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The Heavy Quark Form Factors at Two Loops

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Abstract

We compute the two-loop QCD corrections to the heavy quark form factors in case of the vector, axial-vector, scalar and pseudo-scalar currents up to second order in the dimensional parameter $\varepsilon = (4 - D)/2$. These terms are required in the renormalization of the higher order corrections to these form factors.

1 Introduction

Since its discovery [1, 2] in 1995, the top quark has been studied extensively both in theoretical and experimental premises. These studies of the heaviest particle of the Standard Model (SM) provide as well a detailed probe onto some aspects of electro-weak symmetry breaking (EWSB). Due to its very short lifetime, the top quark decays before hadronizing, and thus provides a window to study its production dynamics widely without accounting for the hadronization effects and therefore as a quark being directly accessible as such.[1] The experiments carried out at the Tevatron and later at the LHC, have already measured many observables which allow to extract the properties of the top quark with remarkable accuracy. Compared to the Tevatron, the LHC offers an abundant rate of top quark pair and single production, hence providing a perfect ground for precision tests. Due to the combined effort from both the theoretical and experimental sides, a striking accuracy has been achieved in many observables, e.g. the uncertainties on the predictions of the inclusive production cross-section of a top quark pair, now are around 5 % at a fixed top quark mass of $m_t = 172.5$ GeV. While these precise measurements provide a strong ground for testing the predictions within the SM, beyond the Standard Model (BSM) physics scenarios can as well hide under those small uncertainties. To find a hint of BSM physics or to rule some hypotheses out, we need more precision and certainly a future linear or circular e^+e^- collider can achieve that. In order to match the experimental accuracy, precise predictions are required on the theoretical side.

In this paper, we focus on perturbative Quantum Chromodynamics (QCD) corrections to the form factors involving heavy quarks which are basic building blocks of various physical quantities concerning top quark pair production. The massive vector and axial vector form factors play an important role in the forward-backward asymmetry of bottom or top quark production at electron-positron colliders. Likewise the decay of a scalar or pseudo-scalar particle to a pair of heavy quarks could also play a very important role in shedding light on the quantum nature of the Higgs boson. There are also static quantities like the anomalous magnetic moment, which receive contributions from such massive form factors. For these reasons, the phenomenology and the perturbative QCD corrections to these form factors have gained much attention during the last decade.

In Refs. [3, 4], the first order QCD corrections were obtained for the vector and axial-vector form factors. A massless approximation was considered to obtain the next-to-next-to-leading order (NNLO) QCD corrections in [5] numerically, later followed by an analytic computation in [6]. Another numerical computation was performed in [7] at NNLO using a different formalism. On the other hand, the next-to-leading order (NLO) contributions to the scalar and pseudo-scalar form factors were known [8–11] for long and NNLO corrections by employing quark mass expansion to various orders, c.f. [12–19]. A series of papers followed obtaining the two-loop QCD corrections for the vector form factor [20], the axial-vector form factor [21], the anomaly contributions [22] and the scalar and pseudo-scalar form factors [23]. An independent cross-check of the vector form factor has been performed in [24] with the addition of the $\mathcal{O}(\varepsilon)$ contribution, where $\varepsilon = (4 - D)/2$ and D is the space-time dimension. Recently, the calculation of a subset of the three-loop master integrals [25] has made it possible to obtain the vector form factor at three loops [26] in the color-planar limit. The large β_0 limit has been considered in [27]. While the main goal is to obtain the complete three-loop corrections for the form factors, the $\mathcal{O}(\varepsilon)$ pieces at two-loop order are important ingredients. In addition, we compute the master integrals to the required order in ε with a different technique.

[1] The reconstruction of the t-quark itself, and in particular its mass, however, requires to study the hadronization effects.

In the present paper, we compute the contributions to the massive form factors up to $\mathcal{O}(\varepsilon^2)$ for different currents, namely, vector, axial-vector, scalar and pseudo-scalar currents, which serve as input for ongoing and future 3- and 4-loop calculations. We also perform the expansion of the exact results in different kinematic regions. In Section 2, we briefly describe the theoretical formalism for all the currents and corresponding form factors, followed by their renormalization procedure and a description of the universal infrared (IR) structure. Section 3 contains the details about the computational technique. Here we describe especially how we have computed the master integrals. In Section 4 we present the results, also expanding the complete expressions in regions, which are kinematically relevant. Finally we conclude in Section 5. Various of the expressions are rather voluminous. A part of it is presented in the appendices and the $O(\varepsilon^2)$ terms are only given in computer readable form in a file attached to this paper.

2 The heavy quark form factors

We consider the decay of a virtual massive boson of momentum q into a pair of heavy quarks of mass m, momenta q_1 and q_2 and color c and d, through a vertex X_{cd}, where $X_{cd} = \Gamma^{\mu}_{V,cd}, \Gamma^{\mu}_{A,cd}, \Gamma_{S,cd}$ and $\Gamma_{P,cd}$ indicates the coupling to a vector, an axial-vector, a scalar and a pseudo-scalar boson, respectively. Here $q^2 = (q_1 + q_2)^2$ is the center of mass energy squared and the dimensionless variable s is defined by

$$s = \frac{q^2}{m^2}\,. \tag{2.1}$$

The amplitude takes the following general form

$$\bar{u}_c(q_1) X_{cd} v_d(q_2)\,, \tag{2.2}$$

where $\bar{u}_c(q_1)$ and $v_d(q_2)$ are the bi-spinors of the quark and the anti-quark, respectively. We denote the corresponding UV renormalized form factors by F_I, with $I = V, A, S, P$. They are expanded in the strong coupling constant $\alpha_s = g_s^2/(4\pi)$ as follows

$$F_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n F_I^{(n)}\,. \tag{2.3}$$

The unrenormalized form factors are denoted by $\hat{F}_I$. In the following sub-sections, we discuss the properties of each current, the corresponding renormalization procedure and their universal infrared structure.

2.1 The vector and axial-vector current

In this section, we consider the spin 1 case, i.e. the vertex Γ^{μ} of a Z-boson or photon coupling to a pair of heavy quarks. The general structure of Γ^{μ} consists of six form factors, two of which are CP odd. We consider only higher order QCD effects and SM neutral current interactions to lowest order. Since CP invariance holds, we only take into account the four CP even form factors, $F_{V,i}(s), F_{A,i}(s)$ $i = 1, 2$ in the following. They can be cast in the following general form

$$\begin{aligned}\Gamma^{\mu}_{cd} &= \Gamma^{\mu}_{V,cd} + \Gamma^{\mu}_{A,cd} \\ &= -i\delta_{cd}\Big[v_Q\Big(\gamma^{\mu}\ F_{V,1} + \frac{i}{2m}\sigma^{\mu\nu}q_{\nu}\ F_{V,2}\Big) + a_Q\Big(\gamma^{\mu}\gamma_5\ F_{A,1} + \frac{1}{2m}q^{\mu}\gamma_5\ F_{A,2}\Big)\Big],\end{aligned} \tag{2.4}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $q = q_1 + q_2$, and v_Q and a_Q are the SM vector and axial-vector coupling constants as defined by

$$v_Q = \frac{e}{\sin\theta_w \cos\theta_w}\Big(\frac{T_3^Q}{2} - \sin^2\theta_w Q_Q\Big)\,, \qquad a_Q = -\frac{e}{\sin\theta_w \cos\theta_w}\frac{T_3^Q}{2}\,. \tag{2.5}$$

e denotes the elementary charge, θ_w the weak mixing angle, T_3^Q the third component of the weak isospin, and Q_Q the charge of the heavy quark.

To extract the form factors $F_{I,i},\ I = V, A$, we multiply Γ^μ by the following projectors and perform a trace over the spinor and color indices

$$\begin{aligned} P_{V,i} &= \frac{i}{v_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\Big(\gamma_\mu g^1_{V,i} + \frac{1}{2m}(q_{2\mu} - q_{1\mu})g^2_{V,i}\Big)\frac{\not{q}_1 + m}{m}\,, \\ P_{A,i} &= \frac{i}{a_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\Big(\gamma_\mu\gamma_5 g^1_{A,i} + \frac{1}{2m}(q_{1\mu} + q_{2\mu})\gamma_5 g^2_{A,i}\Big)\frac{\not{q}_1 + m}{m}\,, \end{aligned} \tag{2.6}$$

where, [2]

$$\begin{aligned} g^1_{V,1} &= -\frac{1}{4(1-\varepsilon)}\frac{1}{(s-4)}\,, & g^2_{V,1} &= \frac{(3-2\varepsilon)}{(1-\varepsilon)}\frac{1}{(s-4)^2}\,, \\ g^1_{V,2} &= \frac{1}{(1-\varepsilon)}\frac{1}{s(s-4)}\,, & g^2_{V,2} &= -\frac{1}{(1-\varepsilon)}\frac{1}{(s-4)^2}\left(\frac{4}{s} + 2 - 2\varepsilon\right)\,, \\ g^1_{A,1} &= -\frac{1}{4(1-\varepsilon)}\frac{1}{(s-4)}\,, & g^2_{A,1} &= -\frac{1}{(1-\varepsilon)}\frac{1}{s(s-4)}\,, \\ g^1_{A,2} &= \frac{1}{(1-\varepsilon)}\frac{1}{s(s-4)}\,, & g^2_{A,2} &= \frac{1}{(1-\varepsilon)}\frac{1}{s^2(s-4)}\Big(4(3-2\varepsilon) - 2s(1-\varepsilon)\Big)\,, \end{aligned} \tag{2.7}$$

and N_c denotes the number of colors. Later on we will also use the Casimir operators $C_A = N_c, C_F = (N_c^2 - 1)/(2N_c), T_F = 1/2$ for $SU(N_c)$, with $N_c = 3$ in the case of QCD.

2.2 The scalar and pseudo-scalar current

We consider the current implied by a general neutral spin 0 particle h that couples to heavy quarks through the Yukawa interaction

$$\mathcal{L}_{int} = -\frac{m}{v}\Big[s_Q\bar{Q}Q + ip_Q\bar{Q}\gamma_5 Q\Big]h, \tag{2.8}$$

where $v = (\sqrt{2}G_F)^{-1/2}$ is the SM Higgs vacuum expectation value, with G_F being the Fermi constant, s_Q and p_Q are the scalar and pseudo-scalar coupling, respectively, and Q and h are the heavy quark and scalar and pseudo-scalar field, respectively. The vertex for $h \to \bar{Q} + Q$, $X_{cd} \equiv \Gamma_{cd}$ consists of two form factors with the following general structure

$$\begin{aligned} \Gamma_{cd} &= \Gamma_{S,cd} + \Gamma_{P,cd} \\ &= -\frac{m}{v}\delta_{cd}\,\Big[s_Q\,F_S + ip_Q\gamma_5\,F_P\Big]\,, \end{aligned} \tag{2.9}$$

[2] In [20], the expression for $g^2_{V,2}$ contains a typographical error.

where F_S and F_P denote the renormalized scalar and pseudo-scalar form factors, respectively. As before, the form factors can be obtained from Γ_{cd} through suitable projectors as given below and performing the trace over the spinor and color indices

$$P_S = \frac{v}{2ms_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\left(-\frac{1}{(s-4)}\right)\frac{\not{q}_1 + m}{m}\,,$$
$$P_P = \frac{v}{2mp_Q}\frac{\delta_{cd}}{N_c}\frac{\not{q}_2 - m}{m}\left(-\frac{i}{s}\gamma_5\right)\frac{\not{q}_1 + m}{m}\,. \tag{2.10}$$

2.3 Anomaly and Ward identities

Since we use dimensional regularization [28] in $D = 4-2\varepsilon$ space-time dimensions, one important point is to define a proper description for the treatment of γ_5. In the case of the axial-vector and the pseudo-scalar form factors, two types of Feynman diagrams contribute: the non-singlet diagrams containing only open fermion lines, and the singlet diagrams where a fermion loop is attached to the axial-vector or pseudo-scalar vertex. It is convenient to separate the two contributions and write,

$$\Gamma^{\mu}_{A,cd} = \Gamma^{\mu,\mathrm{ns}}_{A,cd} + \Gamma^{\mu,\mathrm{s}}_{A,cd}\,, \quad \Gamma_{P,cd} = \Gamma^{\mathrm{ns}}_{P,cd} + \Gamma^{\mathrm{s}}_{P,cd}\,, \tag{2.11}$$

where ns and s denote the non-singlet and the singlet contributions, respectively.

In the non-singlet case, we use an anticommuting γ_5 in D space-time dimensions, with $\gamma_5^2 = 1$, as it does not lead to any spurious singularities. This approach respects chiral invariance and leaves us with the Ward identity

$$q^{\mu}\Gamma^{\mu,\mathrm{ns}}_{A,cd} = 2m\Gamma^{\mathrm{ns}}_{P,cd}\,, \tag{2.12}$$

which in terms of the form factors, takes the form

$$2F^{\mathrm{ns}}_{A,1} + \frac{s}{2}F^{\mathrm{ns}}_{A,2} = 2mF^{\mathrm{ns}}_{P}\,. \tag{2.13}$$

On the other hand, the singlet pieces for the axial-vector and the pseudo-scalar vertex are related to each other through the Adler-Bell-Jackiw (ABJ) anomaly [29,30]. With this constraint, we use the following prescription as presented in [31,32], which mostly followed [28]: For a single γ_5 in a fermion loop, we use

$$\gamma_5 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}, \tag{2.14}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor and all Lorentz indices are taken D-dimensional. Finally, the contraction of two ϵ-tensors is expressed in terms of products of D-dimensional metric tensors. This prescription of γ_5 needs a special treatment during renormalization, as discussed later.

The ABJ anomaly involves the truncated matrix element of the gluonic operator between the vacuum and a pair of heavy quark states. The gluonic operator is given by

$$G(x)\tilde{G}(x) \equiv \varepsilon_{\mu\nu\rho\sigma}G^{a,\mu\nu}(x)G^{a,\rho\sigma}(x)\,, \tag{2.15}$$

where $G^{a,\mu\nu}$ represents the gluonic field strength tensor. Denoting its contribution by $F_{G,Q}$, we can immediately write down the anomalous Ward identity for the singlet case as follows

$$q_{\mu}\Gamma^{\mu,s}_{A,cd} = 2m\Gamma^{s}_{P,cd} - i\left(\frac{\alpha_s}{4\pi}\right)T_F\langle G\tilde{G}\rangle_Q\,, \tag{2.16}$$

which implies

$$2F^{s}_{A,1} + \frac{s}{2}F^{s}_{A,2} = 2mF^{s}_{P} - i\left(\frac{\alpha_s}{4\pi}\right)T_F F_{G,Q}\,. \tag{2.17}$$

2.4 Renormalization

The UV renormalization of the form factors has been performed in a mixed scheme. We renormalize the heavy quark mass and wave function in the on-shell (OS) renormalization scheme, while the strong coupling constant is renormalized in the modified minimal subtraction ($\overline{\text{MS}}$) scheme [33, 34]. The corresponding renormalization constants are well known and are denoted by $Z_{m,\text{OS}}$ [35–39], $Z_{2,\text{OS}}$ [35–37, 40] and Z_{a_s} [41–45] for the heavy quark mass, wave function and strong coupling constant, respectively. All renormalization constants follow a perturbative expansion in α_s

$$Z_I = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n Z_I^{(n)} \,. \tag{2.18}$$

For reference, we present $Z_{m,\text{OS}}^{(n)}$ and $Z_{2,\text{OS}}^{(n)}$ in appendix A up to $n = 2$ and $\mathcal{O}(\varepsilon^2)$.

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F (n_l + n_h) \,. \tag{2.19}$$

Here n_l and n_h denote the number of light and heavy quarks, respectively. In the following we will set $n_h = 1$.

While the renormalization of the heavy-quark wave function and the strong coupling constant can be done multiplicatively, the mass renormalization requires the explicit calculation of counterterm diagrams. Hence, the bare and renormalized vector form factors are at two loops related by

$$F_{V,i} = Z_{2,\text{OS}} \hat{F}_{V,i} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z_{m,\text{OS}}^{(1)} \hat{F}_{V,i}^{ct,(1)} + \mathcal{O}(\alpha_s^3) \,, \tag{2.20}$$

where the unrenormalized form factors $\hat{F}_{V,i}$ are expanded in the unrenormalized strong coupling constant $\hat{\alpha}_s = \alpha_s Z_{a_s}$, and $\hat{F}_{V,i}^{\text{ct},(1)}$ denotes the bare contribution from counterterm diagrams at one loop.

The non-singlet contributions to the axial-vector form factor can be renormalized in the same way. The singlet part requires extra care due to the prescription employed for γ_5. It is infrared finite and the UV pole is renormalized by the multiplicative renormalization constant Z_J as

$$F_{A,i}^s = Z_J Z_5^{\text{fin}} \hat{F}_{A,i}^s \,, \tag{2.21}$$

where Z_5^{fin} is a finite renormalization constant which restores the anomalous Ward identity Eq. (2.16). We would like to remark that the Ward identities are valid for physical quantities. Therefore, it is not reasonable to study them at higher orders in ε, and neither it is to consider ε-dependent pieces of Z_5^{fin}. In the $\overline{\text{MS}}$ scheme for the form factors,

$$Z_J = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{6 C_F T_F}{\varepsilon} + \mathcal{O}(\alpha_s^3) \tag{2.22}$$

and

$$Z_5^{\text{fin}} = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left(3 C_F T_F\right) + \mathcal{O}(\alpha_s^3) \,. \tag{2.23}$$

The remaining finite renormalization has to be carried out later for the corresponding observables of which the form factors form a part.

The renormalization of the quantity $F_{G,Q}$ appearing in Eq. (2.17) involves the mixing of the gluonic operator $G\tilde{G}$ with another operator, namely, $\partial_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi$, as discussed in [32, 46, 47], where ψ indicates all quark flavors including the massive one. We get,

$$F_{G,Q} = Z_{GG} \hat{F}_{G,Q} + Z_{GJ} \hat{F}_{J,Q} \,, \tag{2.24}$$

where $\hat{F}_{J,Q}$ indicates the bare contribution from the second operator, while Z_{GG} and Z_{GJ} are the corresponding renormalization constants.

The renormalization of the scalar and pseudo-scalar (non-singlet) vertices also follows a similar procedure, except for the presence of the heavy quark mass in the Yukawa coupling. Thus the renormalized form factors are given by

$$\begin{aligned} F_{S,i} &= Z_{m,\mathrm{OS}} Z_{2,\mathrm{OS}} \hat{F}_{S,i} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(1)}_{m,\mathrm{OS}} \hat{F}^{ct,(1)}_{S,i} + \mathcal{O}(\alpha_s^3)\,, \\ F^{\mathrm{ns}}_{P,i} &= Z_{m,\mathrm{OS}} Z_{2,\mathrm{OS}} \hat{F}^{\mathrm{ns}}_{P,i} + \left(\frac{\alpha_s}{4\pi}\right)^2 Z^{(1)}_{m,\mathrm{OS}} \hat{F}^{ct,\mathrm{ns},(1)}_{P,i} + \mathcal{O}(\alpha_s^3)\,, \end{aligned} \tag{2.25}$$

On the other hand, the singlet piece of the pseudo-scalar vertex is both IR and UV finite, hence no additional renormalization is necessary.

2.5 The infrared structure

The study of the IR behavior of the form factors has attracted a lot of attention in the past few decades. A plethora of works on massless scattering amplitudes [48–52] has already provided a remarkable understanding on the universal IR pattern characterized by soft and collinear dynamics. In [53], the first step was taken to generalize this for the two-loop scattering amplitudes with massive partons. Later, in [54], following a soft-collinear effective theory (SCET) approach, the general IR structures have been presented.

The IR singularities of the massive form factors can be factorized as a multiplicative renormalization factor. Its structure is constrained by the renormalization group equation (RGE), as follows,

$$F_I = Z(\mu) F^{\mathrm{fin}}_I(\mu)\,, \tag{2.26}$$

where F^{fin}_I is finite as $\varepsilon \to 0$. The RGE for Z reads

$$\frac{d}{d\ln\mu} \ln Z(\varepsilon, x, m, \mu) = -\Gamma(x, m, \mu)\,, \tag{2.27}$$

where Γ is the corresponding anomalous dimension. Notice that Z does not carry any information regarding the vertex. Γ can be identified as the massive cusp anomalous dimension, which is by now available up to the three-loop level [55–58]. Both Z and Γ can be expanded in a perturbative series in α_s as follows

$$Z = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n Z^{(n)}\,, \qquad \Gamma = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} \Gamma_n \tag{2.28}$$

and we find the following solution for Eq. (2.27)

$$Z = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left[\frac{\Gamma_0}{2\varepsilon}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[\frac{1}{\varepsilon^2}\left(\frac{\Gamma_0^2}{8} - \frac{\beta_0\Gamma_0}{4}\right) + \frac{\Gamma_1}{4\varepsilon}\right] + \mathcal{O}(\alpha_s^3)\,. \tag{2.29}$$

Eq. (2.29) correctly predicts the infrared singularities for all massive form factors at the two-loop level.

3 Details of the calculation

The Feynman diagrams were generated using `QGRAF` [59], the output of which was then processed using `Q2e/Exp` [60, 61] and `FORM` [62, 63] in order to express the diagrams in terms of a linear combination of a large set of scalar integrals. These integrals were then reduced to a much smaller set of master integrals (MIs) using integration by parts identities (IBPs) [64–68] with the help of the program `Crusher` [69]. Since all this is common practice, we refrain from going into any detail.

After performing the reductions, all that remains to be done is to calculate the master integrals. In the following sections, we present the methods we used to achieve this.

3.1 The conventional differential equations method

We computed the two-loop master integrals contributing to the massive fermion form factors as Laurent expansions in the dimensional parameter ε by means of the differential equation method [70–76]. This technique has already been applied to massive form factor integrals at two and three loops in [25, 77, 78]. In this work, we calculate the two-loop master integrals up to a sufficiently high order in ε to obtain $\mathcal{O}(\varepsilon^2)$ accuracy in the form factors.

In this section, we briefly review the main steps of this calculation. The master integrals are classified according to their underlying topology. In particular, we distinguish the non-singlet topologies (Figure 1) from the singlet topology (Figure 2) according to whether the external current does or does not couple to the external massive quark.

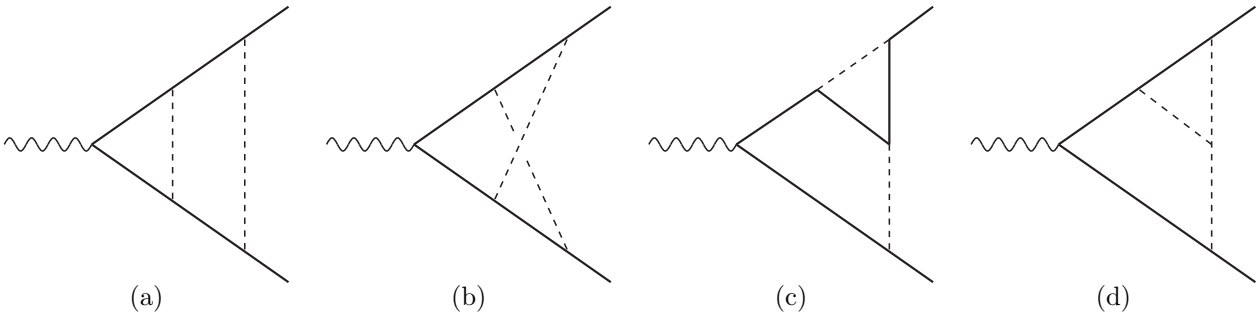

Figure 1: The non-singlet topologies required for the calculation of two-loop form factors. Solid lines represent massive particles in external or internal lines, while dashed lines correspond to massless propagators. The external vector current can also be replaced by an axial-vector, scalar or pseudo-scalar. The master integrals associated to topologies (c) and (d) are a subset of the master integrals required for topologies (a) and (b).

In the case of the non-singlet topologies depicted in Figure 1, it turns out that the master integrals associated to the two topologies on the right of that figure represent a subset of the ones required to calculate the topologies on the left. We therefore concentrate on the topologies in Figures 1(a) and 1(b). The master integrals for both topologies can be expressed in terms of a single integral family with seven propagators given by

$$J(\nu_1, \ldots, \nu_7) = \left((4\pi)^{2-\varepsilon} e^{\varepsilon\gamma_E)}\right)^2 \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{D_1^{\nu_1} \ldots D_7^{\nu_7}}, \tag{3.1}$$

where

$$D_1 = (l_1 + q_1)^2 - m^2, \quad D_2 = (l_2 + q_1)^2 - m^2, \quad D_3 = (l_1 - q_2)^2 - m^2,$$
$$D_4 = (l_2 - q_2)^2 - m^2, \quad D_5 = l_1^2, \quad D_6 = (l_1 - l_2)^2, \quad D_7 = (l_1 - l_2 + q_2)^2 - m^2. \quad (3.2)$$

Here the q_i's with $i = 1, 2$ are the external momenta, which are taken on-shell ($q_1^2 = q_2^2 = m^2$). The MIs are therefore labeled by the exponents $\nu_1, \ldots, \nu_7$ of the denominators $D_1, \ldots, D_7$. The integrals corresponding to the topology in Figure 1(a) will have $\nu_7 = 0$, while the ones corresponding to the topology in Figure 1(b) will have $\nu_3 = 0$. There are several master integrals where ν_3 and ν_7 are both equal to zero, which are therefore common to both topologies[3]. The list of master integrals required to reduce these topologies is given in Table 1. Notice that there are several sets of master integrals that have the same set of non-vanishing positive powers of propagators. Such a set of integrals is called a *sector*. For example, integrals J_6, J_7 and J_8 belong to the same sector, since for all of them $\nu_1, \nu_4, \nu_6 \geq 1$ and $\nu_2 = \nu_3 = \nu_5 = \nu_7 = 0$. A given sector is said to be a subsector of another sector if the propagators in the first sector are a subset of the propagators in the second.

MI	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	k
J_1	1	1	0	0	0	0	0	6
J_2	0	1	0	0	1	1	0	6
J_3	1	0	0	1	0	0	1	4
J_4	1	1	1	0	0	0	0	4
J_5	1	1	1	1	0	0	0	4
J_6	1	0	0	1	0	1	0	4
J_7	1	0	0	2	0	1	0	3
J_8	1	0	0	1	0	2	0	4
J_9	0	1	1	0	1	1	0	3
J_{10}	0	1	0	1	1	1	0	4
J_{11}	1	1	0	1	0	0	0	3
J_{12}	1	0	0	1	0	1	1	3
J_{13}	1	0	0	2	0	1	1	3
J_{14}	1	0	0	1	0	2	1	3
J_{15}	1	1	0	0	1	0	1	3
J_{16}	1	1	0	0	2	0	1	3
J_{17}	1	1	0	0	1	0	2	3
J_{18}	1	1	0	1	0	0	1	3
J_{19}	1	1	0	2	0	0	1	3
J_{20}	1	1	0	1	1	1	0	2
J_{21}	1	1	0	1	1	2	0	4
J_{22}	1	1	0	1	1	1	1	3
J_{23}	1	1	0	1	1	2	1	3

Table 1: The list of the non-singlet master integrals identified by the indices ν_1 to ν_7. In the last column, we indicate the order k in ε to which each integral needs to be expanded in order to calculate the form factors to $\mathcal{O}(\varepsilon^2)$.

[3]This is the reason we chose to group here both topologies within a single integral family, although the actual reductions performed with `Crusher` were done using two separate families.

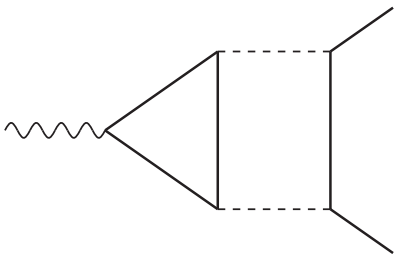

Figure 2: The singlet topology.

In the case of the singlet topology, shown in Figure 2, the master integrals are given by

$$K(\nu_1,\ldots,\nu_6) = \left((4\pi)^{2-\varepsilon} e^{\varepsilon\gamma_E)}\right)^2 \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{{D_1'}^{\nu_1}\ldots {D_6'}^{\nu_6}}, \tag{3.3}$$

where

$$\begin{aligned} &D_1' = (l_1+q_1)^2, \quad D_2' = (l_2+q_1)^2 - m^2, \quad D_3' = (l_1-q_2)^2, \\ &D_4' = (l_2-q_2)^2 - m^2, \quad D_5' = l_1^2 - m^2, \quad D_6' = (l_1-l_2)^2 - m^2. \end{aligned} \tag{3.4}$$

The list of master integrals required in this case is given in Table 2. From now on, instead of using the exponents ν_i to identify the master integrals, we will use the single subindex we have assigned to each integral according to the leftmost columns of Tables 1 and 2.

MI	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	k
K_1	0	1	0	0	1	0	6
K_2	0	1	0	0	1	1	4
K_3	1	1	1	0	0	0	3
K_4	1	1	1	1	0	0	3
K_5	0	1	1	0	0	1	3
K_6	0	1	1	0	0	2	3
K_7	0	1	0	1	1	0	3
K_8	1	1	1	0	1	0	3
K_9	0	1	1	0	1	1	3
K_{10}	0	1	1	0	1	2	3
K_{11}	0	1	0	1	1	1	3
K_{12}	0	1	0	1	1	2	3
K_{13}	1	1	1	1	1	0	2
K_{14}	1	1	1	1	0	1	2
K_{15}	1	1	1	1	1	1	2

Table 2: The list of singlet master integrals identified by the indices ν_1 to ν_6. In the last column, we indicate the order k in ε to which each integral needs to be expanded in order to calculate the form factors to $\mathcal{O}(\varepsilon^2)$.

All of the master integrals can be expressed in terms of harmonic polylogarithms (HPLs) [79]

in the kinematic variable [80, 81]

$$x = \frac{\sqrt{q^2 - 4m^2} - \sqrt{q^2}}{\sqrt{q^2 - 4m^2} + \sqrt{q^2}} \quad \leftrightarrow \quad \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}. \tag{3.5}$$

In particular, we focus on the Euclidean region, $q^2 < 0$, corresponding to x ranging in $(0, 1)$. A large center of mass energy $|q^2| \gg m^2$ is equivalent to the boundary $x \to 0$, while the large-mass limit $m^2 \gg |q^2|$ is mapped to the endpoint $x \to 1$.

We derived a system of coupled linear differential equations for each topology by reducing the derivative with respect to x of each MI to a linear combination of the master integrals themselves, with the help of `Crusher`. The derivative of an integral cannot produce integrals in this linear combination with more propagators than the original one, so they are all either in the same sector as the integral to which one takes the derivative or they belong to a subsector. This means that the integrals in the subsectors need to be solved first, and the differential equations within a given sector will constitute a coupled subsystem. In the case of the singlet master integrals, there are three such coupled subsystems, namely, the subsystems formed by the set of integrals $\{K_5,\, K_6\}$, $\{K_9,\, K_{10}\}$ and $\{K_{11},\, K_{12}\}$. In all other sectors, only one integral is present. The differential equations in each coupled subsystem will therefore need to be decoupled in order to solve them. The strategy for solving the whole system consists then in solving first the simplest sectors (with fewer propagators), and move up in the chain of subsystems, decoupling and solving each one of them until all integrals are obtained. The starting point are the integrals for which the derivative with respect to x equals zero[4], which must therefore be obtained not by the differential equations method but by other means.

In the case of the non-singlet topologies, by expanding the MIs in a Laurent series around $\varepsilon = 0$, the systems greatly simplify, assuming an almost complete block-triangular form. Only one 2×2 coupled subsystem remains after the expansion in ε. We therefore solve the system order-by-order in ε by integrating each equation by quadratures. These steps are automated and results are efficiently simplified using a minimal set of independent HPLs by means of the `Mathematica` packages `Sigma` [82, 83] and `HarmonicSums` [84–89]. This procedure is slightly modified for the aforementioned 2×2 coupled system, which does not assume a triangular form after expanding in ε

$$\frac{d}{dx}\begin{pmatrix} J_{22} \\ J_{23} \end{pmatrix} = \begin{bmatrix} \frac{1+x^2}{x(1-x^2)} & \frac{1-x^2}{x^2} \\ -\frac{1}{1-x^2} & \frac{4(1+x^2)}{x(1-x^2)} \end{bmatrix} \begin{pmatrix} J_{22} \\ J_{23} \end{pmatrix} + \begin{pmatrix} R_1(\varepsilon, x) \\ R_2(\varepsilon, x) \end{pmatrix}, \tag{3.6}$$

where the inhomogeneities $R_1(\varepsilon, x)$, $R_2(\varepsilon, x)$ are determined at each order in ε by subsector MIs. The most general solution of the homogeneous system involves only logarithms and rational functions

$$\begin{aligned} J_{22} &= c_1 \frac{x^2}{(1-x^2)^2} + c_2 \frac{x^2 \ln(x)}{(1-x^2)^2}, \\ J_{23} &= c_1 \frac{x^2(1+x^2)}{(1-x^2)^4} + c_2 \frac{x^3(x^2 \ln(x) + \ln(x) + 2)}{(1-x^2)^4}, \end{aligned} \tag{3.7}$$

therefore we integrate the inhomogeneous system order-by-order in ε using the method of variation of constants. For example, at leading order in ε, the inhomogeneous parts of the system

[4]These are J_1, J_2 and J_3 in the case of the non-singlet topologies, and K_1 and K_2 in the singlet case.

(3.6) are

$$
\begin{aligned}
R_1(\varepsilon,x) &= \frac{1}{\varepsilon}\Bigg\{\frac{x^4+x^3+6x^2+x+1}{(1-x)^2(x+1)^4}\ln(x) \\
&\quad -\frac{x^6+2x^5-25x^4-4x^3-25x^2+2x+1}{16x^2(1-x)(x+1)^3}\Bigg\}+\mathcal{O}(\varepsilon^0), \qquad (3.8)\\
R_2(\varepsilon,x) &= \frac{1}{\varepsilon}\Bigg\{\frac{x\ln(x)}{(1-x)^4(x+1)^6}\left(2x^6+x^5-4x^4+6x^3-4x^2+x+2\right) \\
&\quad -\frac{x^8+2x^7-8x^6-10x^5+22x^4-10x^3-8x^2+2x+1}{4(1-x)^3x(x+1)^5}\Bigg\}+\mathcal{O}(\varepsilon^0). \qquad (3.9)
\end{aligned}
$$

By introducing Eq. (3.7) with $c_i \to c_i(x)$ into the system (3.6), we get first order differential equations for $c_1(x)$ and $c_2(x)$ that can be solved by quadratures using the same automated tools introduced above. The integration constants are fixed by imposing the regularity of the functions J_{22} and J_{23} in the limit of vanishing space-like momentum $q^2 \to 0$, corresponding to $x \to 1$, giving

$$
\begin{aligned}
J_{22} &= \frac{1}{\varepsilon}\Bigg\{-\frac{x^2}{3(1-x^2)^2}[H_0^3(x)-3H_0(x)(2H_{0,-1}(x)-2H_{0,1}(x)-\zeta_2) \\
&\quad +12H_{0,0,-1}(x)-12H_{0,0,1}(x)+3\zeta_3]\Bigg\}+\mathcal{O}(\varepsilon^0), \\
J_{23} &= \frac{1}{\varepsilon}\Bigg\{\frac{x^3(1+x^2)}{(1-x^2)^4}\Bigg[2H_0(x)H_{0,-1}(x)-2H_0(x)H_{0,1}(x)-4H_{0,0,-1}(x)+4H_{0,0,1}(x) \\
&\quad -\zeta_2 H_0(x)-\frac{1}{3}H_0^3(x)-\zeta_3\Bigg]+\frac{x^3}{(1-x^2)^3}[2H_{0,1}(x)-2H_{0,-1}(x)+2H_{-1}(x)H_0(x) \\
&\quad -2H_0(x)H_1(x)-H_0^2(x)-\zeta_2]+\frac{x^6+2x^5-25x^4-4x^3-25x^2+2x+1}{16(1-x)^2(x+1)^4} \\
&\quad -\frac{x^2(x^4+x^3+6x^2+x+1)}{(1-x)^3(x+1)^5}H_0(x)\Bigg\}+\mathcal{O}(\varepsilon^0)\,, \qquad (3.10)
\end{aligned}
$$

where the harmonic polylogarithms, $H_{a_1,a_2,\ldots,a_n}(x)$, are defined by

$$
H_{a_1,a_2,\ldots,a_n}(x) = \int_0^x dy f_{a_1}(y) H_{a_2,\ldots,a_n}(y), \quad H_\emptyset = 1,\ a_i \in \{0,1,-1\}\ , \qquad (3.11)
$$

with

$$
f_0(x)=\frac{1}{x}, \quad f_1(x)=\frac{1}{1-x}, \quad f_{-1}(x)=\frac{1}{1+x}\ , \qquad (3.12)
$$

and

$$
H_{\underbrace{0,\ldots,0}_{\text{n times}}}(x) = \frac{1}{n!}\ln^n(x)\ . \qquad (3.13)
$$

ζ_n denotes the Riemann ζ-function

$$
\zeta_n = \sum_{k=1}^{\infty}\frac{1}{k^n}, n \geq 2, n \in \mathbb{N}.
$$

We proceed similarly at higher orders, getting expansions of the MIs J_{22} and J_{23} up to $\mathcal{O}(\varepsilon^3)$. It should be noticed that solving a system of coupled differential equations is in general far from

trivial. When the solution is written in terms of multiple polylogarithms or related iterated integrals, as in the case of the two-loop massive form factor [77, 78] and of the planar three-loop massive form factor [25], it is possible to properly choose the set of MIs such that the system doesn't have any coupled equation, assuming for example the *canonical* form of [74, 75]. In this work we are interested in applying the methods developed in [76, 90, 91], to solve the systems of coupled differential equations algorithmically, as discussed in the next sections.

In order to solve the differential equations, boundary conditions have to be determined. As it was observed in [77, 78], the analytic structure of the master integrals strongly constrains the choice of the integration constants. In particular, boundary conditions of the master integrals of the non-singlet topologies are completely determined by requiring the regularity of the functions in $x = 1$, as will be discussed in the example above. However, we cannot use the same argument for some of the master integrals of the singlet topology, which are characterized by a branch cut at $x = 1$, as occurs, for example, in the case of the integral $K_{14} = K(1, 1, 1, 1, 0, 1)$ depicted in Figure 3. We fixed the boundary conditions of such integrals by matching the general solutions of

Figure 3: The master integral K_{14}. Massive (massless) propagators are represented by solid (dashed) lines. The presence of a massless cut determines the asymptotic behaviour $K_{14} \sim \log(1 - x)$.

the differential equations with the asymptotic expansions of the corresponding integrals around $x \to 1$. The latter were computed by means of the large-mass expansion $m^2 \gg q^2$ [92], or with a Mellin Barnes representation, as described e.g. in [93].

3.2 Calculation of the master integrals using difference equations

In the following we describe an alternative method for calculating the master integrals. The idea of the method is to write all integrals in terms of series expansions and then use the differential equations obeyed by the MIs to derive difference equations satisfied by the coefficients of these series. In the non-singlet case, this can be done using the fact that the MIs are regular around $x = 1$. In terms of the variable $y = 1 - x$, we can therefore write

$$J_i(y) = \sum_{n=0}^{\infty} \sum_{j=-2}^{r} \varepsilon^j C_{i,j}(n) y^n, \tag{3.14}$$

where we have included the expansion in ε up to $\mathcal{O}(\varepsilon^r)$. We can introduce Eq. (3.14) in the differential equations after rewriting them in terms of the variable y, leading to a system of coupled difference equations for the different $C_{i,j}$'s.

The method of solving differential equations by introducing series expansions and then finding the solutions to the resulting recursion relations is, of course well-known. However, to the best of our knowledge, this method has not been used before to calculate master integrals in perturbative quantum field theory. We propose this method here since it can be useful also for higher loop calculations, and we can take advantage of the powerful mathematical tools implemented in `Sigma` and `HarmonicSums` to solve the difference equations.

Whenever we are given subsystems of differential equations, it is usually more convenient to uncouple them and then insert Eq. (3.14) in the corresponding uncoupled equations, as supposed to inserting Eq. (3.14) first and then uncouple the resulting coupled difference equations. This is so because usually the later approach will lead to difference equations of higher order, although occasionally, this might not be the case and this later approach might turn out to be preferable.

In the singlet case, some of the master integrals will not be regular at $x = 1$, due to the presence of the logarithm $\ln(1-x) = \ln(y)$ discussed in the previous section. We therefore include a formal expansion around powers of this logarithm[5],

$$K_i(y) = \sum_{n=0}^{\infty} \sum_{k=0}^{3} \sum_{j=-2}^{r} \varepsilon^j C_{i,j,k}(n) \ln^k(y) y^n. \tag{3.15}$$

The integrals belonging to the three multi-integral sectors in the singlet case have no logarithmic singularities at $y = 0$, and can therefore all be written as in Eq. (3.14). Only some of the integrals that are the sole representative of their sector in Table 2 need to be written according to Eq. (3.15). Since Eq. (3.14) is a particular case of Eq. (3.15)[6], we will consider for illustration purposes one of the integrals that require an expansion of the form (3.15). The integral $K_4 = K(1,1,1,1,0,0)$ is simple enough (it consists of a product of two one-loop integrals) to be calculated just through Feynman parameters, but precisely because of this simplicity, it allows us to describe the main features of the method without unnecessary complications or long formulae.

The differential equation associated to K_4 is given by

$$\frac{dK_4}{dy} + \left(\frac{2\varepsilon y}{(2-y)(1-y)} + \frac{2(1+2\varepsilon)}{y(2-y)} \right) K_4 = \frac{2-2\varepsilon}{y(2-y)} K_3. \tag{3.16}$$

As we discussed above, in order to solve the differential equations, we must first obtain all of the integrals associated to the subtopologies of the system under consideration. In the case of Eq. (3.16), this means obtaining K_3. In terms of the variable y, it is given by

$$\begin{aligned} K_3(y) &= -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}[-3 - \ln(1-y) + 2\ln(y)] - 7 - 3\ln(1-y) - \frac{1}{2}\ln^2(1-y) \\ &\quad + [6 + 2\ln(1-y)]\ln(y) - 2\ln^2(y) + \varepsilon\Bigg[-15 + \frac{8}{3}\zeta_3 - 7\ln(1-y) \\ &\quad - \frac{3}{2}\ln^2(1-y) - \frac{1}{6}\ln^3(1-y) + [14 + 6\ln(1-y) + \ln^2(1-y)]\ln(y) \\ &\quad - [6 + 2\ln(1-y)]\ln^2(y) + \frac{4}{3}\ln^2(y)\Bigg] + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3.17}$$

which, in expanded form, as in Eq. (3.15), can be written as

$$\begin{aligned} K_3(y) &= -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 + 2\ln(y) + \sum_{n=1}^{\infty} \frac{y^n}{n}\right) - 7 + 6\ln(y) - 2\ln^2(y) \\ &\quad - \sum_{n=1}^{\infty} \frac{2}{n} y^n \ln(y) + \sum_{n=1}^{\infty} \left(\frac{1+3n}{n^2} - \frac{1}{n} S_1(n) \right) y^n \\ &\quad + \varepsilon\Bigg[\frac{4}{3}\ln^3(y) - 6\ln^2(y) + 14\ln(y) - 15 + \frac{8}{3}\zeta_3 \end{aligned}$$

[5] In two particular cases (integrals K_8 and K_{13}), the sum in n actually starts from $n = -1$.

[6] That is, the case where $C_{i,j,k}(n) = 0$ for $k > 0$.

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\frac{2}{n}y^n\ln^2(y) - 2\sum_{n=1}^{\infty}\left(\frac{1+3n}{n^2} - \frac{1}{n}S_1(n)\right)y^n\ln(y)\\
&+\sum_{n=1}^{\infty}\left(\frac{1+3n+7n^2}{n^3} - \frac{1+3n}{n^2}S_1(n) + \frac{1}{2n}S_1^2(n) - \frac{1}{2n}S_2(n)\right)y^n\Bigg]\\
&+\mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{3.18}
$$

where

$$
S_k(n) = \sum_{i=1}^{n}\frac{1}{i^k}. \tag{3.19}
$$

We proceed as follows: First, we clear the denominators in Eq. (3.16), which in this case means multiplying by $(2-y)(1-y)y$. After that, we can then insert the expanded version of K_3 given in Eq. (3.18) together with the generic expanded version of K_4 according to Eq. (3.15). The resulting equation will be satisfied if the coefficients in the expansion in y, for each power in $\ln(y)$, are equal on both sides of the equation. This leads to the following system of difference equations

$$
\begin{aligned}
0 &= (2-n)C_{4,-2,3}(n-2) + (3n-1)C_{4,-2,3}(n-1) - 2(n+1)C_{4,-2,3}(n), && (3.20)\\
0 &= (2-n)C_{4,-2,k}(n-2) + (3n-1)C_{4,-2,k}(n-1) - 2(n+1)C_{4,-2,k}(n)\\
&\quad -(k+1)C_{4,-2,k+1}(n-2) + 3(k+1)C_{4,-2,k+1}(n-1) - 2(k+1)C_{4,-2,k+1}(n)\\
&\quad \text{for}\quad k = 0,1,2, && (3.21)\\
0 &= -2C_{4,j,3}(n-2) + 4C_{4,j,3}(n-1) - 4C_{4,j,3}(n) + (2-n)C_{4,j+1,3}(n-2)\\
&\quad +(3n-1)C_{4,j+1,3}(n-1) - 2(n+1)C_{4,j+1,3}(n)\\
&\quad \text{for}\quad j = -2,-1,0, && (3.22)\\
0 &= -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2)\\
&\quad +(3n-1)C_{4,j+1,k}(n-1) - 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2)\\
&\quad +3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n)\\
&\quad \text{for}\quad j = -2,\ k = 1,2 \quad\text{and}\quad j = -1,\ k = 2, && (3.23)\\
0 &= -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2)\\
&\quad +(3n-1)C_{4,j+1,k}(n-1) - 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2)\\
&\quad +3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n) + \frac{j-2k-7jk}{(n-1)n}\\
&\quad \text{for}\quad j = -2,\ k = 0;\ j = -1,\ k = 1 \quad\text{and}\quad j = 0,\ k = 2, && (3.24)\\
0 &= -2C_{4,j,k}(n-2) + 4C_{4,j,k}(n-1) - 4C_{4,j,k}(n) + (2-n)C_{4,j+1,k}(n-2)\\
&\quad +(3n-1)C_{4,j+1,k}(n-1) - 2(n+1)C_{4,j+1,k}(n) - (k+1)C_{4,j+1,k+1}(n-2)\\
&\quad +3(k+1)C_{4,j+1,k+1}(n-1) - 2(k+1)C_{4,j+1,k+1}(n)\\
&\quad +\frac{2(-1)^k(k+1)}{(n-1)n}S_1(n) - 2(-1)^k(k+1)\frac{3n^2-n-1}{(n-1)^2n^2}\\
&\quad \text{for}\quad j = -1,\ k = 0 \quad\text{and}\quad j = 0,\ k = 1, && (3.25)\\
0 &= -2C_{4,0,0}(n-2) + 4C_{4,0,0}(n-1) - 4C_{4,0,0}(n) + (2-n)C_{4,1,0}(n-2)\\
&\quad +(3n-1)C_{4,1,0}(n-1) - 2(n+1)C_{4,1,0}(n) - C_{4,1,1}(n-2)\\
&\quad +3C_{4,1,1}(n-1) - 2C_{4,1,1}(n) + \frac{2\,(3n^2-n-1)}{(n-1)^2n^2}S_1(n) - \frac{S_1^2(n)}{(n-1)n}
\end{aligned}
$$

$$+\frac{S_2(n)}{(n-1)n} - \frac{2\,(6n^4-6n^3+1)}{(n-1)^3n^3}. \tag{3.26}$$

This system is triangular. So, the coefficients $C_{4,j,k}(n)$ can be obtained successively by solving Eqs. (3.20) to (3.26) one after the other, inserting the results of the $C_{4,j,k}$'s obtained at each step in subsequent equations. We start with the coefficient for which the value of j is the lowest and the value of k is the largest (in this case, $C_{4,-2,3}(n)$), and proceed to obtain the coefficients for lower values of k, keeping j fixed, until all coefficients for $k = 3$ to $k = 0$ are determined. After that, we increase the value of j by one and repeat the procedure until all values of j are exhausted and all coefficients are determined. All of this can be done automatically using `Sigma` and `HarmonicSums`. The results will be given in terms of harmonic sums [94],

$$S_{b,\vec{a}}(n) = \sum_{k=1}^{n} \frac{(\mathrm{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k), \quad S_\emptyset = 1,\ b, a_i\ \in \mathbb{Z}\backslash\{0\}\ , \tag{3.27}$$

and generalized harmonic sums [88, 95],

$$S_{b,\vec{a}}(\{c,\vec{d}\},n) = \sum_{k=1}^{n} \frac{c^k}{k^b} S_{\vec{a}}(\{\vec{d}\},k), \quad b, a_i \in \mathbb{N}\backslash\{0\}, \quad c, d_i \in \mathbb{Z}\backslash\{0\},\ S_\emptyset = 1\ . \tag{3.28}$$

Since all difference equations are of second order, we need at least two initial values in order to solve them. The first few expansion coefficients of $K_4(y)$ are given by

$$\begin{aligned}
K_4(y) &= -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-2 + y + \frac{2}{3}y^2 + \frac{y^3}{2} + 2\ln(y)\right) - 4 + 2y + \frac{5}{6}y^2 + \frac{y^3}{3} \\
&+ \left(4 - 2y - \frac{4}{3}y^2 - y^3\right)\ln(y) - 2\ln^2(y) + \varepsilon\Bigg[-8 + \frac{8}{3}\zeta_3 + 4y + \frac{5}{3}y^2 \\
&+ \frac{5}{6}y^3 + \left(-4 + 2y + \frac{4}{3}y^2 + y^3\right)\ln^2(y) + \left(8 - 4y - \frac{5}{3}y^2 - \frac{2}{3}y^3\right)\ln(y) \\
&+ \frac{4}{3}\ln^3(y)\Bigg] + \mathcal{O}(\varepsilon^2, y^4).
\end{aligned} \tag{3.29}$$

These were obtained as described in the previous section. In general, the solutions to the difference equations (3.20–3.26) for general values of n will be valid starting from a certain value $n = n_0$, and therefore the initial values used to solve the difference equations must also be taken starting from $n \geq n_0$. There are a few cases above where we can start from $n_0 = 0$, but in most cases we must take the initial values starting from $n_0 = 1$. Later, when we formally perform the sum (3.15), the expansion terms for $n < n_0$ will have to be added separately.

We obtain the following results for the expansion coefficients,

$$C_{4,-2,k}(n) = 0 \quad \text{for} \quad k \geq 1,\, n \geq 0 \quad \text{and} \quad k = 0,\, n \geq 1, \tag{3.30}$$

$$C_{4,-1,k}(n) = 0 \quad \text{for} \quad k \geq 2,\, n \geq 0, \quad \text{and} \quad k = 1,\, n \geq 1, \tag{3.31}$$

$$C_{4,-1,0}(n) = \frac{2}{n+1} \quad \text{for} \quad n \geq 1, \tag{3.32}$$

$$C_{4,0,k}(n) = 0 \quad \text{for} \quad k = 3,\, n \geq 0, \quad \text{and} \quad k = 2,\, n \geq 1, \tag{3.33}$$

$$C_{4,0,1}(n) = -\frac{4}{n+1} \quad \text{for} \quad n \geq 1, \tag{3.34}$$

$$C_{4,0,0}(n) = \frac{8}{n+1} + \frac{2(1-2n)}{n(n+1)} S_1(n) - \frac{2^{1-n}}{n(n+1)} S_1(\{2\},n) \quad \text{for} \quad n \geq 1, \tag{3.35}$$

$$
\begin{aligned}
C_{4,1,3}(n) &= 0 \quad \text{for} \quad n \geq 1, && (3.36)\\
C_{4,1,2}(n) &= \frac{4}{n+1} \quad \text{for} \quad n \geq 1, && (3.37)\\
C_{4,1,1}(n) &= -2C_{4,0,0}(n) \quad \text{for} \quad n \geq 1, && (3.38)\\
C_{4,1,0}(n) &= \frac{24}{n+1} + \frac{4n-3}{n(n+1)} S_1^2(n) - \frac{S_2(n)}{n(n+1)} - \frac{8}{n(n+1)2^n} S_1(\{2\},n)\\
&\quad -\frac{4S_1(n)}{n(n+1)} \left[4n - 2 + \frac{1}{2^n} S_1(\{2\},n)\right] - \frac{6}{n(n+1)2^n} S_2(\{2\},n)\\
&\quad -\frac{4(n-1)}{n(n+1)} S_{1,1}(\{\tfrac{1}{2},2\},n) + \frac{10}{n(n+1)2^n} S_{1,1}(\{2,1\},n) \quad \text{for} \quad n \geq 1, && (3.39)
\end{aligned}
$$

with the separate values,

$$
\begin{aligned}
&C_{4,-2,0}(0) = -1, \quad C_{4,-1,1}(0) = 2, \quad C_{4,-1,0}(0) = -2, \quad C_{4,0,2}(0) = -2,\\
&C_{4,0,1}(0) = 4, \quad C_{4,0,0}(0) = -4, \quad C_{4,1,3}(0) = \frac{4}{3}, \quad C_{4,1,2}(0) = -4,\\
&C_{4,1,1}(0) = 8, \quad C_{4,1,0}(0) = \frac{8}{3}\zeta_3 - 8, && (3.40)
\end{aligned}
$$

We can insert the results from Eq. (3.30) to Eq. (3.40) into the expansion (3.15) and perform the sums using `Sigma`, `HarmonicSums`, `EvaluateMultiSums` and `SumProduction` [96–98]. We obtain

$$
\begin{aligned}
K_4(y) &= -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left[\frac{2}{y} H_1(y) + 2\ln(y) - 4\right] - 12 - \frac{2}{y}(y-2) H_{2,1}(y) + \frac{(y-3)}{y} H_1^2(y)\\
&\quad + \frac{8}{y} H_1(y) + \left(8 - \frac{4}{y} H_1(y)\right)\ln(y) - 2\ln^2(y) + \varepsilon\left[\frac{y-2}{y}(-8H_{2,1}(y)\right.\\
&\quad + 4H_{1,2,1}(y) + 6H_{2,1,1}(y) - 4H_{2,2,1}(y)) - \frac{3y-7}{3y} H_1^3(y) + \frac{4}{y}(y-3) H_1^2(y)\\
&\quad + \frac{24}{y} H_1(y) + \frac{8}{3}\zeta_3 - 32 + \left(\frac{4}{y}(y-2) H_{2,1}(y) - \frac{2}{y}(y-3) H_1^2(y) - \frac{16}{y} H_1(y)\right.\\
&\quad \left.\left. + 24\right)\ln(y) + \left(\frac{4}{y} H_1(y) - 8\right)\ln^2(y) + \frac{4}{3}\ln^3(y)\right] + \mathcal{O}(\varepsilon^2). && (3.41)
\end{aligned}
$$

Notice the presence of the letters

$$
\frac{1}{2-y}, \quad \frac{1}{1-y} \text{ and } \quad \frac{1}{y}. \tag{3.42}
$$

After we go back to the original variable $x = 1 - y$, we obtain a representation in terms of the standard harmonic polylogarithms with the letters (3.12),

$$
\begin{aligned}
K_4(x) &= -\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\left[\frac{2}{1-x} H_0(x) + 2H_1(x) + 4\right] - \frac{2(1+x)}{1-x} H_{0,-1}(x) - \frac{2+x}{1-x} H_0^2(x)\\
&\quad + \frac{1}{1-x}(2(1+x)H_{-1}(x) - 4H_1(x) - 8) H_0(x) - 2H_1^2(x) - 8H_1(x) - 12\\
&\quad + \frac{1+x}{1-x}\zeta_2 + \varepsilon\left[\left(\frac{1+x}{1-x}(2\zeta_2 - 4H_{0,-1}(x)) - 24\right) H_1(x) - \frac{4}{3} H_1^3(x)\right.\\
&\quad \left. - \frac{1+x}{1-x}[H_{-1}(x)\left(-4H_{0,-1}(x) - 3H_0^2(x) - (4H_1(x) + 8) H_0(x) + 2\zeta_2\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+2H_0(x)\left(H_{0,-1}(x)-\zeta_2\right)+8H_{0,-1}(x)+4H_{0,-1,-1}(x)+2H_{0,0,-1}(x)\\
&+2H_0(x)H_{-1}^2(x)-4\zeta_2]-\frac{4+3x}{3(1-x)}H_0^3(x)-\frac{2+x}{1-x}\left(2H_1(x)+4\right)H_0^2(x)\\
&-\frac{2}{1-x}\left(2H_1^2(x)+8H_1(x)+12\right)H_0(x)-8H_1^2(x)+\frac{2(7-x)}{3(1-x)}\zeta_3-32\Big]\\
&+\mathcal{O}(\varepsilon^2). \qquad (3.43)
\end{aligned}
$$

4 Results

We calculated the heavy-quark form factors F_I, $I = V, A, S, P$, up to two loops and $\mathcal{O}(\varepsilon^2)$. Due to the length of the expressions we list here only the expansion corresponding to the low-energy ($0 < q^2 \ll m^2$), high-energy ($|q^2| \gg m^2$) and threshold ($|q^2| \approx 4m^2$) region up to $\mathcal{O}(\varepsilon)$. In appendices B, C, D, and E we present the complete analytic expressions up to $\mathcal{O}(\varepsilon)$. The full expressions up to $\mathcal{O}(\varepsilon^2)$ are provided as supplemental material together with this publication. We present renormalized results for all form factors but the singlet contributions to the axial-vector and pseudo-scalar currents $\hat{F}_{A,i}^{\mathrm{s},(2)}$ and $\hat{F}_P^{\mathrm{s},(2)}$ for which we present the bare results as discussed in Section 2.4. The expansions have been obtained with the help of the Mathematica packages `Sigma` and `HarmonicSums`.

For convenience, we collect here the notation used in the presentation of the results. We use the dimensionless variable x, Eq. (3.5). The kinematic regions of interest correspond to $x \to 1$ ($q^2 = 0$), $x \to \pm 0$ ($q^2 = \mp\infty$) and $x \to -1$ ($q^2 = 4m^2$).

Since the region $0 < q^2 < 4m^2$ corresponds to the upper half of the unit circle in the complex plane it is convenient to define the variable ϕ by

$$
x = e^{i\phi} \qquad (4.1)
$$

and to expand around $\phi = 0$ instead.

In the threshold region we use the velocity of the heavy quarks as basic variable

$$
\beta = \sqrt{1-\frac{4m^2}{q^2}} \quad \leftrightarrow \quad x = \frac{\beta-1}{\beta+1} \qquad (4.2)
$$

and expand around $\beta = 0$. This avoids the appearance of square roots.

For the presentation of the complete analytic results in appendices B, C, D and E we introduce the following subsidiary variables

$$
x_+ = \frac{1}{1+x}\,, \quad \eta = \frac{1}{(1-x)(1+x)}\,, \quad \xi = \frac{(1+x^2)}{(1-x)(1+x)}\,. \qquad (4.3)
$$

Furthermore, we use the abbreviations

$$
\begin{aligned}
c_1 &= 12\zeta_2 \ln^2(2) + \ln^4(2) + 24\mathrm{Li}_4\Big(\frac{1}{2}\Big)\\
c_2 &= 26\zeta_2^2 \ln(2) - 20\zeta_2 \ln^3(2) - \ln^5(2) + 120\mathrm{Li}_5\Big(\frac{1}{2}\Big) \qquad (4.4)
\end{aligned}
$$

and

$$
H_{a_1,\ldots,a_n} \equiv H_{a_1,\ldots,a_n}(x)\,, \quad L \equiv H_0(x) = \ln(x)\,, \quad \bar{H}_0(\phi) \equiv H_0(\phi) - \frac{i\pi}{2}\,, \qquad (4.5)
$$

with the harmonic polylogarithms $H_{a_1,\ldots,a_n}(x)$ as defined in Eq. (3.11).

To validate our results we compare them to the existing literature. Up to $\mathcal{O}(\varepsilon^0)$ we agree with all available unrenormalized results for the various form factors. Note that in Refs. [20, 21, 23] a different normalization for the master integrals has been used, resulting in a difference proportional to $(\Gamma(1+\varepsilon)/\exp(\gamma_E\varepsilon))^2$, where γ_E denotes the Euler-Mascheroni constant.

At $\mathcal{O}(\varepsilon)$ we can compare our results for the vector form factors with the results given in Ref. [24] and find a difference

$$- C_F C_A \left\{ \varepsilon \left[\frac{1037 x^3}{(1+x)^6} \right] \right\}, \tag{4.6}$$

which has been reported in [26] already. In addition we compared our analytic results as well as the corresponding expansions with the results for the color-planar limit given in [26] and find agreement.

Comparing the renormalized results, we found that the wave function renormalization in Refs. [20, 21, 23] has been performed incorrectly, resulting in a difference proportional to ζ_2 at $\mathcal{O}(\varepsilon^0)$. For the vector form factor we agree with the renormalized results in the color-planar limit given in Ref. [26] and up to the term mentioned in Eq. (4.6) above with the results given in Ref. [24].

Although we do not present results for the renormalized singlet contributions we like to point out, that we cannot reproduce the result for the singlet axial-vector contribution presented in Ref. [22].

4.1 Low energy region $0 < q^2 \ll m^2$

The low energy limit of the space-like form factors is given by $x \to 1$. To facilitate the expansion of the HPLs in the region, we use the variable x as defined in Eq. (4.1) and expand around $\phi = 0$. In the following we present the series expansion of the one and two-loop form factors, denoted by $\bar{F}$, for all the currents up to 4th order in ϕ.

4.1.1 Vector form factor

For the vector form factors we find

$$\begin{aligned}
\bar{F}^{(1)}_{V,1} &= C_F \left\{ \frac{1}{\varepsilon} \left[-\frac{2}{3}\phi^2 - \frac{2}{45}\phi^4 \right] + \left[-\frac{1}{2}\phi^2 - \frac{17}{120}\phi^4 \right] + \varepsilon \left[\phi^2 \Big(-2 - \frac{\zeta_2}{3} \Big) \right.\right. \\
&\quad \left.\left. + \phi^4 \Big(-\frac{1}{4} - \frac{\zeta_2}{45} \Big) \right] + \varepsilon^2 \left[\phi^2 \Big(-4 - \frac{\zeta_2}{4} + \frac{2\zeta_3}{9} \Big) + \phi^4 \Big(-\frac{2}{3} - \frac{17\zeta_2}{240} + \frac{2\zeta_3}{135} \Big) \right] \right\}. \qquad (4.7) \\
\bar{F}^{(2)}_{V,1} &= C_F^2 \left\{ \frac{2}{9\varepsilon^2}\phi^4 + \frac{1}{3\varepsilon}\phi^4 + \left[\phi^2 \Big(-\frac{47}{36} - \frac{175}{9}\zeta_2 + 48\ln(2)\zeta_2 - 12\zeta_3 \Big) + \phi^4 \Big(\frac{14473}{6480} \right.\right. \\
&\quad \left.\left. - \frac{34243\zeta_2}{5400} + \frac{68}{5}\ln(2)\zeta_2 - \frac{17}{5}\zeta_3 \Big) \right] + \varepsilon \left[\phi^2 \Big(\frac{11713}{216} - 8c_1 - \frac{6763}{54}\zeta_2 + \frac{578}{3}\ln(2)\zeta_2 + \frac{504}{5}\zeta_2^2 \right.\right. \\
&\quad \left.\left. - \frac{1567}{18}\zeta_3 \Big) + \phi^4 \Big(\frac{3508637}{194400} - \frac{34c_1}{15} - \frac{758317\zeta_2}{20250} + \frac{10139}{180}\ln(2)\zeta_2 + \frac{714}{25}\zeta_2^2 - \frac{295457\zeta_3}{10800} \Big) \right] \right\} \\
&\quad + C_F C_A \left\{ \frac{1}{\varepsilon^2} \left[\frac{11}{9}\phi^2 + \frac{11}{135}\phi^4 \right] + \frac{1}{\varepsilon} \left[\phi^2 \Big(-\frac{94}{27} + \frac{4\zeta_2}{3} \Big) + \phi^4 \Big(-\frac{91}{405} + \frac{4\zeta_2}{45} \Big) \right] \right.
\end{aligned}$$

$$
\begin{aligned}
&+\left[\phi^2\Big(-\frac{2579}{324}+\frac{155}{18}\zeta_2-24\ln(2)\zeta_2+\frac{26}{3}\zeta_3\Big)+\phi^4\Big(-\frac{36239}{19440}+\frac{7447\zeta_2}{2160}-\frac{34}{5}\ln(2)\zeta_2\right.\\
&\left.+\frac{169}{90}\zeta_3\Big)\right]+\varepsilon\left[\phi^2\Big(-\frac{134327}{1944}+4c_1+\frac{1297}{27}\zeta_2-\frac{289}{3}\ln(2)\zeta_2-\frac{608}{15}\zeta_2^2+\frac{1487}{36}\zeta_3\Big)\right.\\
&\left.+\phi^4\Big(-\frac{278341}{23328}+\frac{17c_1}{15}+\frac{110029\zeta_2}{6480}-\frac{10139}{360}\ln(2)\zeta_2-\frac{613}{45}\zeta_2^2+\frac{12041}{864}\zeta_3\Big)\right]\Bigg\}\\
&+C_Fn_lT_F\Bigg\{\frac{1}{\varepsilon^2}\left[-\frac{4}{9}\phi^2-\frac{4}{135}\phi^4\right]+\frac{1}{\varepsilon}\left[\frac{20}{27}\phi^2+\frac{4}{81}\phi^4\right]+\left[\phi^2\Big(\frac{283}{81}+\frac{16\zeta_2}{9}\Big)\right.\\
&\left.+\phi^4\Big(\frac{3139}{4860}+\frac{16\zeta_2}{135}\Big)\right]+\varepsilon\left[\phi^2\Big(\frac{8827}{486}+\frac{181\zeta_2}{27}+\frac{32\zeta_3}{9}\Big)+\phi^4\Big(\frac{95527}{29160}+\frac{1777\zeta_2}{1620}+\frac{32\zeta_3}{135}\Big)\right]\Bigg\}\\
&+C_FT_F\Bigg\{\left[\phi^2\Big(-\frac{1099}{81}+9\zeta_2\Big)+\phi^4\Big(-\frac{21019}{4860}+\frac{53\zeta_2}{20}\Big)\right]+\varepsilon\left[\phi^2\Big(-\frac{635}{18}+\frac{937}{27}\zeta_2\right.\\
&\left.-\frac{154}{3}\ln(2)\zeta_2+\frac{1601}{54}\zeta_3\Big)+\phi^4\Big(-\frac{8293}{648}+\frac{9821}{810}\zeta_2-\frac{283}{18}\ln(2)\zeta_2+\frac{29651\zeta_3}{3240}\Big)\right] \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
\bar{F}^{(1)}_{V,2}&=C_F\Bigg\{\left[2+\frac{1}{3}\phi^2+\frac{7}{180}\phi^4\right]+\varepsilon\left[8+\frac{5}{3}\phi^2+\frac{41}{180}\phi^4\right]+\varepsilon^2\left[16+\zeta_2+\phi^2\Big(4+\frac{\zeta_2}{6}\Big)\right.\\
&\left.+\phi^4\Big(\frac{19}{30}+\frac{7\zeta_2}{360}\Big)\right]\Bigg\}. \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\bar{F}^{(2)}_{V,2}&=C_F^2\Bigg\{\frac{1}{\varepsilon}\left[-\frac{4}{3}\phi^2-\frac{14}{45}\phi^4\right]+\left[-31+40\zeta_2-48\ln(2)\zeta_2+12\zeta_3+\phi^2\Big(-\frac{77}{5}+\frac{122}{5}\zeta_2\right.\\
&\left.-\frac{184}{5}\ln(2)\zeta_2+\frac{46}{5}\zeta_3\Big)+\phi^4\Big(-\frac{4931}{1260}+\frac{2963}{350}\zeta_2-\frac{1478}{105}\ln(2)\zeta_2+\frac{739}{210}\zeta_3\Big)\right]+\varepsilon\left[-\frac{1243}{6}\right.\\
&\left.+8c_1+\frac{944}{3}\zeta_2-384\ln(2)\zeta_2-\frac{504}{5}\zeta_2^2+176\zeta_3+\phi^2\Big(-\frac{19666}{225}+\frac{92c_1}{15}+\frac{3704}{25}\zeta_2\right.\\
&\left.-\frac{14164}{75}\ln(2)\zeta_2-\frac{1932}{25}\zeta_2^2+\frac{7201}{75}\zeta_3\Big)+\phi^4\Big(-\frac{9903863}{396900}+\frac{739c_1}{315}+\frac{10057561\zeta_2}{220500}\right.\\
&\left.-\frac{125887\ln(2)\zeta_2}{2205}-\frac{739}{25}\zeta_2^2+\frac{1376111\zeta_3}{44100}\Big)\right]\Bigg\}+C_FC_A\Bigg\{\left[\frac{317}{9}-12\zeta_2+24\ln(2)\zeta_2-6\zeta_3\right.\\
&\left.+\phi^2\Big(\frac{1699}{270}-\frac{137}{15}\zeta_2+\frac{92}{5}\ln(2)\zeta_2-\frac{23}{5}\zeta_3\Big)+\phi^4\Big(\frac{11927}{22680}-\frac{21269\zeta_2}{6300}+\frac{739}{105}\ln(2)\zeta_2\right.\\
&\left.-\frac{739}{420}\zeta_3\Big)\right]+\varepsilon\left[\frac{12881}{54}-4c_1-\frac{313}{3}\zeta_2+192\ln(2)\zeta_2+\frac{252}{5}\zeta_2^2-72\zeta_3\right.\\
&\left.+\phi^2\Big(\frac{485453}{8100}-\frac{46c_1}{15}-\frac{8983}{150}\zeta_2+\frac{7082}{75}\ln(2)\zeta_2+\frac{966}{25}\zeta_2^2-\frac{6281}{150}\zeta_3\Big)\right.\\
&\left.+\phi^4\Big(\frac{50620531}{4762800}-\frac{739c_1}{630}-\frac{4335431\zeta_2}{220500}+\frac{125887\ln(2)\zeta_2}{4410}+\frac{739}{50}\zeta_2^2-\frac{1224967\zeta_3}{88200}\Big)\right]\Bigg\}
\end{aligned}
$$

$$
+ C_F n_l T_F \Bigg\{ \Bigg[-\frac{100}{9} - \frac{62}{27}\phi^2 - \frac{253}{810}\phi^4 \Bigg] + \varepsilon \Bigg[-\frac{1922}{27} - 12\zeta_2
$$

$$
+ \phi^2\Big(-\frac{1405}{81} - 2\zeta_2 \Big) + \phi^4\Big(-\frac{13147}{4860} - \frac{7\zeta_2}{30} \Big) \Bigg] \Bigg\} + C_F T_F \Bigg\{ \Bigg[\frac{476}{9} - 32\zeta_2 + \phi^2\Big(\frac{622}{27} - 14\zeta_2 \Big)
$$

$$
+ \phi^4\Big(\frac{4841}{810} - \frac{109\zeta_2}{30} \Big) \Bigg] + \varepsilon \Bigg[\frac{2254}{27} - \frac{308}{3}\zeta_2 + 192\ln(2)\zeta_2 - 112\zeta_3 + \phi^2\Big(\frac{4247}{81} - \frac{490}{9}\zeta_2
$$

$$
+ 84\ln(2)\zeta_2 - 49\zeta_3 \Big) + \phi^4\Big(\frac{16753}{972} - \frac{2203}{135}\zeta_2 + \frac{109}{5}\ln(2)\zeta_2 - \frac{763}{60}\zeta_3 \Big) \Bigg] \Bigg\} . \tag{4.10}
$$

Note, that $\bar{F}_{V,2}$ is UV and IR finite in this limit and the leading term agrees with the computation of the anomalous magnetic moment in [99].

