HYPERDIRE
HYPERgeometric functions DIfferential REduction: Mathematica-based packages for the differential reduction of generalized hypergeometric functions: Lauricella function F_C of three variables

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Abstract

We present a further extension of the HYPERDIRE project, which is devoted to the creation of a set of Mathematica-based program packages for manipulations with Horn-type hypergeometric functions on the basis of differential equations. Specifically, we present the implementation of the differential reduction for the Lauricella function F_C of three variables.

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PROGRAM SUMMARY

Program title: HYPERDIRE

Version: 3.0.0

Release: 1.0.0

Catalogue identifyer:

Program summary URL:

Program obtainable from: `https://sites.google.com/site/loopcalculations/home`

Licensing provisions: GNU General Public Licence

No. of lines in distributed program, including test data etc.:

No. of bytes in distributed program, including test data etc.:

Distribution format: `tar.gz`

Programming language: Mathematica.

Computer: All computers running Mathematica.

Operating systems: Operating systems running Mathematica.

Classification:

Does the new version supersede the previous version?: No, it significantly extends the previous version.

Keywords: Feynman integrals, Generalized hypergeometric functions, Differential reduction.

Nature of the problem: Reduction of hypergeometric function F_C of three variables to a set of basis functions.

Solution method: Differential reduction.

Restriction on the complexity of the problem: None.

Reasons for new version: The extension package allows the user to handle the Lauricella function F_C of three variable.

Summary of revisions: The previous version goes unchanged.

Running time: Depends on the complexity of the problem.

1 Introduction

Multiloop and/or multileg Feynman diagrams as well as phase space integrals in covariant gauge within dimensional regularization [1] can be written in terms of generalized hypergeometric functions. The creation of the HYPERDIRE program packages [2–5] is motivated by the importance of Horn-type hypergeometric functions for the analytical evaluations of Feynman diagrams, especially at the one-loop level [6]. Possible applications of the differential-reduction algorithm to Feynman diagrams beyond the one-loop level were discussed in Ref. [7].

A Feynman diagram may be written in the form of a Mellin-Barnes integral [8], which depends on external kinematic invariants, the dimension n of space-time, and the powers of the propagators. Upon application of Cauchy's theorem, the Feynman integral can be converted into a linear combination of multiple series:

$$\Phi(n, \vec{x}) \sim \sum_{k_1, \cdots, k_{r+m}=0}^{\infty} \prod_{a,b} \frac{\Gamma(\sum_{i=1}^{m} A_{ai}k_i + B_a)}{\Gamma(\sum_{j=1}^{r} C_{bj}k_j + D_b)} x_1^{k_1} \cdots x_{r+m}^{k_{r+m}} \ , \tag{1}$$

where x_i are some rational functions of Mandelstam variables and A_{ai}, B_a, C_{bj}, D_b are linear functions of the space-time dimension and the propagator powers. The representation of Eq. (1) corresponds to a Horn-type hypergeometric series [9] if the hidden index of the summation is considered as an independent variable.

In general, the multiple series

$$H(\vec{z}) = \sum_{\vec{m}=0}^{\infty} C(\vec{m}) \vec{z}^{\,\vec{m}} \ , \tag{2}$$

where $\vec{m} = (m_1, \ldots, m_r)$ and $\vec{z}^{\,\vec{m}} = z_1^{m_1} \cdots z_r^{m_r}$, are called *Horn-type hypergeometric* if, for each $i = 1, \ldots, r$, the ratio $C(\vec{m} + \vec{e}_j)/C(\vec{m})$, where $\vec{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ is the jth unit vector, is a rational function of the summation indices, i.e.

$$\frac{C(\vec{m} + \vec{e}_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} \ , \tag{3}$$

where $P_j(\vec{m})$ and $Q_j(\vec{m})$ are polynomials [9, 10]. In explicit form, the coefficients $C(\vec{m})$ can then be written as

$$C(\vec{m}) = \prod_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \frac{\prod_{j=1}^{N} \Gamma(\vec{\mu}_j \cdot \vec{m} + \gamma_j)}{\prod_{k=1}^{M} \Gamma(\vec{\nu}_k \cdot \vec{m} + \delta_k)} \ , \tag{4}$$

where $N, M \geq 0$, $\lambda_i, \gamma_j, \delta_k$ are arbitrary complex numbers, $\vec{\mu}_j, \vec{\nu}_k$ are arbitrary integer-valued vectors, and R is an arbitrary rational function.

