer value of the space-time dimension n serves as a regulator which allows us to count the number of independent solutions.

As a by-product, we discovered the following interesting consequences of our analysis:

- As follows from Eq. (4.6), the bases for L-loop sunrise diagrams can be constructed in two equivalent ways. In the first set, sunrise diagrams with higher symmetric derivatives with respect to masses, $\frac{\partial^J}{\partial M_{i_1}^2 \partial M_{i_2}^2 \cdots \partial M_{i_J}^2} J^{(L)}(\vec{M}_j^2; \vec{1}; p^2)$, where $J = L, L+1$ and $i_1 < i_2 < \cdots < i_J$, can be excluded (see Eq. (4.6)). In the second set, the diagram with unit powers of propagators and its first derivatives with respect to masses can be excluded in favor of higher derivatives (see Eq. (4.9)). In both cases, the numbers of basic elements coincide with Eq. (4.5), and the bases do not include the diagrams with powers of propagator larger than 2.

- As follows from Eq. (5.4), the bases for L-loop bubble diagrams have the following structure: for even numbers of loops $L = 2, 4, 6, \cdots$, the basis includes the diagram with unit powers of propagators, $B(\{M_i^2\}; \{1\})$, and its even-order symmetric derivatives with respect to masses (see Eq. (5.7)). For odd numbers of loops $L = 3, 5, 7, \cdots$, the basis includes only odd-order symmetric derivatives with respect to masses (see Eq. (5.8)), but does not include the original diagram.

- As follows from Eq. (4.13), a sunrise diagram with R massive and two or more massless lines, or with one "dressed" massless line, is irreducible, and its holonomic rank near $\vec{z} = 0$ coincides with the holonomic rank of the hypergeometric function $F_C^{(R)}$ with irreducible monodromy. The latter property is relevant for the analysis of special functions generated by the ε expansions of multiloop sunrise diagrams [11, 13, 51].

A L-loop V-type diagram

In recent analyses aiming at finding the full sets of IBP relations for the complete reduction of Feynman diagram to minimal sets of master integrals [14, 19, 49, 52], diagrams of V type (see Fig. 3) have played an important role. In this section, we prove the following theorem: **For generic values of masses and external momenta, the V-type diagrams (see Fig. 3) are reducible to sunrise diagrams with the same masses and momenta.**

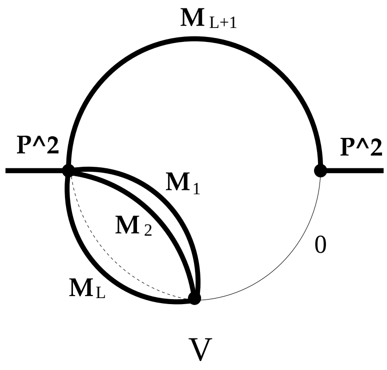

Figure 3. *L*-loop V-type diagram.

Let us consider the L-loop V-type diagram defined in momentum space as

$$V^{(L)}(M_1^2, \cdots, M_{L+1}^2, \alpha_1, \cdots, \alpha_{L+1}; \sigma; p^2)$$
$$= \int \frac{d^n(k_1 \cdots k_L)}{[k_1^2 - M_1^2]^{\alpha_1} \cdots [(k_1 - \cdots - k_L)^2 - M_L^2]^{\alpha_L} [(p-k_L)^2 - M_{L+1}^2]^{\alpha_{L+1}} [k_L^2]^\sigma} \; . \tag{A.1}$$

The Mellin–Barnes representation of this diagram is

$$V^{(L)}(M_1^2, \cdots, M_{L+1}^2, \alpha_1, \cdots, \alpha_{L+1}; \sigma; p^2) = (p^2)^{\frac{n}{2}L - \alpha - \alpha_{L+1} - \beta - \sigma} [i^{1-n} \pi^{n/2}]^L$$
$$\times \int \left\{ \prod_{j=1}^{L+1} dt_j \frac{\Gamma(-t_j)\Gamma\left(\frac{n}{2} - \alpha_j - t_j\right)}{\Gamma(\alpha_j)} \left(-\frac{M_j^2}{p^2} \right)^{t_j} \right\} \frac{\Gamma\left(\frac{n}{2}L - \alpha - \sigma - t\right) \Gamma\left(\alpha - \frac{n}{2}(L-1) + t\right)}{\Gamma\left(\frac{n}{2}L - \alpha - t\right) \Gamma\left(\alpha + \sigma - \frac{n}{2}(L-1) + t\right)}$$
$$\times \frac{\Gamma\left(\alpha + \alpha_{L+1} + \sigma - \frac{n}{2}L + t + t_{L+1}\right)}{\Gamma\left(\frac{n}{2}(L+1) - \alpha - \alpha_{L+1} - \sigma - t - t_{L+1}\right)} \; , \tag{A.2}$$

where

$$L \geq 2 \; , \quad \alpha = \sum_{j=1}^{L} \alpha_j \; , \quad t = \sum_{j=1}^{L} t_j \; .$$

