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α_s from hadron multiplicities via SUSY-like relation between anomalous dimensions

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We recover in QCD an amazingly simple relationship between the anomalous dimensions, resummed through next-to-next-to-leading-logarithmic order, in the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution equations for the first Mellin moments $D_{q,g}(\mu^2)$ of the quark and gluon fragmentation functions, which correspond to the average hadron multiplicities in jets initiated by quarks and gluons, respectively. This relationship, which is independent of the number of quark flavors, dramatically improves previous treatments by allowing for an exact solution of the evolution equations. So far, such relationships have only been known from supersymmetric QCD, where $C_F/C_A = 1$. This also allows us to extend our knowledge of the ratio $D_g^-(\mu^2)/D_q^-(\mu^2)$ of the minus components by one order in $\sqrt{\alpha_s}$. Exploiting available next-to-next-to-next-to-leading-order information on the ratio $D_g^+(\mu^2)/D_q^+(\mu^2)$ of the dominant plus components, we fit the world data of $D_{q,g}(\mu^2)$ for charged hadrons measured in e^+e^- annihilation to obtain $\alpha_s^{(5)}(M_Z) = 0.1205\,^{+0.016}_{-0.0020}$.

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In the parton model of QCD [1], the inclusive production of single hadrons involves the notion of fragmentation functions $D_a(x,\mu^2)$, where μ is the factorization scale. At leading order (LO), their values correspond to the probability for a parton $a = q, g$ produced at short distance $\hbar c/\sqrt{\mu^2}$ to produce a jet that contains a hadron carrying the fraction x of the momentum of parton a. Owing to the factorization theorem, the $D_a(x,\mu^2)$ functions are universal in the sense that they do not depend on the process by which parton a is produced. By local parton-hadron duality [2], there should be a local correspondence between parton and hadron distributions in hard-scattering processes. Yet, $D_a(x,\mu^2)$ are genuinely nonperturbative, which implies that their x dependences at some scale μ_0 cannot be calculated from the QCD Lagrangian using perturbation theory, but need to be determined by fitting experimental data. However, once $D_a(x,\mu_0^2)$ are assumed to be known, their μ^2 dependences are governed by the timelike Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [3, 4]. The anomalous dimensions therein, the $a \to b$ splitting functions $P_{ba}(x)$, are known at next-to-next-to-leading order [5]. The scaling violations, *i.e.*, the μ^2 dependences, of $D_a(x,\mu^2)$ may be exploited in global data fits to extract the strong-coupling constant $\alpha_s = g_s^2/(4\pi)$, leading to very competitive results [6] as for the world average [7].

The DGLAP equations are conveniently solved in Mellin space, where $D_a(N,\mu^2) = \int dx\, x^{N-1} D_a(x,\mu^2)$ with $N = 1, 2, \ldots$ and similarly for $P_{ba}(x)$, because convolutions are converted to products. We have

$$\frac{\mu^2 d}{d\mu^2}\begin{pmatrix} D_s(N,\mu^2) \\ D_g(N,\mu^2)\end{pmatrix} = \begin{pmatrix} P_{qq}(N) & P_{gq}(N) \\ P_{qg}(N) & P_{gg}(N)\end{pmatrix}\begin{pmatrix} D_s(N,\mu^2) \\ D_g(N,\mu^2)\end{pmatrix}, \tag{1}$$

where $D_s = (1/2n_f)\sum_{q=1}^{n_f}(D_q + D_{\bar{q}})$, with n_f being the number of active quark flavors, is the quark singlet component. The quark non-singlet component, which is irrelevant for the following, obeys a decoupled DGLAP equation. After solving the DGLAP equations in Mellin space, one returns to x space via the inverse Mellin transform, analytically continuing N to complex values.

The first Mellin moment $D_a(\mu^2) \equiv D_a(1,\mu^2)$ is of particular interest in its own right because, up to corrections of orders beyond our consideration here, it corresponds to the average hadron multiplicity $\langle n_h \rangle_a$ of jets initiated by parton a. There exists a wealth of experimental data on $\langle n_h \rangle_q$, $\langle n_h \rangle_g$, and their ratio $r = \langle n_h \rangle_g / \langle n_h \rangle_q$ for charged hadrons h taken in e^+e^- annihilation at various center-of-mass energies $\sqrt{s}$, ranging from 10 to 209 GeV (for a comprehensive compilation of experimental publications, see Ref. [8]), which allows for a high-precision determination of α_s [8, 9]. In fact, besides α_s and ignoring power corrections for the time being, there are just two more fit parameters, $D_q(\mu_0^2)$ and $D_g(\mu_0^2)$ at some reference scale μ_0, which have a very clear and simple physical interpretation, while no input from external sources, *e.g.*, parton density functions, is required. This provides a strong motivation for us to deepen our theoretical understanding of D_a within the QCD formalism as much as possible, which is actually limiting the error in the value of α_s thus extracted. The study of D_a is a topic of old vintage; the LO value of r, $C^{-1} = C_A/C_F$ with