From the condition (3) on the coefficients $C(\vec{m})$ of the Horn-type hypergeometric function $H(\vec{z})$, we can derive the following proper system of partial differential equations (PDEs):

$$\left[Q_j(\vec{\theta}) \frac{1}{z_j} - P_j(\vec{\theta}) \right] H(\vec{z}) = 0 \ , \tag{5}$$

where $j = 1, \ldots, r$, $\vec{\theta} = (\theta_1, \ldots, \theta_r)$, and θ_k is the differential operator

$$\theta_k = z_k \frac{\partial}{\partial z_k} \; . \tag{6}$$

In previous publications [2–5], we presented the Mathematica-based [11] package HY-PERDIRE for the differential reduction of Horn-type hypergeometric functions. In Ref. [2], we implemented the reduction of the Horn-type hypergeometric functions $_{p+1}F_p$ of one variable to restricted sets of basis functions and predicted the numbers of such functions. We demonstrated that the differential-reduction algorithm can be used for the reduction of Feynman diagrams without resorting to the integration-by-parts technique. We established and implemented the criterion of reducibility of the Horn-type hypergeometric functions $_{p+1}F_p$ to simpler functions for special values of parameters. Subsequently, we developed the HY-PERDIRE project further to cover the full set of Horn-type hypergeometric functions of two variables [4], including the Appell functions F_1, F_2, F_3, and F_4 [3], and also certain Horn-type hypergeometric functions of three variables, namely F_D and F_S [5]. In this paper, we discuss the case of the Horn-type hypergeometric function F_C of three variable, which appears, e.g., in the calculation of two-loop bubble-type Feynman diagram with different masses. With the implementation of F_C, F_D, and F_S, we start to study the applicability of the differential-reduction method to the set of Lauricella–Saran hypergeometric functions of three variables.

2 Differential reduction of Horn-type hypergeometric functions

Let us now consider the Horn-type hypergeometric function $H(\vec{z}) = H(\vec{\gamma}; \vec{\sigma}; \vec{z})$, which explicitly depends on a set of *contiguous* variables, $\vec{z} = (z_1, \ldots, z_k)$, and two sets of *discrete* variables, $\vec{\gamma} = (\gamma_1, \ldots, \gamma_i)$ and $\vec{\sigma} = (\sigma_1, \ldots, \sigma_j)$, which are called *upper* and *lower* parameters, respectively.

In Refs. [12, 13], it was shown that there exist unique linear differential operators which can generate identities called contiguous or ladder relations between the hypergeometric function $H(\vec{\gamma}; \vec{\sigma}; \vec{z})$ and its counterparts with one of the upper (lower) parameters shifted by unity, namely

$$H(\vec{\gamma} + \vec{e_c}; \vec{\sigma}; \vec{x}) = \frac{1}{\gamma_c} \left(\sum_{a=1}^{r} \mu_{ca} \theta_a + \gamma_c \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) = U_{\gamma_c}^+ H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; , \tag{7}$$

$$H(\vec{\gamma}; \vec{\sigma} - \vec{e_c}; \vec{x}) = \frac{1}{\sigma_c - 1} \left(\sum_{b=1}^{r} \nu_{cb} \theta_b + \sigma_c - 1 \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) = U_{\sigma_c}^- H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; . \tag{8}$$

The direct operators $U_{\gamma_c}^+$ and $U_{\sigma_c}^-$ are called *step-up* and *step-down* operators for the upper and lower indices, respectively. It is possible to construct the inverse differential operators

$U_{\gamma_c}^-$ and $U_{\sigma_c}^+$ satisfying

$$\left[U_{\gamma_c}^- U_{\gamma_c}^+\right] H(\vec{\gamma}; \vec{\sigma}; \vec{x}) = H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \,,$$
$$\left[U_{\sigma_c}^+ U_{\sigma_c}^-\right] H(\vec{\gamma}; \vec{\sigma}; \vec{x}) = H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \,. \tag{9}$$

Once these operators are constructed, we can combine them to shift the parameters of the Horn-type hypergeometric function by any integer, i.e. to obtain contiguous relations of the form

$$\left[U^- U^+\right] H(\vec{\gamma}; \vec{\sigma}; \vec{x}) = H(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{x}) \,. \tag{10}$$

The process of applying $U_{\gamma_c}^\pm$ and $U_{\sigma_c}^\pm$ to a Horn-type hypergeometric function to shift its parameters by integers is called *differential reduction*. In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts,

$$H(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{x}) = \prod_{i,j,m,n} U_{\gamma_i}^+ U_{\gamma_j}^- U_{\sigma_m}^+ U_{\sigma_n}^- H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \,. \tag{11}$$

The development of systematic techniques for the solution of contiguous relations has a long history. It was started by Gauss, who described the reduction of the hypergeometric function $_2F_1$ in 1823 [14]. Numerous papers have since then been published on this problem [15]. An algorithmic solution was found by Takayama in Ref. [13], and this method was later extended in a series of publications [16] (see also Refs. [17]).