4.1.2 Axial-vector form factor

For the axial-vector form factor we present the renormalized results for the non-singlet contributions and the unrenormalized one for the singlet parts.

$$
\bar{F}^{(1),\mathrm{ns}}_{A,1} = C_F \Bigg\{ \frac{1}{\varepsilon} \Bigg[-\frac{2}{3}\phi^2 - \frac{2}{45}\phi^4 \Bigg] + \Bigg[-2 - \frac{5}{6}\phi^2 - \frac{13}{72}\phi^4 \Bigg] + \varepsilon \Bigg[\phi^2\Big(-\frac{7}{3} - \frac{\zeta_2}{3} \Big) + \phi^4\Big(-\frac{29}{90}
$$

$$
- \frac{\zeta_2}{45} \Big) \Bigg] + \varepsilon^2 \Bigg[-\zeta_2 + \phi^2\Big(-4 - \frac{5\zeta_2}{12} + \frac{2\zeta_3}{9} \Big) + \phi^4\Big(-\frac{7}{10} - \frac{13\zeta_2}{144} + \frac{2\zeta_3}{135} \Big) \Bigg] \Bigg\} . \tag{4.11}
$$

$$
\bar{F}^{(2),\mathrm{ns}}_{A,1} = C_F^2 \Bigg\{ \frac{1}{\varepsilon^2} \Bigg[\frac{2}{9}\phi^4 \Bigg] + \frac{1}{\varepsilon} \Bigg[\frac{4}{3}\phi^2 + \frac{29}{45}\phi^4 \Bigg] + \Bigg[-\frac{29}{3} + 32\zeta_2 - 32\ln(2)\zeta_2 + 8\zeta_3 + \phi^2\Big(-\frac{1121}{180}
$$

$$
- \frac{217}{45}\zeta_2 + \frac{128}{5}\ln(2)\zeta_2 - \frac{32}{5}\zeta_3 \Big) + \phi^4\Big(\frac{95341}{45360} - \frac{5423\zeta_2}{5400} + \frac{376}{105}\ln(2)\zeta_2 - \frac{94}{105}\zeta_3 \Big) \Bigg]
$$

$$
+ \varepsilon \Bigg[-\frac{1889}{18} + \frac{16c_1}{3} + 56\zeta_2 - \frac{160}{3}\ln(2)\zeta_2 - \frac{336}{5}\zeta_2^2 + \frac{232}{3}\zeta_3 + \phi^2\Big(-\frac{18997}{5400} - \frac{64c_1}{15}
$$

$$
- \frac{49259\zeta_2}{1350} + \frac{6958}{75}\ln(2)\zeta_2 + \frac{1344}{25}\zeta_2^2 - \frac{14777}{450}\zeta_3 \Big) + \phi^4\Big(-\frac{5022433}{9525600} - \frac{188c_1}{315} - \frac{44327\zeta_2}{20250}
$$

$$
+ \frac{721523\ln(2)\zeta_2}{44100} + \frac{188}{25}\zeta_2^2 - \frac{131151\zeta_3}{19600} \Big) \Bigg] \Bigg\} + C_F C_A \Bigg\{ \frac{1}{\varepsilon^2} \Bigg[\frac{11}{9}\phi^2 + \frac{11}{135}\phi^4 \Bigg] + \frac{1}{\varepsilon} \Bigg[\phi^2\Big(-\frac{94}{27}
$$

$$
+ \frac{4\zeta_2}{3} \Big) + \phi^4\Big(-\frac{91}{405} + \frac{4\zeta_2}{45} \Big) \Bigg] + \Bigg[-\frac{143}{9} - 8\zeta_2 + 16\ln(2)\zeta_2 - 4\zeta_3 + \phi^2\Big(-\frac{19813}{1620}
$$

$$
+ \frac{317}{90}\zeta_2 - \frac{64}{5}\ln(2)\zeta_2 + \frac{88}{15}\zeta_3 \Big) + \phi^4\Big(-\frac{413831}{136080} + \frac{14479\zeta_2}{10800} - \frac{188}{105}\ln(2)\zeta_2 + \frac{197}{315}\zeta_3 \Big) \Bigg]
$$

$$
+ \varepsilon \Bigg[-\frac{887}{54} - \frac{8c_1}{3} - \frac{59}{3}\zeta_2 + \frac{80}{3}\ln(2)\zeta_2 + \frac{168}{5}\zeta_2^2 - \frac{68}{3}\zeta_3 + \phi^2\Big(-\frac{3362201}{48600} + \frac{32c_1}{15}
$$

$$
+ \frac{1753}{150}\zeta_2 - \frac{3479}{75}\ln(2)\zeta_2 - \frac{1276}{75}\zeta_2^2 + \frac{16777}{900}\zeta_3 \Big) + \phi^4\Big(-\frac{281936227}{28576800} + \frac{94c_1}{315} + \frac{583651\zeta_2}{378000}
$$

$$
- \frac{721523\ln(2)\zeta_2}{88200} - \frac{698}{225}\zeta_2^2 + \frac{5002453\zeta_3}{1058400}\Big)\Big]\Big\} + C_F n_l T_F \Big\{ \frac{1}{\varepsilon^2}\Big[- \frac{4}{9}\phi^2 - \frac{4}{135}\phi^4\Big]
$$

$$
+ \frac{1}{\varepsilon}\Big[\frac{20}{27}\phi^2 + \frac{4}{81}\phi^4\Big] + \Big[\frac{28}{9} + \phi^2\Big(\frac{361}{81} + \frac{16\zeta_2}{9}\Big) + \phi^4\Big(\frac{3901}{4860} + \frac{16\zeta_2}{135}\Big)\Big]
$$

$$
+ \varepsilon\Big[- \frac{34}{27} + 12\zeta_2 + \phi^2\Big(\frac{9229}{486} + \frac{235\zeta_2}{27} + \frac{32\zeta_3}{9}\Big) + \phi^4\Big(\frac{105253}{29160} + \frac{431\zeta_2}{324} + \frac{32\zeta_3}{135}\Big)\Big]\Big\}
$$

$$
+ C_F T_F \Big\{ \Big[\frac{460}{9} - 32\zeta_2 + \phi^2\Big(\frac{491}{81} - 3\zeta_2\Big) + \phi^4\Big(\frac{1343}{4860} - \frac{3\zeta_2}{20}\Big)\Big] + \varepsilon\Big[\frac{3998}{27} - \frac{412}{3}\zeta_2
$$

$$
+ 192\ln(2)\zeta_2 - 112\zeta_3 + \phi^2\Big(\frac{4931}{162} - \frac{653}{27}\zeta_2 + \frac{62}{3}\ln(2)\zeta_2 - \frac{667}{54}\zeta_3\Big)
$$

$$
+ \phi^4\Big(\frac{10963}{1944} - \frac{1436}{405}\zeta_2 + \frac{97}{90}\ln(2)\zeta_2 - \frac{2101\zeta_3}{3240}\Big)\Big]\Big\}. \tag{4.12}
$$

$$
\bar{\tilde{F}}^{(2),\mathrm{s}}_{A,1} = C_F T_F \Big\{ - \frac{6}{\varepsilon} + \Big[- \frac{37}{3} + \frac{64}{3}\zeta_2 + \phi^2\Big(- \frac{85}{18} + \frac{136\zeta_2}{45}\Big) + \phi^4\Big(- \frac{341}{540} + \frac{2554\zeta_2}{4725}\Big)\Big]
$$

$$
+ \varepsilon\Big[- \frac{817}{18} + \frac{266}{9}\zeta_2 - 128\ln(2)\zeta_2 + \frac{224}{3}\zeta_3 + \phi^2\Big(- \frac{3659}{540} + \frac{4876}{675}\zeta_2 - \frac{272}{15}\ln(2)\zeta_2
$$

$$
+ \frac{476}{45}\zeta_3\Big) - \frac{1}{2}i\phi^3\zeta_2 + \phi^4\Big(- \frac{14281}{16200} + \frac{985837\zeta_2}{1984500} - \frac{5108\ln(2)\zeta_2}{1575} + \frac{1277}{675}\zeta_3\Big)\Big]\Big\}. \tag{4.13}
$$

$$
\bar{F}^{(1),\mathrm{ns}}_{A,2} = C_F \Big\{ \Big[\frac{14}{3} + \frac{11}{15}\phi^2 + \frac{103}{1260}\phi^4\Big] + \varepsilon\Big[8 + \frac{31}{15}\phi^2 + \frac{389}{1260}\phi^4\Big] + \varepsilon^2\Big[16 + \frac{7}{3}\zeta_2
$$

$$
+ \phi^2\Big(4 + \frac{11\zeta_2}{30}\Big) + \phi^4\Big(\frac{47}{70} + \frac{103\zeta_2}{2520}\Big)\Big]\Big\}. \tag{4.14}
$$

$$
\bar{F}^{(2),\mathrm{ns}}_{A,2} = C_F^2 \Big\{ \frac{1}{\varepsilon}\Big[- \frac{28}{9}\phi^2 - \frac{94}{135}\phi^4\Big] + \Big[- \frac{23}{5} + \frac{176}{5}\zeta_2 - \frac{176}{5}\ln(2)\zeta_2 + \frac{44}{5}\zeta_3 + \phi^2\Big(- \frac{11111}{945}
$$

$$
+ \frac{592}{45}\zeta_2 - \frac{88}{7}\ln(2)\zeta_2 + \frac{22}{7}\zeta_3\Big) + \phi^4\Big(- \frac{251113}{56700} + \frac{208091\zeta_2}{66150} - \frac{298}{105}\ln(2)\zeta_2 + \frac{149}{210}\zeta_3\Big)\Big]
$$

$$
+ \varepsilon\Big[\frac{15527}{450} + \frac{88c_1}{15} + \frac{28688}{225}\zeta_2 - \frac{6512}{25}\ln(2)\zeta_2 - \frac{1848}{25}\zeta_2^2 + \frac{3388}{25}\zeta_3 + \phi^2\Big(\frac{376}{11025} + \frac{44c_1}{21}
$$

$$
+ \frac{3776}{135}\zeta_2 - \frac{235856\ln(2)\zeta_2}{3675} - \frac{132}{5}\zeta_2^2 + \frac{466972\zeta_3}{11025}\Big) + \phi^4\Big(- \frac{47414267}{17860500} + \frac{149c_1}{315}
$$

$$
+ \frac{34009847\zeta_2}{6945750} - \frac{25891\ln(2)\zeta_2}{2205} - \frac{149}{25}\zeta_2^2 + \frac{1220729\zeta_3}{132300}\Big)\Big]\Big\} + C_F C_A \Big\{ \Big[\frac{7663}{135} - \frac{752}{45}\zeta_2
$$

$$
+ \frac{88}{5}\ln(2)\zeta_2 - \frac{22}{5}\zeta_3 + \phi^2\Big(\frac{5039}{378} - \frac{422}{75}\zeta_2 + \frac{44}{7}\ln(2)\zeta_2 - \frac{11}{7}\zeta_3\Big) + \phi^4\Big(\frac{27793}{12600} - \frac{56827\zeta_2}{44100}
$$

$$
+ \frac{149}{105}\ln(2)\zeta_2 - \frac{149}{420}\zeta_3\Big)\Big] + \varepsilon\Big[\frac{871991}{4050} - \frac{44c_1}{15} - \frac{19517}{675}\zeta_2 + \frac{3256}{25}\ln(2)\zeta_2 + \frac{924}{25}\zeta_2^2
$$

$$
\begin{aligned}
&- \frac{14846}{225}\zeta_3 + \phi^2\Big(\frac{20526143}{396900} - \frac{22c_1}{21} - \frac{142673\zeta_2}{15750} + \frac{117928\ln(2)\zeta_2}{3675} + \frac{66}{5}\zeta_2^2 - \frac{70838\zeta_3}{3675}\Big) \\
&+ \phi^4\Big(\frac{548179231}{71442000} - \frac{149c_1}{630} - \frac{5017571\zeta_2}{3087000} + \frac{25891\ln(2)\zeta_2}{4410} + \frac{149}{50}\zeta_2^2 - \frac{118921\zeta_3}{29400}\Big)\Big]\Big\} \\
&+ C_F n_l T_F\Big\{\Big[-\frac{412}{27} - \frac{466}{135}\phi^2 - \frac{2761}{5670}\phi^4\Big] + \varepsilon\Big[-\frac{5630}{81} - 28\zeta_2 + \phi^2\Big(-\frac{7427}{405} - \frac{22\zeta_2}{5}\Big) \\
&+ \phi^4\Big(-\frac{104863}{34020} - \frac{103\zeta_2}{210}\Big)\Big]\Big\} + C_F T_F\Big\{\Big[-\frac{412}{27} + \frac{32}{3}\zeta_2 + \phi^2\Big(-\frac{14}{27} + \frac{2\zeta_2}{5}\Big) + \phi^4\Big(\frac{4601}{5670} \\
&- \frac{103\zeta_2}{210}\Big)\Big] + \varepsilon\Big[-\frac{9230}{81} + \frac{724}{9}\zeta_2 - 64\ln(2)\zeta_2 + \frac{112}{3}\zeta_3 + \phi^2\Big(-\frac{10703}{405} + \frac{146}{9}\zeta_2 \\
&- \frac{12}{5}\ln(2)\zeta_2 + \frac{7}{5}\zeta_3\Big) + \phi^4\Big(-\frac{119839}{34020} + \frac{1238}{945}\zeta_2 + \frac{103}{35}\ln(2)\zeta_2 - \frac{103}{60}\zeta_3\Big)\Big]\Big\}. \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
\bar{\tilde{F}}^{(2),\mathrm{s}}_{A,2} &= C_F T_F\Big\{\Big[\frac{24}{\phi^2} + \frac{20}{3} + \frac{16}{15}\zeta_2 + 4i\phi\zeta_2 + \phi^2\Big(-\frac{8}{15} - \frac{184}{315}\zeta_2 + \frac{4\bar{H}_0(\phi)}{3}\Big) + \frac{13}{15}i\phi^3\zeta_2 \\
&+ \phi^4\Big(-\frac{461}{6300} - \frac{1174\zeta_2}{4725} + \frac{13\bar{H}_0(\phi)}{45}\Big)\Big] + \varepsilon\Big[\frac{1}{\phi^2}\Big(124 - 128\zeta_2\Big) + \frac{3394}{45} - \frac{11264}{225}\zeta_2 \\
&- \frac{32}{5}\ln(2)\zeta_2 + \frac{56}{15}\zeta_3 + i\phi\Big(-10\zeta_2 + 8\ln(2)\zeta_2 - 8\zeta_2\bar{H}_0(\phi)\Big) + \phi^2\Big(\frac{3173}{135} - \frac{221449\zeta_2}{33075} \\
&+ \frac{368}{105}\ln(2)\zeta_2 - \frac{92}{45}\zeta_3 - 4\bar{H}_0(\phi) - \frac{4}{3}\bar{H}_0^2(\phi)\Big) + i\phi^3\Big(-\frac{9}{5}\zeta_2 + \frac{26}{15}\ln(2)\zeta_2 - \frac{26}{15}\zeta_2\bar{H}_0(\phi)\Big) \\
&+ \phi^4\Big(\frac{5566123}{1701000} - \frac{1063927\zeta_2}{1984500} + \frac{2348\ln(2)\zeta_2}{1575} - \frac{587}{675}\zeta_3 - \frac{31\bar{H}_0(\phi)}{45} - \frac{13}{45}\bar{H}_0^2(\phi)\Big)\Big]\Big\}. \qquad (4.16)
\end{aligned}
$$

4.1.3 Scalar form factor

For the scalar form factor in the low energy limit we obtain

$$
\begin{aligned}
\bar{F}_S^{(1)} &= C_F\Big\{\frac{1}{\varepsilon}\Big[-\frac{2}{3}\phi^2 - \frac{2}{45}\phi^4\Big] + \Big[-6 - \frac{1}{3}\phi^2 - \frac{31}{180}\phi^4\Big] + \varepsilon\Big[-8 + \phi^2\Big(-\frac{7}{3} - \frac{\zeta_2}{3}\Big) \\
&+ \phi^4\Big(-\frac{49}{180} - \frac{\zeta_2}{45}\Big)\Big] + \varepsilon^2\Big[-16 - 3\zeta_2 + \phi^2\Big(-4 - \frac{\zeta_2}{6} + \frac{2\zeta_3}{9}\Big) \\
&+ \phi^4\Big(-\frac{7}{10} - \frac{31\zeta_2}{360} + \frac{2\zeta_3}{135}\Big)\Big]\Big\}. \qquad (4.17) \\
\bar{F}_S^{(2)} &= C_F^2\Big\{\frac{2}{9\varepsilon^2}\phi^4 + \frac{1}{\varepsilon}\Big[4\phi^2 + \frac{22}{45}\phi^4\Big] + \Big[33 + \phi^2\Big(\frac{62}{9} - \frac{71}{9}\zeta_2 + 28\ln(2)\zeta_2 - 7\zeta_3\Big) \\
&+ \phi^4\Big(\frac{5743}{1620} - \frac{7001\zeta_2}{5400} + \frac{11}{3}\ln(2)\zeta_2 - \frac{11}{12}\zeta_3\Big)\Big] + \varepsilon\Big[\frac{135}{2} - 120\zeta_2 + 192\ln(2)\zeta_2 - 48\zeta_3 \\
&+ \phi^2\Big(\frac{1297}{27} - \frac{14c_1}{3} - \frac{2723}{54}\zeta_2 + 98\ln(2)\zeta_2 + \frac{294}{5}\zeta_2^2 - \frac{725}{18}\zeta_3\Big) + \phi^4\Big(\frac{193531}{24300} - \frac{11c_1}{18}
\end{aligned}
$$

$$
- \frac{141179\zeta_2}{20250} + \frac{5971}{300}\ln(2)\zeta_2 + \frac{77}{10}\zeta_2^2 - \frac{29381\zeta_3}{3600}\Big)\Big]\Big\} + C_F C_A\Big\{\frac{1}{\varepsilon^2}\Big[\frac{11}{9}\phi^2 + \frac{11}{135}\phi^4\Big]
$$

$$
+ \frac{1}{\varepsilon}\Big[\phi^2\Big(-\frac{94}{27} + \frac{4\zeta_2}{3}\Big) + \phi^4\Big(-\frac{91}{405} + \frac{4\zeta_2}{45}\Big)\Big] + \Big[-\frac{185}{3} + \phi^2\Big(-\frac{650}{81} + \frac{47}{18}\zeta_2
$$

$$
- 14\ln(2)\zeta_2 + \frac{37}{6}\zeta_3\Big) + \phi^4\Big(-\frac{2389}{972} + \frac{11897\zeta_2}{10800} - \frac{11}{6}\ln(2)\zeta_2 + \frac{229}{360}\zeta_3\Big)\Big] + \varepsilon\Big[-\frac{1463}{6}
$$

$$
+ 21\zeta_2 - 96\ln(2)\zeta_2 + 24\zeta_3 + \phi^2\Big(-\frac{32507}{486} + \frac{7c_1}{3} + \frac{1873}{108}\zeta_2 - 49\ln(2)\zeta_2 - \frac{293}{15}\zeta_2^2 + \frac{629}{36}\zeta_3\Big)
$$

$$
+ \phi^4\Big(-\frac{2663879}{291600} + \frac{11c_1}{36} + \frac{212591\zeta_2}{81000} - \frac{5971}{600}\ln(2)\zeta_2 - \frac{2873}{900}\zeta_2^2 + \frac{101327\zeta_3}{21600}\Big)\Big]\Big\}
$$

$$
+ C_F n_l T_F\Big\{\frac{1}{\varepsilon^2}\Big[-\frac{4}{9}\phi^2 - \frac{4}{135}\phi^4\Big] + \frac{1}{\varepsilon}\Big[\frac{20}{27}\phi^2 + \frac{4}{81}\phi^4\Big] + \Big[\frac{52}{3} + \phi^2\Big(\frac{316}{81} + \frac{16\zeta_2}{9}\Big)
$$

$$
+ \phi^4\Big(\frac{883}{1215} + \frac{16\zeta_2}{135}\Big)\Big] + \varepsilon\Big[\frac{206}{3} + 36\zeta_2 + \phi^2\Big(\frac{4808}{243} + \frac{154\zeta_2}{27} + \frac{32\zeta_3}{9}\Big) + \phi^4\Big(\frac{2560}{729} + \frac{1037\zeta_2}{810}
$$

$$
+ \frac{32\zeta_3}{135}\Big)\Big]\Big\} + C_F T_F\Big\{\Big[\frac{52}{3} + 8i\phi\zeta_2 + \phi^2\Big(-\frac{1417}{81} + \frac{91}{9}\zeta_2 + \frac{8}{3}\bar{H}_0(\phi)\Big) + \frac{17}{15}i\phi^3\zeta_2
$$

$$
+ \phi^4\Big(-\frac{25076}{6075} + \frac{6071\zeta_2}{2700} + \frac{17}{45}\bar{H}_0(\phi)\Big)\Big] + \varepsilon\Big[\frac{350}{3} - 60\zeta_2 + i\phi\Big(16\ln(2)\zeta_2 - 16\bar{H}_0(\phi)\zeta_2\Big)
$$

$$
+ \phi^2\Big(-\frac{787}{162} + 23\zeta_2 - 58\ln(2)\zeta_2 + \frac{1811}{54}\zeta_3 + \frac{8}{3}\bar{H}_0(\phi) - \frac{8}{3}\bar{H}_0(\phi)^2\Big) + i\phi^3\Big(\frac{17}{15}\zeta_2
$$

$$
+ \frac{34}{15}\ln(2)\zeta_2 - \frac{34}{15}\zeta_2\bar{H}_0(\phi)\Big) + \phi^4\Big(-\frac{241487}{121500} + \frac{35681\zeta_2}{6750} - \frac{1997}{150}\ln(2)\zeta_2 + \frac{125491\zeta_3}{16200}
$$

$$
+ \frac{6}{5}\bar{H}_0(\phi) - \frac{17}{45}\bar{H}_0(\phi)^2\Big)\Big]\Big\}. \tag{4.18}
$$

4.1.4 Pseudo-scalar form factor

The non-singlet part of the pseudo-scalar form factor can be obtained by using Eq. (2.13)

$$
\begin{aligned}
\bar{F}_P^{(1),\mathrm{ns}} &= \bar{F}_{A,1}^{(1)} + \Big(\frac{\phi^2}{4} - \frac{\phi^4}{48}\Big)\bar{F}_{A,2}^{(1)} + \mathcal{O}(\phi^6)\,,\\
\bar{F}_P^{(2),\mathrm{ns}} &= \bar{F}_{A,1}^{(2)} + \Big(\frac{\phi^2}{4} - \frac{\phi^4}{48}\Big)\bar{F}_{A,2}^{(2)} + \mathcal{O}(\phi^6)\,.
\end{aligned} \tag{4.19}
$$

The unrernormalized singlet contribution to the pseudo-scalar form factor reads

$$
\bar{\hat{F}}_P^{(2),s} = C_F T_F\Big\{\Big[\frac{8}{3} + \frac{64}{3}\zeta_2 + 12i\phi\zeta_2 + \phi^2\Big(-\frac{68}{9} + 4\bar{H}_0(\phi) + \frac{148\zeta_2}{45}\Big) + 2i\phi^3\zeta_2 + \phi^4\Big(-\frac{689}{540}
$$

$$
+ \frac{1759\zeta_2}{4725} + \frac{2}{3}\bar{H}_0(\phi)\Big)\Big] + \varepsilon\Big[\frac{88}{9} - \frac{352}{9}\zeta_2 - 128\ln(2)\zeta_2 + \frac{224}{3}\zeta_3 + i\phi\Big(-36\zeta_2 + 24\ln(2)\zeta_2
$$

$$
- 24\zeta_2\bar{H}_0(\phi)\Big) + \phi^2\Big(\frac{5227}{135} - \frac{3182}{675}\zeta_2 - \frac{296}{15}\ln(2)\zeta_2 + \frac{518}{45}\zeta_3 - 12\bar{H}_0(\phi) - 4\bar{H}_0(\phi)^2\Big)
$$

$$+ i\phi^3\Big(- 7\zeta_2 + 4\ln(2)\zeta_2 - 4\zeta_2\bar{H}_0(\phi)\Big) + \phi^4\Big(\frac{101609}{16200} - \frac{255167\zeta_2}{496125} - \frac{3518\ln(2)\zeta_2}{1575} + \frac{1759\zeta_3}{1350} - 2\bar{H}_0(\phi) - \frac{2}{3}\bar{H}_0(\phi)^2\Big)\Bigg]\Bigg\}. \tag{4.20}$$

4.2 High energy region $\left|q^2\right| \gg m^2$

We now present the expansion of all the form factors in the asymptotic limit i.e. for $x \to 0^+$ up to $\mathcal{O}(x^2)$. The expanded form factors are denoted by $\mathcal{F}_I$. We use the abbreviation $L = \ln(x)$ in the following. The correct analytic continuation to negative values of x is given by $L \to L + i\pi$.

4.2.1 Vector form factor

$$\begin{aligned}
\mathcal{F}^{(1)}_{V,1} = C_F\Bigg\{&\frac{1}{\varepsilon}\Big[- 2 - 2L - 4Lx^2\Big] + \Big[- 4 + 2\zeta_2 - 3L - L^2 + 2(-2+L)x + x^2\Big(1 - 8L - 2L^2 + 4\zeta_2\Big)\Big] \\
&+ \varepsilon\Big[- 8 + 2\zeta_2 + 4\zeta_3 + L\Big(- 8 + \zeta_2\Big) - \frac{3L^2}{2} - \frac{L^3}{3} + x\Big(- 10 - 2\zeta_2 + 6L + L^2\Big) \\
&+ x^2\Big(4 + 8\zeta_2 + 8\zeta_3 + L\Big(- 19 + 2\zeta_2\Big) - 4L^2 - \frac{2L^3}{3}\Big)\Big] + \varepsilon^2\Big[- 16 + 6\zeta_2 + \frac{14}{5}\zeta_2^2 + \frac{20}{3}\zeta_3 \\
&+ L\Big(- 16 + \frac{3\zeta_2}{2} + \frac{14\zeta_3}{3}\Big) + L^2\Big(- 4 + \frac{\zeta_2}{2}\Big) - \frac{L^3}{2} - \frac{L^4}{12} + x\Big(- 26 - 8\zeta_2 - 4\zeta_3 \\
&+ L\Big(16 - \zeta_2\Big) + 3L^2 + \frac{L^3}{3}\Big) + x^2\Big(17 + \frac{39}{2}\zeta_2 + \frac{28}{5}\zeta_2^2 + 16\zeta_3 + L\Big(- 46 + 4\zeta_2 + \frac{28\zeta_3}{3}\Big) \\
&+ L^2\Big(- \frac{19}{2} + \zeta_2\Big) - \frac{4L^3}{3} - \frac{L^4}{6}\Big)\Big]\Bigg\}.
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\mathcal{F}^{(2)}_{V,1} = C_F^2\Bigg\{&\frac{1}{\varepsilon^2}\Big[2 + 4L + 2L^2 + 8L(1+L)x^2\Big] + \frac{1}{\varepsilon}\Big[8 + 14L + 8L^2 + 2L^3 - 4\zeta_2 - 4L\zeta_2 - 4(-2 \\
&+ L)(1+L)x + x^2\Big(- 2 + 30L + 32L^2 + 8L^3 - 8\zeta_2 - 16L\zeta_2\Big)\Big] + \Big[46 + \frac{85L}{2} + \frac{55L^2}{2} \\
&+ \frac{20L^3}{3} + \frac{7L^4}{6} + 39\zeta_2 - 4L^2\zeta_2 - 48\ln(2)\zeta_2 - \frac{118}{5}\zeta_2^2 - 44\zeta_3 - 32L\zeta_3 + x\Big(- 22 + 13L \\
&- 37L^2 - \frac{28L^3}{3} - \frac{L^4}{3} - 30\zeta_2 + 36L\zeta_2 + 8L^2\zeta_2 + 288\ln(2)\zeta_2 + \frac{128}{5}\zeta_2^2 - 88\zeta_3 - 48L\zeta_3\Big) \\
&+ x^2\Big(\frac{1307}{2} - 365L + 214L^2 + 32L^3 + \frac{26L^4}{3} - 980\zeta_2 - 376L\zeta_2 - 84L^2\zeta_2 - 576\ln(2)\zeta_2 \\
&- \frac{1756}{5}\zeta_2^2 + 808\zeta_3 + 496L\zeta_3\Big)\Big] + \varepsilon\Big[4 + 8c_1 + \frac{479L}{4} + \frac{153L^2}{2} + \frac{137L^3}{6} + \frac{11L^4}{3} + \frac{L^5}{2} \\
&+ 163\zeta_2 + 17L\zeta_2 - \frac{8}{3}L^3\zeta_2 - 24\ln(2)\zeta_2 - 160\zeta_2^2 - \frac{106}{5}L\zeta_2^2 - \frac{346}{3}\zeta_3 - \frac{284}{3}L\zeta_3 - \frac{112}{3}L^2\zeta_3 \\
&- 12\zeta_2\zeta_3 - 18\zeta_5 + x\Big(210 - 48c_1 - \frac{335L}{2} - 96L^2 - \frac{131L^3}{3} - 6L^4 - \frac{L^5}{5} - 580\zeta_2 + 282L\zeta_2 \\
&+ 34L^2\zeta_2 + \frac{40}{3}L^3\zeta_2 + 1728\ln(2)\zeta_2 + 44\zeta_2^2 - \frac{668}{5}L\zeta_2^2 - 156\zeta_3 + 300L\zeta_3 + 4L^2\zeta_3 - 208\zeta_2\zeta_3
\end{aligned}$$

$$+ 480\zeta_5\Big) + x^2\Big(- \frac{1951}{4} + 96c_1 + 464L + 185L^2 + 178L^3 + \frac{74L^4}{3} + \frac{58L^5}{15} - 1164\zeta_2$$
$$- 2206L\zeta_2 - 72L^2\zeta_2 - \frac{476}{3}L^3\zeta_2 - 4224\ln(2)\zeta_2 + 3212\zeta_2^2 + \frac{5992}{5}L\zeta_2^2 - 2000\zeta_3 - \frac{7792}{3}L\zeta_3$$
$$- \frac{268}{3}L^2\zeta_3 + 1656\zeta_2\zeta_3 - 6252\zeta_5\Big)\Big]\Big\} + C_FC_A\Big\{\frac{1}{\varepsilon^2}\Big[\frac{11}{3} + \frac{11L}{3} + \frac{22Lx^2}{3}\Big] + \frac{1}{\varepsilon}\Big[- \frac{49}{9} - \frac{67L}{9}$$
$$+ 2\zeta_2 + 2L\zeta_2 - 2\zeta_3 + x^2\Big(4 - \frac{188L}{9} - 4L^2 - \frac{4L^3}{3} - 4\zeta_2 - 8\zeta_3\Big)\Big] + \Big[- \frac{1595}{27} - \frac{2545L}{54}$$
$$- \frac{233L^2}{18} - \frac{11L^3}{9} - \frac{7}{9}\zeta_2 - \frac{22}{3}L\zeta_2 + 2L^2\zeta_2 + 24\ln(2)\zeta_2 - \frac{3}{5}\zeta_2^2 + \frac{134}{3}\zeta_3 + 26L\zeta_3 + x\Big(- \frac{904}{9}$$
$$+ \frac{341L}{9} - \frac{25L^2}{3} + \frac{8L^3}{3} + \frac{L^4}{6} + \frac{494}{3}\zeta_2 + 44L\zeta_2 + 8L^2\zeta_2 - 144\ln(2)\zeta_2 + 28\zeta_2^2 - 200\zeta_3$$
$$- 72L\zeta_3\Big) + x^2\Big(\frac{8723}{18} - \frac{8968L}{27} + \frac{931L^2}{9} - \frac{94L^3}{9} - \frac{7L^4}{3} - \frac{6848}{9}\zeta_2 - \frac{1076}{3}L\zeta_2 - 72L^2\zeta_2$$
$$+ 288\ln(2)\zeta_2 - \frac{1408}{5}\zeta_2^2 + \frac{5188}{3}\zeta_3 + 808L\zeta_3\Big)\Big] + \varepsilon\Big[- \frac{28745}{162} - 4c_1 - \frac{70165L}{324} - \frac{3337L^2}{54}$$
$$- \frac{565L^3}{54} - \frac{11L^4}{12} - \frac{71}{27}\zeta_2 - \frac{575}{18}L\zeta_2 - \frac{11}{2}L^2\zeta_2 + \frac{4}{3}L^3\zeta_2 + 12\ln(2)\zeta_2 + \frac{637}{5}\zeta_2^2 + \frac{88}{5}L\zeta_2^2$$
$$+ \frac{1577}{9}\zeta_3 + \frac{260}{3}L\zeta_3 + 26L^2\zeta_3 - 2\zeta_2\zeta_3 - 157\zeta_5 + x\Big(- \frac{12683}{27} + 24c_1 + \frac{5639L}{54} + \frac{134L^2}{9}$$
$$- \frac{17L^3}{3} + \frac{11L^4}{6} + \frac{L^5}{10} + \frac{4448}{9}\zeta_2 + 115L\zeta_2 + 26L^2\zeta_2 + \frac{16}{3}L^3\zeta_2 - 864\ln(2)\zeta_2 - \frac{2944}{5}\zeta_2^2$$
$$- \frac{482}{5}L\zeta_2^2 + 272\zeta_3 + 44L\zeta_3 - 38L^2\zeta_3 + 8\zeta_2\zeta_3 + 768\zeta_5\Big) + x^2\Big(- \frac{25015}{27} - 48c_1 + \frac{155567L}{162}$$
$$- \frac{18413L^2}{54} + \frac{3260L^3}{27} - \frac{53L^4}{6} - \frac{19L^5}{15} - \frac{41933}{54}\zeta_2 - \frac{6002}{9}L\zeta_2 - 125L^2\zeta_2 - 52L^3\zeta_2$$
$$+ 2112\ln(2)\zeta_2 + \frac{19996}{5}\zeta_2^2 + 1052L\zeta_2^2 + \frac{154}{9}\zeta_3 - \frac{5168}{3}L\zeta_3 + 476L^2\zeta_3 + 148\zeta_2\zeta_3$$
$$- 8178\zeta_5\Big)\Big]\Big\} + C_Fn_lT_F\Big\{\frac{1}{\varepsilon^2}\Big[- \frac{4}{3} - \frac{4L}{3} - \frac{8Lx^2}{3}\Big] + \frac{1}{\varepsilon}\Big[\frac{20}{9} + \frac{20L}{9} + \frac{40Lx^2}{9}\Big]$$
$$+ \Big[\frac{424}{27} + \frac{418L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{28}{9}\zeta_2 + \frac{8}{3}L\zeta_2 - \frac{16}{3}\zeta_3 + x\Big(\frac{200}{9} - \frac{148L}{9} - \frac{4L^2}{3} + \frac{8}{3}\zeta_2\Big)$$
$$+ x^2\Big(- \frac{68}{9} + \frac{1064L}{27} + \frac{88L^2}{9} + \frac{8L^3}{9} - \frac{176}{9}\zeta_2 + \frac{16}{3}L\zeta_2 - \frac{32}{3}\zeta_3\Big)\Big] + \varepsilon\Big[\frac{5204}{81} + \frac{5813L}{81}$$
$$+ \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{176}{27}\zeta_2 + \frac{74}{9}L\zeta_2 + 2L^2\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{280}{9}\zeta_3 - \frac{16}{3}L\zeta_3 + x\Big(\frac{3808}{27}$$
$$- \frac{2654L}{27} - \frac{184L^2}{9} - \frac{4L^3}{3} + \frac{584}{9}\zeta_2 - 4L\zeta_2 + 16\zeta_3\Big) + x^2\Big(- \frac{2698}{27} + \frac{18238L}{81} + \frac{1406L^2}{27}$$
$$+ \frac{224L^3}{27} + \frac{2L^4}{3} - \frac{2974}{27}\zeta_2 + \frac{184}{9}L\zeta_2 + 4L^2\zeta_2 - \frac{192}{5}\zeta_2^2 - \frac{896}{9}\zeta_3 - \frac{32}{3}L\zeta_3\Big)\Big]\Big\}$$
$$+ C_FT_F\Big\{\Big[\frac{1532}{27} + \frac{530L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{8}{3}\zeta_2 + 4L\zeta_2 + x\Big(- \frac{784}{9} - \frac{436L}{9} - \frac{52L^2}{3}$$

$$
\begin{aligned}
&- 88\zeta_2\Big) + x^2\Big(\frac{1568}{9} + \frac{9604L}{27} + \frac{808L^2}{9} - \frac{16L^3}{9} + \frac{2096}{3}\zeta_2 - 8L\zeta_2\Big)\Bigg] + \varepsilon\Bigg[\frac{4138}{27} + \frac{191L}{3} \\
&+ \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{1616}{27}\zeta_2 + 6L\zeta_2 + 2L^2\zeta_2 + \frac{224}{3}\ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 - \frac{184}{3}\zeta_3 - \frac{56}{9}L\zeta_3 \\
&+ x\Big(-\frac{1148}{27} - \frac{5246L}{27} - \frac{472L^2}{9} - 12L^3 - \frac{2192}{9}\zeta_2 - 4L\zeta_2 + 320\ln(2)\zeta_2 - \frac{352}{3}\zeta_3\Big) \\
&+ x^2\Big(\frac{215}{27} + \frac{32696L}{27} + \frac{9452L^2}{27} + \frac{1772L^3}{27} - \frac{4L^4}{3} + \frac{47870}{27}\zeta_2 + 40L\zeta_2 - 4L^2\zeta_2 \\
&- \frac{9344}{3}\ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 + \frac{13120}{9}\zeta_3 + \frac{176}{9}L\zeta_3\Big)\Bigg]\Bigg\}. \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{V,2}^{(1)} = C_F\Bigg\{ &- 4Lx + \varepsilon\Bigg[x\Big(-16L - 2L^2 + 4\zeta_2\Big) + 8(-1+L)x^2\Bigg] + \varepsilon^2\Bigg[x\Big(-8L^2 - \frac{2L^3}{3} \\
&+ \frac{1}{3}L\Big(-96 + 6\zeta_2\Big) + 16\zeta_2 + 8\zeta_3\Big) + x^2\Big(-40 + 32L + 4L^2 - 8\zeta_2\Big)\Bigg]\Bigg\}. \qquad (4.23)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{V,2}^{(2)} = C_F^2\Bigg\{ &\frac{1}{\varepsilon}\Bigg[8L(1+L)x\Bigg] + \Bigg[x\Big(62L + 34L^2 + 8L^3 + 60\zeta_2 - 48L\zeta_2 - 192\ln(2)\zeta_2 + 16\zeta_3\Big) \\
&+ x^2\Big(-232 + 232L - 200L^2 - \frac{64L^3}{3} - \frac{4L^4}{3} + 752\zeta_2 + 208L\zeta_2 + 64L^2\zeta_2 + 384\ln(2)\zeta_2 \\
&+ \frac{1056}{5}\zeta_2^2 - 864\zeta_3 - 448L\zeta_3\Big)\Bigg] + \varepsilon\Bigg[x\Big(-20 + 32c_1 + 249L + 74L^2 + \frac{94L^3}{3} + \frac{14L^4}{3} \\
&+ 492\zeta_2 - 116L\zeta_2 - 16L^2\zeta_2 - 1152\ln(2)\zeta_2 - \frac{1192}{5}\zeta_2^2 + 128\zeta_3 - 192L\zeta_3\Big) + x^2\Big(-640 \\
&- 64c_1 + 392L - 308L^2 - \frac{728L^3}{3} - \frac{40L^4}{3} - \frac{4L^5}{5} + 2200\zeta_2 + 1296L\zeta_2 + 264L^2\zeta_2 \\
&+ \frac{256}{3}L^3\zeta_2 + 3072\ln(2)\zeta_2 - \frac{14176}{5}\zeta_2^2 - \frac{4848}{5}L\zeta_2^2 + 2672\zeta_3 + 1488L\zeta_3 - 80L^2\zeta_3 - 1088\zeta_2\zeta_3 \\
&+ 4608\zeta_5\Big)\Bigg]\Bigg\} + C_FC_A\Bigg\{\Bigg[x\Big(12 - \frac{346L}{9} + \frac{2L^2}{3} - \frac{244}{3}\zeta_2 + 96\ln(2)\zeta_2 + 80\zeta_3\Big) + x^2\Big(-\frac{616}{3} \\
&+ \frac{232L}{3} - 72L^2 + \frac{32L^3}{3} + \frac{2L^4}{3} + 656\zeta_2 + 368L\zeta_2 + 64L^2\zeta_2 - 192\ln(2)\zeta_2 + \frac{1104}{5}\zeta_2^2 \\
&- 1456\zeta_3 - 544L\zeta_3\Big)\Bigg] + \varepsilon\Bigg[x\Big(78 - 16c_1 - \frac{8057L}{27} - \frac{250L^2}{9} + \frac{2L^3}{3} - \frac{1768}{9}\zeta_2 - 38L\zeta_2 \\
&- 8L^2\zeta_2 + 576\ln(2)\zeta_2 + \frac{1504}{5}\zeta_2^2 - 264\zeta_3 + 48L\zeta_3\Big) + x^2\Big(\frac{2672}{9} + 32c_1 - \frac{5912L}{9} - 4L^2 \\
&- \frac{152L^3}{3} + \frac{20L^4}{3} + \frac{2L^5}{5} + 664\zeta_2 + 656L\zeta_2 + 160L^2\zeta_2 + \frac{160}{3}L^3\zeta_2 - 1536\ln(2)\zeta_2 - \frac{16032}{5}\zeta_2^2 \\
&- \frac{4104}{5}L\zeta_2^2 + 608\zeta_3 + 1168L\zeta_3 - 248L^2\zeta_3 - 224\zeta_2\zeta_3 + 5760\zeta_5\Big)\Bigg]\Bigg\} \\
&+ C_Fn_lT_F\Bigg\{\Bigg[-\frac{32}{3}(-1+L)x^2 + x\Big(\frac{200L}{9} + \frac{8L^2}{3} - \frac{16}{3}\zeta_2\Big)\Bigg] + \varepsilon\Bigg[x\Big(\frac{3844L}{27} + \frac{296L^2}{9}
\end{aligned}
$$

$$+\frac{8L^3}{3}-\frac{592}{9}\zeta_2+8L\zeta_2-32\zeta_3\Big)+x^2\Big(\frac{1472}{9}-\frac{1184L}{9}-16L^2+32\zeta_2\Big)\Big]\Big\}$$
$$+C_FT_F\Big\{\Big[x\Big(\frac{272}{3}+\frac{200L}{9}+\frac{8L^2}{3}-16\zeta_2\Big)+x^2\Big(-\frac{544}{3}-\frac{992L}{3}-64L^2-512\zeta_2\Big)\Big]$$
$$+\varepsilon\Big[x\Big(\frac{1528}{9}+\frac{3844L}{27}+\frac{296L^2}{9}+\frac{8L^3}{3}+\frac{256}{9}\zeta_2+8L\zeta_2+128\ln(2)\zeta_2-\frac{256}{3}\zeta_3\Big)$$
$$+x^2\Big(-\frac{2368}{9}-\frac{8672L}{9}-336L^2-\frac{128L^3}{3}-\frac{4000}{3}\zeta_2+2304\ln(2)\zeta_2-1088\zeta_3\Big)\Big]\Big\}. \qquad (4.24)$$

We note that in the asymptotic limit, the magnetic part of the vector form factors vanish.

4.2.2 Axial-vector form factor

$$\mathcal{F}^{(1),\mathrm{ns}}_{A,1}=C_F\Big\{\frac{1}{\varepsilon}\Big[-2-2L-4Lx^2\Big]+\Big[-4-3L-L^2+2\zeta_2+2(-2+3L)x+x^2\Big(1-8L$$
$$-2L^2+4\zeta_2\Big)\Big]+\varepsilon\Big[-8-8L-\frac{3L^2}{2}-\frac{L^3}{3}+2\zeta_2+L\zeta_2+4\zeta_3+x\Big(-10+6L+3L^2$$
$$-6\zeta_2\Big)+x^2\Big(12-27L-4L^2-\frac{2L^3}{3}+8\zeta_2+2L\zeta_2+8\zeta_3\Big)\Big]+\varepsilon^2\Big[-16-16L-4L^2$$
$$-\frac{L^3}{2}-\frac{L^4}{12}+6\zeta_2+\frac{3}{2}L\zeta_2+\frac{1}{2}L^2\zeta_2+\frac{14}{5}\zeta_2^2+\frac{20}{3}\zeta_3+\frac{14}{3}L\zeta_3+x\Big(-26+16L+3L^2$$
$$+L^3-8\zeta_2-3L\zeta_2-12\zeta_3\Big)+x^2\Big(25-46L-\frac{27L^2}{2}-\frac{4L^3}{3}-\frac{L^4}{6}+\frac{55}{2}\zeta_2+4L\zeta_2$$
$$+L^2\zeta_2+\frac{28}{5}\zeta_2^2+16\zeta_3+\frac{28}{3}L\zeta_3\Big)\Big]\Big\} \qquad (4.25)$$
$$\mathcal{F}^{(2),\mathrm{ns}}_{A,1}=C_F^2\Big\{\frac{1}{\varepsilon^2}\Big[2+4L+2L^2+8L(1+L)x^2\Big]+\frac{1}{\varepsilon}\Big[8+14L+8L^2+2L^3-4\zeta_2-4L\zeta_2$$
$$-4(1+L)(-2+3L)x+x^2\Big(-2+30L+32L^2+8L^3-8\zeta_2-16L\zeta_2\Big)\Big]+\Big[46+\frac{85L}{2}$$
$$+\frac{55L^2}{2}+\frac{20L^3}{3}+\frac{7L^4}{6}+39\zeta_2-4L^2\zeta_2-48\ln(2)\zeta_2-\frac{118}{5}\zeta_2^2-44\zeta_3-32L\zeta_3$$
$$+x\Big(-22+19L-19L^2-12L^3+\frac{L^4}{3}-66\zeta_2+44L\zeta_2+8L^2\zeta_2+288\ln(2)\zeta_2+\frac{144}{5}\zeta_2^2$$
$$-104\zeta_3-80L\zeta_3\Big)+x^2\Big(\frac{1195}{2}-357L+182L^2+\frac{64L^3}{3}+\frac{14L^4}{3}-228\zeta_2-216L\zeta_2-52L^2\zeta_2$$
$$-576\ln(2)\zeta_2-\frac{1308}{5}\zeta_2^2+488\zeta_3+432L\zeta_3\Big)\Big]+\varepsilon\Big[4+8c_1+\frac{479L}{4}+\frac{153L^2}{2}+\frac{137L^3}{6}$$
$$+\frac{11L^4}{3}+\frac{L^5}{2}+163\zeta_2+17L\zeta_2-\frac{8}{3}L^3\zeta_2-24\ln(2)\zeta_2-160\zeta_2^2-\frac{106}{5}L\zeta_2^2-\frac{346}{3}\zeta_3-\frac{284}{3}L\zeta_3$$
$$-\frac{112}{3}L^2\zeta_3-12\zeta_2\zeta_3-18\zeta_5+x\Big(190-48c_1-\frac{173L}{2}-26L^2-\frac{53L^3}{3}-\frac{22L^4}{3}+\frac{L^5}{5}-856\zeta_2$$

$$+ 166L\zeta_2 - 2L^2\zeta_2 + \frac{8}{3}L^3\zeta_2 + 1728\ln(2)\zeta_2 + \frac{1124}{5}\zeta_2^2 - 84L\zeta_2^2 - 628\zeta_3 + 228L\zeta_3 - 52L^2\zeta_3$$
$$+ 80\zeta_2\zeta_3 + 864\zeta_5\Big) + x^2\Big(- \frac{543}{4} + 96c_1 + 792L - 163L^2 + \frac{518L^3}{3} + \frac{58L^4}{3} + \frac{22L^5}{15} + 1708\zeta_2$$
$$- 510L\zeta_2 + 136L^2\zeta_2 - \frac{188}{3}L^3\zeta_2 - 2688\ln(2)\zeta_2 + \frac{3596}{5}\zeta_2^2 + \frac{2328}{5}L\zeta_2^2 + 832\zeta_3 - \frac{5008}{3}L\zeta_3$$
$$+ \frac{452}{3}L^2\zeta_3 - 328\zeta_2\zeta_3 - 5868\zeta_5\Big)\Bigg] \Bigg\} + C_FC_A\Bigg\{ \frac{1}{\varepsilon^2}\Bigg[\frac{11}{3} + \frac{11L}{3} + \frac{22Lx^2}{3}\Bigg] + \frac{1}{\varepsilon}\Bigg[- \frac{49}{9} - \frac{67L}{9}$$
$$+ 2\zeta_2 + 2L\zeta_2 - 2\zeta_3 + x^2\Big(4 - \frac{188L}{9} - 4L^2 - \frac{4L^3}{3} - 4\zeta_2 - 8\zeta_3\Big)\Bigg] + \Bigg[- \frac{1595}{27} - \frac{2545L}{54}$$
$$- \frac{233L^2}{18} - \frac{11L^3}{9} - \frac{7}{9}\zeta_2 - \frac{22}{3}L\zeta_2 + 2L^2\zeta_2 + 24\ln(2)\zeta_2 - \frac{3}{5}\zeta_2^2 + \frac{134}{3}\zeta_3 + 26L\zeta_3 + x\Big(- \frac{796}{9}$$
$$+ \frac{241L}{3} - 9L^2 - \frac{L^4}{6} + 158\zeta_2 + 36L\zeta_2 + 8L^2\zeta_2 - 144\ln(2)\zeta_2 + \frac{132}{5}\zeta_2^2 - 168\zeta_3 - 56L\zeta_3\Big)$$
$$+ x^2\Big(\frac{7523}{18} - \frac{9760L}{27} + \frac{283L^2}{9} - \frac{46L^3}{9} - \frac{L^4}{3} - \frac{2528}{9}\zeta_2 - \frac{164}{3}L\zeta_2 - 24L^2\zeta_2 + 288\ln(2)\zeta_2$$
$$- \frac{544}{5}\zeta_2^2 + \frac{1060}{3}\zeta_3 + 328L\zeta_3\Big)\Bigg] + \varepsilon\Bigg[- \frac{28745}{162} - 4c_1 - \frac{70165L}{324} - \frac{3337L^2}{54} - \frac{565L^3}{54}$$
$$- \frac{11L^4}{12} - \frac{71}{27}\zeta_2 - \frac{575}{18}L\zeta_2 - \frac{11}{2}L^2\zeta_2 + \frac{4}{3}L^3\zeta_2 + 12\ln(2)\zeta_2 + \frac{637}{5}\zeta_2^2 + \frac{88}{5}L\zeta_2^2 + \frac{1577}{9}\zeta_3$$
$$+ \frac{260}{3}L\zeta_3 + 26L^2\zeta_3 - 2\zeta_2\zeta_3 - 157\zeta_5 + x\Big(- \frac{10577}{27} + 24c_1 + \frac{3799L}{18} + \frac{112L^2}{3} - 9L^3 + \frac{L^4}{6}$$
$$- \frac{L^5}{10} + \frac{1744}{3}\zeta_2 + 153L\zeta_2 + 42L^2\zeta_2 + \frac{32}{3}L^3\zeta_2 - 864\ln(2)\zeta_2 - \frac{2944}{5}\zeta_2^2 - \frac{606}{5}L\zeta_2^2 + 184\zeta_3$$
$$+ 20L\zeta_3 - 10L^2\zeta_3 - 136\zeta_2\zeta_3 + 576\zeta_5\Big) + x^2\Big(- \frac{12007}{27} - 48c_1 - \frac{19681L}{162} - \frac{22517L^2}{54}$$
$$+ \frac{956L^3}{27} - \frac{37L^4}{6} - \frac{L^5}{15} - \frac{25085}{54}\zeta_2 - \frac{4418}{9}L\zeta_2 + 43L^2\zeta_2 - 36L^3\zeta_2 + 1344\ln(2)\zeta_2 + \frac{8004}{5}\zeta_2^2$$
$$+ 548L\zeta_2^2 + \frac{4042}{9}\zeta_3 - \frac{2720}{3}L\zeta_3 + 164L^2\zeta_3 + 628\zeta_2\zeta_3 - 2994\zeta_5\Big)\Bigg] \Bigg\} + C_Fn_lT_F\Bigg\{ \frac{1}{\varepsilon^2}\Bigg[- \frac{4}{3}$$
$$- \frac{4L}{3} - \frac{8Lx^2}{3}\Bigg] + \frac{1}{\varepsilon}\Bigg[\frac{20}{9} + \frac{20L}{9} + \frac{40Lx^2}{9}\Bigg] + \Bigg[\frac{424}{27} + \frac{418L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{28}{9}\zeta_2$$
$$+ \frac{8}{3}L\zeta_2 - \frac{16}{3}\zeta_3 + x\Big(\frac{200}{9} - \frac{68L}{3} - 4L^2 + 8\zeta_2\Big) + x^2\Big(- \frac{164}{9} + \frac{1352L}{27} + \frac{88L^2}{9} + \frac{8L^3}{9}$$
$$- \frac{176}{9}\zeta_2 + \frac{16}{3}L\zeta_2 - \frac{32}{3}\zeta_3\Big)\Bigg] + \varepsilon\Bigg[\frac{5204}{81} + \frac{5813L}{81} + \frac{562L^2}{27} + \frac{94L^3}{27} + \frac{L^4}{3} - \frac{176}{27}\zeta_2 + \frac{74}{9}L\zeta_2$$
$$+ 2L^2\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{280}{9}\zeta_3 - \frac{16}{3}L\zeta_3 + x\Big(\frac{3808}{27} - \frac{862L}{9} - \frac{80L^2}{3} - 4L^3 + \frac{232}{3}\zeta_2 - 12L\zeta_2$$
$$+ 48\zeta_3\Big) + x^2\Big(- \frac{4234}{27} + \frac{20254L}{81} + \frac{1838L^2}{27} + \frac{224L^3}{27} + \frac{2L^4}{3} - \frac{3838}{27}\zeta_2 + \frac{184}{9}L\zeta_2 + 4L^2\zeta_2$$
$$- \frac{192}{5}\zeta_2^2 - \frac{896}{9}\zeta_3 - \frac{32}{3}L\zeta_3\Big)\Bigg] \Bigg\} + C_FT_F\Bigg\{ \Bigg[\frac{1532}{27} + \frac{530L}{27} + \frac{38L^2}{9} + \frac{4L^3}{9} - \frac{8}{3}\zeta_2 + 4L\zeta_2$$

$$+x\Big(-\frac{160}{9}-\frac{164L}{3}-20L^2-136\zeta_2\Big)+x^2\Big(\frac{320}{9}+\frac{2980L}{27}+\frac{520L^2}{9}-\frac{16L^3}{9}+\frac{1136}{3}\zeta_2$$
$$-8L\zeta_2\Big)\Big]+\varepsilon\Big[\frac{4138}{27}+\frac{191L}{3}+\frac{562L^2}{27}+\frac{94L^3}{27}+\frac{L^4}{3}-\frac{1616}{27}\zeta_2+6L\zeta_2+2L^2\zeta_2$$
$$+\frac{224}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2-\frac{184}{3}\zeta_3-\frac{56}{9}L\zeta_3+x\Big(\frac{3628}{27}-\frac{1726L}{9}-\frac{176L^2}{3}-\frac{44L^3}{3}-400\zeta_2$$
$$-12L\zeta_2+576\ln(2)\zeta_2-256\zeta_3\Big)+x^2\Big(-\frac{10537}{27}+\frac{10904L}{27}+\frac{2972L^2}{27}+\frac{1196L^3}{27}-\frac{4L^4}{3}$$
$$+\frac{25118}{27}\zeta_2+40L\zeta_2-4L^2\zeta_2-\frac{4736}{3}\ln(2)\zeta_2-\frac{8}{5}\zeta_2^2+\frac{6208}{9}\zeta_3+\frac{176}{9}L\zeta_3\Big)\Big]\Big\}. \tag{4.26}$$

Here we observe that for $x=0$, the electric vector and axial-vector form factors are the same, as expected. The bare singlet piece for the electric axial-vector form factor is given by

$$\hat{\mathcal{F}}^{(2),\mathrm{s}}_{A,1}=C_FT_F\Big\{-\frac{6}{\varepsilon}+\Big[-29-12L+8\zeta_2+x\Big(8+24L+8L^2-48\zeta_2-32L\zeta_2+64\zeta_3\Big)$$
$$+x^2\Big(-4-16L-56L^2-\frac{16L^3}{3}+32\zeta_2+32L\zeta_2+32L^2\zeta_2-\frac{64}{5}\zeta_2^2-128\zeta_3-128L\zeta_3\Big)\Big]$$
$$+\varepsilon\Big[-\frac{199}{2}-58L-12L^2+26\zeta_2+16L\zeta_2+8\zeta_3+x\Big(-4+76L+56L^2+8L^3-176\zeta_2$$
$$-48L\zeta_2-48L^2\zeta_2+384\ln(2)\zeta_2+\frac{224}{5}\zeta_2^2-80\zeta_3+32L\zeta_3\Big)+x^2\Big(-274+88L-180L^2$$
$$-\frac{136L^3}{3}-\frac{10L^4}{3}+632\zeta_2-16L\zeta_2+64L^2\zeta_2+32L^3\zeta_2-1536\ln(2)\zeta_2-48\zeta_2^2-\frac{448}{5}L\zeta_2^2$$
$$+400\zeta_3+224L\zeta_3-32L^2\zeta_3+160\zeta_2\zeta_3-144\zeta_5\Big)\Big]\Big\}. \tag{4.27}$$

The magnetic parts of the axial-vector form factor read

$$\mathcal{F}^{(1),\mathrm{ns}}_{A,2}=C_F\Big\{-4(2+3L)x-16(1+L)x^2+\varepsilon\Big[x\Big(-16-24L-6L^2+12\zeta_2\Big)+x^2\Big(-56$$
$$-8L-8L^2+16\zeta_2\Big)\Big]+\varepsilon^2\Big[x\Big(-32-48L-12L^2-2L^3+20\zeta_2+6L\zeta_2+24\zeta_3\Big)$$
$$+x^2\Big(-136-16L-4L^2-\frac{8L^3}{3}+8L\zeta_2+32\zeta_3\Big)\Big]\Big\}. \tag{4.28}$$

$$\mathcal{F}^{(2),\mathrm{ns}}_{A,2}=C_F^2\Big\{\frac{1}{\varepsilon}\Big[8(1+L)(2+3L)x+32(1+L)^2x^2\Big]+\Big[x\Big(68+122L+86L^2+24L^3+132\zeta_2$$
$$-48L\zeta_2-192\ln(2)\zeta_2+48\zeta_3\Big)+x^2\Big(-80+416L+48L^2+32L^3+\frac{4L^4}{3}+16L\zeta_2$$
$$+384\ln(2)\zeta_2+\frac{32}{5}\zeta_2^2-160\zeta_3-64L\zeta_3\Big)\Big]+\varepsilon\Big[x\Big(242+32c_1+335L+250L^2+\frac{218L^3}{3}$$
$$+14L^4+812\zeta_2-28L\zeta_2-48L^2\zeta_2-1152\ln(2)\zeta_2-\frac{1944}{5}\zeta_2^2+176\zeta_3-96L\zeta_3\Big)+x^2\Big(316$$

$$
\begin{aligned}
&- 64c_1 + 548L + 620L^2 + 16L^3 + \frac{56L^4}{3} + \frac{4L^5}{5} - 616\zeta_2 + 384L\zeta_2 - 24L^2\zeta_2 - \frac{64}{3}L^3\zeta_2 \\
&+ 3072\ln(2)\zeta_2 + \frac{96}{5}\zeta_2^2 + \frac{496}{5}L\zeta_2^2 - 2000\zeta_3 - 176L\zeta_3 - 112L^2\zeta_3 + 576\zeta_2\zeta_3 + 768\zeta_5\Big)\Big]\Big\} \\
&+ C_F C_A\Big\{\Big[x\Big(-\frac{968}{9} - \frac{458L}{3} - 22L^2 - 84\zeta_2 + 96\ln(2)\zeta_2 + 48\zeta_3\Big) + x^2\Big(-\frac{2800}{9} - \frac{1936L}{9} \\
&- \frac{304L^2}{3} - \frac{2L^4}{3} + \frac{1184}{3}\zeta_2 + 144L\zeta_2 + 32L^2\zeta_2 - 192\ln(2)\zeta_2 + \frac{528}{5}\zeta_2^2 - 560\zeta_3 - 224L\zeta_3\Big)\Big] \\
&+ \varepsilon\Big[x\Big(-\frac{14872}{27} - 16c_1 - \frac{6853L}{9} - \frac{590L^2}{3} - 22L^3 - \frac{908}{3}\zeta_2 - 82L\zeta_2 + 576\ln(2)\zeta_2 \\
&+ \frac{1248}{5}\zeta_2^2 + 232\zeta_3 + 144L\zeta_3\Big) + x^2\Big(-\frac{61028}{27} + 32c_1 - \frac{18980L}{27} - \frac{5092L^2}{9} - 96L^3 - \frac{2L^5}{5} \\
&+ \frac{12992}{9}\zeta_2 + 568L\zeta_2 + 176L^2\zeta_2 + \frac{128}{3}L^3\zeta_2 - 1536\ln(2)\zeta_2 - \frac{7296}{5}\zeta_2^2 - \frac{2424}{5}L\zeta_2^2 + 1312\zeta_3 \\
&+ 304L\zeta_3 - 40L^2\zeta_3 - 544\zeta_2\zeta_3 + 2304\zeta_5\Big)\Big]\Big\} + C_F n_l T_F\Big\{\Big[x\Big(\frac{304}{9} + \frac{136L}{3} + 8L^2 - 16\zeta_2\Big) \\
&+ x^2\Big(\frac{896}{9} + \frac{320L}{9} + \frac{32L^2}{3} - \frac{64}{3}\zeta_2\Big)\Big] + \varepsilon\Big[x\Big(\frac{4664}{27} + \frac{2036L}{9} + \frac{184L^2}{3} + 8L^3 - \frac{224}{3}\zeta_2 \\
&+ 24L\zeta_2 - 96\zeta_3\Big) + x^2\Big(\frac{18544}{27} + \frac{2704L}{27} + \frac{368L^2}{9} + \frac{32L^3}{3} + \frac{128}{9}\zeta_2 + 32L\zeta_2 - 128\zeta_3\Big)\Big]\Big\} \\
&+ C_F T_F\Big\{\Big[x\Big(-\frac{128}{9} + \frac{136L}{3} + 8L^2 + 80\zeta_2\Big) + x^2\Big(-\frac{1024}{9} - \frac{1408L}{9} - \frac{160L^2}{3} - 320\zeta_2\Big)\Big] \\
&+ \varepsilon\Big[x\Big(-\frac{304}{27} + \frac{2036L}{9} + \frac{184L^2}{3} + 8L^3 + \frac{1072}{3}\zeta_2 + 24L\zeta_2 - 384\ln(2)\zeta_2 + 192\zeta_3\Big) \\
&+ x^2\Big(\frac{5200}{27} - \frac{5936L}{27} - \frac{1360L^2}{9} - 32L^3 - \frac{6080}{9}\zeta_2 + 32L\zeta_2 + 1280\ln(2)\zeta_2 - \frac{1600}{3}\zeta_3\Big)\Big]\Big\}. \qquad (4.29)
\end{aligned}
$$

The bare singlet piece for the magnetic axial-vector form factor is given by

$$
\begin{aligned}
\hat{\mathcal{F}}^{(2),\mathrm{s}}_{A,2} &= C_F T_F\Big\{\Big[x\Big(-80 - 48L - 8L^2 - 32\zeta_2\Big) + x^2\Big(-160 + \frac{4L^4}{3} - 256\zeta_2 - 128L\zeta_2 + 32L^2\zeta_2 \\
&+ \frac{256}{5}\zeta_2^2 + 256\zeta_3 + 192L\zeta_3\Big)\Big] + \varepsilon\Big[x\Big(-360 - 264L - 72L^2 - \frac{32L^3}{3} + 64\zeta_2 + 16L\zeta_2 \\
&- 48\zeta_3\Big) + x^2\Big(-768 - 288L - 32L^2 - \frac{16L^3}{3} - \frac{8L^4}{3} + \frac{16L^5}{15} - 896\zeta_2 - 192L\zeta_2 \\
&- 192L^2\zeta_2 + 16L^3\zeta_2 + 1536\ln(2)\zeta_2 + \frac{256}{5}\zeta_2^2 - 32L\zeta_2^2 - 448\zeta_3 - 512L\zeta_3 \\
&+ 208L^2\zeta_3 + 64\zeta_2\zeta_3 - 704\zeta_5\Big)\Big]\Big\}. \qquad (4.30)
\end{aligned}
$$

Similar to the vector form factor, the magnetic part of the axial-vector form factor also vanishes for $x = 0$.

4.2.3 Scalar form factor

Next, we present the scalar form factor in the asymptotic limit.

$$\begin{aligned}
\mathcal{F}_S^{(1)} &= C_F\Bigg\{\frac{1}{\varepsilon}\Bigg[-2-2L-4Lx^2\Bigg]+\Bigg[-2-L^2+2\zeta_2+4(-1+3L)x+x^2\Big(1-2L-2L^2\\
&+4\zeta_2\Big)\Bigg]+\varepsilon\Bigg[-4-2L-\frac{L^3}{3}-\zeta_2+L\zeta_2+4\zeta_3+x\Big(-4+12L+6L^2-12\zeta_2\Big)+x^2\Big(\frac{45}{2}\\
&-24L-L^2-\frac{2L^3}{3}+2\zeta_2+2L\zeta_2+8\zeta_3\Big)\Bigg]+\varepsilon^2\Bigg[-8-4L-L^2-\frac{L^4}{12}+\frac{1}{2}L^2\zeta_2+\frac{14}{5}\zeta_2^2\\
&+\frac{2}{3}\zeta_3+\frac{14}{3}L\zeta_3+\zeta_2+x\Big(-8+28L+6L^2+2L^3-14\zeta_2-6L\zeta_2-24\zeta_3\Big)+x^2\Big(\frac{193}{4}-34L\\
&-12L^2-\frac{L^3}{3}-\frac{L^4}{6}+\frac{49}{2}\zeta_2+L\zeta_2+L^2\zeta_2+\frac{28}{5}\zeta_2^2+4\zeta_3+\frac{28}{3}L\zeta_3\Big)\Bigg]\Bigg\}. \qquad (4.31)\\
\mathcal{F}_S^{(2)} &= C_F^2\Bigg\{\frac{1}{\varepsilon^2}\Bigg[2+4L+2L^2+8L(1+L)x^2\Bigg]+\frac{1}{\varepsilon}\Bigg[4+4L+2L^2+2L^3-4\zeta_2-4L\zeta_2\\
&-8(1+L)(-1+3L)x+x^2\Big(-2+10L+8L^2+8L^3-8\zeta_2-16L\zeta_2\Big)\Bigg]+\Bigg[29+12L+6L^2\\
&+\frac{2L^3}{3}+\frac{7L^4}{6}+6\zeta_2+12L\zeta_2-4L^2\zeta_2-\frac{118}{5}\zeta_2^2-56\zeta_3-32L\zeta_3+x\Big(32-72L+4L^2-24L^3\\
&-112\zeta_2+64L\zeta_2+8L^2\zeta_2+288\ln(2)\zeta_2+\frac{136}{5}\zeta_2^2-120\zeta_3-64L\zeta_3\Big)+x^2\Big(\frac{1069}{2}-380L\\
&+166L^2-\frac{10L^3}{3}+5L^4-36\zeta_2-132L\zeta_2-36L^2\zeta_2-576\ln(2)\zeta_2-\frac{724}{5}\zeta_2^2+72\zeta_3\\
&+224L\zeta_3\Big)\Bigg]+\varepsilon\Bigg[-\frac{113}{2}+36L+14L^2+\frac{14L^3}{3}+\frac{L^4}{6}+\frac{L^5}{2}-40\zeta_2+24L\zeta_2+12L^2\zeta_2\\
&-\frac{8}{3}L^3\zeta_2+264\ln(2)\zeta_2-\frac{314}{5}\zeta_2^2-\frac{106}{5}L\zeta_2^2-\frac{478}{3}\zeta_3-\frac{212}{3}L\zeta_3-\frac{112}{3}L^2\zeta_3-12\zeta_2\zeta_3-18\zeta_5\\
&+x\Big(232-48c_1-216L-40L^2+12L^3-\frac{41L^4}{3}-1016\zeta_2+24L\zeta_2+12L^2\zeta_2+8L^3\zeta_2\\
&+1536\ln(2)\zeta_2+\frac{1552}{5}\zeta_2^2-\frac{544}{5}L\zeta_2^2-616\zeta_3+240L\zeta_3-24L^2\zeta_3-64\zeta_2\zeta_3+672\zeta_5\Big)\\
&+x^2\Big(-\frac{2149}{4}+96c_1+895L-221L^2+\frac{472L^3}{3}+\frac{37L^4}{6}+\frac{5L^5}{3}+2568\zeta_2-366L\zeta_2\\
&+214L^2\zeta_2-52L^3\zeta_2-2304\ln(2)\zeta_2-\frac{306}{5}\zeta_2^2+100L\zeta_2^2+604\zeta_3-\frac{4240}{3}L\zeta_3+\frac{128}{3}L^2\zeta_3\\
&-520\zeta_2\zeta_3-2916\zeta_5\Big)\Bigg]\Bigg\}+C_FC_A\Bigg\{\frac{1}{\varepsilon^2}\Bigg[\frac{11}{3}+\frac{11L}{3}+\frac{22Lx^2}{3}\Bigg]+\frac{1}{\varepsilon}\Bigg[-\frac{49}{9}-\frac{67L}{9}+2\zeta_2\\
&+2L\zeta_2-2\zeta_3+x^2\Big(4-\frac{188L}{9}-4L^2-\frac{4L^3}{3}-4\zeta_2-8\zeta_3\Big)\Bigg]+\Bigg[-\frac{869}{27}-\frac{242L}{27}-\frac{67L^2}{9}
\end{aligned}$$

$$
- \frac{11L^3}{9} + \frac{182}{9}\zeta_2 - \frac{22}{3}L\zeta_2 + 2L^2\zeta_2 - \frac{3}{5}\zeta_2^2 + \frac{98}{3}\zeta_3 + 26L\zeta_3 + x\Big(- \frac{580}{9} + \frac{416L}{3} + 12L^2
$$
$$
+ 92\zeta_2 - 144\ln(2)\zeta_2 - 44\zeta_3\Big) + x^2\Big(\frac{3463}{36} - \frac{5705L}{54} - \frac{37L^2}{18} - \frac{43L^3}{9} - \frac{L^4}{2} - \frac{611}{9}\zeta_2 + \frac{58}{3}L\zeta_2
$$
$$
+ 10L^2\zeta_2 + 288\ln(2)\zeta_2 - \frac{122}{5}\zeta_2^2 + \frac{364}{3}\zeta_3 + 96L\zeta_3\Big)\Bigg] + \varepsilon\Bigg[- \frac{6437}{162} - \frac{2122L}{81} - \frac{341L^2}{27}
$$
$$
- \frac{134L^3}{27} - \frac{11L^4}{12} + \frac{1972}{27}\zeta_2 - \frac{103}{9}L\zeta_2 - \frac{11}{2}L^2\zeta_2 + \frac{4}{3}L^3\zeta_2 - 132\ln(2)\zeta_2 + 65\zeta_2^2 + \frac{88}{5}L\zeta_2^2
$$
$$
+ \frac{1055}{9}\zeta_3 + \frac{152}{3}L\zeta_3 + 26L^2\zeta_3 - 2\zeta_2\zeta_3 - 157\zeta_5 + x\Big(- \frac{2756}{27} + 24c_1 + \frac{3700L}{9} + \frac{476L^2}{3}
$$
$$
+ \frac{32L^3}{3} - \frac{L^4}{6} + \frac{746}{3}\zeta_2 + 14L\zeta_2 + 2L^2\zeta_2 - 768\ln(2)\zeta_2 - 360\zeta_2^2 + 220\zeta_3 - 16L\zeta_3\Big)
$$
$$
+ x^2\Big(\frac{40951}{216} - 48c_1 - \frac{85217L}{324} - \frac{24019L^2}{108} + \frac{553L^3}{54} - \frac{79L^4}{12} - \frac{L^5}{6} - \frac{13060}{27}\zeta_2 - \frac{188}{9}L\zeta_2
$$
$$
+ 28L^2\zeta_2 + \frac{2}{3}L^3\zeta_2 + 1152\ln(2)\zeta_2 + 841\zeta_2^2 + \frac{798}{5}L\zeta_2^2 - \frac{1916}{9}\zeta_3 - \frac{908}{3}L\zeta_3 + 92L^2\zeta_3
$$
$$
+ 388\zeta_2\zeta_3 - 942\zeta_5\Big)\Bigg]\Bigg\} + C_F n_l T_F\Bigg\{\frac{1}{\varepsilon^2}\Bigg[- \frac{4}{3} - \frac{4L}{3} - \frac{8Lx^2}{3}\Bigg] + \frac{1}{\varepsilon}\Bigg[\frac{20}{9} + \frac{20L}{9} + \frac{40Lx^2}{9}\Bigg]
$$
$$
+ \Bigg[\frac{196}{27} + \frac{112L}{27} + \frac{20L^2}{9} + \frac{4L^3}{9} + \frac{8}{9}\zeta_2 + \frac{8}{3}L\zeta_2 - \frac{16}{3}\zeta_3 + x\Big(\frac{128}{9} - \frac{112L}{3} - 8L^2 + 16\zeta_2\Big)
$$
$$
+ x^2\Big(- \frac{290}{9} + \frac{1064L}{27} + \frac{52L^2}{9} + \frac{8L^3}{9} - \frac{104}{9}\zeta_2 + \frac{16}{3}L\zeta_2 - \frac{32}{3}\zeta_3\Big)\Bigg] + \varepsilon\Bigg[\frac{1706}{81} + \frac{1232L}{81}
$$
$$
+ \frac{148L^2}{27} + \frac{40L^3}{27} + \frac{L^4}{3} + \frac{328}{27}\zeta_2 + \frac{20}{9}L\zeta_2 + 2L^2\zeta_2 - \frac{96}{5}\zeta_2^2 - \frac{64}{9}\zeta_3 - \frac{16}{3}L\zeta_3 + x\Big(\frac{1504}{27}
$$
$$
- \frac{1328L}{9} - \frac{136L^2}{3} - 8L^3 + \frac{344}{3}\zeta_2 - 24L\zeta_2 + 96\zeta_3\Big) + x^2\Big(- \frac{7375}{27} + \frac{16600L}{81} + \frac{1496L^2}{27}
$$
$$
+ \frac{116L^3}{27} + \frac{2L^4}{3} - \frac{3154}{27}\zeta_2 + \frac{76}{9}L\zeta_2 + 4L^2\zeta_2 - \frac{192}{5}\zeta_2^2 - \frac{464}{9}\zeta_3 - \frac{32}{3}L\zeta_3\Big)\Bigg]\Bigg\}
$$
$$
+ C_F T_F\Bigg\{\Bigg[\frac{1628}{27} + \frac{224L}{27} + \frac{20L^2}{9} + \frac{4L^3}{9} - \frac{68}{3}\zeta_2 + 4L\zeta_2 + x\Big(- \frac{160}{9} + \frac{128L}{3} + 8L^2 - \frac{L^4}{3}
$$
$$
+ 16\zeta_2 - 8L^2\zeta_2 - \frac{64}{5}\zeta_2^2 - 48L\zeta_3\Big) + x^2\Big(\frac{2192}{9} - \frac{1088L}{27} - \frac{416L^2}{9} - \frac{136L^3}{9} - \frac{4L^4}{3} + \frac{512}{3}\zeta_2
$$
$$
+ 40L\zeta_2 - 32L^2\zeta_2 - \frac{256}{5}\zeta_2^2 - 256\zeta_3 - 192L\zeta_3\Big)\Bigg] + \varepsilon\Bigg[\frac{4214}{27} + \frac{64L}{9} + \frac{148L^2}{27} + \frac{40L^3}{27} + \frac{L^4}{3}
$$
$$
- \frac{4028}{27}\zeta_2 + 2L^2\zeta_2 + \frac{512}{3}\ln(2)\zeta_2 - \frac{8}{5}\zeta_2^2 - \frac{328}{3}\zeta_3 - \frac{56}{9}L\zeta_3 + x\Big(\frac{2368}{27} + \frac{1696L}{9} + \frac{344L^2}{3}
$$
$$
+ \frac{32L^3}{3} + \frac{L^4}{3} - \frac{4L^5}{15} - \frac{520}{3}\zeta_2 - 88L\zeta_2 - 8L^2\zeta_2 - 4L^3\zeta_2 + 192\ln(2)\zeta_2 + \frac{96}{5}\zeta_2^2 + 8L\zeta_2^2
$$
$$
+ 240\zeta_3 + 112L\zeta_3 - 52L^2\zeta_3 - 16\zeta_2\zeta_3 + 176\zeta_5\Big) + x^2\Big(\frac{2621}{27} + \frac{10172L}{27} - \frac{10276L^2}{27} - \frac{2116L^3}{27}
$$
$$
- 14L^4 - \frac{16L^5}{15} + \frac{29510}{27}\zeta_2 + 148L\zeta_2 + 148L^2\zeta_2 - 16L^3\zeta_2 - \frac{7040}{3}\ln(2)\zeta_2 - \frac{1504}{5}\zeta_2^2 + 32L\zeta_2^2
$$

$$+\frac{3904}{9}\zeta_3-\frac{5296}{9}L\zeta_3-208L^2\zeta_3-64\zeta_2\zeta_3+704\zeta_5\Big)\Big]\Big\}. \tag{4.32}$$

4.2.4 Pseudo-scalar form factor

The non-singlet part of the pseudo-scalar form factor can be obtained in this limit using Eq. (2.13) as

$$\mathcal{F}_P^{(n),\rm ns}=\mathcal{F}_{A,1}^{(n),\rm ns}+\left(-\frac{(1-x)^2}{4x}\right)\mathcal{F}_{A,2}^{(n),\rm ns}\quad\text{for } n=1,2. \tag{4.33}$$

The unrenormalized singlet piece is given by

$$\begin{aligned}\hat{\mathcal{F}}_P^{(2),\rm s}=C_FT_F\Bigg\{\Bigg[&x\Big(8L^2-\frac{L^4}{3}+32\zeta_2-8L^2\zeta_2-\frac{64}{5}\zeta_2^2-48L\zeta_3\Big)+x^2\Big(112-64L+16L^2+16L^3\\&+32\zeta_2+96L\zeta_2\Big)\Bigg]+\varepsilon\Bigg[x\Big(16L^2+8L^3+L^4-\frac{4L^5}{15}+16L\zeta_2+8L^2\zeta_2-4L^3\zeta_2+\frac{224}{5}\zeta_2^2\\&+8L\zeta_2^2+48\zeta_3+208L\zeta_3-52L^2\zeta_3-16\zeta_2\zeta_3+176\zeta_5\Big)+x^2\Big(-400+480L-80L^2\\&-\frac{56L^3}{3}+\frac{32L^4}{3}-64\zeta_2-160L\zeta_2-48L^2\zeta_2-\frac{176}{5}\zeta_2^2+32\zeta_3-128L\zeta_3\Big)\Bigg]\Bigg\}.\end{aligned} \tag{4.34}$$

4.3 Threshold region $q^2\sim 4m^2$

Now we provide the expansion of the form factors in the threshold region $q^2\sim 4m^2$ or $x\to -1$. We expand the form factors around $\beta=0$ up to $\mathcal{O}(\beta^2)$ and denote the form factors by $\tilde{F}$ in this limit.