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color factors $C_F = 4/3$ and $C_A = 3$, was found four decades ago [10]. Subsequent analyses [9, 11] were performed using the generating-functional approach in the modified leading-logarithmic approximation (MLLA) [12].

The description of the μ^2 dependences of D_a at fixed order in perturbation theory are spoiled by the fact that $P_{ba} \equiv P_{ba}(1)$ are ill defined and require resummation, which was performed for the leading logarithms (LL) [13], the next-to-leading logarithms (NLL) [14], and the next-to-next-to-leading logarithms (NNLL) [15]. In Ref. [8], Eq. (1) is first diagonalized for arbitrary value of N at LO, and then the NNLL resummation is incorporated. Unfortunately, this two-step procedure, which has been standard practice in the literature so far [16, 17], fails to fully exploit the available knowledge on the higher-order corrections and yields an approximation, the uncertainty of which is difficult to estimate reliably.

In this Letter, we expose a relationship between the NNLL-resummed expressions for P_{ba}, which has gone unnoticed so far. Its existence in QCD is quite remarkable and interesting in its own right, because a similar relationship is familiar from supersymmetric (SUSY) QCD, where $C = 1$ [4, 12, 15, 18]. Owing to this new relationship, the DGLAP equations may be solved exactly, which greatly consolidates the theoretical foundation for the determination of α_s and thus reduces its theoretical uncertainty.

Our starting point is Eq. (1) for $N = 1$ with NNLL resummation. We have [15]

$$\begin{aligned} P_{aa} &= \gamma_0(\delta_{ag} + K_a^{(1)}\gamma_0 + K_a^{(2)}\gamma_0^2) + \mathscr{O}(\gamma_0^4) \qquad (a = q, g), \\ P_{gq} &= C(P_{gg} + A) + \mathscr{O}(\gamma_0^4), \\ P_{qg} &= C^{-1}(P_{qq} + A) + \mathscr{O}(\gamma_0^4), \end{aligned} \tag{2}$$

where $\gamma_0 = \sqrt{2C_A a_s}$, with $a_s = \alpha_s/(4\pi)$ being the couplant, δ_{ab} is the Kronecker symbol, and

$$\begin{aligned} K_q^{(1)} &= \frac{2}{3}C\varphi, \quad K_q^{(2)} = -\frac{1}{6}C\varphi[17 - 2\varphi(1 - 2C)], \\ K_g^{(1)} &= -\frac{1}{12}[11 + 2\varphi(1 + 6C)], \quad K_g^{(2)} = \frac{1193}{288} - 2\zeta(2) - \frac{5\varphi}{72}(7 - 38C) + \frac{\varphi^2}{72}(1 - 2C)(1 - 18C), \\ A &= K_q^{(1)}\gamma_0^2, \quad \varphi = \frac{2n_f T_R}{C_A}, \quad T_R = \frac{1}{2}. \end{aligned} \tag{3}$$

Eq. (2) is written in a form that allows us to glean a novel relationship:

$$C^{-1}P_{gq} - P_{gg} = CP_{qg} - P_{qq}, \tag{4}$$

which is independent of n_f. Eq. (4) generalizes the case of SUSY QCD [4, 12, 15, 18] from $C = 1$ to $C = 9/4$. The corresponding relation in $\mathscr{N} = 1$ SUSY [4] is known to be violated beyond LO [5]. It will be interesting to see if Eq. (4) also holds beyond $\mathscr{O}(\gamma_0^3)$.