Previously, it was pointed out [7] that the differential-reduction algorithm in Eq. (11) can be applied to the reduction of Feynman diagrams to some subsets of basis hypergeometric functions with well-known analytical properties and that the system of differential equations in Eq. (5) can also be used for the construction of so-called ε expansions of hypergeometric functions about rational values of their parameters via direct solutions of the systems of differential equations.

[5].

3   Lauricella function F_C

The Lauricella function F_C of three variables [18] is defined as a Taylor expansion about the point $\vec{z} = \vec{0}$ as follows:

$$F_C^{(3)}(a, b; c_1, c_2, c_3; z_1, z_2, z_3) = \sum_{m_1, m_2, m_3 = 0}^{\infty} \frac{(a)_{m_1+m_2+m_3} (b)_{m_1+m_2+m_3}}{(c_1)_{m_1} (c_2)_{m_2} (c_3)_{m_3}} \frac{z_1^{m_1} z_2^{m_2} z_3^{m_3}}{m_1! m_2! m_3!} \,, \tag{12}$$

where $(a)_m = (a + m - 1)!/(a - 1)!$ is the Pochhammer symbol. The corresponding PDEs of Eq. (5) read:

$$\frac{1}{z_i} \theta_i (c_i - 1 + \theta_i) F_C(\vec{z}) = (a + \theta_1 + \theta_2 + \theta_3)(b + \theta_1 + \theta_2 + \theta_3) F_C(\vec{z}) \,, \qquad i = 1, 2, 3 \,, \tag{13}$$

where we have used the short-hand notation $F_C(\vec{z}) = F_C^{(3)}(a, b; c_1, c_2, c_3; z_1, z_2, z_3)$. The canonical form of Eq. (13) reads:

$$\theta_1^2 F_C(\vec{z}) = \frac{1}{D_0}\left\{-abz_1 + [(a+b)z_1 + (z_2 + z_3 - 1)(c_1 - 1)]\theta_1\right.$$

$$\left. - z_1 \sum_{i\in\{2,3\}} (1 + a + b - c_i)\theta_i + z_1 \sum_{i\neq j}^3 \theta_i\theta_j\right\} F_C(\vec{z}),$$

$$\theta_2^2 F_C(\vec{z}) = \frac{1}{D_0}\left\{-abz_2 + [(a+b)z_2 + (z_1 + z_3 - 1)(c_2 - 1)]\theta_2\right.$$

$$\left. - z_2 \sum_{i\in\{1,3\}} (1 + a + b - c_i)\theta_i + z_2 \sum_{i\neq j}^3 \theta_i\theta_j\right\} F_C(\vec{z}),$$

$$\theta_3^2 F_C(\vec{z}) = \frac{1}{D_0}\left\{-abz_3 + [(a+b)z_3 + (z_1 + z_3 - 1)(c_3 - 1)]\theta_3\right.$$

$$\left. - z_3 \sum_{i\in\{1,2\}} (1 + a + b - c_i)\theta_i + z_3 \sum_{i\neq j}^3 \theta_i\theta_j\right\} F_C(\vec{z}), \tag{14}$$

where $D_0 = 1 - z_1 - z_2 - z_3$, which can be written in compact form as

$$L_i F_C(\vec{z}) = \theta_i^2 F_C(\vec{z}) = \left(\sum_{i\neq j=1}^3 P_{ij}\theta_i\theta_j + \sum_{m=1}^3 R_{im}\theta_m + S_i\right) F_C(\vec{z}), \qquad i = 1, 2, 3. \tag{15}$$

Here, we can define the conditions of complete integrability,

$$\theta_i \left[\theta_j L_k\right] F_C(\vec{z}) = \theta_j \left[\theta_i L_k\right] F_C(\vec{z}), \qquad i, j, k = 1, 2, 3. \tag{16}$$