4.3.1 Vector form factor

First, we present the electric component of the vector form factor

$$\begin{aligned}\tilde{F}_{V,1}^{(1)}=C_F\Bigg\{&\frac{1}{\varepsilon}\Bigg[\frac{8\beta^2}{3}+i\pi\Big(-\frac{1}{\beta}-\beta\Big)\Bigg]+\Bigg[\frac{6\zeta_2}{\beta}-6+6\beta\zeta_2-\frac{10\beta^2}{9}+i\pi\Big(\frac{1}{\beta}\Big(-1+2\ln(2)\\&+2\log(\beta)\Big)+\beta(-1+2\ln(2)+2\log(\beta))\Big)\Bigg]+\varepsilon\Bigg[\frac{1}{\beta}\Big(6\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta)\Big)+4\\&+\beta\Big(6\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{344}{27}+\frac{4\zeta_2}{3}\Big)+i\pi\Big(\frac{1}{\beta}\Big(-4+\frac{3}{2}\zeta_2+2\ln(2)\\&-2\ln^2(2)+2\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)+\beta\Big(-3+\frac{3}{2}\zeta_2+2\ln(2)-2\ln^2(2)\\&+2\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)\Big)\Bigg]+\varepsilon^2\Bigg[\frac{1}{\beta}\Big(24\zeta_2+3\zeta_2^2-12\zeta_2\ln(2)+12\zeta_2\ln^2(2)\\&-12\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta)\Big)-24-3\zeta_2+\beta\Big(18\zeta_2+3\zeta_2^2-12\zeta_2\ln(2)\\&+12\zeta_2\ln^2(2)-12\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta)\Big)+\beta^2\Big(-\frac{472}{81}-\frac{5\zeta_2}{9}-\frac{8\zeta_3}{9}\Big)\end{aligned}$$

$$
\begin{aligned}
&+ i\pi\Big(\frac{1}{\beta}\Big(- 8 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3 + 8\ln(2) - 3\zeta_2\ln(2) - 2\ln^2(2) + \frac{4\ln^3(2)}{3} + 8\log(\beta)\\
&- 3\zeta_2\log(\beta) - 4\ln(2)\log(\beta) + 4\ln^2(2)\log(\beta) - 2\log^2(\beta) + 4\ln(2)\log^2(\beta) + \frac{4\log^3(\beta)}{3}\Big)\\
&+ \beta\Big(- 4 + \frac{3}{2}\zeta_2 + \frac{7}{3}\zeta_3 + 6\ln(2) - 3\zeta_2\ln(2) - 2\ln^2(2) + \frac{4\ln^3(2)}{3} + 6\log(\beta) - 3\zeta_2\log(\beta)\\
&- 4\ln(2)\log(\beta) + 4\ln^2(2)\log(\beta) - 2\log^2(\beta) + 4\ln(2)\log^2(\beta) + \frac{4\log^3(\beta)}{3}\Big)\Big)\Big]\Big\}, \qquad (4.35)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}^{(2)}_{V,1} = C_F^2\Bigg\{&\frac{1}{\varepsilon^2}\Bigg[- \frac{3\zeta_2}{\beta^2} - 6\zeta_2 - 3\beta^2\zeta_2 + i\pi\Bigg(- \frac{8\beta}{3}\Bigg)\Bigg] + \frac{1}{\varepsilon}\Bigg[\frac{1}{\beta^2}\Big(- 6\zeta_2 + 12\zeta_2\ln(2)\\
&+ 12\zeta_2\log(\beta)\Big) - 12\zeta_2 + 16\beta\zeta_2 + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta) + \beta^2\Big(- 16 + 3\zeta_2 + 12\zeta_2\ln(2)\\
&+ 12\zeta_2\log(\beta)\Big) + i\pi\Bigg(- \frac{6\zeta_2}{\beta^2} + \frac{6}{\beta} - 12\zeta_2 + \frac{8}{9}\beta(5 + 6\ln(2) + 6\log(\beta)) - 6\beta^2\zeta_2\Bigg)\Bigg]\\
&+ \Bigg[\frac{1}{\beta^2}\Big(- 28\zeta_2 + 15\zeta_2^2 + 24\zeta_2\ln(2) - 24\zeta_2\ln^2(2) + 24\zeta_2\log(\beta) - 48\zeta_2\ln(2)\log(\beta)\\
&- 24\zeta_2\log^2(\beta)\Big) - \frac{36\zeta_2}{\beta} + \frac{421}{15} - \frac{1926}{25}\zeta_2 + 30\zeta_2^2 - \frac{81}{5}\zeta_3 + \frac{156}{5}\zeta_2\ln(2) - 48\zeta_2\ln^2(2)\\
&+ \frac{56}{5}\zeta_2\log(\beta) - 96\zeta_2\ln(2)\log(\beta) - 48\zeta_2\log^2(\beta) + \beta\Big(\frac{256}{3}\zeta_2 - 32\zeta_2\ln(2) - 32\zeta_2\log(\beta)\Big)\\
&+ \beta^2\Big(\frac{691}{35} - \frac{51302\zeta_2}{3675} + 15\zeta_2^2 + \frac{1192}{35}\zeta_3 - \frac{108}{35}\zeta_2\ln(2) - 24\zeta_2\ln^2(2) + \frac{412}{35}\zeta_2\log(\beta)\\
&- 48\zeta_2\ln(2)\log(\beta) - 24\zeta_2\log^2(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta^2}\Big(- 12\zeta_2 + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)\Big)\\
&- \frac{1}{\beta}\Big(12(\ln(2) + \log(\beta))\Big) - \frac{28}{5}\zeta_2 + 48\zeta_2\ln(2) + 48\zeta_2\log(\beta) + \beta\Big(- \frac{2498}{27} + \frac{8}{3}\zeta_2\\
&+ \frac{448\ln(2)}{9} - \frac{16}{3}\ln^2(2) + \frac{256\log(\beta)}{9} - \frac{32}{3}\ln(2)\log(\beta) - \frac{16}{3}\log^2(\beta)\Big) + \beta^2\Big(- \frac{281}{35}\zeta_2\\
&+ 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)\Big)\Bigg)\Bigg] + \varepsilon\Bigg[\frac{1}{\beta^2}\Big(- 28\zeta_2 + 30\zeta_2^2 - 106\zeta_2\zeta_3 + 112\zeta_2\ln(2)\\
&- 60\zeta_2^2\ln(2) - 48\zeta_2\ln^2(2) + 32\zeta_2\ln^3(2) + 112\zeta_2\log(\beta) - 60\zeta_2^2\log(\beta) - 96\zeta_2\ln(2)\log(\beta)\\
&+ 96\zeta_2\ln^2(2)\log(\beta) - 48\zeta_2\log^2(\beta) + 96\zeta_2\ln(2)\log^2(\beta) + 32\zeta_2\log^3(\beta)\Big) + \frac{1}{\beta}\Big(72\ln(2)\zeta_2\\
&+ 72\log(\beta)\zeta_2\Big) - \frac{36163}{450} - \frac{10c_1}{3} + \frac{157876}{375}\zeta_2 - \frac{306}{5}\zeta_2^2 - \frac{43756}{75}\zeta_3 - 212\zeta_2\zeta_3\\
&- \frac{32552}{75}\zeta_2\ln(2) - 120\zeta_2^2\ln(2) + \frac{32}{5}\zeta_2\ln^2(2) + 64\zeta_2\ln^3(2) - \frac{7144}{25}\zeta_2\log(\beta)\\
&- 120\zeta_2^2\log(\beta) - \frac{224}{5}\zeta_2\ln(2)\log(\beta) + 192\zeta_2\ln^2(2)\log(\beta) - \frac{112}{5}\zeta_2\log^2(\beta)\\
&+ 192\zeta_2\ln(2)\log^2(\beta) + 64\zeta_2\log^3(\beta) + \beta\Big(\frac{14416}{9}\zeta_2 + 16\zeta_2^2 - \frac{3776}{3}\zeta_2\ln(2) + 32\zeta_2\ln^2(2)\\
&- \frac{2624}{3}\zeta_2\log(\beta) + 64\zeta_2\ln(2)\log(\beta) + 32\zeta_2\log^2(\beta)\Big) + \beta^2\Big(\frac{1880267}{22050} + \frac{16c_1}{21} + \frac{61140199\zeta_2}{385875}
\end{aligned}
$$

$$+\frac{26498}{175}\zeta_2^2+\frac{286234\zeta_3}{3675}-106\zeta_2\zeta_3+\frac{800012\zeta_2\ln(2)}{3675}-60\zeta_2^2\ln(2)+\frac{208}{7}\zeta_2\ln^2(2)$$
$$+32\zeta_2\ln^3(2)+\frac{175684\zeta_2\log(\beta)}{1225}-60\zeta_2^2\log(\beta)-\frac{2368}{35}\zeta_2\ln(2)\log(\beta)+96\zeta_2\ln^2(2)\log(\beta)$$
$$-\frac{824}{35}\zeta_2\log^2(\beta)+96\zeta_2\ln(2)\log^2(\beta)+32\zeta_2\log^3(\beta)\Big)+i\pi\Bigg(\frac{1}{\beta^2}\Big(-56\zeta_2-18\zeta_2^2$$
$$+48\zeta_2\ln(2)-48\zeta_2\ln^2(2)+48\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\Big)$$
$$+\frac{1}{\beta}\Big(36-6\zeta_2+12\ln^2(2)+24\ln(2)\log(\beta)+12\log^2(\beta)\Big)+\frac{3572}{25}\zeta_2-36\zeta_2^2+\frac{112}{5}\zeta_2\ln(2)$$
$$-96\zeta_2\ln^2(2)+\frac{112}{5}\zeta_2\log(\beta)-192\zeta_2\ln(2)\log(\beta)-96\zeta_2\log^2(\beta)+\beta\Big(-\frac{49163}{81}+\frac{1064}{9}\zeta_2$$
$$+\frac{64}{9}\zeta_3+\frac{19408\ln(2)}{27}-\frac{16}{3}\zeta_2\ln(2)-\frac{2656}{9}\ln^2(2)+\frac{32\ln^3(2)}{9}+\frac{14416\log(\beta)}{27}$$
$$-\frac{16}{3}\zeta_2\log(\beta)-\frac{3776}{9}\ln(2)\log(\beta)+\frac{32}{3}\ln^2(2)\log(\beta)-\frac{1312}{9}\log^2(\beta)+\frac{32}{3}\ln(2)\log^2(\beta)$$
$$+\frac{32\log^3(\beta)}{9}\Big)+\beta^2\Big(-\frac{62992\zeta_2}{1225}-18\zeta_2^2-\frac{76}{35}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{674}{35}\zeta_2\log(\beta)$$
$$-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\Big)\Bigg)\Bigg]\Bigg\}+C_FC_A\Bigg\{\frac{1}{\varepsilon^2}\Bigg[-\frac{44\beta^2}{9}+i\pi\Bigg(\frac{11}{6\beta}+\frac{11\beta}{6}\Bigg)\Bigg]$$
$$+\frac{1}{\varepsilon}\Bigg[-8\beta\zeta_2+\beta^2\Big(\frac{376}{27}+\frac{32\zeta_2}{3}\Big)+i\pi\Bigg(-\frac{31}{18\beta}+\frac{1}{6}\beta(-13-32\ln(2)-16\log(\beta))\Bigg)\Bigg]$$
$$+\Bigg[\frac{1}{\beta}\Big(\frac{146}{3}\zeta_2-44\zeta_2\ln(2)-44\zeta_2\log(\beta)\Big)-\frac{379}{15}+\frac{5482}{75}\zeta_2-\frac{166}{5}\zeta_3-\frac{312}{5}\zeta_2\ln(2)$$
$$-\frac{144}{5}\zeta_2\log(\beta)+\beta\Big(-\frac{16}{3}\zeta_2+84\zeta_2\ln(2)+36\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{26779}{2835}+88\zeta_2-\frac{346}{105}\zeta_3$$
$$-\frac{1952}{35}\zeta_2\ln(2)-48\zeta_2\log(\beta)\Big)+i\pi\Bigg(\frac{1}{\beta}\Big(-\frac{394}{27}+\frac{146\ln(2)}{9}-\frac{22\ln^2(2)}{3}+\frac{146\log(\beta)}{9}$$
$$-\frac{44}{3}\ln(2)\log(\beta)-\frac{22\log^2(\beta)}{3}\Big)+\frac{72}{5}\zeta_2+\beta\Big(-\frac{139}{9}-\frac{40}{3}\zeta_2-\frac{152\ln(2)}{9}+\frac{74\ln^2(2)}{3}$$
$$-\frac{16\log(\beta)}{9}+28\ln(2)\log(\beta)+6\log^2(\beta)\Big)+\frac{351}{14}\beta^2\zeta_2\Bigg)\Bigg]+\varepsilon\Bigg[\frac{1}{\beta}\Big(\frac{2422}{9}\zeta_2+99\zeta_2^2$$
$$-\frac{752}{3}\zeta_2\ln(2)+132\zeta_2\ln^2(2)-\frac{752}{3}\zeta_2\log(\beta)+264\zeta_2\ln(2)\log(\beta)+132\zeta_2\log^2(\beta)\Big)$$
$$-\frac{28301}{150}+\frac{28c_1}{5}+\frac{228613\zeta_2}{1125}-\frac{7748}{25}\zeta_2^2+\frac{8539}{75}\zeta_3-\frac{1076}{5}\zeta_2\ln(2)-\frac{256}{5}\zeta_2\ln^2(2)$$
$$-\frac{4144}{25}\zeta_2\log(\beta)+\frac{576}{5}\zeta_2\ln(2)\log(\beta)+\frac{288}{5}\zeta_2\log^2(\beta)+\beta\Big(\frac{976}{9}\zeta_2+59\zeta_2^2+\frac{1612}{3}\zeta_2\ln(2)$$
$$-508\zeta_2\ln^2(2)+\frac{796}{3}\zeta_2\log(\beta)-632\zeta_2\ln(2)\log(\beta)-172\zeta_2\log^2(\beta)\Big)+\beta^2\Big(\frac{39802157}{198450}$$
$$-\frac{284c_1}{105}+\frac{5048597\zeta_2}{9450}-\frac{209161\zeta_2^2}{1050}-\frac{244703\zeta_3}{7350}-\frac{323804\zeta_2\ln(2)}{3675}+\frac{264}{7}\zeta_2\ln^2(2)$$

$$
- \frac{1274}{5}\zeta_2\log(\beta) + 144\zeta_2\ln(2)\log(\beta) + 96\zeta_2\log^2(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(- \frac{13177}{162} + \frac{38}{3}\zeta_2 + \frac{22}{3}\zeta_3
$$

$$
+ \frac{2422\ln(2)}{27} - 11\zeta_2\ln(2) - \frac{376}{9}\ln^2(2) + \frac{44\ln^3(2)}{3} + \frac{2422\log(\beta)}{27} - 11\zeta_2\log(\beta)
$$

$$
- \frac{752}{9}\ln(2)\log(\beta) + 44\ln^2(2)\log(\beta) - \frac{376}{9}\log^2(\beta) + 44\ln(2)\log^2(\beta) + \frac{44\log^3(\beta)}{3}\Big)
$$

$$
+ \frac{2072}{25}\zeta_2 - \frac{288}{5}\zeta_2\ln(2) - \frac{288}{5}\zeta_2\log(\beta) + \beta\Big(- \frac{2950}{27} - \frac{596}{9}\zeta_2 - \frac{194}{3}\zeta_3 - \frac{256\ln(2)}{27}
$$

$$
+ \frac{335}{3}\zeta_2\ln(2) + 150\ln^2(2) - \frac{764}{9}\ln^3(2) + \frac{976\log(\beta)}{27} + 77\zeta_2\log(\beta) + \frac{1612}{9}\ln(2)\log(\beta)
$$

$$
- \frac{508}{3}\ln^2(2)\log(\beta) + \frac{398\log^2(\beta)}{9} - \frac{316}{3}\ln(2)\log^2(\beta) - \frac{172}{9}\log^3(\beta)\Big) + \beta^2\Big(\frac{792}{5}\zeta_2
$$

$$
- \frac{786}{7}\zeta_2\ln(2) - \frac{657}{7}\zeta_2\log(\beta)\Big)\Bigg)\Bigg]\Bigg\} + C_F T_F n_l\Bigg\{\frac{1}{\varepsilon^2}\Bigg[\frac{16\beta^2}{9} + i\pi\Bigg(- \frac{2}{3\beta} - \frac{2\beta}{3}\Bigg)\Bigg]
$$

$$
+ \frac{1}{\varepsilon}\Bigg[- \frac{80\beta^2}{27} + i\pi\Bigg(\frac{10}{9\beta} + \frac{10\beta}{9}\Bigg)\Bigg] + \Bigg[\frac{1}{\beta}\Big(- \frac{64}{3}\zeta_2 + 16\zeta_2\ln(2) + 16\zeta_2\log(\beta)\Big) + 4
$$

$$
+ \beta\Big(- \frac{64}{3}\zeta_2 + 16\zeta_2\ln(2) + 16\zeta_2\log(\beta)\Big) + \beta^2\Big(- \frac{868}{81} - \frac{64\zeta_2}{9}\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(\frac{206}{27} - \frac{64\ln(2)}{9}
$$

$$
+ \frac{8\ln^2(2)}{3} - \frac{64\log(\beta)}{9} + \frac{16}{3}\ln(2)\log(\beta) + \frac{8\log^2(\beta)}{3}\Big) + \beta\Big(\frac{116}{27} - \frac{64\ln(2)}{9} + \frac{8\ln^2(2)}{3}
$$

$$
- \frac{64\log(\beta)}{9} + \frac{16}{3}\ln(2)\log(\beta) + \frac{8\log^2(\beta)}{3}\Big)\Bigg)\Bigg] + \varepsilon\Bigg[\frac{1}{\beta}\Big(- \frac{1112}{9}\zeta_2 - 36\zeta_2^2 + \frac{304}{3}\zeta_2\ln(2)
$$

$$
- 48\zeta_2\ln^2(2) + \frac{304}{3}\zeta_2\log(\beta) - 96\zeta_2\ln(2)\log(\beta) - 48\zeta_2\log^2(\beta)\Big) + \frac{574}{9} + 36\zeta_2
$$

$$
+ \beta\Big(- \frac{680}{9}\zeta_2 - 36\zeta_2^2 + \frac{304}{3}\zeta_2\ln(2) - 48\zeta_2\ln^2(2) + \frac{304}{3}\zeta_2\log(\beta) - 96\zeta_2\ln(2)\log(\beta)
$$

$$
- 48\zeta_2\log^2(\beta)\Big) + \beta^2\Big(- \frac{518}{81} - \frac{220\zeta_2}{27} - \frac{128\zeta_3}{9}\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(\frac{3211}{81} - \frac{16}{3}\zeta_2 - \frac{8}{3}\zeta_3 - \frac{1112\ln(2)}{27}
$$

$$
+ 4\zeta_2\ln(2) + \frac{152\ln^2(2)}{9} - \frac{16}{3}\ln^3(2) - \frac{1112\log(\beta)}{27} + 4\zeta_2\log(\beta) + \frac{304}{9}\ln(2)\log(\beta)
$$

$$
- 16\ln^2(2)\log(\beta) + \frac{152\log^2(\beta)}{9} - 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta)\Big) + \beta\Big(\frac{934}{81} - \frac{16}{3}\zeta_2 - \frac{8}{3}\zeta_3
$$

$$
- \frac{680\ln(2)}{27} + 4\zeta_2\ln(2) + \frac{152\ln^2(2)}{9} - \frac{16}{3}\ln^3(2) - \frac{680\log(\beta)}{27} + 4\zeta_2\log(\beta)
$$

$$
+ \frac{304}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) + \frac{152\log^2(\beta)}{9} - 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta)\Big)\Bigg)\Bigg]\Bigg\}
$$

$$
+ C_F T_F\Bigg\{\Bigg[\frac{148}{3} - \frac{416}{15}\zeta_2 + \beta^2\Big(- \frac{1372}{81} + \frac{52\zeta_2}{5}\Big) + i\pi\Bigg(\frac{2\zeta_2}{3\beta} + \beta\Big(\frac{16}{15} + \frac{2\zeta_2}{3}\Big)\Bigg)\Bigg]
$$

$$
+ \varepsilon\Bigg[- \frac{4\zeta_2^2}{\beta} + \frac{2102}{15} - \frac{28616}{225}\zeta_2 - \frac{1456}{15}\zeta_3 + \frac{832}{5}\zeta_2\ln(2) + \beta\Big(- \frac{32\zeta_2}{5} - 4\zeta_2^2\Big)
$$

$$
\begin{aligned}
&+\beta^2\Big(-\frac{47902}{1215}+\frac{475514\zeta_2}{11025}+\frac{5104}{135}\zeta_3-\frac{6232}{105}\zeta_2\ln(2)+\frac{24}{7}\zeta_2\log(\beta)\Big)\\
&+i\pi\Bigg(\frac{1}{\beta}\Big(\frac{2}{3}\zeta_2-\frac{4}{9}\zeta_3-\frac{4}{3}\zeta_2\ln(2)-\frac{4}{3}\zeta_2\log(\beta)\Big)+\beta\Big(\frac{16}{15}+\frac{2}{3}\zeta_2-\frac{4}{9}\zeta_3-\frac{32\ln(2)}{15}\\
&-\frac{4}{3}\zeta_2\ln(2)-\frac{32\log(\beta)}{15}-\frac{4}{3}\zeta_2\log(\beta)\Big)\Bigg)\Bigg]\Bigg\}. \qquad (4.36)
\end{aligned}
$$

Next, we present the magnetic component of the vector form factor

$$
\begin{aligned}
\tilde{F}^{(1)}_{V,2}=C_F\Bigg\{&-2+\frac{4\beta^2}{3}+i\pi\Big(\frac{1}{\beta}-\beta\Big)+\varepsilon\Bigg[-\frac{6\zeta_2}{\beta}-4+6\beta\zeta_2+\frac{28\beta^2}{9}+i\pi\Bigg(-\frac{1}{\beta}\Big(2(-2\\
&+\ln(2)+\log(\beta))\Big)+\beta(-5+2\ln(2)+2\log(\beta))\Bigg)\Bigg]+\varepsilon^2\Bigg[\frac{1}{\beta}\Big(-24\zeta_2+12\zeta_2\ln(2)\\
&+12\zeta_2\log(\beta)\Big)-8-\zeta_2+\beta\Big(30\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{160}{27}+\frac{2\zeta_2}{3}\Big)\\
&+i\pi\Bigg(\frac{1}{\beta}\Big(8-\frac{3}{2}\zeta_2-8\ln(2)+2\ln^2(2)-8\log(\beta)+4\ln(2)\log(\beta)+2\log^2(\beta)\Big)\\
&+\beta\Big(-12+\frac{3}{2}\zeta_2+10\ln(2)-2\ln^2(2)+10\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)\Bigg)\Bigg]\Bigg\}. \qquad (4.37)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}^{(2)}_{V,2}=C_F^2\Bigg\{&\frac{1}{\varepsilon}\Bigg[\frac{6\zeta_2}{\beta^2}+\beta^2\Big(-\frac{16}{3}-6\zeta_2\Big)+i\pi\Big(\frac{2}{\beta}+\frac{10\beta}{3}\Big)\Bigg]+\Bigg[\frac{1}{\beta^2}\Big(28\zeta_2-24\zeta_2\ln(2)\\
&-24\zeta_2\log(\beta)\Big)+\frac{269}{15}-\frac{4922}{75}\zeta_2-\frac{12\zeta_2}{\beta}-60\beta\zeta_2+\frac{41}{5}\zeta_3+\frac{404}{5}\zeta_2\ln(2)+\frac{24}{5}\zeta_2\log(\beta)\\
&+\beta^2\Big(\frac{3658}{315}+\frac{103928\zeta_2}{3675}-\frac{471}{35}\zeta_3-\frac{1628}{35}\zeta_2\ln(2)+\frac{512}{35}\zeta_2\log(\beta)\Big)+i\pi\Bigg(\frac{12\zeta_2}{\beta^2}-\frac{1}{\beta}\Big(4(\ln(2)\\
&+\log(\beta))\Big)-\frac{12}{5}\zeta_2-\frac{2}{9}\beta(-73+90\ln(2)+90\log(\beta))-\frac{181}{35}\beta^2\zeta_2\Bigg)\Bigg]+\varepsilon\Bigg[\frac{1}{\beta^2}\Big(28\zeta_2-30\zeta_2^2\\
&-112\zeta_2\ln(2)+48\zeta_2\ln^2(2)-112\zeta_2\log(\beta)+96\zeta_2\ln(2)\log(\beta)+48\zeta_2\log^2(\beta)\Big)\\
&+\frac{1}{\beta}\Big(24\ln(2)\zeta_2+24\log(\beta)\zeta_2\Big)+\frac{20713}{450}-\frac{38c_1}{3}-\frac{529628\zeta_2}{1125}+\frac{1342}{5}\zeta_2^2-\frac{15944}{75}\zeta_3\\
&+\frac{27752}{75}\zeta_2\ln(2)+\frac{448}{5}\zeta_2\ln^2(2)+\frac{344}{25}\zeta_2\log(\beta)-\frac{96}{5}\zeta_2\ln(2)\log(\beta)-\frac{48}{5}\zeta_2\log^2(\beta)\\
&+\beta\Big(-\frac{200}{3}\zeta_2+280\zeta_2\ln(2)+280\zeta_2\log(\beta)\Big)+\beta^2\Big(-\frac{4863529}{33075}+\frac{250c_1}{21}+\frac{115201589\zeta_2}{385875}\\
&-\frac{42283}{175}\zeta_2^2+\frac{330121\zeta_3}{1225}-\frac{749248\zeta_2\ln(2)}{3675}-\frac{3672}{35}\zeta_2\ln^2(2)+\frac{245324\zeta_2\log(\beta)}{1225}\\
&-\frac{1328}{35}\zeta_2\ln(2)\log(\beta)-\frac{1024}{35}\zeta_2\log^2(\beta)\Big)+i\pi\Bigg(\frac{1}{\beta^2}\Big(56\zeta_2-48\zeta_2\ln(2)-48\zeta_2\log(\beta)\Big)\\
&+\frac{1}{\beta}\Big(-4-2\zeta_2+4\ln^2(2)+8\ln(2)\log(\beta)+4\log^2(\beta)\Big)-\frac{172}{25}\zeta_2+\frac{48}{5}\zeta_2\ln(2)+\frac{48}{5}\zeta_2\log(\beta)
\end{aligned}
$$

$$+\beta\Big(-\frac{241}{9}-\frac{50}{3}\zeta_2-\frac{8\ln(2)}{9}+\frac{140\ln^2(2)}{3}-\frac{200\log(\beta)}{9}+\frac{280}{3}\ln(2)\log(\beta)+\frac{140\log^2(\beta)}{3}\Big)$$
$$+\beta^2\Big(-\frac{147512\zeta_2}{1225}+\frac{1924}{35}\zeta_2\ln(2)+\frac{1174}{35}\zeta_2\log(\beta)\Big)\Big)\Big]\Big\}+C_FC_A\Big\{\Big[-\frac{28\zeta_2}{\beta}-\frac{373}{45}$$
$$+\frac{1156}{25}\zeta_2-\frac{94}{5}\zeta_3-\frac{328}{5}\zeta_2\ln(2)-\frac{96}{5}\zeta_2\log(\beta)+60\beta\zeta_2+\beta^2\Big(-\frac{1234}{315}-\frac{1577}{25}\zeta_2+\frac{288}{35}\zeta_3$$
$$+\frac{2344}{35}\zeta_2\ln(2)+\frac{32}{5}\zeta_2\log(\beta)\Big)+i\pi\Big(-\frac{1}{9\beta}\Big(4(-25+21\ln(2)+21\log(\beta))\Big)+\frac{48}{5}\zeta_2$$
$$+\frac{2}{3}\beta(-49+38\ln(2)+30\log(\beta))-\frac{299}{70}\beta^2\zeta_2\Big)\Big]+\varepsilon\Big[\frac{1}{\beta}\Big(-\frac{682}{3}\zeta_2+168\zeta_2\ln(2)$$
$$+168\zeta_2\log(\beta)\Big)-\frac{181841}{1350}+\frac{116c_1}{15}+\frac{104129}{375}\zeta_2-\frac{6652}{25}\zeta_2^2+\frac{6236}{75}\zeta_3-\frac{4112}{15}\zeta_2\ln(2)$$
$$-\frac{224}{5}\zeta_2\ln^2(2)-\frac{1856}{25}\zeta_2\log(\beta)+\frac{384}{5}\zeta_2\ln(2)\log(\beta)+\frac{192}{5}\zeta_2\log^2(\beta)+\beta\Big(\frac{1822}{3}\zeta_2$$
$$-520\zeta_2\ln(2)-424\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{16374871}{99225}-\frac{128c_1}{21}-\frac{975547\zeta_2}{5250}+\frac{82843}{350}\zeta_2^2-\frac{458029\zeta_3}{3675}$$
$$-\frac{71626\zeta_2\ln(2)}{3675}+\frac{4056}{35}\zeta_2\ln^2(2)-\frac{638}{25}\zeta_2\log(\beta)+\frac{112}{5}\zeta_2\ln(2)\log(\beta)-\frac{64}{5}\zeta_2\log^2(\beta)\Big)$$
$$+i\pi\Big(\frac{1}{\beta}\Big(\frac{4015}{54}-\frac{15}{2}\zeta_2-\frac{682\ln(2)}{9}+28\ln^2(2)-\frac{682\log(\beta)}{9}+56\ln(2)\log(\beta)+28\log^2(\beta)\Big)$$
$$+\frac{928}{25}\zeta_2-\frac{192}{5}\zeta_2\ln(2)-\frac{192}{5}\zeta_2\log(\beta)+\beta\Big(-\frac{12113}{54}+\frac{79}{2}\zeta_2+\frac{2270\ln(2)}{9}-108\ln^2(2)$$
$$+\frac{1822\log(\beta)}{9}-\frac{520}{3}\ln(2)\log(\beta)-\frac{212}{3}\log^2(\beta)\Big)+\beta^2\Big(-\frac{456}{25}\zeta_2+\frac{1018}{35}\zeta_2\ln(2)$$
$$+\frac{373}{35}\zeta_2\log(\beta)\Big)\Big)\Big]\Big\}+C_FT_Fn_l\Big\{\Big[\frac{8\zeta_2}{\beta}+\frac{52}{9}-8\beta\zeta_2-\frac{40\beta^2}{9}+i\pi\Big(\frac{1}{9\beta}\Big(2(-25+12\ln(2)$$
$$+12\log(\beta))\Big)-\frac{2}{9}\beta(-31+12\ln(2)+12\log(\beta))\Big)\Big]+\varepsilon\Big[\frac{1}{\beta}\Big(\frac{296}{3}\zeta_2-48\zeta_2\ln(2)$$
$$-48\zeta_2\log(\beta)\Big)+\frac{1010}{27}+12\zeta_2+\beta\Big(-\frac{368}{3}\zeta_2+48\zeta_2\ln(2)+48\zeta_2\log(\beta)\Big)+\beta^2\Big(-\frac{2228}{81}$$
$$-8\zeta_2\Big)+i\pi\Big(\frac{1}{\beta}\Big(-\frac{961}{27}+2\zeta_2+\frac{296\ln(2)}{9}-8\ln^2(2)+\frac{296\log(\beta)}{9}-16\ln(2)\log(\beta)$$
$$-8\log^2(\beta)\Big)+\beta\Big(\frac{1405}{27}-2\zeta_2-\frac{368\ln(2)}{9}+8\ln^2(2)-\frac{368\log(\beta)}{9}+16\ln(2)\log(\beta)$$
$$+8\log^2(\beta)\Big)\Big)\Big]\Big\}+C_FT_F\Big\{\Big[-\frac{92}{9}+\frac{32}{5}\zeta_2+\beta^2\Big(\frac{248}{27}-\frac{36\zeta_2}{5}\Big)\Big]+\varepsilon\Big[\frac{82}{135}+\frac{324}{25}\zeta_2$$
$$+\frac{112}{5}\zeta_3-\frac{192}{5}\zeta_2\ln(2)+\beta^2\Big(\frac{772}{405}-\frac{171322\zeta_2}{11025}-\frac{96}{5}\zeta_3+\frac{1032}{35}\zeta_2\ln(2)-\frac{24}{7}\zeta_2\log(\beta)\Big)$$
$$+i\pi\Big(-\frac{2\zeta_2}{3\beta}+\beta\Big(\frac{16}{15}+\frac{2\zeta_2}{3}\Big)\Big)\Big]\Big\}. \tag{4.38}$$

4.3.2 Axial-vector form factor

In the following, we provide the non-singlet part of the axial-vector form factors in the threshold region

$$\tilde{F}_{A,1}^{(1),\mathrm{ns}} = C_F\Bigg\{\frac{1}{\varepsilon}\Bigg[\frac{8\beta^2}{3}+i\pi\Bigg(-\frac{1}{\beta}-\beta\Bigg)\Bigg]+\Bigg[\frac{6\zeta_2}{\beta}-4+6\beta\zeta_2-\frac{22\beta^2}{9}+i\pi\Bigg(\frac{1}{\beta}\Big(2(-1+\ln(2)$$
$$+\log(\beta))\Big)+2\beta(\ln(2)+\log(\beta))\Bigg)\Bigg]+\varepsilon\Bigg[\frac{1}{\beta}\Big(12\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta)\Big)$$
$$+\beta\Big(-12\ln(2)\zeta_2-12\log(\beta)\zeta_2\Big)+\beta^2\Big(\frac{404}{27}+\frac{4\zeta_2}{3}\Big)+i\pi\Bigg(\frac{1}{\beta}\Big(-4+\frac{3}{2}\zeta_2+4\ln(2)$$
$$-2\ln^2(2)+4\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)+\beta\Big(-2+\frac{3}{2}\zeta_2-2\ln^2(2)$$
$$-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)\Bigg)\Bigg]+\varepsilon^2\Bigg[\frac{1}{\beta}\Big(24\zeta_2+3\zeta_2^2-24\zeta_2\ln(2)+12\zeta_2\ln^2(2)$$
$$-24\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta)\Big)-16-2\zeta_2+\beta\Big(12\zeta_2+3\zeta_2^2$$
$$+12\zeta_2\ln^2(2)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta)\Big)+\beta^2\Big(-\frac{808}{81}-\frac{11\zeta_2}{9}-\frac{8\zeta_3}{9}\Big)$$
$$+i\pi\Bigg(\frac{1}{\beta}\Big(-8+3\zeta_2+\frac{7}{3}\zeta_3+8\ln(2)-3\zeta_2\ln(2)-4\ln^2(2)+\frac{4\ln^3(2)}{3}+8\log(\beta)$$
$$-3\zeta_2\log(\beta)-8\ln(2)\log(\beta)+4\ln^2(2)\log(\beta)-4\log^2(\beta)+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\Big)$$
$$+\beta\Big(-4+\frac{7}{3}\zeta_3+4\ln(2)-3\zeta_2\ln(2)+\frac{4\ln^3(2)}{3}+4\log(\beta)-3\zeta_2\log(\beta)+4\ln^2(2)\log(\beta)$$
$$+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\Big)\Bigg)\Bigg]\Bigg\}. \tag{4.39}$$

$$\tilde{F}_{A,1}^{(2),\mathrm{ns}} = C_F^2\Bigg\{\frac{1}{\varepsilon^2}\Bigg[-\frac{3\zeta_2}{\beta^2}-6\zeta_2-3\beta^2\zeta_2+i\pi\Bigg(-\frac{8\beta}{3}\Bigg)\Bigg]+\frac{1}{\varepsilon}\Bigg[\frac{1}{\beta^2}\Big(-12\zeta_2+12\zeta_2\ln(2)$$
$$+12\zeta_2\log(\beta)\Big)-12\zeta_2+16\beta\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta)+\beta^2\Big(-\frac{32}{3}+9\zeta_2+12\zeta_2\ln(2)$$
$$+12\zeta_2\log(\beta)\Big)+i\pi\Bigg(-\frac{6\zeta_2}{\beta^2}+\frac{4}{\beta}-12\zeta_2+\frac{2}{9}\beta(5+24\ln(2)+24\log(\beta))-6\beta^2\zeta_2\Bigg)\Bigg]$$
$$+\Bigg[\frac{1}{\beta^2}\Big(-24\zeta_2+15\zeta_2^2+48\zeta_2\ln(2)-24\zeta_2\ln^2(2)+48\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)$$
$$-24\zeta_2\log^2(\beta)\Big)+\frac{46}{3}-32\zeta_2-\frac{24\zeta_2}{\beta}+30\zeta_2^2-27\zeta_3+84\zeta_2\ln(2)-48\zeta_2\ln^2(2)+8\zeta_2\log(\beta)$$
$$-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)+\beta\Big(-\frac{20}{3}\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{247}{9}$$
$$-\frac{1508}{75}\zeta_2+15\zeta_2^2+50\zeta_3-\frac{172}{5}\zeta_2\ln(2)-24\zeta_2\ln^2(2)-\frac{4}{5}\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)$$

$$
- 24\zeta_2 \log^2(\beta)\Big) + i\pi\left(\frac{1}{\beta^2}\Big(- 24\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta)\Big) - \frac{1}{\beta}\Big(8(-1 + \ln(2)\right.
$$

$$
\left. + \log(\beta))\Big) - 4\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta) + \beta\Big(- \frac{956}{27} + \frac{8}{3}\zeta_2 - \frac{20\ln(2)}{9} - \frac{16}{3}\ln^2(2)\right.
$$

$$
\left. - \frac{20\log(\beta)}{9} - \frac{32}{3}\ln(2)\log(\beta) - \frac{16}{3}\log^2(\beta)\Big) + \beta^2\Big(\frac{2}{5}\zeta_2 + 24\zeta_2 \ln(2) + 24\zeta_2 \log(\beta)\Big)\right)\Bigg]
$$

$$
+ \varepsilon\Bigg[\frac{1}{\beta^2}\Big(48\zeta_2 + 60\zeta_2^2 - 106\zeta_2\zeta_3 + 96\zeta_2 \ln(2) - 60\zeta_2^2 \ln(2) - 96\zeta_2 \ln^2(2) + 32\zeta_2 \ln^3(2)
$$

$$
+ 96\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) - 192\zeta_2 \ln(2)\log(\beta) + 96\zeta_2 \ln^2(2)\log(\beta) - 96\zeta_2 \log^2(\beta)
$$

$$
+ 96\zeta_2 \ln(2)\log^2(\beta) + 32\zeta_2 \log^3(\beta)\Big) + \frac{1}{\beta}\Big(- 48\zeta_2 + 48\zeta_2 \ln(2) + 48\zeta_2 \log(\beta)\Big) - \frac{313}{9}
$$

$$
- \frac{38c_1}{3} + \frac{940}{3}\zeta_2 + \frac{282}{5}\zeta_2^2 - \frac{1202}{3}\zeta_3 - 212\zeta_2\zeta_3 - \frac{424}{3}\zeta_2 \ln(2) - 120\zeta_2^2 \ln(2) + 32\zeta_2 \ln^2(2)
$$

$$
+ 64\zeta_2 \ln^3(2) - 304\zeta_2 \log(\beta) - 120\zeta_2^2 \log(\beta) - 32\zeta_2 \ln(2)\log(\beta) + 192\zeta_2 \ln^2(2)\log(\beta)
$$

$$
- 16\zeta_2 \log^2(\beta) + 192\zeta_2 \ln(2)\log^2(\beta) + 64\zeta_2 \log^3(\beta) + \beta\Big(\frac{3964}{9}\zeta_2 + 16\zeta_2^2 + \frac{40}{3}\zeta_2 \ln(2)
$$

$$
+ 32\zeta_2 \ln^2(2) + \frac{40}{3}\zeta_2 \log(\beta) + 64\zeta_2 \ln(2)\log(\beta) + 32\zeta_2 \log^2(\beta)\Big) + \beta^2\Big(- \frac{35237}{270} + \frac{28c_1}{5}
$$

$$
- \frac{117812\zeta_2}{1125} + \frac{3639}{25}\zeta_2^2 + \frac{724}{15}\zeta_3 - 106\zeta_2\zeta_3 - \frac{624}{5}\zeta_2 \ln(2) - 60\zeta_2^2 \ln(2) + \frac{488}{5}\zeta_2 \ln^2(2)
$$

$$
+ 32\zeta_2 \ln^3(2) + \frac{3696}{25}\zeta_2 \log(\beta) - 60\zeta_2^2 \log(\beta) + \frac{16}{5}\zeta_2 \ln(2)\log(\beta) + 96\zeta_2 \ln^2(2)\log(\beta)
$$

$$
+ \frac{8}{5}\zeta_2 \log^2(\beta) + 96\zeta_2 \ln(2)\log^2(\beta) + 32\zeta_2 \log^3(\beta)\Big) + i\pi\left(\frac{1}{\beta^2}\Big(- 48\zeta_2 - 18\zeta_2^2 + 96\zeta_2 \ln(2)\right.
$$

$$
\left. - 48\zeta_2 \ln^2(2) + 96\zeta_2 \log(\beta) - 96\zeta_2 \ln(2)\log(\beta) - 48\zeta_2 \log^2(\beta)\Big) + \frac{1}{\beta}\Big(32 - 4\zeta_2 - 16\ln(2)\right.
$$

$$
\left. + 8\ln^2(2) - 16\log(\beta) + 16\ln(2)\log(\beta) + 8\log^2(\beta)\Big) + 152\zeta_2 - 36\zeta_2^2 + 16\zeta_2 \ln(2)\right.
$$

$$
\left. - 96\zeta_2 \ln^2(2) + 16\zeta_2 \log(\beta) - 192\zeta_2 \ln(2)\log(\beta) - 96\zeta_2 \log^2(\beta) + \beta\Big(- \frac{18635}{81} - \frac{10}{9}\zeta_2\right.
$$

$$
\left. + \frac{64}{9}\zeta_3 + \frac{5116\ln(2)}{27} - \frac{16}{3}\zeta_2 \ln(2) + \frac{20\ln^2(2)}{9} + \frac{32\ln^3(2)}{9} + \frac{3964\log(\beta)}{27} - \frac{16}{3}\zeta_2 \log(\beta)\right.
$$

$$
\left. + \frac{40}{9}\ln(2)\log(\beta) + \frac{32}{3}\ln^2(2)\log(\beta) + \frac{20\log^2(\beta)}{9} + \frac{32}{3}\ln(2)\log^2(\beta) + \frac{32\log^3(\beta)}{9}\Big)\right.
$$

$$
\left. + \beta^2\Big(- \frac{1848}{25}\zeta_2 - 18\zeta_2^2 - \frac{8}{5}\zeta_2 \ln(2) - 48\zeta_2 \ln^2(2) - \frac{8}{5}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2)\log(\beta)\right.
$$

$$
\left. - 48\zeta_2 \log^2(\beta)\Big)\right)\Bigg]\Bigg\} + C_F C_A\Bigg\{\frac{1}{\varepsilon^2}\left[- \frac{44\beta^2}{9} + i\pi\left(\frac{11}{6\beta} + \frac{11\beta}{6}\right)\right] + \frac{1}{\varepsilon}\Bigg[- 8\beta\zeta_2 + \beta^2\Big(\frac{376}{27}\Bigg.
$$

$$
\Bigg. + \frac{32\zeta_2}{3}\Big) + i\pi\left(- \frac{31}{18\beta} + \frac{1}{6}\beta(-13 - 32\ln(2) - 16\log(\beta))\right)\Bigg] + \Bigg[\frac{1}{\beta}\Big(\frac{194}{3}\zeta_2 - 44\zeta_2 \ln(2)\Bigg.
$$

$$
\Bigg. - 44\zeta_2 \log(\beta)\Big) - \frac{202}{9} + \frac{178}{3}\zeta_2 - 18\zeta_3 - 72\zeta_2 \ln(2) - 16\zeta_2 \log(\beta) + \beta\Big(\frac{8}{3}\zeta_2 + 84\zeta_2 \ln(2)
$$

$$
+ 36\zeta_2 \log(\beta)\Big) + \beta^2\Big(\frac{1159}{405} + \frac{6263}{75}\zeta_2 - \frac{346}{15}\zeta_3 - \frac{216}{5}\zeta_2 \ln(2) - \frac{336}{5}\zeta_2 \log(\beta)\Big)
$$

$$+ i\pi\Bigg(\frac{1}{\beta}\Big(-\frac{478}{27}+\frac{194\ln(2)}{9}-\frac{22\ln^2(2)}{3}+\frac{194\log(\beta)}{9}-\frac{44}{3}\ln(2)\log(\beta)-\frac{22\log^2(\beta)}{3}\Big)$$
$$+ 8\zeta_2+\beta\Big(-\frac{143}{9}-\frac{40}{3}\zeta_2-\frac{80\ln(2)}{9}+\frac{74\ln^2(2)}{3}+\frac{8\log(\beta)}{9}+28\ln(2)\log(\beta)+6\log^2(\beta)\Big)$$
$$+\frac{168}{5}\beta^2\zeta_2\Bigg)\Bigg]+\varepsilon\Bigg[\frac{1}{\beta}\Big(\frac{2704}{9}\zeta_2+99\zeta_2^2-\frac{1040}{3}\zeta_2\ln(2)+132\zeta_2\ln^2(2)-\frac{1040}{3}\zeta_2\log(\beta)$$
$$+264\zeta_2\ln(2)\log(\beta)+132\zeta_2\log^2(\beta)\Big)-\frac{4103}{27}+\frac{28c_1}{3}+\frac{2294}{9}\zeta_2-\frac{1404}{5}\zeta_2^2+73\zeta_3$$
$$-\frac{940}{3}\zeta_2\ln(2)-64\zeta_2\ln^2(2)-144\zeta_2\log(\beta)+64\zeta_2\ln(2)\log(\beta)+32\zeta_2\log^2(\beta)+\beta\Big(\frac{1702}{9}\zeta_2$$
$$+59\zeta_2^2+\frac{988}{3}\zeta_2\ln(2)-508\zeta_2\ln^2(2)+\frac{460}{3}\zeta_2\log(\beta)-632\zeta_2\ln(2)\log(\beta)-172\zeta_2\log^2(\beta)\Big)$$
$$+\beta^2\Big(\frac{361877}{1350}-4c_1+\frac{2123531\zeta_2}{3375}-\frac{17228}{75}\zeta_2^2-\frac{8419}{150}\zeta_3-\frac{6966}{25}\zeta_2\ln(2)+\frac{576}{5}\zeta_2\ln^2(2)$$
$$-\frac{9536}{25}\zeta_2\log(\beta)+\frac{1344}{5}\zeta_2\ln(2)\log(\beta)+\frac{672}{5}\zeta_2\log^2(\beta)\Big)+i\pi\Bigg(\frac{1}{\beta}\Big(-\frac{6338}{81}+\frac{97}{6}\zeta_2+\frac{22}{3}\zeta_3$$
$$+\frac{2704\ln(2)}{27}-11\zeta_2\ln(2)-\frac{520}{9}\ln^2(2)+\frac{44\ln^3(2)}{3}+\frac{2704\log(\beta)}{27}-11\zeta_2\log(\beta)$$
$$-\frac{1040}{9}\ln(2)\log(\beta)+44\ln^2(2)\log(\beta)-\frac{520}{9}\log^2(\beta)+44\ln(2)\log^2(\beta)+\frac{44\log^3(\beta)}{3}\Big)$$
$$+72\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)+\beta\Big(-\frac{9149}{54}-\frac{727}{18}\zeta_2-\frac{194}{3}\zeta_3+\frac{1238\ln(2)}{27}+\frac{335}{3}\zeta_2\ln(2)$$
$$+94\ln^2(2)-\frac{764}{9}\ln^3(2)+\frac{1702\log(\beta)}{27}+77\zeta_2\log(\beta)+\frac{988}{9}\ln(2)\log(\beta)-\frac{508}{3}\ln^2(2)\log(\beta)$$
$$+\frac{230\log^2(\beta)}{9}-\frac{316}{3}\ln(2)\log^2(\beta)-\frac{172}{9}\log^3(\beta)\Big)+\beta^2\Big(\frac{4768}{25}\zeta_2-\frac{672}{5}\zeta_2\ln(2)$$
$$-\frac{672}{5}\zeta_2\log(\beta)\Big)\Bigg)\Bigg]\Bigg\}+C_FT_Fn_l\Bigg\{\frac{1}{\varepsilon^2}\Bigg[\frac{16\beta^2}{9}+i\pi\Bigg(-\frac{2}{3\beta}-\frac{2\beta}{3}\Bigg)\Bigg]+\frac{1}{\varepsilon}\Bigg[-\frac{80\beta^2}{27}$$
$$+i\pi\Bigg(\frac{10}{9\beta}+\frac{10\beta}{9}\Bigg)\Bigg]+\Bigg[\frac{1}{\beta}\Big(-\frac{88}{3}\zeta_2+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\Big)+\frac{56}{9}+\beta\Big(-\frac{40}{3}\zeta_2$$
$$+16\zeta_2\ln(2)+16\zeta_2\log(\beta)\Big)+\beta^2\Big(-\frac{940}{81}-\frac{64\zeta_2}{9}\Big)+i\pi\Bigg(\frac{1}{\beta}\Big(\frac{248}{27}-\frac{88\ln(2)}{9}+\frac{8\ln^2(2)}{3}$$
$$-\frac{88\log(\beta)}{9}+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\Big)+\beta\Big(\frac{38}{27}-\frac{40\ln(2)}{9}+\frac{8\ln^2(2)}{3}-\frac{40\log(\beta)}{9}$$
$$+\frac{16}{3}\ln(2)\log(\beta)+\frac{8\log^2(\beta)}{3}\Big)\Bigg)\Bigg]+\varepsilon\Bigg[\frac{1}{\beta}\Big(-\frac{1280}{9}\zeta_2-36\zeta_2^2+\frac{448}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)$$
$$+\frac{448}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\Big)+\frac{1228}{27}+24\zeta_2+\beta\Big(-\frac{296}{9}\zeta_2-36\zeta_2^2$$
$$+\frac{160}{3}\zeta_2\ln(2)-48\zeta_2\ln^2(2)+\frac{160}{3}\zeta_2\log(\beta)-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)\Big)+\beta^2\Big(\frac{22}{9}$$
$$-\frac{4\zeta_2}{27}-\frac{128\zeta_3}{9}\Big)+i\pi\Bigg(\frac{1}{\beta}\Big(\frac{3160}{81}-\frac{22}{3}\zeta_2-\frac{8}{3}\zeta_3-\frac{1280\ln(2)}{27}+4\zeta_2\ln(2)+\frac{224\ln^2(2)}{9}$$

$$
- \frac{16}{3}\ln^3(2) - \frac{1280\log(\beta)}{27} + 4\zeta_2\log(\beta) + \frac{448}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) + \frac{224\log^2(\beta)}{9}
$$
$$
- 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta)\Big) + \beta\Big(\frac{733}{81} - \frac{10}{3}\zeta_2 - \frac{8}{3}\zeta_3 - \frac{296\ln(2)}{27} + 4\zeta_2\ln(2) + \frac{80\ln^2(2)}{9}
$$
$$
- \frac{16}{3}\ln^3(2) - \frac{296\log(\beta)}{27} + 4\zeta_2\log(\beta) + \frac{160}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) + \frac{80\log^2(\beta)}{9}
$$
$$
- 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta)\Big)\Big)\Big]\Big\} + C_F T_F\Big\{\Big[\frac{320}{9} - \frac{64}{3}\zeta_2 + \beta^2\Big(-\frac{388}{81} + \frac{16\zeta_2}{5}\Big)
$$
$$
+ i\pi\left(\frac{2\zeta_2}{3\beta} + \frac{2}{3}\beta\zeta_2\right)\Big] + \varepsilon\Big[-\frac{4\zeta_2^2}{\beta} + \frac{2944}{27} - \frac{884}{9}\zeta_2 - \frac{224}{3}\zeta_3 + 128\zeta_2\ln(2) - 4\beta\zeta_2^2
$$
$$
+ \beta^2\Big(-\frac{18106}{1215} + \frac{1256}{75}\zeta_2 + \frac{2512}{135}\zeta_3 - \frac{448}{15}\zeta_2\ln(2)\Big) + i\pi\Big(\frac{1}{\beta}\Big(\frac{4}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2)
$$
$$
- \frac{4}{3}\zeta_2\log(\beta)\Big) + \beta\Big(-\frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) - \frac{4}{3}\zeta_2\log(\beta)\Big)\Big)\Big]\Big\}. \tag{4.40}
$$

The unrenormalized singlet part is given by

$$
\tilde{\tilde{F}}_{A,1}^{(2),\mathrm{s}} = C_F T_F\Big\{-\frac{6}{\varepsilon} - \frac{6}{\beta^2} - \frac{53}{3} - \frac{76}{3}\zeta_2 + 64\zeta_2\ln(2) + \beta^2\Big(\frac{32}{3} + \frac{44}{15}\zeta_2 - \frac{256}{5}\zeta_2\ln(2)\Big)
$$
$$
+ \varepsilon\Big[\frac{1}{\beta^2}\Big(-15 + 20\zeta_2 - 42\zeta_3 + 16\ln(2) + 48\zeta_2\ln(2)\Big) - \frac{48\zeta_2}{\beta} - \frac{1169}{18} - \frac{32c_1}{3} - \frac{1424}{9}\zeta_2
$$
$$
+ \frac{944}{5}\zeta_2^2 - \frac{98}{3}\zeta_3 + \frac{32\ln(2)}{3} - \frac{88}{3}\zeta_2\ln(2) + 64\zeta_2\ln^2(2) + 48\beta\zeta_2 + \beta^2\Big(\frac{2528}{45} + \frac{128c_1}{15}
$$
$$
+ \frac{15374}{225}\zeta_2 - \frac{3776}{25}\zeta_2^2 - \frac{14}{15}\zeta_3 - \frac{176\ln(2)}{15} - \frac{4936}{75}\zeta_2\ln(2) - \frac{256}{5}\zeta_2\ln^2(2)\Big)
$$
$$
+ i\pi\Big(\frac{1}{\beta^2}\Big(-8 + 12\zeta_2\Big) - \frac{1}{\beta}\Big(16(-1 + \ln(2))\Big) - \frac{16}{3} - 16\zeta_2 + \beta\Big(-\frac{64}{3} + 16\ln(2)\Big)
$$
$$
+ \beta^2\Big(\frac{88}{15} + \frac{16\zeta_2}{5}\Big)\Big)\Big]\Big\}, \tag{4.41}
$$

$$
\tilde{F}_{A,2}^{(1),\mathrm{ns}} = C_F\Big\{-2 - \frac{4\beta^2}{3} + i\pi\left(\frac{2}{\beta} - \beta\right) + \varepsilon\Big[-\frac{12\zeta_2}{\beta} + 4 + 6\beta\zeta_2 - \frac{28\beta^2}{9} + i\pi\Big(-\frac{1}{\beta}\Big(4(-1
$$
$$
+ \ln(2) + \log(\beta))\Big) + 2\beta(-2 + \ln(2) + \log(\beta))\Big)\Big] + \varepsilon^2\Big[\frac{1}{\beta}\Big(-24\zeta_2 + 24\zeta_2\ln(2)
$$
$$
+ 24\zeta_2\log(\beta)\Big) - 8 - \zeta_2 + \beta\Big(24\zeta_2 - 12\zeta_2\ln(2) - 12\zeta_2\log(\beta)\Big) + \beta^2\Big(-\frac{160}{27} - \frac{2\zeta_2}{3}\Big)
$$
$$
+ i\pi\Big(\frac{1}{\beta}\Big(8 - 3\zeta_2 - 8\ln(2) + 4\ln^2(2) - 8\log(\beta) + 8\ln(2)\log(\beta) + 4\log^2(\beta)\Big)
$$
$$
+ \beta\Big(-8 + \frac{3}{2}\zeta_2 + 8\ln(2) - 2\ln^2(2) + 8\log(\beta) - 4\ln(2)\log(\beta) - 2\log^2(\beta)\Big)\Big)\Big]\Big\}. \tag{4.42}
$$

$$\tilde{F}^{\mathrm{ns}(2)}_{A,2} = C_F^2\Bigg\{\frac{1}{\varepsilon}\Bigg[\frac{12\zeta_2}{\beta^2} + 6\zeta_2 + \beta^2\Big(-\frac{16}{3} - 12\zeta_2\Big) + i\pi\Bigg(\frac{2}{\beta} + \frac{26\beta}{3}\Bigg)\Bigg] + \Bigg[\frac{1}{\beta^2}\Big(24\zeta_2 - 48\zeta_2\ln(2)$$
$$- 48\zeta_2\log(\beta)\Big) - \frac{12\zeta_2}{\beta} + \frac{41}{3} - 18\zeta_2 - 45\zeta_3 - 60\zeta_2\ln(2) - 80\zeta_2\log(\beta) + 20\beta\zeta_2 + \beta^2\Big(\frac{218}{9}$$
$$- \frac{3292}{75}\zeta_2 - 7\zeta_3 - \frac{268}{5}\zeta_2\ln(2) + \frac{264}{5}\zeta_2\log(\beta)\Big) + i\pi\Bigg(\frac{24\zeta_2}{\beta^2} - \frac{1}{\beta}\Big(4(3 + \ln(2) + \log(\beta))\Big)$$
$$+ 40\zeta_2 + \frac{2}{9}\beta(-265 + 126\ln(2) + 30\log(\beta)) - \frac{132}{5}\beta^2\zeta_2\Bigg)\Bigg] + \varepsilon\Bigg[\frac{1}{\beta^2}\Big(-48\zeta_2 - 60\zeta_2^2$$
$$- 96\zeta_2\ln(2) + 96\zeta_2\ln^2(2) - 96\zeta_2\log(\beta) + 192\zeta_2\ln(2)\log(\beta) + 96\zeta_2\log^2(\beta)\Big) + \frac{1}{\beta}\Big(72\zeta_2$$
$$+ 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)\Big) - \frac{823}{18} - \frac{10c_1}{3} + \frac{776}{3}\zeta_2 - \frac{1572}{5}\zeta_2^2 - \frac{673}{3}\zeta_3 - \frac{1052}{3}\zeta_2\ln(2)$$
$$+ 112\zeta_2\ln^2(2) - 128\zeta_2\log(\beta) + 320\zeta_2\ln(2)\log(\beta) + 160\zeta_2\log^2(\beta) + \beta\Big(\frac{3608}{3}\zeta_2$$
$$- 968\zeta_2\ln(2) - 584\zeta_2\log(\beta)\Big) + \beta^2\Big(\frac{6661}{135} + \frac{266c_1}{15} - \frac{193438\zeta_2}{1125} - \frac{6534}{25}\zeta_2^2 + \frac{3811}{15}\zeta_3$$
$$+ \frac{9292}{15}\zeta_2\ln(2) - \frac{1248}{5}\zeta_2\ln^2(2) + \frac{9704}{25}\zeta_2\log(\beta) - \frac{1056}{5}\zeta_2\ln(2)\log(\beta) - \frac{528}{5}\zeta_2\log^2(\beta)\Big)$$
$$+ i\pi\Bigg(\frac{1}{\beta^2}\Big(48\zeta_2 - 96\zeta_2\ln(2) - 96\zeta_2\log(\beta)\Big) + \frac{1}{\beta}\Big(-8 - 2\zeta_2 + 24\ln(2) + 4\ln^2(2) + 24\log(\beta)$$
$$+ 8\ln(2)\log(\beta) + 4\log^2(\beta)\Big) + 64\zeta_2 - 160\zeta_2\ln(2) - 160\zeta_2\log(\beta) + \beta\Big(-\frac{1031}{3} + \frac{302}{3}\zeta_2$$
$$+ \frac{5080\ln(2)}{9} - \frac{740}{3}\ln^2(2) + \frac{3608\log(\beta)}{9} - \frac{968}{3}\ln(2)\log(\beta) - \frac{292}{3}\log^2(\beta)\Big)$$
$$+ \beta^2\Big(-\frac{4852}{25}\zeta_2 + \frac{528}{5}\zeta_2\ln(2) + \frac{528}{5}\zeta_2\log(\beta)\Big)\Bigg)\Bigg]\Bigg\} + C_F C_A\Bigg\{\Bigg[-\frac{44\zeta_2}{\beta} + \frac{151}{9} + \frac{104}{3}\zeta_2$$
$$- 30\zeta_3 - 72\zeta_2\ln(2) - 32\zeta_2\log(\beta) + 84\beta\zeta_2 + \beta^2\Big(-\frac{2666}{45} - \frac{5138}{75}\zeta_2 + \frac{252}{5}\zeta_3 + \frac{656}{5}\zeta_2\ln(2)$$
$$+ \frac{256}{5}\zeta_2\log(\beta)\Big) + i\pi\Bigg(-\frac{1}{9\beta}\Big(4(-32 + 33\ln(2) + 33\log(\beta))\Big) + 16\zeta_2 + \frac{14}{9}\beta(-23 + 18\ln(2)$$
$$+ 18\log(\beta)) - \frac{128}{5}\beta^2\zeta_2\Bigg)\Bigg] + \varepsilon\Bigg[\frac{1}{\beta}\Big(-\frac{776}{3}\zeta_2 + 264\zeta_2\ln(2) + 264\zeta_2\log(\beta)\Big) - \frac{1433}{54} + \frac{20c_1}{3}$$
$$+ \frac{415}{9}\zeta_2 - \frac{1644}{5}\zeta_2^2 + 46\zeta_3 - \frac{104}{3}\zeta_2\ln(2) - 32\zeta_2\ln^2(2) - 96\zeta_2\log(\beta) + 128\zeta_2\ln(2)\log(\beta)$$
$$+ 64\zeta_2\log^2(\beta) + \beta\Big(\frac{1990}{3}\zeta_2 - 504\zeta_2\ln(2) - 504\zeta_2\log(\beta)\Big) + \beta^2\Big(-\frac{145603}{2025} - \frac{40c_1}{3}$$
$$- \frac{202052\zeta_2}{1125} + \frac{14536}{25}\zeta_2^2 - \frac{2001}{25}\zeta_3 + \frac{9148}{75}\zeta_2\ln(2) + \frac{384}{5}\zeta_2\ln^2(2) + \frac{4736}{25}\zeta_2\log(\beta)$$
$$- \frac{1024}{5}\zeta_2\ln(2)\log(\beta) - \frac{512}{5}\zeta_2\log^2(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(\frac{1924}{27} - 11\zeta_2 - \frac{776\ln(2)}{9} + 44\ln^2(2)$$
$$- \frac{776\log(\beta)}{9} + 88\ln(2)\log(\beta) + 44\log^2(\beta)\Big) + 48\zeta_2 - 64\zeta_2\ln(2) - 64\zeta_2\log(\beta)$$

$$
\begin{aligned}
&+\beta\Big(-\frac{10195}{54}+\frac{157}{6}\zeta_2+\frac{2182\ln(2)}{9}-84\ln^2(2)+\frac{1990\log(\beta)}{9}-168\ln(2)\log(\beta)\\
&-84\log^2(\beta)\Big)+\beta^2\Big(-\frac{2368}{25}\zeta_2+\frac{512}{5}\zeta_2\ln(2)+\frac{512}{5}\zeta_2\log(\beta)\Big)\Big)\Big]\Big\}\\
&+C_FT_Fn_l\Big\{\Big[\frac{16\zeta_2}{\beta}-\frac{44}{9}-8\beta\zeta_2+8\beta^2+i\pi\Big(\frac{1}{9\beta}\Big(16(-4+3\ln(2)+3\log(\beta))\Big)-\frac{2}{9}\beta(-25\\
&+12\ln(2)+12\log(\beta))\Big)\Big]+\varepsilon\Big[\frac{1}{\beta}\Big(\frac{352}{3}\zeta_2-96\zeta_2\ln(2)-96\zeta_2\log(\beta)\Big)+\frac{338}{27}+12\zeta_2\\
&+\beta\Big(-\frac{296}{3}\zeta_2+48\zeta_2\ln(2)+48\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{3044}{81}+8\zeta_2\Big)+i\pi\Big(\frac{1}{\beta}\Big(-\frac{944}{27}+4\zeta_2\\
&+\frac{352\ln(2)}{9}-16\ln^2(2)+\frac{352\log(\beta)}{9}-32\ln(2)\log(\beta)-16\log^2(\beta)\Big)+\beta\Big(\frac{889}{27}-2\zeta_2\\
&-\frac{296\ln(2)}{9}+8\ln^2(2)-\frac{296\log(\beta)}{9}+16\ln(2)\log(\beta)+8\log^2(\beta)\Big)\Big)\Big]\Big\}\\
&+C_FT_F\Big\{\Big[\frac{196}{9}-\frac{32}{3}\zeta_2+\beta^2\Big(-\frac{632}{27}+\frac{64\zeta_2}{5}\Big)+i\pi\Big(\frac{16\beta}{15}\Big)\Big]+\varepsilon\Big[\frac{1322}{27}-\frac{412}{9}\zeta_2-\frac{112}{3}\zeta_3\\
&+64\zeta_2\ln(2)-\frac{32}{5}\beta\zeta_2+\beta^2\Big(-\frac{13348}{405}+\frac{9232}{225}\zeta_2+\frac{224}{5}\zeta_3-\frac{384}{5}\zeta_2\ln(2)\Big)+i\pi\Big(-\frac{4\zeta_2}{3\beta}\\
&+\beta\Big(\frac{32}{15}+\frac{2}{3}\zeta_2-\frac{32\ln(2)}{15}-\frac{32\log(\beta)}{15}\Big)\Big)\Big]\Big\}. \qquad (4.43)
\end{aligned}
$$

The unrenormalized singlet contribution is

$$
\begin{aligned}
\tilde{\hat{F}}^{(2),\mathrm{s}}_{A,2}&=C_FT_F\Big\{\Big[\frac{6}{\beta^2}+\frac{8}{3}+\frac{136}{3}\zeta_2-42\zeta_3+16\ln(2)-16\zeta_2\ln(2)-48\beta\zeta_2+\beta^2\Big(-30-\frac{904}{15}\zeta_2\\
&+70\zeta_3+\frac{16\ln(2)}{3}+\frac{176}{5}\zeta_2\ln(2)\Big)+i\pi\Big(-8+12\zeta_2+\beta(16-16\ln(2))+\beta^2\Big(-\frac{8}{3}\\
&-20\zeta_2\Big)\Big)\Big]+\varepsilon\Big[\frac{1}{\beta^2}\Big(15-20\zeta_2+42\zeta_3-16\ln(2)-48\zeta_2\ln(2)\Big)+\frac{48\zeta_2}{\beta}+\frac{364}{9}-4c_1\\
&+\frac{2684}{9}\zeta_2-\frac{976}{5}\zeta_2^2+\frac{182}{3}\zeta_3+\frac{112\ln(2)}{3}-\frac{416}{3}\zeta_2\ln(2)-56\zeta_3\ln(2)-16\ln^2(2)\\
&+200\zeta_2\ln^2(2)+\beta\Big(-240\zeta_2+288\zeta_2\ln(2)+96\zeta_2\log(\beta)\Big)+\beta^2\Big(-\frac{427}{5}+\frac{236c_1}{45}\\
&-\frac{101524}{225}\zeta_2+\frac{26288}{75}\zeta_2^2-\frac{1666}{15}\zeta_3-\frac{3392\ln(2)}{45}+\frac{33536}{75}\zeta_2\ln(2)+\frac{280}{3}\zeta_3\ln(2)-\frac{16}{3}\ln^2(2)\\
&-\frac{1624}{5}\zeta_2\ln^2(2)\Big)+i\pi\Big(\frac{1}{\beta^2}\Big(8-12\zeta_2\Big)+\frac{1}{\beta}\Big(16(-1+\ln(2))\Big)-\frac{56}{3}+28\zeta_2+28\zeta_3+16\ln(2)\\
&-24\zeta_2\ln(2)+\beta\Big(\frac{160}{3}-80\zeta_2-80\ln(2)+48\ln^2(2)-32\log(\beta)+32\ln(2)\log(\beta)\Big)
\end{aligned}
$$

$$+\beta^2\Big(\frac{1696}{45}-\frac{428}{15}\zeta_2-\frac{140}{3}\zeta_3+\frac{16\ln(2)}{3}+40\zeta_2\ln(2)\Big)\Big)\Big]\Big\}. \tag{4.44}$$

4.3.3 Scalar form factor

The scalar form factor is in this limit given by

$$\begin{aligned}
\tilde{F}_S^{(1)} &= C_F\Big\{\frac{1}{\varepsilon}\Big[\frac{8\beta^2}{3}+i\pi\Big(-\frac{1}{\beta}-\beta\Big)\Big]+\Big[\frac{6\zeta_2}{\beta}-2+6\beta\zeta_2-\frac{76\beta^2}{9}+i\pi\Big(\frac{1}{\beta}\Big(2(-1+\ln(2)\\
&+\log(\beta))\Big)+\beta(3+2\ln(2)+2\log(\beta))\Big)\Big]+\varepsilon\Big[\frac{1}{\beta}\Big(12\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta)\Big)+4\\
&+\beta\Big(-18\zeta_2-12\zeta_2\ln(2)-12\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{404}{27}+\frac{4\zeta_2}{3}\Big)+i\pi\Big(\frac{1}{\beta}\Big(-4+\frac{3}{2}\zeta_2+4\ln(2)\\
&-2\ln^2(2)+4\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)+\beta\Big(4+\frac{3}{2}\zeta_2-6\ln(2)-2\ln^2(2)\\
&-6\log(\beta)-4\ln(2)\log(\beta)-2\log^2(\beta)\Big)\Big)\Big]+\varepsilon^2\Big[\frac{1}{\beta}\Big(24\zeta_2+3\zeta_2^2-24\zeta_2\ln(2)+12\zeta_2\ln^2(2)\\
&-24\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta)\Big)-8-\zeta_2+\beta\Big(-24\zeta_2+3\zeta_2^2+36\zeta_2\ln(2)\\
&+12\zeta_2\ln^2(2)+36\zeta_2\log(\beta)+24\zeta_2\ln(2)\log(\beta)+12\zeta_2\log^2(\beta)\Big)+\beta^2\Big(-\frac{2752}{81}-\frac{38\zeta_2}{9}-\frac{8\zeta_3}{9}\Big)\\
&+i\pi\Big(\frac{1}{\beta}\Big(-8+3\zeta_2+\frac{7}{3}\zeta_3+8\ln(2)-3\zeta_2\ln(2)-4\ln^2(2)+\frac{4\ln^3(2)}{3}+8\log(\beta)-3\zeta_2\log(\beta)\\
&-8\ln(2)\log(\beta)+4\ln^2(2)\log(\beta)-4\log^2(\beta)+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\Big)+\beta\Big(8-\frac{9}{2}\zeta_2\\
&+\frac{7}{3}\zeta_3-8\ln(2)-3\zeta_2\ln(2)+6\ln^2(2)+\frac{4\ln^3(2)}{3}-8\log(\beta)-3\zeta_2\log(\beta)+12\ln(2)\log(\beta)\\
&+4\ln^2(2)\log(\beta)+6\log^2(\beta)+4\ln(2)\log^2(\beta)+\frac{4\log^3(\beta)}{3}\Big)\Big)\Big]\Big\}. \qquad (4.45)\\
\tilde{F}_S^{(2)} &= C_F^2\Big\{\frac{1}{\varepsilon^2}\Big[-\frac{3\zeta_2}{\beta^2}-6\zeta_2-3\beta^2\zeta_2+i\pi\Big(-\frac{8\beta}{3}\Big)\Big]+\frac{1}{\varepsilon}\Big[\frac{1}{\beta^2}\Big(-12\zeta_2+12\zeta_2\ln(2)\\
&+12\zeta_2\log(\beta)\Big)+6\zeta_2+24\zeta_2\ln(2)+24\zeta_2\log(\beta)+16\beta\zeta_2+\beta^2\Big(-\frac{16}{3}+27\zeta_2+12\zeta_2\ln(2)\\
&+12\zeta_2\log(\beta)\Big)+i\pi\Big(-\frac{6\zeta_2}{\beta^2}+\frac{2}{\beta}-12\zeta_2+\frac{2}{9}\beta(23+24\ln(2)+24\log(\beta))-6\beta^2\zeta_2\Big)\Big]\\
&+\Big[\frac{1}{\beta^2}\Big(-24\zeta_2+15\zeta_2^2+48\zeta_2\ln(2)-24\zeta_2\ln^2(2)+48\zeta_2\log(\beta)-48\zeta_2\ln(2)\log(\beta)\\
&-24\zeta_2\log^2(\beta)\Big)-\frac{12\zeta_2}{\beta}+5+98\zeta_2+30\zeta_2^2-44\zeta_3-40\zeta_2\ln(2)-48\zeta_2\ln^2(2)-88\zeta_2\log(\beta)\\
&-96\zeta_2\ln(2)\log(\beta)-48\zeta_2\log^2(\beta)+\beta\Big(-\frac{92}{3}\zeta_2-32\zeta_2\ln(2)-32\zeta_2\log(\beta)\Big)+\beta^2\Big(\frac{1684}{45}
\end{aligned}$$

$$
- \frac{2446}{25}\zeta_2 + 15\zeta_2^2 - \frac{58}{5}\zeta_3 - \frac{964}{5}\zeta_2\ln(2) - 24\zeta_2\ln^2(2) - \frac{684}{5}\zeta_2\log(\beta) - 48\zeta_2\ln(2)\log(\beta)
$$

$$
- 24\zeta_2\log^2(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta^2}\Big(- 24\zeta_2 + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)\Big) - \frac{4(\ln(2)+\log(\beta))}{\beta}
$$

$$
+ 44\zeta_2 + 48\zeta_2\ln(2) + 48\zeta_2\log(\beta) + \beta\Big(- \frac{632}{27} + \frac{8}{3}\zeta_2 - \frac{92\ln(2)}{9} - \frac{16}{3}\ln^2(2) - \frac{92\log(\beta)}{9}
$$

$$
- \frac{32}{3}\ln(2)\log(\beta) - \frac{16}{3}\log^2(\beta)\Big) + \beta^2\Big(\frac{342}{5}\zeta_2 + 24\zeta_2\ln(2) + 24\zeta_2\log(\beta)\Big)\Bigg)\Bigg]
$$

$$
+ \varepsilon\Bigg[\frac{1}{\beta^2}\Big(48\zeta_2 + 60\zeta_2^2 - 106\zeta_2\zeta_3 + 96\zeta_2\ln(2) - 60\zeta_2^2\ln(2) - 96\zeta_2\ln^2(2) + 32\zeta_2\ln^3(2)
$$

$$
+ 96\zeta_2\log(\beta) - 60\zeta_2^2\log(\beta) - 192\zeta_2\ln(2)\log(\beta) + 96\zeta_2\ln^2(2)\log(\beta) - 96\zeta_2\log^2(\beta)
$$

$$
+ 96\zeta_2\ln(2)\log^2(\beta) + 32\zeta_2\log^3(\beta)\Big) + \frac{1}{\beta}\Big(24\ln(2)\zeta_2 + 24\log(\beta)\zeta_2\Big) - \frac{451}{6} - 8c_1 + \frac{2368}{3}\zeta_2
$$

$$
- \frac{1278}{5}\zeta_2^2 - 291\zeta_3 - 212\zeta_2\zeta_3 - 740\zeta_2\ln(2) - 120\zeta_2^2\ln(2) + 176\zeta_2\ln^2(2) + 64\zeta_2\ln^3(2)
$$

$$
- 688\zeta_2\log(\beta) - 120\zeta_2^2\log(\beta) + 352\zeta_2\ln(2)\log(\beta) + 192\zeta_2\ln^2(2)\log(\beta) + 176\zeta_2\log^2(\beta)
$$

$$
+ 192\zeta_2\ln(2)\log^2(\beta) + 64\zeta_2\log^3(\beta) + \beta\Big(\frac{3316}{9}\zeta_2 + 16\zeta_2^2 + \frac{184}{3}\zeta_2\ln(2) + 32\zeta_2\ln^2(2)
$$

$$
+ \frac{184}{3}\zeta_2\log(\beta) + 64\zeta_2\ln(2)\log(\beta) + 32\zeta_2\log^2(\beta)\Big) + \beta^2\Big(- \frac{100966}{675} + \frac{28c_1}{3} - \frac{61918}{125}\zeta_2
$$

$$
- \frac{10939}{25}\zeta_2^2 - \frac{15298}{75}\zeta_3 - 106\zeta_2\zeta_3 - \frac{5392}{75}\zeta_2\ln(2) - 60\zeta_2^2\ln(2) + \frac{1144}{5}\zeta_2\ln^2(2) + 32\zeta_2\ln^3(2)
$$

$$
+ \frac{4776}{25}\zeta_2\log(\beta) - 60\zeta_2^2\log(\beta) + \frac{2736}{5}\zeta_2\ln(2)\log(\beta) + 96\zeta_2\ln^2(2)\log(\beta) + \frac{1368}{5}\zeta_2\log^2(\beta)
$$

$$
+ 96\zeta_2\ln(2)\log^2(\beta) + 32\zeta_2\log^3(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta^2}\Big(- 48\zeta_2 - 18\zeta_2^2 + 96\zeta_2\ln(2) - 48\zeta_2\ln^2(2)
$$

$$
+ 96\zeta_2\log(\beta) - 96\zeta_2\ln(2)\log(\beta) - 48\zeta_2\log^2(\beta)\Big) + \frac{1}{\beta}\Big(8 - 2\zeta_2 + 4\ln^2(2) + 8\ln(2)\log(\beta)
$$

$$
+ 4\log^2(\beta)\Big) + 344\zeta_2 - 36\zeta_2^2 - 176\zeta_2\ln(2) - 96\zeta_2\ln^2(2) - 176\zeta_2\log(\beta) - 192\zeta_2\ln(2)\log(\beta)
$$

$$
- 96\zeta_2\log^2(\beta) + \beta\Big(- \frac{11021}{81} - \frac{46}{9}\zeta_2 + \frac{64}{9}\zeta_3 + \frac{4468\ln(2)}{27} - \frac{16}{3}\zeta_2\ln(2) + \frac{92\ln^2(2)}{9}
$$

$$
+ \frac{32\ln^3(2)}{9} + \frac{3316\log(\beta)}{27} - \frac{16}{3}\zeta_2\log(\beta) + \frac{184}{9}\ln(2)\log(\beta) + \frac{32}{3}\ln^2(2)\log(\beta) + \frac{92\log^2(\beta)}{9}
$$

$$
+ \frac{32}{3}\ln(2)\log^2(\beta) + \frac{32\log^3(\beta)}{9}\Big) + \beta^2\Big(- \frac{2388}{25}\zeta_2 - 18\zeta_2^2 - \frac{1368}{5}\zeta_2\ln(2) - 48\zeta_2\ln^2(2)
$$

$$
- \frac{1368}{5}\zeta_2\log(\beta) - 96\zeta_2\ln(2)\log(\beta) - 48\zeta_2\log^2(\beta)\Big)\Bigg)\Bigg]\Bigg\} + C_F C_A\Bigg\{\frac{1}{\varepsilon^2}\Bigg[- \frac{44\beta^2}{9} + i\pi\Bigg(\frac{11}{6\beta}
$$

$$
+ \frac{11\beta}{6}\Bigg)\Bigg] + \frac{1}{\varepsilon}\Bigg[- 8\beta\zeta_2 + \beta^2\Big(\frac{376}{27} + \frac{32\zeta_2}{3}\Big) + i\pi\Bigg(- \frac{31}{18\beta} + \frac{1}{6}\beta(-13 - 32\ln(2)
$$

$$
- 16\log(\beta))\Bigg)\Bigg] + \Bigg[\frac{1}{\beta}\Big(\frac{194}{3}\zeta_2 - 44\zeta_2\ln(2) - 44\zeta_2\log(\beta)\Big) + \frac{49}{9} + 38\zeta_2 - 20\zeta_3 - 64\zeta_2\ln(2)
$$

$$
- 16\zeta_2\log(\beta) + \beta\Big(- \frac{100}{3}\zeta_2 + 84\zeta_2\ln(2) + 36\zeta_2\log(\beta)\Big) + \beta^2\Big(- \frac{16472}{405} + \frac{1679}{15}\zeta_2 - \frac{472}{15}\zeta_3
$$
$$
- \frac{336}{5}\zeta_2\ln(2) - 80\zeta_2\log(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(- \frac{478}{27} + \frac{194\ln(2)}{9} - \frac{22\ln^2(2)}{3} + \frac{194\log(\beta)}{9}
$$
$$
- \frac{44}{3}\ln(2)\log(\beta) - \frac{22\log^2(\beta)}{3}\Big) + 8\zeta_2 + \beta\Big(\frac{43}{9} - \frac{40}{3}\zeta_2 - \frac{188\ln(2)}{9} + \frac{74\ln^2(2)}{3}
$$
$$
- \frac{100\log(\beta)}{9} + 28\ln(2)\log(\beta) + 6\log^2(\beta)\Big) + 40\beta^2\zeta_2\Bigg)\Bigg] + \varepsilon\Bigg[\frac{1}{\beta}\Big(\frac{2704}{9}\zeta_2 + 99\zeta_2^2
$$
$$
- \frac{1040}{3}\zeta_2\ln(2) + 132\zeta_2\ln^2(2) - \frac{1040}{3}\zeta_2\log(\beta) + 264\zeta_2\ln(2)\log(\beta) + 132\zeta_2\log^2(\beta)\Big)
$$
$$
- \frac{383}{54} + 8c_1 + \frac{409}{3}\zeta_2 - 264\zeta_2^2 - \zeta_3 - 220\zeta_2\ln(2) - 64\zeta_2\ln^2(2) - 144\zeta_2\log(\beta)
$$
$$
+ 64\zeta_2\ln(2)\log(\beta) + 32\zeta_2\log^2(\beta) + \beta\Big(- \frac{692}{9}\zeta_2 + 59\zeta_2^2 + \frac{1636}{3}\zeta_2\ln(2) - 508\zeta_2\ln^2(2)
$$
$$
+ \frac{1108}{3}\zeta_2\log(\beta) - 632\zeta_2\ln(2)\log(\beta) - 172\zeta_2\log^2(\beta)\Big) + \beta^2\Big(\frac{17767}{675} - \frac{32c_1}{15} + \frac{450622}{675}\zeta_2
$$
$$
- \frac{25112}{75}\zeta_2^2 + \frac{2717}{150}\zeta_3 - \frac{26306}{75}\zeta_2\ln(2) + \frac{608}{5}\zeta_2\ln^2(2) - 416\zeta_2\log(\beta) + 320\zeta_2\ln(2)\log(\beta)
$$
$$
+ 160\zeta_2\log^2(\beta)\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(- \frac{6338}{81} + \frac{97}{6}\zeta_2 + \frac{22}{3}\zeta_3 + \frac{2704\ln(2)}{27} - 11\zeta_2\ln(2) - \frac{520}{9}\ln^2(2)
$$
$$
+ \frac{44\ln^3(2)}{3} + \frac{2704\log(\beta)}{27} - 11\zeta_2\log(\beta) - \frac{1040}{9}\ln(2)\log(\beta) + 44\ln^2(2)\log(\beta)
$$
$$
- \frac{520}{9}\log^2(\beta) + 44\ln(2)\log^2(\beta) + \frac{44\log^3(\beta)}{3}\Big) + 72\zeta_2 - 32\zeta_2\ln(2) - 32\zeta_2\log(\beta)
$$
$$
+ \beta\Big(- \frac{2317}{27} - \frac{422}{9}\zeta_2 - \frac{194}{3}\zeta_3 - \frac{1156\ln(2)}{27} + \frac{335}{3}\zeta_2\ln(2) + 130\ln^2(2) - \frac{764}{9}\ln^3(2)
$$
$$
- \frac{692\log(\beta)}{27} + 77\zeta_2\log(\beta) + \frac{1636}{9}\ln(2)\log(\beta) - \frac{508}{3}\ln^2(2)\log(\beta) + \frac{554\log^2(\beta)}{9}
$$
$$
- \frac{316}{3}\ln(2)\log^2(\beta) - \frac{172}{9}\log^3(\beta)\Big) + \beta^2\Big(208\zeta_2 - 160\zeta_2\ln(2) - 160\zeta_2\log(\beta)\Big)\Bigg)\Bigg]\Bigg\}
$$
$$
+ C_F T_F n_l\Bigg\{\frac{1}{\varepsilon^2}\Bigg[\frac{16\beta^2}{9} + i\pi\Big(- \frac{2}{3\beta} - \frac{2\beta}{3}\Big)\Bigg] + \frac{1}{\varepsilon}\Bigg[- \frac{80\beta^2}{27} + i\pi\Big(\frac{10}{9\beta} + \frac{10\beta}{9}\Big)\Bigg]
$$
$$
+ \Bigg[\frac{1}{\beta}\Big(- \frac{88}{3}\zeta_2 + 16\zeta_2\ln(2) + 16\zeta_2\log(\beta)\Big) - \frac{20}{9} + \beta\Big(\frac{32}{3}\zeta_2 + 16\zeta_2\ln(2) + 16\zeta_2\log(\beta)\Big)
$$
$$
+ \beta^2\Big(- \frac{400}{81} - \frac{64\zeta_2}{9}\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(\frac{248}{27} - \frac{88\ln(2)}{9} + \frac{8\ln^2(2)}{3} - \frac{88\log(\beta)}{9} + \frac{16}{3}\ln(2)\log(\beta)
$$
$$
+ \frac{8\log^2(\beta)}{3}\Big) + \beta\Big(- \frac{268}{27} + \frac{32\ln(2)}{9} + \frac{8\ln^2(2)}{3} + \frac{32\log(\beta)}{9} + \frac{16}{3}\ln(2)\log(\beta)
$$
$$
+ \frac{8\log^2(\beta)}{3}\Big)\Bigg)\Bigg] + \varepsilon\Bigg[\frac{1}{\beta}\Big(- \frac{1280}{9}\zeta_2 - 36\zeta_2^2 + \frac{448}{3}\zeta_2\ln(2) - 48\zeta_2\ln^2(2) + \frac{448}{3}\zeta_2\log(\beta)
$$
$$
- 96\zeta_2\ln(2)\log(\beta) - 48\zeta_2\log^2(\beta)\Big) + \frac{62}{27} + 12\zeta_2 + \beta\Big(\frac{1360}{9}\zeta_2 - 36\zeta_2^2 - \frac{272}{3}\zeta_2\ln(2)
$$

$$
\begin{aligned}
&- 48\zeta_2 \ln^2(2) - \frac{272}{3}\zeta_2 \log(\beta) - 96\zeta_2 \ln(2)\log(\beta) - 48\zeta_2 \log^2(\beta)\Big) + \beta^2\Big(\frac{800}{9} + \frac{968\zeta_2}{27} \\
&- \frac{128\zeta_3}{9}\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(\frac{3160}{81} - \frac{22}{3}\zeta_2 - \frac{8}{3}\zeta_3 - \frac{1280\ln(2)}{27} + 4\zeta_2\ln(2) + \frac{224\ln^2(2)}{9} - \frac{16}{3}\ln^3(2) \\
&- \frac{1280\log(\beta)}{27} + 4\zeta_2\log(\beta) + \frac{448}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) + \frac{224\log^2(\beta)}{9} \\
&- 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta)\Big) + \beta\Big(-\frac{3848}{81} + \frac{8}{3}\zeta_2 - \frac{8}{3}\zeta_3 + \frac{1360\ln(2)}{27} + 4\zeta_2\ln(2) \\
&- \frac{136}{9}\ln^2(2) - \frac{16}{3}\ln^3(2) + \frac{1360\log(\beta)}{27} + 4\zeta_2\log(\beta) - \frac{272}{9}\ln(2)\log(\beta) - 16\ln^2(2)\log(\beta) \\
&- \frac{136}{9}\log^2(\beta) - 16\ln(2)\log^2(\beta) - \frac{16}{3}\log^3(\beta)\Big)\Bigg)\Bigg]\Bigg\} + C_F T_F\Bigg\{\Bigg[\frac{580}{9} - \frac{212}{3}\zeta_2 - 16\ln(2) \\
&+ 64\zeta_2\ln(2) + \beta^2\Big(-\frac{712}{81} + \frac{252}{5}\zeta_2 - 42\zeta_3 + \frac{32\ln(2)}{3} - \frac{336}{5}\zeta_2\ln(2)\Big) + i\pi\Bigg(\frac{2\zeta_2}{3\beta} + 8 + \frac{2}{3}\beta\zeta_2 \\
&+ \beta^2\Big(-\frac{16}{3} + 12\zeta_2\Big)\Bigg)\Bigg] + \varepsilon\Bigg[-\frac{4\zeta_2^2}{\beta} + \frac{5186}{27} - \frac{32c_1}{3} - \frac{2800}{9}\zeta_2 + \frac{944}{5}\zeta_2^2 - \frac{616}{3}\zeta_3 \\
&- \frac{224\ln(2)}{3} + \frac{584}{3}\zeta_2\ln(2) + 16\ln^2(2) + 64\zeta_2\ln^2(2) - 4\beta\zeta_2^2 + \beta^2\Big(-\frac{35116}{1215} + \frac{68c_1}{15} \\
&+ \frac{18992}{75}\zeta_2 - \frac{8656}{25}\zeta_2^2 + \frac{5914}{135}\zeta_3 + \frac{3584\ln(2)}{45} - \frac{8696}{75}\zeta_2\ln(2) - 56\zeta_3\ln(2) - \frac{32}{3}\ln^2(2) \\
&+ \frac{744}{5}\zeta_2\ln^2(2)\Big) + i\pi\Bigg(\frac{1}{\beta}\Big(\frac{4}{3}\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) - \frac{4}{3}\zeta_2\log(\beta)\Big) + \frac{112}{3} - 12\zeta_2 - 16\ln(2) \\
&+ \beta\Big(-2\zeta_2 - \frac{4}{9}\zeta_3 - \frac{4}{3}\zeta_2\ln(2) - \frac{4}{3}\zeta_2\log(\beta)\Big) + \beta^2\Big(-\frac{1792}{45} + 40\zeta_2 + 28\zeta_3 + \frac{32\ln(2)}{3} \\
&- 24\zeta_2\ln(2)\Big)\Bigg)\Bigg]\Bigg\}. \qquad (4.46)
\end{aligned}
$$

4.3.4 Pseudo-scalar form factor

The non-singlet part of the pseudo-scalar form factor can be obtained using the chiral Ward identity Eq. (2.13) as follows

$$
\tilde{F}_P^{(n),\mathrm{ns}} = \tilde{F}_{A,1}^{(n),\mathrm{ns}} + \Big(1 + \beta^2 + \beta^4\Big)\tilde{F}_{A,2}^{(n),\mathrm{ns}} + \mathcal{O}(\beta^3)\,. \qquad (4.47)
$$

The bare singlet piece is given by

$$
\begin{aligned}
\tilde{\hat{F}}_P^{(2),\mathrm{s}} = C_F T_F\Bigg\{&\Bigg[20\zeta_2 - 42\zeta_3 + 48\zeta_2\ln(2) - 48\beta\zeta_2 + \beta^2\Big(-\frac{40}{3} - 12\zeta_2 + 28\zeta_3 + 16\ln(2) \\
&- 32\zeta_2\ln(2)\Big) + i\pi\Bigg(12\zeta_2 + \beta(16 - 16\ln(2)) + \beta^2\Big(-8 - 8\zeta_2\Big)\Bigg)\Bigg] + \varepsilon\Bigg[-\frac{44c_1}{3} + 88\zeta_2 \\
&- \frac{32}{5}\zeta_2^2 + 112\zeta_3 - 264\zeta_2\ln(2) - 56\zeta_3\ln(2) + 264\zeta_2\ln^2(2) + \beta\Big(-144\zeta_2 + 288\zeta_2\ln(2) \\
&+ 96\zeta_2\log(\beta)\Big) + \beta^2\Big(\frac{256}{9} + \frac{88c_1}{9} - 108\zeta_2 + \frac{64}{15}\zeta_2^2 - \frac{112}{3}\zeta_3 - 48\ln(2) + \frac{680}{3}\zeta_2\ln(2)
\end{aligned}
$$

$$+\frac{112}{3}\zeta_3\ln(2) - 16\ln^2(2) - 176\zeta_2\ln^2(2)\Big) + i\pi\Bigg(- 12\zeta_2 + 28\zeta_3 - 24\zeta_2\ln(2)$$
$$+\beta\Big(16 - 80\zeta_2 - 48\ln(2) + 48\ln^2(2) - 32\log(\beta) + 32\ln(2)\log(\beta)\Big)$$
$$+\beta^2\Big(24 - \frac{4}{3}\zeta_2 - \frac{56}{3}\zeta_3 + 16\ln(2) + 16\zeta_2\ln(2)\Big)\Bigg)\Bigg]\Bigg\}. \tag{4.48}$$

5 Conclusion

The massive form factors are basic building blocks to many observables in heavy quark physics. The precision study of these objects will both shed light on the physical structure of the top quark itself and also on important aspects of the mechanism to create the fermion masses. A future electron-positron collider can achieve high precision and hence an equal or better theory prediction is indispensable. In a similar way this also applies to the LHC for its high luminosity phase. In the present paper, we have computed the heavy quark form factors for vector, axial-vector, scalar and pseudo-scalar currents at two-loop level up to the $\mathcal{O}(\varepsilon^2)$ contributions. These contributions constitute important ingredients to renormalize the three-loop form factors and do also contribute to potential future 4-loop calculations. In addition, they serve as a cross-check of earlier results available in the literature. In the calculation we have used both traditional techniques in solving the differential equations for the master integrals, as well as a more recent automated method, based on coupled difference equations. Both methods play a role in computing higher than second order corrections to the different form factors.