For $\sigma = 0$, the V-type diagram coincides with the L-loop sunrise diagram. The integral in Eq. (A.2) includes the following ratio of Γ functions with arguments differing by integers:

$$\frac{\Gamma\left(\frac{n}{2}L - \alpha - \sigma - t\right)}{\Gamma\left(\frac{n}{2}L - \alpha - t\right)} \times \frac{\Gamma\left(\alpha - \frac{n}{2}(L-1) + t\right)}{\Gamma\left(\alpha + \sigma - \frac{n}{2}(L-1) + t\right)} \; .$$

Let us introduce two differential operators, K_1 and K_2, defined as

$$K_1 : \prod_{j=0}^{\sigma-1} [(\sum_{k=1}^{L} \theta_k + b + j)] \; , \quad K_2 : \prod_{j=1}^{\sigma} [(-\sum_{k=1}^{L} \theta_k + a - j)] \; ,$$

where $a = \frac{n}{2}L - \alpha$ and $b = \alpha - \frac{n}{2}(L-1)$. Applying these operators to Eq. (A.2), we get

$$(K_1 \circ K_2)\, V^{(L)} = J^{(L)},$$

where $J^{(L)}$ is the sunrise diagram defined by Eq. (4.1). Indeed, we have

$$K_1 \circ K_2 \left(\prod_{j=1}^{L} z_j^{t_j} \left(\frac{\Gamma\,(b{+}t)}{\Gamma\,(b{+}\sigma{+}t)} \right) \left(\frac{\Gamma\,(a{-}\sigma{-}t)}{\Gamma\,(a{-}t)} \right) \right) = \prod_{j=1}^{L} z_j^{t_j} \,.$$

The operators K_1 and K_2 are products of differential operators of the first order, so that their inverse operators correspond to one-fold integrals over linear forms. Integrals of this type convert Puiseux-type solutions of diagram $J^{(L)}$ into Puiseux-type solutions of diagram $V^{(L)}$. As a consequence, the dimension of the space of nontrivial solutions of the differential operators related to the V-type diagram coincides with the dimension of the space of nontrivial solutions of the differential operators related to the sunrise diagram. □

B Sunrise diagram and Bessel functions

Let us consider the one-fold integral representation of the sunrise diagram according to Ref. [4] (see also Ref. [6]). Using the Fourier transform of the massive propagator in Euclidean space-time,

$$\begin{aligned} \Delta(x,M) \equiv \int d^n k \frac{\exp\,(-ikx)}{(k^2+M^2)^\alpha} &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \left(\frac{x}{2}\right)^{\alpha-\frac{n}{2}} M^{\frac{n}{2}-\alpha} K_{\frac{n}{2}-\alpha}(Mx) \\ &\sim \left(\frac{M}{x}\right)^{\frac{n}{2}-\alpha} K_{\frac{n}{2}-\alpha}(Mx)\,, \end{aligned} \tag{B.1}$$

where $K_\nu(z)$ is the MacDonald function (for details, see Section 3.7 in Ref. [53]) and n is non-integer, and performing the angular integration in n dimensions,

$$\int d^n\hat{x}\, \exp\,(-ikx) = 2\pi^{\frac{n}{2}} \left(\frac{x\sqrt{-p^2}}{2} \right)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(x\sqrt{-p^2})\,,$$

where $J_\nu(z)$ is the Bessel function, we find the one-fold integral representation of the massive sunrise diagram in Euclidean space-time to be

$$\begin{aligned} J^E(\vec{M_j^2};\vec{\alpha_j};p^2) &= \int \prod_{j=1}^{L} \frac{d^n k_j}{[k_j^2{+}M_j^2]^{\alpha_j}} \times \frac{1}{[(p{-}k_1{-}\cdots{-}k_L)^2{+}M_{L+1}^2]^{\alpha_{L+1}}}\,, \\ &\sim \int_0^\infty \frac{dt}{t^{\frac{n}{2}L-\sum_{k=1}^{L+1}\alpha_k}} \times J_{\frac{n}{2}-1}(t\sqrt{-p^2}) \times \prod_{j=1}^{L+1} K_{\frac{n}{2}-\alpha_j}(M_j t)\,. \end{aligned} \tag{B.2}$$