We now solve Eq. (1) exactly by exploiting Eq. (4). To this end, we diagonalize the NNLL DGLAP evolution kernel as

$$U^{-1}\begin{pmatrix} P_{qq} & P_{gq} \\ P_{qg} & P_{gg} \end{pmatrix} U = \begin{pmatrix} P_{--} & 0 \\ 0 & P_{++} \end{pmatrix}, \tag{5}$$

by means of the matrices [16]

$$U = \begin{pmatrix} 1 & -1 \\ \frac{1-\alpha}{\varepsilon} & \frac{\alpha}{\varepsilon} \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} \alpha & \varepsilon \\ \alpha - 1 & \varepsilon \end{pmatrix}, \tag{6}$$

where

$$\alpha = \frac{P_{qq} - P_{++}}{P_{--} - P_{++}}, \qquad \varepsilon = \frac{P_{gq}}{P_{--} - P_{++}}, \tag{7}$$

$$P_{\pm\pm} = \frac{1}{2}\left[P_{qq} + P_{gg} \pm \sqrt{(P_{qq} - P_{gg})^2 + 4P_{qg}P_{gq}}\right]. \tag{8}$$

Eq. (1) thus assumes the form

$$\frac{\mu^2 d}{d\mu^2}\begin{pmatrix} D_- \\ D_+ \end{pmatrix} = \left[\begin{pmatrix} P_{--} & 0 \\ 0 & P_{++} \end{pmatrix} - U^{-1}\frac{\mu^2 d}{d\mu^2}U\right]\begin{pmatrix} D_- \\ D_+ \end{pmatrix}, \tag{9}$$

where the second term contained within the square brackets stems from the commutator of $\mu^2 d/d\mu^2$ and U, and

$$\begin{pmatrix} D_- \\ D_+ \end{pmatrix} = U^{-1} \begin{pmatrix} D_s \\ D_g \end{pmatrix} = \begin{pmatrix} (\alpha-1)D_s + \varepsilon D_g \\ \alpha D_s + \varepsilon D_g \end{pmatrix}. \tag{10}$$

Owing to Eq. (4), the square root in Eq. (8) disappears, and we have

$$P_{--} = -A, \qquad P_{++} = P_{qq} + P_{gg} + A, \tag{11}$$

$$\alpha = \frac{P_{gg}+A}{P_{qq}+P_{gg}+2A}, \qquad \varepsilon = -C\alpha\,. \tag{12}$$

Inserting the second equality of Eq. (12) in Eq. (6), we have

$$U^{-1}\frac{\mu^2 d}{d\mu^2}U = -\frac{1}{\alpha}\,\frac{\mu^2 d}{d\mu^2}\alpha \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

Using the QCD β function,

$$\frac{\mu^2 d}{d\mu^2}a_s = \beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 + \mathscr{O}(a_s^4), \tag{14}$$

with one- and two-loop coefficients

$$\beta_0 = \frac{C_A}{3}(11-2\varphi), \quad \beta_1 = \frac{2C_A^2}{3}[17-\varphi(5+3C)], \tag{15}$$

we may convert the differential operator as

$$\frac{\mu^2 d}{d\mu^2} = \frac{C_A}{\gamma_0}\beta\left(\frac{\gamma_0^2}{2C_A}\right)\frac{d}{d\gamma_0}. \tag{16}$$

Inserting Eqs. (2) and (3) in the first equality of Eq. (12), we have $\alpha = 1 - 4C\varphi\gamma_0/3 + \mathscr{O}(\gamma_0^2)$, so that

$$\frac{1}{\alpha}\,\frac{\mu^2 d}{d\mu^2}\alpha = \frac{C\varphi\beta_0}{3C_A}\gamma_0^3 + \mathscr{O}(\gamma_0^4). \tag{17}$$

Inserting Eqs. (11), (13), and (17) in Eq. (9), we may cast Eq. (1) in its final form,

$$\frac{\mu^2 d}{d\mu^2}\begin{pmatrix} D_- \\ D_+ \end{pmatrix} = \begin{pmatrix} \frac{C\varphi\beta_0}{3C_A}\gamma_0^3 - A & 0 \\ \frac{C\varphi\beta_0}{3C_A}\gamma_0^3 & P_{gg}+P_{qq}+A \end{pmatrix}\begin{pmatrix} D_- \\ D_+ \end{pmatrix}. \tag{18}$$

The initial conditions are given by Eq. (10) for $\mu = \mu_0$ in terms of the three constants $\alpha_s(\mu_0^2)$, $D_s(\mu_0^2)$, and $D_g(\mu_0^2)$.