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A Renormalization Constants

In this appendix, we present corresponding renormalization constants up to relevant order in ε. The wave function renormalization constants in OS scheme up to two-loop [35–37, 40] are

$$Z_{2,OS}^{(1)} = C_F\bigg[-\frac{3}{\varepsilon}-4+\varepsilon\bigg\{-8-\frac{3}{2}\zeta_2\bigg\}+\varepsilon^2\bigg\{-16-2\zeta_2+\zeta_3\bigg\}+\varepsilon^3\bigg\{-32-4\zeta_2 - \frac{27}{40}\zeta_2^2+\frac{4}{3}\zeta_3\bigg\}\bigg], \tag{A.1}$$

$$\begin{aligned}Z_{2,OS}^{(2)} = {}& C_F^2\bigg[\frac{1}{\varepsilon^2}\bigg\{\frac{9}{2}\bigg\}+\frac{1}{\varepsilon}\bigg\{\frac{51}{4}\bigg\}+\bigg\{\frac{433}{8}-\frac{147}{2}\zeta_2-24\zeta_3+96\zeta_2\ln(2)\bigg\}+\varepsilon\bigg\{\frac{211}{16}-16c_1 \\ & -\frac{1017}{4}\zeta_2+\frac{1008}{5}\zeta_2^2-297\zeta_3+552\zeta_2\ln(2)\bigg\}+\varepsilon^2\bigg\{\frac{4889}{32}-92c_1-\frac{96c_2}{5}-\frac{8851}{8}\zeta_2 \\ & +\frac{11703}{20}\zeta_2^2-\frac{2069}{2}\zeta_3+264\zeta_2\zeta_3+2436\zeta_5+1968\zeta_2\ln(2)\bigg\}\bigg]+C_AC_F\bigg[\frac{1}{\varepsilon^2}\bigg\{\frac{11}{2}\bigg\} \\ & +\frac{1}{\varepsilon}\bigg\{-\frac{127}{12}\bigg\}+\bigg\{-\frac{1705}{24}+30\zeta_2+12\zeta_3-48\zeta_2\ln(2)\bigg\}+\varepsilon\bigg\{-\frac{9907}{48}+8c_1+\frac{769}{12}\zeta_2 \\ & -\frac{504}{5}\zeta_2^2+129\zeta_3-276\zeta_2\ln(2)\bigg\}+\varepsilon^2\bigg\{-\frac{79225}{96}+46c_1+\frac{48c_2}{5}+\frac{6367}{24}\zeta_2-\frac{14359}{40}\zeta_2^2 \\ & +\frac{7595}{18}\zeta_3-132\zeta_2\zeta_3-1218\zeta_5-984\zeta_2\ln(2)\bigg\}\bigg]+C_FT_Fn_l\bigg[\frac{1}{\varepsilon^2}\bigg\{-2\bigg\}+\frac{1}{\varepsilon}\bigg\{\frac{11}{3}\bigg\} \\ & +\bigg\{\frac{113}{6}+8\zeta_2\bigg\}+\varepsilon\bigg\{\frac{851}{12}+\frac{127}{3}\zeta_2+16\zeta_3\bigg\}+\varepsilon^2\bigg\{\frac{5753}{24}+\frac{853}{6}\zeta_2+\frac{597}{10}\zeta_2^2+\frac{610}{9}\zeta_3\bigg\}\bigg] \\ & +C_FT_F\bigg[\frac{1}{\varepsilon}+\bigg\{\frac{947}{18}-30\zeta_2\bigg\}+\varepsilon\bigg\{\frac{17971}{108}-\frac{445}{3}\zeta_2-\frac{340}{3}\zeta_3+192\zeta_2\ln(2)\bigg\} \\ & +\varepsilon^2\bigg\{\frac{422747}{648}-32c_1-\frac{8605}{18}\zeta_2+\frac{1683}{10}\zeta_2^2-\frac{4810}{9}\zeta_3+912\zeta_2\ln(2)\bigg\}\bigg].\end{aligned} \tag{A.2}$$

The heavy quark mass renormalization constants in OS scheme are [35–39]

$$Z_{m,OS}^{(1)} = C_F\bigg[-\frac{3}{\varepsilon}-4+\varepsilon\bigg\{-8-\frac{3}{2}\zeta_2\bigg\}+\varepsilon^2\bigg\{-16-2\zeta_2+\zeta_3\bigg\}+\varepsilon^3\bigg\{-32-4\zeta_2 - \frac{27}{40}\zeta_2^2+\frac{4}{3}\zeta_3\bigg\}\bigg], \tag{A.3}$$

$$\begin{aligned}Z_{m,OS}^{(2)} = {}& C_F^2\bigg[\frac{1}{\varepsilon^2}\bigg\{\frac{9}{2}\bigg\}+\frac{1}{\varepsilon}\bigg\{\frac{45}{4}\bigg\}+\bigg\{\frac{199}{8}-\frac{51}{2}\zeta_2-12\zeta_3+48\zeta_2\ln(2)\bigg\}+\varepsilon\bigg\{\frac{677}{16}-8c_1-\frac{615}{4}\zeta_2 \\ & +\frac{504}{5}\zeta_2^2-135\zeta_3+288\zeta_2\ln(2)\bigg\}+\varepsilon^2\bigg\{\frac{1167}{32}-48c_1-\frac{48c_2}{5}-\frac{4821}{8}\zeta_2+\frac{7719}{20}\zeta_2^2\end{aligned}$$

$$
- \frac{1203}{2}\zeta_3 + 132\zeta_2\zeta_3 + 1218\zeta_5 + 1056\zeta_2\ln(2)\Bigg\}\Bigg] + C_A C_F\Bigg[\frac{1}{\varepsilon^2}\Bigg\{\frac{11}{2}\Bigg\} + \frac{1}{\varepsilon}\Bigg\{-\frac{97}{12}\Bigg\}
$$

$$
+ \Bigg\{-\frac{1111}{24} + 8\zeta_2 + 6\zeta_3 - 24\zeta_2\ln(2)\Bigg\} + \varepsilon\Bigg\{-\frac{8581}{48} + 4c_1 + \frac{271}{12}\zeta_2 - \frac{252}{5}\zeta_2^2 + 52\zeta_3
$$

$$
- 144\zeta_2\ln(2)\Bigg\} + \varepsilon^2\Bigg\{-\frac{58543}{96} + 24c_1 + \frac{24c_2}{5} + \frac{1537}{24}\zeta_2 - \frac{9783}{40}\zeta_2^2 + \frac{3929}{18}\zeta_3 - 66\zeta_2\zeta_3
$$

$$
- 609\zeta_5 - 528\zeta_2\ln(2)\Bigg\}\Bigg] + C_F T_F n_l\Bigg[\frac{1}{\varepsilon^2}\Bigg\{-2\Bigg\} + \frac{1}{\varepsilon}\Bigg\{\frac{5}{3}\Bigg\} + \Bigg\{\frac{71}{6} + 8\zeta_2\Bigg\}
$$

$$
+ \varepsilon\Bigg\{\frac{581}{12} + \frac{97}{3}\zeta_2 + 16\zeta_3\Bigg\} + \varepsilon^2\Bigg\{\frac{4079}{24} + \frac{643}{6}\zeta_2 + \frac{597}{10}\zeta_2^2 + \frac{478}{9}\zeta_3\Bigg\}\Bigg] + C_F T_F\Bigg[\frac{1}{\varepsilon^2}\Bigg\{-2\Bigg\}
$$

$$
+ \frac{1}{\varepsilon}\Bigg\{\frac{5}{3}\Bigg\} + \Bigg\{\frac{143}{6} - 16\zeta_2\Bigg\} + \varepsilon\Bigg\{\frac{1133}{12} - \frac{227}{3}\zeta_2 - 56\zeta_3 + 96\zeta_2\ln(2)\Bigg\}
$$

$$
+ \varepsilon^2\Bigg\{\frac{8135}{24} - 16c_1 - \frac{1553}{6}\zeta_2 + \frac{837}{10}\zeta_2^2 - \frac{2546}{9}\zeta_3 + 480\zeta_2\ln(2)\Bigg\}\Bigg]. \tag{A.4}
$$

B The vector form factors up to two-loop

In this appendix, we present the vector form factors $F_{V,1}^{(n)}$ and $F_{V,2}^{(n)}$ up to two loops and $\mathcal{O}(\varepsilon)$.

$$
\begin{aligned}
F_{V,1}^{(1)} = C_F \Bigg[& \frac{1}{\varepsilon}\Big\{ -2-2\xi H_0 \Big\} + \Big\{ -4+\xi\Big(4H_{-1}H_0 - H_0^2 - 4H_{0,-1} + 2\zeta_2\Big) + \Big(-3-2x-3x^2\Big)\eta H_0 \Big\} \\
& + \varepsilon\Bigg\{ -8+\eta\Big(2\Big(3+2x+3x^2\Big)H_{-1}H_0 + \frac{1}{2}\Big(-3-2x-3x^2\Big)H_0^2 - 2\Big(3+2x+3x^2\Big)H_{0,-1} + 2\Big(1 \\
& + x + 2x^2\Big)\zeta_2\Big) + \xi\Big(\Big(-8-4H_{-1}^2+\zeta_2\Big)H_0 + 2H_{-1}H_0^2 - \frac{1}{3}H_0^3 + 8H_{-1}H_{0,-1} - 4H_{0,0,-1} - 8H_{0,-1,-1} \\
& - 4H_{-1}\zeta_2 + 4\zeta_3\Big)\Big\} + \varepsilon^2\Bigg\{ -16+\eta\Big(\Big(-2\Big(3+2x+3x^2\Big)H_{-1}^2 + \frac{1}{2}\Big(3+2x+3x^2\Big)\zeta_2\Big)H_0 + \Big(3+2x \\
& + 3x^2\Big)H_{-1}H_0^2 + \frac{1}{6}\Big(-3-2x-3x^2\Big)H_0^3 + 4\Big(3+2x+3x^2\Big)H_{-1}H_{0,-1} - 2\Big(3+2x+3x^2\Big)H_{0,0,-1} \\
& - 4\Big(3+2x+3x^2\Big)H_{0,-1,-1} + 2\Big(3+5x^2\Big)\zeta_2 - 2\Big(3+2x+3x^2\Big)H_{-1}\zeta_2 + \frac{4}{3}\Big(5+3x+4x^2\Big)\zeta_3\Big) + \xi\Big(\Big(\\
& -16+\Big(16-2\zeta_2\Big)H_{-1} + \frac{8}{3}H_{-1}^3 + \frac{14}{3}\zeta_3\Big)H_0 + \Big(-4-2H_{-1}^2+\frac{1}{2}\zeta_2\Big)H_0^2 + \frac{2}{3}H_{-1}H_0^3 - \frac{1}{12}H_0^4 + \Big(-16 \\
& - 8H_{-1}^2 - 2\zeta_2\Big)H_{0,-1} + 8H_{-1}H_{0,0,-1} + 16H_{-1}H_{0,-1,-1} - 4H_{0,0,0,-1} - 8H_{0,0,-1,-1} - 16H_{0,-1,-1,-1} \\
& + 4H_{-1}^2\zeta_2 + \frac{14}{5}\zeta_2^2 - 8H_{-1}\zeta_3\Big)\Big\} + \varepsilon^3\Bigg\{ -32+\eta\Big(\Big(\frac{4}{3}\Big(3+2x+3x^2\Big)H_{-1}^3 + \Big(-3-2x-3x^2\Big)H_{-1}\zeta_2 \\
& + \frac{7}{3}\Big(3+2x+3x^2\Big)\zeta_3\Big)H_0 + \Big(\Big(-3-2x-3x^2\Big)H_{-1}^2 + \frac{1}{4}\Big(3+2x+3x^2\Big)\zeta_2 + \Big(-1-x^2\Big)H_{-1}\zeta_2\Big)H_0^2 \\
& + \frac{1}{3}\Big(3+2x+3x^2\Big)H_{-1}H_0^3 + \frac{1}{24}\Big(-3-2x-3x^2\Big)H_0^4 + \Big(-4\Big(3+2x+3x^2\Big)H_{-1}^2 + \Big(-3 \\
& - 2x - 3x^2\Big)\zeta_2\Big)H_{0,-1} + 4\Big(3+2x+3x^2\Big)H_{-1}H_{0,0,-1} + 8\Big(3+2x+3x^2\Big)H_{-1}H_{0,-1,-1} - 2\Big(3+2x \\
& + 3x^2\Big)H_{0,0,0,-1} - 4\Big(3+2x+3x^2\Big)H_{0,0,-1,-1} - 8\Big(3+2x+3x^2\Big)H_{0,-1,-1,-1} + 4\Big(3+5x^2\Big)\zeta_2 + 2\Big(3 \\
& + 2x + 3x^2\Big)H_{-1}^2\zeta_2 + \frac{1}{20}\Big(75+56x+93x^2\Big)\zeta_2^2 + \frac{4}{3}\Big(13+11x^2\Big)\zeta_3 - 4\Big(3+2x+3x^2\Big)H_{-1}\zeta_3\Big) \\
& + \xi\Big(\Big(-32+\Big(32-\frac{28\zeta_3}{3}\Big)H_{-1} + \Big(-16+2\zeta_2\Big)H_{-1}^2 - \frac{4}{3}H_{-1}^4 + 4\zeta_2 + \frac{47}{20}\zeta_2^2\Big)H_0 + \Big(-8+8H_{-1} \\
& + \frac{4}{3}H_{-1}^3 + \frac{7}{3}\zeta_3\Big)H_0^2 + \Big(\frac{1}{6}\Big(-8+\zeta_2\Big) - \frac{2}{3}H_{-1}^2\Big)H_0^3 + \frac{1}{6}H_{-1}H_0^4 - \frac{1}{60}H_0^5 + \Big(-16\zeta_2 - \frac{28\zeta_2^2}{5}\Big)H_{-1} \\
& + \Big(-32+\Big(32+4\zeta_2\Big)H_{-1} + \frac{16}{3}H_{-1}^3 + \frac{4}{3}\zeta_3\Big)H_{0,-1} + \Big(-16-8H_{-1}^2-2\zeta_2\Big)H_{0,0,-1} + \Big(-32 \\
& - 16H_{-1}^2 - 4\zeta_2\Big)H_{0,-1,-1} + 8H_{-1}H_{0,0,0,-1} + 16H_{-1}H_{0,0,-1,-1} + 32H_{-1}H_{0,-1,-1,-1} - 4H_{0,0,0,0,-1} \\
& - 8H_{0,0,0,-1,-1} - 16H_{0,0,-1,-1,-1} - 32H_{0,-1,-1,-1,-1} - \frac{8}{3}H_{-1}^3\zeta_2 + 8H_{-1}^2\zeta_3 - \frac{8}{3}\zeta_2\zeta_3 + 12\zeta_5\Big)\Big\}\Bigg] .
\end{aligned}
\tag{B.1}
$$

$$
F_{V,1}^{(2)} = C_F^2 \Bigg[\frac{1}{\varepsilon^2}\Big\{ 2+4\xi H_0 + 2\xi^2 H_0^2 \Big\} + \frac{1}{\varepsilon}\Big\{ 8+\xi\Big(-8H_{-1}H_0 + 4\Big(2+x+x^2\Big)\eta H_0^2 + 8H_{0,-1} - 4\zeta_2\Big)
$$

$$+\xi^2\Big(-8H_{-1}H_0^2+2H_0^3+8H_0H_{0,-1}-4H_0\zeta_2\Big)+2\Big(7+2x+7x^2\Big)\eta H_0\Big\}+\Big\{46+\eta^2\Big(\Big($$
$$-4H_{-1}H_0^2P_{103}+8H_0H_{0,-1}P_{121}-8H_0H_{0,1}P_{124}+8H_{0,0,1}P_{135}-8H_{0,0,-1}P_{136}+\frac{1}{2}H_0^2P_{161}$$
$$-4P_{115}\zeta_3\Big)x_+^2+\Big(\frac{16}{3}H_0^3H_1P_{176}-4H_0^2H_{0,-1}P_{180}-8H_0H_{0,0,-1}P_{193}+48H_{0,0,0,-1}P_{194}$$
$$-8H_0^2H_{0,1}P_{195}-\frac{4}{3}H_0^3P_{204}+16H_0H_{0,0,1}P_{216}-16H_{0,0,0,1}P_{242}+\frac{1}{6}H_0^4P_{274}+\Big(-4xH_0P_{134}$$
$$+16H_0H_1P_{174}-16H_{0,1}P_{174}-8H_{0,-1}P_{218}+4H_0^2P_{219}\Big)\zeta_2+\frac{2}{5}P_{308}\zeta_2^2\Big)x_+^3\Big)+\xi\Big(8H_{-1}^2H_0$$
$$-16H_{-1}H_{0,-1}+16H_{0,-1,-1}+48(-1+x)\ln(2)x_+\zeta_2\Big)+\xi^2\Big(16H_{-1}^2H_0^2-8H_{-1}H_0^3+\Big(32H_0H_1$$
$$-32H_{0,-1}\Big)H_{0,1}+16H_{0,1}^2-32H_{-1}H_0H_{0,-1}-32H_0H_1H_{0,-1}+8H_{0,-1}^2-64H_1H_{0,0,1}+64H_1H_{0,0,-1}$$
$$-64H_0H_{0,1,1}+32H_0H_{0,1,-1}+32H_0H_{0,-1,1}+16H_0H_{0,-1,-1}+64H_{0,0,1,1}+32H_{0,-1,0,1}$$
$$+16H_{-1}H_0\zeta_2+16H_1\zeta_3\Big)+\eta\Big(x_+^3\zeta_2\Big(39+7x-750x^2+1074x^3-185x^4-57x^5-8H_{-1}P_{51}\Big)$$
$$+\frac{1}{2}\Big(85-18x+85x^2\Big)H_0+\Big(2H_{-1}H_0P_{37}-2H_{0,-1}P_{37}-32\Big(2+x+10x^2+x^3+2x^4\Big)H_0H_1$$
$$+32\Big(2+x+10x^2+x^3+2x^4\Big)H_{0,1}\Big)x_+^2-16H_0P_{99}x_+^4\zeta_3\Big)+4H_0^2H_1P_{24}x_+^4+192x\ln(2)x_+^2\zeta_2\Big\}$$
$$+\varepsilon\Big\{\ln(2)x_+^4\zeta_2\Big(-24P_4+48H_1P_{12}+48H_{-1}P_{12}\Big)+\xi\Big(\eta\Big(\ln(2)x_+^3\zeta_2\Big(-192(-1+x)x^2H_{0,1}$$
$$-192(-1+x)x^2H_{0,-1}\Big)-\frac{4}{3}H_0^3P_{66}x_+^3\zeta_2\Big)-\frac{16}{3}H_{-1}^3H_0-32x\Big(2-29x+2x^2\Big)x_+^4H_0H_{0,1}^2$$
$$+16H_{-1}^2H_{0,-1}-32H_{-1}H_{0,-1,-1}+32H_{0,-1,-1,-1}-4(-1+x)x_+\Big)+\eta^2\Big(16H_0^2H_{0,1,1}P_{53}x_+$$
$$+\Big(H_{-1}H_0^2P_{83}+16H_0H_{0,1,1}P_{95}+4H_{-1}^2H_0^2P_{100}+8H_{0,1}^2P_{119}+16H_0H_{0,1,-1}P_{129}+16H_0H_{0,-1,1}P_{129}$$
$$+16H_{0,-1,0,1}P_{132}+16H_{0,0,1,-1}P_{133}+16H_{0,0,-1,1}P_{133}-16H_{0,0,-1,-1}P_{137}+8H_{0,-1}^2P_{141}+\frac{1}{2}H_0^2P_{163}$$
$$+\Big(32H_0P_{88}+32H_{-1}H_0P_{109}-16H_{0,-1}P_{140}\Big)H_{0,1}+\Big(-16H_{-1}H_0P_{125}-8H_0P_{149}\Big)H_{0,-1}+\Big($$
$$-16H_{-1}P_{133}-4P_{156}\Big)H_{0,0,1}+\Big(16H_{-1}P_{137}+2P_{164}\Big)H_{0,0,-1}+\Big(4H_{-1}P_{145}-\frac{2P_{165}}{3}\Big)\zeta_3\Big)x_+^2+\Big($$
$$-16H_0^2H_{0,1,-1}P_{111}-16H_0^2H_{0,-1,1}P_{111}+\frac{16}{3}H_0^3H_1^2P_{186}+32H_{0,0,1,0,1}P_{189}-16H_0H_{0,0,-1,-1}P_{199}$$
$$-32H_{0,0,1,0,-1}P_{203}-32H_0H_{0,0,1,-1}P_{213}-32H_0H_{0,0,-1,1}P_{213}+32H_{0,0,-1,0,1}P_{220}+8H_0^2H_{0,-1,-1}P_{230}$$
$$-32H_{0,0,0,1,-1}P_{238}-32H_{0,0,0,-1,1}P_{238}+\frac{1}{3}H_0^4P_{241}+32H_{0,0,-1,0,-1}P_{249}+96H_{0,0,0,-1,-1}P_{250}$$
$$-32H_0H_{0,0,1,1}P_{256}+32H_{0,0,0,1,1}P_{287}+\frac{1}{30}H_0^5P_{290}-16H_{0,0,0,0,1}P_{305}+16H_{0,0,0,0,-1}P_{306}$$
$$+\frac{1}{6}H_0^3P_{309}+\Big(2H_0^4P_{175}-\frac{32}{3}H_{-1}H_0^3P_{186}\Big)H_1+\Big(\frac{4}{3}H_0^3P_{196}-\frac{2}{3}H_0^4P_{231}\Big)H_{-1}+\Big(\frac{8}{3}H_0^3P_{190}$$
$$+16H_{-1}H_0^2P_{198}-4H_0^2P_{284}\Big)H_{0,1}+\Big(-16xH_{-1}H_0^2P_{98}+16H_0^2H_1P_{178}-\frac{4}{3}H_0^3P_{257}$$
$$+4H_0^2P_{292}\Big)H_{0,-1}+\Big(32H_{0,1}P_{170}-64H_{-1}H_0P_{179}+256H_0H_1P_{185}-8H_0P_{228}-8H_0^2P_{232}\Big)H_{0,0,1}$$

$$+\Big(32H_{-1}H_0P_{117}-8H_0P_{225}-32H_{0,1}P_{227}+4H_0^2P_{239}-16H_{0,-1}P_{278}\Big)H_{0,0,-1}+\Big(32H_{-1}P_{182}$$
$$-64H_1P_{248}+16H_0P_{293}+8P_{300}\Big)H_{0,0,0,1}+\Big(-96H_{-1}P_{112}+32H_1P_{285}-8H_0P_{297}-8P_{303}\Big)H_{0,0,0,-1}$$
$$+\Big(2xH_0^2P_{154}+H_0P_{166}+32H_0H_1^2P_{177}+64H_{0,1,1}P_{186}-32H_{0,1,-1}P_{223}-32H_{0,-1,1}P_{224}$$
$$-16H_{0,0,1}P_{254}+16H_{0,-1,-1}P_{275}+8H_{0,0,-1}P_{294}+\Big(-64H_{-1}H_0P_{177}-8H_0^2P_{236}\Big)H_1+\Big($$
$$-8H_0^2P_{188}+8H_0P_{266}\Big)H_{-1}+\Big(64H_{-1}P_{177}-32H_1P_{197}-8P_{268}\Big)H_{0,1}+\Big(-32H_{-1}P_{184}-8P_{200}$$
$$-16H_0P_{207}+32H_1P_{224}\Big)H_{0,-1}-4P_{289}\zeta_3\Big)\zeta_2+\Big(-\frac{8}{5}H_{-1}P_{183}-\frac{8}{5}H_1P_{296}+\frac{4P_{307}}{5}-\frac{2}{5}H_0P_{310}\Big)\zeta_2^2$$
$$+\Big(-16H_0H_1P_{209}-16H_{0,-1}P_{245}+16H_{0,1}P_{253}-\frac{4}{3}H_0^2P_{291}-\frac{4}{3}H_0P_{295}\Big)\zeta_3+6P_{299}\zeta_5\Big)x_+^3\Big)$$
$$+\eta\Big(\ln(2)x_+^4\zeta_2\Big(-3072x^3H_{0,1}-3072x^3H_{0,-1}\Big)+\Big(4H_{-1}H_0P_{40}-2H_{-1}^2H_0P_{41}-4H_{0,-1,-1}P_{41}$$
$$+\frac{1}{4}H_0P_{43}+32\Big(2+x+10x^2+x^3+2x^4\Big)\zeta_2H_1+\Big(-4H_0P_{36}+192\Big(2+x+10x^2+x^3$$
$$+2x^4\Big)H_{-1}H_0\Big)H_1-32\Big(2+x+10x^2+x^3+2x^4\Big)H_0H_1^2+\Big(4P_{36}+64\Big(2+x+10x^2+x^3$$
$$+2x^4\Big)H_1-192\Big(2+x+10x^2+x^3+2x^4\Big)H_{-1}\Big)H_{0,1}+\Big(-4P_{40}+4H_{-1}P_{41}-192\Big(2+x+10x^2$$
$$+x^3+2x^4\Big)H_1\Big)H_{0,-1}-64\Big(2+x+10x^2+x^3+2x^4\Big)H_{0,1,1}+192\Big(2+x+10x^2+x^3+2x^4\Big)H_{0,1,-1}$$
$$+192\Big(2+x+10x^2+x^3+2x^4\Big)H_{0,-1,1}\Big)x_+^2+\Big(-2H_0^2H_1P_{71}+\Big(163+251x-3084x^2+4532x^3$$
$$-847x^4-183x^5+8H_{-1}^2P_{63}-2H_{-1}P_{72}\Big)\zeta_2\Big)x_+^3+\Big(384H_0H_1H_{0,0,-1}P_{92}-48H_0H_{0,-1}^2P_{97}$$
$$+32H_0H_{0,-1,0,1}P_{122}-32H_{0,-1}H_{0,0,1}P_{127}-\frac{4}{3}H_0^3H_1P_{151}+\Big(32H_0^2H_1P_{104}-32H_0H_{0,-1}P_{106}\Big)H_{0,1}$$
$$+\Big(-16H_0H_{0,1}P_{128}-8H_0H_1P_{142}\Big)\zeta_2-32H_{-1}H_0P_{99}\zeta_3\Big)x_+^4\Big)+\xi^2\Big(-\frac{64}{3}H_{-1}^3H_0^2+16H_{-1}^2H_0^3$$
$$+\Big(128H_{-1}H_0H_{0,-1}-96H_{0,-1}^2\Big)H_1+\Big(\Big(-128H_{-1}H_0+64H_{0,-1}\Big)H_1+64H_0H_1^2$$
$$+128H_{-1}H_{0,-1}\Big)H_{0,1}+\Big(32H_1-64H_{-1}\Big)H_{0,1}^2+64H_{-1}^2H_0H_{0,-1}-64H_0H_1^2H_{0,-1}-32H_{-1}H_{0,-1}^2$$
$$+\Big(256H_{-1}H_1-128H_1^2\Big)H_{0,0,1}+\Big(-256H_{-1}H_1+128H_1^2\Big)H_{0,0,-1}+\Big(256H_{-1}H_0-192H_0H_1$$
$$-256H_{0,-1}\Big)H_{0,1,1}+\Big(-128H_{-1}H_0+64H_0H_1-64H_{0,1}+192H_{0,-1}\Big)H_{0,1,-1}+\Big(-128H_{-1}H_0$$
$$+64H_0H_1+192H_{0,1}+192H_{0,-1}\Big)H_{0,-1,1}+\Big(-64H_{-1}H_0+64H_0H_1-320H_{0,1}+32H_{0,-1}\Big)H_{0,-1,-1}$$
$$+\Big(256H_1-256H_{-1}\Big)H_{0,0,1,1}-256H_1H_{0,0,1,-1}-256H_1H_{0,0,-1,1}+256H_1H_{0,0,-1,-1}+192H_0H_{0,1,1,1}$$
$$-64H_0H_{0,1,-1,-1}-128H_{-1}H_{0,-1,0,1}-64H_0H_{0,-1,1,-1}-64H_0H_{0,-1,-1,1}+32H_0H_{0,-1,-1,-1}$$
$$-384H_{0,0,1,1,1}+768H_{0,0,1,1,-1}+256H_{0,0,1,-1,1}-256H_{0,0,-1,1,1}-64H_{0,1,0,1,1}+256H_{0,1,0,1,-1}$$
$$-256H_{0,-1,0,1,1}+128H_{0,-1,0,1,-1}+128H_{0,-1,0,-1,1}-256H_{0,-1,1,0,1}+320H_{0,-1,-1,0,1}-32H_{-1}^2H_0\zeta_2$$
$$+\Big(-64H_{-1}H_1+32H_1^2\Big)\zeta_3\Big)+28xx_+^2+8c_1\Big(1-4x+x^2\Big)x_+^2+\Big(-8H_{-1}H_0^2H_1P_{12}$$
$$-32H_0H_{0,-1,-1}P_{19}+16H_1H_{0,0,1}P_{20}-16H_{0,0,1,1}P_{20}-16H_0H_1H_{0,-1}P_{21}-16H_0H_1H_{0,1}P_{27}$$
$$+4H_0^2H_1^2P_{29}+16H_1H_{0,0,-1}P_{30}+\Big(-48H_{-1}H_1P_{12}+48H_{-1,1}P_{12}\Big)\zeta_2-4H_1P_{39}\zeta_3\Big)x_+^4\Bigg\}\Bigg]$$

$$+C_FC_A\left[\frac{1}{\varepsilon^2}\left\{\frac{11}{3}+\frac{11\xi H_0}{3}\right\}+\frac{1}{\varepsilon}\left\{-\frac{49}{9}+\xi\Big(\eta\Big(-\frac{4}{3}x^2H_0^3-2\Big(-1+3x^2\Big)H_0\zeta_2\Big)\right.\right.$$
$$-\frac{67}{9}H_0+4H_{-1}H_0-4H_0H_1+4H_{0,1}-4H_{0,-1}\Big)+\eta\Big(-4x^2H_0^2-2\Big(-1+3x^2\Big)\zeta_2\Big)$$
$$\left.+\xi^2\Big(-4H_0H_{0,1}+4H_0H_{0,-1}+8H_{0,0,1}-8H_{0,0,-1}-2\zeta_3\Big)\right\}+\left\{\frac{1}{27}\Big(-1595-3514x$$
$$-1595x^2\Big)x_+^2+\eta^2x_+^3\Big(-32xH_0H_{0,0,1}P_{94}-\frac{1}{6}xH_0^4P_{123}+\frac{4}{3}H_0^3H_1P_{169}+4H_0^2H_{0,-1}P_{211}$$
$$-8H_{0,0,0,-1}P_{237}+8H_{0,0,0,1}P_{251}+\Big(16H_0H_1P_{172}+2H_0^2P_{173}-4H_{0,1}P_{192}+4H_{0,-1}P_{217}\Big)\zeta_2$$
$$+\frac{1}{5}P_{168}\zeta_2^2-2H_0P_{244}\zeta_3\Big)+\xi^2\Big(\Big(-8H_1+8H_{-1}\Big)\zeta_3+\Big(16H_{-1}H_0-16H_0H_1-8H_{0,-1}\Big)H_{0,1}$$
$$-4H_{0,1}^2-16H_{-1}H_0H_{0,-1}+16H_0H_1H_{0,-1}+12H_{0,-1}^2+\Big(32H_1-32H_{-1}\Big)H_{0,0,1}+\Big(-32H_1$$
$$+32H_{-1}\Big)H_{0,0,-1}+24H_0H_{0,1,1}-8H_0H_{0,1,-1}-8H_0H_{0,-1,1}-8H_0H_{0,-1,-1}-32H_{0,0,1,1}$$
$$+32H_{0,0,1,-1}+32H_{0,0,-1,1}-32H_{0,0,-1,-1}\Big)+\eta\Big(\Big(\frac{1}{54}H_0P_3+4H_0H_1P_{11}-4H_{0,1}P_{11}+\frac{4}{9}H_{-1}H_0P_{35}$$
$$-\frac{4}{9}H_{0,-1}P_{35}\Big)x_+^2+\Big(4H_0^2H_1P_{49}+8H_{0,0,1}P_{57}+\frac{2}{3}H_{-1}H_0^2P_{67}-\frac{4}{3}H_{0,0,-1}P_{73}+\frac{1}{18}H_0^2P_{77}$$
$$+\Big(\frac{4}{3}H_{-1}P_{56}+\frac{P_{79}}{9}\Big)\zeta_2-\frac{2}{3}P_{50}\zeta_3\Big)x_+^3+\Big(\frac{1}{9}H_0^3P_{84}+4H_0^2H_{0,1}P_{96}-\frac{2}{3}H_0P_{143}\zeta_2\Big)x_+^4\Big)+\xi\Big(\eta\Big(\frac{4}{3}\Big(3$$
$$+x^2\Big)H_{-1}H_0^3+16H_{-1}H_0\zeta_2\Big)-\frac{104}{3}H_{-1}^2H_0+24H_{-1}H_0H_1-4H_0H_1^2+\Big(8H_1-24H_{-1}\Big)H_{0,1}$$
$$+\Big(-24H_1+\frac{208H_{-1}}{3}\Big)H_{0,-1}-8H_{0,1,1}+24H_{0,1,-1}+24H_{0,-1,1}-\frac{208}{3}H_{0,-1,-1}+4H_1\zeta_2$$
$$-24(-1+x)\ln(2)x_+\zeta_2\Big)+\Big(24H_0H_{0,1}P_9-4H_0H_{0,-1}P_{17}\Big)x_+^4-8(-1+x)H_0H_{0,0,-1}P_8x_+^5$$
$$\left.-96x\ln(2)x_+^2\zeta_2\right\}+\varepsilon\left\{\frac{1}{162}\Big(-28745-70126x-28745x^2\Big)x_+^2+\ln(2)x_+^4\zeta_2\Big(12P_4-24H_1P_{12}\right.$$
$$-24H_{-1}P_{12}\Big)+\eta\Big(\ln(2)x_+^4\zeta_2\Big(1536x^3H_{0,1}+1536x^3H_{0,-1}\Big)+\Big(\frac{1}{324}H_0P_1+4H_0H_1^2P_{11}+8H_{0,1,1}P_{11}$$
$$-24H_{0,1,-1}P_{11}-24H_{0,-1,1}P_{11}+\frac{4}{9}H_{-1}^2H_0P_{34}+\frac{8}{9}H_{0,-1,-1}P_{34}+\frac{1}{27}H_{-1}H_0P_{45}+\Big(-24H_{-1}H_0P_{11}$$
$$+4H_0P_{15}\Big)H_1+\Big(-8H_1P_{11}+24H_{-1}P_{11}-4P_{15}\Big)H_{0,1}+\Big(\frac{P_2}{27}+24H_1P_{11}-\frac{8}{9}H_{-1}P_{34}\Big)H_{0,-1}$$
$$-4H_1P_{11}\zeta_2\Big)x_+^2+\Big(-8H_{0,0,1,1}P_{58}+2H_0^2H_1^2P_{59}+8H_{0,0,1,-1}P_{62}+8H_{0,0,-1,1}P_{62}-2H_{-1}^2H_0^2P_{68}$$
$$-8H_{0,0,-1,-1}P_{70}+\frac{1}{9}H_{-1}H_0^2P_{74}+\frac{1}{54}H_0^2P_{78}+\Big(4H_{-1}H_0^2P_{55}+2H_0^2P_{64}\Big)H_1+\Big(8H_1P_{58}-4P_{60}$$
$$-8H_{-1}P_{62}\Big)H_{0,0,1}+\Big(8H_1P_{54}+8H_{-1}P_{70}+\frac{2P_{80}}{9}\Big)H_{0,0,-1}+\Big(-48H_{-1}H_1P_{48}-12H_{-1}^2P_{52}$$
$$-\frac{2}{9}H_{-1}P_{76}+\frac{P_{81}}{27}\Big)\zeta_2+\Big(\frac{P_{46}}{9}-2H_{-1}P_{65}-2H_1P_{69}\Big)\zeta_3\Big)x_+^3+\Big(\frac{1}{12}H_0^4P_{85}+32H_0H_{0,0,1,-1}P_{87}$$
$$+32H_0H_{0,0,-1,1}P_{87}+48H_0H_{0,-1}^2P_{93}+16H_0H_{0,1}^2P_{101}-16H_0H_{0,-1,0,1}P_{110}+4H_0H_{0,0,1}P_{116}$$
$$-16H_0H_{0,0,-1,-1}P_{120}+\frac{2}{3}H_0^3H_1P_{146}+2H_0^2H_{0,-1}P_{150}+4H_{0,0,0,1}P_{152}+\frac{2}{3}H_{-1}H_0^3P_{153}$$

$$
- 4H_{0,0,0,-1}P_{155} + \frac{1}{54}H_0^3P_{162} + \Big(8H_{-1}H_0^2P_{96} - 16H_0^2H_1P_{104} - 16H_0H_{0,-1}P_{108} - 2H_0^2P_{147}\Big)H_{0,1}
$$
$$
+ \Big(- 32H_0H_1P_{105} - 4H_0P_{148}\Big)H_{0,0,-1} + \Big(\frac{1}{18}H_0P_{82} + 2H_{-1}H_0P_{126} - 2H_{0,-1}P_{130} - 4H_{0,1}P_{131}
$$
$$
+ 4H_0H_1P_{139} + \frac{1}{2}H_0^2P_{159}\Big)\zeta_2 + \frac{1}{5}P_{160}\zeta_2^2 + \frac{4}{3}H_0P_{144}\zeta_3\Big)x_+^4\Big) + \xi\Big(\eta\Big(\ln(2)x_+^3\zeta_2\Big(96(-1+x)x^2H_{0,1}
$$
$$
+ 96(-1+x)x^2H_{0,-1}\Big) - \frac{4}{3}\Big(3+5x^2\Big)H_{-1}^2H_0^3 - 8\Big(1+3x^2\Big)H_{-1}^2H_0\zeta_2\Big) + 80H_{-1}^3H_0 - 72H_{-1}^2H_0H_1
$$
$$
+ 24H_{-1}H_0H_1^2 - \frac{8}{3}H_0H_1^3 + \Big(- 48H_{-1}H_1 + 8H_1^2 + 72H_{-1}^2\Big)H_{0,1} + \Big(144H_{-1}H_1 - 24H_1^2
$$
$$
- 240H_{-1}^2\Big)H_{0,-1} + \Big(- 16H_1 + 48H_{-1}\Big)H_{0,1,1} + \Big(48H_1 - 144H_{-1}\Big)H_{0,1,-1} + \Big(48H_1
$$
$$
- 144H_{-1}\Big)H_{0,-1,1} + \Big(- 144H_1 + 480H_{-1}\Big)H_{0,-1,-1} + 16H_{0,1,1,1} - 48H_{0,1,1,-1} - 48H_{0,1,-1,1}
$$
$$
+ 144H_{0,1,-1,-1} - 48H_{0,-1,1,1} + 144H_{0,-1,1,-1} + 144H_{0,-1,-1,1} - 480H_{0,-1,-1,-1} + 4H_1^2\zeta_2\Big)
$$
$$
+ \eta^2\Big(\Big(- \frac{1}{30}xH_0^5P_{138} + 128H_0H_{0,0,1,1}P_{187} - 8H_0^2H_{0,1,1}P_{202} + 8H_0^2H_{0,1,-1}P_{205} + 8H_0^2H_{0,-1,1}P_{205}
$$
$$
+ \frac{1}{3}H_{-1}H_0^4P_{212} - \frac{4}{3}H_0^3H_1^2P_{215} - \frac{8}{3}H_0^3H_{0,1}P_{229} - 16H_{0,0,1,0,1}P_{246} - 8H_0^2H_{0,-1,-1}P_{247}
$$
$$
- 32H_{0,0,-1,0,-1}P_{252} + 16H_{0,0,-1,0,1}P_{255} + 16H_{0,0,1,0,-1}P_{262} + 32H_{0,0,0,1,-1}P_{283} + 32H_{0,0,0,-1,1}P_{283}
$$
$$
- 16H_{0,0,0,1,1}P_{298} + 8H_{0,0,0,0,1}P_{301} - 8H_{0,0,0,0,-1}P_{302} - 16H_{0,0,0,-1,-1}P_{304} + \Big(\frac{1}{3}H_0^4P_{86}
$$
$$
+ \frac{8}{3}H_{-1}H_0^3P_{215}\Big)H_1 + \Big(- 8H_{-1}H_0^2P_{118} - 8H_0^2H_1P_{201} + \frac{4}{3}H_0^3P_{288}\Big)H_{0,-1} + \Big(32H_{-1}H_0P_{171}
$$
$$
- 32H_0H_1P_{210} + 16H_{0,1}P_{221} - 16H_{0,-1}P_{263} + 4H_0^2P_{269}\Big)H_{0,0,1} + \Big(- 16H_{0,1}P_{233} + 32H_{0,-1}P_{243}
$$
$$
- 4H_0^2P_{281}\Big)H_{0,0,-1} + \Big(- 16H_{-1}P_{214} + 16H_1P_{259} - 8H_0P_{286}\Big)H_{0,0,0,1} + \Big(16H_{-1}P_{226} - 16H_1P_{240}
$$
$$
+ 8H_0P_{280}\Big)H_{0,0,0,-1} + \Big(- 8x^2H_{-1}H_0^2P_{47} - 8H_0H_1^2P_{206} - 8H_{0,1,1}P_{234} + \frac{4}{3}H_0^3P_{261} - 16H_{0,0,-1}P_{264}
$$
$$
+ 8H_{0,1,-1}P_{270} + 8H_{0,-1,1}P_{271} - 8H_{0,-1,-1}P_{276} + 8H_{0,0,1}P_{282} + \Big(4xH_0^2P_{107} + 16H_{-1}H_0P_{206}\Big)H_1
$$
$$
+ \Big(32xH_1P_{102} - 32xH_{-1}P_{102} - 4H_0P_{279}\Big)H_{0,1} + \Big(16H_{-1}P_{208} - 16H_1P_{235} + 4H_0P_{277}\Big)H_{0,-1}
$$
$$
+ 2P_{267}\zeta_3\Big)\zeta_2 + \Big(\frac{8}{5}H_{-1}P_{222} - \frac{2}{5}H_0P_{260} - \frac{4}{5}H_1P_{265}\Big)\zeta_2^2 + \Big(16H_0H_1P_{181} - 16H_{-1}H_0P_{191}
$$
$$
- 4H_{0,1}P_{258} + 2H_0^2P_{272} + 4H_{0,-1}P_{273}\Big)\zeta_3 + P_{167}\zeta_5\Big)x_+^3 - 8H_{-1}H_0^2x_+^2\zeta_2\Big) + \xi^2\Big(\Big(- 64H_{-1}H_0H_{0,-1}
$$
$$
+ 48H_{0,-1}^2\Big)H_1 + \Big(- 32H_{-1}^2H_0 + \Big(64H_{-1}H_0 - 32H_{0,-1}\Big)H_1 - 32H_0H_1^2 + 32H_{-1}H_{0,-1}\Big)H_{0,1}
$$
$$
+ \Big(- 16H_1 + 16H_{-1}\Big)H_{0,1}^2 + 32H_{-1}^2H_0H_{0,-1} + 32H_0H_1^2H_{0,-1} - 48H_{-1}H_{0,-1}^2 + \Big(- 128H_{-1}H_1
$$
$$
+ 64H_1^2 + 64H_{-1}^2\Big)H_{0,0,1} + \Big(128H_{-1}H_1 - 64H_1^2 - 64H_{-1}^2\Big)H_{0,0,-1} + \Big(- 96H_{-1}H_0 + 96H_0H_1
$$
$$
+ 16H_{0,1} + 80H_{0,-1}\Big)H_{0,1,1} + \Big(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\Big)H_{0,1,-1} + \Big(32H_{-1}H_0
$$
$$
- 32H_0H_1 - 48H_{0,1} - 48H_{0,-1}\Big)H_{0,-1,1} + \Big(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\Big)H_{0,-1,-1}
$$
$$
+ \Big(- 128H_1 + 128H_{-1}\Big)H_{0,0,1,1} + \Big(128H_1 - 128H_{-1}\Big)H_{0,0,1,-1} + \Big(128H_1 - 128H_{-1}\Big)H_{0,0,-1,1}
$$
$$
+ \Big(- 128H_1 + 128H_{-1}\Big)H_{0,0,-1,-1} - 112H_0H_{0,1,1,1} + 16H_0H_{0,1,1,-1} + 16H_0H_{0,1,-1,1}
$$

$$
\begin{aligned}
&+16H_0H_{0,1,-1,-1}+16H_0H_{0,-1,1,1}+16H_0H_{0,-1,1,-1}+16H_0H_{0,-1,-1,1}+16H_0H_{0,-1,-1,-1}\\
&+128H_{0,0,1,1,1}-256H_{0,0,1,1,-1}-128H_{0,0,1,-1,1}+128H_{0,0,1,-1,-1}+128H_{0,0,-1,1,-1}\\
&+128H_{0,0,-1,-1,1}+256H_{0,0,-1,-1,-1}-64H_{0,1,0,1,-1}+64H_{0,-1,0,1,1}+192H_{0,-1,0,-1,-1}\\
&+64H_{0,-1,1,0,1}+\Big(32H_{-1}H_1-16H_1^2-16H_{-1}^2\Big)\zeta_3\Big)-4c_1\Big(1-4x+x^2\Big)x_+^2+\Big(8H_{-1}H_0H_{0,-1}P_5\\
&-8H_0H_{0,1,1}P_{10}-8H_{0,1}^2P_{14}-24H_{0,-1,0,1}P_{16}-16H_0H_{0,1,-1}P_{22}-16H_0H_{0,-1,1}P_{22}\\
&+8H_0H_1H_{0,-1}P_{23}+16H_0H_{0,-1}P_{25}+8H_0H_{0,-1,-1}P_{31}-4H_{0,-1}^2P_{33}+\Big(-16H_{-1}H_0P_{13}+4H_0P_{18}\\
&+8H_0H_1P_{26}+16H_{0,-1}P_{28}\Big)H_{0,1}-24H_{-1,1}P_{12}\zeta_2\Big)x_+^4-16(-1+x)H_{-1}H_0H_{0,0,-1}P_8x_+^5\Big\}\Big]\\
&+C_Fn_lT_F\Big[\frac{1}{\varepsilon^2}\Big\{-\frac{4}{3}-\frac{4\xi H_0}{3}\Big\}+\frac{1}{\varepsilon}\Big\{\frac{20}{9}+\frac{20\xi H_0}{9}\Big\}+\Big\{\frac{424}{27}+\eta\Big(\frac{2}{27}\Big(209+6x+209x^2\Big)H_0-\frac{8}{9}\Big(19\\
&+6x+19x^2\Big)H_{-1}H_0+\frac{2}{9}\Big(19+6x+19x^2\Big)H_0^2+\frac{8}{9}\Big(19+6x+19x^2\Big)H_{0,-1}-\frac{4}{9}\Big(7+6x+31x^2\Big)\zeta_2\Big)\\
&+\xi\Big(\frac{16}{3}H_{-1}^2H_0-\frac{8}{3}H_{-1}H_0^2+\frac{4}{9}H_0^3-\frac{32}{3}H_{-1}H_{0,-1}+\frac{16}{3}H_{0,0,-1}+\frac{32}{3}H_{0,-1,-1}+\Big(\frac{8}{3}H_0+\frac{16}{3}H_{-1}\Big)\zeta_2\\
&-\frac{16\zeta_3}{3}\Big)\Big\}+\varepsilon\Big\{\frac{5204}{81}+\xi\Big(\Big(-\frac{16H_0}{3}+32H_{-1}\Big)\zeta_3-\frac{32}{3}H_{-1}^3H_0+8H_{-1}^2H_0^2-\frac{8}{3}H_{-1}H_0^3+32H_{-1}^2H_{0,-1}\\
&-32H_{-1}H_{0,0,-1}-64H_{-1}H_{0,-1,-1}+16H_{0,0,0,-1}+32H_{0,0,-1,-1}+64H_{0,-1,-1,-1}+\Big(-8H_{-1}H_0+2H_0^2\\
&-16H_{-1}^2+24H_{0,-1}\Big)\zeta_2-\frac{96\zeta_2^2}{5}\Big)+\eta\Big(\frac{1}{81}\Big(5813-1218x+5813x^2\Big)H_0+\frac{8}{9}\Big(47+18x+47x^2\Big)H_{-1}^2H_0\\
&+\frac{2}{27}\Big(281+6x+281x^2\Big)H_0^2+\frac{2}{27}\Big(47+18x+47x^2\Big)H_0^3+\frac{1}{3}\Big(1+x^2\Big)H_0^4+\Big(-\frac{8}{27}\Big(281+6x\\
&+281x^2\Big)H_0-\frac{4}{9}\Big(47+18x+47x^2\Big)H_0^2\Big)H_{-1}+\Big(\frac{8}{27}\Big(281+6x+281x^2\Big)-\frac{16}{9}\Big(47+18x\\
&+47x^2\Big)H_{-1}\Big)H_{0,-1}+\frac{8}{9}\Big(47+18x+47x^2\Big)H_{0,0,-1}+\frac{16}{9}\Big(47+18x+47x^2\Big)H_{0,-1,-1}+\Big(-\frac{8}{27}\Big(22+3x\\
&+259x^2\Big)+\frac{2}{9}\Big(37+18x+37x^2\Big)H_0+\frac{8}{9}\Big(47+18x+47x^2\Big)H_{-1}\Big)\zeta_2-\frac{8}{9}\Big(35+18x+59x^2\Big)\zeta_3\Big)\Big\}\Big]\\
&+C_FT_F\Big[\Big\{\frac{4}{27}\Big(383+178x+383x^2\Big)x_+^2+\eta\Big(\frac{2}{27}H_0P_{42}x_+^2+\Big(\frac{8}{3}xH_0^3P_6+4H_0P_{90}\zeta_2\Big)x_+^4\Big)\\
&-\frac{4}{9}(-1+x)\xi\eta H_0^3P_6x_+^3+\Big(\frac{2}{9}H_0^2P_{32}-\frac{8}{3}P_7\zeta_2\Big)x_+^4\Big\}+\varepsilon\Big\{\frac{2}{27}\Big(2069+1990x+2069x^2\Big)x_+^2\\
&+\frac{32}{3}(-1+x)^2\Big(7+72x+7x^2\Big)\ln(2)x_+^4\zeta_2+\xi\eta x_+^3\Big(-\frac{8}{9}(-1+x)H_{-1}H_0^3P_6-\frac{1}{3}(-1+x)H_0^4P_6\\
&-\frac{16}{3}(-1+x)H_0^2H_{0,1}P_6+\frac{16}{3}(-1+x)H_0^2H_{0,-1}P_6+\frac{64}{3}(-1+x)H_0H_{0,0,1}P_6\\
&-\frac{32}{3}(-1+x)H_0H_{0,0,-1}P_6-32(-1+x)H_{0,0,0,1}P_6+\frac{16}{3}(-1+x)H_{0,0,0,-1}P_6\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\eta\Big(\Big(-\frac{4}{27}H_{-1}H_0P_{42}+\frac{4}{27}H_{0,-1}P_{42}+\frac{1}{27}H_0P_{44}\Big)x_+^2+\Big(-\frac{2}{27}H_0^2P_{61}+\frac{4P_{75}\zeta_2}{27}\Big)x_+^3\\
&+\Big(\frac{16}{3}xH_{-1}H_0^3P_6+2xH_0^4P_6+32xH_0^2H_{0,1}P_6-32xH_0^2H_{0,-1}P_6-128xH_0H_{0,0,1}P_6\\
&+64xH_0H_{0,0,-1}P_6+192xH_{0,0,0,1}P_6-32xH_{0,0,0,-1}P_6-\frac{2}{27}H_0^3P_{157}+\Big(\frac{8}{3}H_{-1}H_0P_{89}\\
&+2H_0^2P_{90}-\frac{8}{3}H_{0,-1}P_{113}-\frac{2}{9}H_0P_{158}\Big)\zeta_2-\frac{8}{5}P_{91}\zeta_2^2-\frac{8}{9}H_0P_{114}\zeta_3\Big)x_+^4\Big)+\Big(-\frac{4}{9}H_{-1}H_0^2P_{32}\\
&+\frac{8}{9}H_0^2H_1P_{32}-\frac{16}{9}H_0H_{0,1}P_{32}+\frac{16}{9}H_{0,0,1}P_{32}+\frac{8}{9}H_{0,0,-1}P_{32}-\frac{8}{3}H_{-1}P_{32}\zeta_2-\frac{8}{9}P_{38}\zeta_3\Big)x_+^4\Big\}\Big].
\end{aligned}
$$

The polynomials are defined as

$$
\begin{aligned}
P_1&=-70165x^4-177188x^3-77966x^2-177188x-70165,\\
P_2&=-3623x^4-23056x^3+6062x^2-23056x-3623,\\
P_3&=-2545x^4-6812x^3-4646x^2-6812x-2545,\\
P_4&=x^4-56x^3-42x^2-56x+1,\\
P_5&=x^4-16x^3+26x^2-16x+1,\\
P_6&=x^4-2x^3+12x^2-2x+1,\\
P_7&=x^4+37x^3-124x^2+37x+1,\\
P_8&=2x^4-9x^3+122x^2-9x+2,\\
P_9&=2x^4+9x^3+18x^2+9x+2,\\
P_{10}&=3x^4-94x^3+322x^2-94x+3,\\
P_{11}&=3x^4-8x^3+26x^2-8x+3,\\
P_{12}&=3x^4+16x^3+62x^2+16x+3,\\
P_{13}&=4x^4-13x^3+140x^2-13x+4,\\
P_{14}&=4x^4+27x^3+10x^2+27x+4,\\
P_{15}&=5x^4-90x^3+138x^2-90x+5,\\
P_{16}&=5x^4+26x^3+46x^2+26x+5,\\
P_{17}&=5x^4+79x^3-146x^2+79x+5,\\
P_{18}&=7x^4-42x^3+234x^2-42x+7,\\
P_{19}&=7x^4-3x^3+349x^2-3x+7,\\
P_{20}&=7x^4+7x^3-48x^2+7x+7,\\
P_{21}&=7x^4+11x^3+284x^2+11x+7,\\
P_{22}&=7x^4+85x^3-96x^2+85x+7,\\
P_{23}&=7x^4+104x^3-142x^2+104x+7,\\
P_{24}&=8x^4+15x^3+194x^2+15x+8,\\
P_{25}&=10x^4+46x^3+29x^2+46x+10,\\
P_{26}&=11x^4-40x^3+342x^2-40x+11,\\
P_{27}&=11x^4+22x^3+58x^2+22x+11,\\
P_{28}&=11x^4+72x^3+44x^2+72x+11,\\
P_{29}&=15x^4+37x^3+164x^2+37x+15,\\
P_{30}&=17x^4+38x^3+630x^2+38x+17,
\end{aligned}
$$

$P_{31} = 18x^4 + 227x^3 - 272x^2 + 227x + 18,$
$P_{32} = 19x^4 - 2x^3 + 206x^2 - 2x + 19,$
$P_{33} = 19x^4 + 211x^3 - 246x^2 + 211x + 19,$
$P_{34} = 28x^4 - 187x^3 + 110x^2 - 187x + 28,$
$P_{35} = 31x^4 + 59x^3 + 164x^2 + 59x + 31,$
$P_{36} = 43x^4 + 20x^3 + 130x^2 + 20x + 43,$
$P_{37} = 55x^4 - 48x^3 + 562x^2 - 48x + 55,$
$P_{38} = 69x^4 + 408x^3 - 698x^2 + 408x + 69,$
$P_{39} = 79x^4 + 142x^3 + 1698x^2 + 142x + 79,$
$P_{40} = 81x^4 - 104x^3 + 766x^2 - 104x + 81,$
$P_{41} = 257x^4 - 48x^3 + 2078x^2 - 48x + 257,$
$P_{42} = 265x^4 - 124x^3 + 3494x^2 - 124x + 265,$
$P_{43} = 479x^4 + 704x^3 - 30x^2 + 704x + 479,$
$P_{44} = 1719x^4 + 452x^3 + 20170x^2 + 452x + 1719,$
$P_{45} = 3623x^4 + 23056x^3 - 6062x^2 + 23056x + 3623,$
$P_{46} = -1045x^5 - 8139x^4 + 3518x^3 + 1634x^2 + 8151x + 1577,$
$P_{47} = x^5 + 5x^4 - 9x^3 + 60x^2 - 46x + 19,$
$P_{48} = 2x^5 + 8x^4 + 25x^3 - 21x^2 - 5x - 1,$
$P_{49} = 2x^5 + 13x^4 - 5x^3 - 11x^2 - 25x - 6,$
$P_{50} = 5x^5 - 285x^4 + 1592x^3 - 1840x^2 + 99x - 67,$
$P_{51} = 5x^5 - 30x^4 + 53x^3 - 61x^2 + 24x - 7,$
$P_{52} = 5x^5 - 26x^4 + 87x^3 - 119x^2 + 2x - 13,$
$P_{53} = 5x^5 + 8x^4 + 6x^3 - 22x^2 - 16x - 13,$
$P_{54} = 5x^5 + 171x^4 - 494x^3 + 398x^2 - 243x - 29,$
$P_{55} = 9x^5 + 23x^4 + 2x^3 + 94x^2 + 49x + 15,$
$P_{56} = 10x^5 - 105x^4 + 139x^3 - 203x^2 + 57x - 26,$
$P_{57} = 10x^5 + 29x^4 + 59x^3 - 43x^2 - 17x - 6,$
$P_{58} = 11x^5 - 81x^4 + 480x^3 - 448x^2 + 105x - 3,$
$P_{59} = 11x^5 - 21x^4 + 284x^3 - 316x^2 - 3x - 19,$
$P_{60} = 13x^5 + 61x^4 - 316x^3 + 460x^2 - 101x + 11,$
$P_{61} = 16x^5 - 120x^4 - 287x^3 - 3083x^2 - 21x - 281,$
$P_{62} = 25x^5 - 45x^4 + 614x^3 - 518x^2 + 117x - 1,$
$P_{63} = 26x^5 - 45x^4 + 305x^3 - 313x^2 + 39x - 28,$
$P_{64} = 27x^5 - 37x^4 + 236x^3 - 92x^2 - 3x - 3,$
$P_{65} = 41x^5 + 423x^4 - 482x^3 + 1106x^2 + 45x + 115,$
$P_{66} = 42x^5 + 136x^4 + 87x^3 + 89x^2 - 4x + 2,$
$P_{67} = 53x^5 - 12x^4 + 851x^3 - 571x^2 + 222x + 17,$
$P_{68} = 56x^5 + 43x^4 + 695x^3 - 383x^2 + 191x + 22,$
$P_{69} = 57x^5 + 183x^4 + 430x^3 - 606x^2 - 315x - 101,$
$P_{70} = 58x^5 + 9x^4 + 779x^3 - 467x^2 + 225x + 20,$

$$
\begin{aligned}
P_{71} &= 77x^5 + 139x^4 + 406x^3 + 298x^2 + 53x + 51,\\
P_{72} &= 83x^5 - 217x^4 + 2594x^3 - 646x^2 + 59x + 47,\\
P_{73} &= 83x^5 + 432x^4 - 499x^3 + 779x^2 - 222x - 13,\\
P_{74} &= 97x^5 + 1533x^4 + 3340x^3 - 4976x^2 - 357x - 533,\\
P_{75} &= 139x^5 + 2247x^4 - 9395x^3 + 6025x^2 - 2388x - 404,\\
P_{76} &= 187x^5 + 399x^4 - 5192x^3 + 9388x^2 - 843x + 349,\\
P_{77} &= 217x^5 + 81x^4 - 860x^3 + 616x^2 - 621x - 233,\\
P_{78} &= 254x^5 - 25098x^4 - 1801x^3 - 10009x^2 - 10761x - 3337,\\
P_{79} &= 347x^5 - 1737x^4 + 5228x^3 - 3040x^2 + 1737x - 7,\\
P_{80} &= 1343x^5 + 3651x^4 - 5788x^3 + 7424x^2 - 4827x - 907,\\
P_{81} &= 4774x^5 - 4044x^4 + 30811x^3 - 3449x^2 + 12363x - 71,\\
P_{82} &= -1367x^6 + 1822x^5 - 25085x^4 + 19772x^3 - 1829x^2 + 2110x - 575,\\
P_{83} &= -399x^6 - 124x^5 - 2335x^4 - 1448x^3 + 2075x^2 - 124x + 307,\\
P_{84} &= -89x^6 + 154x^5 - 1547x^4 + 1262x^3 - 29x^2 + 4x - 11,\\
P_{85} &= -29x^6 + 188x^5 - 335x^4 + 974x^3 + 67x^2 + 2x - 11,\\
P_{86} &= -9x^6 + 55x^5 - 422x^4 + 254x^3 - 175x^2 - 63x - 24,\\
P_{87} &= x^6 - 31x^5 + 213x^4 - 428x^3 + 213x^2 - 31x + 1,\\
P_{88} &= x^6 - 12x^5 + 14x^4 - 38x^3 + 14x^2 - 12x + 1,\\
P_{89} &= x^6 + 4x^5 - 5x^4 + 128x^3 - 5x^2 + 4x + 1,\\
P_{90} &= x^6 + 4x^5 + 3x^4 + 48x^3 + 3x^2 + 4x + 1,\\
P_{91} &= x^6 + 4x^5 + 6x^4 + 18x^3 + 6x^2 + 4x + 1,\\
P_{92} &= x^6 + 4x^5 + 11x^4 + 11x^2 + 4x + 1,\\
P_{93} &= x^6 + 6x^5 - 4x^4 + 21x^3 - 4x^2 + 6x + 1,\\
P_{94} &= x^6 + 7x^5 - 23x^4 + 57x^3 - 50x^2 + 28x - 4,\\
P_{95} &= 2x^6 - 11x^5 - 114x^4 + 182x^3 - 114x^2 - 11x + 2,\\
P_{96} &= 2x^6 - 5x^5 + 92x^4 - 162x^3 + 92x^2 - 5x + 2,\\
P_{97} &= 2x^6 + 6x^5 + 33x^4 - 68x^3 + 33x^2 + 6x + 2,\\
P_{98} &= 2x^6 + 9x^5 - 21x^4 + 102x^3 - 88x^2 + 31x - 3,\\
P_{99} &= 2x^6 + 11x^5 - 9x^4 + 48x^3 - 9x^2 + 11x + 2,\\
P_{100} &= 2x^6 + 13x^5 + 246x^4 - 274x^3 + 248x^2 + 17x + 4,\\
P_{101} &= 3x^6 - 4x^5 + 115x^4 - 302x^3 + 115x^2 - 4x + 3,\\
P_{102} &= 3x^6 + 4x^5 + 47x^4 - 116x^3 + 137x^2 - 32x + 5,\\
P_{103} &= 3x^6 + 5x^5 + 153x^4 - 202x^3 + 155x^2 + 9x + 5,\\
P_{104} &= 3x^6 + 13x^5 + 22x^4 + 12x^3 + 22x^2 + 13x + 3,\\
P_{105} &= 3x^6 + 24x^5 - 37x^4 + 196x^3 - 37x^2 + 24x + 3,\\
P_{106} &= 4x^6 + 6x^5 + 141x^4 - 220x^3 + 141x^2 + 6x + 4,\\
P_{107} &= 4x^6 + 41x^5 - 173x^4 + 608x^3 - 580x^2 + 193x - 29,\\
P_{108} &= 5x^6 - 24x^5 + 291x^4 - 698x^3 + 291x^2 - 24x + 5,\\
P_{109} &= 5x^6 + 14x^5 - 19x^4 + 64x^3 - 19x^2 + 14x + 5,\\
P_{110} &= 5x^6 + 59x^5 - 175x^4 + 508x^3 - 175x^2 + 59x + 5,
\end{aligned}
$$

$P_{111} = 6x^6 - 57x^5 + 167x^4 - 223x^3 + 17x^2 - 30x - 8,$
$P_{112} = 6x^6 - 53x^5 + 267x^4 - 239x^3 + 73x^2 + 6x + 4,$
$P_{113} = 7x^6 + 28x^5 + x^4 + 536x^3 + x^2 + 28x + 7,$
$P_{114} = 7x^6 + 28x^5 + 13x^4 + 416x^3 + 13x^2 + 28x + 7,$
$P_{115} = 7x^6 + 40x^5 - 163x^4 + 280x^3 - 159x^2 + 48x + 11,$
$P_{116} = 7x^6 + 132x^5 + 71x^4 - 2312x^3 + 1121x^2 - 396x - 15,$
$P_{117} = 9x^6 - 84x^5 + 355x^4 - 341x^3 + 94x^2 - 3x + 2,$
$P_{118} = 9x^6 - 57x^5 + 232x^4 - 176x^3 + 97x^2 + 15x + 8,$
$P_{119} = 9x^6 + 11x^5 + 139x^4 - 254x^3 + 139x^2 + 11x + 9,$
$P_{120} = 10x^6 - 25x^5 + 470x^4 - 874x^3 + 470x^2 - 25x + 10,$
$P_{121} = 11x^6 - 8x^5 + 339x^4 - 556x^3 + 339x^2 - 8x + 11,$
$P_{122} = 11x^6 + 35x^5 + 195x^4 - 324x^3 + 195x^2 + 35x + 11,$
$P_{123} = 12x^6 + 37x^5 + 49x^4 + 42x^3 + 42x^2 + 11x - 1,$
$P_{124} = 13x^6 + 3x^5 + 231x^4 - 430x^3 + 231x^2 + 3x + 13,$
$P_{125} = 13x^6 + 14x^5 + 111x^4 - 20x^3 + 111x^2 + 14x + 13,$
$P_{126} = 13x^6 + 34x^5 - 61x^4 + 908x^3 - 873x^2 + 138x - 63,$
$P_{127} = 13x^6 + 57x^5 + 76x^4 - 56x^3 + 76x^2 + 57x + 13,$
$P_{128} = 14x^6 + 52x^5 + 153x^4 - 76x^3 + 153x^2 + 52x + 14,$
$P_{129} = 15x^6 - 9x^5 + 517x^4 - 982x^3 + 517x^2 - 9x + 15,$
$P_{130} = 15x^6 + 706x^5 - 55x^4 + 2888x^3 - 159x^2 + 198x - 41,$
$P_{131} = 17x^6 - 150x^5 + 1153x^4 - 1852x^3 + 883x^2 - 174x + 11,$
$P_{132} = 17x^6 + 25x^5 + 231x^4 - 418x^3 + 231x^2 + 25x + 17,$
$P_{133} = 17x^6 + 46x^5 - 109x^4 + 348x^3 - 109x^2 + 46x + 17,$
$P_{134} = 18x^6 - x^5 - 85x^4 + 150x^3 + 76x^2 + 99x - 1,$
$P_{135} = 18x^6 + 7x^5 + 290x^4 - 502x^3 + 290x^2 + 7x + 18,$
$P_{136} = 19x^6 - 21x^5 + 525x^4 - 910x^3 + 523x^2 - 25x + 17,$
$P_{137} = 24x^6 + 15x^5 - 24x^4 + 234x^3 - 26x^2 + 11x + 22,$
$P_{138} = 24x^6 + 75x^5 + 91x^4 + 118x^3 + 50x^2 + 29x - 3,$
$P_{139} = 25x^6 - 126x^5 + 1233x^4 - 1820x^3 + 859x^2 - 166x + 11,$
$P_{140} = 25x^6 + 19x^5 + 479x^4 - 854x^3 + 479x^2 + 19x + 25,$
$P_{141} = 27x^6 - 20x^5 + 835x^4 - 1428x^3 + 835x^2 - 20x + 27,$
$P_{142} = 28x^6 + 129x^5 - 244x^4 + 892x^3 - 334x^2 + 27x + 14,$
$P_{143} = 29x^6 - 352x^5 + 839x^4 - 1502x^3 + 317x^2 - 46x + 11,$
$P_{144} = 29x^6 + 665x^5 - 2515x^4 + 4174x^3 - 715x^2 + 269x + 65,$
$P_{145} = 37x^6 - 94x^5 + 223x^4 - 108x^3 + 231x^2 - 78x + 45,$
$P_{146} = 37x^6 - 10x^5 + 1175x^4 - 1924x^3 + 735x^2 - 386x - 27,$
$P_{147} = 39x^6 + 20x^5 + 1021x^4 - 2324x^3 + 865x^2 - 376x - 21,$
$P_{148} = 44x^6 + 265x^5 + 704x^4 - 2880x^3 + 1696x^2 - 529x - 20,$
$P_{149} = 45x^6 - 80x^5 + 239x^4 - 792x^3 + 239x^2 - 80x + 45,$
$P_{150} = 50x^6 + 119x^5 + 610x^4 - 2060x^3 + 1254x^2 - 435x - 2,$

$P_{151} = 53x^6 + 173x^5 + 357x^4 + 892x^3 - 935x^2 - 17x - 11,$
$P_{152} = 59x^6 - 326x^5 + 1675x^4 + 1888x^3 - 1503x^2 + 446x + 9,$
$P_{153} = 63x^6 + 9x^5 + 1145x^4 + 762x^3 - 1073x^2 + 385x - 11,$
$P_{154} = 72x^6 + 381x^5 - 659x^4 + 1986x^3 - 786x^2 + 9x + 21,$
$P_{155} = 81x^6 - 429x^5 + 863x^4 + 3222x^3 - 2399x^2 + 667x + 43,$
$P_{156} = 93x^6 - 130x^5 + 491x^4 - 716x^3 - 21x^2 - 194x - 35,$
$P_{157} = 105x^6 - 30x^5 + 1065x^4 - 56x^3 - 431x^2 - 14x - 47,$
$P_{158} = 125x^6 - 94x^5 + 1295x^4 - 220x^3 - 201x^2 - 78x - 27,$
$P_{159} = 133x^6 - 332x^5 + 4847x^4 - 2420x^3 + 159x^2 + 24x - 11,$
$P_{160} = 145x^6 - 738x^5 + 15433x^4 - 15038x^3 + 12285x^2 - 404x + 637,$
$P_{161} = 229x^6 - 60x^5 + 1517x^4 - 968x^3 + 247x^2 + 4x + 55,$
$P_{162} = 443x^6 + 2186x^5 - 12793x^4 + 14416x^3 + 2471x^2 - 766x - 565,$
$P_{163} = 649x^6 - 416x^5 + 4063x^4 - 2272x^3 + 975x^2 - 80x + 153,$
$P_{164} = 759x^6 - 516x^5 + 4247x^4 - 4888x^3 - 163x^2 - 516x + 53,$
$P_{165} = 1043x^6 - 224x^5 + 10999x^4 - 14384x^3 + 2761x^2 + 400x + 173,$
$P_{166} = -337x^7 - 125x^6 - 3325x^5 + 2351x^4 - 431x^3 - 1203x^2 - 19x + 17,$
$P_{167} = -117x^7 - 1119x^6 + 4661x^5 - 17313x^4 + 15395x^3 - 6031x^2 + 297x - 157,$
$P_{168} = -71x^7 - 353x^6 + 621x^5 - 2761x^4 + 2243x^3 - 991x^2 + 131x - 3,$
$P_{169} = x^7 - 7x^6 + 69x^5 - 267x^4 + 239x^3 - 89x^2 - 5x - 5,$
$P_{170} = x^7 - 6x^6 + 115x^5 - 150x^4 + 206x^3 - 75x^2 + 30x + 7,$
$P_{171} = x^7 - 5x^6 + 61x^5 - 93x^4 + 121x^3 - 41x^2 + 17x + 3,$
$P_{172} = x^7 - 2x^6 + 37x^5 - 130x^4 + 123x^3 - 42x^2 - x - 2,$
$P_{173} = x^7 - x^6 + 33x^5 - 99x^4 + 113x^3 - 23x^2 + 7x + 1,$
$P_{174} = x^7 + x^6 + 29x^5 - 39x^4 + 53x^3 - 19x^2 + 5x + 1,$
$P_{175} = x^7 + 2x^6 + 16x^5 + 41x^4 + 15x^3 + 24x^2 + 22x + 7,$
$P_{176} = x^7 + 2x^6 + 17x^5 - 16x^4 + 30x^3 - 7x^2 + 4x + 1,$
$P_{177} = x^7 + 4x^6 - 7x^5 + 30x^4 - 16x^3 + 17x^2 + 2x + 1,$
$P_{178} = x^7 + 6x^6 - 26x^5 - 17x^4 - 39x^3 - 14x^2 - 30x - 9,$
$P_{179} = x^7 + 7x^6 - 31x^5 + 215x^4 - 173x^3 + 61x^2 + 11x + 5,$
$P_{180} = x^7 + 9x^6 - 57x^5 + 183x^4 - 197x^3 + 47x^2 - 15x - 3,$
$P_{181} = x^7 + 9x^6 - 33x^5 + 113x^4 - 120x^3 + 28x^2 - 12x - 2,$
$P_{182} = x^7 + 14x^6 - 91x^5 + 692x^4 - 552x^3 + 191x^2 + 46x + 19,$
$P_{183} = x^7 + 35x^6 - 220x^5 + 1700x^4 - 846x^3 + 830x^2 + 331x + 121,$
$P_{184} = 2x^7 + 3x^6 + 19x^5 - 76x^4 + 62x^3 - 29x^2 - 9x - 4,$
$P_{185} = 2x^7 + 6x^6 + 13x^5 - 13x^4 + 20x^3 - 8x^2 - 3x - 1,$
$P_{186} = 2x^7 + 7x^6 - 2x^5 + 37x^4 - 9x^3 + 22x^2 + 5x + 2,$
$P_{187} = 2x^7 + 7x^6 + 5x^5 + 14x^4 - 7x^3 - 4x - 1,$
$P_{188} = 2x^7 + 8x^6 - x^5 + 83x^4 + x^3 + 61x^2 + 28x + 10,$
$P_{189} = 2x^7 + 23x^6 - 172x^5 + 363x^4 - 433x^3 + 122x^2 - 53x - 12,$
$P_{190} = 2x^7 + 24x^6 - 157x^5 + 447x^4 - 587x^3 + 57x^2 - 84x - 22,$