Eq. (16) does not provide new independent conditions between the differential operators θ_i. Thus Eq. (15) can be reduced to the following Pfaff system of eight independent differential equations:

$$d\vec{f} = R\vec{f}, \tag{17}$$

where $\vec{f} = (F_C(\vec{z}), \theta_1 F_C(\vec{z}), \theta_2 F_C(\vec{z}), \theta_3 F_C(\vec{z}), \theta_1\theta_2 F_C(\vec{z}), \theta_1\theta_3 F_C(\vec{z}), \theta_2\theta_3 F_C(\vec{z}), \theta_1\theta_2\theta_3 F_C(\vec{z}))$.

3.1 Differential reduction of F_C

In the case of the Lauricella function $F_C(\vec{z})$, the direct differential operators for the upper parameters in Eq. (7) read:

$$F_C(a + 1, b; c_1, c_2, c_3; \vec{z}) = U_a^+ F_C(\vec{z}) = \frac{1}{a}(a + \theta_1 + \theta_2 + \theta_3) F_C(\vec{z}),$$

$$F_C(a, b + 1; c_1, c_2, c_3; \vec{z}) = U_b^+ F_C(\vec{z}) = \frac{1}{b}(b + \theta_1 + \theta_2 + \theta_3) F_C(\vec{z}), \tag{18}$$

and those for the lower parameters in Eq. (8) read:

$$F_C(a, b; c_1 - 1, c_2, c_3; \vec{z}) = U_{c_1}^- F_C(\vec{z}) = \frac{1}{c_1 - 1}(c_1 - 1 + \theta_1) F_C(\vec{z}) ,$$

$$F_C(a, b; c_1, c_2 - 1, c_3; \vec{z}) = U_{c_2}^- F_C(\vec{z}) = \frac{1}{c_2 - 1}(c_2 - 1 + \theta_2) F_C(\vec{z}) ,$$

$$F_C(a, b; c_1, c_2, c_3 - 1; \vec{z}) = U_{c_3}^- F_C(\vec{z}) = \frac{1}{c_3 - 1}(c_3 - 1 + \theta_3) F_C(\vec{z}) . \tag{19}$$

As explained above, we can determine the corresponding inverse differential operators, U_a^- and U_b^-, through eight independent solutions of Eq. (17),

$$\begin{aligned} F_C(a - 1, b; c_1, c_2, c_3; \vec{z}) &= U_a^- F_C(\vec{z}) , \\ F_C(a, b - 1; c_1, c_2, c_3; \vec{z}) &= U_b^- F_C(\vec{z}) , \end{aligned} \tag{20}$$

where

$$U_x^- = A_x + B_x\theta_1 + C_x\theta_2 + D_x\theta_3 + E_x\theta_1\theta_2 + F_x\theta_1\theta_3 + G_x\theta_2\theta_3 + H_x\theta_1\theta_2\theta_3 , \tag{21}$$

with $x = a, b$. Similar solutions can be obtained for the inverse differential operators $U_{c_i}^+$ ($i = 1, 2, 3$) with eight independent functions in the form of Eq. (21).

By using the definitions of the inverse operators in Eqs. (9) and (21), we can explicitly obtain the following equation for the coefficients $\vec{A}_x = (A_x, B_x, C_x, D_x, E_x, F_x, G_x, H_x)$:

$$f_0(\vec{A}_x) + \sum_{i=1}^{3} f_i(\vec{A}_x)\theta_i + \sum_{i,j=1}^{3} f_{ij}(\vec{A}_x)\theta_i\theta_j + \sum_{i \neq j=1}^{3} f_{ijj}(\vec{A}_x)\theta_i\theta_j^2 + f_{123}(\vec{A}_x)\theta_1\theta_2\theta_3$$

$$+ \sum_{i=1}^{3} f_{123i}(\vec{A}_x)\theta_1\theta_2\theta_3\theta_i = 1 , \tag{22}$$

where the coefficients $f_{\ldots}(\vec{A}_x)$ are linear maps of $\vec{A}_x$ and rational functions of the discrete and continuous variables of $F_C(\vec{z})$.