The solution of Eq. (18) is greatly facilitated by the fact that one entry of the matrix on its right-hand side is zero. We may thus obtain D_- as the general solution of a homogeneous differential equation,

$$\begin{aligned}\frac{D_-(\mu^2)}{D_-(\mu_0^2)} &= \exp\left[\int_{\mu_0^2}^{\mu^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\left(\frac{C\varphi\beta_0}{3C_A}\gamma_0^3 - A\right)\right] \\ &= \frac{T_-(\gamma_0(\mu^2))}{T_-(\gamma_0(\mu_0^2))},\end{aligned} \tag{19}$$

where, with the help of Eq. (16),

$$\begin{aligned}T_-(\gamma_0) &= \exp\left[\frac{4C\varphi}{3}\int d\gamma_0\left(\frac{2C_A}{\beta_0\gamma_0} - 1\right)\right] \\ &= \gamma_0^{d_-}\exp\left(-\frac{4}{3}C\varphi\gamma_0\right),\end{aligned} \tag{20}$$

with $d_- = 8C_A C\varphi/(3\beta_0)$. The small-$x$ correction $\propto \gamma_0$ in Eq. (20) originates from the extra term in Eq. (9) and represents a novel feature of our approach. In Ref. [8] and analogous analyses for parton distribution functions [19], the minus components do not participate in the resummation.

We are then left with an inhomogeneous differential equation for D_+. The general solution $\tilde{D}_+$ of its homogeneous part reads

$$\begin{aligned}\frac{\tilde{D}_+(\mu^2)}{\tilde{D}_+(\mu_0^2)} &= \exp\left[\int_{\mu_0^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\gamma_0\left(1+K_+^{(1)}\gamma_0+K_+^{(2)}\gamma_0^2\right)\right] \\ &= \frac{T_+(\gamma_0(\mu^2))}{T_+(\gamma_0(\mu_0^2))},\end{aligned} \tag{21}$$

where

$$\begin{aligned}K_+^{(1)} &= 2K_q^{(1)}+K_g^{(1)} = -\frac{1}{12}[11+2\varphi(1-2C)], \\ K_+^{(2)} &= K_q^{(2)}+K_g^{(2)} = \frac{1193}{288}-2\zeta(2)-\frac{7\varphi}{72}(5+2C)+\frac{\varphi^2}{72}(1-2C)(1+6C), \\ T_+(\gamma_0) &= \exp\left[-\frac{4C_A}{\beta_0}\int\frac{d\gamma_0}{\gamma_0^2}\,\frac{1+K_+^{(1)}\gamma_0+K_+^{(2)}\gamma_0^2}{1+b_1\gamma_0^2}\right] \\ &= \gamma_0^{d_+}\exp\left[\frac{4C_A}{\beta_0\gamma_0}-\frac{4C_A}{\beta_0}\left(K_+^{(2)}-b_1\right)\gamma_0\right],\end{aligned} \tag{22}$$

with $d_+ = -4C_AK_+^{(1)}/\beta_0$ and $b_1 = \beta_1/(2C_A\beta_0)$. Adding to $\tilde{D}_+$ a special solution of the inhomogeneous differential equation for D_+, we find its general solution to be

$$D_+(\mu^2) = \left[\frac{D_+(\mu_0^2)}{T_+(\gamma_0(\mu_0^2))}-\frac{4}{3}C\varphi\frac{D_-(\mu_0^2)}{T_-(\gamma_0(\mu_0^2))}\int_{\gamma_0(\mu_0^2)}^{\gamma_0(\mu^2)}\frac{d\gamma_0}{1+b_1\gamma_0^2}\,\frac{T_-(\gamma_0)}{T_+(\gamma_0)}\right]T_+(\gamma_0(\mu^2)). \tag{23}$$

The final expressions for D_- and D_+ in Eqs. (19) and (23), respectively, are fully renormalization group improved because all μ dependence resides in γ_0.

The NLL approximation is recovered by omitting the exponential factor multiplying $\gamma_0^{d_-}$ in Eq. (20), putting $K_+^{(2)} = b_1 = 0$ in Eq. (22), and omitting the second term within the square brackets in Eq. (23). The LL approximation then emerges from the NLL one by also putting $d_- = d_+ = 0$ in Eqs. (20) and (22), respectively. Hence follows the large-μ^2 asymptotic behavior $D_-/D_+ \propto \exp\{-[(8C_A/\beta_0)\ln(\mu^2/\Lambda^2)]^{1/2}\}$, where Λ is the asymptotic scale parameter, which implies a strong fading of D_-.