$P_{191} = 3x^7 + 76x^5 - 247x^4 + 240x^3 - 81x^2 - 3x - 4,$
$P_{192} = 3x^7 - 11x^6 + 143x^5 - 527x^4 + 485x^3 - 173x^2 - 7x - 9,$
$P_{193} = 3x^7 - 9x^6 + 183x^5 - 689x^4 + 703x^3 - 173x^2 + 15x - 1,$
$P_{194} = 3x^7 + 3x^6 + 68x^5 - 246x^4 + 260x^3 - 58x^2 + 3x - 1,$
$P_{195} = 3x^7 + 4x^6 + 63x^5 - 182x^4 + 210x^3 - 43x^2 + 8x + 1,$
$P_{196} = 3x^7 + 6x^6 - 400x^5 + 317x^4 + 389x^3 - 590x^2 + 19,$
$P_{197} = 3x^7 + 11x^6 - 9x^5 + 67x^4 - 25x^3 + 39x^2 + 7x + 3,$
$P_{198} = 3x^7 + 14x^6 - 33x^5 + 224x^4 - 168x^3 + 73x^2 + 10x + 5,$
$P_{199} = 3x^7 + 59x^6 - 477x^5 + 1807x^4 - 1849x^3 + 447x^2 - 77x - 9,$
$P_{200} = 3x^7 + 177x^6 + 507x^5 + 365x^4 - 1153x^3 + 413x^2 - 93x + 37,$
$P_{201} = 4x^7 + 3x^6 + 77x^5 - 316x^4 + 260x^3 - 117x^2 - 27x - 12,$
$P_{202} = 4x^7 + 33x^6 - 113x^5 + 462x^4 - 490x^3 + 93x^2 - 45x - 8,$
$P_{203} = 4x^7 + 35x^6 - 212x^5 + 557x^4 - 655x^3 + 142x^2 - 77x - 18,$
$P_{204} = 5x^7 - 7x^6 - 94x^5 + 45x^4 + 7x^3 - 5x^2 - 10x - 5,$
$P_{205} = 5x^7 + 3x^6 + 103x^5 - 269x^4 + 241x^3 - 123x^2 - 15x - 9,$
$P_{206} = 5x^7 + 5x^6 + 89x^5 - 239x^4 + 267x^3 - 69x^2 + 7x - 1,$
$P_{207} = 5x^7 + 9x^6 + 32x^5 - 314x^4 + 188x^3 - 122x^2 - 63x - 23,$
$P_{208} = 5x^7 + 12x^6 + 32x^5 - 61x^4 + 75x^3 - 22x^2 - 6x - 3,$
$P_{209} = 5x^7 + 15x^6 + 37x^5 - 73x^4 + 59x^3 - 47x^2 - 21x - 7,$
$P_{210} = 5x^7 + 27x^6 - 51x^5 + 247x^4 - 219x^3 + 71x^2 - 15x - 1,$
$P_{211} = 6x^7 + 9x^6 + 87x^5 - 190x^4 + 218x^3 - 67x^2 + 3x - 2,$
$P_{212} = 6x^7 + 17x^6 + 41x^5 + 84x^4 + 84x^3 + 79x^2 + 55x + 18,$
$P_{213} = 7x^7 - 5x^6 + 293x^5 - 1119x^4 + 1133x^3 - 283x^2 + 11x - 5,$
$P_{214} = 7x^7 + 2x^6 + 174x^5 - 21x^4 + 245x^3 - 14x^2 + 94x + 25,$
$P_{215} = 7x^7 + 11x^6 + 99x^5 - 225x^4 + 281x^3 - 59x^2 + 13x + 1,$
$P_{216} = 7x^7 + 13x^6 + 107x^5 - 367x^4 + 409x^3 - 77x^2 + 5x - 1,$
$P_{217} = 7x^7 + 15x^6 + 49x^5 - 143x^4 + 129x^3 - 59x^2 - 21x - 9,$
$P_{218} = 7x^7 + 15x^6 + 53x^5 - 131x^4 + 145x^3 - 43x^2 - 9x - 5,$
$P_{219} = 7x^7 + 19x^6 + 46x^5 - 20x^4 + 62x^3 - 16x^2 - x - 1,$
$P_{220} = 8x^7 - 5x^6 + 351x^5 - 878x^4 + 1004x^3 - 261x^2 + 59x + 10,$
$P_{221} = 10x^7 - 15x^6 + 345x^5 - 1186x^4 + 1158x^3 - 365x^2 + 3x - 14,$
$P_{222} = 10x^7 - 5x^6 + 294x^5 - 496x^4 + 755x^3 - 109x^2 + 116x + 27,$
$P_{223} = 10x^7 + 29x^6 + 29x^5 - 40x^4 + 54x^3 - 19x^2 - 23x - 8,$
$P_{224} = 10x^7 + 29x^6 + 35x^5 - 142x^4 + 156x^3 - 25x^2 - 23x - 8,$
$P_{225} = 11x^7 - 15x^6 + 131x^5 + 1575x^4 - 3873x^3 + 2265x^2 - 13x + 47,$
$P_{226} = 11x^7 - 3x^6 + 317x^5 - 487x^4 + 711x^3 - 157x^2 + 99x + 21,$
$P_{227} = 11x^7 + 12x^6 + 294x^5 - 665x^4 + 777x^3 - 214x^2 + 36x + 5,$
$P_{228} = 11x^7 + 13x^6 + 403x^5 - 1715x^4 + 2781x^3 - 1669x^2 - 43x - 37,$
$P_{229} = 11x^7 + 15x^6 + 174x^5 - 413x^4 + 420x^3 - 169x^2 - 12x - 10,$
$P_{230} = 11x^7 + 55x^6 - 183x^5 + 653x^4 - 639x^3 + 193x^2 - 49x - 9,$

$$P_{231} = 12x^7 + 35x^6 + 69x^5 + 120x^4 + 118x^3 + 101x^2 + 67x + 22,$$
$$P_{232} = 12x^7 + 71x^6 - 253x^5 + 1158x^4 - 1200x^3 + 223x^2 - 89x - 18,$$
$$P_{233} = 13x^7 - 48x^6 + 636x^5 - 1993x^4 + 1965x^3 - 656x^2 + 36x - 17,$$
$$P_{234} = 13x^7 + 19x^6 + 193x^5 - 457x^4 + 555x^3 - 123x^2 + 23x + 1,$$
$$P_{235} = 13x^7 + 26x^6 + 138x^5 - 441x^4 + 455x^3 - 128x^2 - 20x - 11,$$
$$P_{236} = 13x^7 + 32x^6 + 136x^5 - 289x^4 + 317x^3 - 116x^2 - 20x - 9,$$
$$P_{237} = 13x^7 + 75x^6 - 197x^5 + 655x^4 - 543x^3 + 277x^2 - 27x + 3,$$
$$P_{238} = 13x^7 + 119x^6 - 727x^5 + 2363x^4 - 2517x^3 + 617x^2 - 185x - 35,$$
$$P_{239} = 13x^7 + 121x^6 - 755x^5 + 2461x^4 - 2755x^3 + 545x^2 - 247x - 55,$$
$$P_{240} = 13x^7 + 128x^6 - 498x^5 + 2121x^4 - 1897x^3 + 658x^2 - 32x + 19,$$
$$P_{241} = 13x^7 + 132x^6 + 185x^5 + 187x^4 - 215x^3 - 10x^2 + 17x + 11,$$
$$P_{242} = 14x^7 + 31x^6 + 166x^5 - 587x^4 + 657x^3 - 116x^2 - x - 4,$$
$$P_{243} = 14x^7 + 40x^6 + 71x^5 - 165x^4 + 207x^3 - 41x^2 - 22x - 8,$$
$$P_{244} = 15x^7 + 9x^6 + 319x^5 - 967x^4 + 981x^3 - 309x^2 - 3x - 13,$$
$$P_{245} = 15x^7 + 49x^6 - 2x^5 + 34x^4 - 20x^3 + 12x^2 - 43x - 13,$$
$$P_{246} = 16x^7 - 13x^6 + 487x^5 - 1568x^4 + 1540x^3 - 507x^2 + x - 20,$$
$$P_{247} = 16x^7 + 11x^6 + 319x^5 - 862x^4 + 890x^3 - 299x^2 + x - 12,$$
$$P_{248} = 17x^7 + 49x^6 + 127x^5 - 89x^4 + 201x^3 - 47x^2 - x - 1,$$
$$P_{249} = 17x^7 + 63x^6 - 63x^5 + 167x^4 - 209x^3 + 33x^2 - 81x - 23,$$
$$P_{250} = 17x^7 + 69x^6 - 116x^5 + 434x^4 - 448x^3 + 106x^2 - 75x - 19,$$
$$P_{251} = 17x^7 + 70x^6 - 54x^5 + 189x^4 - 77x^3 + 134x^2 - 22x - 1,$$
$$P_{252} = 18x^7 + 30x^6 + 241x^5 - 635x^4 + 677x^3 - 211x^2 - 12x - 12,$$
$$P_{253} = 18x^7 + 43x^6 + 185x^5 - 710x^4 + 752x^3 - 155x^2 - 25x - 12,$$
$$P_{254} = 18x^7 + 53x^6 + 81x^5 - 204x^4 + 190x^3 - 91x^2 - 59x - 20,$$
$$P_{255} = 19x^7 - 46x^6 + 778x^5 - 2375x^4 + 2347x^3 - 798x^2 + 34x - 23,$$
$$P_{256} = 19x^7 + 65x^6 + 47x^5 + 109x^4 - 67x^3 - 17x^2 - 47x - 13,$$
$$P_{257} = 19x^7 + 91x^6 - 279x^5 + 593x^4 - 859x^3 + 89x^2 - 205x - 57,$$
$$P_{258} = 19x^7 + 95x^6 - 173x^5 + 7x^4 - 77x^3 + 123x^2 - 125x - 29,$$
$$P_{259} = 19x^7 + 113x^6 - 261x^5 + 1317x^4 - 1093x^3 + 421x^2 - 17x + 13,$$
$$P_{260} = 20x^7 - 181x^6 + 1831x^5 - 5450x^4 + 5282x^3 - 1951x^2 + 109x - 44,$$
$$P_{261} = 20x^7 + 56x^6 + 131x^5 - 118x^4 + 265x^3 - 26x^2 + 7x + 1,$$
$$P_{262} = 21x^7 - 4x^6 + 554x^5 - 1661x^4 + 1633x^3 - 574x^2 - 8x - 25,$$
$$P_{263} = 21x^7 + 50x^6 + 188x^5 - 565x^4 + 593x^3 - 168x^2 - 38x - 17,$$
$$P_{264} = 21x^7 + 55x^6 + 131x^5 - 375x^4 + 445x^3 - 81x^2 - 25x - 11,$$
$$P_{265} = 22x^7 - 179x^6 + 1753x^5 - 4744x^4 + 5262x^3 - 1383x^2 + 401x + 52,$$
$$P_{266} = 22x^7 - x^6 + 405x^5 - 124x^4 - 50x^3 + 221x^2 - x + 40,$$
$$P_{267} = 23x^7 + 65x^6 + 25x^5 - 129x^4 + 283x^3 + 85x^2 + x - 1,$$
$$P_{268} = 23x^7 + 92x^6 - 436x^5 + 1149x^4 - 1213x^3 + 298x^2 - 22x - 19,$$
$$P_{269} = 24x^7 + 11x^6 + 543x^5 - 1222x^4 + 1236x^3 - 533x^2 - 5x - 22,$$
$$P_{270} = 25x^7 + 49x^6 + 259x^5 - 685x^4 + 699x^3 - 249x^2 - 43x - 23,$$

$$
\begin{aligned}
P_{271} &= 25x^7 + 49x^6 + 271x^5 - 889x^4 + 903x^3 - 261x^2 - 43x - 23,\\
P_{272} &= 25x^7 + 94x^6 - 4x^5 + 567x^4 - 301x^3 + 194x^2 + 20x + 13,\\
P_{273} &= 27x^7 + 17x^6 + 525x^5 - 1825x^4 + 1671x^3 - 635x^2 - 83x - 49,\\
P_{274} &= 27x^7 + 83x^6 + 117x^5 + 117x^4 + 121x^3 + 53x^2 + 19x + 7,\\
P_{275} &= 29x^7 + 57x^6 + 247x^5 - 901x^4 + 887x^3 - 257x^2 - 63x - 31,\\
P_{276} &= 31x^7 + 63x^6 + 253x^5 - 899x^4 + 885x^3 - 263x^2 - 69x - 33,\\
P_{277} &= 31x^7 + 69x^6 + 263x^5 - 783x^4 + 825x^3 - 233x^2 - 51x - 25,\\
P_{278} &= 31x^7 + 85x^6 + 183x^5 - 763x^4 + 749x^3 - 193x^2 - 91x - 33,\\
P_{279} &= 31x^7 + 121x^6 - 53x^5 + 505x^4 - 379x^3 + 143x^2 - 67x - 13,\\
P_{280} &= 32x^7 - 57x^6 + 1229x^5 - 3252x^4 + 3154x^3 - 1299x^2 + 15x - 46,\\
P_{281} &= 34x^7 - 7x^6 + 899x^5 - 2460x^4 + 2446x^3 - 909x^2 + x - 36,\\
P_{282} &= 34x^7 + 101x^6 + 153x^5 - 116x^4 + 270x^3 - 43x^2 - 35x - 12,\\
P_{283} &= 35x^7 - 5x^6 + 918x^5 - 2502x^4 + 2572x^3 - 868x^2 + 35x - 25,\\
P_{284} &= 37x^7 + 78x^6 - 64x^5 + 1211x^4 - 2191x^3 + 1360x^2 + 50x + 31,\\
P_{285} &= 37x^7 + 104x^6 + 326x^5 - 271x^4 + 495x^3 - 166x^2 - 8x - 5,\\
P_{286} &= 38x^7 + 29x^6 + 807x^5 - 1678x^4 + 1748x^3 - 757x^2 + x - 28,\\
P_{287} &= 40x^7 + 167x^6 - 262x^5 + 911x^4 - 897x^3 + 272x^2 - 161x - 38,\\
P_{288} &= 42x^7 + 71x^6 + 557x^5 - 1378x^4 + 1490x^3 - 477x^2 - 23x - 26,\\
P_{289} &= 43x^7 + 77x^6 + 485x^5 - 1245x^4 + 1567x^3 - 255x^2 + 61x + 3,\\
P_{290} &= 43x^7 + 135x^6 + 177x^5 + 261x^4 + 145x^3 + 113x^2 + 39x + 15,\\
P_{291} &= 46x^7 + 141x^6 + 284x^5 + 193x^4 + 325x^3 + 86x^2 + 81x + 28,\\
P_{292} &= 47x^7 + 3x^6 + 543x^5 + 259x^4 - 1821x^3 + 1571x^2 + 15x + 23,\\
P_{293} &= 50x^7 + 209x^6 - 257x^5 + 2042x^4 - 1776x^3 + 447x^2 - 95x - 12,\\
P_{294} &= 51x^7 + 121x^6 + 375x^5 - 1683x^4 + 1613x^3 - 425x^2 - 151x - 61,\\
P_{295} &= 55x^7 + 216x^6 - 2082x^5 + 6547x^4 - 5893x^3 + 1446x^2 + 24x + 71,\\
P_{296} &= 56x^7 + 101x^6 + 925x^5 - 3030x^4 + 2512x^3 - 1295x^2 - 323x - 130,\\
P_{297} &= 65x^7 + 345x^6 - 1061x^5 + 5123x^4 - 4969x^3 + 1171x^2 - 279x - 43,\\
P_{298} &= 67x^7 + 74x^6 + 1200x^5 - 3387x^4 + 3527x^3 - 1100x^2 - 14x - 47,\\
P_{299} &= 77x^7 + 151x^6 + 1175x^5 - 2699x^4 + 3217x^3 - 805x^2 + 71x - 3,\\
P_{300} &= 91x^7 + 153x^6 + 833x^5 - 2047x^4 + 3597x^3 - 1845x^2 + 15x - 29,\\
P_{301} &= 96x^7 + 179x^6 + 1307x^5 - 2262x^4 + 2682x^3 - 1007x^2 + x - 36,\\
P_{302} &= 98x^7 + 115x^6 + 1791x^5 - 3676x^4 + 3984x^3 - 1571x^2 + 17x - 54,\\
P_{303} &= 111x^7 + 60x^6 + 836x^5 - 3631x^4 + 6545x^3 - 2672x^2 + 84x - 53,\\
P_{304} &= 119x^7 + 177x^6 + 1763x^5 - 4297x^4 + 4773x^3 - 1423x^2 + 27x - 51,\\
P_{305} &= 139x^7 + 512x^6 - 90x^5 + 3131x^4 - 2207x^3 + 750x^2 - 116x - 7,\\
P_{306} &= 141x^7 + 544x^6 - 346x^5 + 4169x^4 - 3273x^3 + 986x^2 - 160x - 13,\\
P_{307} &= 146x^7 + 1297x^6 - 1553x^5 + 8418x^4 - 9754x^3 + 3745x^2 - 563x - 200,\\
P_{308} &= 181x^7 + 479x^6 + 1355x^5 - 2119x^4 + 2973x^3 - 745x^2 - 113x - 59,\\
P_{309} &= 543x^7 + 587x^6 + 2403x^5 + 3623x^4 - 3279x^3 + 277x^2 - 195x + 137,\\
P_{310} &= 649x^7 + 1613x^6 + 5451x^5 - 6985x^4 + 11899x^3 - 1941x^2 + 493x + 53\,. \qquad \text{(B.2)}
\end{aligned}
$$

$$F_{V,2}^{(1)} = C_F\Bigg[\Big\{ -4x\eta H_0\Big\} + \varepsilon\Big\{\eta\Big(-16xH_0 + 8xH_{-1}H_0 - 2xH_0^2 - 8xH_{0,-1} + 4x\zeta_2\Big)\Big\}$$
$$+ \varepsilon^2\Big\{\eta\Big(\Big(-32x + 32xH_{-1} - 8xH_{-1}^2 + 2x\zeta_2\Big)H_0 + \Big(-8x + 4xH_{-1}\Big)H_0^2 - \frac{2}{3}xH_0^3 + \Big($$
$$-32x + 16xH_{-1}\Big)H_{0,-1} - 8xH_{0,0,-1} - 16xH_{0,-1,-1} + 16x\zeta_2 - 8xH_{-1}\zeta_2 + 8x\zeta_3\Big)\Big\} + \varepsilon^3\Big\{\eta\Big(\Big($$
$$-64x + \Big(64x - 4x\zeta_2\Big)H_{-1} - 32xH_{-1}^2 + \frac{16}{3}xH_{-1}^3 + 8x\zeta_2 + \frac{28}{3}x\zeta_3\Big)H_0 + \Big(-16x + 16xH_{-1}$$
$$-4xH_{-1}^2 + x\zeta_2\Big)H_0^2 + \Big(-\frac{8x}{3} + \frac{4xH_{-1}}{3}\Big)H_0^3 - \frac{1}{6}xH_0^4 + \Big(-32x\zeta_2 - 16x\zeta_3\Big)H_{-1} + \Big(-64x$$
$$+64xH_{-1} - 16xH_{-1}^2 - 4x\zeta_2\Big)H_{0,-1} + \Big(-32x + 16xH_{-1}\Big)H_{0,0,-1} + \Big(-64x + 32xH_{-1}\Big)H_{0,-1,-1}$$
$$-8xH_{0,0,0,-1} - 16xH_{0,0,-1,-1} - 32xH_{0,-1,-1,-1} + 32x\zeta_2 + 8xH_{-1}^2\zeta_2 + \frac{28}{5}x\zeta_2^2 + 32x\zeta_3\Big)\Big\}\Bigg]. \tag{B.3}$$

$$F_{V,2}^{(2)} = C_F^2\Bigg[\frac{1}{\varepsilon}\Big\{8x\eta H_0 + 8x\xi\eta H_0^2\Big\} + \Big\{\xi^2\eta^3\Big(-\frac{4}{3}(-1+x)^2x^2H_0^4 - 192(-1+x)^2x^2H_{0,-1}\zeta_2\Big)$$
$$+\eta^3x_+^2\Big(384x^2H_{0,0,0,-1}P_{358} + 32x^2H_0^2H_{0,-1}P_{362} + \frac{16}{3}x^4H_0^4 + \Big(32x^2H_0^2P_{334} - 1536x^4H_{0,-1}\Big)\zeta_2$$
$$+\frac{96}{5}x^2P_{372}\zeta_2^2\Big) + \xi\Big(\eta^3x_+\Big(4(-1+x)x^3H_0^4 - 1728(-1+x)x^3H_{0,-1}\zeta_2\Big) + \eta\Big(\Big(128(-1+x)xH_0H_1$$
$$-128(-1+x)xH_{0,1}\Big)x_+ + \Big(-64(-1+x)x^2H_0^3H_1 + 1344(-1+x)x^2H_{0,0,0,1} + \Big($$
$$-384(-1+x)x^2H_0H_1 + 384(-1+x)x^2H_{0,1}\Big)\zeta_2 + 448(-1+x)x^2H_0\zeta_3\Big)x_+^3\Big)\Big)$$
$$+\eta^2\Big(\Big(32xH_{0,0,1}P_{313} - 8xH_{-1}H_0^2P_{314} + 48xH_0H_{0,-1}P_{315} - 80xH_{0,0,-1}P_{317} - 16xH_0^2H_1P_{327}$$
$$+2xH_0^2P_{450} + 48xH_{-1}P_{355}\zeta_2 + 16xP_{316}\zeta_3\Big)x_+^2 + \Big(-\frac{8}{3}xH_0^3P_{471} - 16xH_0P_{473}\zeta_2\Big)x_+^3\Big)$$
$$+\eta\Big(-4x\Big(-15 - 173x + 323x^2 + x^3\Big)x_+^3\zeta_2 + 62xH_0 + \Big(24x\Big(13 - 38x + 13x^2\Big)H_{-1}H_0$$
$$+128x^2H_0H_1 - 128x^2H_{0,1} - 24x\Big(13 - 38x + 13x^2\Big)H_{0,-1}\Big)x_+^2 + \Big(-64x^3H_0^3H_1$$
$$-32x^2\Big(11 - 41x + 11x^2\Big)H_0^2H_{0,1} + 512x^2\Big(2 - 11x + 2x^2\Big)H_0H_{0,0,1} - 64x^2\Big(19 - 73x$$
$$+19x^2\Big)H_0H_{0,0,-1} + 9408x^3H_{0,0,0,1} + \Big(-384x^3H_0H_1 + 384x^3H_{0,1}\Big)\zeta_2 + 448x^3H_0\zeta_3\Big)x_+^4\Big)$$
$$+1824x^2H_0H_{0,1}x_+^4 - 192x\log(2)x_+^2\zeta_2\Big\} + \varepsilon\Big\{\xi^2\eta^3\Big(-\frac{8}{3}(-1+x)^2x^2H_{-1}H_0^4$$
$$-4096(-1+x)^2x^2H_{0,0,-1,0,-1} + \Big(-384(-1+x)^2x^2H_{-1}H_{0,-1} + 384(-1+x)^2x^2H_{0,1,-1}\Big)\zeta_2\Big)$$
$$+\xi\Big(\eta^2x_+\Big(-64(-1+x)x\Big(3 + 22x + 3x^2\Big)H_1H_{0,0,1} - 32(-1+x)x\Big(23 - 20x + 23x^2\Big)H_{-1}H_{0,0,-1}$$
$$+64(-1+x)x\Big(3 + 22x + 3x^2\Big)H_{0,0,1,1} + 32(-1+x)x\Big(23 - 20x + 23x^2\Big)H_{0,0,-1,-1}\Big)$$

$$+\eta^3 x_+\Big(8(-1+x)x^3H_{-1}H_0^4+\frac{4}{15}(-1+x)x^2\Big(3-4x+3x^2\Big)H_0^5+256(-1+x)x^2\Big(29$$
$$-61x+29x^2\Big)H_0H_{0,0,1,-1}+256(-1+x)x^2\Big(29-61x+29x^2\Big)H_0H_{0,0,-1,1}$$
$$-8192(-1+x)x^3H_{0,0,-1,0,-1}+\Big(32(-1+x)x^2\Big(19-31x+19x^2\Big)H_0^2H_1$$
$$-3456(-1+x)x^3H_{-1}H_{0,-1}-2688(-1+x)x^3H_{0,1,-1}\Big)\zeta_2\Big)+\eta\Big(\Big(-768(-1+x)xH_{-1}H_0H_1$$
$$+128(-1+x)xH_0H_1^2+\Big(-256(-1+x)xH_1+768(-1+x)xH_{-1}\Big)H_{0,1}+768(-1+x)xH_1H_{0,-1}$$
$$+256(-1+x)xH_{0,1,1}-768(-1+x)xH_{0,1,-1}-768(-1+x)xH_{0,-1,1}-128(-1+x)xH_1\zeta_2\Big)x_+$$
$$+\Big(-128(-1+x)x^2H_{-1}H_0^3H_1+64(-1+x)x^2H_0^3H_1^2-128(-1+x)x^2H_0^2H_1H_{0,1}$$
$$-1024(-1+x)x^2H_0H_1H_{0,0,1}+1024(-1+x)x^2H_0H_1H_{0,0,-1}+2688(-1+x)x^2H_{-1}H_{0,0,0,1}$$
$$-1024(-1+x)x^2H_0H_{0,0,1,1}+\Big(-768(-1+x)x^2H_{-1}H_0H_1+384(-1+x)x^2H_0H_1^2+\Big($$
$$-768(-1+x)x^2H_1+768(-1+x)x^2H_{-1}\Big)H_{0,1}+768(-1+x)x^2H_{0,1,1}\Big)\zeta_2+\Big(896(-1+x)x^2H_{-1}H_0$$
$$+256(-1+x)x^2H_0H_1\Big)\zeta_3\Big)x_+^3\Big)\Big)+\eta^3\Big(\log(2)x_+^2\zeta_2\Big(1536(-2+x)x^3(-1+2x)H_{0,1}$$
$$+1536(-2+x)x^3(-1+2x)H_{0,-1}\Big)+64x^2\Big(3-5x+3x^2\Big)H_0^2H_{0,1,1}+\Big(384x^2H_0H_{0,-1}^2P_{333}$$
$$+384x^2H_0^2H_{0,1,-1}P_{335}+384x^2H_0^2H_{0,-1,1}P_{335}-256x^2H_0H_{0,1}^2P_{349}+\frac{8}{3}x^2H_0^4H_1P_{364}$$
$$+1536x^2H_{0,0,0,1,-1}P_{383}+1536x^2H_{0,0,0,-1,1}P_{383}-768x^2H_{0,0,0,-1,-1}P_{396}-128x^2H_0H_{0,-1,0,1}P_{400}$$
$$-64x^2H_0^2H_{0,-1,-1}P_{404}-128x^2H_{0,0,1,0,1}P_{415}+128x^2H_{0,0,1,0,-1}P_{423}+128x^2H_{0,0,-1,0,1}P_{436}$$
$$+64x^2H_{0,0,0,0,1}P_{455}-64x^2H_{0,0,0,0,-1}P_{460}+\frac{2}{3}xH_0^4P_{492}-\frac{2}{3}xH_0^3P_{506}+\frac{32}{3}x^4H_{-1}H_0^4+\frac{4}{15}x^3\Big(3$$
$$-4x+3x^2\Big)H_0^5+\Big(256x^2H_0H_{0,-1}P_{380}-\frac{64}{3}x^2H_0^3P_{391}+16xH_0^2P_{485}\Big)H_{0,1}+\Big(64x^2H_{-1}H_0^2P_{362}$$
$$-64x^2H_0^2H_1P_{363}+\frac{32}{3}x^2H_0^3P_{414}-8xH_0^2P_{495}\Big)H_{0,-1}+\Big(-128x^2H_{0,-1}P_{375}+128x^2H_{0,1}P_{405}$$
$$+32x^2H_0^2P_{430}-64xH_0P_{478}\Big)H_{0,0,1}+\Big(-512x^2H_{0,-1}P_{318}-128x^2H_{0,1}P_{427}-32x^2H_0^2P_{441}$$
$$-16xH_0P_{487}\Big)H_{0,0,-1}+\Big(-64x^2H_0P_{443}-32xP_{491}\Big)H_{0,0,0,1}+\Big(768x^2H_{-1}P_{358}+384x^2H_1P_{378}$$
$$+64x^2H_0P_{452}+16xP_{502}\Big)H_{0,0,0,-1}+1024x^3\Big(29-61x+29x^2\Big)H_0H_{0,0,1,-1}+1024x^3\Big(29$$
$$-61x+29x^2\Big)H_0H_{0,0,-1,1}+128x^2\Big(5-11x+5x^2\Big)\Big(11-34x+11x^2\Big)H_0H_{0,0,-1,-1}$$
$$-384x^2\Big(5-6x+5x^2\Big)\Big(7-13x+7x^2\Big)H_{0,0,0,1,1}-4096x^4H_{0,0,-1,0,-1}+\Big(-64x^2H_{0,0,1}P_{312}$$
$$+384x^2H_{0,-1,1}P_{325}+512x^2H_{0,0,-1}P_{332}+64x^2H_{-1}H_0^2P_{334}+384x^2H_{0,-1,-1}P_{353}$$
$$+128x^2H_0H_{0,1}P_{354}+\frac{32}{3}x^2H_0^3P_{369}-8xH_0^2P_{484}+4xH_0P_{507}+128x^3\Big(19-31x+19x^2\Big)H_0^2H_1$$
$$+\Big(-384x^2H_1P_{325}-128x^2H_0P_{331}-16xP_{499}-3072x^4H_{-1}\Big)H_{0,-1}-3072x^4H_{0,1,-1}$$
$$-64x^2P_{386}\zeta_3\Big)\zeta_2+\Big(\frac{192}{5}x^2H_{-1}P_{372}-\frac{96}{5}x^2H_1P_{426}-\frac{16}{5}x^2H_0P_{462}+\frac{8xP_{496}}{5}\Big)\zeta_2^2+\Big($$
$$-16x^2H_0^2P_{356}+128x^2H_{0,-1}P_{370}+64x^2H_{0,1}P_{394}+16xH_0P_{479}\Big)\zeta_3+96x^2P_{419}\zeta_5\Big)x_+^2\Big)$$

$$+\eta^2\Big(\log(2)x_+^2\zeta_2\Big(-384xH_1P_{324}-384xH_{-1}P_{324}-384xP_{344}\Big)+\Big(8xH_{-1}^2H_0^2P_{311}$$
$$+16xH_0^2H_1^2P_{319}-64xH_{0,-1,0,1}P_{328}+128xH_{0,0,1,-1}P_{336}+128xH_{0,0,-1,1}P_{336}+64xH_0H_{0,1,1}P_{337}$$
$$-64xH_0H_{0,1,-1}P_{339}-64xH_0H_{0,-1,1}P_{339}+16xH_{0,-1}^2P_{340}+32xH_0H_{0,-1,-1}P_{371}-4xH_{-1}H_0^2P_{459}$$
$$+2xH_0^2P_{466}+\Big(64xH_{-1}H_0^2P_{324}+8xH_0^2P_{401}\Big)H_1+\Big(-128xH_0H_1P_{322}+64xH_{-1}H_0P_{326}$$
$$+64xH_{0,-1}P_{330}+64xH_0P_{343}\Big)H_{0,1}+\Big(-384xH_{-1}H_0P_{323}+64xH_0H_1P_{338}-16xH_0P_{413}\Big)H_{0,-1}$$
$$+\Big(-128xH_{-1}P_{336}-16xP_{420}-64x^2\Big(3+22x+3x^2\Big)H_1\Big)H_{0,0,1}+\Big(-128xH_1P_{346}+8xP_{464}$$
$$+128x^2\Big(23-20x+23x^2\Big)H_{-1}\Big)H_{0,0,-1}+64x^2\Big(3+22x+3x^2\Big)H_{0,0,1,1}-128x^2\Big(23-20x$$
$$+23x^2\Big)H_{0,0,-1,-1}+\Big(384xH_{-1}H_1P_{324}-384xH_{-1,1}P_{324}-48xH_{-1}^2P_{373}+24xH_{-1}P_{425}$$
$$+4xP_{428}\Big)\zeta_2+\Big(32xH_1P_{347}+32xH_{-1}P_{390}-16xP_{440}\Big)\zeta_3\Big)x_+^2+\Big(-\frac{16}{3}xH_0^3H_1P_{470}$$
$$+\frac{8}{3}xH_{-1}H_0^3P_{472}+\Big(16xH_{-1}H_0P_{476}-32(-2+x)x\Big(5+39x-129x^2+101x^3\Big)H_{0,1}\Big)\zeta_2\Big)x_+^3\Big)$$
$$+\eta\Big(\Big(3x\Big(83+206x+83x^2\Big)H_0+32x\Big(41-60x+41x^2\Big)H_{-1}H_0-24x\Big(51-122x$$
$$+51x^2\Big)H_{-1}^2H_0+\Big(-176x\Big(3+2x+3x^2\Big)H_0-768x^2H_{-1}H_0\Big)H_1+128x^2H_0H_1^2+\Big(176x\Big(3$$
$$+2x+3x^2\Big)-256x^2H_1+768x^2H_{-1}\Big)H_{0,1}+\Big(-32x\Big(41-60x+41x^2\Big)+768x^2H_1$$
$$+48x\Big(51-122x+51x^2\Big)H_{-1}\Big)H_{0,-1}+256x^2H_{0,1,1}-768x^2H_{0,1,-1}-768x^2H_{0,-1,1}-48x\Big(51$$
$$-122x+51x^2\Big)H_{0,-1,-1}-128x^2H_1\zeta_2\Big)x_+^2+\Big(-128x^3H_{-1}H_0^3H_1+64x^3H_0^3H_1^2+\Big($$
$$-64x^2\Big(11-41x+11x^2\Big)H_{-1}H_0^2+640x^3H_0^2H_1\Big)H_{0,1}+\Big(1024x^2\Big(2-11x+2x^2\Big)H_{-1}H_0$$
$$-4096x^3H_0H_1\Big)H_{0,0,1}+\Big(-128x^2\Big(19-73x+19x^2\Big)H_{-1}H_0+4096x^3H_0H_1\Big)H_{0,0,-1}+\Big($$
$$-1536(-2+x)x^2(-1+2x)H_1+18816x^3H_{-1}\Big)H_{0,0,0,1}-1024x^3H_0H_{0,0,1,1}+\Big(-32xH_0H_1P_{329}$$
$$-768x^3H_{-1}H_0H_1+384x^3H_0H_1^2+\Big(-768x^3H_1+768x^3H_{-1}\Big)H_{0,1}+768x^3H_{0,1,1}\Big)\zeta_2$$
$$+\Big(896x^3H_{-1}H_0+1024x^3H_0H_1\Big)\zeta_3\Big)x_+^4\Big)-20xx_+^2+32c_1xx_+^2-1056x^2H_{0,1}^2x_+^4\Big\}\Big]$$
$$+C_FC_A\Big[\Big\{\eta^2x_+^2\Big(-4xH_{-1}H_0^2P_{382}+8xH_0H_{0,-1}P_{392}-8xH_{0,0,-1}P_{397}-\frac{2}{3}xH_0^2P_{402}+\Big($$
$$-24xH_{-1}P_{355}-\frac{4xP_{451}}{3}\Big)\zeta_2\Big)+\xi^2\eta^3\Big(\frac{2}{3}(-1+x)^2x^2H_0^4+96(-1+x)^2x^2H_{0,-1}\zeta_2\Big)+\xi\eta^3x_+\Big($$
$$-2(-1+x)x^3H_0^4+864(-1+x)x^3H_{0,-1}\zeta_2\Big)+\eta^3x_+^2\Big(16x^2H_0^2H_{0,-1}P_{389}+96x^2H_{0,0,0,-1}P_{398}$$
$$-\frac{4}{3}x^2H_0^3P_{474}-\frac{8}{3}x^4H_0^4+\Big(8x^2H_0^2P_{367}-8x^2H_0P_{475}+768x^4H_{0,-1}\Big)\zeta_2+\frac{24}{5}x^2P_{417}\zeta_2^2\Big)+\eta\Big(\Big($$
$$-\frac{2}{9}x\Big(173+670x+173x^2\Big)H_0+\frac{8}{3}x\Big(5-8x+5x^2\Big)H_{-1}H_0-\frac{8}{3}x\Big(5-8x+5x^2\Big)H_{0,-1}\Big)x_+^2$$
$$+\Big(-32x^2\Big(17-47x+17x^2\Big)\zeta_3H_0+32x^2\Big(3-8x+3x^2\Big)H_0^3H_1-80x^2\Big(5-8x+5x^2\Big)H_0^2H_{0,1}$$
$$+128x^2\Big(8-5x+8x^2\Big)H_0H_{0,0,1}-32x^2\Big(41-62x+41x^2\Big)H_0H_{0,0,-1}-96x^2\Big(13+4x+13x^2\Big)H_{0,0,0,1}$$

$$+\Big(192x^2\Big(3-8x+3x^2\Big)H_0H_1-192x^2\Big(3-8x+3x^2\Big)H_{0,1}\Big)\zeta_2\Big)x_+^4\Big)+12xx_+^2+\Big($$
$$-48(-1+x)xH_0H_1+48(-1+x)xH_{0,1}\Big)x_+^3+\Big(16x\Big(2+x+2x^2\Big)H_0^2H_1-32x\Big(2+7x$$
$$+2x^2\Big)H_0H_{0,1}+32x\Big(2+13x+2x^2\Big)H_{0,0,1}+16x\Big(5-71x+5x^2\Big)\zeta_3\Big)x_+^4+96x\log(2)x_+^2\zeta_2\Bigg\}$$
$$+\varepsilon\Bigg\{\xi^2\Big(\eta^3\Big(\frac{4}{3}(-1+x)^2x^2H_{-1}H_0^4+192(-1+x)^2x^2H_{-1}H_{0,-1}\zeta_2\Big)+64(-1+x)^2x\eta^2H_{-1}H_0H_{0,-1}\Big)$$
$$+\eta^2\Big(\log(2)x_+^2\zeta_2\Big(192xH_1P_{324}+192xH_{-1}P_{324}+192xP_{344}\Big)+\Big(-32xH_0^2H_1^2P_{320}$$
$$+128xH_{0,0,1,1}P_{341}-128xH_0H_{0,1,1}P_{342}+64xH_{0,-1,0,1}P_{345}-64xH_{0,0,1,-1}P_{357}-64xH_{0,0,-1,1}P_{357}$$
$$+8xH_{-1}H_0^2P_{384}+16xH_{0,0,-1,-1}P_{411}+8xH_{0,-1}^2P_{418}-16xH_0H_{0,-1,-1}P_{421}-\frac{2}{9}xH_0^2P_{469}$$
$$-32x^2\Big(31-64x+31x^2\Big)H_{-1}^2H_0^2+\Big(-32xH_{-1}H_0^2P_{324}-32x^2\Big(-4+23x-16x^2+6x^3\Big)H_0^2\Big)H_1$$
$$+\Big(64xH_0H_1P_{348}-32xH_{-1}H_0P_{352}+32xH_0P_{361}-32xH_{0,-1}P_{379}\Big)H_{0,1}+\Big(-8xH_0P_{408}$$
$$+768x^3H_{-1}H_0\Big)H_{0,-1}+\Big(-64xP_{321}-128xH_1P_{341}+64xH_{-1}P_{357}\Big)H_{0,0,1}+\Big(64xH_1P_{376}$$
$$+16xP_{381}-16xH_{-1}P_{411}\Big)H_{0,0,-1}+\Big(-192xH_{-1}H_1P_{324}+192xH_{-1,1}P_{324}+24xH_{-1}^2P_{373}$$
$$-16xH_{-1}P_{374}-\frac{4xP_{468}}{9}\Big)\zeta_2+\Big(-16xH_1P_{377}-16xH_{-1}P_{395}-8xP_{429}\Big)\zeta_3\Big)x_+^2\Big)+\eta^3\Big(\log(2)x_+^2\zeta_2\Big($$
$$-768(-2+x)x^3(-1+2x)H_{0,1}-768(-2+x)x^3(-1+2x)H_{0,-1}\Big)+\Big(96x^2H_0^2H_{0,1,-1}P_{366}$$
$$+96x^2H_0^2H_{0,-1,1}P_{366}+96x^2H_0H_{0,-1}^2P_{368}-64x^2H_0H_{0,1}^2P_{407}+32x^2H_0^2H_{0,1,1}P_{410}$$
$$-128x^2H_{0,0,-1,0,-1}P_{416}-128x^2H_0H_{0,0,1,-1}P_{431}-128x^2H_0H_{0,0,-1,1}P_{431}-32x^2H_0^2H_{0,-1,-1}P_{432}$$
$$+384x^2H_{0,0,0,1,-1}P_{433}+384x^2H_{0,0,0,-1,1}P_{433}-64x^2H_0H_{0,-1,0,1}P_{434}-192x^2H_{0,0,0,1,1}P_{439}$$
$$-192x^2H_{0,0,0,-1,-1}P_{444}-64x^2H_{0,0,1,0,1}P_{445}+64x^2H_0H_{0,0,-1,-1}P_{447}+64x^2H_{0,0,1,0,-1}P_{449}$$
$$+64x^2H_{0,0,-1,0,1}P_{457}+32x^2H_{0,0,0,0,0,1}P_{461}-32x^2H_{0,0,0,0,0,-1}P_{465}-\frac{1}{3}x^2H_0^4P_{477}+\frac{2}{3}xH_0^3P_{481}-\frac{2}{15}x^3\Big(3$$
$$-4x+3x^2\Big)H_0^5+\Big(\frac{4}{3}x^2H_0^4P_{387}+\frac{16}{3}xH_0^3P_{480}\Big)H_1+\Big(\frac{4}{3}xH_0^3P_{500}-\frac{16}{3}x^4H_0^4\Big)H_{-1}+\Big(-\frac{16}{3}x^2H_0^3P_{435}$$
$$+64x^2H_0H_{0,-1}P_{437}-16xH_0^2P_{482}\Big)H_{0,1}+\Big(-32x^2H_0^2H_1P_{388}+32x^2H_{-1}H_0^2P_{389}+\frac{16}{3}x^2H_0^3P_{442}$$
$$-4xH_0^2P_{488}\Big)H_{0,-1}+\Big(-64x^2H_{0,-1}P_{399}+64x^2H_{0,1}P_{438}+16x^2H_0^2P_{448}-32xH_0P_{483}\Big)H_{0,0,1}$$
$$+\Big(256x^2H_{0,-1}P_{318}-64x^2H_{0,1}P_{454}-16x^2H_0^2P_{458}+8xH_0P_{497}\Big)H_{0,0,-1}+\Big(-32x^2H_0P_{456}$$
$$+32xP_{494}\Big)H_{0,0,0,1}+\Big(192x^2H_{-1}P_{398}+192x^2H_1P_{422}+32x^2H_0P_{463}-8xP_{504}\Big)H_{0,0,0,-1}$$
$$+\Big(32x^2H_{0,0,1}P_{312}-256x^2H_{0,0,-1}P_{332}-192x^2H_{0,-1,-1}P_{353}+192x^2H_{0,-1,1}P_{359}+192x^2H_{0,1,-1}P_{360}$$
$$+\frac{8}{3}x^2H_0^3P_{393}+4xH_0^2P_{498}-2xH_0P_{505}+\Big(-16x^2H_0^2P_{424}+32xH_0P_{489}\Big)H_1+\Big(16x^2H_0^2P_{367}$$
$$+8xH_0P_{486}\Big)H_{-1}+\Big(32x^2H_0P_{403}-32xP_{490}\Big)H_{0,1}+\Big(-192x^2H_1P_{359}+32x^2H_0P_{385}+8xP_{501}$$
$$+1536x^4H_{-1}\Big)H_{0,-1}-32x^2P_{365}\zeta_3\Big)\zeta_2+\Big(\frac{48}{5}x^2H_{-1}P_{417}-\frac{48}{5}x^2H_1P_{453}-\frac{8}{5}x^2H_0P_{467}+\frac{4xP_{503}}{5}\Big)\zeta_2^2$$

$$+\Big(64x^2H_{0,-1}P_{406}-8x^2H_0^2P_{409}+32x^2H_{0,1}P_{412}-16xH_0P_{493}\Big)\zeta_3+48x^2P_{446}\zeta_5\Big)x_+^2\Big)$$
$$+\xi\Big(\eta^2x_+\Big(-4(-1+x)x\Big(31-64x+31x^2\Big)H_{-1}^2H_0^2+448(-1+x)x^2H_{-1}H_0H_{0,-1}\Big)$$
$$+\eta x_+^3\Big(64(-1+x)x^2H_0^2H_1H_{0,1}+512(-1+x)x^2H_0H_{0,0,1,1}\Big)+\eta^3x_+\Big(-4(-1+x)x^3H_{-1}H_0^4$$
$$-\frac{2}{15}(-1+x)x^2\Big(3-4x+3x^2\Big)H_0^5+1728(-1+x)x^3H_{-1}H_{0,-1}\zeta_2\Big)+\Big(-576(-1+x)xH_0H_{0,1,-1}$$
$$-576(-1+x)xH_0H_{0,-1,1}\Big)x_+^3\Big)+\eta\Big(\Big(-\frac{7}{27}x\Big(1151+3922x+1151x^2\Big)H_0-\frac{4}{9}x\Big(749-2246x$$
$$+749x^2\Big)H_{-1}H_0+8x\Big(7-16x+7x^2\Big)H_{-1}^2H_0+32x\Big(11-19x+11x^2\Big)H_0H_1-32x\Big(11$$
$$-19x+11x^2\Big)H_{0,1}+\Big(\frac{4}{9}x\Big(749-2246x+749x^2\Big)-16x\Big(7-16x+7x^2\Big)H_{-1}\Big)H_{0,-1}$$
$$+16x\Big(7-16x+7x^2\Big)H_{0,-1,-1}\Big)x_+^2+\Big(64x^2\Big(3-8x+3x^2\Big)H_{-1}H_0^3H_1-32x^2\Big(3-8x$$
$$+3x^2\Big)H_0^3H_1^2+\Big(-160x^2\Big(5-8x+5x^2\Big)H_{-1}H_0^2-320x^3H_0^2H_1\Big)H_{0,1}+\Big(256x^2\Big(8-5x$$
$$+8x^2\Big)H_{-1}H_0+256x^2\Big(10-19x+10x^2\Big)H_0H_1\Big)H_{0,0,1}+\Big(-64x^2\Big(41-62x+41x^2\Big)H_{-1}H_0$$
$$-256x^2\Big(10-19x+10x^2\Big)H_0H_1\Big)H_{0,0,-1}+\Big(-384x^2\Big(16-35x+16x^2\Big)H_1-192x^2\Big(13+4x$$
$$+13x^2\Big)H_{-1}\Big)H_{0,0,0,1}+512x^3H_0H_{0,0,1,1}+\Big(384x^2\Big(3-8x+3x^2\Big)H_{-1}H_0H_1-192x^2\Big(3-8x$$
$$+3x^2\Big)H_0H_1^2+\Big(384x^2\Big(3-8x+3x^2\Big)H_1-384x^2\Big(3-8x+3x^2\Big)H_{-1}\Big)H_{0,1}-384x^2\Big(3-8x$$
$$+3x^2\Big)H_{0,1,1}\Big)\zeta_2+\Big(-64x^2\Big(17-47x+17x^2\Big)H_{-1}H_0-64x^2\Big(10-19x+10x^2\Big)H_0H_1\Big)\zeta_3\Big)x_+^4\Big)$$
$$+78xx_+^2-16c_1xx_+^2+\Big(288(-1+x)xH_{-1}H_0H_1-48(-1+x)xH_0H_1^2+\Big(96(-1+x)xH_1$$
$$-288(-1+x)xH_{-1}\Big)H_{0,1}-288(-1+x)xH_1H_{0,-1}-96(-1+x)xH_{0,1,1}+288(-1+x)xH_{0,1,-1}$$
$$+288(-1+x)xH_{0,-1,1}+48(-1+x)xH_1\zeta_2\Big)x_+^3+\Big(32(-2+x)x(-1+2x)H_{0,1}^2$$
$$-32x\Big(11-62x+11x^2\Big)H_0H_1H_{0,-1}-2880x^2H_0H_{0,1,-1}-2880x^2H_0H_{0,-1,1}\Big)x_+^4\Big\}\Big]$$
$$+C_Fn_lT_F\Big[\eta\Big\{\frac{200}{9}xH_0-\frac{32}{3}xH_{-1}H_0+\frac{8}{3}xH_0^2+\frac{32}{3}xH_{0,-1}-\frac{16x\zeta_2}{3}\Big\}+\varepsilon\eta\Big\{\frac{3844}{27}xH_0$$
$$+32xH_{-1}^2H_0+\frac{296}{9}xH_0^2+\frac{8}{3}xH_0^3+\Big(-\frac{1184}{9}xH_0-16xH_0^2\Big)H_{-1}+\Big(\frac{1184x}{9}$$
$$-64xH_{-1}\Big)H_{0,-1}+32xH_{0,0,-1}+64xH_{0,-1,-1}+\Big(-\frac{592x}{9}+8xH_0+32xH_{-1}\Big)\zeta_2-32x\zeta_3\Big\}\Big]$$
$$+C_FT_F\Big[\Big\{\xi x_+^3\Big(-\frac{8}{3}(-1+x)xH_0^2+16(-1+x)x\zeta_2\Big)+\eta\Big(\frac{8}{9}x\Big(25-322x+25x^2\Big)x_+^2H_0$$
$$+\Big(-32x^3H_0^3-192x^3H_0\zeta_2\Big)x_+^4\Big)+\frac{272xx_+^2}{3}+\Big(-\frac{160}{3}x^2H_0^2-576x^2\zeta_2\Big)x_+^4\Big\}$$

$$
\begin{aligned}
&+\varepsilon\Bigg\{\eta\Big(\Big(\frac{4}{27}x\Big(961-4138x+961x^2\Big)H_0-\frac{16}{9}x\Big(25-322x+25x^2\Big)H_{-1}H_0+\frac{16}{9}x\Big(25\\
&-322x+25x^2\Big)H_{0,-1}\Big)x_+^2+\Big(-\frac{16}{9}x\Big(-16+684x-981x^2+41x^3\Big)\zeta_2-\frac{8}{9}x\Big(-37\\
&+273x+24x^2+12x^3\Big)H_0^2\Big)x_+^3+\Big(-\frac{8}{9}xH_0^3P_{351}-64x^3H_{-1}H_0^3-24x^3H_0^4-384x^3H_0^2H_{0,1}\\
&+384x^3H_0^2H_{0,-1}+1536x^3H_0H_{0,0,1}-768x^3H_0H_{0,0,-1}-2304x^3H_{0,0,0,1}+384x^3H_{0,0,0,-1}\\
&+\Big(-\frac{8}{3}xH_0P_{350}-384x^3H_{-1}H_0-96x^3H_0^2+1536x^3H_{0,-1}\Big)\zeta_2+\frac{96}{5}x^3\zeta_2^2+384x^3H_0\zeta_3\Big)x_+^4\Big)\\
&+\xi\Big(\Big(\frac{16}{3}(-1+x)xH_{-1}H_0^2-\frac{32}{3}(-1+x)xH_0^2H_1+\frac{64}{3}(-1+x)xH_0H_{0,1}-\frac{64}{3}(-1+x)xH_{0,0,1}\\
&-\frac{32}{3}(-1+x)xH_{0,0,-1}+32(-1+x)xH_{-1}\zeta_2\Big)x_+^3-128(-1+x)x\log(2)x_+^3\zeta_2\Big)\\
&+\frac{1528xx_+^2}{9}+\Big(\frac{320}{3}x^2H_{-1}H_0^2-\frac{640}{3}x^2H_0^2H_1+\frac{1280}{3}x^2H_0H_{0,1}-\frac{1280}{3}x^2H_{0,0,1}\\
&-\frac{640}{3}x^2H_{0,0,-1}+640x^2H_{-1}\zeta_2-\frac{64}{3}x\Big(4+67x+4x^2\Big)\zeta_3\Big)x_+^4+2816x^2\log(2)x_+^4\zeta_2\Bigg\}\Bigg].
\end{aligned}
$$

The polynomials are given by

$$
\begin{aligned}
P_{311}&=x^4-96x^3+178x^2-96x+1,\\
P_{312}&=x^4-69x^3+106x^2-69x+1,\\
P_{313}&=x^4-67x^3+140x^2-67x+1,\\
P_{314}&=x^4-64x^3+122x^2-64x+1,\\
P_{315}&=x^4-54x^3+102x^2-54x+1,\\
P_{316}&=x^4-52x^3+62x^2-52x+1,\\
P_{317}&=x^4-52x^3+98x^2-52x+1,\\
P_{318}&=x^4-29x^3+51x^2-29x+1,\\
P_{319}&=x^4-27x^3+64x^2-27x+1,\\
P_{320}&=x^4-20x^3+39x^2-20x+1,\\
P_{321}&=x^4-18x^3+61x^2-30x+7,\\
P_{322}&=x^4-2x^3+12x^2-2x+1,\\
P_{323}&=x^4-x^3+6x^2-x+1,\\
P_{324}&=x^4+5x^3-10x^2+5x+1,\\
P_{325}&=x^4+15x^3-26x^2+15x+1,\\
P_{326}&=x^4+22x^3-18x^2+22x+1,\\
P_{327}&=x^4+47x^3-88x^2+47x+1,\\
P_{328}&=x^4+53x^3-112x^2+53x+1,\\
P_{329}&=x^4+93x^3-222x^2+91x+9,\\
P_{330}&=x^4+115x^3-228x^2+115x+1,\\
P_{331}&=2x^4-44x^3+61x^2-44x+2,\\
P_{332}&=2x^4-33x^3+53x^2-33x+2,\\
P_{333}&=2x^4-16x^3+27x^2-16x+2,\\
P_{334}&=2x^4-10x^3+9x^2-10x+2,
\end{aligned}
$$

$$P_{335} = 2x^4 - 10x^3 + 17x^2 - 10x + 2,$$
$$P_{336} = 2x^4 + 27x^3 - 28x^2 + 27x + 2,$$
$$P_{337} = 2x^4 + 29x^3 - 42x^2 + 29x + 2,$$
$$P_{338} = 2x^4 + 75x^3 - 130x^2 + 75x + 2,$$
$$P_{339} = 2x^4 + 137x^3 - 246x^2 + 137x + 2,$$
$$P_{340} = 3x^4 - 410x^3 + 770x^2 - 410x + 3,$$
$$P_{341} = 3x^4 - 35x^3 + 65x^2 - 35x + 3,$$
$$P_{342} = 3x^4 - 32x^3 + 59x^2 - 32x + 3,$$
$$P_{343} = 3x^4 - 26x^3 - 12x^2 - 26x + 3,$$
$$P_{344} = 3x^4 - 4x^3 + 4x^2 - 4x + 3,$$
$$P_{345} = 3x^4 + 2x^3 - 8x^2 + 2x + 3,$$
$$P_{346} = 3x^4 + 80x^3 - 140x^2 + 80x + 3,$$
$$P_{347} = 3x^4 + 202x^3 - 388x^2 + 202x + 3,$$
$$P_{348} = 4x^4 - 55x^3 + 104x^2 - 55x + 4,$$
$$P_{349} = 4x^4 - 8x^3 + 9x^2 - 8x + 4,$$
$$P_{350} = 5x^4 - 172x^3 - 54x^2 - 12x - 3,$$
$$P_{351} = 5x^4 - 124x^3 - 24x^2 + 36x - 3,$$
$$P_{352} = 5x^4 - 88x^3 + 162x^2 - 88x + 5,$$
$$P_{353} = 5x^4 - 53x^3 + 70x^2 - 53x + 5,$$
$$P_{354} = 5x^4 - 41x^3 + 61x^2 - 41x + 5,$$
$$P_{355} = 5x^4 - 12x^3 + 26x^2 - 12x + 5,$$
$$P_{356} = 5x^4 - 9x^3 + 20x^2 - 9x + 5,$$
$$P_{357} = 6x^4 - 83x^3 + 152x^2 - 83x + 6,$$
$$P_{358} = 6x^4 - 40x^3 + 67x^2 - 40x + 6,$$
$$P_{359} = 7x^4 - 49x^3 + 78x^2 - 49x + 7,$$
$$P_{360} = 7x^4 - 41x^3 + 58x^2 - 41x + 7,$$
$$P_{361} = 7x^4 - 34x^3 + 84x^2 - 34x + 7,$$
$$P_{362} = 7x^4 - 31x^3 + 50x^2 - 31x + 7,$$
$$P_{363} = 7x^4 - 7x^3 + 2x^2 - 7x + 7,$$
$$P_{364} = 7x^4 + 9x^3 - 26x^2 + 9x + 7,$$
$$P_{365} = 7x^4 + 21x^3 - 11x^2 + 21x + 7,$$
$$P_{366} = 8x^4 - 31x^3 + 44x^2 - 31x + 8,$$
$$P_{367} = 8x^4 - 31x^3 + 60x^2 - 31x + 8,$$
$$P_{368} = 8x^4 - 19x^3 + 24x^2 - 19x + 8,$$
$$P_{369} = 8x^4 - 12x^3 + 11x^2 - 12x + 8,$$
$$P_{370} = 8x^4 - 4x^3 + 9x^2 - 4x + 8,$$
$$P_{371} = 9x^4 + 398x^3 - 698x^2 + 398x + 9,$$
$$P_{372} = 11x^4 - 77x^3 + 109x^2 - 77x + 11,$$
$$P_{373} = 11x^4 - 56x^3 + 118x^2 - 56x + 11,$$
$$P_{374} = 11x^4 - 35x^3 + 267x^2 - 91x + 40,$$

$$P_{375} = 11x^4 + 29x^3 - 84x^2 + 29x + 11,$$
$$P_{376} = 12x^4 - 79x^3 + 136x^2 - 79x + 12,$$
$$P_{377} = 12x^4 + 43x^3 - 112x^2 + 43x + 12,$$
$$P_{378} = 13x^4 - 49x^3 + 74x^2 - 49x + 13,$$
$$P_{379} = 13x^4 - 38x^3 + 54x^2 - 38x + 13,$$
$$P_{380} = 14x^4 - 62x^3 + 97x^2 - 62x + 14,$$
$$P_{381} = 14x^4 - 55x^3 + 325x^2 - 107x + 39,$$
$$P_{382} = 15x^4 - 140x^3 + 254x^2 - 140x + 15,$$
$$P_{383} = 15x^4 - 67x^3 + 106x^2 - 67x + 15,$$
$$P_{384} = 15x^4 + 17x^3 - 115x^2 + 69x - 10,$$
$$P_{385} = 16x^4 - 139x^3 + 200x^2 - 139x + 16,$$
$$P_{386} = 17x^4 - 123x^3 + 167x^2 - 123x + 17,$$
$$P_{387} = 17x^4 - 111x^3 + 182x^2 - 111x + 17,$$
$$P_{388} = 17x^4 - 95x^3 + 154x^2 - 95x + 17,$$
$$P_{389} = 17x^4 - 71x^3 + 106x^2 - 71x + 17,$$
$$P_{390} = 17x^4 - 9x^3 + 50x^2 - 9x + 17,$$
$$P_{391} = 18x^4 - 82x^3 + 133x^2 - 82x + 18,$$
$$P_{392} = 19x^4 - 148x^3 + 254x^2 - 148x + 19,$$
$$P_{393} = 20x^4 - 129x^3 + 212x^2 - 129x + 20,$$
$$P_{394} = 21x^4 - 221x^3 + 398x^2 - 221x + 21,$$
$$P_{395} = 21x^4 - 119x^3 + 230x^2 - 119x + 21,$$
$$P_{396} = 22x^4 - 72x^3 + 115x^2 - 72x + 22,$$
$$P_{397} = 23x^4 - 156x^3 + 254x^2 - 156x + 23,$$
$$P_{398} = 24x^4 - 73x^3 + 100x^2 - 73x + 24,$$
$$P_{399} = 25x^4 - 182x^3 + 318x^2 - 182x + 25,$$
$$P_{400} = 25x^4 - 165x^3 + 286x^2 - 165x + 25,$$
$$P_{401} = 25x^4 - 24x^3 + 54x^2 + 40x - 39,$$
$$P_{402} = 25x^4 + 8x^3 - 240x^2 + 64x - 1,$$
$$P_{403} = 26x^4 - 71x^3 + 112x^2 - 71x + 26,$$
$$P_{404} = 27x^4 - 107x^3 + 170x^2 - 107x + 27,$$
$$P_{405} = 27x^4 - 71x^3 + 92x^2 - 71x + 27,$$
$$P_{406} = 28x^4 - 149x^3 + 225x^2 - 149x + 28,$$
$$P_{407} = 28x^4 - 137x^3 + 216x^2 - 137x + 28,$$
$$P_{408} = 29x^4 - 38x^3 + 210x^2 - 38x + 29,$$
$$P_{409} = 31x^4 - 144x^3 + 214x^2 - 144x + 31,$$
$$P_{410} = 33x^4 - 154x^3 + 238x^2 - 154x + 33,$$
$$P_{411} = 39x^4 - 368x^3 + 686x^2 - 368x + 39,$$
$$P_{412} = 39x^4 - 34x^3 - 8x^2 - 34x + 39,$$
$$P_{413} = 41x^4 - 274x^3 + 130x^2 - 274x + 41,$$
$$P_{414} = 41x^4 - 121x^3 + 186x^2 - 121x + 41,$$

$$P_{415} = 43x^4 - 143x^3 + 204x^2 - 143x + 43,$$
$$P_{416} = 44x^4 - 223x^3 + 342x^2 - 223x + 44,$$
$$P_{417} = 46x^4 - 135x^3 + 224x^2 - 135x + 46,$$
$$P_{418} = 47x^4 - 316x^3 + 534x^2 - 316x + 47,$$
$$P_{419} = 48x^4 - 250x^3 + 391x^2 - 250x + 48,$$
$$P_{420} = 49x^4 - 232x^3 - 42x^2 - 168x - 15,$$
$$P_{421} = 51x^4 - 344x^3 + 590x^2 - 344x + 51,$$
$$P_{422} = 51x^4 - 223x^3 + 342x^2 - 223x + 51,$$
$$P_{423} = 53x^4 - 205x^3 + 312x^2 - 205x + 53,$$
$$P_{424} = 53x^4 - 199x^3 + 306x^2 - 199x + 53,$$
$$P_{425} = 53x^4 - 174x^3 + 356x^2 + 6x + 15,$$
$$P_{426} = 57x^4 - 325x^3 + 490x^2 - 325x + 57,$$
$$P_{427} = 57x^4 - 245x^3 + 386x^2 - 245x + 57,$$
$$P_{428} = 59x^4 + 1366x^3 - 2290x^2 + 710x + 123,$$
$$P_{429} = 61x^4 - 128x^3 + 134x^2 - 76x + 33,$$
$$P_{430} = 67x^4 - 367x^3 + 614x^2 - 367x + 67,$$
$$P_{431} = 67x^4 - 231x^3 + 322x^2 - 231x + 67,$$
$$P_{432} = 69x^4 - 301x^3 + 454x^2 - 301x + 69,$$
$$P_{433} = 70x^4 - 291x^3 + 438x^2 - 291x + 70,$$
$$P_{434} = 71x^4 - 243x^3 + 338x^2 - 243x + 71,$$
$$P_{435} = 72x^4 - 295x^3 + 436x^2 - 295x + 72,$$
$$P_{436} = 73x^4 - 317x^3 + 498x^2 - 317x + 73,$$
$$P_{437} = 80x^4 - 335x^3 + 508x^2 - 335x + 80,$$
$$P_{438} = 81x^4 - 388x^3 + 610x^2 - 388x + 81,$$
$$P_{439} = 81x^4 - 386x^3 + 606x^2 - 386x + 81,$$
$$P_{440} = 89x^4 - 453x^3 + 539x^2 - 143x - 8,$$
$$P_{441} = 91x^4 - 419x^3 + 684x^2 - 419x + 91,$$
$$P_{442} = 103x^4 - 491x^3 + 750x^2 - 491x + 103,$$
$$P_{443} = 107x^4 - 591x^3 + 986x^2 - 591x + 107,$$
$$P_{444} = 112x^4 - 519x^3 + 784x^2 - 519x + 112,$$
$$P_{445} = 113x^4 - 520x^3 + 810x^2 - 520x + 113,$$
$$P_{446} = 120x^4 - 464x^3 + 701x^2 - 464x + 120,$$
$$P_{447} = 125x^4 - 474x^3 + 686x^2 - 474x + 125,$$
$$P_{448} = 125x^4 - 449x^3 + 634x^2 - 449x + 125,$$
$$P_{449} = 127x^4 - 560x^3 + 858x^2 - 560x + 127,$$
$$P_{450} = 127x^4 - 314x^3 + 256x^2 - 54x + 17,$$
$$P_{451} = 143x^4 - 620x^3 + 732x^2 - 460x + 61,$$
$$P_{452} = 151x^4 - 795x^3 + 1318x^2 - 795x + 151,$$
$$P_{453} = 151x^4 - 559x^3 + 862x^2 - 559x + 151,$$
$$P_{454} = 159x^4 - 673x^3 + 1018x^2 - 673x + 159,$$

$$P_{455} = 163x^4 - 827x^3 + 1350x^2 - 827x + 163,$$
$$P_{456} = 181x^4 - 633x^3 + 886x^2 - 633x + 181,$$
$$P_{457} = 191x^4 - 805x^3 + 1218x^2 - 805x + 191,$$
$$P_{458} = 209x^4 - 856x^3 + 1266x^2 - 856x + 209,$$
$$P_{459} = 215x^4 - 306x^3 - 148x^2 + 522x - 187,$$
$$P_{460} = 221x^4 - 1153x^3 + 1914x^2 - 1153x + 221,$$
$$P_{461} = 245x^4 - 907x^3 + 1302x^2 - 907x + 245,$$
$$P_{462} = 303x^4 - 1763x^3 + 2306x^2 - 1763x + 303,$$
$$P_{463} = 305x^4 - 1143x^3 + 1646x^2 - 1143x + 305,$$
$$P_{464} = 379x^4 - 1402x^3 + 372x^2 - 574x - 23,$$
$$P_{465} = 379x^4 - 1397x^3 + 1986x^2 - 1397x + 379,$$
$$P_{466} = 497x^4 - 686x^3 + 638x^2 - 294x + 37,$$
$$P_{467} = 513x^4 - 1705x^3 + 2998x^2 - 1705x + 513,$$
$$P_{468} = 1277x^4 - 3170x^3 + 1683x^2 - 2680x + 442,$$
$$P_{469} = 1666x^4 - 3184x^3 - 189x^2 + 430x + 125,$$
$$P_{470} = x^5 + 64x^4 + 185x^3 - 443x^2 + 238x + 11,$$
$$P_{471} = 8x^5 + 46x^4 - 23x^3 + x^2 - x - 3,$$
$$P_{472} = 9x^5 + 237x^4 - 112x^3 - 160x^2 + 331x + 31,$$
$$P_{473} = 14x^5 + 43x^4 - 2x^3 + 22x^2 - 4x + 3,$$
$$P_{474} = 19x^5 - 176x^4 + 382x^3 - 356x^2 + 91x - 8,$$
$$P_{475} = 19x^5 - 138x^4 + 314x^3 - 356x^2 + 159x - 46,$$
$$P_{476} = 23x^5 - 137x^4 + 208x^3 + 160x^2 - 43x + 45,$$
$$P_{477} = 31x^5 - 220x^4 + 488x^3 - 560x^2 + 153x - 20,$$
$$P_{478} = x^6 + 33x^5 - 265x^4 + 592x^3 - 557x^2 + 199x + 5,$$
$$P_{479} = 2x^6 + 159x^5 - 718x^4 + 1124x^3 - 616x^2 + 69x - 12,$$
$$P_{480} = 3x^6 - 158x^5 + 520x^4 - 742x^3 + 507x^2 - 172x + 18,$$
$$P_{481} = 3x^6 + 360x^5 - 1283x^4 + 1288x^3 - 5x^2 - 12x + 1,$$
$$P_{482} = 4x^6 - 121x^5 + 529x^4 - 846x^3 + 594x^2 - 177x + 21,$$
$$P_{483} = 4x^6 - 51x^5 - 262x^4 + 760x^3 - 684x^2 + 203x - 26,$$
$$P_{484} = 5x^6 + 323x^5 - 654x^4 + 738x^3 - 185x^2 - 29x + 2,$$
$$P_{485} = 6x^6 + 7x^5 - 316x^4 + 906x^3 - 906x^2 + 303x + 8,$$
$$P_{486} = 7x^6 + 44x^5 - 207x^4 + 712x^3 - 739x^2 + 324x - 45,$$
$$P_{487} = 9x^6 + 210x^5 + 697x^4 - 2900x^3 + 3103x^2 - 1134x - 17,$$
$$P_{488} = 9x^6 + 232x^5 - 1443x^4 + 3096x^3 - 2669x^2 + 1032x - 113,$$
$$P_{489} = 10x^6 - 187x^5 + 555x^4 - 742x^3 + 472x^2 - 143x + 11,$$
$$P_{490} = 10x^6 - 181x^5 + 543x^4 - 742x^3 + 484x^2 - 149x + 11,$$
$$P_{491} = 13x^6 - 114x^5 + 763x^4 - 1462x^3 + 1305x^2 - 512x - 17,$$
$$P_{492} = 16x^6 - 14x^5 + 33x^4 - 180x^3 + 64x^2 - 6x + 7,$$
$$P_{493} = 18x^6 - 301x^5 + 883x^4 - 980x^3 + 490x^2 - 79x - 3,$$
$$P_{494} = 21x^6 - 358x^5 + 281x^4 + 484x^3 - 777x^2 + 250x - 33,$$

$$
\begin{aligned}
P_{495} &= 23x^6 - 342x^5 + 199x^4 + 1128x^3 - 1799x^2 + 726x + 17,\\
P_{496} &= 38x^6 - 1856x^5 + 4771x^4 - 9562x^3 + 6836x^2 - 2070x - 149,\\
P_{497} &= 39x^6 + 12x^5 - 1857x^4 + 4240x^3 - 3727x^2 + 1204x - 135,\\
P_{498} &= 45x^6 - 704x^5 + 1622x^4 - 1268x^3 + 327x^2 + 36x - 2,\\
P_{499} &= 47x^6 + 66x^5 - 173x^4 - 816x^3 + 697x^2 - 378x + 29,\\
P_{500} &= 65x^6 - 604x^5 + 891x^4 + 712x^3 - 1837x^2 + 972x - 103,\\
P_{501} &= 77x^6 - 464x^5 + 909x^4 - 1576x^3 + 703x^2 - 288x + 15,\\
P_{502} &= 105x^6 - 168x^5 + 2339x^4 - 5364x^3 + 4403x^2 - 1524x - 31,\\
P_{503} &= 147x^6 - 3672x^5 + 12192x^4 - 15040x^3 + 9997x^2 - 3256x + 376,\\
P_{504} &= 155x^6 - 1264x^5 - 351x^4 + 4144x^3 - 5011x^2 + 1488x - 169,\\
P_{505} &= 159x^6 - 750x^5 + 1921x^4 - 2228x^3 + 469x^2 - 294x + 19,\\
P_{506} &= 165x^6 - 710x^5 - 709x^4 + 1724x^3 - 665x^2 + 306x - 47,\\
P_{507} &= 327x^6 - 458x^5 + 1457x^4 - 1020x^3 - 499x^2 + 158x - 29
\end{aligned}
\tag{B.4}
$$

C The axial-vector form factor up to two-loop

In this appendix, we present the axial-vector form factors $F_{A,1}^{(n)}$ and $F_{A,2}^{(n)}$ up to two loops and $\mathcal{O}(\varepsilon)$.

$$
\begin{aligned}
F_{A,1}^{(1)} = C_F \Bigg[& \frac{1}{\varepsilon}\Big\{ -2-2\xi H_0 \Big\} + \Big\{ -4+\xi\Big(4H_{-1}H_0 - H_0^2 - 4H_{0,-1} + 2\zeta_2\Big) + \Big(-3+2x-3x^2\Big)\eta H_0 \Big\} \\
& + \varepsilon\Big\{ -8 + \eta\Big(2\Big(3-2x+3x^2\Big)H_{-1}H_0 + \frac{1}{2}\Big(-3+2x-3x^2\Big)H_0^2 - 2\Big(3-2x+3x^2\Big)H_{0,-1} \\
& + 2\Big(1-x+2x^2\Big)\zeta_2\Big) + \xi\Big(\Big(-8-4H_{-1}^2+\zeta_2\Big)H_0 + 2H_{-1}H_0^2 - \frac{1}{3}H_0^3 + 8H_{-1}H_{0,-1} \\
& - 4H_{0,0,-1} - 8H_{0,-1,-1} - 4H_{-1}\zeta_2 + 4\zeta_3\Big)\Big\} + \varepsilon^2\Big\{ -16 + \eta\Big(\Big(-2\Big(3-2x+3x^2\Big)H_{-1}^2 \\
& + \frac{1}{2}\Big(3-2x+3x^2\Big)\zeta_2\Big)H_0 + \Big(3-2x+3x^2\Big)H_{-1}H_0^2 + \frac{1}{6}\Big(-3+2x-3x^2\Big)H_0^3 + 4\Big(3-2x \\
& + 3x^2\Big)H_{-1}H_{0,-1} - 2\Big(3-2x+3x^2\Big)H_{0,0,-1} - 4\Big(3-2x+3x^2\Big)H_{0,-1,-1} + 2\Big(3+5x^2\Big)\zeta_2 - 2\Big(3 \\
& - 2x+3x^2\Big)H_{-1}\zeta_2 + \frac{4}{3}\Big(5-3x+4x^2\Big)\zeta_3\Big) + \xi\Big(\Big(-16+\Big(16-2\zeta_2\Big)H_{-1} + \frac{8}{3}H_{-1}^3 + \frac{14}{3}\zeta_3\Big)H_0 \\
& + \Big(-4-2H_{-1}^2+\frac{1}{2}\zeta_2\Big)H_0^2 + \frac{2}{3}H_{-1}H_0^3 - \frac{1}{12}H_0^4 + \Big(-16-8H_{-1}^2-2\zeta_2\Big)H_{0,-1} + 8H_{-1}H_{0,0,-1} \\
& + 16H_{-1}H_{0,-1,-1} - 4H_{0,0,0,-1} - 8H_{0,0,-1,-1} - 16H_{0,-1,-1,-1} + 4H_{-1}^2\zeta_2 + \frac{14}{5}\zeta_2^2 - 8H_{-1}\zeta_3\Big)\Big\} \\
& + \varepsilon^3\Big\{ -32 + \eta\Big(\Big(\frac{4}{3}\Big(3-2x+3x^2\Big)H_{-1}^3 + \Big(-3+2x-3x^2\Big)H_{-1}\zeta_2 + \frac{7}{3}\Big(3-2x+3x^2\Big)\zeta_3\Big)H_0 + \Big(\Big(\\
& - 3+2x-3x^2\Big)H_{-1}^2 + \frac{1}{4}\Big(3-2x+3x^2\Big)\zeta_2 + \Big(-1-x^2\Big)H_{-1}\zeta_2\Big)H_0^2 + \frac{1}{3}\Big(3-2x+3x^2\Big)H_{-1}H_0^3 \\
& + \frac{1}{24}\Big(-3+2x-3x^2\Big)H_0^4 + \Big(-4\Big(3-2x+3x^2\Big)H_{-1}^2 + \Big(-3+2x-3x^2\Big)\zeta_2\Big)H_{0,-1} + 4\Big(3 \\
& - 2x+3x^2\Big)H_{-1}H_{0,0,-1} + 8\Big(3-2x+3x^2\Big)H_{-1}H_{0,-1,-1} - 2\Big(3-2x+3x^2\Big)H_{0,0,0,-1} - 4\Big(3 \\
& - 2x+3x^2\Big)H_{0,0,-1,-1} - 8\Big(3-2x+3x^2\Big)H_{0,-1,-1,-1} + 4\Big(3+5x^2\Big)\zeta_2 + 2\Big(3-2x+3x^2\Big)H_{-1}^2\zeta_2 \\
& + \frac{1}{20}\Big(75-56x+93x^2\Big)\zeta_2^2 + \frac{4}{3}\Big(13+11x^2\Big)\zeta_3 - 4\Big(3-2x+3x^2\Big)H_{-1}\zeta_3\Big) + \xi\Big(\Big(-32+\Big(32 \\
& - \frac{28\zeta_3}{3}\Big)H_{-1} + \Big(-16+2\zeta_2\Big)H_{-1}^2 - \frac{4}{3}H_{-1}^4 + 4\zeta_2 + \frac{47}{20}\zeta_2^2\Big)H_0 + \Big(-8+8H_{-1} + \frac{4}{3}H_{-1}^3 \\
& + \frac{7}{3}\zeta_3\Big)H_0^2 + \Big(\frac{1}{6}\Big(-8+\zeta_2\Big) - \frac{2}{3}H_{-1}^2\Big)H_0^3 + \frac{1}{6}H_{-1}H_0^4 - \frac{1}{60}H_0^5 + \Big(-16\zeta_2 - \frac{28\zeta_2^2}{5}\Big)H_{-1} + \Big(\\
& - 32 + \Big(32+4\zeta_2\Big)H_{-1} + \frac{16}{3}H_{-1}^3 + \frac{4}{3}\zeta_3\Big)H_{0,-1} + \Big(-16-8H_{-1}^2-2\zeta_2\Big)H_{0,0,-1} + \Big(-32 \\
& - 16H_{-1}^2 - 4\zeta_2\Big)H_{0,-1,-1} + 8H_{-1}H_{0,0,0,-1} + 16H_{-1}H_{0,0,-1,-1} + 32H_{-1}H_{0,-1,-1,-1} - 4H_{0,0,0,0,-1} \\
& - 8H_{0,0,0,-1,-1} - 16H_{0,0,-1,-1,-1} - 32H_{0,-1,-1,-1,-1} - \frac{8}{3}H_{-1}^3\zeta_2 + 8H_{-1}^2\zeta_3 - \frac{8}{3}\zeta_2\zeta_3 + 12\zeta_5\Big)\Big\}\Bigg] .
\end{aligned}
\tag{C.1}
$$

$$F_{A,1}^{(2),\mathrm{ns}} = C_F^2\Bigg[\frac{1}{\varepsilon^2}\Big\{2+4\xi H_0+2\xi^2H_0^2\Big\}+\frac{1}{\varepsilon}\Big\{8+\xi\Big(-8H_{-1}H_0+4\Big(2-x+x^2\Big)\eta H_0^2+8H_{0,-1}-4\zeta_2\Big)$$
$$+\xi^2\Big(-8H_{-1}H_0^2+2H_0^3+8H_0H_{0,-1}-4H_0\zeta_2\Big)+2\Big(7-2x+7x^2\Big)\eta H_0\Big\}+\Big\{46+\eta\Big(\frac{1}{2}\Big(85$$
$$-6x+85x^2\Big)H_0+2\Big(55-82x+55x^2\Big)H_{-1}H_0-32\Big(2-x+2x^2\Big)H_0H_1+32\Big(2-x$$
$$+2x^2\Big)H_{0,1}-2\Big(55-82x+55x^2\Big)H_{0,-1}\Big)+\xi\Big(8H_{-1}^2H_0-16H_{-1}H_{0,-1}+16H_{0,-1,-1}$$
$$+48(-1+x)\log(2)x_+\zeta_2\Big)+\eta^2\Big(-4H_{-1}H_0^2P_{528}+4H_0^2H_1P_{552}+8H_0H_{0,-1}P_{559}-8H_0H_{0,1}P_{563}$$
$$+8H_{0,0,1}P_{574}-8H_{0,0,-1}P_{576}+\frac{1}{2}H_0^2P_{603}+\Big(\frac{16}{3}H_0^3H_1P_{614}+\Big(16H_0H_1P_{612}-16H_{0,1}P_{612}\Big)\zeta_2\Big)x_+$$
$$+\Big(39-146x-46x^3+57x^4+8H_{-1}P_{536}\Big)\zeta_2-4P_{547}\zeta_3\Big)+\eta^3\Big(\frac{1}{6}H_0^4P_{643}+4H_0^2H_{0,-1}P_{658}$$
$$+8H_0H_{0,0,-1}P_{670}+8H_0^2H_{0,1}P_{673}-48H_{0,0,0,-1}P_{674}+\frac{4}{3}H_0^3P_{690}-16H_0H_{0,0,1}P_{698}+16H_{0,0,0,1}P_{727}$$
$$+\Big(4xH_0P_{632}-4H_0^2P_{699}+8H_{0,-1}P_{701}\Big)\zeta_2-\frac{2}{5}P_{811}\zeta_2^2-16H_0P_{667}\zeta_3\Big)+\xi^2\Big(16H_{-1}^2H_0^2-8H_{-1}H_0^3$$
$$+\Big(32H_0H_1-32H_{0,-1}\Big)H_{0,1}+16H_{0,1}^2-32H_{-1}H_0H_{0,-1}-32H_0H_1H_{0,-1}+8H_{0,-1}^2-64H_1H_{0,0,1}$$
$$+64H_1H_{0,0,-1}-64H_0H_{0,1,1}+32H_0H_{0,1,-1}+32H_0H_{0,-1,1}+16H_0H_{0,-1,-1}+64H_{0,0,1,1}$$
$$+32H_{0,-1,0,1}+16H_{-1}H_0\zeta_2+16H_1\zeta_3\Big)+192x\log(2)x_+^2\zeta_2\Big\}+\varepsilon\Big\{\xi\Big(\eta^2\log(2)\zeta_2\Big(192x^2H_{0,1}$$
$$+192x^2H_{0,-1}\Big)-\frac{16}{3}H_{-1}^3H_0+\Big(32(-1+x^2)x\Big(10-21x+10x^2\Big)\eta^3\Big)H_0H_{0,1}^2+16H_{-1}^2H_{0,-1}$$
$$-32H_{-1}H_{0,-1,-1}+32H_{0,-1,-1,-1}-4(-1+x)x_+\Big)+\eta\Big(\frac{1}{4}\Big(479+70x+479x^2\Big)H_0+4\Big(81$$
$$-146x+81x^2\Big)H_{-1}H_0-2\Big(257-286x+257x^2\Big)H_{-1}^2H_0+\Big(-4\Big(43-18x+43x^2\Big)H_0$$
$$+192\Big(2-x+2x^2\Big)H_{-1}H_0\Big)H_1-32\Big(2-x+2x^2\Big)H_0H_1^2+\Big(4\Big(43-18x+43x^2\Big)+64\Big(2$$
$$-x+2x^2\Big)H_1-192\Big(2-x+2x^2\Big)H_{-1}\Big)H_{0,1}+\Big(-4\Big(81-146x+81x^2\Big)-192\Big(2-x$$
$$+2x^2\Big)H_1+4\Big(257-286x+257x^2\Big)H_{-1}\Big)H_{0,-1}-64\Big(2-x+2x^2\Big)H_{0,1,1}+192\Big(2-x$$
$$+2x^2\Big)H_{0,1,-1}+192\Big(2-x+2x^2\Big)H_{0,-1,1}-4\Big(257-286x+257x^2\Big)H_{0,-1,-1}+32\Big(2$$
$$-x+2x^2\Big)H_1\zeta_2\Big)+\eta^2\Big(H_{-1}H_0^2P_{509}+4H_{-1}^2H_0^2P_{522}+16H_0H_{0,1,1}P_{525}-32H_0H_{0,-1,-1}P_{542}$$
$$-16H_{0,0,1,1}P_{544}+8H_{0,1}^2P_{553}+16H_0H_{0,1,-1}P_{567}+16H_0H_{0,-1,1}P_{567}+4H_0^2H_1^2P_{568}+16H_{0,-1,0,1}P_{573}$$
$$-16H_{0,0,-1,-1}P_{581}+8H_{0,-1}^2P_{585}+\frac{1}{2}H_0^2P_{605}+\log(2)\zeta_2\Big(-24P_{512}+48\Big(3-2x^2+3x^4\Big)H_1+48\Big(3$$
$$-2x^2+3x^4\Big)H_{-1}\Big)+\Big(2H_0^2P_{596}-8\Big(3-2x^2+3x^4\Big)H_{-1}H_0^2\Big)H_1+\Big(32H_0P_{520}-16H_0H_1P_{560}$$
$$-16H_{0,-1}P_{583}+160\Big(1-4x^2+x^4\Big)H_{-1}H_0\Big)H_{0,1}+\Big(-16H_0H_1P_{543}-16H_{-1}H_0P_{564}$$
$$-8H_0P_{590}\Big)H_{0,-1}+\Big(16H_1P_{544}-4P_{600}-16\Big(17-78x^2+17x^4\Big)H_{-1}\Big)H_{0,0,1}+\Big(16H_1P_{571}$$
$$+16H_{-1}P_{581}+2P_{606}\Big)H_{0,0,-1}+16\Big(17-78x^2+17x^4\Big)H_{0,0,1,-1}+16\Big(17-78x^2+17x^4\Big)H_{0,0,-1,1}$$