Multiplying the PDEs for $F_C(\vec{z})$ in Eq. (15) by different powers of θ_i, we can eliminate higher powers of θ_i in Eq. (22) and write it using a minimal set of eight independent terms,

$$F_0(\vec{A}_x) + \sum_{i=1}^{3} F_i(\vec{A}_x)\theta_i + \sum_{i \neq j=1}^{3} F_{ij}(\vec{A}_x)\theta_i\theta_j + F_{123}(\vec{A}_x)\theta_1\theta_2\theta_3 = 1 . \tag{23}$$

Setting in turn $F_0(\vec{A}_x) = 1$, $F_i(\vec{A}_x) = 0$, $F_{ij}(\vec{A}_x) = 0$, and $F_{123}(\vec{A}_x) = 0$, we obtain eight equations for the variables $\vec{A}_x$, and, by solving this system, we can obtain the inverse operators in the form

$$U_x^{\text{inv}} = \frac{1}{D_x^{\text{discr}} D_x^{\text{cont}}}(A_x' + B_x'\theta_1 + C_x'\theta_2 + D_x'\theta_3 + E_x'\theta_1\theta_2 + F_x'\theta_1\theta_3 + G_x'\theta_2\theta_3 + H_x'\theta_1\theta_2\theta_3) , \tag{24}$$

7

where D_x^{discr} and D_x^{cont} are polynomials in the discrete variables a, b, c_1, c_2, c_3 and the continuous variables $\vec{z}$, respectively, and $\vec{A}'_x$ are some rather cumbersome polynomials in the variables of $F_C(\vec{z})$. Specifically, we have

$$
\begin{aligned}
D_a^{\text{discr}} &= \prod_{i\neq j=1}^{3} (1 + a - c_i)(2 + a - c_i - c_j)(3 + a - c_1 - c_2 - c_3) \,, \\
D_b^{\text{discr}} &= D_a^{\text{discr}}|_{a\to b} \,, \\
D_{c_i}^{\text{discr}} &= \prod_{p\in\{a,b\}} \prod_{j\neq i=1}^{3} (1 + p - c_i)(2 + p - c_i - c_j)(3 + p - c_1 - c_2 - c_3) \,.
\end{aligned}
\tag{25}
$$

The denominator D_a^{cont} coincides with the surfaces of the singularities of the PDE system for $F_C(\vec{z})$ with three variables,

$$
\begin{aligned}
D_a^{\text{cont}} &= \left[-1 + \sum_{i}^{3}(3z_i - 3z_i^2 + z_i^3) - \sum_{i\neq j=1}^{3}(z_i^2 z_j + z_i z_j) + 10 z_1 z_2 z_3 \right] (1 - z_1 - z_2 - z_3) \,, \\
D_a^{\text{cont}} &= D_b^{\text{cont}} = D_{c_i}^{\text{cont}}, \qquad i = 1, 2, 3 \,.
\end{aligned}
\tag{26}
$$

It is well known that, in the limit $z_i \to 0$, $F_C(\vec{z})$ degenerates to the Lauricella function of two variables, $F_4(a, b, c_1, c_2, \vec{z})$. So, by taking the limit $z_i \to 0$ in Eq. (24), we can obtain the corresponding inverse operators [3] for F_4, with a reduced number of independent functions,

$$
U_x^{\text{inv}}|_{z_i\to 0} = \frac{1}{D_{i,x}^{\text{discr}} D_{i,x}^{\text{cont}}} \left(A'_{i,x} + \sum_{j\neq i=1}^{3} B'_{i,x}\theta_j + \sum_{j\neq i\neq k=1}^{3} E'_{i,j,x}\theta_j\theta_k \right) \,.
\tag{27}
$$

In Eq. (27), the expressions for $D_{i,x}^{\text{discr}}$ are obtained from Eq. (25) by putting all the factors involving the variable c_i to unity, and $D_{i,x}^{\text{cont}} = 1 - \sum_{j\neq i=1}^{3} x_j$.

Using the explicit forms of the direct and inverse operators in Eqs. (18), (19), and (24) and eliminating the higher powers of θ_i via the same procedure as in Eq. (23), we can write the results of the differential reduction according to Eq. (11) in the following form:

$$
\begin{aligned}
&F_C(a + n_1, b + n_2; c_1 + m_1, c_2 + m_2, c_3 + m_3; \vec{z}) \\
&= \left[S_0(\vec{z}) + \sum_{i} S_i(\vec{z})\frac{\partial}{\partial z_i} + \sum_{i\neq j} S_{ij}(\vec{z})\frac{\partial^2}{\partial z_i \partial z_j} + S_{123}(\vec{z})\frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} \right] F_C(\vec{z}) \,,
\end{aligned}
\tag{28}
$$

where n_i and m_i is a set of integers, and S, S_j, S_{ij} are polynomials in z_i and the discrete variables of $F_C(\vec{z})$.