Using Eqs. (6) and (10), we now return to the parton basis, where it is useful to decompose $D_a = D_a^+ + D_a^-$ into the large and small components $D_a^\pm$ proportional to $D_\pm$, respectively. Defining $r_\pm = D_g^\pm/D_s^\pm$ and using Eqs. (2), (3), and (12), we then have $D_s^\pm = \mp D_\pm$ and

$$\begin{aligned}r_+ &= -\frac{\alpha}{\epsilon} = \frac{1}{C}+\mathscr{O}(\gamma_0^2), \\ r_- &= \frac{1-\alpha}{\epsilon} = -\frac{4}{3}\varphi\gamma_0+\frac{\varphi}{18}[29-2\varphi(5-2C)]\gamma_0^2+\mathscr{O}(\gamma_0^3).\end{aligned} \tag{24}$$

Recalling that $\langle n_h\rangle_q = D_s$ and $\langle n_h\rangle_g = D_g$, we thus have

$$r = \frac{r_+ + r_-D_s^-/D_s^+}{1+D_s^-/D_s^+}. \tag{25}$$

Eq. (24) differs from Eqs. (53) and (54) in Ref. [8],

$$\begin{aligned}\bar{r}_+ &= \frac{1}{C}\left\{1-\frac{\gamma_0}{3}[1+\phi(1-2C)]+\mathscr{O}(\gamma_0^2)\right\}, \\ \bar{r}_- &= -\frac{2}{3}\phi\gamma_0+\mathscr{O}(\gamma_0^2).\end{aligned} \tag{26}$$

On the other hand, $\bar{r}_+$ in Eq. (26) agrees with the result for r obtained in Ref. [20] in the approximation of putting $D_a^- = 0$ and extended to through $\mathscr{O}(\gamma_0^3)$ in Refs. [21, 22], which is in line with the reasoning in Chapter 7 of Ref. [12].

By the same token, we may accommodate the higher-order corrections [21, 22] by including within the curly brackets in Eq. (26) the terms $\bar{c}_2\gamma_0^2 + \bar{c}_3\gamma_0^3$, where

$$\bar{c}_2 = \frac{179}{72} - \frac{20}{9}\zeta(2) - \frac{355}{1944}n_f + \frac{43}{26244}n_f^2,$$
$$\bar{c}_3 = -\frac{5213}{1152} - \frac{8}{3}\zeta(2) + \frac{40}{9}\zeta(3) + \left(-\frac{9761}{31104} + \frac{14}{27}\zeta(2)\right)n_f + \frac{15595}{314928}n_f^2 - \frac{4799}{17006112}n_f^3. \tag{27}$$

For $n_f = 5$,

$$\bar{r}_+ = 2.250 - 0.889\,\gamma_0 - 4.593\,\gamma_0^2 + 0.740\,\gamma_0^3 + \mathscr{O}(\gamma_0^4). \tag{28}$$

The difference between $r_\pm$ and $\bar{r}_\pm$ is an artifact of the different diagonalization procedures adopted here and in Ref. [8]. In fact, taking the limit $N \to 1$ in $D_a(N, \mu^2)$ and diagonalizing the DGLAP equations are noncommuting operations. Consequently, our components D_a differ from those in Ref. [8], $\overline{D}_a$ with $\bar{r}_\pm = \overline{D}_g^\pm/\overline{D}_s^\pm$, by terms of $\mathscr{O}(\gamma_0)$. Specifically, we have

$$D_a^\pm = \sum_{b=s,g} M_{ab}\overline{D}_b^\pm, \tag{29}$$

where

$$M_{ss} = 1 - \frac{4}{3}C\varphi\gamma_0, \qquad M_{sg} = -\frac{C}{3}\gamma_0[1 + \varphi(1-6C)],$$
$$M_{gs} = -\frac{2}{3}\varphi\gamma_0, \qquad M_{gg} = 1 + \frac{2}{3}C\varphi\gamma_0. \tag{30}$$

In fact, this transformation converts $\bar{r}_\pm$ into $r_\pm$ and, by exploiting Eq. (27), allows us to extend our result for r_+ through $\mathscr{O}(\gamma_0^3)$; the counterpart of Eq. (28) reads

$$r_+ = 2.250 - 4.505\,\gamma_0^2 - 0.586\,\gamma_0^3 + \mathscr{O}(\gamma_0^4). \tag{31}$$

Note that our advanced procedure to solve Eq. (1) allows us to determine r_- through $\mathscr{O}(\gamma_0^2)$, while $\bar{r}_-$ from Ref. [8] is limited to $\mathscr{O}(\gamma_0)$. We denote the approximation of using Eq. (31) on top of Eqs. (24) and (25) as NNNLO$_{\text{approx}}$+NNLL.