$$+\Big(-\frac{32}{3}H_{-1}H_0^3H_1P_{618}+\frac{16}{3}H_0^3H_1^2P_{618}+\Big(-64H_{-1}H_0H_1P_{616}+32H_0H_1^2P_{616}+64H_{0,1,1}P_{618}$$
$$+\Big(64H_{-1}P_{616}-32H_1P_{620}\Big)H_{0,1}\Big)\zeta_2\Big)x_+ +\Big(163-514x+82x^2-74x^3+183x^4-8H_{-1}^2P_{584}$$
$$+2H_{-1}P_{598}-48\Big(3-2x^2+3x^4\Big)H_{-1}H_1+48\Big(3-2x^2+3x^4\Big)H_{-1,1}\Big)\zeta_2+\Big(4H_{-1}P_{587}-4H_1P_{597}$$
$$-\frac{2P_{607}}{3}\Big)\zeta_3\Big)+\xi^2\Big(-\frac{64}{3}H_{-1}^3H_0^2+16H_{-1}^2H_0^3+\Big(128H_{-1}H_0H_{0,-1}-96H_{0,-1}^2\Big)H_1+\Big(\Big(-128H_{-1}H_0$$
$$+64H_{0,-1}\Big)H_1+64H_0H_1^2+128H_{-1}H_{0,-1}\Big)H_{0,1}+\Big(32H_1-64H_{-1}\Big)H_{0,1}^2+64H_{-1}^2H_0H_{0,-1}$$
$$-64H_0H_1^2H_{0,-1}-32H_{-1}H_{0,-1}^2+\Big(256H_{-1}H_1-128H_1^2\Big)H_{0,0,1}+\Big(-256H_{-1}H_1+128H_1^2\Big)H_{0,0,-1}$$
$$+\Big(256H_{-1}H_0-192H_0H_1-256H_{0,-1}\Big)H_{0,1,1}+\Big(-128H_{-1}H_0+64H_0H_1-64H_{0,1}$$
$$+192H_{0,-1}\Big)H_{0,1,-1}+\Big(-128H_{-1}H_0+64H_0H_1+192H_{0,1}+192H_{0,-1}\Big)H_{0,-1,1}+\Big(-64H_{-1}H_0$$
$$+64H_0H_1-320H_{0,1}+32H_{0,-1}\Big)H_{0,-1,-1}+\Big(256H_1-256H_{-1}\Big)H_{0,0,1,1}-256H_1H_{0,0,1,-1}$$
$$-256H_1H_{0,0,-1,1}+256H_1H_{0,0,-1,-1}+192H_0H_{0,1,1,1}-64H_0H_{0,1,-1,-1}-128H_{-1}H_{0,-1,0,1}$$
$$-64H_0H_{0,-1,1,-1}-64H_0H_{0,-1,-1,1}+32H_0H_{0,-1,-1,-1}-384H_{0,0,1,1,1}+768H_{0,0,1,1,-1}$$
$$+256H_{0,0,1,-1,1}-256H_{0,0,-1,1,1}-64H_{0,1,0,1,1}+256H_{0,1,0,1,-1}-256H_{0,-1,0,1,1}+128H_{0,-1,0,1,-1}$$
$$+128H_{0,-1,0,-1,1}-256H_{0,-1,1,0,1}+320H_{0,-1,-1,0,1}-32H_{-1}^2H_0\zeta_2+\Big(-64H_{-1}H_1+32H_1^2\Big)\zeta_3\Big)$$
$$+\eta^3\Big(16H_0^2H_{0,1,-1}P_{623}+16H_0^2H_{0,-1,1}P_{623}+\frac{1}{6}H_0^3P_{639}+\frac{1}{30}H_0^5P_{641}+\frac{1}{3}H_0^4P_{644}-48H_0H_{0,-1}^2P_{663}$$
$$-32H_{0,0,1,0,1}P_{669}+16H_0H_{0,0,-1,-1}P_{682}+32H_{0,0,1,0,-1}P_{686}-16H_0^2H_{0,1,1}P_{688}+32H_0H_{0,0,1,-1}P_{696}$$
$$+32H_0H_{0,0,-1,1}P_{696}-32H_{0,0,-1,0,1}P_{703}+32H_0H_{0,-1,0,1}P_{712}-8H_0^2H_{0,-1,-1}P_{715}+32H_{0,0,0,1,-1}P_{726}$$
$$+32H_{0,0,0,-1,1}P_{726}-32H_{0,0,-1,0,-1}P_{738}-96H_{0,0,0,-1,-1}P_{739}+32H_0H_{0,0,1,1}P_{744}-32H_{0,0,0,1,1}P_{780}$$
$$+16H_{0,0,0,0,1}P_{807}-16H_{0,0,0,0,-1}P_{808}+\Big(-\frac{2}{3}H_0^4P_{675}-\frac{4}{3}H_0^3P_{790}\Big)H_1+\Big(-\frac{4}{3}H_0^3P_{680}$$
$$+\frac{2}{3}H_0^4P_{716}\Big)H_{-1}+\Big(-\frac{8}{3}H_0^3P_{668}+32H_0^2H_1P_{677}-16H_{-1}H_0^2P_{679}-32H_0H_{0,-1}P_{683}+4H_0^2P_{776}\Big)H_{0,1}$$
$$+128x^2\Big(10-21x+10x^2\Big)H_0H_{0,1}^2+\Big(16xH_{-1}H_0^2P_{617}-16H_0^2H_1P_{655}+\frac{4}{3}H_0^3P_{747}-4H_0^2P_{786}\Big)H_{0,-1}$$
$$+\Big(-32H_{0,1}P_{647}+64H_{-1}H_0P_{654}-256H_0H_1P_{662}+8H_0P_{714}+8H_0^2P_{717}-32H_{0,-1}P_{722}\Big)H_{0,0,1}$$
$$+\Big(-32H_{-1}H_0P_{629}+128H_0H_1P_{676}+8H_0P_{709}+32H_{0,1}P_{710}-4H_0^2P_{725}+16H_{0,-1}P_{768}\Big)H_{0,0,-1}$$
$$+\Big(-32H_{-1}P_{657}+64H_1P_{736}-16H_0P_{788}-8P_{800}\Big)H_{0,0,0,1}+\Big(96H_{-1}P_{624}-32H_1P_{777}+8H_0P_{795}$$
$$+8P_{803}\Big)H_{0,0,0,-1}+\Big(-2xH_0^2P_{635}+32H_{0,1,-1}P_{707}+32H_{0,-1,1}P_{708}+16H_{0,0,1}P_{742}-16H_{0,-1,-1}P_{765}$$
$$+\frac{4}{3}H_0^3P_{782}-8H_{0,0,-1}P_{789}+H_0P_{812}+\Big(8H_0^2P_{719}-8H_0P_{763}\Big)H_1+\Big(8H_0^2P_{665}-8H_0P_{754}\Big)H_{-1}+\Big($$
$$-16H_0P_{728}+8P_{755}\Big)H_{0,1}+\Big(32H_{-1}P_{666}+8P_{681}+16H_0P_{694}-32H_1P_{708}\Big)H_{0,-1}+4P_{783}\zeta_3\Big)\zeta_2$$
$$+\Big(\frac{8}{5}H_{-1}P_{659}+\frac{8}{5}H_1P_{792}-\frac{4P_{810}}{5}+\frac{2}{5}H_0P_{814}\Big)\zeta_2^2+\Big(-32H_{-1}H_0P_{667}+16H_0H_1P_{691}+16H_{0,-1}P_{731}$$
$$-16H_{0,1}P_{741}+\frac{4}{3}H_0^2P_{785}+\frac{4}{3}H_0P_{791}\Big)\zeta_3-6P_{798}\zeta_5\Big)+8xx_+^2+8c_1\Big(1-4x+x^2\Big)x_+^2\Bigg\}\Bigg]$$

$$+C_FC_A\left[\frac{1}{\varepsilon^2}\left\{\frac{11}{3}+\frac{11\xi H_0}{3}\right\}+\frac{1}{\varepsilon}\left\{-\frac{49}{9}+\xi\Big(\eta\Big(-\frac{4}{3}x^2H_0^3-2\Big(-1+3x^2\Big)H_0\zeta_2\Big)-\frac{67}{9}H_0+4H_{-1}H_0\right.\right.$$
$$-4H_0H_1+4H_{0,1}-4H_{0,-1}\Big)+\eta\Big(-4x^2H_0^2-2\Big(-1+3x^2\Big)\zeta_2\Big)+\xi^2\Big(-4H_0H_{0,1}+4H_0H_{0,-1}+8H_{0,0,1}$$
$$\left.\left.-8H_{0,0,-1}-2\zeta_3\Big)\right\}+\left\{-\frac{1595}{27}+\eta\Big(-\frac{5}{54}\Big(509-114x+509x^2\Big)H_0+\frac{4}{9}\Big(31-105x+31x^2\Big)H_{-1}H_0\right.\right.$$
$$+4\Big(3-2x+3x^2\Big)H_0H_1-4\Big(3-2x+3x^2\Big)H_{0,1}-\frac{4}{9}\Big(31-105x+31x^2\Big)H_{0,-1}\Big)+\xi^2\Big(\Big(-8H_1$$
$$+8H_{-1}\Big)\zeta_3+\Big(16H_{-1}H_0-16H_0H_1-8H_{0,-1}\Big)H_{0,1}-4H_{0,1}^2-16H_{-1}H_0H_{0,-1}+16H_0H_1H_{0,-1}$$
$$+12H_{0,-1}^2+\Big(32H_1-32H_{-1}\Big)H_{0,0,1}+\Big(-32H_1+32H_{-1}\Big)H_{0,0,-1}+24H_0H_{0,1,1}-8H_0H_{0,1,-1}$$
$$-8H_0H_{0,-1,1}-8H_0H_{0,-1,-1}-32H_{0,0,1,1}+32H_{0,0,1,-1}+32H_{0,0,-1,1}-32H_{0,0,-1,-1}\Big)+\xi\Big(\eta\Big(\frac{4}{3}\Big(3$$
$$+x^2\Big)H_{-1}H_0^3+16H_{-1}H_0\zeta_2\Big)-\frac{104}{3}H_{-1}^2H_0+24H_{-1}H_0H_1-4H_0H_1^2+\Big(8H_1-24H_{-1}\Big)H_{0,1}$$
$$+\Big(-24H_1+\frac{208H_{-1}}{3}\Big)H_{0,-1}-8H_{0,1,1}+24H_{0,1,-1}+24H_{0,-1,1}-\frac{208}{3}H_{0,-1,-1}+4H_1\zeta_2$$
$$-24(-1+x)\log(2)x_+\zeta_2\Big)+\eta^2\Big(-4H_0^2H_1P_{526}-4H_0H_{0,-1}P_{538}-8H_{0,0,1}P_{555}-\frac{2}{3}H_{-1}H_0^2P_{591}$$
$$+\frac{4}{3}H_{0,0,-1}P_{599}-\frac{1}{18}(1+x)(233-299x+65x^2+217x^3)H_0^2+8(1+x)^2\Big(6-11x+6x^2\Big)H_0H_{0,1}$$
$$+\Big(\frac{4}{3}H_0^3H_1P_{611}+\Big(16H_0H_1P_{613}-4H_{0,1}P_{619}\Big)\zeta_2\Big)x_++\Big(\frac{P_{510}}{9}-\frac{4}{3}H_{-1}P_{554}\Big)\zeta_2+\frac{2P_{534}\zeta_3}{3}\Big)$$
$$+\eta^3\Big(32xH_0H_{0,0,1}P_{615}+\frac{1}{6}xH_0^4P_{630}+\frac{1}{9}H_0^3P_{640}+8H_0H_{0,0,-1}P_{660}+4H_0^2H_{0,1}P_{661}-4H_0^2H_{0,-1}P_{695}$$
$$+8H_{0,0,0,-1}P_{723}-8H_{0,0,0,1}P_{737}+\Big(-2H_0^2P_{650}-4H_{0,-1}P_{700}-\frac{2}{3}H_0P_{764}\Big)\zeta_2+\frac{1}{5}P_{797}\zeta_2^2$$
$$\left.+2H_0P_{730}\zeta_3\Big)-96x\log(2)x_+^2\zeta_2\right\}+\varepsilon\left\{\eta\Big(-\frac{5}{324}\Big(14033+462x+14033x^2\Big)H_0+\frac{1}{27}\Big(3623+534x\right.$$
$$+3623x^2\Big)H_{-1}H_0+\frac{4}{9}\Big(28+279x+28x^2\Big)H_{-1}^2H_0+\frac{1}{9}\Big(-533-659x-677x^2+97x^3\Big)x_+H_{-1}H_0^2$$
$$+\Big(4\Big(5-32x+5x^2\Big)H_0-24\Big(3-2x+3x^2\Big)H_{-1}H_0\Big)H_1+4\Big(3-2x+3x^2\Big)H_0H_1^2+\Big(-4\Big(5-32x$$
$$+5x^2\Big)-8\Big(3-2x+3x^2\Big)H_1+24\Big(3-2x+3x^2\Big)H_{-1}\Big)H_{0,1}+\Big(\frac{1}{27}\Big(-3623-534x-3623x^2\Big)$$
$$+24\Big(3-2x+3x^2\Big)H_1-\frac{8}{9}\Big(28+279x+28x^2\Big)H_{-1}\Big)H_{0,-1}+8\Big(3-2x+3x^2\Big)H_{0,1,1}-24\Big(3-2x$$
$$+3x^2\Big)H_{0,1,-1}-24\Big(3-2x+3x^2\Big)H_{0,-1,1}+\frac{8}{9}\Big(28+279x+28x^2\Big)H_{0,-1,-1}-4\Big(3-2x+3x^2\Big)H_1\zeta_2\Big)$$
$$+\xi\Big(\eta^2\log(2)\zeta_2\Big(-96x^2H_{0,1}-96x^2H_{0,-1}\Big)+\eta\Big(-\frac{4}{3}\Big(3+5x^2\Big)H_{-1}^2H_0^3-8\Big(1+3x^2\Big)H_{-1}^2H_0\zeta_2\Big)$$
$$+80H_{-1}^3H_0-72H_{-1}^2H_0H_1+24H_{-1}H_0H_1^2-\frac{8}{3}H_0H_1^3+\Big(-48H_{-1}H_1+8H_1^2+72H_{-1}^2\Big)H_{0,1}$$
$$+\Big(144H_{-1}H_1-24H_1^2-240H_{-1}^2\Big)H_{0,-1}+\Big(-16H_1+48H_{-1}\Big)H_{0,1,1}+\Big(48H_1-144H_{-1}\Big)H_{0,1,-1}$$
$$+\Big(48H_1-144H_{-1}\Big)H_{0,-1,1}+\Big(-144H_1+480H_{-1}\Big)H_{0,-1,-1}+16H_{0,1,1,1}-48H_{0,1,1,-1}$$

$$
\begin{aligned}
&- 48H_{0,1,-1,1} + 144H_{0,1,-1,-1} - 48H_{0,-1,1,1} + 144H_{0,-1,1,-1} + 144H_{0,-1,-1,1} - 480H_{0,-1,-1,-1} \\
&+ 4H_1^2\zeta_2\Big) + \eta^2\Big(\frac{1}{54}H_0^2P_{511} - 24H_0H_{0,1,1}P_{513} - 8H_{0,1}^2P_{533} - 16H_0H_{0,1,-1}P_{548} - 16H_0H_{0,-1,1}P_{548} \\
&+ 8H_{0,0,1,1}P_{556} - 2H_0^2H_1^2P_{558} - 8H_{0,-1,0,1}P_{569} + 8H_0H_{0,-1,-1}P_{575} - 4H_{0,-1}^2P_{578} - 8H_{0,0,1,-1}P_{582} \\
&- 8H_{0,0,-1,1}P_{582} + 2H_{-1}^2H_0^2P_{592} + 8H_{0,0,-1,-1}P_{594} + \log(2)\zeta_2\Big(12P_{512} - 24\Big(3 - 2x^2 + 3x^4\Big)H_1 \\
&- 24\Big(3 - 2x^2 + 3x^4\Big)H_{-1}\Big) + \Big(-2H_0^2P_{586} - 4\Big(-15 + 2x^2 + 9x^4\Big)H_{-1}H_0^2\Big)H_1 + \Big(\\
&- 16H_{-1}H_0P_{531} + 4H_0P_{541} + 8H_0H_1P_{557} + 16H_{0,-1}P_{561}\Big)H_{0,1} + \Big(8H_{-1}H_0P_{514} + 8H_0H_1P_{549} \\
&+ 8H_0P_{580}\Big)H_{0,-1} + \Big(-8H_1P_{556} + 4P_{565} + 8H_{-1}P_{582}\Big)H_{0,0,1} + \Big(-8H_1P_{539} - 8H_{-1}P_{594} \\
&- \frac{2P_{609}}{9}\Big)H_{0,0,-1} + \Big(\frac{8}{3}H_{-1}H_0^3H_1P_{627} - \frac{4}{3}H_0^3H_1^2P_{627} + \Big(16H_{-1}H_0H_1P_{622} - 8H_0H_1^2P_{622} \\
&- 8H_{0,1,1}P_{631} + \Big(32x\Big(3 - 8x + 14x^2 + 3x^4\Big)H_1 - 32x\Big(3 - 8x + 14x^2 + 3x^4\Big)H_{-1}\Big)H_{0,1}\Big)\zeta_2\Big)x_+ \\
&+ \Big(\frac{P_{508}}{27} + 12H_{-1}^2P_{535} + \frac{2}{9}H_{-1}P_{602} + 48\Big(1 - x^2 + 2x^4\Big)H_{-1}H_1 - 24\Big(3 - 2x^2 + 3x^4\Big)H_{-1,1}\Big)\zeta_2 \\
&+ \Big(2H_{-1}P_{589} + 2H_1P_{593} + \frac{P_{608}}{9}\Big)\zeta_3\Big) + \xi^2\Big(\Big(-64H_{-1}H_0H_{0,-1} + 48H_{0,-1}^2\Big)H_1 + \Big(-32H_{-1}^2H_0 \\
&+ \Big(64H_{-1}H_0 - 32H_{0,-1}\Big)H_1 - 32H_0H_1^2 + 32H_{-1}H_{0,-1}\Big)H_{0,1} + \Big(-16H_1 + 16H_{-1}\Big)H_{0,1}^2 \\
&+ 32H_{-1}^2H_0H_{0,-1} + 32H_0H_1^2H_{0,-1} - 48H_{-1}H_{0,-1}^2 + \Big(-128H_{-1}H_1 + 64H_1^2 + 64H_{-1}^2\Big)H_{0,0,1} \\
&+ \Big(128H_{-1}H_1 - 64H_1^2 - 64H_{-1}^2\Big)H_{0,0,-1} + \Big(-96H_{-1}H_0 + 96H_0H_1 + 16H_{0,1} + 80H_{0,-1}\Big)H_{0,1,1} \\
&+ \Big(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\Big)H_{0,1,-1} + \Big(32H_{-1}H_0 - 32H_0H_1 - 48H_{0,1} \\
&- 48H_{0,-1}\Big)H_{0,-1,1} + \Big(32H_{-1}H_0 - 32H_0H_1 + 16H_{0,1} - 48H_{0,-1}\Big)H_{0,-1,-1} + \Big(-128H_1 \\
&+ 128H_{-1}\Big)H_{0,0,1,1} + \Big(128H_1 - 128H_{-1}\Big)H_{0,0,1,-1} + \Big(128H_1 - 128H_{-1}\Big)H_{0,0,-1,1} + \Big(-128H_1 \\
&+ 128H_{-1}\Big)H_{0,0,-1,-1} - 112H_0H_{0,1,1,1} + 16H_0H_{0,1,1,-1} + 16H_0H_{0,1,-1,1} + 16H_0H_{0,1,-1,-1} \\
&+ 16H_0H_{0,-1,1,1} + 16H_0H_{0,-1,1,-1} + 16H_0H_{0,-1,-1,1} + 16H_0H_{0,-1,-1,-1} + 128H_{0,0,1,1,1} \\
&- 256H_{0,0,1,1,-1} - 128H_{0,0,1,-1,1} + 128H_{0,0,1,-1,-1} + 128H_{0,0,-1,1,-1} + 128H_{0,0,-1,-1,1} \\
&+ 256H_{0,0,-1,-1,-1} - 64H_{0,1,0,1,-1} + 64H_{0,-1,0,1,1} + 192H_{0,-1,0,-1,-1} + 64H_{0,-1,1,0,1} + \Big(32H_{-1}H_1 \\
&- 16H_1^2 - 16H_{-1}^2\Big)\zeta_3\Big) + \eta^3\Big(\frac{1}{30}xH_0^5P_{634} + \frac{1}{12}H_0^4P_{642} + 32H_0H_{0,0,1,-1}P_{646} + 32H_0H_{0,0,-1,1}P_{646} \\
&+ 48H_0H_{0,-1}^2P_{651} - 80H_0H_{0,-1,0,1}P_{656} - 128H_0H_{0,0,1,1}P_{664} + 16H_0H_{0,1}^2P_{671} + 8H_0^2H_{0,1,1}P_{685} \\
&- 8H_0^2H_{0,1,-1}P_{689} - 8H_0^2H_{0,-1,1}P_{689} - 16H_0H_{0,0,-1,-1}P_{704} + 16H_{0,0,1,0,1}P_{733} + 8H_0^2H_{0,-1,-1}P_{734} \\
&+ 32H_{0,0,-1,0,-1}P_{740} - 16H_{0,0,-1,0,1}P_{743} - 16H_{0,0,1,0,-1}P_{750} - 32H_{0,0,0,1,-1}P_{774} - 32H_{0,0,0,-1,1}P_{774} \\
&+ 16H_{0,0,0,1,1}P_{796} - 8H_{0,0,0,0,1}P_{801} + 8H_{0,0,0,0,-1}P_{802} + 16H_{0,0,0,-1,-1}P_{805} + \frac{1}{54}H_0^3P_{813} + \Big(\frac{1}{3}H_0^4P_{626} \\
&+ \frac{2}{3}H_0^3P_{775}\Big)H_1 + \Big(\frac{1}{3}H_0^4P_{645} + \frac{2}{3}H_0^3P_{794}\Big)H_{-1} + \Big(8H_{-1}H_0^2P_{661} - 16H_0^2H_1P_{677} - 16H_0H_{0,-1}P_{687} \\
&+ \frac{8}{3}H_0^3P_{713} - 2H_0^2P_{779}\Big)H_{0,1} + \Big(8H_{-1}H_0^2P_{625} + 8H_0^2H_1P_{684} - \frac{4}{3}H_0^3P_{781} + 2H_0^2P_{787}\Big)H_{0,-1} \\
&+ \Big(-32H_{-1}H_0P_{649} + 32H_0H_1P_{693} + 4H_0P_{702} - 16H_{0,1}P_{706} + 16H_{0,-1}P_{751} - 4H_0^2P_{757}\Big)H_{0,0,1}
\end{aligned}
$$

$$+\Big(16H_{-1}H_0P_{660}-32H_0H_1P_{678}+16H_{0,1}P_{718}-32H_{0,-1}P_{729}+4H_0^2P_{772}-4H_0P_{784}\Big)H_{0,0,-1}$$
$$+\Big(16H_{-1}P_{697}-16H_1P_{745}+8H_0P_{778}+4P_{793}\Big)H_{0,0,0,1}+\Big(-16H_{-1}P_{711}+16H_1P_{724}$$
$$-8H_0P_{771}-4P_{799}\Big)H_{0,0,0,-1}+\Big(\frac{1}{18}H_0P_{638}-\frac{4}{3}H_0^3P_{749}+16H_{0,0,-1}P_{752}-8H_{0,1,-1}P_{759}$$
$$-8H_{0,-1,1}P_{760}+8H_{0,-1,-1}P_{770}-8H_{0,0,1}P_{773}+\frac{1}{2}H_0^2P_{806}+\Big(-4xH_0^2P_{621}+4H_0P_{758}\Big)H_1$$
$$+\Big(8H_0^2P_{653}+2H_0P_{721}\Big)H_{-1}+\Big(-4P_{735}+4H_0P_{769}\Big)H_{0,1}+\Big(-16H_{-1}P_{692}$$
$$+16H_1P_{720}-2P_{732}-4H_0P_{767}\Big)H_{0,-1}-2P_{756}\zeta_3\Big)\zeta_2+\Big(-\frac{8}{5}H_{-1}P_{705}+\frac{2}{5}H_0P_{748}$$
$$+\frac{4}{5}H_1P_{753}+\frac{P_{809}}{5}\Big)\zeta_2^2+\Big(-16H_0H_1P_{652}+16H_{-1}H_0P_{672}+4H_{0,1}P_{746}-2H_0^2P_{761}$$
$$-4H_{0,-1}P_{762}+\frac{4}{3}H_0P_{766}\Big)\zeta_3+P_{804}\zeta_5\Big)-\frac{28745}{162}(1+x)^2x_+^2-4c_1\Big(1-4x+x^2\Big)x_+^2\Big\}\Big]$$
$$+C_Fn_lT_F\Big[\frac{1}{\varepsilon^2}\Big\{-\frac{4}{3}-\frac{4\xi H_0}{3}\Big\}+\frac{1}{\varepsilon}\Big\{\frac{20}{9}+\frac{20\xi H_0}{9}\Big\}+\Big\{\frac{424}{27}+\eta\Big(\frac{2}{27}\Big(209-78x+209x^2\Big)H_0$$
$$-\frac{8}{9}\Big(19-6x+19x^2\Big)H_{-1}H_0+\frac{2}{9}\Big(19-6x+19x^2\Big)H_0^2+\frac{8}{9}\Big(19-6x+19x^2\Big)H_{0,-1}-\frac{4}{9}\Big(7$$
$$-6x+31x^2\Big)\zeta_2\Big)+\xi\Big(\frac{16}{3}H_{-1}^2H_0-\frac{8}{3}H_{-1}H_0^2+\frac{4}{9}H_0^3-\frac{32}{3}H_{-1}H_{0,-1}+\frac{16}{3}H_{0,0,-1}+\frac{32}{3}H_{0,-1,-1}$$
$$+\Big(\frac{8}{3}H_0+\frac{16}{3}H_{-1}\Big)\zeta_2-\frac{16\zeta_3}{3}\Big)\Big\}+\varepsilon\Big\{\frac{5204}{81}+\xi\Big(\Big(-\frac{16H_0}{3}+32H_{-1}\Big)\zeta_3-\frac{32}{3}H_{-1}^3H_0+8H_{-1}^2H_0^2$$
$$-\frac{8}{3}H_{-1}H_0^3+32H_{-1}^2H_{0,-1}-32H_{-1}H_{0,0,-1}-64H_{-1}H_{0,-1,-1}+16H_{0,0,0,-1}+32H_{0,0,-1,-1}$$
$$+64H_{0,-1,-1,-1}+\Big(-8H_{-1}H_0+2H_0^2-16H_{-1}^2+24H_{0,-1}\Big)\zeta_2-\frac{96\zeta_2^2}{5}\Big)+\eta\Big(\frac{1}{81}\Big(5813$$
$$-1014x+5813x^2\Big)H_0+\frac{8}{9}\Big(47-18x+47x^2\Big)H_{-1}^2H_0+\frac{2}{27}\Big(281-78x+281x^2\Big)H_0^2$$
$$+\frac{2}{27}\Big(47-18x+47x^2\Big)H_0^3+\frac{1}{3}\Big(1+x^2\Big)H_0^4+\Big(-\frac{8}{27}\Big(281-78x+281x^2\Big)H_0-\frac{4}{9}\Big(47$$
$$-18x+47x^2\Big)H_0^2\Big)H_{-1}+\Big(\frac{8}{27}\Big(281-78x+281x^2\Big)-\frac{16}{9}\Big(47-18x+47x^2\Big)H_{-1}\Big)H_{0,-1}$$
$$+\frac{8}{9}\Big(47-18x+47x^2\Big)H_{0,0,-1}+\frac{16}{9}\Big(47-18x+47x^2\Big)H_{0,-1,-1}+\Big(-\frac{8}{27}\Big(22-39x+259x^2\Big)$$
$$+\frac{2}{9}\Big(37-18x+37x^2\Big)H_0+\frac{8}{9}\Big(47-18x+47x^2\Big)H_{-1}\Big)\zeta_2-\frac{8}{9}\Big(35-18x+59x^2\Big)\zeta_3\Big)\Big\}\Big]$$
$$+C_FT_F\Big[\Big\{\frac{4}{27}\Big(383+646x+383x^2\Big)x_+^2+\eta x_+^2\Big(\frac{4}{9}H_0^3P_{517}+\frac{2}{27}H_0P_{604}+4H_0P_{518}\zeta_2\Big)$$
$$+\Big(\frac{2}{9}H_0^2P_{577}-\frac{8}{3}P_{521}\zeta_2\Big)x_+^4\Big\}+\varepsilon\Big\{\frac{2}{27}\Big(2069+4378x+2069x^2\Big)x_+^2+\eta\Big(\frac{1}{27}\Big(1719$$
$$-2918x+1719x^2\Big)H_0+\Big(\frac{8}{9}H_{-1}H_0^3P_{517}+\frac{1}{3}H_0^4P_{517}+\frac{16}{3}H_0^2H_{0,1}P_{517}-\frac{64}{3}H_0H_{0,0,1}P_{517}$$
$$+\frac{32}{3}H_0H_{0,0,-1}P_{517}+32H_{0,0,0,1}P_{517}-\frac{16}{3}H_{0,0,0,-1}P_{517}-\frac{2}{27}H_0^2P_{570}-\frac{4}{27}H_{-1}H_0P_{604}$$

$$
\begin{aligned}
&- \frac{8}{5}x\Big(2+2x^2+x^3\Big)\zeta_2^2 + \Big(-\frac{16}{3}H_0^2P_{517} + \frac{4P_{604}}{27}\Big)H_{0,-1} + \Big(\frac{8}{3}H_{-1}H_0P_{516} + 2H_0^2P_{518}\\
&- \frac{8}{3}H_{0,-1}P_{545} + \frac{4P_{601}}{27}\Big)\zeta_2 - \frac{8}{9}H_0P_{546}\zeta_3\Big)x_+^2 + \Big(-\frac{2}{27}H_0^3P_{636} - \frac{2}{9}H_0P_{637}\zeta_2\Big)x_+^3\Big)\\
&+ \Big(-\frac{4}{9}H_{-1}H_0^2P_{577} + \frac{8}{9}H_0^2H_1P_{577} - \frac{16}{9}H_0H_{0,1}P_{577} + \frac{16}{9}H_{0,0,1}P_{577} + \frac{8}{9}H_{0,0,-1}P_{577}\\
&- \frac{8}{3}H_{-1}P_{577}\zeta_2 - \frac{8}{9}P_{595}\zeta_3\Big)x_+^4 + \frac{32}{3}\log(2)P_{550}x_+^4\zeta_2 + \frac{8}{5}(-1+x)\xi\eta x_+\zeta_2^2\Bigg\}\Bigg].
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\hat{F}_{A,1}^{(2),\mathrm{s}} = C_FT_F\Bigg[&\frac{1}{\varepsilon}\Big\{-6\Big\} + \Big\{\eta^3\Big(-32x\zeta_2\Big(1-x+4x^2-x^3+x^4\Big)H_0 - \frac{16}{3}x^2\Big(1+4x-3x^2+2x^3\Big)H_0^3\Big)\\
&+ \eta\Big(-4\Big(3-2x+3x^2\Big)H_0 + \Big(\frac{64}{3}x^2H_0^3H_1 + 64x^2H_0^2H_{0,1} - 512x^2H_0H_{0,0,1} + 256x^2H_0H_{0,0,-1}\\
&+ 1024x^2H_{0,0,0,1} - 768x^2H_{0,0,0,-1} + \Big(32x^2H_0^2 + 128x^2H_0H_1 - 128x^2H_{0,1}\Big)\zeta_2 - \frac{64}{5}x^2\zeta_2^2\\
&- 128x^2H_0\zeta_3\Big)x_+^2\Big) + \Big(-32(-1+x)H_{-1}H_0 + 16(-1+x)H_0H_1 - 16(-1+x)H_{0,1}\\
&+ 32(-1+x)H_{0,-1}\Big)x_+ + \Big(-29-34x-29x^2+8x(1+3x)H_0^2 - 64xH_0^2H_1\\
&+ 192xH_0H_{0,1} - 128xH_0H_{0,-1} - 256xH_{0,0,1} + 256xH_{0,0,-1} + 64x\zeta_3\Big)x_+^2 - 48x\xi\eta\zeta_2\\
&+ 8\xi^2\zeta_2\Big\} + \varepsilon\Big\{\log(2)x_+^2\zeta_2\Big(384x - 384xH_1 - 384xH_{-1}\Big) + \eta\Big(\log(2)x_+^2\zeta_2\Big(768x^2H_{0,1}\\
&+ 768x^2H_{0,-1}\Big) - 2\Big(29+3x+15x^2+25x^3\Big)x_+H_0 + 8\Big(19-42x+19x^2\Big)H_{-1}H_0\\
&- 32(-2+x)(-1+2x)H_0H_1 - 8\Big(19-42x+19x^2\Big)H_{0,-1} + \Big(\Big(\frac{128}{3}x^2H_{-1}H_0^3 + 16x^2H_0^4\Big)H_1\\
&- \frac{64}{3}x^2H_0^3H_1^2 + \Big(16P_{532} + 128x^2H_{-1}H_0^2 + \frac{64}{3}x^2H_0^3 + 128x^2H_0^2H_1 + 256x^2H_0H_{0,-1}\Big)H_{0,1}\\
&- 512x^2H_0H_{0,1}^2 + \Big(\frac{256}{3}x^2H_0^3 - 256x^2H_0^2H_1\Big)H_{0,-1} - 384x^2H_0H_{0,-1}^2 + \Big(\Big(-64x\Big(5-14x\\
&+ 3x^2\Big) - 1024x^2H_{-1}\Big)H_0 - 448x^2H_0^2 + 1280x^2H_{0,1} - 1792x^2H_{0,-1}\Big)H_{0,0,1} + \Big(\Big(-128x\Big(-5\\
&+ 3x + 3x^2\Big) + 512x^2H_{-1}\Big)H_0 + 128x^2H_0^2 - 512x^2H_{0,1} + 1792x^2H_{0,-1}\Big)H_{0,0,-1} + 256x^2H_0^2H_{0,1,1}\\
&- 128x^2H_0^2H_{0,1,-1} - 128x^2H_0^2H_{0,-1,1} - 128x^2H_0^2H_{0,-1,-1} + \Big(1536x^2H_0 - 512x^2H_1\\
&+ 2048x^2H_{-1}\Big)H_{0,0,0,1} + \Big(-1536x^2H_0 + 1280x^2H_1 - 1536x^2H_{-1}\Big)H_{0,0,0,-1} + 2048x^2H_0H_{0,0,1,-1}\\
&+ 2048x^2H_0H_{0,0,-1,1} - 1024x^2H_0H_{0,0,-1,-1} + 1024x^2H_0H_{0,-1,0,1} - 1664x^2H_{0,0,0,0,1}\\
&+ 2560x^2H_{0,0,0,0,-1} - 3328x^2H_{0,0,0,1,1} + 256x^2H_{0,0,0,1,-1} + 256x^2H_{0,0,0,-1,1} - 2304x^2H_{0,0,0,-1,-1}\\
&- 1280x^2H_{0,0,1,0,1} + 768x^2H_{0,0,1,0,-1} + 512x^2H_{0,0,-1,0,1} - 1280x^2H_{0,0,-1,0,-1} + \Big(64x^2H_{-1}H_0^2\\
&+ 32x^2H_0^3 + 256x^2H_{-1}H_0H_1 - 128x^2H_0H_1^2 + \Big(-128x^2H_0 + 256x^2H_1 - 256x^2H_{-1}\Big)H_{0,1}\\
&+ \Big(384x^2H_0 - 1024x^2H_1\Big)H_{0,-1} + 384x^2H_{0,0,1} - 1536x^2H_{0,0,-1} - 256x^2H_{0,1,1} + 256x^2H_{0,1,-1}\\
&+ 1024x^2H_{0,-1,1} - 768x^2H_{0,-1,-1} + 160x^2\zeta_3\Big)\zeta_2 + \Big(-\frac{448}{5}x^2H_0 + \frac{64}{5}x^2H_1 - \frac{128}{5}x^2H_{-1}\Big)\zeta_2^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-32x\Big(-1-9x+5x^2\Big)H_0-256x^2H_{-1}H_0-32x^2H_0^2-1344x^2H_{0,1}+448x^2H_{0,-1}\Big)\zeta_3\\
&-144x^2\zeta_5\Big)x_+^2\Big)+\eta^3\Big(\frac{32}{3}xH_{-1}H_0^3P_{524}-64xH_0^2H_{0,1}P_{530}+64xH_0^2H_{0,-1}P_{537}+\frac{32}{3}xH_0^3H_1P_{551}\\
&-64xH_{0,0,0,-1}P_{566}+32xH_{0,0,0,1}P_{588}-\frac{8}{3}xH_0^3P_{628}-\frac{2}{3}x^2\Big(5+26x-19x^2+12x^3\Big)H_0^4+\Big(\\
&-64xH_{0,-1}P_{515}-64xH_{0,1}P_{523}+64xH_{-1}H_0P_{524}+64xH_0H_1P_{529}+16H_0P_{648}\Big)\zeta_2-\frac{16}{5}xP_{579}\zeta_2^2\Big)\\
&+\eta^2\Big(16H_0H_{0,1}P_{527}-16H_{-1}H_0^2P_{540}+32H_{0,0,-1}P_{562}+32\Big(-1+2x^2-4x^3+x^4\Big)H_0^2H_1-32\Big(3\\
&-x-x^3+3x^4\Big)H_0H_{0,-1}-16x\Big(21-26x+5x^2+8x^3\Big)H_{0,0,1}+\Big(4H_0^2P_{633}+2P_{610}\zeta_2\Big)x_+\\
&-16H_{-1}P_{519}\zeta_2-8P_{572}\zeta_3\Big)+\Big(128(-1+x)H_{-1}^2H_0-96(-1+x)H_{-1}H_0H_1+16(-1+x)H_0H_1^2\\
&+\Big(-32(-1+x)H_1+96(-1+x)H_{-1}\Big)H_{0,1}+\Big(96(-1+x)H_1-256(-1+x)H_{-1}\Big)H_{0,-1}\\
&+32(-1+x)H_{0,1,1}-96(-1+x)H_{0,1,-1}-96(-1+x)H_{0,-1,1}+256(-1+x)H_{0,-1,-1}\\
&-16(-1+x)H_1\zeta_2\Big)x_++\Big(\frac{1}{2}\Big(-199-278x-199x^2\Big)+\Big(672xH_1-96xH_{-1}\Big)\zeta_3+32xH_{-1}^2H_0^2+\Big(\\
&-8(-1+x)^2+64xH_{-1}H_0^2\Big)H_1-96xH_0^2H_1^2+\Big(-256xH_{-1}H_0+384xH_0H_1+640xH_{0,-1}\Big)H_{0,1}\\
&-64xH_{0,1}^2+\Big(128xH_{-1}H_0+128xH_0H_1\Big)H_{0,-1}-384xH_{0,-1}^2+\Big(-384xH_1+384xH_{-1}\Big)H_{0,0,1}\\
&+\Big(-384xH_1-384xH_{-1}\Big)H_{0,0,-1}-256xH_0H_{0,1,1}-384xH_0H_{0,1,-1}-384xH_0H_{0,-1,1}\\
&+640xH_0H_{0,-1,-1}+384xH_{0,0,1,1}-384xH_{0,0,1,-1}-384xH_{0,0,-1,1}+384xH_{0,0,-1,-1}\\
&-384xH_{0,-1,0,1}+\Big(384xH_{-1}H_1+192xH_{-1}^2-384xH_{-1,1}\Big)\zeta_2\Big)x_+^2-16x(3+5x)H_0^2x_+^3\zeta_2\Bigg\}\Bigg].
\end{aligned}
\tag{C.3}
$$

The polynomials are defined as

$$
\begin{aligned}
P_{508}&=-4774x^4+21306x^3-28365x^2+14928x-71\,,\\
P_{509}&=-399x^4+486x^3-660x^2-182x+307\,,\\
P_{510}&=-347x^4+2262x^3-3174x^2+1698x-7\,,\\
P_{511}&=-254x^4+4452x^3-4779x^2+462x-3337\,,\\
P_{512}&=x^4-60x^3+110x^2-60x+1\,,\\
P_{513}&=x^4-30x^3+54x^2-30x+1\,,\\
P_{514}&=x^4-16x^3+26x^2-16x+1\,,\\
P_{515}&=x^4-6x^3-16x^2+14x-9\,,\\
P_{516}&=x^4+2x^3-10x^2+2x+1\,,\\
P_{517}&=x^4+2x^3-4x^2+2x+1\,,\\
P_{518}&=x^4+2x^3-2x^2+2x+1\,,\\
P_{519}&=x^4+10x^3-42x^2+2x+5\,,\\
P_{520}&=x^4+10x^3-35x^2+10x+1\,,\\
P_{521}&=x^4+55x^3+68x^2+55x+1\,,\\
P_{522}&=2x^4-21x^3+142x^2-21x+4\,,\\
P_{523}&=2x^4-7x^3+14x^2-7x+2\,,
\end{aligned}
$$

$$P_{524} = 2x^4 - 5x^3 - 4x^2 + 7x - 4\,,$$
$$P_{525} = 2x^4 - x^3 - 70x^2 - x + 2\,,$$
$$P_{526} = 2x^4 + 5x^3 - 14x^2 + 5x + 6\,,$$
$$P_{527} = 2x^4 + 11x^3 - 18x^2 + 11x + 2\,,$$
$$P_{528} = 3x^4 - 15x^3 + 76x^2 - 15x + 5\,,$$
$$P_{529} = 3x^4 - 9x^3 + 14x^2 - 5x + 1\,,$$
$$P_{530} = 3x^4 - 4x^3 - 2x^2 + 8x - 3\,,$$
$$P_{531} = 4x^4 - 27x^3 + 44x^2 - 27x + 4\,,$$
$$P_{532} = 4x^4 - 3x^3 - 10x^2 - 3x + 4\,,$$
$$P_{533} = 4x^4 + 9x^3 - 22x^2 + 9x + 4\,,$$
$$P_{534} = 5x^4 - 252x^3 + 408x^2 - 252x + 67\,,$$
$$P_{535} = 5x^4 - 27x^3 + 34x^2 - 27x + 13\,,$$
$$P_{536} = 5x^4 - 27x^3 + 36x^2 - 27x + 7\,,$$
$$P_{537} = 5x^4 - 9x^3 + 4x^2 + 7x - 3\,,$$
$$P_{538} = 5x^4 + 49x^3 - 98x^2 + 49x + 5\,,$$
$$P_{539} = 5x^4 + 128x^3 - 254x^2 + 128x + 29\,,$$
$$P_{540} = 6x^4 - 10x^3 - 7x^2 + 12x - 5\,,$$
$$P_{541} = 7x^4 - 102x^3 + 162x^2 - 102x + 7\,,$$
$$P_{542} = 7x^4 - 39x^3 + 135x^2 - 39x + 7\,,$$
$$P_{543} = 7x^4 - 23x^3 + 96x^2 - 23x + 7\,,$$
$$P_{544} = 7x^4 - 11x^3 - 52x^2 - 11x + 7\,,$$
$$P_{545} = 7x^4 + 14x^3 - 34x^2 + 14x + 7\,,$$
$$P_{546} = 7x^4 + 14x^3 - 22x^2 + 14x + 7\,,$$
$$P_{547} = 7x^4 + 30x^3 - 142x^2 + 30x + 11\,,$$
$$P_{548} = 7x^4 + 47x^3 - 96x^2 + 47x + 7\,,$$
$$P_{549} = 7x^4 + 64x^3 - 126x^2 + 64x + 7\,,$$
$$P_{550} = 7x^4 + 82x^3 + 110x^2 + 82x + 7\,,$$
$$P_{551} = 8x^4 - 19x^3 + 14x^2 + 5x - 4\,,$$
$$P_{552} = 8x^4 - 19x^3 + 46x^2 - 19x + 8\,,$$
$$P_{553} = 9x^4 - 13x^3 + 40x^2 - 13x + 9\,,$$
$$P_{554} = 10x^4 - 81x^3 + 108x^2 - 81x + 26\,,$$
$$P_{555} = 10x^4 - 3x^3 - 6x^2 - 3x + 6\,,$$
$$P_{556} = 11x^4 - 98x^3 + 170x^2 - 98x + 3\,,$$
$$P_{557} = 11x^4 - 72x^3 + 118x^2 - 72x + 11\,,$$
$$P_{558} = 11x^4 - 46x^3 + 66x^2 - 46x + 19\,,$$
$$P_{559} = 11x^4 - 42x^3 + 136x^2 - 42x + 11\,,$$
$$P_{560} = 11x^4 - 14x^3 - 30x^2 - 14x + 11\,,$$
$$P_{561} = 11x^4 + 20x^3 - 52x^2 + 20x + 11\,,$$
$$P_{562} = 12x^4 - 12x^3 - 7x^2 + 10x + 1\,,$$
$$P_{563} = 13x^4 - 29x^3 + 64x^2 - 29x + 13\,,$$

$$P_{564} = 13x^4 - 24x^3 + 50x^2 - 24x + 13\,,$$
$$P_{565} = 13x^4 + 126x^3 - 218x^2 + 142x - 11\,,$$
$$P_{566} = 14x^4 - 41x^3 + 68x^2 - 29x + 8\,,$$
$$P_{567} = 15x^4 - 53x^3 + 188x^2 - 53x + 15\,,$$
$$P_{568} = 15x^4 - 17x^3 - 8x^2 - 17x + 15\,,$$
$$P_{569} = 15x^4 + 10x^3 - 38x^2 + 10x + 15\,,$$
$$P_{570} = 16x^4 - 136x^3 - 7x^2 + 344x - 281\,,$$
$$P_{571} = 17x^4 - 46x^3 + 190x^2 - 46x + 17\,,$$
$$P_{572} = 17x^4 - 26x^3 + 24x^2 + 10x - 1\,,$$
$$P_{573} = 17x^4 - 23x^3 + 56x^2 - 23x + 17\,,$$
$$P_{574} = 18x^4 - 39x^3 + 82x^2 - 39x + 18\,,$$
$$P_{575} = 18x^4 + 125x^3 - 256x^2 + 125x + 18\,,$$
$$P_{576} = 19x^4 - 69x^3 + 196x^2 - 69x + 17\,,$$
$$P_{577} = 19x^4 - 14x^3 + 14x^2 - 14x + 19\,,$$
$$P_{578} = 19x^4 + 109x^3 - 230x^2 + 109x + 19\,,$$
$$P_{579} = 20x^4 - 53x^3 + 36x^2 + 15x - 14\,,$$
$$P_{580} = 20x^4 + 23x^3 - 58x^2 + 23x + 20\,,$$
$$P_{581} = 24x^4 - 27x^3 - 42x^2 - 27x + 22\,,$$
$$P_{582} = 25x^4 - 108x^3 + 178x^2 - 108x + 1\,,$$
$$P_{583} = 25x^4 - 53x^3 + 148x^2 - 53x + 25\,,$$
$$P_{584} = 26x^4 - 81x^3 + 102x^2 - 81x + 28\,,$$
$$P_{585} = 27x^4 - 102x^3 + 320x^2 - 102x + 27\,,$$
$$P_{586} = 27x^4 - 78x^3 + 106x^2 - 62x + 3\,,$$
$$P_{587} = 37x^4 - 108x^3 + 66x^2 - 108x + 45\,,$$
$$P_{588} = 38x^4 - 127x^3 + 158x^2 - 55x + 2\,,$$
$$P_{589} = 41x^4 + 216x^3 - 322x^2 + 216x - 115\,,$$
$$P_{590} = 45x^4 - 8x^3 - 186x^2 - 8x + 45\,,$$
$$P_{591} = 53x^4 - 141x^3 + 228x^2 - 141x - 17\,,$$
$$P_{592} = 56x^4 - 103x^3 + 162x^2 - 103x - 22\,,$$
$$P_{593} = 57x^4 - 8x^3 - 82x^2 - 8x + 101\,,$$
$$P_{594} = 58x^4 - 135x^3 + 214x^2 - 135x - 20\,,$$
$$P_{595} = 69x^4 + 564x^3 + 790x^2 + 564x + 69\,,$$
$$P_{596} = 77x^4 - 132x^3 + 230x^2 - 68x - 51\,,$$
$$P_{597} = 79x^4 - 182x^3 + 362x^2 - 182x + 79\,,$$
$$P_{598} = 83x^4 - 70x^3 - 180x^2 + 310x - 47\,,$$
$$P_{599} = 83x^4 + 153x^3 - 360x^2 + 153x + 13\,,$$
$$P_{600} = 93x^4 + 28x^3 - 330x^2 + 92x - 35\,,$$
$$P_{601} = 139x^4 + 3248x^3 - 7x^2 - 3040x - 404\,,$$
$$P_{602} = 187x^4 - 1314x^3 + 810x^2 + 234x - 349\,,$$
$$P_{603} = 229x^4 - 266x^3 + 244x^2 - 70x + 55\,,$$

$$P_{604} = 265x^4 - 208x^3 + 14x^2 - 208x + 265\,,$$
$$P_{605} = 649x^4 - 902x^3 + 1050x^2 - 246x + 153\,,$$
$$P_{606} = 759x^4 - 550x^3 - 828x^2 + 118x + 53\,,$$
$$P_{607} = 1043x^4 - 618x^3 - 1024x^2 + 762x + 173\,,$$
$$P_{608} = 1045x^4 + 5364x^3 - 8166x^2 + 2628x + 1577\,,$$
$$P_{609} = 1343x^4 + 2430x^3 - 4194x^2 + 1530x + 907\,,$$
$$P_{610} = x^5 - 15x^4 - 70x^3 + 18x^2 - 43x + 13\,,$$
$$P_{611} = x^5 - 5x^4 + 18x^3 - 26x^2 + x - 5\,,$$
$$P_{612} = x^5 - 5x^4 + 34x^3 - 30x^2 + 7x + 1\,,$$
$$P_{613} = x^5 - 2x^4 + 10x^3 - 12x^2 + x - 2\,,$$
$$P_{614} = x^5 - 2x^4 + 18x^3 - 14x^2 + 4x + 1\,,$$
$$P_{615} = x^5 + 4x^4 - 15x^3 + 26x^2 - 16x + 4\,,$$
$$P_{616} = x^5 + 4x^4 - 14x^3 + 18x^2 - 2x + 1\,,$$
$$P_{617} = 2x^5 + 5x^4 - 34x^3 + 52x^2 - 36x + 5\,,$$
$$P_{618} = 2x^5 + 5x^4 - 12x^3 + 20x^2 - x + 2\,,$$
$$P_{619} = 3x^5 - 9x^4 + 38x^3 - 50x^2 + 3x - 9\,,$$
$$P_{620} = 3x^5 + 9x^4 - 26x^3 + 38x^2 - 3x + 3\,,$$
$$P_{621} = 4x^5 + 19x^4 - 72x^3 + 96x^2 - 76x + 19\,,$$
$$P_{622} = 5x^5 - x^4 + 26x^3 - 18x^2 + 5x - 1\,,$$
$$P_{623} = 6x^5 - 51x^4 + 56x^3 - 43x^2 + 6x + 8\,,$$
$$P_{624} = 6x^5 - 47x^4 + 80x^3 - 51x^2 + 6x - 4\,,$$
$$P_{625} = 7x^5 - 24x^4 + 56x^3 - 32x^2 + 7x - 8\,,$$
$$P_{626} = 7x^5 - 22x^4 + 88x^3 - 46x^2 + 7x - 24\,,$$
$$P_{627} = 7x^5 + x^4 + 30x^3 - 14x^2 + 7x + 1\,,$$
$$P_{628} = 10x^5 - 5x^4 - 57x^3 + 66x^2 - 15x - 3\,,$$
$$P_{629} = 11x^5 - 83x^4 + 132x^3 - 85x^2 + 11x - 2\,,$$
$$P_{630} = 12x^5 - x^4 + 10x^3 - 28x^2 - 2x - 1\,,$$
$$P_{631} = 13x^5 + x^4 + 58x^3 - 30x^2 + 13x + 1\,,$$
$$P_{632} = 18x^5 - 37x^4 - 72x^3 + 118x^2 - 78x + 3\,,$$
$$P_{633} = 24x^5 - 34x^4 - 43x^3 + 25x^2 - x - 3\,,$$
$$P_{634} = 24x^5 - 3x^4 + 22x^3 - 52x^2 - 2x - 3\,,$$
$$P_{635} = 72x^5 + 115x^4 - 762x^3 + 644x^2 - 90x - 3\,,$$
$$P_{636} = 105x^5 - 195x^4 + 156x^3 - 68x^2 + 69x - 47\,,$$
$$P_{637} = 125x^5 - 279x^4 + 182x^3 - 42x^2 - 15x - 27\,,$$
$$P_{638} = -1367x^6 + 2934x^5 - 4861x^4 + 8604x^3 - 9253x^2 + 5094x - 575\,,$$
$$P_{639} = -543x^6 + 702x^5 - 157x^4 - 660x^3 + 1035x^2 - 450x + 137\,,$$
$$P_{640} = -89x^6 + 402x^5 - 499x^4 + 354x^3 - 37x^2 + 24x - 11\,,$$
$$P_{641} = -43x^6 + 6x^5 - 59x^4 + 104x^3 - x^2 + 6x + 15\,,$$
$$P_{642} = -29x^6 + 240x^5 - 287x^4 + 170x^3 + 19x^2 + 26x - 11\,,$$
$$P_{643} = -27x^6 + 2x^5 - 27x^4 + 56x^3 + 7x^2 + 2x + 7\,,$$

$$P_{644} = -13x^6 - 59x^5 + 14x^4 + 33x^3 - 30x^2 - 20x + 11\,,$$
$$P_{645} = -6x^6 - x^5 - 8x^4 - 28x^3 + 16x^2 - x + 18\,,$$
$$P_{646} = x^6 - 29x^5 + 117x^4 - 204x^3 + 117x^2 - 29x + 1\,,$$
$$P_{647} = x^6 - 27x^5 + 174x^4 - 276x^3 + 166x^2 - 27x - 7\,,$$
$$P_{648} = x^6 - 13x^5 + 52x^4 - 42x^3 + 8x^2 - 3x + 1\,,$$
$$P_{649} = x^6 - 8x^5 + 33x^4 - 52x^3 + 29x^2 - 8x - 3\,,$$
$$P_{650} = x^6 - 4x^5 + 17x^4 - 14x^3 + 15x^2 - 4x - 1\,,$$
$$P_{651} = x^6 + 2x^5 - 8x^4 + 11x^3 - 8x^2 + 2x + 1\,,$$
$$P_{652} = x^6 + 2x^5 - 7x^4 + 2x^3 - 6x^2 + 2x + 2\,,$$
$$P_{653} = x^6 + 2x^5 - 7x^4 + 7x^3 - 9x^2 + 2x - 1\,,$$
$$P_{654} = x^6 + 4x^5 - 31x^4 + 56x^3 - 37x^2 + 4x - 5\,,$$
$$P_{655} = x^6 + 5x^5 - 35x^4 + 36x^3 - 27x^2 + 5x + 9\,,$$
$$P_{656} = x^6 + 5x^5 - 19x^4 + 28x^3 - 19x^2 + 5x + 1\,,$$
$$P_{657} = x^6 + 9x^5 - 76x^4 + 148x^3 - 96x^2 + 9x - 19\,,$$
$$P_{658} = x^6 + 10x^5 - 71x^4 + 104x^3 - 69x^2 + 10x + 3\,,$$
$$P_{659} = x^6 + 36x^5 - 208x^4 + 664x^3 - 330x^2 + 36x - 121\,,$$
$$P_{660} = 2x^6 - 19x^5 + 78x^4 - 138x^3 + 78x^2 - 19x + 2\,,$$
$$P_{661} = 2x^6 - 11x^5 + 44x^4 - 78x^3 + 44x^2 - 11x + 2\,,$$
$$P_{662} = 2x^6 - 2x^5 + 13x^4 - 26x^3 + 12x^2 - 2x + 1\,,$$
$$P_{663} = 2x^6 - 2x^5 + 17x^4 - 32x^3 + 17x^2 - 2x + 2\,,$$
$$P_{664} = 2x^6 - x^5 + 6x^4 - 16x^3 + 5x^2 - x + 1\,,$$
$$P_{665} = 2x^6 + 2x^5 - 7x^4 + 40x^3 - 19x^2 + 2x - 10\,,$$
$$P_{666} = 2x^6 + 3x^5 - 16x^4 + 12x^3 - 14x^2 + 3x + 4\,,$$
$$P_{667} = 2x^6 + 5x^5 - 33x^4 + 48x^3 - 33x^2 + 5x + 2\,,$$
$$P_{668} = 2x^6 + 18x^5 - 147x^4 + 168x^3 - 127x^2 + 18x + 22\,,$$
$$P_{669} = 2x^6 + 35x^5 - 235x^4 + 356x^3 - 225x^2 + 35x + 12\,,$$
$$P_{670} = 3x^6 - 22x^5 + 169x^4 - 264x^3 + 167x^2 - 22x + 1\,,$$
$$P_{671} = 3x^6 - 8x^5 + 35x^4 - 58x^3 + 35x^2 - 8x + 3\,,$$
$$P_{672} = 3x^6 - 7x^5 + 31x^4 - 60x^3 + 32x^2 - 7x + 4\,,$$
$$P_{673} = 3x^6 - 7x^5 + 54x^4 - 84x^3 + 50x^2 - 7x - 1\,,$$
$$P_{674} = 3x^6 - 6x^5 + 50x^4 - 80x^3 + 48x^2 - 6x + 1\,,$$
$$P_{675} = 3x^6 - 5x^5 + 41x^4 - 20x^3 + 17x^2 - 5x - 21\,,$$
$$P_{676} = 3x^6 - 4x^5 + 25x^4 - 52x^3 + 25x^2 - 4x + 3\,,$$
$$P_{677} = 3x^6 - x^5 + 6x^4 - 20x^3 + 6x^2 - x + 3\,,$$
$$P_{678} = 3x^6 + 4x^5 - 13x^4 + 4x^3 - 13x^2 + 4x + 3\,,$$
$$P_{679} = 3x^6 + 7x^5 - 48x^4 + 84x^3 - 56x^2 + 7x - 5\,,$$
$$P_{680} = 3x^6 + 35x^5 - 207x^4 - 118x^3 + 355x^2 - x - 19\,,$$
$$P_{681} = 3x^6 + 68x^5 + 139x^4 - 76x^3 - 81x^2 + 152x - 37\,,$$
$$P_{682} = 3x^6 + 70x^5 - 519x^4 + 776x^3 - 513x^2 + 70x + 9\,,$$
$$P_{683} = 4x^6 - 26x^5 + 173x^4 - 296x^3 + 173x^2 - 26x + 4\,,$$

$$P_{684} = 4x^6 - 7x^5 + 28x^4 - 72x^3 + 36x^2 - 7x + 12\,,$$
$$P_{685} = 4x^6 + 3x^5 - 12x^4 - 18x^3 - 8x^2 + 3x + 8\,,$$
$$P_{686} = 4x^6 + 37x^5 - 261x^4 + 372x^3 - 247x^2 + 37x + 18\,,$$
$$P_{687} = 5x^6 - 28x^5 + 115x^4 - 190x^3 + 115x^2 - 28x + 5\,,$$
$$P_{688} = 5x^6 - 15x^5 + 82x^4 - 180x^3 + 90x^2 - 15x + 13\,,$$
$$P_{689} = 5x^6 - 12x^5 + 47x^4 - 106x^3 + 51x^2 - 12x + 9\,,$$
$$P_{690} = 5x^6 - 10x^5 - 60x^4 + 59x^3 - 16x^2 - 7x + 5\,,$$
$$P_{691} = 5x^6 - 8x^5 + 49x^4 - 104x^3 + 51x^2 - 8x + 7\,,$$
$$P_{692} = 5x^6 + 3x^5 - 15x^4 + 12x^3 - 17x^2 + 3x + 3\,,$$
$$P_{693} = 5x^6 + 4x^5 - 11x^4 + 4x^3 - 15x^2 + 4x + 1\,,$$
$$P_{694} = 5x^6 + 10x^5 - 58x^4 + 32x^3 - 40x^2 + 10x + 23\,,$$
$$P_{695} = 6x^6 - 7x^5 + 30x^4 - 56x^3 + 26x^2 - 7x + 2\,,$$
$$P_{696} = 7x^6 - 38x^5 + 287x^4 - 456x^3 + 285x^2 - 38x + 5\,,$$
$$P_{697} = 7x^6 - 21x^5 + 95x^4 - 122x^3 + 63x^2 - 21x - 25\,,$$
$$P_{698} = 7x^6 - 8x^5 + 71x^4 - 112x^3 + 65x^2 - 8x + 1\,,$$
$$P_{699} = 7x^6 - 2x^5 + 16x^4 - 40x^3 + 10x^2 - 2x + 1\,,$$
$$P_{700} = 7x^6 + 6x^5 - 33x^4 + 24x^3 - 31x^2 + 6x + 9\,,$$
$$P_{701} = 7x^6 + 6x^5 - 29x^4 + 24x^3 - 31x^2 + 6x + 5\,,$$
$$P_{702} = 7x^6 + 76x^5 + 231x^4 - 952x^3 + 961x^2 - 292x - 15\,,$$
$$P_{703} = 8x^6 - 59x^5 + 410x^4 - 644x^3 + 392x^2 - 59x - 10\,,$$
$$P_{704} = 10x^6 - 55x^5 + 230x^4 - 422x^3 + 230x^2 - 55x + 10\,,$$
$$P_{705} = 10x^6 - 33x^5 + 143x^4 - 127x^3 + 106x^2 - 33x - 27\,,$$
$$P_{706} = 10x^6 - 27x^5 + 116x^4 - 210x^3 + 120x^2 - 27x + 14\,,$$
$$P_{707} = 10x^6 + 9x^5 - 52x^4 + 52x^3 - 54x^2 + 9x + 8\,,$$
$$P_{708} = 10x^6 + 9x^5 - 46x^4 + 52x^3 - 48x^2 + 9x + 8\,,$$
$$P_{709} = 11x^6 - 96x^5 - 53x^4 + 946x^3 - 911x^2 + 166x - 47\,,$$
$$P_{710} = 11x^6 - 51x^5 + 349x^4 - 564x^3 + 333x^2 - 51x - 5\,,$$
$$P_{711} = 11x^6 - 36x^5 + 165x^4 - 246x^3 + 133x^2 - 36x - 21\,,$$
$$P_{712} = 11x^6 - 23x^5 + 163x^4 - 292x^3 + 163x^2 - 23x + 11\,,$$
$$P_{713} = 11x^6 - 18x^5 + 76x^4 - 159x^3 + 75x^2 - 18x + 10\,,$$
$$P_{714} = 11x^6 + 32x^5 + 107x^4 - 728x^3 + 597x^2 - 72x + 37\,,$$
$$P_{715} = 11x^6 + 42x^5 - 285x^4 + 424x^3 - 287x^2 + 42x + 9\,,$$
$$P_{716} = 12x^6 + x^5 + 12x^4 + 28x^3 - 22x^2 + x - 22\,,$$
$$P_{717} = 12x^6 + 29x^5 - 242x^4 + 300x^3 - 236x^2 + 29x + 18\,,$$
$$P_{718} = 13x^6 - 57x^5 + 233x^4 - 408x^3 + 237x^2 - 57x + 17\,,$$
$$P_{719} = 13x^6 - 17x^5 + 117x^4 - 228x^3 + 113x^2 - 17x + 9\,,$$
$$P_{720} = 13x^6 - 3x^5 + 17x^4 - 56x^3 + 15x^2 - 3x + 11\,,$$
$$P_{721} = 13x^6 + 2x^5 - 37x^4 + 236x^3 - 337x^2 + 282x - 63\,,$$
$$P_{722} = 13x^6 + 7x^5 - 36x^4 + 28x^3 - 36x^2 + 7x + 13\,,$$
$$P_{723} = 13x^6 + 36x^5 - 141x^4 + 246x^3 - 157x^2 + 36x - 3\,,$$

$$P_{724} = 13x^6 + 39x^5 - 133x^4 + 196x^3 - 165x^2 + 39x - 19\,,$$
$$P_{725} = 13x^6 + 118x^5 - 869x^4 + 1208x^3 - 827x^2 + 118x + 55\,,$$
$$P_{726} = 13x^6 + 120x^5 - 859x^4 + 1264x^3 - 837x^2 + 120x + 35\,,$$
$$P_{727} = 14x^6 - 9x^5 + 91x^4 - 148x^3 + 81x^2 - 9x + 4\,,$$
$$P_{728} = 14x^6 - 8x^5 + 57x^4 - 144x^3 + 57x^2 - 8x + 14\,,$$
$$P_{729} = 14x^6 + 2x^5 - 3x^4 - 16x^3 - 9x^2 + 2x + 8\,,$$
$$P_{730} = 15x^6 - 28x^5 + 127x^4 - 240x^3 + 125x^2 - 28x + 13\,,$$
$$P_{731} = 15x^6 + 20x^5 - 122x^4 + 152x^3 - 124x^2 + 20x + 13\,,$$
$$P_{732} = 15x^6 + 426x^5 - 543x^4 + 320x^3 - 71x^2 + 230x - 41\,,$$
$$P_{733} = 16x^6 - 43x^5 + 186x^4 - 346x^3 + 190x^2 - 43x + 20\,,$$
$$P_{734} = 16x^6 - 27x^5 + 114x^4 - 220x^3 + 110x^2 - 27x + 12\,,$$
$$P_{735} = 17x^6 - 166x^5 + 489x^4 - 716x^3 + 475x^2 - 158x + 11\,,$$
$$P_{736} = 17x^6 - 18x^5 + 121x^4 - 220x^3 + 105x^2 - 18x + 1\,,$$
$$P_{737} = 17x^6 + 21x^5 - 71x^4 + 122x^3 - 87x^2 + 21x + 1\,,$$
$$P_{738} = 17x^6 + 28x^5 - 183x^4 + 224x^3 - 177x^2 + 28x + 23\,,$$
$$P_{739} = 17x^6 + 34x^5 - 230x^4 + 304x^3 - 228x^2 + 34x + 19\,,$$
$$P_{740} = 18x^6 - 16x^5 + 73x^4 - 158x^3 + 67x^2 - 16x + 12\,,$$
$$P_{741} = 18x^6 - 9x^5 + 94x^4 - 172x^3 + 88x^2 - 9x + 12\,,$$
$$P_{742} = 18x^6 + x^5 - 68x^3 + 2x^2 + x + 20\,,$$
$$P_{743} = 19x^6 - 73x^5 + 303x^4 - 544x^3 + 307x^2 - 73x + 23\,,$$
$$P_{744} = 19x^6 - 8x^5 + 43x^4 - 128x^3 + 37x^2 - 8x + 13\,,$$
$$P_{745} = 19x^6 + 24x^5 - 73x^4 + 100x^3 - 105x^2 + 24x - 13\,,$$
$$P_{746} = 19x^6 + 42x^5 - 147x^4 + 196x^3 - 137x^2 + 42x + 29\,,$$
$$P_{747} = 19x^6 + 62x^5 - 433x^4 + 536x^3 - 395x^2 + 62x + 57\,,$$
$$P_{748} = 20x^6 - 303x^5 + 1214x^4 - 1692x^3 + 1238x^2 - 303x + 44\,,$$
$$P_{749} = 20x^6 - 8x^5 + 51x^4 - 101x^3 + 30x^2 - 8x - 1\,,$$
$$P_{750} = 21x^6 - 53x^5 + 227x^4 - 438x^3 + 231x^2 - 53x + 25\,,$$
$$P_{751} = 21x^6 - 11x^5 + 59x^4 - 134x^3 + 55x^2 - 11x + 17\,,$$
$$P_{752} = 21x^6 + 8x^5 - 25x^4 + 8x^3 - 35x^2 + 8x + 11\,,$$
$$P_{753} = 22x^6 - 171x^5 + 708x^4 - 892x^3 + 634x^2 - 171x - 52\,,$$
$$P_{754} = 22x^6 - 61x^5 + 162x^4 - 118x^3 - 12x^2 + 95x - 40\,,$$
$$P_{755} = 23x^6 + 37x^5 - 429x^4 + 596x^3 - 277x^2 - 17x + 19\,,$$
$$P_{756} = 23x^6 + 68x^5 - 295x^4 + 448x^3 - 317x^2 + 68x + 1\,,$$
$$P_{757} = 24x^6 - 67x^5 + 282x^4 - 564x^3 + 280x^2 - 67x + 22\,,$$
$$P_{758} = 25x^6 - 174x^5 + 489x^4 - 716x^3 + 467x^2 - 150x + 11\,,$$
$$P_{759} = 25x^6 - 6x^5 + 21x^4 - 112x^3 + 19x^2 - 6x + 23\,,$$
$$P_{760} = 25x^6 - 6x^5 + 33x^4 - 112x^3 + 31x^2 - 6x + 23\,,$$
$$P_{761} = 25x^6 + 5x^5 - 5x^4 + 22x^3 - 43x^2 + 5x - 13\,,$$
$$P_{762} = 27x^6 - 32x^5 + 129x^4 - 344x^3 + 151x^2 - 32x + 49\,,$$
$$P_{763} = 28x^6 + 27x^5 - 416x^4 + 596x^3 - 290x^2 - 7x + 14\,,$$

$$P_{764} = 29x^6 - 348x^5 + 511x^4 - 354x^3 + 85x^2 - 78x + 11\,,$$
$$P_{765} = 29x^6 + 30x^5 - 139x^4 + 120x^3 - 137x^2 + 30x + 31\,,$$
$$P_{766} = 29x^6 + 435x^5 - 1475x^4 + 1674x^3 - 731x^2 - 9x + 65\,,$$
$$P_{767} = 31x^6 + 8x^5 - 29x^4 - 44x^3 - 35x^2 + 8x + 25\,,$$
$$P_{768} = 31x^6 + 8x^5 - 13x^4 - 64x^3 - 11x^2 + 8x + 33\,,$$
$$P_{769} = 31x^6 + 20x^5 - 69x^4 + 36x^3 - 87x^2 + 20x + 13\,,$$
$$P_{770} = 31x^6 + 30x^5 - 141x^4 + 120x^3 - 139x^2 + 30x + 33\,,$$
$$P_{771} = 32x^6 - 151x^5 + 636x^4 - 1240x^3 + 650x^2 - 151x + 46\,,$$
$$P_{772} = 34x^6 - 91x^5 + 382x^4 - 746x^3 + 384x^2 - 91x + 36\,,$$
$$P_{773} = 34x^6 + x^5 + 8x^4 - 68x^3 - 14x^2 + x + 12\,,$$
$$P_{774} = 35x^6 - 90x^5 + 388x^4 - 718x^3 + 378x^2 - 90x + 25\,,$$
$$P_{775} = 37x^6 - 118x^5 + 343x^4 - 716x^3 + 639x^2 - 206x - 27\,,$$
$$P_{776} = 37x^6 - 59x^5 - 233x^4 + 714x^3 - 541x^2 + 73x - 31\,,$$
$$P_{777} = 37x^6 - 57x^5 + 371x^4 - 668x^3 + 339x^2 - 57x + 5\,,$$
$$P_{778} = 38x^6 - 107x^5 + 466x^4 - 932x^3 + 456x^2 - 107x + 28\,,$$
$$P_{779} = 39x^6 - 112x^5 + 413x^4 - 900x^3 + 769x^2 - 228x - 21\,,$$
$$P_{780} = 40x^6 + 69x^5 - 463x^4 + 628x^3 - 465x^2 + 69x + 38\,,$$
$$P_{781} = 42x^6 - 41x^5 + 182x^4 - 380x^3 + 166x^2 - 41x + 26\,,$$
$$P_{782} = 42x^6 + 2x^5 + 3x^4 - 40x^3 - 41x^2 + 2x - 2\,,$$
$$P_{783} = 43x^6 + 20x^5 - 27x^4 + 16x^3 - 73x^2 + 20x - 3\,,$$
$$P_{784} = 44x^6 + 67x^5 + 304x^4 - 1272x^3 + 1264x^2 - 371x - 20\,,$$
$$P_{785} = 46x^6 - 39x^5 + 271x^4 - 420x^3 + 197x^2 - 39x - 28\,,$$
$$P_{786} = 47x^6 - 154x^5 + 53x^4 + 580x^3 - 701x^2 + 110x - 23\,,$$
$$P_{787} = 50x^6 - 55x^5 + 170x^4 - 732x^3 + 814x^2 - 333x - 2\,,$$
$$P_{788} = 50x^6 + 37x^5 - 310x^4 + 364x^3 - 348x^2 + 37x + 12\,,$$
$$P_{789} = 51x^6 + 32x^5 - 117x^4 + 32x^3 - 107x^2 + 32x + 61\,,$$
$$P_{790} = 53x^6 - 33x^5 - 279x^4 + 596x^3 - 427x^2 + 53x - 11\,,$$
$$P_{791} = 55x^6 + 105x^5 - 1711x^4 + 2400x^3 - 889x^2 + 135x - 71\,,$$
$$P_{792} = 56x^6 - 141x^5 + 930x^4 - 1916x^3 + 1004x^2 - 141x + 130\,,$$
$$P_{793} = 59x^6 - 446x^5 + 203x^4 + 872x^3 - 1215x^2 + 398x + 9\,,$$
$$P_{794} = 63x^6 - 205x^5 + 273x^4 + 118x^3 - 537x^2 + 347x - 11\,,$$
$$P_{795} = 65x^6 + 154x^5 - 1187x^4 + 1624x^3 - 1209x^2 + 154x + 43\,,$$
$$P_{796} = 67x^6 - 105x^5 + 485x^4 - 938x^3 + 465x^2 - 105x + 47\,,$$
$$P_{797} = 71x^6 + 132x^5 - 461x^4 + 508x^3 - 535x^2 + 132x - 3\,,$$
$$P_{798} = 77x^6 - 144x^5 + 1043x^4 - 1760x^3 + 969x^2 - 144x + 3\,,$$
$$P_{799} = 81x^6 - 571x^5 - 129x^4 + 1738x^3 - 1887x^2 + 461x + 43\,,$$
$$P_{800} = 91x^6 - 48x^5 - 99x^4 - 638x^3 + 595x^2 - 50x + 29\,,$$
$$P_{801} = 96x^6 - 163x^5 + 758x^4 - 1528x^3 + 698x^2 - 163x + 36\,,$$
$$P_{802} = 98x^6 - 221x^5 + 988x^4 - 1976x^3 + 944x^2 - 221x + 54\,,$$
$$P_{803} = 111x^6 - 139x^5 + 111x^4 - 1216x^3 + 985x^2 - 169x + 53\,,$$

$$
\begin{aligned}
P_{804} &= 117x^6 + 576x^5 - 2249x^4 + 3792x^3 - 2523x^2 + 576x - 157\,,\\
P_{805} &= 119x^6 - 132x^5 + 603x^4 - 1194x^3 + 535x^2 - 132x + 51\,,\\
P_{806} &= 133x^6 - 760x^5 + 1463x^4 - 1204x^3 + 231x^2 + 100x - 11\,,\\
P_{807} &= 139x^6 + 41x^5 - 335x^4 + 308x^3 - 467x^2 + 41x + 7\,,\\
P_{808} &= 141x^6 + 79x^5 - 621x^4 + 724x^3 - 749x^2 + 79x + 13\,,\\
P_{809} &= 145x^6 - 634x^5 + 3441x^4 - 7530x^3 + 6725x^2 - 2952x + 637\,,\\
P_{810} &= 146x^6 + 877x^5 - 3462x^4 + 3876x^3 - 1458x^2 - 263x + 200\,,\\
P_{811} &= 181x^6 - 72x^5 + 599x^4 - 1328x^3 + 477x^2 - 72x + 59\,,\\
P_{812} &= 337x^6 - 154x^5 - 165x^4 - 68x^3 + 283x^2 - 186x + 17\,,\\
P_{813} &= 443x^6 + 2898x^5 - 5681x^4 + 4104x^3 - 2225x^2 + 1314x - 565\,,\\
P_{814} &= 649x^6 - 210x^5 + 2025x^4 - 3960x^3 + 1323x^2 - 210x - 53\,. \qquad \text{(C.4)}
\end{aligned}
$$

$$
\begin{aligned}
F_{A,2}^{(1)} = C_F \Bigg[& \Bigg\{ -8x(1+x)^2\eta^2 - 4x(1+x)^2\Big(3-2x+3x^2\Big)\eta^3 H_0 \Bigg\} + \varepsilon \Bigg\{ -16x(1+x)^2\eta^2 \\
& + \eta^3\Big(-8x(1+x)^2\Big(3-2x+3x^2\Big)H_0 + 8x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}H_0 - 2x(1+x)^2\Big(3-2x \\
& +3x^2\Big)H_0^2 - 8x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,-1} + 4x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2\Big)\Bigg\} + \varepsilon^2\Bigg\{ -32x(1+x)^2\eta^2 \\
& + \eta^3\Big(\Big(-16x(1+x)^2\Big(3-2x+3x^2\Big) + 16x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1} - 8x(1+x)^2\Big(3-2x \\
& +3x^2\Big)H_{-1}^2 + 2x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2\Big)H_0 + \Big(-4x(1+x)^2\Big(3-2x+3x^2\Big) + 4x(1+x)^2\Big(3-2x \\
& +3x^2\Big)H_{-1}\Big)H_0^2 - \frac{2}{3}x(1+x)^2\Big(3-2x+3x^2\Big)H_0^3 + \Big(-16x(1+x)^2\Big(3-2x+3x^2\Big) + 16x(1+x)^2\Big(3 \\
& -2x+3x^2\Big)H_{-1}\Big)H_{0,-1} - 8x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,0,-1} - 16x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,-1,-1} \\
& + 4x(1+x)^2\Big(5-4x+7x^2\Big)\zeta_2 - 8x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}\zeta_2 + 8x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_3\Big)\Bigg\} \\
& + \varepsilon^3\Bigg\{ -64x(1+x)^2\eta^2 + \eta^3\Big(\Big(-32x(1+x)^2\Big(3-2x+3x^2\Big) + \Big(32x(1+x)^2\Big(3-2x+3x^2\Big) \\
& - 4x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2\Big)H_{-1} - 16x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}^2 + \frac{16}{3}x(1+x)^2\Big(3-2x \\
& +3x^2\Big)H_{-1}^3 + 4x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2 + \frac{28}{3}x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_3\Big)H_0 + \Big(-8x(1+x)^2\Big(3 \\
& -2x+3x^2\Big) + 8x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1} - 4x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}^2 + x(1+x)^2\Big(3 \\
& -2x+3x^2\Big)\zeta_2\Big)H_0^2 + \Big(-\frac{4}{3}x(1+x)^2\Big(3-2x+3x^2\Big) + \frac{4}{3}x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}\Big)H_0^3 \\
& - \frac{1}{6}x(1+x)^2\Big(3-2x+3x^2\Big)H_0^4 + \Big(-16x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2 - 16x(1+x)^2\Big(3-2x \\
& +3x^2\Big)\zeta_3\Big)H_{-1} + \Big(-32x(1+x)^2\Big(3-2x+3x^2\Big) + 32x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1} - 16x(1+x)^2\Big(3 \\
& -2x+3x^2\Big)H_{-1}^2 - 4x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2\Big)H_{0,-1} + \Big(-16x(1+x)^2\Big(3-2x+3x^2\Big) \\
& + 16x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}\Big)H_{0,0,-1} + \Big(-32x(1+x)^2\Big(3-2x+3x^2\Big) + 32x(1+x)^2\Big(3-2x \\
& +3x^2\Big)H_{-1}\Big)H_{0,-1,-1} - 8x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,0,0,-1} - 16x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,0,-1,-1} \\
& - 32x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,-1,-1,-1} + 8x(1+x)^2\Big(5-4x+7x^2\Big)\zeta_2 + 8x(1+x)^2\Big(3 \\
& - 2x+3x^2\Big)H_{-1}^2\zeta_2 + \frac{28}{5}x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2^2 + \frac{8}{3}x(1+x)^2\Big(19-12x+17x^2\Big)\zeta_3\Big)\Bigg\}\Bigg] .
\end{aligned}
\tag{C.5}
$$

$$
F_{A,2}^{(2),\mathrm{ns}} = C_F^2\Bigg[\frac{1}{\varepsilon}\Bigg\{16x(1+x)^2\eta^2 + 8x(1+x)^2\Big(5-2x+5x^2\Big)\eta^3 H_0 + 8x(1+x)^2\Big(3-2x+3x^2\Big)\xi\eta^3 H_0^2\Bigg\}
$$