It is easy to see that, if one of the factors in the denominator D_x^{discr} is equal to zero, we obtain from Eq. (22) some new PDE identities,

$$
(A'_x + B'_x\theta_1 + C'_x\theta_2 + D'_x\theta_3 + E'_x\theta_1\theta_2 + F'_x\theta_1\theta_3 + G'_x\theta_2\theta_3 + H'_x\theta_1\theta_2\theta_3)F_C(\vec{z}) = 0 \,.
\tag{29}
$$

Eq. (29) means that the hypergeometric functions entering Eq. (17) are expressible in terms of simpler hypergeometric functions, e.g. Gauss hypergeometric functions, and corresponds to the condition of reducibility of the monodromy group of $F_C(\vec{z})$. As a consequence, the inverse operators in Eq. (24) and the differential-reduction algorithm in Eq. (28) can be expressed in a simpler form involving just seven and six independent functions, respectively.

4 FcFunction — Mathematica-based program for the differential reduction of the Lauricella function F_C

In this section, we present the Mathematica-based[3] program package **FcFunction** for the differential reduction of the Lauricella function $F_C(\vec{z})$ of three variables, which is freely available from Ref. [19]. It allows one to automatically perform the differential reduction in accordance with Eq. (28). Its current version only handles non-exceptional parameter values.

The file `readme.txt` provides a brief description of the installiation and usage of the program package **FcFunction**. The main package file `FcFunction.m` contains the general definitions of the differential-reduction formulas. All the cumbersome formulas needed for shifting the values of single parameters are accommodated in additional files that are gzipped and end with `*.m.gz`. The file `example-FcFunction.m` includes the example calculations explained in subsection 4.3.

4.1 Input format

The program package **FcFunction** may be loaded in the standard way:

$$<< \text{"FcFunction.m"}$$

It includes the following basic routines for the Lauricella function $F_C(\vec{z})$:

$$\textbf{FcIndexChange}[\text{changingVector}, \text{parameterVector}] \tag{30}$$

and

$$\textbf{FcSeries}[\ldots], \tag{31}$$

where "parameterVector" defines the list of parameters of that function and "changingVector" defines the set of integers by which the values of these parameters are to be shifted, i.e. the vector pairs $(\vec{\gamma}, \vec{\sigma})$ and $(\vec{k}, \vec{l})$ in Eq. (11), respectively. For example, the operator:

$$\textbf{FcIndexChange}[\{1, -1, 0, 0, 2\}, \{a, b, c_1, c_2, c_3, z_1, z_2, z_3\}] \tag{32}$$

shifts the arguments of the function $F_C(a, b; c_1, c_2, c_3; z_1, z_2, z_3)$ so as to generate $F_C(a+1, b-1; c_1, c_2, c_3 + 2; z_1, z_2, z_3)$.

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[3]This program package was tested using Mathematica 8.0 [11].

The function **FcSeries**[...] is designed for the numerical evaluation of $F_C(\vec{z})$ and its derivatives. It returns the values of the Taylor series of $F_C(\vec{z})$ in Eq. (12) and its derivatives upon the commands:

$$\textbf{FcSeries}[\text{vectorInit}, \text{numbSer}],$$
$$\textbf{FcSeries}[\text{numberOfvariable}, \text{vectorInit}, \text{numbSer}], \tag{33}$$

respectively, where "numberOfvariable" is the list of the variables with respect to which to differentiate, "vectorInit" is the set of parameters of $F_C(\vec{z})$, and "numbSer" is the number of terms to be retained in the Taylor expansion.

4.2   Output format

The output structure of all the operators of the program package **FcFunction** in Eq. (30) is as follows:

$$\{\{Q_0, Q_1, Q_2, Q_3, Q_{12}, Q_{13}, Q_{23}, Q_{123}\}, \{\text{parameterVectorNew}\}\}, \tag{34}$$

where "parameterVectorNew" is the new set of parameters of $F_C(\vec{z})$, i.e. $(\vec{\gamma} + \vec{k}, \vec{\sigma} + \vec{l})$ in Eq. (11), and $Q_0, Q_1, Q_2, \ldots, Q_{123}$ are the rational coefficient functions of the differential operators in Eq. (21), so that

$$
\begin{aligned}
F_C(\vec{\gamma}; \vec{\sigma}; \vec{z}) &= (Q_0 + Q_1\theta_1 + Q_2\theta_2 + Q_3\theta_3 + Q_{12}\theta_1\theta_2 + Q_{13}\theta_1\theta_3 + Q_{23}\theta_2\theta_3 \\
&\quad + Q_{123}\theta_1\theta_2\theta_3)F_C(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{z}).
\end{aligned}
\tag{35}
$$