Power-like corrections were found to be indispensable for a realistic description of the experimental data of $\langle n_h\rangle_q$, $\langle n_h\rangle_g$, and r [22, 23]. Following Refs. [22, 23], we include them by multiplying r_+ in Eq. (31) with the factor

$$1 + (1 + \frac{n_f}{27})\frac{\mu_{\text{cr}}}{\mu}\gamma_0, \tag{32}$$

where μ_{cr} is a critical scale parameter to be fitted. In the MLLA approach, $\mu_{\text{cr}} = K_{\text{cr}}\Lambda_{\text{QCD}}$ usually serves as the initial point of the evolution, which is implemented with the basic variables $Y = \ln(\mu/\mu_0)$ and $\lambda = \ln K_{\text{cr}}$. The most frequent choice, $\lambda = 0$, corresponds to the *limiting-spectrum* approximation [2]. Other recent choices include $\lambda = 1.4$ and $\lambda = 2.0$ [9]. Since logarithmic and powerlike corrections become comparable at small values of μ^2, a judicious choice of μ is important to prevent strong correlations. Motivated by Refs. [10, 24, 25], we choose $\mu^2 = R^2Q^2 + 4M_{\text{eff}}^2$, where R is the jet radius, $Q^2 = \sqrt{s}$, and M_{eff} is the effective gluon mass. We adopt $R = 0.3$ as a typical value from Ref. [24] and $M_{\text{eff}}(Q^2) = m^2/[1 + (Q^2/M^2)^\gamma]$ with $m = 0.375$ GeV, $M = 0.557$ GeV, and $\gamma = 1.06$ from Ref. [25].

We are now in a position to perform a global fit to the available measurements of $\langle n_h\rangle_q$ and $\langle n_h\rangle_g$ for changed hadrons h in e^+e^- annihilation, which were carefully compiled in Ref. [8]. They include 58 and 35 data points, respectively, and come from CLASSE CESR with $\sqrt{s} = 10$ GeV, SLAC PEP with 29 GeV, DESY PETRA with 12–47 GeV, KEK TRISTAN with 50–61 GeV, SLAC SLC with 91 GeV, CERN LEP1 with 91 GeV, and CERN LEP2 with 130–209 GeV. The jet algorithms adopted in these experimental analyses are mutually compatible [26]. As in Ref. [8], we choose the reference scale to be $Q_0 = 50$ GeV, which roughly corresponds to the geometric mean of the smallest and largest of the occurring $\sqrt{s}$ values, and put $n_f = 5$ throughout our analysis. As may be seen in Fig. 1, our NNNLO$_{\text{approx}}$ + NNLL fit yields an excellent description of the experimental data included in it, with a χ^2 per degree of freedom of $\chi^2_{\text{dof}} = 1.32$. The fit parameters are determined to be $\langle n_h(Q_0^2)\rangle_q = 16.38 \pm 0.05$, $\langle n_h(Q_0^2)\rangle_g = 23.87 \pm 0.07$, $K_{\text{cr}} = 7.09^{+1.71}_{-1.21}$, and

$$\alpha_s^{(5)}(M_Z^2) = 0.1205\begin{array}{l}+0.0016\\-0.0020\end{array}, \tag{33}$$