$$+\Big\{68x(1+x)^2\eta^2+\eta^3\Big(2x(1+x)^2\Big(61-22x+61x^2\Big)H_0+8x(1+x)^2\Big(25-86x+25x^2\Big)H_{-1}H_0$$
$$+\Big(128x^2(1+x)^2H_0-\frac{1088}{3}x^3H_0^3\Big)H_1-128x^2(1+x)^2H_{0,1}-8x(1+x)^2\Big(25-86x$$
$$+25x^2\Big)H_{0,-1}+\Big(-2176x^3H_0H_1+2176x^3H_{0,1}\Big)\zeta_2\Big)+\xi^2\Big(-192(-1+x)x^2(1+x)^3\eta^4\zeta_2H_{0,-1}$$
$$-144(-1+x)x(1+x)^3\eta^3H_{-1}\zeta_2\Big)+\xi\Big(\eta^3\Big(\Big(128(-1+x)x(1+x)^3H_0-\frac{64}{3}(-1+x)x^2(1+x)H_0^3\Big)H_1$$
$$-128(-1+x)x(1+x)^3H_{0,1}+\Big(-128(-1+x)x^2(1+x)H_0H_1-576x^2(1+x)^2H_{-1}$$
$$+128(-1+x)x^2(1+x)H_{0,1}\Big)\zeta_2\Big)-960x^3(1+x)^2\eta^4H_{0,-1}\zeta_2\Big)+\eta^5\Big(-64x^2(1+x)^2H_0H_{0,0,-1}P_{817}$$
$$-32x^2(1+x)^2H_0^2H_{0,1}P_{818}+32x^2(1+x)^2H_0^2H_{0,-1}P_{820}+\frac{4}{3}x^2(1+x)^2H_0^4P_{835}$$
$$+64x^2(1+x)^2H_{0,0,0,1}P_{843}-\frac{8}{3}x(1+x)^2H_0^3P_{1018}+\frac{32}{5}x^2(1+x)^2P_{816}\zeta_2^2-1024x^3(1+x)^2\Big(5$$
$$-7x+5x^2\Big)H_0H_{0,0,1}-384x^3(1+x)^2\Big(28-41x+28x^2\Big)H_{0,0,0,-1}+\Big(16x(1+x)^2H_0P_{1037}$$
$$-96x^3(1+x)^2\Big(2-9x+2x^2\Big)H_0^2\Big)\zeta_2-64x^2(1+x)^2H_0P_{821}\zeta_3\Big)+\eta^4\Big(16x(1+x)^2H_0^2H_1P_{868}$$
$$-32x(1+x)^2H_0H_{0,1}P_{900}+32x(1+x)^2H_{0,0,1}P_{930}-8x(1+x)^2H_{-1}H_0^2P_{948}$$
$$+16x(1+x)^2H_0H_{0,-1}P_{962}-16x(1+x)^2H_{0,0,-1}P_{968}+2x(1+x)^2H_0^2P_{1002}+4x(1+x)^2P_{985}\zeta_2$$
$$+16x(1+x)^2P_{862}\zeta_3\Big)-192x\log(2)x_+^2\zeta_2\Big\}+\varepsilon\Big\{\eta^2\Big(32c_1x(x-1)^2+242x(x+1)^2\Big)$$
$$+\xi\Big(\eta^4\Big(\log(2)\zeta_2\Big(-384(-1+x)x^2(1+x)^3H_1-384(-1+x)x^2(1+x)^3H_{-1}\Big)$$
$$+64(-1+x)x^2(1+x)^3H_{-1}H_0^2H_1+\Big(384(-1+x)x^2(1+x)^3H_{-1}H_1-1920x^3(1+x)^2H_{-1}H_{0,-1}$$
$$-384(-1+x)x^2(1+x)^3H_{-1,1}\Big)\zeta_2\Big)+\eta^3\Big(\Big(-768(-1+x)x(1+x)^3H_{-1}H_0$$
$$-\frac{128}{3}(-1+x)x^2(1+x)H_{-1}H_0^3\Big)H_1+\Big(128(-1+x)x(1+x)^3H_0+\frac{64}{3}(-1+x)x^2(1+x)H_0^3\Big)H_1^2$$
$$+\Big(-256(-1+x)x(1+x)^3H_1+768(-1+x)x(1+x)^3H_{-1}\Big)H_{0,1}+768(-1+x)x(1+x)^3H_1H_{0,-1}$$
$$+256(-1+x)x(1+x)^3H_{0,1,1}-768(-1+x)x(1+x)^3H_{0,1,-1}-768(-1+x)x(1+x)^3H_{0,-1,1}+\Big(\Big($$
$$-256(-1+x)x^2(1+x)H_{-1}H_1+128(-1+x)x^2(1+x)H_1^2\Big)H_0-128(-1+x)x(1+x)^3H_1+\Big($$
$$-256(-1+x)x^2(1+x)H_1+256(-1+x)x^2(1+x)H_{-1}\Big)H_{0,1}+256(-1+x)x^2(1+x)H_{0,1,1}\Big)\zeta_2\Big)$$
$$+384(-1+x)x^2(1+x)^3\Big(5-x+5x^2\Big)\eta^5\zeta_2H_{0,-1,-1}\Big)+\eta^3\Big(5x(1+x)^2\Big(67-18x+67x^2\Big)H_0$$
$$+16x(1+x)^2\Big(75-166x+75x^2\Big)H_{-1}H_0-8x(1+x)^2\Big(127-298x+127x^2\Big)H_{-1}^2H_0$$
$$+\Big(-208x(1+x)^2\Big(3-2x+3x^2\Big)H_0+\Big(-768x^2(1+x)^2H_0-\frac{2176}{3}x^3H_0^3\Big)H_{-1}\Big)H_1$$
$$+\Big(128x^2(1+x)^2H_0+\frac{1088}{3}x^3H_0^3\Big)H_1^2+\Big(208x(1+x)^2\Big(3-2x+3x^2\Big)-256x^2(1+x)^2H_1$$
$$+768x^2(1+x)^2H_{-1}\Big)H_{0,1}+\Big(-16x(1+x)^2\Big(75-166x+75x^2\Big)+768x^2(1+x)^2H_1$$

$$
+ 16x(1+x)^2\Big(127-298x+127x^2\Big)H_{-1}\Big)H_{0,-1} + 256x^2(1+x)^2H_{0,1,1} - 768x^2(1+x)^2H_{0,1,-1}
- 768x^2(1+x)^2H_{0,-1,1} - 16x(1+x)^2\Big(127-298x+127x^2\Big)H_{0,-1,-1} + \Big(\Big(- 4352x^3H_{-1}H_1
+ 2176x^3H_1^2\Big)H_0 - 128x^2(1+x)^2H_1 + \Big(- 4352x^3H_1 + 4352x^3H_{-1}\Big)H_{0,1} + 4352x^3H_{0,1,1}\Big)\zeta_2\Big)
+ \eta^4\Big(384x(1+x)^2H_{0,0,1,-1}P_{837} + 384x(1+x)^2H_{0,0,-1,1}P_{837} + 192x(1+x)^2H_{0,0,1,1}P_{840}
+ 16x(1+x)^2H_0^2H_1^2P_{867} + 192x(1+x)^2H_{0,-1,0,1}P_{869} + 192x(1+x)^2H_0H_{0,1,-1}P_{883}
+ 192x(1+x)^2H_0H_{0,-1,1}P_{883} + 32x(1+x)^2H_{0,1}^2P_{903} - 64x(1+x)^2H_0H_{0,1,1}P_{904}
- 96x(1+x)^2H_{0,0,-1,-1}P_{916} - 32x(1+x)^2H_0H_{0,-1,-1}P_{960} + 8x(1+x)^2H_{-1}^2H_0^2P_{969}
+ 16x(1+x)^2H_{0,-1}^2P_{983} - 4x(1+x)^2H_{-1}H_0^2P_{1008} + 2x(1+x)^2H_0^2P_{1012} + \log(2)\zeta_2\Big(
- 384x(1+x)^2P_{865} + 1536x^3(1+x)^2H_1 + 1536x^3(1+x)^2H_{-1}\Big) + \Big(8x(1+x)^2H_0^2P_{984}
- 256x^3(1+x)^2H_{-1}H_0^2\Big)H_1 + \Big(64x(1+x)^2H_{-1}H_0P_{870} - 32x(1+x)^2H_0P_{911}
-64x(1+x)^2H_{0,-1}P_{947}+256x^2(1+x)^2\Big(2+15x+2x^2\Big)H_0H_1\Big)H_{0,1}+\Big(-128x(1+x)^2H_{-1}H_0P_{901}
- 64x(1+x)^2H_0H_1P_{902} - 16x(1+x)^2H_0P_{949}\Big)H_{0,-1} + \Big(- 384x(1+x)^2H_{-1}P_{837}
- 192x(1+x)^2H_1P_{840} - 16x(1+x)^2P_{957}\Big)H_{0,0,1} + \Big(768x(1+x)^2H_1P_{838} + 96x(1+x)^2H_{-1}P_{916}
+ 8x(1+x)^2P_{1009}\Big)H_{0,0,-1} + \Big(- 48x(1+x)^2H_{-1}^2P_{929} + 8x(1+x)^2H_{-1}P_{997} + 4x(1+x)^2P_{1006}
- 1536x^3(1+x)^2H_{-1}H_1 + 1536x^3(1+x)^2H_{-1,1}\Big)\zeta_2 + \Big(- 192x(1+x)^2H_1P_{850}
+ 96x(1+x)^2H_{-1}P_{879} - 16x(1+x)^2P_{992}\Big)\zeta_3\Big) + \eta^5\Big(- 768x^2(1+x)^2H_0H_{0,0,1,-1}P_{819}
- 768x^2(1+x)^2H_0H_{0,0,-1,1}P_{819} - 1024x^2(1+x)^2H_0H_{0,0,1,1}P_{833} + \frac{4}{5}x^2(1+x)^2H_0^5P_{834}
- 512x^2(1+x)^2H_{0,0,-1,0,-1}P_{845} - 256x^2(1+x)^2H_0H_{0,1}^2P_{846} - 768x^2(1+x)^2H_{0,0,0,-1,-1}P_{874}
+ 128x^2(1+x)^2H_0H_{0,0,-1,-1}P_{886} + 512x^2(1+x)^2H_{0,0,0,1,-1}P_{887} + 512x^2(1+x)^2H_{0,0,0,-1,1}P_{887}
- 64x^2(1+x)^2H_0^2H_{0,-1,-1}P_{888} + 128x^2(1+x)^2H_{0,0,1,0,-1}P_{905} - 128x^2(1+x)^2H_0H_{0,-1,0,1}P_{907}
- 128x^2(1+x)^2H_{0,0,1,0,1}P_{921} + 64x^2(1+x)^2H_0^2H_{0,1,1}P_{927} - 128x^2(1+x)^2H_{0,0,0,1,1}P_{934}
+ 128x^2(1+x)^2H_{0,0,-1,0,1}P_{943} + 64x^2(1+x)^2H_{0,0,0,0,-1}P_{963} - 64x^2(1+x)^2H_{0,0,0,0,1}P_{970}
- \frac{2}{3}x(1+x)^2H_0^4P_{1045} - \frac{2}{3}x(1+x)^2H_0^3P_{1060} + \log(2)\zeta_2\Big(1536x^3(1+x)^2\Big(1+x+x^2\Big)H_{0,1}
+ 1536x^3(1+x)^2\Big(1+x+x^2\Big)H_{0,-1}\Big) + \Big(\frac{8}{3}x^2(1+x)^2H_0^4P_{825} - \frac{16}{3}x(1+x)^2H_0^3P_{1043}\Big)H_1
+ \Big(\frac{8}{3}x^2(1+x)^2H_0^4P_{835} + \frac{8}{3}x(1+x)^2H_0^3P_{1050}\Big)H_{-1} + \Big(- 64x^2(1+x)^2H_{-1}H_0^2P_{818}
- 128x^2(1+x)^2H_0^2H_1P_{832} + 256x^2(1+x)^2H_0H_{0,-1}P_{875} + 16x(1+x)^2H_0^2P_{1044}
+ \frac{64}{3}x^3(1+x)^2\Big(74-79x+74x^2\Big)H_0^3\Big)H_{0,1} + \Big(64x^2(1+x)^2H_{-1}H_0^2P_{820} - 64x^2(1+x)^2H_0^2H_1P_{823}
+ \frac{32}{3}x^2(1+x)^2H_0^3P_{906} - 8x(1+x)^2H_0^2P_{1054}\Big)H_{0,-1} - 384x^3(1+x)^2\Big(10-17x+10x^2\Big)H_0H_{0,-1}^2
$$

$$
+\Big(-128x^2(1+x)^2H_{0,-1}P_{826}+1024x^2(1+x)^2H_0H_1P_{831}-32x^2(1+x)^2H_0^2P_{873}
+128x^2(1+x)^2H_{0,1}P_{922}-64x(1+x)^2H_0P_{1031}-2048x^3(1+x)^2\Big(5-7x+5x^2\Big)H_{-1}H_0\Big)H_{0,0,1}
+\Big(-128x^2(1+x)^2H_{-1}H_0P_{817}-1024x^2(1+x)^2H_0H_1P_{831}+512x^2(1+x)^2H_{0,-1}P_{839}
-96x^2(1+x)^2H_0^2P_{853}-128x^2(1+x)^2H_{0,1}P_{944}+16x(1+x)^2H_0P_{1049}\Big)H_{0,0,-1}-128x^3(1+x)^2\Big(26
-25x+26x^2\Big)H_0^2H_{0,1,-1}-128x^3(1+x)^2\Big(26-25x+26x^2\Big)H_0^2H_{0,-1,1}+\Big(128x^2(1+x)^2H_{-1}P_{843}
-1024x^2(1+x)^2H_1P_{847}+64x^2(1+x)^2H_0P_{937}-32x(1+x)^2P_{1038}\Big)H_{0,0,0,1}+\Big(64x^2(1+x)^2H_0P_{815}
+128x^2(1+x)^2H_1P_{965}+16x(1+x)^2P_{1055}-768x^3(1+x)^2\Big(28-41x+28x^2\Big)H_{-1}\Big)H_{0,0,0,-1}
+\Big(64x^2(1+x)^2H_{0,0,1}P_{841}-512x^2(1+x)^2H_{0,0,-1}P_{851}-\frac{32}{3}x^2(1+x)^2H_0^3P_{852}
+128x^2(1+x)^2H_{0,1,-1}P_{890}+128x^2(1+x)^2H_{0,-1,1}P_{891}-8x(1+x)^2H_0^2P_{1047}+4x(1+x)^2H_0P_{1059}
+\Big(-32x^2(1+x)^2H_0^2P_{889}-32x(1+x)^2H_0P_{1035}\Big)H_1+\Big(-16x(1+x)^2H_0P_{1048}
-192x^3(1+x)^2\Big(2-9x+2x^2\Big)H_0^2\Big)H_{-1}+\Big(128x^2(1+x)^2H_0P_{822}+32x(1+x)^2P_{1032}\Big)H_{0,1}
+\Big(128x^2(1+x)^2H_0P_{877}-128x^2(1+x)^2H_1P_{891}+16x(1+x)^2P_{1026}\Big)H_{0,-1}+1536x^3(1+x)^2\Big(5
-x+5x^2\Big)H_{0,-1,-1}+64x^2(1+x)^2P_{931}\zeta_3\Big)\zeta_2+\Big(\frac{64}{5}x^2(1+x)^2H_{-1}P_{816}+\frac{16}{5}x^2(1+x)^2H_0P_{975}
-\frac{32}{5}x^2(1+x)^2H_1P_{977}-\frac{8}{5}x(1+x)^2P_{1056}\Big)\zeta_2^2+\Big(-128x^2(1+x)^2H_{-1}H_0P_{821}
-256x^2(1+x)^2H_0H_1P_{831}-64x^2(1+x)^2H_{0,1}P_{844}+128x^2(1+x)^2H_{0,-1}P_{876}-16x^2(1+x)^2H_0^2P_{909}
-16x(1+x)^2H_0P_{1017}\Big)\zeta_3+96x^2(1+x)^2P_{918}\zeta_5\Big)-384(-1+x)x^2(1+x)^3\xi^2\eta^4\zeta_2H_{-1}H_{0,-1}\Bigg\}\Bigg]
+C_FC_A\Bigg[\Bigg\{-\frac{968}{9}x(1+x)^2\eta^2+\eta\Big(16xH_0H_1-16xH_{0,1}\Big)+\eta^3\Big(-\frac{2}{9}x(1+x)^2\Big(687
+26x+687x^2\Big)H_0+\frac{8}{3}x(1+x)^2\Big(3-88x+3x^2\Big)H_{-1}H_0+\frac{32}{3}x^2\Big(3-8x+3x^2\Big)H_0^3H_1
-\frac{8}{3}x(1+x)^2\Big(3-88x+3x^2\Big)H_{0,-1}+\Big(64x^2\Big(3-8x+3x^2\Big)H_0H_1-64x^2\Big(3-8x+3x^2\Big)H_{0,1}\Big)\zeta_2\Big)
+\xi^2\Big(96(-1+x)x^2(1+x)^3\eta^4\zeta_2H_{0,-1}+72(-1+x)x(1+x)^3\eta^3H_{-1}\zeta_2\Big)+\xi\Big(288x^2(1+x)^2\eta^3H_{-1}\zeta_2
+480x^3(1+x)^2\eta^4H_{0,-1}\zeta_2\Big)+\eta^4\Big(-8x(1+x)^2H_{0,0,-1}P_{923}+8x(1+x)^2H_0H_{0,-1}P_{924}
-4x(1+x)^2H_{-1}H_0^2P_{926}-\frac{2}{3}x(1+x)^2H_0^2P_{978}-48x^2(1+x)^2\Big(3-2x+3x^2\Big)H_0^2H_1+32x^2(1+x)^2\Big(7
-2x+7x^2\Big)H_0H_{0,1}-32x^2(1+x)^2\Big(5+2x+5x^2\Big)H_{0,0,1}-\frac{4}{3}x(1+x)^2P_{1000}\zeta_2+16x(1+x)^2P_{858}\zeta_3\Big)
+\eta^5\Big(-\frac{2}{3}x^2(1+x)^2H_0^4P_{835}+128x^2(1+x)^2H_0H_{0,0,1}P_{881}+16x^2(1+x)^2H_0^2H_{0,-1}P_{915}
-\frac{4}{3}x^3(1+x)^2H_0^3P_{925}-16x^2(1+x)^2H_0^2H_{0,1}P_{936}+96x^2(1+x)^2H_{0,0,0,-1}P_{941}
-32x^2(1+x)^2H_0H_{0,0,-1}P_{956}-32x^2(1+x)^2H_{0,0,0,1}P_{961}+\frac{8}{5}x^2(1+x)^2P_{994}\zeta_2^2
+\Big(8x^2(1+x)^2H_0^2P_{878}-8x^2(1+x)^2H_0P_{1016}\Big)\zeta_2-32x^2(1+x)^2H_0P_{914}\zeta_3\Big)+96x\log(2)x_+^2\zeta_2\Bigg\}
$$

$$+\varepsilon\Bigg\{\eta^2\Big(-16c_1x(x-1)^2-\frac{14872}{27}x(x+1)^2\Big)+\xi\Big(\eta^4\Big(\log(2)\zeta_2\Big(192(-1+x)x^2(1+x)^3H_1$$
$$+192(-1+x)x^2(1+x)^3H_{-1}\Big)-32(-1+x)x^2(1+x)^3H_{-1}H_0^2H_1-64(-1+x)x(1+x)^3\Big(3-5x$$
$$+3x^2\Big)H_0H_1H_{0,1}+\Big(-192(-1+x)x^2(1+x)^3H_{-1}H_1+960x^3(1+x)^2H_{-1}H_{0,-1}$$
$$+192(-1+x)x^2(1+x)^3H_{-1,1}\Big)\zeta_2\Big)+\eta^3\Big(-8(-1+x)x(1+x)^3H_{-1}^2H_0$$
$$+16(-1+x)x(1+x)^3H_{-1}H_{0,-1}-16(-1+x)x(1+x)^3H_{0,-1,-1}+624x^2(1+x)^2H_1\zeta_3\Big)$$
$$-192(-1+x)x^2(1+x)^3\Big(5-x+5x^2\Big)\eta^5\zeta_2H_{0,-1,-1}\Big)+\eta^3\Big(-\frac{1}{27}x(1+x)^2\Big(20559$$
$$+1586x+20559x^2\Big)H_0+\frac{4}{9}x(1+x)^2\Big(753+634x+753x^2\Big)H_{-1}H_0+640x^2(1+x)^2H_{-1}^2H_0$$
$$+\Big(32x(1+x)^2\Big(3-19x+3x^2\Big)H_0+\frac{64}{3}x^2\Big(3-8x+3x^2\Big)H_{-1}H_0^3\Big)H_1-\frac{32}{3}x^2\Big(3-8x$$
$$+3x^2\Big)H_0^3H_1^2-32x(1+x)^2\Big(3-19x+3x^2\Big)H_{0,1}+\Big(-\frac{4}{9}x(1+x)^2\Big(753+634x+753x^2\Big)$$
$$-1280x^2(1+x)^2H_{-1}\Big)H_{0,-1}+1280x^2(1+x)^2H_{0,-1,-1}+\Big(128x^2\Big(3-8x+3x^2\Big)H_{-1}H_0H_1$$
$$-64x^2\Big(3-8x+3x^2\Big)H_0H_1^2+\Big(128x^2\Big(3-8x+3x^2\Big)H_1-128x^2\Big(3-8x+3x^2\Big)H_{-1}\Big)H_{0,1}$$
$$-128x^2\Big(3-8x+3x^2\Big)H_{0,1,1}\Big)\zeta_2\Big)+\eta\Big(-96xH_{-1}H_0H_1+16xH_0H_1^2+\Big(-32xH_1+96xH_{-1}\Big)H_{0,1}$$
$$+96xH_1H_{0,-1}+32xH_{0,1,1}-96xH_{0,1,-1}-96xH_{0,-1,1}-16xH_1\zeta_2\Big)+\eta^4\Big(192x(1+x)^2H_{0,0,1,1}P_{830}$$
$$+64x(1+x)^2H_0H_{0,1,-1}P_{856}+64x(1+x)^2H_0H_{0,-1,1}P_{856}-64x(1+x)^2H_{0,0,1,-1}P_{857}$$
$$-64x(1+x)^2H_{0,0,-1,1}P_{857}-64x(1+x)^2H_0H_{0,1,1}P_{859}-16x(1+x)^2H_0^2H_1^2P_{861}$$
$$-48x(1+x)^2H_0H_{0,-1,-1}P_{908}+12x(1+x)^2H_{-1}^2H_0^2P_{910}-8x(1+x)^2H_{-1}H_0^2P_{946}$$
$$+8x(1+x)^2H_{0,-1}^2P_{959}-\frac{2}{9}x(1+x)^2H_0^2P_{1011}+\log(2)\zeta_2\Big(192x(1+x)^2P_{865}-768x^3(1+x)^2H_1$$
$$-768x^3(1+x)^2H_{-1}\Big)+\Big(-16x(1+x)^2H_0^2P_{892}+128x^3(1+x)^2H_{-1}H_0^2\Big)H_1+\Big(64x(1+x)^2H_0P_{824}$$
$$-96x(1+x)^2H_{-1}H_0P_{828}-32x(1+x)^2H_{0,-1}P_{855}-512x^2(1+x)^2\Big(3-5x+3x^2\Big)H_0H_1\Big)H_{0,1}$$
$$-32x^2(1+x)^2\Big(11-10x+11x^2\Big)H_{0,1}^2+\Big(-32x(1+x)^2H_0H_1P_{854}+8x(1+x)^2H_0P_{991}$$
$$-64x^2(1+x)^2\Big(5-4x+5x^2\Big)H_{-1}H_0\Big)H_{0,-1}+\Big(-192x(1+x)^2H_1P_{830}+32x(1+x)^2P_{842}$$
$$+64x(1+x)^2H_{-1}P_{857}\Big)H_{0,0,1}+\Big(192x(1+x)^2H_1P_{827}-16x(1+x)^2P_{980}-16x(1+x)^2\Big(3-4x$$
$$+3x^2\Big)\Big(7-68x+7x^2\Big)H_{-1}\Big)H_{0,0,-1}+16x(1+x)^2\Big(3-4x+3x^2\Big)\Big(7-68x+7x^2\Big)H_{0,0,-1,-1}$$
$$-128x^2(1+x)^2\Big(5-x+5x^2\Big)H_{0,-1,0,1}+\Big(-16x(1+x)^2H_{-1}P_{899}+24x(1+x)^2H_{-1}^2P_{929}$$
$$-\frac{4}{9}x(1+x)^2P_{1014}+768x^3(1+x)^2H_{-1}H_1-768x^3(1+x)^2H_{-1,1}\Big)\zeta_2+\Big(-16x(1+x)^2H_{-1}P_{938}$$
$$+8x(1+x)^2P_{954}\Big)\zeta_3\Big)+\xi^2\Big(192(-1+x)x^2(1+x)^3\eta^4\zeta_2H_{-1}H_{0,-1}-48(-1+x)x(1+x)^3\eta^3H_1\zeta_3\Big)$$

$$
+\eta^5\Big(512x^2(1+x)^2H_0H_{0,0,1,1}P_{833}-\frac{2}{5}x^2(1+x)^2H_0^5P_{834}+32x^2(1+x)^2H_0^2H_{0,1,1}P_{863}
+\frac{1}{3}x^3(1+x)^2H_0^4P_{872}+96x^2(1+x)^2H_0H_{0,-1}^2P_{880}-320x^2(1+x)^2H_0H_{0,-1,0,1}P_{894}
-64x^2(1+x)^2H_0H_{0,1}^2P_{919}+32x^2(1+x)^2H_0^2H_{0,1,-1}P_{940}+32x^2(1+x)^2H_0^2H_{0,-1,1}P_{940}
-448x^2(1+x)^2H_{0,0,0,1,1}P_{945}-128x^2(1+x)^2H_{0,0,-1,0,-1}P_{951}-32x^2(1+x)^2H_0^2H_{0,-1,-1}P_{967}
-128x^2(1+x)^2H_0H_{0,0,1,-1}P_{972}-128x^2(1+x)^2H_0H_{0,0,-1,1}P_{972}-64x^2(1+x)^2H_{0,0,1,0,1}P_{981}
-192x^2(1+x)^2H_{0,0,0,-1,-1}P_{982}+64x^2(1+x)^2H_{0,0,1,0,-1}P_{988}+64x^2(1+x)^2H_0H_{0,0,-1,-1}P_{989}
+64x^2(1+x)^2H_{0,0,-1,0,1}P_{996}+128x^2(1+x)^2H_{0,0,0,1,-1}P_{998}+128x^2(1+x)^2H_{0,0,0,-1,1}P_{998}
+32x^2(1+x)^2H_{0,0,0,0,1}P_{1004}-32x^2(1+x)^2H_{0,0,0,0,-1}P_{1007}-\frac{2}{3}x(1+x)^2H_0^3P_{1051}+\log(2)\zeta_2\Big(
-768x^3(1+x)^2\Big(1+x+x^2\Big)H_{0,1}-768x^3(1+x)^2\Big(1+x+x^2\Big)H_{0,-1}\Big)+\Big(\frac{4}{3}x^2(1+x)^2H_0^4P_{912}
+\frac{16}{3}x(1+x)^2H_0^3P_{1030}\Big)H_1+\Big(-\frac{4}{3}x^2(1+x)^2H_0^4P_{835}+\frac{4}{3}x(1+x)^2H_0^3P_{1052}\Big)H_{-1}
+\Big(64x^2(1+x)^2H_0^2H_1P_{832}-32x^2(1+x)^2H_{-1}H_0^2P_{936}+64x^2(1+x)^2H_0H_{0,-1}P_{971}
-\frac{16}{3}x^2(1+x)^2H_0^3P_{976}+16x(1+x)^2H_0^2P_{1022}\Big)H_{0,1}+\Big(-32x^2(1+x)^2H_0^2H_1P_{913}
+32x^2(1+x)^2H_{-1}H_0^2P_{915}+\frac{16}{3}x^2(1+x)^2H_0^3P_{979}-4x(1+x)^2H_0^2P_{1039}\Big)H_{0,-1}
+\Big(256x^2(1+x)^2H_0H_1P_{849}+256x^2(1+x)^2H_{-1}H_0P_{881}-64x^2(1+x)^2H_{0,-1}P_{935}
+64x^2(1+x)^2H_{0,1}P_{966}+16x^2(1+x)^2H_0^2P_{995}-32x(1+x)^2H_0P_{1033}\Big)H_{0,0,1}+\Big(
-256x^2(1+x)^2H_{0,-1}P_{839}-256x^2(1+x)^2H_0H_1P_{849}-64x^2(1+x)^2H_{-1}H_0P_{956}
-64x^2(1+x)^2H_{0,1}P_{990}-16x^2(1+x)^2H_0^2P_{999}+8x(1+x)^2H_0P_{1053}\Big)H_{0,0,-1}+\Big(
-128x^2(1+x)^2H_1P_{939}-64x^2(1+x)^2H_{-1}P_{961}-32x^2(1+x)^2H_0P_{1001}+32x(1+x)^2P_{1042}\Big)H_{0,0,0,1}
+\Big(192x^2(1+x)^2H_{-1}P_{941}+32x^2(1+x)^2H_0P_{1003}-8x(1+x)^2P_{1057}+64x^2(1+x)^2\Big(3-11x
+3x^2\Big)\Big(13-8x+13x^2\Big)H_1\Big)H_{0,0,0,-1}+\Big(-32x^2(1+x)^2H_{0,0,1}P_{841}+256x^2(1+x)^2H_{0,0,-1}P_{851}
+64x^2(1+x)^2H_{0,-1,1}P_{860}+64x^2(1+x)^2H_{0,1,-1}P_{866}+\frac{8}{3}x^2(1+x)^2H_0^3P_{950}+4x^2(1+x)^2H_0^2P_{1023}
-2x(1+x)^2H_0P_{1058}+\Big(-16x^2(1+x)^2H_0^2P_{955}+32x(1+x)^2H_0P_{1028}\Big)H_1
+\Big(16x^2(1+x)^2H_0^2P_{878}+8x(1+x)^2H_0P_{1036}\Big)H_{-1}+\Big(-32x(1+x)^2P_{1029}+32x^2(1+x)^2\Big(10
-37x-37x^3+10x^4\Big)H_0\Big)H_{0,1}+\Big(-64x^2(1+x)^2H_1P_{860}-32x^2(1+x)^2H_0P_{882}
+8x(1+x)^2P_{1046}\Big)H_{0,-1}-768x^3(1+x)^2\Big(5-x+5x^2\Big)H_{0,-1,-1}-32x^2(1+x)^2P_{953}\zeta_3\Big)\zeta_2
+\Big(\frac{16}{5}x^2(1+x)^2H_{-1}P_{994}-\frac{16}{5}x^2(1+x)^2H_1P_{1005}-\frac{8}{5}x^2(1+x)^2H_0P_{1010}+\frac{4}{5}x(1+x)^2P_{1041}\Big)\zeta_2^2+\Big(
-64x^2(1+x)^2H_0H_1P_{849}-8x^2(1+x)^2H_0^2P_{893}-64x^2(1+x)^2H_{-1}H_0P_{914}+96x^2(1+x)^2H_{0,1}P_{917}
+64x^2(1+x)^2H_{0,-1}P_{920}-16x(1+x)^2H_0P_{1027}\Big)\zeta_3+144x^2(1+x)^2P_{952}\zeta_5\Big)\Big\}\Big]
$$

$$
+C_F n_l T_F\Bigg[\Bigg\{\frac{304}{9}x(1+x)^2\eta^2+\eta^3\Big(-\frac{16}{3}x(1+x)^2\Big(3-2x+3x^2\Big)\zeta_2+\frac{8}{9}x(1+x)^2\Big(51
-26x+51x^2\Big)H_0-\frac{32}{3}x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}H_0+\frac{8}{3}x(1+x)^2\Big(3-2x+3x^2\Big)H_0^2
+\frac{32}{3}x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,-1}\Big)\Bigg\}+\varepsilon\Bigg\{\frac{4664}{27}x(1+x)^2\eta^2+\eta^3\Big(-32x(1+x)^2\Big(3
-2x+3x^2\Big)\zeta_3+\frac{4}{27}x(1+x)^2\Big(1527-722x+1527x^2\Big)H_0+32x(1+x)^2\Big(3-2x
+3x^2\Big)H_{-1}^2H_0+\frac{8}{9}x(1+x)^2\Big(69-38x+69x^2\Big)H_0^2+\frac{8}{3}x(1+x)^2\Big(3-2x+3x^2\Big)H_0^3
+\Big(-\frac{32}{9}x(1+x)^2\Big(69-38x+69x^2\Big)H_0-16x(1+x)^2\Big(3-2x+3x^2\Big)H_0^2\Big)H_{-1}
+\Big(\frac{32}{9}x(1+x)^2\Big(69-38x+69x^2\Big)-64x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}\Big)H_{0,-1}+32x(1+x)^2\Big(3
-2x+3x^2\Big)H_{0,0,-1}+64x(1+x)^2\Big(3-2x+3x^2\Big)H_{0,-1,-1}+\Big(-\frac{32}{9}x(1+x)^2\Big(21-19x
+48x^2\Big)+8x(1+x)^2\Big(3-2x+3x^2\Big)H_0+32x(1+x)^2\Big(3-2x+3x^2\Big)H_{-1}\Big)\zeta_2\Big)\Bigg\}\Bigg]
$$

$$
+C_F T_F\Bigg[\Bigg\{\frac{128}{9}(-1+x)x(1+x)\xi\eta^2+\eta^2\Big(-\frac{1024x^2}{9}+\Big(\frac{8}{3}xH_0^2P_{864}+16xP_{895}\zeta_2\Big)x_+^2\Big)
+\eta^3\Big(\frac{8}{9}xH_0P_{987}-\frac{32}{3}x^3H_0^3-64x^3H_0\zeta_2\Big)\Bigg\}+\varepsilon\Bigg\{-128x\Big(3+2x+3x^2\Big)\log(2)x_+^4\zeta_2+\eta^2\Big(-\frac{16}{27}x\Big(19
-10x+19x^2\Big)+\Big(-\frac{16}{3}xH_{-1}H_0^2P_{864}+\frac{32}{3}xH_0^2H_1P_{864}-\frac{64}{3}xH_0H_{0,1}P_{864}+\frac{64}{3}xH_{0,0,1}P_{864}
+\frac{32}{3}xH_{0,0,-1}P_{864}-32xH_{-1}P_{864}\zeta_2+\frac{64}{3}xP_{932}\zeta_3\Big)x_+^2\Big)+\eta^3\Big(\frac{4}{27}xH_0P_{1013}-8x^3H_0^4+\Big(
-\frac{16}{9}xH_0P_{987}-\frac{64}{3}x^3H_0^3\Big)H_{-1}-128x^3H_0^2H_{0,1}+\Big(\frac{16xP_{987}}{9}+128x^3H_0^2\Big)H_{0,-1}+512x^3H_0H_{0,0,1}
-256x^3H_0H_{0,0,-1}-768x^3H_{0,0,0,1}+128x^3H_{0,0,0,-1}+\Big(-\frac{8}{9}xH_0^3P_{1020}-\frac{8}{9}xH_0^2P_{1021}+\Big(-\frac{8}{3}xH_0P_{1019}
-\frac{16xP_{1025}}{9}\Big)\zeta_2\Big)x_+-128x^3H_{-1}H_0-32x^3H_0^2+512x^3H_{0,-1}\Big)\zeta_2+\frac{32}{5}x^3\zeta_2^2+128x^3H_0\zeta_3\Big)\Bigg\}\Bigg]. \tag{C.6}
$$

$$
\hat{F}_{A,2}^{(2),\mathrm{s}}=C_F T_F\Bigg[\Bigg\{\xi\eta^3\Big(-256(-1+x)x^2(1+x)H_0H_{0,0,1}-32(-1+x)x^2(1+x)H_0^2\zeta_2\Big)
+\eta^4\Big(32x(1+x)^2\zeta_2\Big(-1-6x-6x^2-14x^3+3x^4\Big)+8x(1+x)^2\Big(-1+2x-4x^2-26x^3
+13x^4\Big)H_0^2\Big)+\eta^5\Big(-128x^2(1+x)^2\Big(1+x+8x^2+x^3+x^4\Big)\zeta_2H_0-\frac{64}{3}x^3(1+x)^2\Big(3+8x
-x^2+2x^3\Big)H_0^3\Big)+\eta\Big(128xH_{-1}H_0-64xH_0H_1+32xH_{0,1}-128xH_{0,-1}+\Big(-\frac{64}{3}x^2H_0^3H_1
+\Big(-128x^2H_0H_1+128x^2H_{0,1}\Big)\zeta_2\Big)x_+^2\Big)+\eta^3\Big(-16x(1+x)^2\Big(3-2x+3x^2\Big)H_0+64x^2\Big(3
-2x+3x^2\Big)\zeta_3H_0+\frac{4}{3}x^2(1+x)^2H_0^4+256x^3H_0^2H_{0,1}-1536x^3H_0H_{0,0,1}+1024x^3H_0H_{0,0,-1}
$$

$$
\begin{aligned}
&- 64(-7+x)x^2(-1+7x)H_{0,0,0,1} - 3072x^3H_{0,0,0,-1} + 192x^3H_0^2\zeta_2 + \frac{256}{5}x^2\Big(1+x\\
&+x^2\Big)\zeta_2^2\Big) + \eta^2\Big(-16x\Big(5+4x+5x^2\Big) - 256x^2H_0^2H_1 + 768x^2H_0H_{0,1} - 512x^2H_0H_{0,-1}\\
&- 1024x^2H_{0,0,1} + 1024x^2H_{0,0,-1} + 256x^2\zeta_3\Big)\Big\} + \varepsilon\Big\{\xi\eta^3\Big(-256(-1+x)x^2(1+x)H_0H_{0,1}^2 + \Big(\\
&-512(-1+x)x^2(1+x)H_{-1}H_0 - 128(-1+x)x^2(1+x)H_0^2\Big)H_{0,0,1} - 128(-1+x)x^2(1+x)H_0^2H_{0,0,-1}\\
&+ 128(-1+x)x^2(1+x)H_0^2H_{0,1,1} + 128(-1+x)x^2(1+x)H_0^2H_{0,1,-1}\\
&+ 128(-1+x)x^2(1+x)H_0^2H_{0,-1,1} + 1024(-1+x)x^2(1+x)H_0H_{0,0,1,-1}\\
&+ 1024(-1+x)x^2(1+x)H_0H_{0,0,-1,1} + 256(-1+x)x^2(1+x)H_0H_{0,-1,0,1}\\
&- 128(-1+x)x^2(1+x)H_{0,0,1,0,-1} + \Big(-64(-1+x)x^2(1+x)H_{-1}H_0^2 - 16(-1+x)x^2(1+x)H_0^3\\
&+ 256(-1+x)x^2(1+x)H_0H_{0,-1} - 1024(-1+x)x^2(1+x)H_{0,0,-1} - 64(-1+x)x^2(1+x)\zeta_3\Big)\zeta_2\Big)\\
&+ \eta^5\Big(\frac{128}{3}x^2(1+x)^2H_{-1}H_0^3P_{848} + 256x^2(1+x)^2H_0^2H_{0,-1}P_{896} - 128x^2(1+x)^2H_0^2H_{0,1}P_{898}\\
&+ \frac{128}{3}x^2(1+x)^2H_0^3H_1P_{933} - \frac{8}{3}x^2(1+x)^2H_0^4P_{942} - 256x^2(1+x)^2H_{0,0,0,-1}P_{958}\\
&+ 128x^2(1+x)^2H_{0,0,0,1}P_{993} - \frac{16}{3}x(1+x)^2H_0^3P_{1040} - \frac{64}{5}x^2(1+x)^2P_{974}\zeta_2^2 + \Big(\\
&- 256x^2(1+x)^2H_{0,-1}P_{829} + 256x^2(1+x)^2H_{-1}H_0P_{848} - 256x^2(1+x)^2H_{0,1}P_{884}\\
&+ 256x^2(1+x)^2H_0H_1P_{897} + 16x(1+x)^2H_0P_{1034}\Big)\zeta_2\Big) + \eta\Big(-512xH_{-1}^2H_0 + 384xH_{-1}H_0H_1\\
&- 64xH_0H_1^2 + \Big(64xH_1 - 448xH_{-1}\Big)H_{0,1} + \Big(-384xH_1 + 1024xH_{-1}\Big)H_{0,-1} - 64xH_{0,1,1}\\
&+ 448xH_{0,1,-1} + 448xH_{0,-1,1} - 1024xH_{0,-1,-1} + \Big(-\frac{128}{3}x^2H_{-1}H_0^3H_1 + \frac{64}{3}x^2H_0^3H_1^2\\
&- 256x^2H_0H_{0,-1}H_{0,1} + 256x^2H_0^2H_1H_{0,-1} - 512x^2H_{0,0,-1,0,1} + \Big(-256x^2H_{-1}H_0H_1\\
&+ 128x^2H_0H_1^2 + \Big(-256x^2H_1 + 256x^2H_{-1}\Big)H_{0,1} + 1024x^2H_1H_{0,-1} + 256x^2H_{0,1,1}\\
&- 1024x^2H_{0,-1,1}\Big)\zeta_2\Big)x_+^2 - 64xH_1\zeta_2\Big) + \eta^4\Big(32x(1+x)^2H_0H_{0,1}P_{871} + 16x(1+x)^2H_0^2H_1P_{928}\\
&- 16x(1+x)^2H_{-1}H_0^2P_{964} + 32x(1+x)^2H_{0,0,-1}P_{986} + 8x(1+x)H_0^2P_{1024} + \Big(64x(1+x)^2H_{-1}P_{836}\\
&- 32x(1+x)P_{1015}\Big)\zeta_2 - 16x(1+x)^2P_{973}\zeta_3\Big) + \eta^2\Big(-120x\Big(3+4x+3x^2\Big) + \log(2)\zeta_2\Big(1536x^2\\
&- 1536x^2H_1 - 1536x^2H_{-1}\Big) + 128x^2H_{-1}^2H_0^2 + 256x^2H_{-1}H_0^2H_1 - 384x^2H_0^2H_1^2 + \Big(\\
&- 1024x^2H_{-1}H_0 + 1536x^2H_0H_1 + 2560x^2H_{0,-1}\Big)H_{0,1} - 256x^2H_{0,1}^2 + \Big(\Big(-32x\Big(11+16x\\
&+13x^2\Big) + 512x^2H_{-1}\Big)H_0 + 512x^2H_0H_1\Big)H_{0,-1} - 1536x^2H_{0,-1}^2 + \Big(-64x^2(19+8x)\\
&- 1536x^2H_1 + 1536x^2H_{-1}\Big)H_{0,0,1} + \Big(-1536x^2H_1 - 1536x^2H_{-1}\Big)H_{0,0,-1} - 1024x^2H_0H_{0,1,1}\\
&- 1536x^2H_0H_{0,1,-1} - 1536x^2H_0H_{0,-1,1} + 2560x^2H_0H_{0,-1,-1} + 1536x^2H_{0,0,1,1}\\
&- 1536x^2H_{0,0,1,-1} - 1536x^2H_{0,0,-1,1} + 1536x^2H_{0,0,-1,-1} - 1536x^2H_{0,-1,0,1} + \Big(1536x^2H_{-1}H_1\\
&+ 768x^2H_{-1}^2 - 1536x^2H_{-1,1}\Big)\zeta_2 - 64x^2(3+5x)H_0^2x_+\zeta_2 + \Big(2688x^2H_1 - 384x^2H_{-1}\Big)\zeta_3\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\eta^3\Big(-64x^2\Big(11+31x+11x^2\Big)\zeta_5+\log(2)\zeta_2\Big(3072x^3H_{0,1}+3072x^3H_{0,-1}\Big)-8x(1+x)\Big(33\\
&+15x+27x^2+29x^3\Big)H_0+\frac{16}{15}x^2(1+x)^2H_0^5+\Big(-128(-2+x)x(1+x)^2(-1+2x)H_0\\
&-\frac{8}{3}x^2\Big(5-14x+5x^2\Big)H_0^4\Big)H_1+\Big(32x(1+x)^2\Big(19-42x+19x^2\Big)H_0+\frac{8}{3}x^2(1+x)^2H_0^4\Big)H_{-1}\\
&+\Big(64xP_{885}+512x^3H_{-1}H_0^2+\frac{64}{3}x^2(3+x)(1+3x)H_0^3+512x^3H_0^2H_1\Big)H_{0,1}-1536x^3H_0H_{0,1}^2+\Big(\\
&-32x(1+x)^2\Big(19-42x+19x^2\Big)-\frac{32}{3}x^2\Big(3-26x+3x^2\Big)H_0^3\Big)H_{0,-1}-1536x^3H_0H_{0,-1}^2+\Big(\\
&-256x^2\Big(11-2x+9x^2\Big)H_0-3072x^3H_{-1}H_0-1536x^3H_0^2-512x^2(1+x)^2H_0H_1-256x^2\Big(3-14x\\
&+3x^2\Big)H_{0,1}+128x^2\Big(3-50x+3x^2\Big)H_{0,-1}\Big)H_{0,0,1}+\Big(-512x^2(2+x)(-3+2x)H_0+2048x^3H_{-1}H_0\\
&+768x^3H_0^2+512x^2(1+x)^2H_0H_1+256x^2\Big(3-2x+3x^2\Big)H_{0,1}+7168x^3H_{0,-1}\Big)H_{0,0,-1}\\
&+768x^3H_0^2H_{0,1,1}-768x^3H_0^2H_{0,1,-1}-768x^3H_0^2H_{0,-1,1}-512x^3H_0^2H_{0,-1,-1}+\Big(-128x^2\Big(11-26x\\
&+11x^2\Big)H_0+128x^2\Big(13+10x+13x^2\Big)H_1-128(-7+x)x^2(-1+7x)H_{-1}\Big)H_{0,0,0,1}+\Big(128x^2\Big(5\\
&-38x+5x^2\Big)H_0-256x^2\Big(11+2x+11x^2\Big)H_1-6144x^3H_{-1}\Big)H_{0,0,0,-1}+512x^2(1+x)^2H_0H_{0,0,1,1}\\
&+6144x^3H_0H_{0,0,1,-1}+6144x^3H_0H_{0,0,-1,1}-4096x^3H_0H_{0,0,-1,-1}+3584x^3H_0H_{0,-1,0,1}+64x^2\Big(55\\
&+6x+55x^2\Big)H_{0,0,0,0,1}-64x^2\Big(53-54x+53x^2\Big)H_{0,0,0,0,-1}+128x^2\Big(5-94x+5x^2\Big)H_{0,0,0,1,1}\\
&+256x^2\Big(5+14x+5x^2\Big)H_{0,0,0,1,-1}+256x^2\Big(5+14x+5x^2\Big)H_{0,0,0,-1,1}-9216x^3H_{0,0,0,-1,-1}\\
&+256x^2\Big(3-14x+3x^2\Big)H_{0,0,1,0,1}+3328x^3H_{0,0,1,0,-1}-5120x^3H_{0,0,-1,0,-1}+\Big(384x^3H_{-1}H_0^2\\
&+160x^3H_0^3+128x^2(1+x)^2H_0^2H_1+128x^2\Big(3+2x+3x^2\Big)H_0H_{0,1}+1024x^3H_0H_{0,-1}-128x^2\Big(7+2x\\
&+7x^2\Big)H_{0,0,1}-4096x^3H_{0,0,-1}-1024x^2\Big(1+x+x^2\Big)H_{0,1,-1}-3072x^3H_{0,-1,-1}+768x^3\zeta_3\Big)\zeta_2+\Big(\\
&-\frac{32}{5}x^2\Big(5+66x+5x^2\Big)H_0+\frac{64}{5}x^2\Big(5+14x+5x^2\Big)H_1+\frac{512}{5}x^2\Big(1+x+x^2\Big)H_{-1}\Big)\zeta_2^2+\Big(-128x^2\Big(4\\
&+x+10x^2\Big)H_0+128x^2\Big(3-2x+3x^2\Big)H_{-1}H_0+16x^2\Big(13+18x+13x^2\Big)H_0^2+128x^2(1+x)^2H_0H_1\\
&+128x^2\Big(5-32x+5x^2\Big)H_{0,1}-128x^2\Big(3-8x+3x^2\Big)H_{0,-1}\Big)\zeta_3\Big)-32xH_1x_+^2\Big\}\Big]\,.
\end{aligned}
\tag{C.7}
$$

The polynomials are given by

$$
\begin{aligned}
P_{815}&=x^4-661x^3+786x^2-661x+1\,,\\
P_{816}&=x^4-145x^3+405x^2-145x+1\,,\\
P_{817}&=x^4-93x^3+136x^2-93x+1\,,\\
P_{818}&=x^4-57x^3+88x^2-57x+1\,,\\
P_{819}&=x^4-53x^3+78x^2-53x+1\,,\\
P_{820}&=x^4-37x^3+54x^2-37x+1\,,\\
P_{821}&=x^4-33x^3+52x^2-33x+1\,,\\
P_{822}&=x^4-31x^3+87x^2-31x+1\,,
\end{aligned}
$$

$$P_{823} = x^4 - 29x^3 + 38x^2 - 29x + 1\,,$$
$$P_{824} = x^4 - 26x^3 + 41x^2 - 26x + 1\,,$$
$$P_{825} = x^4 - 25x^3 + 18x^2 - 25x + 1\,,$$
$$P_{826} = x^4 - 25x^3 + 36x^2 - 25x + 1\,,$$
$$P_{827} = x^4 - 23x^3 + 26x^2 - 23x + 1\,,$$
$$P_{828} = x^4 - 16x^3 + 26x^2 - 16x + 1\,,$$
$$P_{829} = x^4 - 14x^3 - 32x^2 + 6x - 9\,,$$
$$P_{830} = x^4 - 14x^3 + 20x^2 - 14x + 1\,,$$
$$P_{831} = x^4 - 13x^3 + 30x^2 - 13x + 1\,,$$
$$P_{832} = x^4 - 7x^3 + 24x^2 - 7x + 1\,,$$
$$P_{833} = x^4 - 5x^3 + 20x^2 - 5x + 1\,,$$
$$P_{834} = x^4 - 5x^3 + 22x^2 - 5x + 1\,,$$
$$P_{835} = x^4 - 5x^3 + 38x^2 - 5x + 1\,,$$
$$P_{836} = x^4 - 2x^3 + 66x^2 + 14x - 7\,,$$
$$P_{837} = x^4 - x^3 - 22x^2 - x + 1\,,$$
$$P_{838} = x^4 - x^3 + 25x^2 - x + 1\,,$$
$$P_{839} = x^4 + 2x^3 - 9x^2 + 2x + 1\,,$$
$$P_{840} = x^4 + 3x^3 + 36x^2 + 3x + 1\,,$$
$$P_{841} = x^4 + 7x^3 - 94x^2 + 7x + 1\,,$$
$$P_{842} = x^4 + 70x^3 - 108x^2 + 78x - 3\,,$$
$$P_{843} = x^4 + 107x^3 - 144x^2 + 107x + 1\,,$$
$$P_{844} = x^4 + 117x^3 - 170x^2 + 117x + 1\,,$$
$$P_{845} = 2x^4 - 45x^3 + 56x^2 - 45x + 2\,,$$
$$P_{846} = 2x^4 - 30x^3 + 59x^2 - 30x + 2\,,$$
$$P_{847} = 2x^4 - 29x^3 + 63x^2 - 29x + 2\,,$$
$$P_{848} = 2x^4 - 7x^3 - 8x^2 + 5x - 4\,,$$
$$P_{849} = 2x^4 - 7x^3 - 2x^2 - 7x + 2\,,$$
$$P_{850} = 2x^4 - 6x^3 + 39x^2 - 6x + 2\,,$$
$$P_{851} = 2x^4 - 4x^3 + x^2 - 4x + 2\,,$$
$$P_{852} = 2x^4 + 8x^3 + 31x^2 + 8x + 2\,,$$
$$P_{853} = 3x^4 - 151x^3 + 204x^2 - 151x + 3\,,$$
$$P_{854} = 3x^4 - 68x^3 + 82x^2 - 68x + 3\,,$$
$$P_{855} = 3x^4 - 58x^3 + 50x^2 - 58x + 3\,,$$
$$P_{856} = 3x^4 - 53x^3 + 64x^2 - 53x + 3\,,$$
$$P_{857} = 3x^4 - 49x^3 + 74x^2 - 49x + 3\,,$$
$$P_{858} = 3x^4 - 41x^3 + 64x^2 - 41x + 3\,,$$
$$P_{859} = 3x^4 - 40x^3 + 56x^2 - 40x + 3\,,$$
$$P_{860} = 3x^4 - 25x^3 + 62x^2 - 25x + 3\,,$$
$$P_{861} = 3x^4 - 16x^3 + 32x^2 - 16x + 3\,,$$
$$P_{862} = 3x^4 - 16x^3 + 170x^2 - 16x + 3\,,$$

$$P_{863} = 3x^4 - 14x^3 - 38x^2 - 14x + 3\,,$$
$$P_{864} = 3x^4 - 14x^3 - 2x^2 - 14x + 3\,,$$
$$P_{865} = 3x^4 - 14x^3 + 16x^2 - 14x + 3\,,$$
$$P_{866} = 3x^4 - 13x^3 + 74x^2 - 13x + 3\,,$$
$$P_{867} = 3x^4 - 7x^3 - 12x^2 - 7x + 3\,,$$
$$P_{868} = 3x^4 - 7x^3 + 64x^2 - 7x + 3\,,$$
$$P_{869} = 3x^4 - 3x^3 + 28x^2 - 3x + 3\,,$$
$$P_{870} = 3x^4 - 2x^3 - 62x^2 - 2x + 3\,,$$
$$P_{871} = 3x^4 + 22x^3 - 34x^2 + 22x + 3\,,$$
$$P_{872} = 3x^4 + 176x^3 - 160x^2 + 408x - 3\,,$$
$$P_{873} = 3x^4 + 269x^3 - 274x^2 + 269x + 3\,,$$
$$P_{874} = 4x^4 - 118x^3 + 153x^2 - 118x + 4\,,$$
$$P_{875} = 4x^4 - 90x^3 + 163x^2 - 90x + 4\,,$$
$$P_{876} = 4x^4 - 61x^3 + 75x^2 - 61x + 4\,,$$
$$P_{877} = 4x^4 - 22x^3 + 9x^2 - 22x + 4\,,$$
$$P_{878} = 4x^4 - 21x^3 + 4x^2 - 21x + 4\,,$$
$$P_{879} = 4x^4 - 13x^3 - 16x^2 - 13x + 4\,,$$
$$P_{880} = 4x^4 - 13x^3 + 24x^2 - 13x + 4\,,$$
$$P_{881} = 4x^4 - 13x^3 + 30x^2 - 13x + 4\,,$$
$$P_{882} = 4x^4 - 11x^3 - 40x^2 - 11x + 4\,,$$
$$P_{883} = 4x^4 - 7x^3 + 86x^2 - 7x + 4\,,$$
$$P_{884} = 4x^4 - 5x^3 + 14x^2 - 5x + 4\,,$$
$$P_{885} = 4x^4 - 3x^3 - 10x^2 - 3x + 4\,,$$
$$P_{886} = 5x^4 - 281x^3 + 396x^2 - 281x + 5\,,$$
$$P_{887} = 5x^4 - 224x^3 + 330x^2 - 224x + 5\,,$$
$$P_{888} = 5x^4 - 149x^3 + 222x^2 - 149x + 5\,,$$
$$P_{889} = 5x^4 - 121x^3 + 262x^2 - 121x + 5\,,$$
$$P_{890} = 5x^4 - 53x^3 + 42x^2 - 53x + 5\,,$$
$$P_{891} = 5x^4 - 41x^3 + 54x^2 - 41x + 5\,,$$
$$P_{892} = 5x^4 - 34x^3 + 56x^2 - 26x + 1\,,$$
$$P_{893} = 5x^4 - 20x^3 + 18x^2 - 20x + 5\,,$$
$$P_{894} = 5x^4 - 17x^3 + 30x^2 - 17x + 5\,,$$
$$P_{895} = 5x^4 - 10x^3 + 2x^2 - 10x + 5\,,$$
$$P_{896} = 5x^4 - 7x^3 + 8x^2 + 9x - 3\,,$$
$$P_{897} = 5x^4 - 7x^3 + 14x^2 - 3x + 3\,,$$
$$P_{898} = 5x^4 - 6x^3 + 2x^2 + 18x - 7\,,$$
$$P_{899} = 5x^4 + 49x^3 + 69x^2 - 55x + 16\,,$$
$$P_{900} = 6x^4 - 13x^3 + 78x^2 - 13x + 6\,,$$
$$P_{901} = 6x^4 - 9x^3 + 16x^2 - 9x + 6\,,$$
$$P_{902} = 6x^4 - 7x^3 + 146x^2 - 7x + 6\,,$$

$$P_{903} = 6x^4 - 5x^3 + 62x^2 - 5x + 6\,,$$
$$P_{904} = 6x^4 + 3x^3 + 122x^2 + 3x + 6\,,$$
$$P_{905} = 7x^4 - 263x^3 + 392x^2 - 263x + 7\,,$$
$$P_{906} = 7x^4 - 211x^3 + 270x^2 - 211x + 7\,,$$
$$P_{907} = 7x^4 - 179x^3 + 314x^2 - 179x + 7\,,$$
$$P_{908} = 7x^4 - 96x^3 + 118x^2 - 96x + 7\,,$$
$$P_{909} = 7x^4 - 79x^3 + 156x^2 - 79x + 7\,,$$
$$P_{910} = 7x^4 - 64x^3 + 94x^2 - 64x + 7\,,$$
$$P_{911} = 7x^4 - 54x^3 + 90x^2 - 54x + 7\,,$$
$$P_{912} = 7x^4 - 41x^3 + 98x^2 - 41x + 7\,,$$
$$P_{913} = 7x^4 - 37x^3 + 78x^2 - 37x + 7\,,$$
$$P_{914} = 7x^4 - 33x^3 + 64x^2 - 33x + 7\,,$$
$$P_{915} = 7x^4 - 29x^3 + 62x^2 - 29x + 7\,,$$
$$P_{916} = 7x^4 - 16x^3 - 82x^2 - 16x + 7\,,$$
$$P_{917} = 7x^4 - 16x^3 + 40x^2 - 16x + 7\,,$$
$$P_{918} = 8x^4 - 267x^3 + 485x^2 - 267x + 8\,,$$
$$P_{919} = 8x^4 - 39x^3 + 56x^2 - 39x + 8\,,$$
$$P_{920} = 8x^4 - 38x^3 + 99x^2 - 38x + 8\,,$$
$$P_{921} = 9x^4 - 233x^3 + 388x^2 - 233x + 9\,,$$
$$P_{922} = 9x^4 - 169x^3 + 308x^2 - 169x + 9\,,$$
$$P_{923} = 9x^4 - 140x^3 + 178x^2 - 140x + 9\,,$$
$$P_{924} = 9x^4 - 116x^3 + 154x^2 - 116x + 9\,,$$
$$P_{925} = 9x^4 - 112x^3 + 146x^2 - 196x + 9\,,$$
$$P_{926} = 9x^4 - 92x^3 + 130x^2 - 92x + 9\,,$$
$$P_{927} = 9x^4 - 85x^3 + 212x^2 - 85x + 9\,,$$
$$P_{928} = 9x^4 - 38x^3 + 8x^2 - 2x - 9\,,$$
$$P_{929} = 9x^4 - 32x^3 + 34x^2 - 32x + 9\,,$$
$$P_{930} = 9x^4 - 19x^3 + 92x^2 - 19x + 9\,,$$
$$P_{931} = 9x^4 - 8x^3 - 5x^2 - 8x + 9\,,$$
$$P_{932} = 9x^4 - 7x^3 + 8x^2 - 7x + 9\,,$$
$$P_{933} = 10x^4 - 17x^3 + 14x^2 + 7x - 2\,,$$
$$P_{934} = 11x^4 - 467x^3 + 660x^2 - 467x + 11\,,$$
$$P_{935} = 11x^4 - 74x^3 + 138x^2 - 74x + 11\,,$$
$$P_{936} = 11x^4 - 42x^3 + 86x^2 - 42x + 11\,,$$
$$P_{937} = 11x^4 + 377x^3 - 302x^2 + 377x + 11\,,$$
$$P_{938} = 12x^4 - 85x^3 + 92x^2 - 85x + 12\,,$$
$$P_{939} = 12x^4 - 49x^3 + 38x^2 - 49x + 12\,,$$
$$P_{940} = 12x^4 - 47x^3 + 124x^2 - 47x + 12\,,$$
$$P_{941} = 12x^4 - 43x^3 + 92x^2 - 43x + 12\,,$$
$$P_{942} = 13x^4 - 7x^3 + 48x^2 + 17x + 1\,,$$

$$P_{943} = 15x^4 - 419x^3 + 694x^2 - 419x + 15\,,$$
$$P_{944} = 15x^4 - 355x^3 + 614x^2 - 355x + 15\,,$$
$$P_{945} = 15x^4 - 70x^3 + 146x^2 - 70x + 15\,,$$
$$P_{946} = 15x^4 - 65x^3 + 37x^2 + 3x + 22\,,$$
$$P_{947} = 15x^4 - 23x^3 + 196x^2 - 23x + 15\,,$$
$$P_{948} = 15x^4 - 16x^3 + 198x^2 - 16x + 15\,,$$
$$P_{949} = 15x^4 + 70x^3 - 114x^2 + 70x + 15\,,$$
$$P_{950} = 16x^4 - 83x^3 + 236x^2 - 83x + 16\,,$$
$$P_{951} = 16x^4 - 75x^3 + 178x^2 - 75x + 16\,,$$
$$P_{952} = 16x^4 - 65x^3 + 109x^2 - 65x + 16\,,$$
$$P_{953} = 17x^4 - 74x^3 + 111x^2 - 74x + 17\,,$$
$$P_{954} = 17x^4 + 312x^3 - 154x^2 + 124x + 29\,,$$
$$P_{955} = 19x^4 - 77x^3 + 86x^2 - 77x + 19\,,$$
$$P_{956} = 19x^4 - 72x^3 + 154x^2 - 72x + 19\,,$$
$$P_{957} = 19x^4 + 92x^3 - 182x^2 + 156x - 45\,,$$
$$P_{958} = 20x^4 - 31x^3 + 76x^2 - 19x + 14\,,$$
$$P_{959} = 21x^4 - 268x^3 + 338x^2 - 268x + 21\,,$$
$$P_{960} = 21x^4 - 62x^3 + 710x^2 - 62x + 21\,,$$
$$P_{961} = 21x^4 - 58x^3 + 146x^2 - 58x + 21\,,$$
$$P_{962} = 21x^4 - 42x^3 + 326x^2 - 42x + 21\,,$$
$$P_{963} = 21x^4 + 775x^3 - 590x^2 + 775x + 21\,,$$
$$P_{964} = 23x^4 - 46x^3 - 44x^2 + 38x - 19\,,$$
$$P_{965} = 25x^4 - 361x^3 + 762x^2 - 361x + 25\,,$$
$$P_{966} = 27x^4 - 128x^3 + 214x^2 - 128x + 27\,,$$
$$P_{967} = 27x^4 - 115x^3 + 242x^2 - 115x + 27\,,$$
$$P_{968} = 27x^4 - 68x^3 + 454x^2 - 68x + 27\,,$$
$$P_{969} = 27x^4 - 24x^3 + 374x^2 - 24x + 27\,,$$
$$P_{970} = 27x^4 + 461x^3 - 178x^2 + 461x + 27\,,$$
$$P_{971} = 28x^4 - 117x^3 + 196x^2 - 117x + 28\,,$$
$$P_{972} = 29x^4 - 105x^3 + 230x^2 - 105x + 29\,,$$
$$P_{973} = 29x^4 - 30x^3 + 72x^2 + 22x + 3\,,$$
$$P_{974} = 30x^4 - 51x^3 + 20x^2 + 17x - 4\,,$$
$$P_{975} = 31x^4 + 877x^3 - 2470x^2 + 877x + 31\,,$$
$$P_{976} = 36x^4 - 149x^3 + 364x^2 - 149x + 36\,,$$
$$P_{977} = 37x^4 - 1021x^3 + 2202x^2 - 1021x + 37\,,$$
$$P_{978} = 39x^4 + 184x^3 - 48x^2 + 32x + 33\,,$$
$$P_{979} = 41x^4 - 185x^3 + 426x^2 - 185x + 41\,,$$
$$P_{980} = 42x^4 + 191x^3 - 219x^2 + 123x + 35\,,$$
$$P_{981} = 43x^4 - 196x^3 + 366x^2 - 196x + 43\,,$$
$$P_{982} = 44x^4 - 193x^3 + 448x^2 - 193x + 44\,,$$

$$
\begin{aligned}
P_{983} &= 45x^4 - 98x^3 + 774x^2 - 98x + 45\,,\\
P_{984} &= 47x^4 - 124x^3 + 178x^2 - 60x - 17\,,\\
P_{985} &= 47x^4 + 6x^3 - 272x^2 - 102x + 33\,,\\
P_{986} &= 49x^4 - 66x^3 - 60x^2 + 26x + 3\,,\\
P_{987} &= 51x^4 - 176x^3 - 70x^2 - 176x + 51\,,\\
P_{988} &= 53x^4 - 232x^3 + 478x^2 - 232x + 53\,,\\
P_{989} &= 55x^4 - 214x^3 + 474x^2 - 214x + 55\,,\\
P_{990} &= 57x^4 - 239x^3 + 430x^2 - 239x + 57\,,\\
P_{991} &= 57x^4 + 126x^3 - 182x^2 + 126x + 57\,,\\
P_{992} &= 60x^4 - 95x^3 + 219x^2 + 147x - 11\,,\\
P_{993} &= 61x^4 - 119x^3 + 128x^2 - 47x + 25\,,\\
P_{994} &= 66x^4 - 271x^3 + 176x^2 - 271x + 66\,,\\
P_{995} &= 67x^4 - 259x^3 + 654x^2 - 259x + 67\,,\\
P_{996} &= 73x^4 - 307x^3 + 582x^2 - 307x + 73\,,\\
P_{997} &= 73x^4 - 214x^3 + 276x^2 + 190x + 11\,,\\
P_{998} &= 90x^4 - 377x^3 + 790x^2 - 377x + 90\,,\\
P_{999} &= 91x^4 - 372x^3 + 838x^2 - 372x + 91\,,\\
P_{1000} &= 93x^4 - 700x^3 + 588x^2 - 476x + 63\,,\\
P_{1001} &= 107x^4 - 415x^3 + 1090x^2 - 415x + 107\,,\\
P_{1002} &= 125x^4 - 230x^3 + 312x^2 - 26x + 43\,,\\
P_{1003} &= 151x^4 - 593x^3 + 1418x^2 - 593x + 151\,,\\
P_{1004} &= 163x^4 - 661x^3 + 1794x^2 - 661x + 163\,,\\
P_{1005} &= 171x^4 - 695x^3 + 814x^2 - 695x + 171\,,\\
P_{1006} &= 215x^4 - 98x^3 - 54x^2 - 554x + 203\,,\\
P_{1007} &= 221x^4 - 875x^3 + 2310x^2 - 875x + 221\,,\\
P_{1008} &= 233x^4 - 510x^3 + 356x^2 + 182x - 117\,,\\
P_{1009} &= 293x^4 - 230x^3 - 100x^2 + 462x - 57\,,\\
P_{1010} &= 303x^4 - 1367x^3 + 1474x^2 - 1367x + 303\,,\\
P_{1011} &= 348x^4 - 1604x^3 + 1989x^2 + 398x + 885\,,\\
P_{1012} &= 581x^4 - 910x^3 + 1242x^2 - 142x + 125\,,\\
P_{1013} &= 1527x^4 - 440x^3 - 2398x^2 - 440x + 1527\,,\\
P_{1014} &= 1866x^4 - 7378x^3 + 7947x^2 - 5276x + 681\,,\\
P_{1015} &= x^5 + 7x^4 + 21x^3 - x^2 + 14x - 2\,,\\
P_{1016} &= 9x^5 - 94x^4 + 110x^3 - 196x^2 + 45x - 18\,,\\
P_{1017} &= 13x^5 + 620x^4 - 884x^3 + 254x^2 - x + 6\,,\\
P_{1018} &= 14x^5 + 175x^4 - 56x^3 + 14x^2 + 6x - 9\,,\\
P_{1019} &= 15x^5 - 157x^4 + 18x^3 - 78x^2 - 21x - 9\,,\\
P_{1020} &= 15x^5 - 109x^4 + 36x^3 - 60x^2 + 27x - 9\,,\\
P_{1021} &= 18x^5 + 6x^4 + 87x^3 + 159x^2 + 119x - 69\,,\\
P_{1022} &= 25x^5 - 251x^4 + 518x^3 - 318x^2 + 105x - 3\,,
\end{aligned}
$$

$$
\begin{aligned}
P_{1023} &= 33x^5 - 376x^4 + 764x^3 - 604x^2 + 107x + 44\,,\\
P_{1024} &= 45x^5 - 81x^4 - 110x^3 + 26x^2 - 15x - 9\,,\\
P_{1025} &= 252x^5 - 250x^4 - 345x^3 + 99x^2 + 125x - 201\,,\\
P_{1026} &= 3x^6 + 86x^5 + 575x^4 + 416x^3 - 131x^2 + 146x - 39\,,\\
P_{1027} &= 6x^6 - 131x^5 + 441x^4 - 572x^3 + 262x^2 - x - 9\,,\\
P_{1028} &= 6x^6 - 77x^5 + 251x^4 - 402x^3 + 212x^2 - 65x + 3\,,\\
P_{1029} &= 6x^6 - 75x^5 + 247x^4 - 402x^3 + 216x^2 - 67x + 3\,,\\
P_{1030} &= 6x^6 - 38x^5 + 215x^4 - 402x^3 + 248x^2 - 104x + 3\,,\\
P_{1031} &= 6x^6 + 3x^5 - 8x^4 + 352x^3 - 370x^2 + 21x - 12\,,\\
P_{1032} &= 6x^6 + 9x^5 - 410x^4 + 588x^3 - 320x^2 - 29x + 12\,,\\
P_{1033} &= 9x^6 - 65x^5 - 181x^4 + 536x^3 - 409x^2 + 121x - 3\,,\\
P_{1034} &= 9x^6 - 14x^5 + 259x^4 - 156x^3 + 27x^2 - 14x + 1\,,\\
P_{1035} &= 9x^6 + 3x^5 - 399x^4 + 588x^3 - 331x^2 - 23x + 9\,,\\
P_{1036} &= 9x^6 + 28x^5 - 89x^4 + 392x^3 - 221x^2 + 196x - 27\,,\\
P_{1037} &= 12x^6 - 27x^5 - 129x^4 + 56x^3 - 60x^2 + 7x - 3\,,\\
P_{1038} &= 15x^6 - 86x^5 - 307x^4 - 678x^3 + 815x^2 - 44x + 21\,,\\
P_{1039} &= 15x^6 - 40x^5 - 517x^4 + 1832x^3 - 1323x^2 + 648x - 39\,,\\
P_{1040} &= 19x^6 - 35x^5 - 150x^4 + 130x^3 - 19x^2 - 3x + 2\,,\\
P_{1041} &= 21x^6 - 936x^5 + 4040x^4 - 7744x^3 + 6563x^2 - 2448x + 312\,,\\
P_{1042} &= 21x^6 - 232x^5 - 5x^4 + 456x^3 - 521x^2 + 152x - 3\,,\\
P_{1043} &= 21x^6 - 19x^5 - 179x^4 + 588x^3 - 551x^2 - x - 3\,,\\
P_{1044} &= 24x^6 - 29x^5 - 178x^4 + 674x^3 - 652x^2 + 27x - 18\,,\\
P_{1045} &= 24x^6 + 70x^5 + 173x^4 + 156x^3 + 8x^2 + 14x - 21\,,\\
P_{1046} &= 27x^6 - 336x^5 + 219x^4 - 920x^3 + 25x^2 - 80x + 9\,,\\
P_{1047} &= 27x^6 + 39x^5 - 786x^4 + 722x^3 - 31x^2 - 9x + 6\,,\\
P_{1048} &= 33x^6 - 80x^5 + 279x^4 - 112x^3 + 99x^2 + 120x - 51\,,\\
P_{1049} &= 33x^6 - 74x^5 + 177x^4 + 1940x^3 - 2185x^2 + 134x - 57\,,\\
P_{1050} &= 33x^6 - 20x^5 + 627x^4 + 112x^3 - 1005x^2 - 20x - 15\,,\\
P_{1051} &= 35x^6 - 448x^5 + 653x^4 - 400x^3 + 235x^2 - 12x + 33\,,\\
P_{1052} &= 39x^6 - 396x^5 + 501x^4 + 392x^3 - 811x^2 + 620x - 57\,,\\
P_{1053} &= 45x^6 - 284x^5 - 1011x^4 + 2752x^3 - 2165x^2 + 716x - 21\,,\\
P_{1054} &= 69x^6 - 166x^5 + 205x^4 + 952x^3 - 1677x^2 + 86x - 45\,,\\
P_{1055} &= 75x^6 - 256x^5 - 543x^4 - 3076x^3 + 2529x^2 - 124x + 51\,,\\
P_{1056} &= 90x^6 + 968x^5 - 6009x^4 + 8498x^3 - 3484x^2 - 498x + 243\,,\\
P_{1057} &= 129x^6 - 1128x^5 - 981x^4 + 3152x^3 - 3337x^2 + 824x - 3\,,\\
P_{1058} &= 173x^6 - 402x^5 + 963x^4 - 892x^3 + 735x^2 - 426x + 41\,,\\
P_{1059} &= 245x^6 - 222x^5 + 531x^4 + 92x^3 - 41x^2 - 22x - 7\,,\\
P_{1060} &= 247x^6 - 522x^5 + 225x^4 + 516x^3 - 1091x^2 + 158x - 109\,. \qquad \text{(C.8)}
\end{aligned}
$$

D The scalar form factors up to two-loop

In this appendix, we present the scalar form factors $F_S^{(n)}$ up to two loops and $\mathcal{O}(\varepsilon)$.