4.3   Examples

Example 1:[4] Reduction of the Lauricella function $F_C(a, b; c_1, c_2, c_3; z_1, z_2, z_3)$.
FcIndexChange$[\{-1,0,1,0,0\}, \{a,b,c_1,c_2,c_3,z_1,z_2,z_3\}]$

$$
\begin{aligned}
&\left\{\left\{1 - \frac{bz_1}{c_1(z_1 + z_2 + z_3 - 1)}, \frac{(c_1 - b)z_1 + (a - 1)(z_2 + z_3 - 1)}{(a-1)c_1(z_1 + z_2 + z_3 - 1)}, \frac{1 - \frac{(a+b-c_2)z_1}{c_1(z_1+z_2+z_3-1)}}{a-1}, \right.\right. \\
&\frac{1 - \frac{(a+b-c_3)z_1}{c_1(z_1+z_2+z_3-1)}}{a-1}, \frac{-z_1 + z_2 + z_3 - 1}{(a-1)c_1(z_1 + z_2 + z_3 - 1)}, \frac{-z_1 + z_2 + z_3 - 1}{(a-1)c_1(z_1 + z_2 + z_3 - 1)}, \\
&\left.\left. -\frac{2z_1}{(a-1)c_1(z_1 + z_2 + z_3 - 1)}, 0\right\}, \{a - 1, b, c_1 + 1, c_2, c_3, z_1, z_2, z_3\}\right\}
\end{aligned}
\tag{36}
$$

---

[4]All functions in the program package HYPERDIRE generate output without additional simplification for maximum efficiency of the algorithm. To get the output in a simpler form, we recommend to use the command `Simplify` in addition.

In explicit form, this reads:

$$
\begin{aligned}
F_c&(a, b; c_1, c_2, c_3; z_1, z_2, z_3) \\
&= \left[1 - \frac{bz_1}{c_1(z_1+z_2+z_3-1)} + \frac{(c_1-b)z_1+(a-1)(z_2+z_3-1)}{(a-1)c_1(z_1+z_2+z_3-1)}\theta_1 \right. \\
&\quad + \frac{1-\frac{(a+b-c_2)z_1}{c_1(z_1+z_2+z_3-1)}}{a-1}\theta_2 + \frac{1-\frac{(a+b-c_3)z_1}{c_1(z_1+z_2+z_3-1)}}{a-1}\theta_3 + \frac{-z_1+z_2+z_3-1}{(a-1)c_1(z_1+z_2+z_3-1)}\theta_1\theta_2 \\
&\quad \left. + \frac{-z_1+z_2+z_3-1}{(a-1)c_1(z_1+z_2+z_3-1)}\theta_1\theta_3 - \frac{2z_1}{(a-1)c_1(z_1+z_2+z_3-1)}\theta_2\theta_3 \right] \\
&\quad \times F_c(a-1, b; c_1+1, c_2, c_3; z_1, z_2, z_3) .
\end{aligned}
\tag{37}
$$

The functions in Eq. (33) allow us to expand the results as formal Taylor series in the variables z_i about zero and to analytically check the results of the differential reduction in Eq. (37). For example, $F_C(a-1, b; c_1+1, c_2, c_3; z_1, z_2, z_3)$ and $\theta_1\theta_2 F_C(a-1, b; c_1+1, c_2, c_3; z_1, z_2, z_3)$ may be Taylor expanded through order ten as:

$$
\begin{aligned}
&\mathbf{FcSeries}[a-1, b, c_1+1, c_2, c_3; z_1, z_2, z_3, 10] , \\
&\mathbf{FcSeries}[\{1, 1, 0\}, a-1, b, c_1+1, c_2, c_3; z_1, z_2, z_3, 10] ,
\end{aligned}
\tag{38}
$$