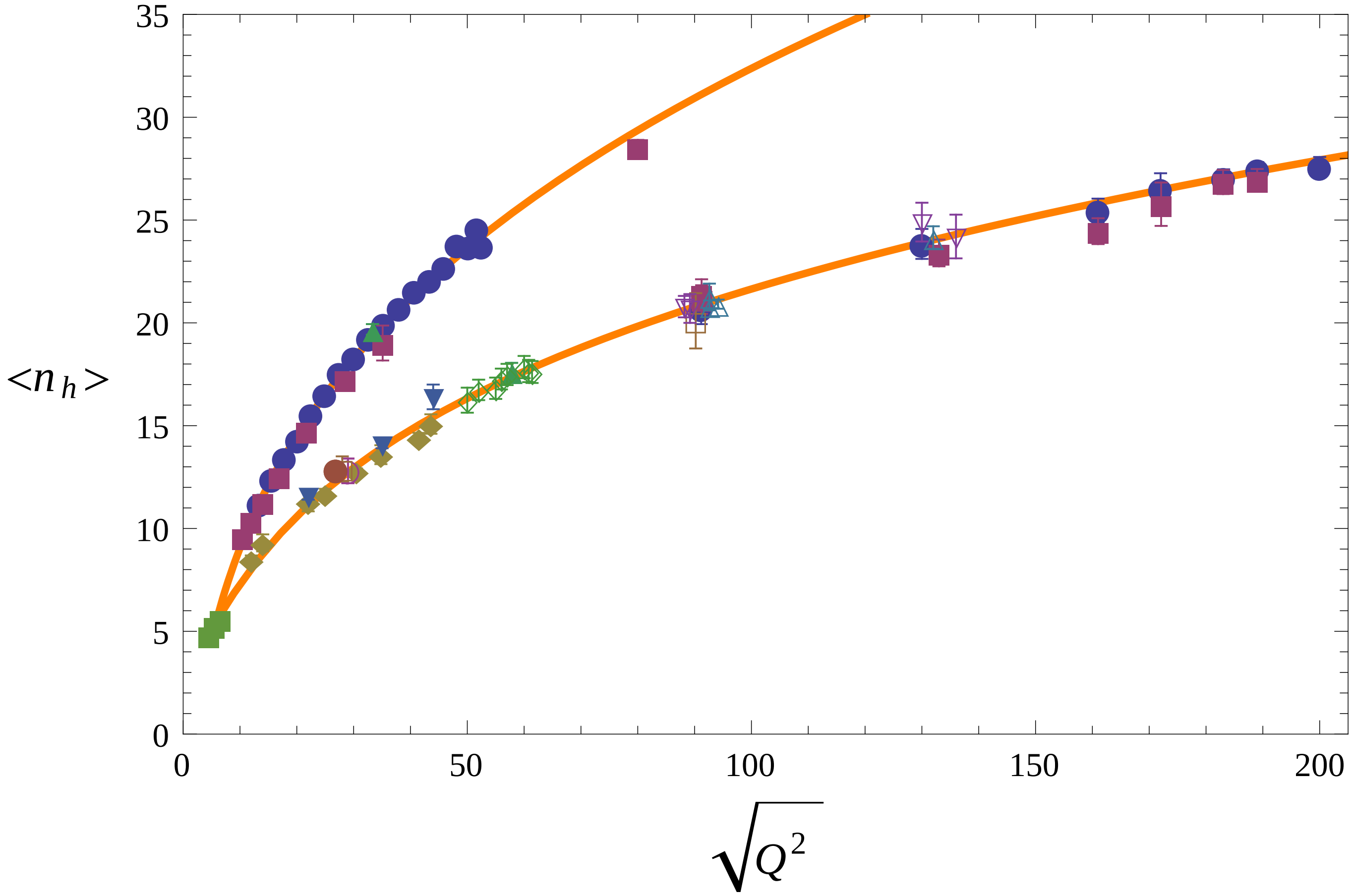

FIG. 1: Comparison of the experimental data of $\langle n_h(\mu^2)\rangle_q$ (lower curves) and $\langle n_h(\mu^2)\rangle_g$ (upper curves) with the $\mathrm{NNNLO_{approx}}+$ NNLL fit to them.

which nicely agrees with the present world average, $\alpha_s^{(5)}(M_Z^2) = 0.1181 \pm 0.0011$ [7]. Our fit results turn out to be very insensitive to the precise choice of Q_0. The power corrections turn out to be sizeable, with $\lambda = 1.96^{+0.21}_{-0.19}$, in agreement with Ref. [9].

In Fig. 2, we compare our $\mathrm{NNNLO_{approx}} + \mathrm{NNLL}$ prediction for r with the experimental data compiled in Ref. [8], which did not enter our fit. They were collected at CESR with $\sqrt{s} = 10$ GeV, DESY DORIS II with 10 GeV, PEP with 29 GeV, PETRA with 22–35 GeV, LEP1 with 91 GeV, LEP2 with 130–209 GeV, and FNAL Tevatron with 1.8 TeV. The agreement is very satisfactory and reassures us of the validity of our analysis.

In summary, we unraveled an unexpected, SUSY-like relationship between the NNLL-resummed first Mellin moments of the timelike DGLAP splitting functions in real QCD, Eq. (4), which is n_f independent, and exploited it to find an exact solution of the DGLAP evolution equation, Eq. (1), bypassing the approximate two-step diagonalization procedure used so far in the literature. This also allowed us push our knowledge of r_- by one order of γ_0. Also incorporating the appropriately transformed $\mathscr{O}(\gamma_0^2)$ and $\mathscr{O}(\gamma_0^3)$ corrections to r_+ as well as power-like corrections, we performed a global fit to the world data of charged-hadron multiplicities in quark and gluon jets produced by e^+e^- annihilation and so extracted the competitive new value of $\alpha_s^{(5)}(M_Z^2)$ in Eq. (33), which nicely agrees with the present world average. Our analysis only relies on first principles of QCD and avoids additional model assumptions, including those inherent to the MLLA. On top of the physical advantages mentioned above, Eq. (4) renders the otherwise complicated formalism aesthetically pleasing and prompts one to speculate if there is some unknown higher reason for it.