For the massless propagator, we have

$$\begin{aligned} \lim_{z\to 0} K_\nu(z) &\to \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu} \Gamma(\nu) + O(z^{-\nu+1})\,, \\ \Delta(x,0) &\equiv \int d^n k \frac{\exp\,(-ikx)}{(k^2)^\alpha} \sim \left(\frac{1}{x^2}\right)^{\frac{n}{2}-\alpha}\,, \end{aligned}$$

so that

$$
\begin{aligned}
J_R^E(\vec{M}_j^2;\vec{\alpha_j},\vec{\beta_i};p^2) &= \int \prod_{j=1}^{R} \frac{d^n(k_1\cdots k_L)}{[k_j^2+M_k^2]^{\alpha_j}[k_{R+1}^2]^{\beta_1}\cdots[(p-k_1-\cdots-k_L)^2]^{\beta_{L+1-R}}} \\
&\sim \int_0^\infty \frac{dt}{t^{\frac{d}{2}L-\sum_{k=1}^R \alpha_k-\sum_{j=1}^{L+1-R}\beta_j}} \times J_{\frac{n}{2}-1}(t\sqrt{-p^2}) \times \prod_{j=1}^{R} K_{\frac{n}{2}-\alpha_j}(M_j t) \, .
\end{aligned}
\tag{B.3}
$$

The recurrence relations for the MacDonald functions,

$$
K_{\nu-1}(z) = K_{\nu+1}(z) - \frac{2}{z}\nu K_\nu(z) \, , \quad \frac{d}{dz}K_\nu(z) = -\frac{1}{2}\left(K_{\nu-1}(z)+K_{\nu+1}(z)\right) \, ,
$$

in combination with recurrence relations for the Bessel functions,

$$
J_{\nu-1}(z) = -J_{\nu+1}(z) + \frac{2}{z}\nu J_\nu(z) \, , \quad \frac{d}{dz}J_\nu(z) = \frac{1}{2}\left(J_{\nu-1}(z)-J_{\nu+1}(z)\right) \, ,
$$

allow us to change the power of t and the orders of the Bessel and MacDonald functions to any integer values. The result of Section 3 can be applied to evaluate the dimension of the basis within this reduction. If all the values of α_j and β_k are integer, then the results of Section 3 are valid. In particular, the difference between the reducible and irreducible monodromies of the integrals defined by Eqs. (B.3) and (B.3), respectively, is an extra integer power of the variable t. There are a few other cases where the results of Section 3 are applicable, namely, different combinations of integer and non-integer values of α_i and β_j. In the following, we present a few examples.

1. Under the conditions that

   - $\frac{n}{2} \notin \mathbb{Z}$,
   - $\alpha_j \notin \mathbb{Z}$, for $\forall j = 1,\cdots,L+1$,
   - $\frac{n}{2} - \alpha_a \notin \mathbb{Z}$, for $\forall a = 1,\cdots,L+1$,
   - $\sum_S \alpha_k \notin Z$, for any subset of $\forall k = 1,\cdots,R$,

   there is an invariant subspace of dimension 1 for the integral defined by Eq. (B.3). Indeed, in this case, only one equation, $\frac{n}{2}(L+1) - \sum_{S_2}\frac{n}{2} = 0 \pmod{\mathbb{Z}}$, is valid. In this case, the number of irreducible integrals defined by Eq. (B.3) is equal to $2^{L+1}-1$.

2. Under the conditions that

   - $n \in \mathbb{Z}$,
   - $\beta_a \in \mathbb{Z}$, $\forall a = 1,\cdots,L+1-R$,
   - $\alpha_j \notin \mathbb{Z}$, for $\forall j = 1,\cdots,R$
   - $\sum_S \alpha_k \notin Z$, for any subset of $\forall k = 1,\cdots,R$,

   there is an invariant subspace of dimension 1 for the integral defined by Eq. (B.3).

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