respectively. Eq. (33) is also useful for numerical estimations of the Lauricella function $F_C(\vec{z})$ and its derivatives near the point $\vec{z} = \vec{0}$. However, the user has to control the convergence of the Taylor series and the accuracy of the numerical evaluation. Specifically, he has to ensure that the condition $\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} < 1$ is satisfied. Here are two examples:

$$
\begin{aligned}
&\mathbf{FcSeries}[\{1+\varepsilon, 2+\varepsilon, 4+3\varepsilon, 6+7\varepsilon, 3+3\varepsilon, 0.1, 0.2, 0.15\}, 10]//.\varepsilon \to 0.1 \\
&\mathbf{FcSeries}[\{1, 0, 0\}\{1+\varepsilon, 2+\varepsilon, 4+3\varepsilon, 6+7\varepsilon, 3+3\varepsilon, 0.1, 0.2, 0.15\}, 10]//.\varepsilon \to 0.1 \\
&1.34179 \\
&1.35774
\end{aligned}
\tag{39}
$$

Example 2: Reduction of the Lauricella function $F_C(1+\varepsilon, 1+2\varepsilon; 3\varepsilon, 4\varepsilon, 5\varepsilon; z_1, z_2, z_3)$.
$\mathbf{FcIndexChange}[\{0,0,2,0,0\}, \{1+\varepsilon, 1+2\varepsilon, 3\varepsilon, 4\varepsilon, 5\varepsilon, z_1, z_2, z_3\}]$

$$\left\{\left\{1 - \frac{(\epsilon+1)(2\epsilon+1)z_1}{3\epsilon(3\epsilon+1)(z_1+z_2+z_3-1)}, \frac{3 - \frac{z_1}{\epsilon(z_1+z_2+z_3-1)}}{3(3\epsilon+1)}, \frac{(\epsilon-3)z_1}{3\epsilon(3\epsilon+1)(z_1+z_2+z_3-1)},\right.\right.$$

$$\frac{(2\epsilon-3)z_1}{3\epsilon(3\epsilon+1)(z_1+z_2+z_3-1)}, -\frac{2z_1}{3(3\epsilon^2+\epsilon)(z_1+z_2+z_3-1)},$$

$$\left.-\frac{2z_1}{3(3\epsilon^2+\epsilon)(z_1+z_2+z_3-1)}, -\frac{2z_1}{3(3\epsilon^2+\epsilon)(z_1+z_2+z_3-1)}, 0\right\},$$

$$\left.\{\epsilon+1, 2\epsilon+1, 3\epsilon+2, 4\epsilon, 5\epsilon, z_1, z_2, z_3\}\right\} \tag{40}$$

This corresponds to the following mathematical formula:

$$F_C(1+\varepsilon, 1+2\varepsilon; 3\varepsilon, 4\varepsilon, 5\varepsilon; z_1, z_2, z_3)$$

$$= \left[1 - \frac{(\epsilon+1)(2\epsilon+1)z_1}{3\epsilon(3\epsilon+1)(z_1+z_2+z_3-1)} + \frac{3 - \frac{z_1}{\epsilon(z_1+z_2+z_3-1)}}{3(3\epsilon+1)}\theta_1\right.$$

$$+ \frac{(\epsilon-3)z_1}{3\epsilon(3\epsilon+1)(z_1+z_2+z_3-1)}\theta_2 + \frac{(2\epsilon-3)z_1}{3\epsilon(3\epsilon+1)(z_1+z_2+z_3-1)}\theta_3$$

$$- \frac{2z_1}{3(3\epsilon^2+\epsilon)(z_1+z_2+z_3-1)}\theta_1\theta_2 - \frac{2z_1}{3(3\epsilon^2+\epsilon)(z_1+z_2+z_3-1)}\theta_1\theta_3$$

$$\left.- \frac{2z_1}{3(3\epsilon^2+\epsilon)(z_1+z_2+z_3-1)}\theta_2\theta_3\right] F_c(1+\varepsilon, 1+2\varepsilon; 2+3\varepsilon, 4\varepsilon, 5\varepsilon; z_1, z_2, z_3). \tag{41}$$

5 Conclusions

The differential-reduction algorithm [12] allows one to relate Horn-type hypergeometric functions with parameters whose values differ by integers. In this paper, we presented a further extension of the Mathematica-based [11] program package HYPERDIRE [2–5] for the differential reduction of generalized hypergeometric functions to sets of basis functions by including the Lauricella function $F_C(\vec{z})$ [18] of three variables. We intend to complete the treatment of the Lauricella functions of three variables in the future.

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