$$
\begin{aligned}
F_S^{(1)} = C_F\Bigg[&\frac{1}{\varepsilon}\Big\{-2-2\xi H_0\Big\}+\Big\{-2+\xi\Big(4H_{-1}H_0-H_0^2-4H_{0,-1}+2\zeta_2\Big)+8x\eta H_0\Big\}\\
&+\varepsilon\Big\{-4+\eta\Big(12xH_0-16xH_{-1}H_0+4xH_0^2+16xH_{0,-1}+\Big(-1-8x+x^2\Big)\zeta_2\Big)+\xi\Big(\Big(-2-4H_{-1}^2\\
&+\zeta_2\Big)H_0+2H_{-1}H_0^2-\frac{1}{3}H_0^3+8H_{-1}H_{0,-1}-4H_{0,0,-1}-8H_{0,-1,-1}-4H_{-1}\zeta_2+4\zeta_3\Big)\Big\}+\varepsilon^2\Big\{-8\\
&+\eta\Big(\Big(24x-24xH_{-1}+16xH_{-1}^2-4x\zeta_2\Big)H_0+\Big(-1+6x-x^2-8xH_{-1}\Big)H_0^2+\frac{4}{3}xH_0^3+\Big(24x\\
&-32xH_{-1}\Big)H_{0,-1}+16xH_{0,0,-1}+32xH_{0,-1,-1}+\Big(1-12x+3x^2\Big)\zeta_2+16xH_{-1}\zeta_2-\frac{2}{3}\Big(-1+24x\\
&+x^2\Big)\zeta_3\Big)+\xi\Big(\Big(-4+\Big(4-2\zeta_2\Big)H_{-1}+\frac{8}{3}H_{-1}^3+\frac{14}{3}\zeta_3\Big)H_0+\Big(-2H_{-1}^2+\frac{\zeta_2}{2}\Big)H_0^2+\frac{2}{3}H_{-1}H_0^3\\
&-\frac{1}{12}H_0^4+\Big(-4-8H_{-1}^2-2\zeta_2\Big)H_{0,-1}+8H_{-1}H_{0,0,-1}+16H_{-1}H_{0,-1,-1}-4H_{0,0,0,-1}-8H_{0,0,-1,-1}\\
&-16H_{0,-1,-1,-1}+4H_{-1}^2\zeta_2+\frac{14}{5}\zeta_2^2-8H_{-1}\zeta_3\Big)\Big\}+\varepsilon^3\Big\{-16+\eta\Big(\Big(48x+\Big(-48x+8x\zeta_2\Big)H_{-1}\\
&+24xH_{-1}^2-\frac{32}{3}xH_{-1}^3-6x\zeta_2-\frac{56}{3}x\zeta_3\Big)H_0+\Big(12x+\Big(-12x+\Big(-1-x^2\Big)\zeta_2\Big)H_{-1}+8xH_{-1}^2\\
&-2x\zeta_2\Big)H_0^2+\Big(\frac{1}{3}\Big(-1+6x-x^2\Big)-\frac{8}{3}xH_{-1}\Big)H_0^3+\frac{1}{3}xH_0^4+\Big(24x\zeta_2+32x\zeta_3\Big)H_{-1}+\Big(48x-48xH_{-1}\\
&+32xH_{-1}^2+8x\zeta_2\Big)H_{0,-1}+\Big(24x-32xH_{-1}\Big)H_{0,0,-1}+\Big(48x-64xH_{-1}\Big)H_{0,-1,-1}+16xH_{0,0,0,-1}\\
&+32xH_{0,0,-1,-1}+64xH_{0,-1,-1,-1}+2\Big(1-12x+3x^2\Big)\zeta_2-16xH_{-1}^2\zeta_2+\frac{1}{20}\Big(-9-224x+9x^2\Big)\zeta_2^2\\
&+\frac{2}{3}(-7+x)(-1+5x)\zeta_3\Big)+\xi\Big(\Big(-8+\Big(8-\frac{28\zeta_3}{3}\Big)H_{-1}+\Big(-4+2\zeta_2\Big)H_{-1}^2-\frac{4}{3}H_{-1}^4+\frac{47}{20}\zeta_2^2\\
&+\zeta_2\Big)H_0+\Big(-2+2H_{-1}+\frac{4}{3}H_{-1}^3+\frac{7}{3}\zeta_3\Big)H_0^2+\Big(-\frac{2}{3}H_{-1}^2+\frac{1}{6}\zeta_2\Big)H_0^3+\frac{1}{6}H_{-1}H_0^4-\frac{1}{60}H_0^5+\Big(-4\zeta_2\\
&-\frac{28\zeta_2^2}{5}\Big)H_{-1}+\Big(-8+\Big(8+4\zeta_2\Big)H_{-1}+\frac{16}{3}H_{-1}^3+\frac{4}{3}\zeta_3\Big)H_{0,-1}+\Big(-4-8H_{-1}^2-2\zeta_2\Big)H_{0,0,-1}+\Big(\\
&-8-16H_{-1}^2-4\zeta_2\Big)H_{0,-1,-1}+8H_{-1}H_{0,0,0,-1}+16H_{-1}H_{0,0,-1,-1}+32H_{-1}H_{0,-1,-1,-1}-4H_{0,0,0,0,-1}\\
&-8H_{0,0,0,-1,-1}-16H_{0,0,-1,-1,-1}-32H_{0,-1,-1,-1,-1}-\frac{8}{3}H_{-1}^3\zeta_2+8H_{-1}^2\zeta_3-\frac{8}{3}\zeta_2\zeta_3+12\zeta_5\Big)\Big\}\Bigg].
\end{aligned}
\tag{D.1}
$$

$$
\begin{aligned}
F_S^{(2)} = C_F^2\Bigg[&\frac{1}{\varepsilon^2}\Big\{2+4\xi H_0+2\xi^2H_0^2\Big\}+\frac{1}{\varepsilon}\Big\{4+\xi\Big(\eta\Big(-4(-1+x)(1+x)H_0-2\Big(-1+8x+x^2\Big)H_0^2\Big)\\
&-8H_{-1}H_0+8H_{0,-1}-4\zeta_2\Big)+\xi^2\Big(-8H_{-1}H_0^2+2H_0^3+8H_0H_{0,-1}-4H_0\zeta_2\Big)-16x\eta H_0\Big\}
\end{aligned}
$$

$$+\Bigg\{29+\eta^2\Big(4H_0H_{0,-1}P_{1063}+4H_0^2H_1P_{1100}-8H_0H_{0,1}P_{1115}+8H_{0,0,1}P_{1121}+2H_{-1}H_0^2P_{1122}$$
$$-4H_{0,0,-1}P_{1123}+2H_0^2P_{1142}+\Big(\frac{16}{3}H_0^3H_1P_{1171}-4H_0^2H_{0,-1}P_{1178}-8H_0H_{0,0,-1}P_{1194}$$
$$+48H_{0,0,0,-1}P_{1195}-8H_0^2H_{0,1}P_{1196}+16H_0H_{0,0,1}P_{1214}-32H_{0,0,0,1}P_{1215}-\frac{2}{3}H_0^3P_{1223}+\frac{1}{6}H_0^4P_{1281}$$
$$+\Big(16H_0H_1P_{1170}-16H_{0,1}P_{1170}-4H_0P_{1210}+4H_0^2P_{1216}-8H_{0,-1}P_{1218}\Big)\zeta_2+\frac{2}{5}P_{1314}\zeta_2^2\Big)x_+$$
$$-8P_{1105}\zeta_3\Big)+\xi^2\Big(-32(-1+x)^2(1+x)^2\eta x_+^2\zeta_3H_0+16H_{-1}^2H_0^2-8H_{-1}H_0^3+\Big(32H_0H_1$$
$$-32H_{0,-1}\Big)H_{0,1}+16H_{0,1}^2-32H_{-1}H_0H_{0,-1}-32H_0H_1H_{0,-1}+8H_{0,-1}^2-64H_1H_{0,0,1}+64H_1H_{0,0,-1}$$
$$-64H_0H_{0,1,1}+32H_0H_{0,1,-1}+32H_0H_{0,-1,1}+16H_0H_{0,-1,-1}+64H_{0,0,1,1}+32H_{0,-1,0,1}+16H_{-1}H_0\zeta_2$$
$$+16H_1\zeta_3\Big)+\eta\Big(x_+\zeta_2\Big(-2\Big(-3+75x-53x^2+5x^3\Big)-4\Big(-5+49x-53x^2+x^3\Big)H_{-1}\Big)$$
$$+4(-3+x)(-1+3x)H_0+12\Big(5-14x+5x^2\Big)H_{-1}H_0-12\Big(5-14x+5x^2\Big)H_{0,-1}+160x^2H_0x_+^2\zeta_3\Big)$$
$$+\xi\Big(8H_{-1}^2H_0-16H_{-1}H_{0,-1}+16H_{0,-1,-1}+128(-1+x)x(1+x)\eta H_0x_+^2\zeta_3\Big)+\Big(32(-1+x)H_0H_1$$
$$-32(-1+x)H_{0,1}\Big)x_++288x\log(2)x_+^2\zeta_2\Bigg\}+\varepsilon\Bigg\{\log(2)x_+^2\zeta_2\Big(24\Big(11+74x+11x^2\Big)+48\Big(3+8x$$
$$+3x^2\Big)H_1+48\Big(3+8x+3x^2\Big)H_{-1}\Big)+\xi\Big(\eta\Big(16(-1+x)(1+x)H_0H_1-16(-1+x)(1+x)H_{0,1}$$
$$+256(-1+x)x(1+x)H_{-1}H_0x_+^2\zeta_3\Big)-\frac{16}{3}H_{-1}^3H_0+16H_{-1}^2H_{0,-1}-32H_{-1}H_{0,-1,-1}$$
$$+32H_{0,-1,-1,-1}\Big)+\eta\Big(\log(2)x_+^2\zeta_2\Big(-192x^2H_{0,1}-192x^2H_{0,-1}\Big)+12(-3+x)(-1+3x)H_0$$
$$+8\Big(3-86x+3x^2\Big)H_{-1}H_0-4\Big(65-174x+65x^2\Big)H_{-1}^2H_0+256xH_0H_1-256xH_{0,1}+\Big(-8\Big(3$$
$$-86x+3x^2\Big)+8\Big(65-174x+65x^2\Big)H_{-1}\Big)H_{0,-1}-8\Big(65-174x+65x^2\Big)H_{0,-1,-1}+\Big(2\Big(95$$
$$-61x-205x^2+83x^3\Big)H_{-1}H_0^2-4\Big(-21+93x-101x^2+13x^3\Big)\zeta_3H_{-1}-8\Big(-23+43x$$
$$-39x^2+27x^3\Big)H_{-1}H_{0,0,-1}+8\Big(-23+43x-39x^2+27x^3\Big)H_{0,0,-1,-1}+\Big(8\Big(-5-90x+67x^2$$
$$+4x^3\Big)-4\Big(29-115x-7x^2+5x^3\Big)H_{-1}+4\Big(-29+121x-125x^2+25x^3\Big)H_{-1}^2\Big)\zeta_2\Big)x_+$$
$$+\Big(-48H_0H_{0,-1}^2P_{1081}+128H_0H_1H_{0,0,-1}P_{1085}+32H_0H_{0,-1,0,1}P_{1125}-32H_{0,-1}H_{0,0,1}P_{1131}$$
$$+\Big(96H_0^2H_1P_{1067}-32H_0H_{0,-1}P_{1095}\Big)H_{0,1}-32x\Big(6-29x+6x^2\Big)H_0H_{0,1}^2-16H_0H_{0,1}P_{1134}\zeta_2$$
$$+320x^2H_{-1}H_0\zeta_3\Big)x_+^2\Big)+\eta^2\Big(48H_0H_{0,1,-1}P_{1066}+48H_0H_{0,-1,1}P_{1066}+8H_{0,1}^2P_{1084}+32H_{0,-1,0,1}P_{1094}$$
$$+32H_0H_{0,1,1}P_{1096}-8H_0H_{0,-1,-1}P_{1114}+4H_{0,-1}^2P_{1120}+4H_0^2H_1P_{1135}+2H_0^2P_{1136}-2H_{-1}^2H_0^2P_{1139}$$
$$+8\Big(2-3x+2x^2\Big)\Big(3+2x+3x^2\Big)H_0^2H_1^2+\Big(-32H_{0,-1}P_{1099}-16H_0H_1P_{1124}+8H_0P_{1126}\Big)H_{0,1}$$
$$+\Big(-16H_{-1}H_0P_{1064}-16H_0H_1P_{1065}-4H_0P_{1151}\Big)H_{0,-1}+\Big(-8P_{1144}+32\Big(1+x+x^2\Big)\Big(5$$
$$-2x+5x^2\Big)H_1\Big)H_{0,0,1}+\Big(16H_1P_{1098}+4P_{1157}\Big)H_{0,0,-1}-32\Big(1+x+x^2\Big)\Big(5-2x+5x^2\Big)H_{0,0,1,1}$$

$$
+\Big(-16H_0^2H_{0,1,-1}P_{1112}-16H_0^2H_{0,-1,1}P_{1112}+\frac{32}{3}H_0^3H_1^2P_{1174}+64H_{0,0,1,0,1}P_{1180}
-48H_0H_{0,0,-1,-1}P_{1181}-64H_{0,0,1,0,-1}P_{1191}+\frac{1}{6}H_0^4P_{1193}+16H_0^2H_{0,1,1}P_{1203}-32H_0H_{0,0,1,-1}P_{1213}
-32H_0H_{0,0,-1,1}P_{1213}+32H_{0,0,-1,0,1}P_{1220}+8H_0^2H_{0,-1,-1}P_{1230}-32H_{0,0,0,1,-1}P_{1240}
-32H_{0,0,0,-1,1}P_{1240}+32H_{0,0,-1,0,-1}P_{1250}+96H_{0,0,0,-1,-1}P_{1252}-32H_0H_{0,0,1,1}P_{1258}
+64H_{0,0,0,1,1,1}P_{1265}+\frac{1}{30}H_0^5P_{1298}+\frac{2}{3}H_0^3P_{1306}-16H_{0,0,0,0,1}P_{1311}+16H_{0,0,0,0,-1}P_{1312}
+\Big(-\frac{64}{3}H_{-1}H_0^3P_{1174}+\frac{2}{3}H_0^4P_{1197}+\frac{8}{3}H_0^3P_{1245}\Big)H_1+\Big(-\frac{2}{3}H_0^4P_{1232}+\frac{2}{3}H_0^3P_{1293}\Big)H_{-1}
+\Big(\frac{8}{3}H_0^3P_{1192}+16H_{-1}H_0^2P_{1200}-4H_0^2P_{1234}\Big)H_{0,1}+\Big(-16xH_{-1}H_0^2P_{1083}+16H_0^2H_1P_{1176}
-\frac{4}{3}H_0^3P_{1261}+2H_0^2P_{1276}\Big)H_{0,-1}+\Big(32H_{0,1}P_{1164}-64H_{-1}H_0P_{1177}+256H_0H_1P_{1184}
-8H_0^2P_{1233}-8H_0P_{1274}\Big)H_{0,0,1}+\Big(64H_{-1}H_0P_{1101}-32H_{0,1}P_{1228}+4H_0P_{1231}+4H_0^2P_{1241}
-16H_{0,-1}P_{1285}\Big)H_{0,0,-1}+\Big(32H_{-1}P_{1179}-64H_1P_{1247}+16P_{1292}+16H_0P_{1300}\Big)H_{0,0,0,1}+\Big(
-96H_{-1}P_{1113}+32H_1P_{1290}-8H_0P_{1304}-4P_{1313}\Big)H_{0,0,0,-1}+\Big(128H_{0,1,1}P_{1174}+32H_0H_1^2P_{1175}
-32H_{0,-1,1}P_{1224}-32H_{0,1,-1}P_{1225}-16H_{0,0,1}P_{1255}-4H_0P_{1271}+16H_{0,-1,-1}P_{1282}-\frac{4}{3}H_0^3P_{1296}
+2H_0^2P_{1299}+8H_{0,0,-1}P_{1301}+\Big(-64H_{-1}H_0P_{1175}-8H_0^2P_{1235}+8H_0P_{1260}\Big)H_1+\Big(-8H_0^2P_{1190}
+4H_0P_{1227}\Big)H_{-1}+\Big(64H_{-1}P_{1175}-32H_1P_{1199}-8P_{1251}\Big)H_{0,1}+\Big(-32H_{-1}P_{1185}-4P_{1201}
-16H_0P_{1209}+32H_1P_{1224}\Big)H_{0,-1}-4P_{1297}\zeta_3\Big)\zeta_2+\Big(-\frac{8}{5}H_{-1}P_{1182}-\frac{8}{5}H_1P_{1303}+\frac{2P_{1315}}{5}
-\frac{2}{5}H_0P_{1316}\Big)\zeta_2^2+\Big(-16H_0H_1P_{1204}-16H_{0,-1}P_{1244}+16H_{0,1}P_{1254}-\frac{8}{3}H_0^2P_{1272}-\frac{4}{3}H_0P_{1302}\Big)\zeta_3
+6P_{1307}\zeta_5\Big)x_+\Big(-4H_1P_{1148}-\frac{2P_{1159}}{3}\Big)\zeta_3\Big)+\xi^2\Big(-64(-1+x)^2(1+x)^2\eta x_+^2\zeta_3H_{-1}H_0
-\frac{64}{3}H_{-1}^3H_0^2+16H_{-1}^2H_0^3+\Big(128H_{-1}H_0H_{0,-1}-96H_{0,-1}^2\Big)H_1+\Big(\Big(-128H_{-1}H_0+64H_{0,-1}\Big)H_1
+64H_0H_1^2+128H_{-1}H_{0,-1}\Big)H_{0,1}+\Big(32H_1-64H_{-1}\Big)H_{0,1}^2+64H_{-1}^2H_0H_{0,-1}-64H_0H_1^2H_{0,-1}
-32H_{-1}H_{0,-1}^2+\Big(256H_{-1}H_1-128H_1^2\Big)H_{0,0,1}+\Big(-256H_{-1}H_1+128H_1^2\Big)H_{0,0,-1}
+\Big(256H_{-1}H_0-192H_0H_1-256H_{0,-1}\Big)H_{0,1,1}+\Big(-128H_{-1}H_0+64H_0H_1-64H_{0,1}
+192H_{0,-1}\Big)H_{0,1,-1}+\Big(-128H_{-1}H_0+64H_0H_1+192H_{0,1}+192H_{0,-1}\Big)H_{0,-1,1}+\Big(-64H_{-1}H_0
+64H_0H_1-320H_{0,1}+32H_{0,-1}\Big)H_{0,-1,-1}+\Big(256H_1-256H_{-1}\Big)H_{0,0,1,1}-256H_1H_{0,0,1,-1}
-256H_1H_{0,0,-1,1}+256H_1H_{0,0,-1,-1}+192H_0H_{0,1,1,1}-64H_0H_{0,1,-1,-1}-128H_{-1}H_{0,-1,0,1}
-64H_0H_{0,-1,1,-1}-64H_0H_{0,-1,-1,1}+32H_0H_{0,-1,-1,-1}-384H_{0,0,1,1,1}+768H_{0,0,1,1,-1}
+256H_{0,0,1,-1,1}-256H_{0,0,-1,1,1}-64H_{0,1,0,1,1}+256H_{0,1,0,1,-1}-256H_{0,-1,0,1,1}+128H_{0,-1,0,1,-1}
+128H_{0,-1,0,-1,1}-256H_{0,-1,1,0,1}+320H_{0,-1,-1,0,1}-32H_{-1}^2H_0\zeta_2+\Big(-64H_{-1}H_1+32H_1^2\Big)\zeta_3\Big)
$$

$$+\Big(-192(-1+x)H_{-1}H_0H_1+32(-1+x)H_0H_1^2+\Big(-64(-1+x)H_1+192(-1+x)H_{-1}\Big)H_{0,1}$$
$$+192(-1+x)H_1H_{0,-1}+64(-1+x)H_{0,1,1}-192(-1+x)H_{0,1,-1}-192(-1+x)H_{0,-1,1}$$
$$-32(-1+x)H_1\zeta_2\Big)x_+-48c_1xx_+^2+\Big(-\frac{113}{2}(1+x)^2-8\Big(3+8x+3x^2\Big)H_{-1}H_0^2H_1+16\Big(7+12x$$
$$+7x^2\Big)H_{-1}H_0H_{0,1}-16\Big(11+16x+11x^2\Big)H_{-1}H_{0,0,1}+16\Big(11+16x+11x^2\Big)H_{0,0,1,-1}+16\Big(11$$
$$+16x+11x^2\Big)H_{0,0,-1,1}+\Big(-48\Big(3+8x+3x^2\Big)H_{-1}H_1+48\Big(3+8x+3x^2\Big)H_{-1,1}\Big)\zeta_2\Big)x_+^2\Bigg\}\Bigg]$$
$$+C_FC_A\Bigg[\frac{1}{\varepsilon^2}\Bigg\{\frac{11}{3}+\frac{11\xi H_0}{3}\Bigg\}+\frac{1}{\varepsilon}\Bigg\{-\frac{49}{9}+\xi\Big(\eta\Big(-\frac{4}{3}x^2H_0^3-2\Big(-1+3x^2\Big)H_0\zeta_2\Big)-\frac{67}{9}H_0$$
$$+4H_{-1}H_0-4H_0H_1+4H_{0,1}-4H_{0,-1}\Big)+\eta\Big(-4x^2H_0^2-2\Big(-1+3x^2\Big)\zeta_2\Big)+\xi^2\Big(-4H_0H_{0,1}$$
$$+4H_0H_{0,-1}+8H_{0,0,1}-8H_{0,0,-1}-2\zeta_3\Big)\Bigg\}+\Bigg\{-\frac{869}{27}+\eta^2x_+\Big(4H_0^2H_{0,-1}P_{1211}-8H_{0,0,0,-1}P_{1236}$$
$$+8H_{0,0,0,1}P_{1248}-\frac{1}{2}x^2\Big(1+7x+4x^2+4x^3\Big)H_0^4-32x^2\Big(7-5x+x^2+x^3\Big)H_0H_{0,0,1}+\Big(4H_{0,-1}P_{1217}$$
$$+2\Big(1+x+3x^2+x^3+x^4+x^5\Big)H_0^2\Big)\zeta_2+\frac{1}{5}P_{1162}\zeta_2^2-2H_0P_{1243}\zeta_3\Big)+\xi^2\Big(\Big(-8H_1+8H_{-1}\Big)\zeta_3$$
$$+\Big(16H_{-1}H_0-16H_0H_1-8H_{0,-1}\Big)H_{0,1}-4H_{0,1}^2-16H_{-1}H_0H_{0,-1}+16H_0H_1H_{0,-1}+12H_{0,-1}^2$$
$$+\Big(32H_1-32H_{-1}\Big)H_{0,0,1}+\Big(-32H_1+32H_{-1}\Big)H_{0,0,-1}+24H_0H_{0,1,1}-8H_0H_{0,1,-1}-8H_0H_{0,-1,1}$$
$$-8H_0H_{0,-1,-1}-32H_{0,0,1,1}+32H_{0,0,1,-1}+32H_{0,0,-1,1}-32H_{0,0,-1,-1}\Big)+\eta\Big(-\frac{22}{27}\Big(11-114x$$
$$+11x^2\Big)H_0+\frac{2}{9}\Big(53-102x+53x^2\Big)H_{-1}H_0+8\Big(-3+x^2\Big)H_0^2H_1-\frac{2}{9}\Big(53-102x+53x^2\Big)H_{0,-1}$$
$$+\Big(\frac{2}{3}\Big(49-17x+79x^2+13x^3\Big)\zeta_3+\frac{1}{9}\Big(-67+74x+11x^2+50x^3\Big)H_0^2+\frac{1}{3}\Big(61+175x-35x^2$$
$$+79x^3\Big)H_{-1}H_0^2+16\Big(-3-x+3x^2+5x^3\Big)H_{0,0,1}-\frac{2}{3}\Big(-53-95x+235x^2+193x^3\Big)H_{0,0,-1}+\Big(\frac{2}{9}\Big(91$$
$$+562x-539x^2+34x^3\Big)-\frac{2}{3}\Big(25-137x+169x^2+7x^3\Big)H_{-1}\Big)\zeta_2\Big)x_++\Big(\frac{1}{9}H_0^3P_{1061}+8H_0^2H_{0,1}P_{1069}$$
$$+16H_0H_{0,0,-1}P_{1070}-\frac{2}{3}H_0P_{1079}\zeta_2\Big)x_+^2\Big)+\xi\Big(8H_0H_1\Big(1+3H_{-1}\Big)+\eta\Big(\frac{4}{3}\Big(3+x^2\Big)H_{-1}H_0^3+\frac{4}{3}\Big(-5$$
$$+x^2\Big)H_0^3H_1+\Big(16H_{-1}H_0+16\Big(-2+x^2\Big)H_0H_1-12\Big(-3+x^2\Big)H_{0,1}\Big)\zeta_2\Big)-\frac{104}{3}H_{-1}^2H_0-4H_0H_1^2$$
$$+\Big(-8+8H_1-24H_{-1}\Big)H_{0,1}+\Big(-24H_1+\frac{208H_{-1}}{3}\Big)H_{0,-1}-8H_{0,1,1}+24H_{0,1,-1}+24H_{0,-1,1}$$
$$-\frac{208}{3}H_{0,-1,-1}+4H_1\zeta_2\Big)+\Big(16\Big(3+5x+3x^2\Big)H_0H_{0,1}-2\Big(19+64x+19x^2\Big)H_0H_{0,-1}\Big)x_+^2$$
$$-144x\log(2)x_+^2\zeta_2\Bigg\}+\varepsilon\Bigg\{\log(2)x_+^2\zeta_2\Big(-12\Big(11+74x+11x^2\Big)-24\Big(3+8x+3x^2\Big)H_1-24\Big(3$$
$$+8x+3x^2\Big)H_{-1}\Big)+\eta^2x_+\Big(H_{-1}H_0^4P_{1187}+128H_0H_{0,0,1,1}P_{1188}-16H_0^2H_{0,1,1}P_{1189}+8H_0^2H_{0,1,-1}P_{1208}$$
$$+8H_0^2H_{0,-1,1}P_{1208}-32H_{0,0,1,0,1}P_{1221}-64H_{0,0,-1,0,-1}P_{1222}-8H_0^2H_{0,-1,-1}P_{1246}+16H_{0,0,-1,0,1}P_{1257}$$

$$+16H_{0,0,1,0,-1}P_{1268}-\frac{4}{3}H_0^3H_{0,1}P_{1269}+32H_{0,0,0,1,-1}P_{1289}+32H_{0,0,0,-1,1}P_{1289}-16H_{0,0,0,1,1}P_{1305}$$
$$+8H_{0,0,0,0,1}P_{1308}-8H_{0,0,0,0,-1}P_{1309}-16H_{0,0,0,-1,-1}P_{1310}-\frac{1}{30}x^2\Big(5+43x+24x^2+24x^3\Big)H_0^5+\Big(-8$$
$$-8x-11x^2-5x^3\Big)H_0^4H_1+\Big(-8H_0^2H_1P_{1202}+\frac{4}{3}H_0^3P_{1295}-8\Big(8+8x+15x^2+x^3\Big)H_{-1}H_0^2\Big)H_{0,-1}$$
$$+\Big(32H_{-1}H_0P_{1173}+32H_{0,1}P_{1206}-32H_0H_1P_{1207}-16H_{0,-1}P_{1267}+4H_0^2P_{1275}\Big)H_{0,0,1}$$
$$+\Big(-16H_{0,1}P_{1238}+32H_{0,-1}P_{1242}-8H_0^2P_{1249}\Big)H_{0,0,-1}+\Big(-16H_{-1}P_{1219}+16H_1P_{1259}$$
$$-8H_0P_{1291}\Big)H_{0,0,0,1}+\Big(16H_{-1}P_{1229}-16H_1P_{1239}+8H_0P_{1287}\Big)H_{0,0,0,-1}+\Big(-4H_{-1}H_0^2P_{1186}$$
$$-16H_{0,0,-1}P_{1266}+8H_{0,-1,1}P_{1277}+8H_{0,1,-1}P_{1278}-8H_{0,-1,-1}P_{1283}-4H_0H_{0,1}P_{1286}+8H_{0,0,1}P_{1288}$$
$$+\frac{2}{3}H_0^3P_{1294}+4x^2\Big(3+5x+4x^2+4x^3\Big)H_0^2H_1+\Big(16H_{-1}P_{1205}-16H_1P_{1237}+4H_0P_{1284}\Big)H_{0,-1}$$
$$+2P_{1273}\zeta_3\Big)\zeta_2+\Big(\frac{4}{5}H_{-1}P_{1263}-\frac{2}{5}H_0P_{1264}-\frac{4}{5}H_1P_{1270}\Big)\zeta_2^2+\Big(16H_0H_1P_{1172}-16H_{-1}H_0P_{1198}$$
$$-4H_{0,1}P_{1256}+2H_0^2P_{1279}+4H_{0,-1}P_{1280}\Big)\zeta_3+P_{1161}\zeta_5\Big)+\xi\Big(8H_0H_1^2\Big(1+3H_{-1}\Big)-24H_0H_1H_{-1}\Big(2$$
$$+3H_{-1}\Big)+4\Big(-2+H_1\Big)H_1\zeta_2+\eta\Big(-\frac{4}{3}\Big(3+5x^2\Big)H_{-1}^2H_0^3+\Big(4(-1+x)(1+x)H_0+\frac{8}{3}\Big(1$$
$$+7x^2\Big)H_{-1}H_0^3\Big)H_1-\frac{4}{3}\Big(1+7x^2\Big)H_0^3H_1^2-4(-1+x)(1+x)H_{0,1}+\Big(-8\Big(1+3x^2\Big)H_{-1}^2H_0+16\Big($$
$$-1+5x^2\Big)H_{-1}H_0H_1-8\Big(-1+5x^2\Big)H_0H_1^2+\Big(96x^2H_1-96x^2H_{-1}\Big)H_{0,1}-8\Big(1+13x^2\Big)H_{0,1,1}\Big)\zeta_2\Big)$$
$$+80H_{-1}^3H_0-\frac{8}{3}H_0H_1^3+\Big(-16H_1\Big(1+3H_{-1}\Big)+8H_1^2+48H_{-1}+72H_{-1}^2\Big)H_{0,1}+\Big(48H_1\Big(1+3H_{-1}\Big)$$
$$-24H_1^2-240H_{-1}^2\Big)H_{0,-1}+\Big(16-16H_1+48H_{-1}\Big)H_{0,1,1}+\Big(-48+48H_1-144H_{-1}\Big)H_{0,1,-1}$$
$$+\Big(-48+48H_1-144H_{-1}\Big)H_{0,-1,1}+\Big(-144H_1+480H_{-1}\Big)H_{0,-1,-1}+16H_{0,1,1,1}-48H_{0,1,1,-1}$$
$$-48H_{0,1,-1,1}+144H_{0,1,-1,-1}-48H_{0,-1,1,1}+144H_{0,-1,1,-1}+144H_{0,-1,-1,1}-480H_{0,-1,-1,-1}+\Big($$
$$-12(-1+x)(1+x)H_0H_{0,1}-8(-1+x)(1+x)H_{-1}H_0H_{0,-1}-80(-1+x)(1+x)H_0H_1H_{0,-1}\Big)x_+^2\Big)$$
$$+\eta\Big(\log(2)x_+^2\zeta_2\Big(96x^2H_{0,1}+96x^2H_{0,-1}\Big)-\frac{2}{81}\Big(1061-16710x+1061x^2\Big)H_0+\frac{4}{27}\Big(341-1266x$$
$$+341x^2\Big)H_{-1}H_0+\frac{2}{9}\Big(47+306x+47x^2\Big)H_{-1}^2H_0-24xH_0H_1+24xH_{0,1}+\Big(-\frac{4}{27}\Big(341-1266x$$
$$+341x^2\Big)-\frac{4}{9}\Big(47+306x+47x^2\Big)H_{-1}\Big)H_{0,-1}+\frac{4}{9}\Big(47+306x+47x^2\Big)H_{0,-1,-1}+\Big(\frac{1}{27}\Big(-341$$
$$+2599x-1127x^2-395x^3\Big)H_0^2+\frac{1}{9}\Big(-137-1037x+115x^2-173x^3\Big)H_{-1}H_0^2+\Big(-65-155x$$
$$-x^2-91x^3\Big)H_{-1}^2H_0^2+\Big(2\Big(-1+61x-45x^2+17x^3\Big)H_0^2+4\Big(15+17x+7x^2+9x^3\Big)H_{-1}H_0^2\Big)H_1$$
$$+2\Big(-13+13x-21x^2+5x^3\Big)H_0^2H_1^2+\Big(-4\Big(5-65x+81x^2+11x^3\Big)+8\Big(3+45x-37x^2$$
$$+5x^3\Big)H_1-8\Big(5+43x-19x^2+19x^3\Big)H_{-1}\Big)H_{0,0,1}+\Big(\frac{2}{9}\Big(-277-1285x+2207x^2+587x^3\Big)$$
$$+8\Big(-35-77x+53x^2+11x^3\Big)H_1+4\Big(61+191x-35x^2+95x^3\Big)H_{-1}\Big)H_{0,0,-1}-8\Big(3+45x$$
$$-37x^2+5x^3\Big)H_{0,0,1,1}+8\Big(5+43x-19x^2+19x^3\Big)H_{0,0,1,-1}+8\Big(5+43x-19x^2+19x^3\Big)H_{0,0,-1,1}$$

$$
-4\Big(61+191x-35x^2+95x^3\Big)H_{0,0,-1,-1}+\Big(-\frac{4}{27}\Big(-493-1480x+2783x^2+206x^3\Big)-48\Big(
-1-2x+3x^2+2x^3\Big)H_{-1}H_1-\frac{2}{9}\Big(61+349x-623x^2+277x^3\Big)H_{-1}-6\Big(-17+33x-49x^2
+x^3\Big)H_{-1}^2\Big)\zeta_2+\Big(\frac{1}{9}\Big(1055+3683x-4591x^2-739x^3\Big)-2\Big(-95-53x+9x^2+51x^3\Big)H_1-2\Big(91
-55x+211x^2+65x^3\Big)H_{-1}\Big)\zeta_3\Big)x_+ +\Big(\frac{1}{12}H_0^4P_{1062}+32H_0H_{0,0,1,-1}P_{1071}+32H_0H_{0,0,-1,1}P_{1071}
+24H_0H_{0,-1}^2P_{1082}+16H_0H_{0,1}^2P_{1086}-16H_0H_{0,-1,0,1}P_{1102}-32H_0H_{0,0,-1,-1}P_{1104}+4H_{0,0,0,1}P_{1137}
+\frac{2}{3}H_0^3H_1P_{1140}+4H_0H_{0,0,1}P_{1141}-2H_{0,0,0,-1}P_{1143}+\frac{1}{3}H_{-1}H_0^3P_{1152}+H_0^2H_{0,-1}P_{1154}+\frac{1}{27}H_0^3P_{1158}
+\Big(-48H_0^2H_1P_{1067}+16H_{-1}H_0^2P_{1069}-16H_0H_{0,-1}P_{1103}-2H_0^2P_{1145}\Big)H_{0,1}+\Big(-96H_0H_1P_{1068}
+32H_{-1}H_0P_{1070}-2H_0P_{1156}\Big)H_{0,0,-1}+\Big(4H_{-1}H_0P_{1080}-4H_{0,1}P_{1097}+4H_0H_1P_{1130}-4H_{0,-1}P_{1138}
+\frac{1}{2}H_0^2P_{1150}+\frac{1}{9}H_0P_{1153}\Big)\zeta_2+\frac{1}{5}P_{1155}\zeta_2^2+\frac{4}{3}H_0P_{1146}\zeta_3\Big)x_+^2\Big)+\xi^2\Big(\Big(-64H_{-1}H_0H_{0,-1}
+48H_{0,-1}^2\Big)H_1+\Big(-32H_{-1}^2H_0+\Big(64H_{-1}H_0-32H_{0,-1}\Big)H_1-32H_0H_1^2+32H_{-1}H_{0,-1}\Big)H_{0,1}
+\Big(-16H_1+16H_{-1}\Big)H_{0,1}^2+32H_{-1}^2H_0H_{0,-1}+32H_0H_1^2H_{0,-1}-48H_{-1}H_{0,-1}^2+\Big(-128H_{-1}H_1
+64H_1^2+64H_{-1}^2\Big)H_{0,0,1}+\Big(128H_{-1}H_1-64H_1^2-64H_{-1}^2\Big)H_{0,0,-1}+\Big(-96H_{-1}H_0+96H_0H_1
+16H_{0,1}+80H_{0,-1}\Big)H_{0,1,1}+\Big(32H_{-1}H_0-32H_0H_1+16H_{0,1}-48H_{0,-1}\Big)H_{0,1,-1}+\Big(32H_{-1}H_0
-32H_0H_1-48H_{0,1}-48H_{0,-1}\Big)H_{0,-1,1}+\Big(32H_{-1}H_0-32H_0H_1+16H_{0,1}-48H_{0,-1}\Big)H_{0,-1,-1}
+\Big(-128H_1+128H_{-1}\Big)H_{0,0,1,1}+\Big(128H_1-128H_{-1}\Big)H_{0,0,1,-1}+\Big(128H_1-128H_{-1}\Big)H_{0,0,-1,1}
+\Big(-128H_1+128H_{-1}\Big)H_{0,0,-1,-1}-112H_0H_{0,1,1,1}+16H_0H_{0,1,1,-1}+16H_0H_{0,1,-1,1}
+16H_0H_{0,1,-1,-1}+16H_0H_{0,-1,1,1}+16H_0H_{0,-1,1,-1}+16H_0H_{0,-1,-1,1}+16H_0H_{0,-1,-1,-1}
+128H_{0,0,1,1,1}-256H_{0,0,1,1,-1}-128H_{0,0,1,-1,1}+128H_{0,0,1,-1,-1}+128H_{0,0,-1,1,-1}+128H_{0,0,-1,-1,1}
+256H_{0,0,-1,-1,-1}-64H_{0,1,0,1,-1}+64H_{0,-1,0,1,1}+192H_{0,-1,0,-1,-1}+64H_{0,-1,1,0,1}+\Big(32H_{-1}H_1
-16H_1^2-16H_{-1}^2\Big)\zeta_3\Big)+24c_1xx_+^2+\Big(-\frac{6437}{162}(1+x)^2+\Big(-240xH_0-8\Big(5-8x+5x^2\Big)H_{-1}H_0
+8\Big(5-24x+5x^2\Big)H_0H_1+40\Big(5+12x+5x^2\Big)H_{0,-1}\Big)H_{0,1}-16(2+x)(1+2x)H_{0,1}^2+2\Big(23+152x
+23x^2\Big)H_0H_{0,-1}-64xH_{-1}H_0H_{0,-1}+320xH_0H_1H_{0,-1}-2\Big(59+176x+59x^2\Big)H_{0,-1}^2+8\Big(3
+44x+3x^2\Big)H_0H_{0,1,1}-32\Big(5+17x+5x^2\Big)H_0H_{0,1,-1}-32\Big(5+17x+5x^2\Big)H_0H_{0,-1,1}+12\Big(19
+64x+19x^2\Big)H_0H_{0,-1,-1}-8\Big(15+32x+15x^2\Big)H_{0,-1,0,1}-24\Big(3+8x+3x^2\Big)H_{-1,1}\zeta_2\Big)x_+^2\Big\}\Big]
+C_Fn_lT_F\Big[\frac{1}{\varepsilon^2}\Big\{-\frac{4}{3}-\frac{4\xi H_0}{3}\Big\}+\frac{1}{\varepsilon}\Big\{\frac{20}{9}+\frac{20\xi H_0}{9}\Big\}+\Big\{\frac{196}{27}+\eta\Big(\frac{16}{27}\Big(7-48x+7x^2\Big)H_0-\frac{16}{9}\Big(5
-12x+5x^2\Big)H_{-1}H_0+\frac{4}{9}\Big(5-12x+5x^2\Big)H_0^2+\frac{16}{9}\Big(5-12x+5x^2\Big)H_{0,-1}-\frac{8}{9}\Big(-1-12x+11x^2\Big)\zeta_2\Big)
$$

$$+\xi\Big(\frac{16}{3}H_{-1}^2H_0-\frac{8}{3}H_{-1}H_0^2+\frac{4}{9}H_0^3-\frac{32}{3}H_{-1}H_{0,-1}+\frac{16}{3}H_{0,0,-1}+\frac{32}{3}H_{0,-1,-1}+\Big(\frac{8}{3}H_0+\frac{16}{3}H_{-1}\Big)\zeta_2$$
$$-\frac{16\zeta_3}{3}\Big)\Big\}+\varepsilon\Big\{\frac{1706}{81}+\xi\Big(\Big(-\frac{16H_0}{3}+32H_{-1}\Big)\zeta_3-\frac{32}{3}H_{-1}^3H_0+8H_{-1}^2H_0^2-\frac{8}{3}H_{-1}H_0^3$$
$$+32H_{-1}^2H_{0,-1}-32H_{-1}H_{0,0,-1}-64H_{-1}H_{0,-1,-1}+16H_{0,0,0,-1}+32H_{0,0,-1,-1}+64H_{0,-1,-1,-1}+\Big($$
$$-8H_{-1}H_0+2H_0^2-16H_{-1}^2+24H_{0,-1}\Big)\zeta_2-\frac{96\zeta_2^2}{5}\Big)+\eta\Big(\frac{16}{81}\Big(77-636x+77x^2\Big)H_0+\frac{32}{9}\Big(5-18x$$
$$+5x^2\Big)H_{-1}^2H_0+\frac{4}{27}\Big(37-246x+37x^2\Big)H_0^2+\frac{8}{27}\Big(5-18x+5x^2\Big)H_0^3+\frac{1}{3}\Big(1+x^2\Big)H_0^4+\Big(-\frac{16}{27}\Big(37$$
$$-246x+37x^2\Big)H_0-\frac{16}{9}\Big(5-18x+5x^2\Big)H_0^2\Big)H_{-1}+\Big(\frac{16}{27}\Big(37-246x+37x^2\Big)-\frac{64}{9}\Big(5-18x$$
$$+5x^2\Big)H_{-1}\Big)H_{0,-1}+\frac{32}{9}\Big(5-18x+5x^2\Big)H_{0,0,-1}+\frac{64}{9}\Big(5-18x+5x^2\Big)H_{0,-1,-1}+\Big(-\frac{8}{27}\Big(-41$$
$$-246x+115x^2\Big)+\frac{4}{9}\Big(5-36x+5x^2\Big)H_0+\frac{32}{9}\Big(5-18x+5x^2\Big)H_{-1}\Big)\zeta_2-\frac{64}{9}\Big(1-9x+4x^2\Big)\zeta_3\Big)\Big\}\Big]$$
$$+C_FT_F\Big[\Big\{\frac{4}{27}\Big(407+694x+407x^2\Big)x_+^2+\frac{64}{5}(-1+x)x(1+x)\xi^2\eta^2\zeta_2^2+\eta\Big(-128xH_{-1}H_0$$
$$+64xH_0H_1-32xH_{0,1}+128xH_{0,-1}+\Big(\frac{32}{27}H_0P_{1118}+32x^2H_0^3H_1+\Big(192x^2H_0H_1$$
$$-192x^2H_{0,1}\Big)\zeta_2\Big)x_+^2+\frac{4}{3}P_{1253}x_+^3\zeta_2\Big)+\xi\Big(\eta x_+^2\Big(-\frac{16}{3}(-1+x)x(1+x)H_0^3H_1+\Big($$
$$-32(-1+x)x(1+x)H_0H_1+32(-1+x)x(1+x)H_{0,1}\Big)\zeta_2\Big)+4H_0\zeta_2-\frac{256}{5}x^2\eta^2\zeta_2^2\Big)$$
$$+\eta^2\Big(-\frac{4}{9}\Big(-1+48x^2+x^4\Big)H_0^3+256x^2H_0^2H_1-768x^2H_0H_{0,1}+512x^2H_0H_{0,-1}$$
$$+1024x^2H_{0,0,1}-1024x^2H_{0,0,-1}+\frac{4}{9}H_0^2P_{1319}x_+^2-256x^2\zeta_3\Big)+\eta^3\Big(-64xH_0H_{0,0,1}P_{1073}$$
$$+16xH_{0,0,0,1}P_{1116}-256x^3H_0^2H_{0,1}-1024x^3H_0H_{0,0,-1}+3072x^3H_{0,0,0,-1}-\frac{xH_0^4}{3x_+^4}$$
$$-8xH_0^2P_{1075}\zeta_2-16xH_0P_{1091}\zeta_3\Big)\Big\}+\varepsilon\Big\{\xi\Big(\eta x_+^2\Big(-\frac{32}{3}(-1+x)x(1+x)H_{-1}H_0^3H_1$$
$$+\frac{16}{3}(-1+x)x(1+x)H_0^3H_1^2-64(-1+x)x(1+x)H_0H_{0,-1}H_{0,1}+64(-1+x)x(1+x)H_0^2H_1H_{0,-1}$$
$$-128(-1+x)x(1+x)H_{0,0,-1,0,1}+\Big(-64(-1+x)x(1+x)H_{-1}H_0H_1+32(-1+x)x(1+x)H_0H_1^2$$
$$+\Big(-64(-1+x)x(1+x)H_1+64(-1+x)x(1+x)H_{-1}\Big)H_{0,1}+256(-1+x)x(1+x)H_1H_{0,-1}$$
$$+64(-1+x)x(1+x)H_{0,1,1}-256(-1+x)x(1+x)H_{0,-1,1}\Big)\zeta_2\Big)+\eta^2\Big(1024x^2H_{0,1,-1}\zeta_2$$
$$-\frac{512}{5}x^2H_{-1}\zeta_2^2\Big)-256(-1+x)x^2(1+x)\eta^3H_0H_{0,-1,0,1}\Big)+\xi^2\Big(\eta^2\Big(-256(-1+x)x(1+x)H_{0,1,-1}\zeta_2$$
$$+\frac{128}{5}(-1+x)x(1+x)H_{-1}\zeta_2^2\Big)+64(-1+x)^2x(1+x)^2\eta^3H_0H_{0,-1,0,1}\Big)+\eta\Big(\Big(64xH_1+96xH_{0,1}\Big)\zeta_2$$
$$+\frac{32}{27}\Big(6+203x+257x^2+6x^3\Big)x_+H_0+512xH_{-1}^2H_0+\Big(512x-384xH_{-1}\Big)H_0H_1+64xH_0H_1^2$$

$$+\Big(-64xH_1+448xH_{-1}\Big)H_{0,1}+\Big(384xH_1-1024xH_{-1}\Big)H_{0,-1}+64xH_{0,1,1}-448xH_{0,1,-1}$$
$$-448xH_{0,-1,1}+1024xH_{0,-1,-1}+\Big(-\frac{64}{27}H_{-1}H_0P_{1119}+64x^2H_{-1}H_0^3H_1-32x^2H_0^3H_1^2+\Big($$
$$-64x\Big(7+12x+7x^2\Big)+384x^2H_0H_{0,-1}\Big)H_{0,1}+\Big(\frac{64P_{1119}}{27}-384x^2H_0^2H_1\Big)H_{0,-1}+768x^2H_{0,0,-1,0,1}$$
$$+\Big(\frac{4P_{1160}}{27}+384x^2H_{-1}H_0H_1-192x^2H_0H_1^2+\Big(384x^2H_1-384x^2H_{-1}\Big)H_{0,1}-1536x^2H_1H_{0,-1}$$
$$-384x^2H_{0,1,1}+1536x^2H_{0,-1,1}\Big)\zeta_2\Big)x_+^2\Big)+\eta^2\Big(-128x^2H_{-1}^2H_0^2-\frac{8}{9}\Big(-1-144x^2+x^4\Big)H_{-1}H_0^3+\Big($$
$$-256x^2H_{-1}H_0^2+16x\Big(-1+16x+x^2\Big)H_0^3\Big)H_1+384x^2H_0^2H_1^2+\Big(1024x^2H_{-1}H_0-1536x^2H_0H_1$$
$$-2560x^2H_{0,-1}\Big)H_{0,1}+256x^2H_{0,1}^2+\Big(\Big(64x\Big(1+27x+2x^2\Big)-512x^2H_{-1}\Big)H_0+\frac{16}{3}\Big(-1$$
$$+192x^2+x^4\Big)H_0^2-512x^2H_0H_1\Big)H_{0,-1}+1536x^2H_{0,-1}^2+\Big(1536x^2H_1-1536x^2H_{-1}\Big)H_{0,0,1}$$
$$+\Big(1536x^2H_1+1536x^2H_{-1}\Big)H_{0,0,-1}+1024x^2H_0H_{0,1,1}+1536x^2H_0H_{0,1,-1}+1536x^2H_0H_{0,-1,1}$$
$$-2560x^2H_0H_{0,-1,-1}-1536x^2H_{0,0,1,1}+1536x^2H_{0,0,1,-1}+1536x^2H_{0,0,-1,1}-1536x^2H_{0,0,-1,-1}$$
$$+1536x^2H_{0,-1,0,1}+\Big(\Big(1536x^2H_1+1536x^2H_{-1}\Big)\zeta_2+\frac{64}{3}P_{1324}x_+^2\zeta_2\Big)\log(2)-\frac{4}{27}H_0^2P_{1262}x_+$$
$$+\Big(\frac{16}{9}H_{0,0,-1}P_{1317}-\frac{8}{9}H_{-1}H_0^2P_{1318}+\frac{16}{9}H_0^2H_1P_{1320}-\frac{32}{9}H_0H_{0,1}P_{1321}+\frac{32}{9}H_{0,0,1}P_{1322}$$
$$+\frac{8}{27}H_0^3P_{1325}+\Big(\frac{16}{9}xH_0P_{1226}-\frac{16}{3}H_{-1}P_{1323}\Big)\zeta_2-\frac{8}{9}P_{1326}\zeta_3\Big)x_+^2+\Big(-\frac{8}{3}\Big(-1-288x^2+x^4\Big)H_{-1}H_0$$
$$+\Big(32x(3+x)(-1+3x)H_0-1536x^2H_{-1}\Big)H_1-768x^2H_{-1}^2+\frac{8}{3}\Big(-7-480x^2+7x^4\Big)H_{0,-1}$$
$$+1536x^2H_{-1,1}\Big)\zeta_2+\Big(-2688x^2H_1+384x^2H_{-1}\Big)\zeta_3\Big)+\eta^3\Big(-64xH_0H_{0,1}^2P_{1073}+32xH_0^2H_{0,1,1}P_{1073}$$
$$+256xH_0H_{0,0,1,-1}P_{1073}+256xH_0H_{0,0,-1,1}P_{1073}+32xH_0^2H_{0,1,-1}P_{1075}+32xH_0^2H_{0,-1,1}P_{1075}$$
$$-32xH_{0,0,1,0,-1}P_{1078}+\frac{8}{3}xH_0^3H_{0,-1}P_{1088}-64xH_{0,0,1,0,1}P_{1089}-32xH_{0,0,0,1,1}P_{1106}+\frac{2}{3}xH_0^4H_1P_{1109}$$
$$-64xH_{0,0,0,1,-1}P_{1110}-64xH_{0,0,0,-1,1}P_{1110}+16xH_{0,0,0,0,-1}P_{1147}-16xH_{0,0,0,0,1}P_{1149}+\log(2)\zeta_2\Big($$
$$-3072x^3H_{0,1}-3072x^3H_{0,-1}\Big)+\frac{1}{3}(1+x)\Big(1-53x^2+43x^3+x^5\Big)H_0^4+\Big(-\frac{16}{3}xH_0^3P_{1093}$$
$$+\frac{16}{3}(1+x)H_0^2P_{1168}-512x^3H_0^2H_1\Big)H_{0,1}-512x^3H_{-1}H_0^2H_{0,1}+1536x^3H_0H_{0,-1}^2+\Big($$
$$-32xH_0^2P_{1072}-128xH_{-1}H_0P_{1073}-32xH_{0,-1}P_{1087}+64xH_{0,1}P_{1089}-\frac{64}{3}(1+x)H_0P_{1165}\Big)H_{0,0,1}$$
$$+\Big(-32xH_0^2P_{1075}-64xH_{0,1}P_{1091}+\frac{32}{3}(1+x)H_0P_{1167}-2048x^3H_{-1}H_0-7168x^3H_{0,-1}\Big)H_{0,0,-1}$$
$$+512x^3H_0^2H_{0,-1,-1}+\Big(32xH_{-1}P_{1116}+32xH_0P_{1127}-32xH_1P_{1132}+16(1+x)P_{1183}\Big)H_{0,0,0,1}+\Big($$
$$-32xH_0P_{1107}+64xH_1P_{1128}-\frac{16}{3}(1+x)P_{1163}+6144x^3H_{-1}\Big)H_{0,0,0,-1}+4096x^3H_0H_{0,0,-1,-1}$$
$$-3840x^3H_0H_{0,-1,0,1}+9216x^3H_{0,0,0,-1,-1}+5120x^3H_{0,0,-1,0,-1}+(1+x)^4\Big(-\frac{2}{3}xH_{-1}H_0^4$$
$$-\frac{4}{15}xH_0^5+128xH_0H_1H_{0,0,1}-128xH_0H_1H_{0,0,-1}-128xH_0H_{0,0,1,1}-32xH_0^2H_1\zeta_2-32xH_0H_1\zeta_3\Big)$$

$$+\Big(64xH_0H_{0,-1}P_{1074}-256xH_{0,0,-1}P_{1074}-16xH_{-1}H_0^2P_{1075}-4xH_0^3P_{1076}-32xH_0H_{0,1}P_{1092}$$
$$+32xH_{0,0,1}P_{1117}+2(1+x)H_0^2P_{1169}+3072x^3H_{0,-1,-1}-16xP_{1077}\zeta_3\Big)\zeta_2+\Big(-\frac{16}{5}xH_1P_{1110}$$
$$+\frac{8}{5}xH_0P_{1111}-\frac{8}{5}(1+x)P_{1166}\Big)\zeta_2^2+\Big(32xH_{0,-1}P_{1090}-32xH_{-1}H_0P_{1091}-32xH_{0,1}P_{1108}-4xH_0^2P_{1133}$$
$$-\frac{8}{9}(1+x)H_0P_{1212}\Big)\zeta_3+16xP_{1129}\zeta_5\Big)+\Big(\frac{2}{27}\Big(2107+4454x+2107x^2\Big)-64xH_1\Big)x_+^2\Big\}\Big]\,. \tag{D.2}$$

The polynomials are listed below

$$\begin{aligned}
P_{1061} &= -62x^4+74x^3+53x^2+2x-11\,,\\
P_{1062} &= -32x^4+56x^3+49x^2-11\,,\\
P_{1063} &= x^4-102x^3+42x^2-102x+1\,,\\
P_{1064} &= x^4-42x^3+18x^2-42x+1\,,\\
P_{1065} &= x^4-28x^3+6x^2-28x+1\,,\\
P_{1066} &= x^4-20x^3+6x^2-20x+1\,,\\
P_{1067} &= x^4+2x^3+4x^2+2x+1\,,\\
P_{1068} &= x^4+2x^3+6x^2+2x+1\,,\\
P_{1069} &= x^4+2x^3+8x^2+2x+1\,,\\
P_{1070} &= x^4+2x^3+14x^2+2x+1\,,\\
P_{1071} &= x^4+2x^3+35x^2+2x+1\,,\\
P_{1072} &= x^4+4x^3-50x^2+4x+1\,,\\
P_{1073} &= x^4+4x^3-26x^2+4x+1\,,\\
P_{1074} &= x^4+4x^3-18x^2+4x+1\,,\\
P_{1075} &= x^4+4x^3+22x^2+4x+1\,,\\
P_{1076} &= x^4+4x^3+38x^2+4x+1\,,\\
P_{1077} &= x^4+4x^3+46x^2+4x+1\,,\\
P_{1078} &= x^4+4x^3+102x^2+4x+1\,,\\
P_{1079} &= 2x^4-194x^3-113x^2-2x+11\,,\\
P_{1080} &= 2x^4-19x^3+26x^2+61x-18\,,\\
P_{1081} &= 2x^4+2x^3+7x^2+2x+2\,,\\
P_{1082} &= 2x^4+4x^3-x^2+4x+2\,,\\
P_{1083} &= 2x^4+6x^3-11x^2+15x-4\,,\\
P_{1084} &= 3x^4-20x^3+2x^2-20x+3\,,\\
P_{1085} &= 3x^4+4x^3+20x^2+4x+3\,,\\
P_{1086} &= 3x^4+6x^3+x^2+6x+3\,,\\
P_{1087} &= 3x^4+12x^3-206x^2+12x+3\,,\\
P_{1088} &= 3x^4+12x^3-110x^2+12x+3\,,\\
P_{1089} &= 3x^4+12x^3-62x^2+12x+3\,,\\
P_{1090} &= 3x^4+12x^3-38x^2+12x+3\,,\\
P_{1091} &= 3x^4+12x^3-14x^2+12x+3\,,
\end{aligned}$$

$$P_{1092} = 3x^4 + 12x^3 + 2x^2 + 12x + 3\,,$$
$$P_{1093} = 3x^4 + 12x^3 + 34x^2 + 12x + 3\,,$$
$$P_{1094} = 4x^4 - 15x^3 - 2x^2 - 15x + 4\,,$$
$$P_{1095} = 4x^4 - 10x^3 + 73x^2 - 10x + 4\,,$$
$$P_{1096} = 4x^4 + 9x^3 + 6x^2 + 9x + 4\,,$$
$$P_{1097} = 5x^4 - 62x^3 + 74x^2 - 70x + 5\,,$$
$$P_{1098} = 5x^4 - 54x^3 + 2x^2 - 54x + 5\,,$$
$$P_{1099} = 5x^4 - 31x^3 + 4x^2 - 31x + 5\,,$$
$$P_{1100} = 5x^4 - 16x^3 + 6x^2 - 16x + 5\,,$$
$$P_{1101} = 5x^4 - 12x^3 + 14x^2 - 4x + 1\,,$$
$$P_{1102} = 5x^4 + 10x^3 + x^2 + 10x + 5\,,$$
$$P_{1103} = 5x^4 + 10x^3 + 17x^2 + 10x + 5\,,$$
$$P_{1104} = 5x^4 + 10x^3 + 43x^2 + 10x + 5\,,$$
$$P_{1105} = 5x^4 + 17x^3 - 22x^2 + 17x + 7\,,$$
$$P_{1106} = 5x^4 + 20x^3 - 386x^2 + 20x + 5\,,$$
$$P_{1107} = 5x^4 + 20x^3 - 162x^2 + 20x + 5\,,$$
$$P_{1108} = 5x^4 + 20x^3 - 138x^2 + 20x + 5\,,$$
$$P_{1109} = 5x^4 + 20x^3 - 66x^2 + 20x + 5\,,$$
$$P_{1110} = 5x^4 + 20x^3 + 46x^2 + 20x + 5\,,$$
$$P_{1111} = 5x^4 + 20x^3 + 254x^2 + 20x + 5\,,$$
$$P_{1112} = 6x^4 - 17x^3 + x^2 - 14x - 8\,,$$
$$P_{1113} = 6x^4 - 9x^3 + 17x^2 - 2x + 4\,,$$
$$P_{1114} = 7x^4 - 146x^3 + 54x^2 - 146x + 7\,,$$
$$P_{1115} = 7x^4 - 28x^3 + 10x^2 - 28x + 7\,,$$
$$P_{1116} = 7x^4 + 28x^3 - 214x^2 + 28x + 7\,,$$
$$P_{1117} = 7x^4 + 28x^3 - 6x^2 + 28x + 7\,,$$
$$P_{1118} = 7x^4 + 50x^3 + 146x^2 + 50x + 7\,,$$
$$P_{1119} = 7x^4 + 482x^3 + 1010x^2 + 482x + 7\,,$$
$$P_{1120} = 9x^4 - 230x^3 + 90x^2 - 230x + 9\,,$$
$$P_{1121} = 9x^4 - 40x^3 + 14x^2 - 40x + 9\,,$$
$$P_{1122} = 9x^4 + 58x^3 - 2x^2 + 58x + 5\,,$$
$$P_{1123} = 11x^4 - 146x^3 + 82x^2 - 146x + 7\,,$$
$$P_{1124} = 11x^4 - 2x^3 + 14x^2 - 2x + 11\,,$$
$$P_{1125} = 11x^4 + 6x^3 + 77x^2 + 6x + 11\,,$$
$$P_{1126} = 11x^4 + 22x^3 - 82x^2 + 22x + 11\,,$$
$$P_{1127} = 11x^4 + 44x^3 - 126x^2 + 44x + 11\,,$$
$$P_{1128} = 11x^4 + 44x^3 - 14x^2 + 44x + 11\,,$$
$$P_{1129} = 11x^4 + 44x^3 + 102x^2 + 44x + 11\,,$$
$$P_{1130} = 13x^4 - 50x^3 + 82x^2 - 66x + 5\,,$$
$$P_{1131} = 13x^4 + 32x^3 + 12x^2 + 32x + 13\,,$$

$$P_{1132} = 13x^4 + 52x^3 + 14x^2 + 52x + 13\,,$$
$$P_{1133} = 13x^4 + 52x^3 + 46x^2 + 52x + 13\,,$$
$$P_{1134} = 14x^4 + 22x^3 + 57x^2 + 22x + 14\,,$$
$$P_{1135} = 15x^4 - 60x^3 + 66x^2 + 4x - 17\,,$$
$$P_{1136} = 17x^4 - 254x^3 + 232x^2 - 18x + 7\,,$$
$$P_{1137} = 17x^4 - 64x^3 - 188x^2 + 256x + 15\,,$$
$$P_{1138} = 21x^4 + 185x^3 + 131x^2 + 73x - 16\,,$$
$$P_{1139} = 23x^4 + 102x^3 + 10x^2 + 102x + 19\,,$$
$$P_{1140} = 25x^4 - 20x^3 + 56x^2 - 148x - 33\,,$$
$$P_{1141} = 25x^4 + 24x^3 + 148x^2 - 200x - 21\,,$$
$$P_{1142} = 26x^4 - 73x^3 + 51x^2 - 15x + 3\,,$$
$$P_{1143} = 33x^4 - 344x^3 - 1006x^2 + 616x + 89\,,$$
$$P_{1144} = 37x^4 - 16x^3 - 98x^2 + 48x + 5\,,$$
$$P_{1145} = 39x^4 - 4x^3 + 104x^2 - 164x - 27\,,$$
$$P_{1146} = 47x^4 + 268x^3 - 263x^2 + 40x + 38\,,$$
$$P_{1147} = 53x^4 + 212x^3 - 322x^2 + 212x + 53\,,$$
$$P_{1148} = 55x^4 - 138x^3 + 70x^2 - 138x + 55\,,$$
$$P_{1149} = 55x^4 + 220x^3 - 86x^2 + 220x + 55\,,$$
$$P_{1150} = 67x^4 - 98x^3 + 88x^2 - 2x - 11\,,$$
$$P_{1151} = 75x^4 - 50x^3 - 98x^2 - 50x + 75\,,$$
$$P_{1152} = 87x^4 + 4x^3 + 206x^2 + 388x + 35\,,$$
$$P_{1153} = 95x^4 - 404x^3 + 496x^2 + 820x - 103\,,$$
$$P_{1154} = 115x^4 + 100x^3 + 116x^2 - 364x - 43\,,$$
$$P_{1155} = 124x^4 + 570x^3 + 801x^2 - 1158x + 325\,,$$
$$P_{1156} = 133x^4 + 216x^3 + 520x^2 - 440x - 61\,,$$
$$P_{1157} = 233x^4 - 388x^3 - 52x^2 + 56x + 55\,,$$
$$P_{1158} = 379x^4 + 263x^3 + 569x^2 + 515x - 134\,,$$
$$P_{1159} = 683x^4 - 672x^3 - 1138x^2 + 744x + 239\,,$$
$$P_{1160} = 895x^4 + 3806x^3 + 2288x^2 - 2878x - 1007\,,$$
$$P_{1161} = -117x^5 - 117x^4 + 80x^3 - 628x^2 - 157x - 157\,,$$
$$P_{1162} = -71x^5 - 71x^4 - 32x^3 - 116x^2 - 3x - 3\,,$$
$$P_{1163} = x^5 - 73x^4 - 216x^3 + 72x^2 - 73x + 1\,,$$
$$P_{1164} = x^5 - 17x^4 + 96x^3 - 80x^2 + 25x + 7\,,$$
$$P_{1165} = x^5 - 16x^4 + 15x^3 - 9x^2 - 16x + 1\,,$$
$$P_{1166} = x^5 - 13x^4 - 100x^3 + 172x^2 - 13x + 1\,,$$
$$P_{1167} = x^5 - 13x^4 + 180x^3 - 204x^2 - 13x + 1\,,$$
$$P_{1168} = x^5 - 7x^4 + 138x^3 - 150x^2 - 7x + 1\,,$$
$$P_{1169} = x^5 - 5x^4 - 156x^3 + 100x^2 - 5x + 1\,,$$
$$P_{1170} = x^5 - 3x^4 + 22x^3 - 18x^2 + 5x + 1\,,$$
$$P_{1171} = x^5 - x^4 + 12x^3 - 8x^2 + 3x + 1\,,$$

$$P_{1172} = x^5 + x^4 + 5x^3 - 7x^2 - 2x - 2\,,$$
$$P_{1173} = x^5 + x^4 + 16x^3 - 8x^2 + 3x + 3\,,$$
$$P_{1174} = x^5 + 2x^4 - 3x^3 + 7x^2 + 1\,,$$
$$P_{1175} = x^5 + 3x^4 - 8x^3 + 12x^2 - x + 1\,,$$
$$P_{1176} = x^5 + 5x^4 - 21x^3 + 5x^2 - 13x - 9\,,$$
$$P_{1177} = x^5 + 5x^4 - 2x^3 + 14x^2 + x + 5\,,$$
$$P_{1178} = x^5 + 9x^4 - 28x^3 + 24x^2 - 11x - 3\,,$$
$$P_{1179} = x^5 + 11x^4 + 6x^3 + 34x^2 + 9x + 19\,,$$
$$P_{1180} = x^5 + 14x^4 - 57x^3 + 47x^2 - 19x - 6\,,$$
$$P_{1181} = x^5 + 21x^4 - 58x^3 + 54x^2 - 23x - 3\,,$$
$$P_{1182} = x^5 + 35x^4 - 27x^3 + 271x^2 + 87x + 121\,,$$
$$P_{1183} = 2x^5 - 41x^4 - 141x^3 + 147x^2 - 41x + 2\,,$$
$$P_{1184} = 2x^5 + x^4 + 9x^3 - 7x^2 - 1\,,$$
$$P_{1185} = 2x^5 + 2x^4 - 11x^3 + 7x^2 - 4x - 4\,,$$
$$P_{1186} = 2x^5 + 2x^4 + 5x^3 + 3x^2 + 2x + 2\,,$$
$$P_{1187} = 2x^5 + 2x^4 + 5x^3 + 11x^2 + 6x + 6\,,$$
$$P_{1188} = 2x^5 + 2x^4 + 7x^3 - 5x^2 - x - 1\,,$$
$$P_{1189} = 2x^5 + 2x^4 + 15x^3 - 19x^2 - 4x - 4\,,$$
$$P_{1190} = 2x^5 + 4x^4 + x^3 + 23x^2 + 8x + 10\,,$$
$$P_{1191} = 2x^5 + 17x^4 - 56x^3 + 42x^2 - 24x - 9\,,$$
$$P_{1192} = 2x^5 + 20x^4 - 51x^3 + 11x^2 - 40x - 22\,,$$
$$P_{1193} = 2x^5 + 34x^4 - 39x^3 - 145x^2 - 77x + 1\,,$$
$$P_{1194} = 3x^5 - 17x^4 + 54x^3 - 50x^2 + 19x - 1\,,$$
$$P_{1195} = 3x^5 - 3x^4 + 15x^3 - 11x^2 + 5x - 1\,,$$
$$P_{1196} = 3x^5 - 3x^4 + 22x^3 - 14x^2 + 7x + 1\,,$$
$$P_{1197} = 3x^5 - x^4 + 35x^3 + 13x^2 + 25x + 21\,,$$
$$P_{1198} = 3x^5 + 3x^4 + 5x^3 - 7x^2 - 4x - 4\,,$$
$$P_{1199} = 3x^5 + 7x^4 - 14x^3 + 26x^2 - x + 3\,,$$
$$P_{1200} = 3x^5 + 9x^4 - 10x^3 + 26x^2 - x + 5\,,$$
$$P_{1201} = 3x^5 + 129x^4 - 40x^3 - 176x^2 - 271x + 35\,,$$
$$P_{1202} = 4x^5 + 4x^4 - x^3 - 15x^2 - 12x - 12\,,$$
$$P_{1203} = 5x^5 - x^4 + 56x^3 - 72x^2 - 7x - 13\,,$$
$$P_{1204} = 5x^5 + x^4 + 30x^3 - 34x^2 - 3x - 7\,,$$
$$P_{1205} = 5x^5 + 5x^4 - 7x^3 + 11x^2 - 3x - 3\,,$$
$$P_{1206} = 5x^5 + 5x^4 - x^3 - 3x^2 - 7x - 7\,,$$
$$P_{1207} = 5x^5 + 5x^4 + 16x^3 - 8x^2 - x - 1\,,$$
$$P_{1208} = 5x^5 + 5x^4 + 17x^3 - 25x^2 - 9x - 9\,,$$
$$P_{1209} = 5x^5 + 7x^4 - 47x^3 + 11x^2 - 25x - 23\,,$$
$$P_{1210} = 6x^5 - 44x^4 - 27x^3 + 47x^2 - 11x - 3\,,$$
$$P_{1211} = 6x^5 + 6x^4 + 11x^3 - 3x^2 - 2x - 2\,,$$

$$P_{1212} = 7x^5 - 133x^4 - 234x^3 + 630x^2 - 133x + 7\,,$$
$$P_{1213} = 7x^5 - 25x^4 + 96x^3 - 92x^2 + 27x - 5\,,$$
$$P_{1214} = 7x^5 - x^4 + 22x^3 - 10x^2 + 7x - 1\,,$$
$$P_{1215} = 7x^5 + 2x^4 + 12x^3 - 2x^2 + 3x - 2\,,$$
$$P_{1216} = 7x^5 + 5x^4 + 17x^3 - 5x^2 + x - 1\,,$$
$$P_{1217} = 7x^5 + 7x^4 - 20x^3 + 16x^2 - 9x - 9\,,$$
$$P_{1218} = 7x^5 + 7x^4 - 16x^3 + 20x^2 - 5x - 5\,,$$
$$P_{1219} = 7x^5 + 7x^4 + 68x^3 - 4x^2 + 25x + 25\,,$$
$$P_{1220} = 8x^5 - 36x^4 + 191x^3 - 155x^2 + 54x + 10\,,$$
$$P_{1221} = 8x^5 + 8x^4 + 11x^3 - 15x^2 - 10x - 10\,,$$
$$P_{1222} = 9x^5 + 9x^4 + 10x^3 - 4x^2 - 6x - 6\,,$$
$$P_{1223} = 10x^5 - 9x^3 + 65x^2 + 31x - 1\,,$$
$$P_{1224} = 10x^5 + 14x^4 - 27x^3 + 31x^2 - 12x - 8\,,$$
$$P_{1225} = 10x^5 + 14x^4 - 21x^3 + 25x^2 - 12x - 8\,,$$
$$P_{1226} = 10x^5 + 197x^4 - 200x^3 - 442x^2 + 54x - 51\,,$$
$$P_{1227} = 11x^5 - 131x^4 + 90x^3 - 58x^2 - 197x + 29\,,$$
$$P_{1228} = 11x^5 - 25x^4 + 173x^3 - 141x^2 + 41x + 5\,,$$
$$P_{1229} = 11x^5 + 11x^4 + 83x^3 - 19x^2 + 21x + 21\,,$$
$$P_{1230} = 11x^5 + 43x^4 - 112x^3 + 116x^2 - 41x - 9\,,$$
$$P_{1231} = 11x^5 + 245x^4 + 736x^3 - 384x^2 + 261x - 37\,,$$
$$P_{1232} = 12x^5 + 12x^4 + 25x^3 + 43x^2 + 22x + 22\,,$$
$$P_{1233} = 12x^5 + 44x^4 - 45x^3 + 33x^2 - 50x - 18\,,$$
$$P_{1234} = 13x^5 - 81x^4 - 280x^3 + 136x^2 - 57x + 13\,,$$
$$P_{1235} = 13x^5 + x^4 + 65x^3 - 57x^2 + 3x - 9\,,$$
$$P_{1236} = 13x^5 + 13x^4 - 35x^3 + 67x^2 + 3x + 3\,,$$
$$P_{1237} = 13x^5 + 13x^4 - 7x^3 + 11x^2 - 11x - 11\,,$$
$$P_{1238} = 13x^5 + 13x^4 + 19x^3 - 27x^2 - 17x - 17\,,$$
$$P_{1239} = 13x^5 + 13x^4 + 83x^3 - 19x^2 + 19x + 19\,,$$
$$P_{1240} = 13x^5 + 113x^4 - 330x^3 + 286x^2 - 135x - 35\,,$$
$$P_{1241} = 13x^5 + 113x^4 - 322x^3 + 238x^2 - 155x - 55\,,$$
$$P_{1242} = 14x^5 + 14x^4 - x^3 + 13x^2 - 8x - 8\,,$$
$$P_{1243} = 15x^5 + 15x^4 + 26x^3 - 22x^2 - 13x - 13\,,$$
$$P_{1244} = 15x^5 + 27x^4 - 53x^3 + 57x^2 - 25x - 13\,,$$
$$P_{1245} = 16x^5 - 7x^4 - 101x^3 + 55x^2 - 15x + 4\,,$$
$$P_{1246} = 16x^5 + 16x^4 + 27x^3 - 19x^2 - 12x - 12\,,$$
$$P_{1247} = 17x^5 + 7x^4 + 84x^3 - 52x^2 + 9x - 1\,,$$
$$P_{1248} = 17x^5 + 17x^4 - 20x^3 + 52x^2 - x - 1\,,$$
$$P_{1249} = 17x^5 + 17x^4 + 54x^3 - 56x^2 - 18x - 18\,,$$
$$P_{1250} = 17x^5 + 37x^4 - 78x^3 + 66x^2 - 43x - 23\,,$$
$$P_{1251} = 17x^5 + 41x^4 - 162x^3 + 150x^2 + 25x - 7\,,$$

$$P_{1252} = 17x^5 + 43x^4 - 87x^3 + 83x^2 - 45x - 19\,,$$
$$P_{1253} = 17x^5 + 135x^4 + 134x^3 + 154x^2 - 39x - 17\,,$$
$$P_{1254} = 18x^5 + 8x^4 + 21x^3 - 9x^2 - 2x - 12\,,$$
$$P_{1255} = 18x^5 + 18x^4 + 7x^3 - 11x^2 - 20x - 20\,,$$
$$P_{1256} = 19x^5 + 19x^4 - 80x^3 + 60x^2 - 29x - 29\,,$$
$$P_{1257} = 19x^5 + 19x^4 + 43x^3 - 51x^2 - 23x - 23\,,$$
$$P_{1258} = 19x^5 + 19x^4 + 54x^3 - 42x^2 - 13x - 13\,,$$
$$P_{1259} = 19x^5 + 19x^4 + 68x^3 - 4x^2 + 13x + 13\,,$$
$$P_{1260} = 19x^5 + 31x^4 - 170x^3 + 142x^2 + 15x - 5\,,$$
$$P_{1261} = 19x^5 + 67x^4 - 192x^3 + 116x^2 - 105x - 57\,,$$
$$P_{1262} = 19x^5 + 4963x^4 + 5538x^3 - 1062x^2 - 781x - 37\,,$$
$$P_{1263} = 20x^5 + 20x^4 + 95x^3 + 53x^2 + 54x + 54\,,$$
$$P_{1264} = 20x^5 + 20x^4 + 263x^3 - 311x^2 - 44x - 44\,,$$
$$P_{1265} = 20x^5 + 49x^4 - 87x^3 + 89x^2 - 48x - 19\,,$$
$$P_{1266} = 21x^5 + 21x^4 - 20x^3 + 40x^2 - 11x - 11\,,$$
$$P_{1267} = 21x^5 + 21x^4 + 14x^3 - 6x^2 - 17x - 17\,,$$
$$P_{1268} = 21x^5 + 21x^4 + 48x^3 - 56x^2 - 25x - 25\,,$$
$$P_{1269} = 22x^5 + 22x^4 + 61x^3 - 57x^2 - 20x - 20\,,$$
$$P_{1270} = 22x^5 + 22x^4 + 95x^3 + 53x^2 + 52x + 52\,,$$
$$P_{1271} = 23x^5 - 52x^4 + 50x^3 - 74x^2 + 43x - 6\,,$$
$$P_{1272} = 23x^5 + 14x^4 + 109x^3 - 35x^2 + 23x + 14\,,$$
$$P_{1273} = 23x^5 + 23x^4 - 152x^3 + 196x^2 - x - 1\,,$$
$$P_{1274} = 23x^5 + 113x^4 + 300x^3 - 156x^2 + 53x - 13\,,$$
$$P_{1275} = 24x^5 + 24x^4 + 123x^3 - 119x^2 - 22x - 22\,,$$
$$P_{1276} = 25x^5 - 329x^4 - 524x^3 + 164x^2 - 233x + 1\,,$$
$$P_{1277} = 25x^5 + 25x^4 - 16x^3 + 20x^2 - 23x - 23\,,$$
$$P_{1278} = 25x^5 + 25x^4 - 4x^3 + 8x^2 - 23x - 23\,,$$
$$P_{1279} = 25x^5 + 25x^4 + 56x^3 + 20x^2 + 13x + 13\,,$$
$$P_{1280} = 27x^5 + 27x^4 - 12x^3 - 32x^2 - 49x - 49\,,$$
$$P_{1281} = 27x^5 + 27x^4 + 52x^3 + 16x^2 + 7x + 7\,,$$
$$P_{1282} = 29x^5 + 29x^4 - 104x^3 + 100x^2 - 31x - 31\,,$$
$$P_{1283} = 31x^5 + 31x^4 - 104x^3 + 100x^2 - 33x - 33\,,$$
$$P_{1284} = 31x^5 + 31x^4 - 32x^3 + 44x^2 - 25x - 25\,,$$
$$P_{1285} = 31x^5 + 31x^4 - 30x^3 + 26x^2 - 33x - 33\,,$$
$$P_{1286} = 31x^5 + 31x^4 + 28x^3 + 8x^2 - 13x - 13\,,$$
$$P_{1287} = 32x^5 + 32x^4 + 223x^3 - 251x^2 - 46x - 46\,,$$
$$P_{1288} = 34x^5 + 34x^4 + 31x^3 + 13x^2 - 12x - 12\,,$$
$$P_{1289} = 35x^5 + 35x^4 + 106x^3 - 86x^2 - 25x - 25\,,$$
$$P_{1290} = 37x^5 + 5x^4 + 243x^3 - 179x^2 + 27x - 5\,,$$
$$P_{1291} = 38x^5 + 38x^4 + 229x^3 - 209x^2 - 28x - 28\,,$$

$$
\begin{aligned}
P_{1292} &= 38x^5 + 55x^4 + 131x^3 - 85x^2 + 9x - 4\,,\\
P_{1293} &= 39x^5 + 189x^4 + 258x^3 + 110x^2 + 119x + 53\,,\\
P_{1294} &= 40x^5 + 40x^4 + 87x^3 - 3x^2 + 2x + 2\,,\\
P_{1295} &= 42x^5 + 42x^4 + 59x^3 - 27x^2 - 26x - 26\,,\\
P_{1296} &= 42x^5 + 48x^4 + 59x^3 + 29x^2 - 4x + 2\,,\\
P_{1297} &= 43x^5 + 27x^4 - 48x^3 + 140x^2 + 19x + 3\,,\\
P_{1298} &= 43x^5 + 43x^4 + 96x^3 + 20x^2 + 15x + 15\,,\\
P_{1299} &= 45x^5 + 101x^4 - 211x^3 + 107x^2 + 16x + 6\,,\\
P_{1300} &= 50x^5 + 98x^4 + 17x^3 + 59x^2 - 60x - 12\,,\\
P_{1301} &= 51x^5 + 51x^4 - 130x^3 + 110x^2 - 61x - 61\,,\\
P_{1302} &= 55x^5 + 211x^4 - 576x^3 + 732x^2 - 91x + 53\,,\\
P_{1303} &= 56x^5 - 48x^4 + 425x^3 - 573x^2 - 26x - 130\,,\\
P_{1304} &= 65x^5 + 217x^4 - 264x^3 + 308x^2 - 195x - 43\,,\\
P_{1305} &= 67x^5 + 67x^4 + 134x^3 - 94x^2 - 47x - 47\,,\\
P_{1306} &= 74x^5 - 99x^4 - 152x^3 + 192x^2 - 70x + 7\,,\\
P_{1307} &= 77x^5 - 35x^4 + 516x^3 - 368x^2 + 109x - 3\,,\\
P_{1308} &= 96x^5 + 96x^4 + 437x^3 - 317x^2 - 36x - 36\,,\\
P_{1309} &= 98x^5 + 98x^4 + 495x^3 - 407x^2 - 54x - 54\,,\\
P_{1310} &= 119x^5 + 119x^4 + 203x^3 - 67x^2 - 51x - 51\,,\\
P_{1311} &= 139x^5 + 207x^4 + 169x^3 + 95x^2 - 75x - 7\,,\\
P_{1312} &= 141x^5 + 241x^4 + 79x^3 + 177x^2 - 113x - 13\,,\\
P_{1313} &= 147x^5 - 63x^4 + 894x^3 - 550x^2 + 203x - 55\,,\\
P_{1314} &= 181x^5 + 113x^4 + 420x^3 - 176x^2 + 9x - 59\,,\\
P_{1315} &= 202x^5 + 1240x^4 - 3283x^3 + 679x^2 + 583x - 157\,,\\
P_{1316} &= 649x^5 + 377x^4 + 1488x^3 - 84x^2 + 325x + 53\,,\\
P_{1317} &= 5x^6 - 530x^5 - 3241x^4 - 4744x^3 - 1585x^2 + 298x + 5\,,\\
P_{1318} &= 5x^6 - 386x^5 - 1009x^4 - 640x^3 + 503x^2 + 370x + 5\,,\\
P_{1319} &= 5x^6 - 224x^5 - 577x^4 - 604x^3 - 73x^2 + 28x + 5\,,\\
P_{1320} &= 5x^6 - 170x^5 - 19x^4 + 314x^3 + 305x^2 - 8x + 5\,,\\
P_{1321} &= 5x^6 - 116x^5 + 323x^4 + 728x^3 + 323x^2 - 116x + 5\,,\\
P_{1322} &= 5x^6 - 44x^5 + 755x^4 + 1304x^3 + 467x^2 - 188x + 5\,,\\
P_{1323} &= 5x^6 + 40x^5 - 157x^4 - 640x^3 - 349x^2 - 56x + 5\,,\\
P_{1324} &= 8x^6 + 25x^5 - 100x^4 - 154x^3 - 100x^2 + 25x + 8\,,\\
P_{1325} &= 15x^6 - 183x^5 - 1323x^4 - 1604x^3 - 121x^2 + 43x + 5\,,\\
P_{1326} &= 123x^6 - 492x^5 - 2159x^4 - 1820x^3 - 1223x^2 - 24x + 123\,. \qquad \text{(D.3)}
\end{aligned}
$$

E The pseudo-scalar form factors up to two-loop

The non-singlet part of the pseudo-scalar form factor can be obtained using Eq. (2.13). The unrenormalized singlet contribution is given up to $\mathcal{O}(\varepsilon)$ by

$$\begin{aligned}
\hat{F}_P^{(2),\mathrm{s}} = C_F T_F \Bigg[& \Bigg\{ \eta^2 (8x(1+x)^2 H_0^2 + 32x(1+x)^2\zeta_2) + \eta^3 (\frac{32}{3}x^2(1+x)^2 H_0^3 + 64x^2(1+x)^2 H_0\zeta_2) + \eta(\\
& - \frac{1}{3}xH_0^4 + \frac{16}{3}xH_0^3 H_1 - 64xH_0 H_{0,0,1} + 112xH_{0,0,0,1} + (- 8xH_0^2 + 32xH_0 H_1 - 32xH_{0,1})\zeta_2 \\
& - \frac{64}{5}x\zeta_2^2 - 48xH_0\zeta_3) \Bigg\} + \varepsilon \Bigg\{ \xi\eta^3 (- (-1+x)x(1+x)^3 H_0^4 + 32(-1+x)x(1+x)^3 H_0^2 H_{0,1}) \\
& + \eta^3 (\frac{32}{5}x(1+x)^2(7-13x+7x^2)\zeta_2^2 - \frac{8}{3}x(1+x)^2(-3+11x^2)H_0^3 + \frac{64}{3}x^2(1+x)^2 H_{-1}H_0^3 \\
& + 6x^2(1+x)^2 H_0^4 - \frac{16}{3}x(1+x)^2(5-6x+5x^2)H_0^3 H_1 + 128x^2(1+x)^2 H_0^2 H_{0,1} \\
& - 128x^2(1+x)^2 H_0^2 H_{0,-1} - 16x(1+x)^2(53-90x+53x^2)H_{0,0,0,1} + 128x(1+x)^2(3 \\
& - x + 3x^2)H_{0,0,0,-1} + (- 16x(1+x)^2(-1+3x)(1+3x)H_0 + 128x^2(1+x)^2 H_{-1}H_0 \\
& - 32x(1+x)^2(5-6x+5x^2)H_0 H_1 + 32x(1+x)^2(5-6x+5x^2)H_{0,1} - 512x^2(1+x)^2 H_{0,-1})\zeta_2) \\
& + \eta^2 (16x(1+x)^2 H_0^2 - 32x(1+x)^2 H_{-1}H_0^2 + 16x(1+x)^2 H_0^2 H_1 - 32x(1+x)^2 H_0 H_{0,-1} \\
& - 32x(1+x)^2 H_{0,0,1} + 128x(1+x)^2 H_{0,0,-1} - 192x(1+x)^2 H_{-1}\zeta_2 + 48x(1+x)^2\zeta_3) \\
& + \eta(- \frac{2}{3}xH_{-1}H_0^4 - \frac{4}{15}xH_0^5 + (\frac{32}{3}xH_{-1}H_0^3 + \frac{10}{3}xH_0^4)H_1 - \frac{16}{3}xH_0^3 H_1^2 + (- 16xH_0^3 \\
& + 64xH_0 H_{0,-1})H_{0,1} - 64xH_0 H_{0,1}^2 + (8xH_0^3 - 64xH_0^2 H_1)H_{0,-1} + ((448x - 128xH_{-1})H_0 \\
& - 32xH_0^2 + 128xH_0 H_1 + 192xH_{0,1} - 96xH_{0,-1})H_{0,0,1} + (- 128xH_0 - 32xH_0^2 - 128xH_0 H_1 \\
& - 192xH_{0,1})H_{0,0,-1} + 32xH_0^2 H_{0,1,1} + 32xH_0^2 H_{0,1,-1} + 32xH_0^2 H_{0,-1,1} + (352xH_0 - 416xH_1 \\
& + 224xH_{-1})H_{0,0,0,1} + (- 160xH_0 + 704xH_1)H_{0,0,0,-1} - 128xH_0 H_{0,0,1,1} + 256xH_0 H_{0,0,1,-1} \\
& + 256xH_0 H_{0,0,-1,1} + 64xH_0 H_{0,-1,0,1} - 880xH_{0,0,0,0,1} + 848xH_{0,0,0,0,-1} - 160xH_{0,0,0,1,1} \\
& - 320xH_{0,0,0,1,-1} - 320xH_{0,0,0,-1,1} - 192xH_{0,0,1,0,1} - 32xH_{0,0,1,0,-1} + 128xH_{0,0,-1,0,1} \\
& + (8xH_0^2 - 16xH_{-1}H_0^2 - 4xH_0^3 + (64xH_{-1}H_0 - 32xH_0^2)H_1 - 32xH_0 H_1^2 + (- 96xH_0 \\
& + 64xH_1 - 64xH_{-1})H_{0,1} + (64xH_0 - 256xH_1)H_{0,-1} + 224xH_{0,0,1} - 256xH_{0,0,-1} \\
& - 64xH_{0,1,1} + 256xH_{0,1,-1} + 256xH_{0,-1,1} - 16x\zeta_3)\zeta_2 + (8xH_0 - 16xH_1 - \frac{128}{5}xH_{-1})\zeta_2^2 \\
& + (208xH_0 - 96xH_{-1}H_0 - 52xH_0^2 - 32xH_0 H_1 - 160xH_{0,1} + 96xH_{0,-1})\zeta_3 + 176x\zeta_5) \Bigg\} \Bigg] .
\end{aligned} \tag{E.1}$$

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