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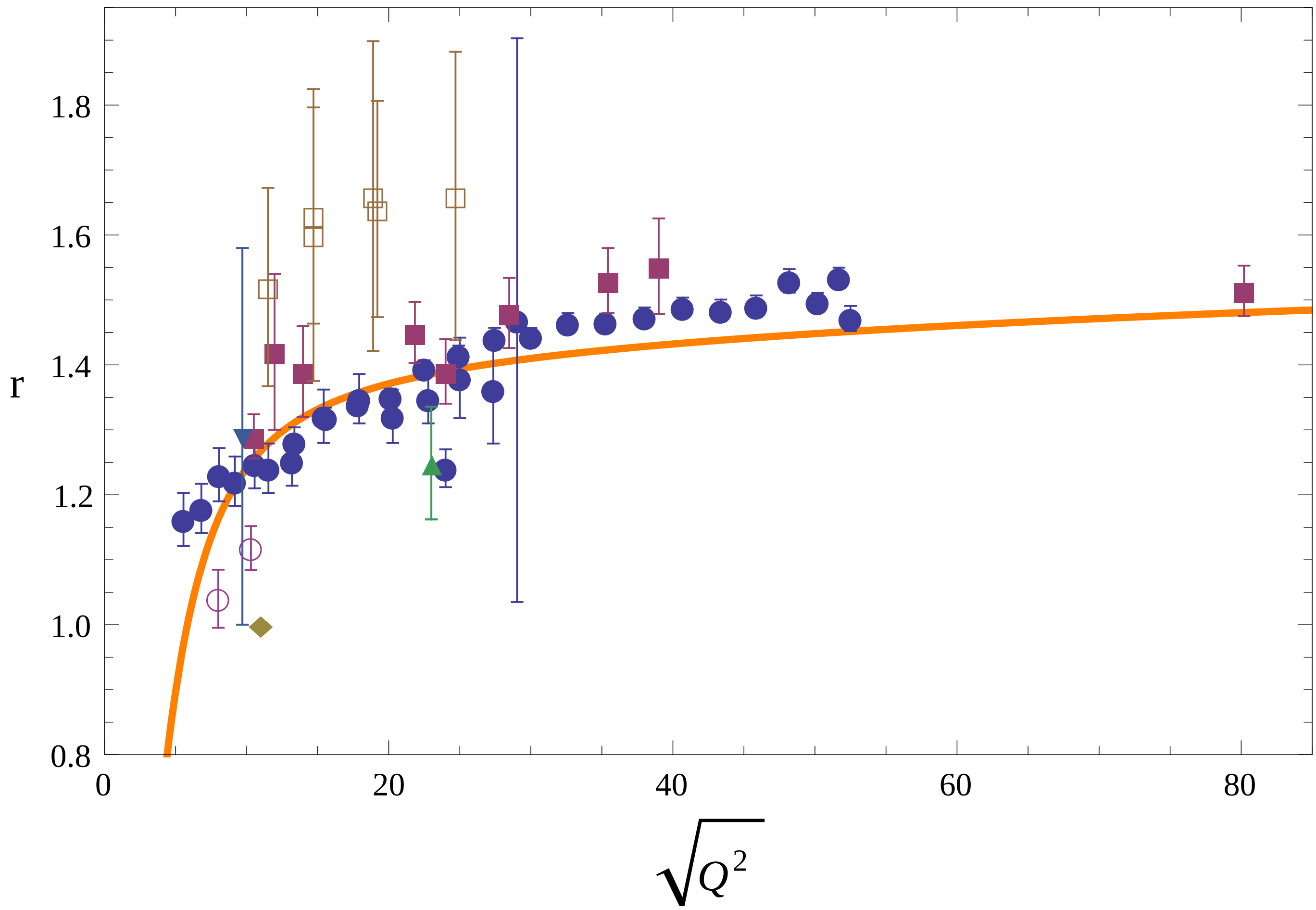

FIG. 2: Comparison of our $\mathrm{NNNLO_{approx}} + \mathrm{NNLL}$ prediction of $r(\mu^2)$ with experimental data excluded from the fit.

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